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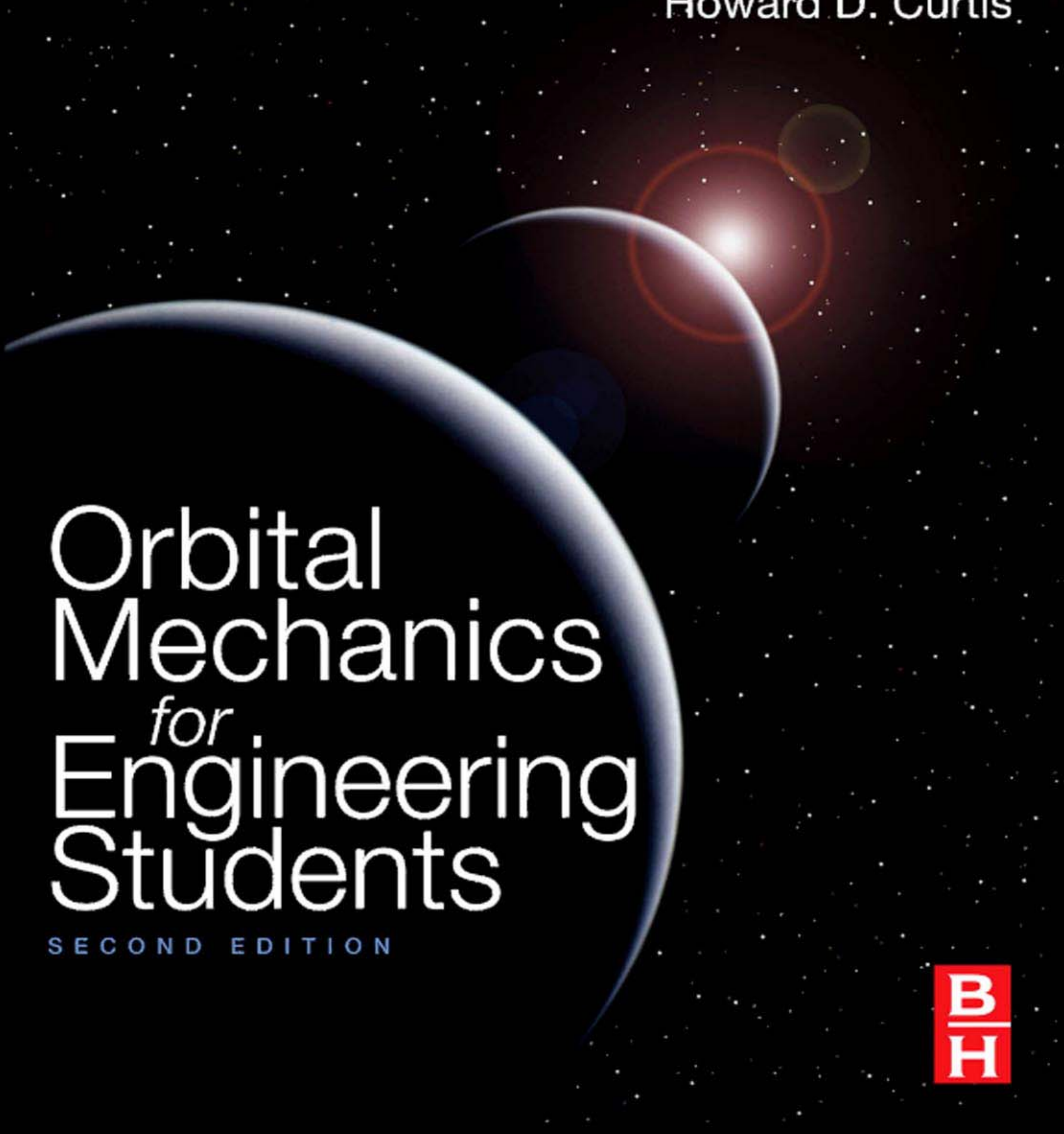

ELSEVIER

Howard D. Curtis

Orbital Mechanics *for* Engineering Students

SECOND EDITION

Orbital Mechanics for Engineering Students

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Orbital Mechanics for Engineering Students

Second Edition

Howard D. Curtis

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To my parents, Rondo and Geraldine.

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Contents

Preface . . . xi
Acknowledgments . . . xv

CHAPTER 1 Dynamics of point masses . . . 1
1.1 Introduction . . . 1
1.2 Vectors . . . 2
1.3 Kinematics . . . 10
1.4 Mass, force and Newton's law of gravitation . . . 15
1.5 Newton's law of motion . . . 19
1.6 Time derivatives of moving vectors . . . 24
1.7 Relative motion . . . 29
1.8 Numerical integration . . . 38
1.8.1 Runge-Kutta methods . . . 42
1.8.2 Heun's Predictor-Corrector method . . . 48
1.8.3 Runge-Kutta with variable step size . . . 50
Problems . . . 54
List of Key Terms . . . 59

CHAPTER 2 The two-body problem . . . 61
2.1 Introduction . . . 61
2.2 Equations of motion in an inertial frame . . . 62
2.3 Equations of relative motion . . . 70
2.4 Angular momentum and the orbit formulas . . . 74
2.5 The energy law . . . 82
2.6 Circular orbits ($e = 0$) . . . 83
2.7 Elliptical orbits ($0 < e < 1$) . . . 89
2.8 Parabolic trajectories ($e = 1$) . . . 100
2.9 Hyperbolic trajectories ($e > 1$) . . . 104
2.10 Perifocal frame . . . 113
2.11 The lagrange coefficients . . . 117
2.12 Restricted three-body problem . . . 129
2.12.1 Lagrange points . . . 133
2.12.2 Jacobi constant . . . 139
Problems . . . 146
List of Key Terms . . . 152

CHAPTER 3 Orbital position as a function of time **155**
3.1 Introduction 155
3.2 Time since periapsis 155
3.3 Circular orbits ($e = 0$) 156
3.4 Elliptical orbits ($e < 1$) 157
3.5 Parabolic trajectories ($e = 1$) 172
3.6 Hyperbolic trajectories ($e < 1$) 174
3.7 Universal variables 182
Problems 194
List of Key Terms 197

CHAPTER 4 Orbits in three dimensions **199**
4.1 Introduction 199
4.2 Geocentric right ascension-declination frame 200
4.3 State vector and the geocentric equatorial frame 203
4.4 Orbital elements and the state vector 208
4.5 Coordinate transformation 216
4.6 Transformation between geocentric equatorial and perifocal frames 229
4.7 Effects of the Earth's oblateness 233
4.8 Ground tracks 244
Problems 249
List of Key Terms 254

CHAPTER 5 Preliminary orbit determination **255**
5.1 Introduction 255
5.2 Gibbs method of orbit determination from three position vectors 256
5.3 Lambert's problem 263
5.4 Sidereal time 275
5.5 Topocentric coordinate system 280
5.6 Topocentric equatorial coordinate system 283
5.7 Topocentric horizon coordinate system 284
5.8 Orbit determination from angle and range measurements 289
5.9 Angles only preliminary orbit determination 297
5.10 Gauss method of preliminary orbit determination 297
Problems 312
List of Key Terms 317

CHAPTER 6 Orbital maneuvers **319**
6.1 Introduction 319
6.2 Impulsive maneuvers 320
6.3 Hohmann transfer 321
6.4 Bi-elliptic Hohmann transfer 328
6.5 Phasing maneuvers 332
6.6 Non-Hohmann transfers with a common apse line 338
6.7 Apse line rotation 343
6.8 Chase maneuvers 350

6.9 Plane change maneuvers ... 355
6.10 Nonimpulsive orbital maneuvers ... 368
Problems ... 374
List of Key Terms ... 390

CHAPTER 7 Relative motion and rendezvous ... **391**
7.1 Introduction ... 391
7.2 Relative motion in orbit ... 392
7.3 Linearization of the equations of relative motion in orbit ... 400
7.4 Clohessy-Wiltshire equations ... 407
7.5 Two-impulse rendezvous maneuvers ... 411
7.6 Relative motion in close-proximity circular orbits ... 419
Problems ... 421
List of Key Terms ... 427

CHAPTER 8 Interplanetary trajectories ... **429**
8.1 Introduction ... 429
8.2 Interplanetary Hohmann transfers ... 430
8.3 Rendezvous Opportunities ... 432
8.4 Sphere of influence ... 437
8.5 Method of patched conics ... 441
8.6 Planetary departure ... 442
8.7 Sensitivity analysis ... 448
8.8 Planetary rendezvous ... 451
8.9 Planetary flyby ... 458
8.10 Planetary ephemeris ... 470
8.11 Non-Hohmann interplanetary trajectories ... 475
Problems ... 482
List of Key Terms ... 483

CHAPTER 9 Rigid-body dynamics ... **485**
9.1 Introduction ... 485
9.2 Kinematics ... 486
9.3 Equations of translational motion ... 495
9.4 Equations of rotational motion ... 497
9.5 Moments of inertia ... 501
9.5.1 Parallel axis theorem ... 517
9.6 Euler's equations ... 524
9.7 Kinetic energy ... 530
9.8 The spinning top ... 533
9.9 Euler angles ... 538
9.10 Yaw, pitch and roll angles ... 549
9.11 Quaternions ... 552
Problems ... 561
List of Key Terms ... 571

CHAPTER 10 Satellite attitude dynamics **573**
10.1 Introduction 573
10.2 Torque-free motion 574
10.3 Stability of torque-free motion 584
10.4 Dual-spin spacecraft 589
10.5 Nutation damper 593
10.6 Coning maneuver 601
10.7 Attitude control thrusters 605
10.8 Yo-yo despin mechanism 608
10.8.1 Radial release 613
10.9 Gyroscopic attitude control 615
10.10 Gravity gradient stabilization 631
Problems 644
List of Key Terms 653

CHAPTER 11 Rocket vehicle dynamics **655**
11.1 Introduction 655
11.2 Equations of motion 656
11.3 The thrust equation 658
11.4 Rocket performance 660
11.5 Restricted staging in field-free space 667
11.6 Optimal staging 678
11.6.1 Lagrange multiplier 678
Problems 686
List of Key Terms 688

Appendix A Physical data **689**

Appendix B A road map **691**

Appendix C Numerical intergration of the *n*-body equations of motion **693**

Appendix D MATLAB® algorithms **701**

Appendix E Gravitational potential energy of a sphere **703**

References **707**

Index **709**

Preface

The purpose of this book, like the first edition, is to provide an introduction to space mechanics for undergraduate engineering students. It is not directed towards graduate students, researchers and experienced practitioners, who may nevertheless find useful review material within the book's contents. The intended readers are those who are studying the subject for the first time and have completed courses in physics, dynamics and mathematics through differential equations and applied linear algebra. I have tried my best to make the text readable and understandable to that audience. In pursuit of that objective I have included a large number of example problems that are explained and solved in detail. Their purpose is not to overwhelm but to elucidate. I find that students like the "teach by example" method. I always assume that the material is being seen for the first time and, wherever possible, I provide solution details so as to leave little to the reader's imagination. The numerous figures throughout the book are also intended to aid comprehension. All of the more labor-intensive computational procedures are implemented in MATLAB® code.

CHANGES TO THE SECOND EDITION

Most of the content and style of the first edition has been retained. Some topics have been revised, rearranged or relocated. I have corrected all of the errors that I discovered or that were reported to me by students, teachers, reviewers and other readers. Key terms are now listed at the end of each chapter. The answers in the example problems are boxed instead of underlined. The homework problems at the end of each chapter have been grouped by applicable section. There are many new example problems and homework problems.

Chapter 1, which is a review of particle dynamics, begins with a new section on vectors, which are used throughout the book. Therefore, I thought a brief review of basic vector concepts and operations was appropriate. The chapter concludes with a new section on the numerical integration of ordinary differential equations (ODEs). These Runge-Kutta and predictor-corrector methods, which I implemented in the MATLAB codes *rk1_4.m*, *rkf45.m* and *heun.m*, facilitate the investigation and simulation of space mechanics problems for which analytical, closed-form solutions are not available. Many of the book's new example problems illustrate applications of this kind. Throughout the text I mostly use the ODE solvers *heun.m* (fixed time step) and *rkf45.m* (variable time step) because they work well and the scripts (see Appendix D) are short and easy to read. In every case I checked their results against two of MATLAB's own suite of ODE solvers, primarily *ode23.m* and *ode45.m*. These general-purpose codes are far more elegant (and lengthy) than the ones mentioned above. They may be listed by issuing the MATLAB `type` command.

I have added two algorithms to Chapter 2 for numerically integrating the two-body equations of motion: an algorithm for propagating a state vector as a function of true anomaly, and an algorithm for finding the roots of a function by the bisection method. The last one is useful for determining the Lagrange points in the restricted three-body problem.

Chapter 4 now includes the material on coordinate transformations previously found in this and other chapters. Section 4.5 includes a more general treatment of the Euler elementary rotation sequences, with emphasis on the classical (3-1-3) Euler sequence and the yaw-pitch-roll (3-2-1) sequence. Algorithms were added to calculate the right ascension and declination from the position vector and to calculate the classical Euler angles and the yaw, pitch and roll angles from the direction cosine matrix. I also moved all discussion of ground tracks into Chapter 4 and offer an algorithm for obtaining the ground track of a satellite from its orbital elements.

Chapter 6 concludes with a new section on nonimpulsive (finite burn time) orbital change maneuvers, including MATLAB simulations.

Chapter 7 now includes an algorithm to find the position, velocity and acceleration of a spacecraft relative to an LVLH frame. Also new to this chapter is the derivation of the linearized equations of relative motion for an elliptical (not necessarily circular) reference orbit.

New to Chapter 9 is a discussion of quaternions and associated algorithms for use in numerically solving Euler's equations of rigid body motion to obtain the evolution of spacecraft attitude. Quaternions can be used with MATLAB's `rotate` command to produce simple animations of spacecraft motion.

Appendices C and D have changed. The MATLAB script in Appendix C was revised. Appendix D no longer contains the listings of MATLAB codes. Instead, the algorithms are listed along with the world wide web addresses from which they may be downloaded. This edition contains over twice the number of MATLAB M-files as did the first.

ORGANIZATION

The organization of the book remains the same as that of the first edition. Chapter 1 is a review of vector kinematics in three dimensions and of Newton's laws of motion and gravitation. It also focuses on the issue of relative motion, crucial to the topics of rendezvous and satellite attitude dynamics. The new material on ordinary differential equation solvers will be useful for students who are expected to code numerical simulations in MATLAB or other programming languages. Chapter 2 presents the vector-based solution of the classical two-body problem, resulting in a host of practical formulas for the analysis of orbits and trajectories of elliptical, parabolic and hyperbolic shape. The restricted three-body problem is covered in order to introduce the notion of Lagrange points and to present the numerical solution of a lunar trajectory problem. Chapter 3 derives Kepler's equations, which relate position to time for the different kinds of orbits. The universal variable formulation is also presented. Chapter 4 is devoted to describing orbits in three dimensions. Coordinate transformations and the Euler elementary rotation sequences are defined. Procedures for transforming back and forth between the state vector and the classical orbital elements are addressed. The effect of the earth's oblateness on the motion of an orbit's ascending node and eccentricity vector is examined. Chapter 5 is an introduction to preliminary orbit determination, including Gibbs's and Gauss's methods and the solution of Lambert's problem. Auxiliary topics include topocentric coordinate systems, Julian day numbering and sidereal time. Chapter 6 presents the common means of transferring from one orbit to another by impulsive delta-v maneuvers, including Hohmann transfers, phasing orbits and plane changes. Chapter 7 is a brief introduction to relative motion in general and to the two-impulse rendezvous problem in particular. The latter is analyzed using the Clohessy-Wiltshire equations, which are derived in this chapter. Chapter 8 is an introduction to interplanetary mission design using patched conics. Chapter 9 presents those elements of rigid-body dynamics required to characterize the attitude of a space vehicle. Euler's equations of rotational motion are derived and applied in a number of example problems. Euler angles, yaw-pitch-roll angles and quaternions are presented as ways to describe the attitude of rigid body. Chapter 10 describes the methods of controlling, changing and stabilizing the attitude of spacecraft by means of thrusters, gyros and other devices. Finally, Chapter 11 is a brief introduction to the characteristics and design of multi-stage launch vehicles.

Chapters 1 through 4 form the core of a first orbital mechanics course. The time devoted to Chapter 1 depends on the background of the student. It might be surveyed briefly and used thereafter simply as a reference. What follows Chapter 4 depends on the objectives of the course.

Chapters 5 through 8 carry on with the subject of orbital mechanics. Chapter 6 on orbital maneuvers should be included in any case. Coverage of Chapters 5, 7 and 8 is optional. However, if all of Chapter 8 on interplanetary missions is to form a part of the course, then the solution of Lambert's problem (Section 5.3) must be studied beforehand.

Chapters 9 and 10 must be covered if the course objectives include an introduction to spacecraft dynamics. In that case Chapters 5, 7 and 8 would probably not be covered in depth.

Chapter 11 is optional if the engineering curriculum requires a separate course in propulsion, including rocket dynamics.

The important topic of spacecraft control systems is omitted. However, the material in this book and a course in control theory provide the basis for the study of spacecraft attitude control.

To understand the material and to solve problems requires using a lot of undergraduate mathematics. Mathematics, of course, is the language of engineering. Students must not forget that Sir Isaac Newton had to invent calculus so he could solve orbital mechanics problems in more than just a heuristic way. Newton (1642–1727) was an English physicist and mathematician, whose 1687 publication *Mathematical Principles of Natural Philosophy* ("the *Principia*") is one of the most influential scientific works of all time. It must be noted that the German mathematician Gottfried Wilhelm von Leibnitz (1646–1716), is credited with inventing infinitesimal calculus independently of Newton in the 1670s.

In addition to honing their math skills, students are urged to take advantage of computers (which, incidentally, use the binary numeral system developed by Leibnitz). There are many commercially available mathematics software packages for personal computers. Wherever possible they should be used to relieve the burden of repetitive and tedious calculations. Computer programming skills can and should be put to good use in the study of orbital mechanics. The elementary MATLAB programs referred to in Appendix D of this book illustrate how many of the procedures developed in the text can be implemented in software. All of the scripts were developed and tested using MATLAB version 7.7. Information about MATLAB, which is a registered trademark of The MathWorks, Inc., may be obtained from

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www.mathworks.com

Appendix A presents some tables of physical data and conversion factors. Appendix B is a road map through the first three chapters, showing how the most fundamental equations of orbital mechanics are related. Appendix C shows how to set up the *n*-body equations of motion and program them in MATLAB. Appendix D contains the web locations of the M-files of all of the MATLAB-implemented algorithms and example problems presented in the text. Appendix E shows that the gravitational field of a spherically symmetric body is the same as if the mass were concentrated at its center.

The field of astronautics is rich and vast. References cited throughout this text are listed at the end of the book. Also listed are other books on the subject that might be of interest to those seeking additional insights.

SUPPLEMENTS TO THE TEXT

For purchasers of this book:

Copies of the MATLAB M-files listed in Appendix D can be freely downloaded from the companion website accompanying this book. To access these files please visit www.elsevierdirect.com/9780123747785 and click on the "companion site" link.

For instructors using this book as text for their course:

Please visit www.textbooks.elsevier.com to register for access to the solutions manual, PowerPoint® lecture slides and other resources.

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Acknowledgments

Since the publication of the first edition and during the preparation of this one, I have received helpful criticism, suggestions and advice from many sources locally and worldwide. I thank them all and regret that time and space limitations prohibited the inclusion of some recommended additional topics that would have enhanced the book. I am especially indebted to those who reviewed the proposed revision plan and second edition manuscript for the publisher for their many suggestions on how the book could be improved. Thanks to:

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Howard D. Curtis
Embry-Riddle Aeronautical University
Daytona Beach, Florida

CHAPTER

Dynamics of point masses

1

Chapter outline

1.1	Introduction	1
1.2	Vectors	2
1.3	Kinematics	10
1.4	Mass, force and Newton's law of gravitation	15
1.5	Newton's law of motion	19
1.6	Time derivatives of moving vectors	24
1.7	Relative motion	29
1.8	Numerical integration	38

1.1 INTRODUCTION

This chapter serves as a self-contained reference on the kinematics and dynamics of point masses as well as some basic vector operations and numerical integration methods. The notation and concepts summarized here will be used in the following chapters. Those familiar with the vector-based dynamics of particles can simply page through the chapter and then refer back to it later as necessary. Those who need a bit more in the way of review will find the chapter contains all of the material they need in order to follow the development of orbital mechanics topics in the upcoming chapters.

We begin with a review of vectors and some vector operations after which we proceed to the problem of describing the curvilinear motion of particles in three dimensions. The concepts of force and mass are considered next, along with Newton's inverse-square law of gravitation. This is followed by a presentation of Newton's second law of motion ("force equals mass times acceleration") and the important concept of angular momentum.

As a prelude to describing motion relative to moving frames of reference, we develop formulas for calculating the time derivatives of moving vectors. These are applied to the computation of relative velocity and acceleration. Example problems illustrate the use of these results, as does a detailed consideration of how the earth's rotation and curvature influence our measurements of velocity and acceleration. This brings in the curious concept of Coriolis force. Embedded in exercises at the end of the chapter is practice in verifying several fundamental vector identities that will be employed frequently throughout the book.

The chapter concludes with an introduction to numerical integration methods, which can be called upon to solve the equations of motion when an analytical solution is not possible.

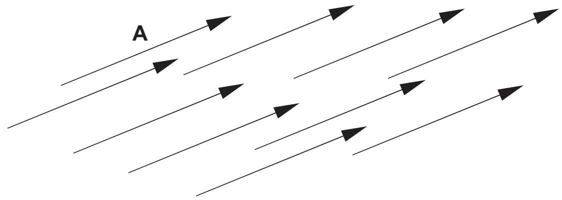


FIGURE 1.1

All of these vectors may be denoted **A**, since their magnitudes and directions are the same.

1.2 VECTORS

A vector is an object that is specified by both a magnitude and a direction. We represent a vector graphically by a directed line segment, that is, an arrow pointing in the direction of the vector. The end opposite the arrow is called the tail. The length of the arrow is proportional to the magnitude of the vector. Velocity is a good example of a vector. We say that a car is traveling east at eighty kilometers per hour. The direction is east and the magnitude, or speed, is 80 km/h. We will use boldface type to represent vector quantities and plain type to denote scalars. Thus, whereas B is a scalar, $\mathbf{B}$ is a vector.

Observe that a vector is specified solely by its magnitude and direction. If $\mathbf{A}$ is a vector, then all vectors having the same physical dimensions, the same length and pointing in the same direction as $\mathbf{A}$ are denoted $\mathbf{A}$, regardless of their line of action, as illustrated in Figure 1.1. Shifting a vector parallel to itself does not mathematically change the vector. However, parallel shift of a vector might produce a different physical effect. For example, an upward 5 kN load (force vector) applied to the tip of an airplane wing gives rise to quite a different stress and deflection pattern in the wing than the same load acting at the wing's mid-span.

The magnitude of a vector $\mathbf{A}$ is denoted $\|\mathbf{A}\|$, or, simply A.

Multiplying a vector $\mathbf{B}$ by the reciprocal of its magnitude produces a vector which points in the direction of $\mathbf{B}$, but it is dimensionless and has a magnitude of one. Vectors having unit dimensionless magnitude are called unit vector*s*. We put a hat (^) over the letter representing a unit vector. Then we can tell simply by inspection that, for example, $\hat{\mathbf{u}}$ is a unit vector, as are $\hat{\mathbf{B}}$ and $\hat{\mathbf{e}}$.

It is convenient to denote the unit vector in the direction of the vector $\mathbf{A}$ as $\hat{\mathbf{u}}_A$. As pointed out above, we obtain this vector from $\mathbf{A}$ as follows

$$\hat{\mathbf{u}}_A = \frac{\mathbf{A}}{A} \tag{1.1}$$

Likewise, $\hat{\mathbf{u}}_C = \mathbf{C}/C$, $\hat{\mathbf{u}}_F = \mathbf{F}/F$, etc.

The sum or ***resultant*** of two vectors is defined by the parallelogram rule (Figure 1.2). Let $\mathbf{C}$ be the sum of the two vectors $\mathbf{A}$ and $\mathbf{B}$. To form that sum using the parallelogram rule, the vectors $\mathbf{A}$ and $\mathbf{B}$ are shifted parallel to themselves (leaving them unaltered) until the tail of $\mathbf{A}$ touches the tail of $\mathbf{B}$. Drawing dotted lines through the head of each vector parallel to the other completes a parallelogram. The diagonal from the tails of $\mathbf{A}$ and $\mathbf{B}$ to the opposite corner is the resultant $\mathbf{C}$. By construction, vector addition is commutative, that is,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.2}$$

A ***Cartesian coordinate system*** in three dimensions consists of three axes, labeled x, y and z, which intersect at the origin O. We will always use a right-handed Cartesian coordinate system, which means if you wrap the fingers of your right hand around the z axis, with the thumb pointing in the positive z direction,

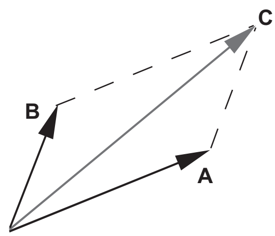


FIGURE 1.2

Parallelogram rule of vector addition.

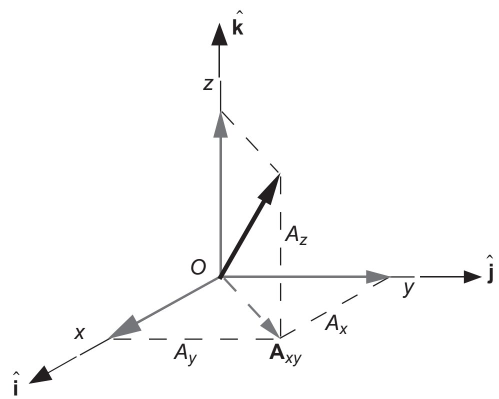


FIGURE 1.3

Three-dimensional, right-handed Cartesian coordinate system.

your fingers will be directed from the x axis towards the y axis. Figure 1.3 illustrates such a system. Note that the unit vectors along the x, y and z-axes are, respectively, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

In terms of its Cartesian components, and in accordance with the above summation rule, a vector **A** is written in terms of its components A_x, A_y and A_z as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{1.3}$$

The projection of **A** on the xy plane is denoted $\mathbf{A}_{xy}$. It follows that

$$\mathbf{A}_{xy} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$

According to the Pythagorean theorem, the magnitude of **A** in terms of its Cartesian components is

$$A = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2} \tag{1.4}$$

From Equations 1.1 and 1.3, the unit vector in the direction of **A** is

$$\hat{\mathbf{u}}_{\mathrm{A}} = \cos\theta_x\hat{\mathbf{i}} + \cos\theta_y\hat{\mathbf{j}} + \cos\theta_z\hat{\mathbf{k}} \tag{1.5}$$

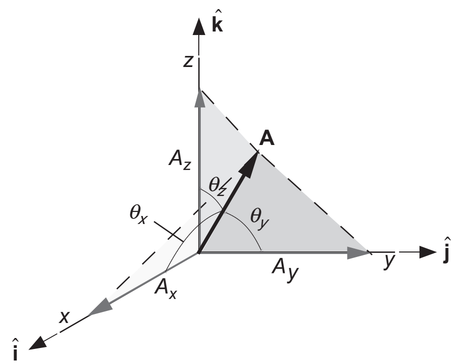


FIGURE 1.4

Direction angles in three dimensions.

where

$$\cos\theta_x = \frac{A_x}{A} \qquad \cos\theta_y = \frac{A_y}{A} \qquad \cos\theta_z = \frac{A_z}{A} \tag{1.6}$$

The direction angles θ_x, θ_y and θ_z are illustrated in Figure 1.4, and are measured between the vector and the positive coordinate axes. Note carefully that the sum of θ_x, θ_y and θ_z is not in general known a priori and cannot be assumed to be, say, 180 degrees.

Example 1.1

Calculate the direction angles of the vector $\mathbf{A} = \hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$.

Solution

First, compute the magnitude of **A** by means of Equation 1.4:

$$A = \sqrt{1^2 + (-4)^2 + 8^2} = 9$$

Then Equations 1.6 yield

$$\theta_x = \cos^{-1}\left(\frac{A_x}{A}\right) = \cos^{-1}\left(\frac{1}{9}\right) \Rightarrow \boxed{\theta_x = 83.62^\circ}$$

$$\theta_y = \cos^{-1}\left(\frac{A_y}{A}\right) = \cos^{-1}\left(\frac{-4}{9}\right) \Rightarrow \boxed{\theta_y = 116.4^\circ}$$

$$\theta_z = \cos^{-1}\left(\frac{A_z}{A}\right) = \cos^{-1}\left(\frac{8}{9}\right) \Rightarrow \boxed{\theta_z = 27.27^\circ}$$

Observe that $\theta_x + \theta_y + \theta_z = 227.3^\circ$.

Multiplication and division of two vectors are undefined operations. There are no rules for computing the product **AB** and the ratio **A/B**. However, there are two well-known binary operations on vectors: the dot product and the cross product. The ***dot product*** of two vectors is a scalar defined as follows,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \tag{1.7}$$

where θ is the angle between the heads of the two vectors, as shown in Figure 1.5. Clearly,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \tag{1.8}$$

If two vectors are perpendicular to each other, then the angle between them is 90°. It follows from Equation 1.7 that their dot product is zero. Since the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ of a Cartesian coordinate system are mutually orthogonal and of magnitude one, Equation 1.7 implies that

$$\begin{aligned}\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1\\ \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0\end{aligned} \tag{1.9}$$

Using these properties it is easy to show that the dot product of the vectors **A** and **B** may be found in terms of their Cartesian components as

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{1.10}$$

If we set $\mathbf{B} = \mathbf{A}$, then it follows from Equations 1.4 and 1.10 that

$$A = \sqrt{\mathbf{A}\cdot\mathbf{A}} \tag{1.11}$$

The dot product operation is used to project one vector onto the line of action of another. We can imagine bringing the vectors tail to tail for this operation, as illustrated in Figure 1.6. If we drop a perpendicular

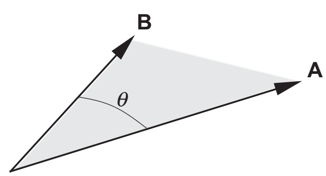


FIGURE 1.5

The angle between two vectors brought tail to tail by parallel shift.

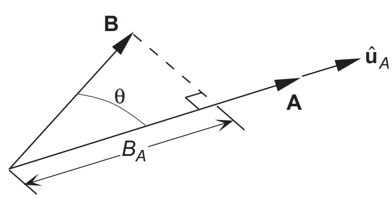


FIGURE 1.6

Projecting the vector **B** onto the direction of **A**.

line from the tip of **B** onto the direction of **A**, then the line segment B_A is the orthogonal projection of **B** onto line of action of **A**. B_A stands for the scalar projection of **B** onto **A**. From trigonometry, it is obvious from the figure that

$$B_A = B\cos\theta$$

Let $\hat{\mathbf{u}}_A$ be the unit vector in the direction of **A**. Then

$$\mathbf{B}\cdot\hat{\mathbf{u}}_A = \|\mathbf{B}\|\overbrace{\|\hat{\mathbf{u}}_A\|}^{1}\cos\theta = B\cos\theta$$

Comparing this expression with the preceding one leads to the conclusion that

$$B_A = \mathbf{B}\cdot\hat{\mathbf{u}}_A = \mathbf{B}\cdot\frac{\mathbf{A}}{A} \tag{1.12}$$

where $\hat{\mathbf{u}}_A$ is given by Equation 1.1. Likewise, the projection of **A** onto **B** is given by

$$A_B = \mathbf{A}\cdot\frac{\mathbf{B}}{B}$$

Observe that $A_B = B_A$ only if **A** and **B** have the same magnitude.

Example 1.2

Let $\mathbf{A} = \hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 18\hat{\mathbf{k}}$ and $\mathbf{B} = 42\hat{\mathbf{i}} - 69\hat{\mathbf{j}} + 98\hat{\mathbf{k}}$. Calculate

(a) The angle between **A** and **B**;
(b) The projection of **B** in the direction of **A**;
(c) The projection of **A** in the direction of **B**.

Solution

First we make the following individual calculations.

$$\mathbf{A}\cdot\mathbf{B} = (1)(42) + (6)(-69) + (18)(98) = 1392 \tag{a}$$

$$A = \sqrt{(1)^2 + (6)^2 + (18)^2} = 19 \tag{b}$$

$$B = \sqrt{(42)^2 + (-69)^2 + (98)^2} = 127 \tag{c}$$

(a) According to Equation 1.7, the angle between **A** and **B** is

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$

Substituting (a), (b) and (c) yields

$$\theta = \cos^{-1}\left(\frac{1392}{19\cdot 127}\right) = \boxed{54.77°}$$

(b) From Equation 1.12 we find the projection of **B** onto **A**:

$$B_A = \mathbf{B} \cdot \frac{\mathbf{A}}{A} = \frac{\mathbf{A} \cdot \mathbf{B}}{A}$$

Substituting (a) and (b) we get

$$B_A = \frac{1392}{19} = \boxed{73.26}$$

(c) The projection of **A** onto **B** is

$$A_B = \mathbf{A} \cdot \frac{\mathbf{B}}{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{B}$$

Substituting (a) and (c) we obtain

$$A_B = \frac{1392}{127} = \boxed{10.96}$$

The ***cross product*** of two vectors yields another vector, which is computed as follows,

$$\mathbf{A} \times \mathbf{B} = (AB\sin\theta)\hat{\mathbf{n}}_{AB} \tag{1.13}$$

where θ is the angle between the heads of A and B, and $\hat{\mathbf{n}}_{AB}$ is the unit vector normal to the plane defined by the two vectors. The direction of $\hat{\mathbf{n}}_{AB}$ is determined by the right hand rule. That is, curl the fingers of the right hand from the first vector (**A**) towards the second vector (**B**), and the thumb shows the direction of $\hat{\mathbf{n}}_{AB}$. See Figure 1.7. If we use Equation 1.13 to compute $\mathbf{B} \times \mathbf{A}$, then $\hat{\mathbf{n}}_{AB}$ points in the opposite direction, which means

$$\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}) \tag{1.14}$$

Therefore, unlike the dot product, the cross product is not commutative.

The cross product is obtained analytically by resolving the vectors into Cartesian components.

$$\mathbf{A} \times \mathbf{B} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) \tag{1.15}$$

Since the set $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ is a mutually perpendicular triad of unit vectors, Equation 1.13 implies that

$$\begin{array}{lll} \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0} & \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \mathbf{0} & \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0} \\ \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} & \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} & \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \end{array} \tag{1.16}$$

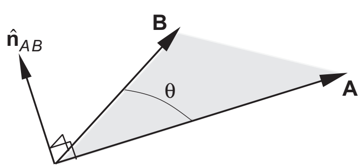


FIGURE 1.7

$\hat{\mathbf{n}}_{AB}$ is normal to both A and B and defines the direction of the cross product $\mathbf{A} \times \mathbf{B}$.

Expanding the right side of Equation 1.15, substituting Equation 1.16 and making use of Equation 1.14 leads to

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\hat{\mathbf{i}} - (A_x B_z - A_z B_x)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}} \tag{1.17}$$

It may be seen that the right-hand side is the determinant of the matrix

$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

Thus, Equation 1.17 can be written

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.18}$$

where the two vertical bars stand for determinant. Obviously the rule for computing the cross product, though straightforward, is a bit lengthier than that for the dot product. Remember that the dot product yields a scalar whereas the cross product yields a vector.

The cross product provides an easy way to compute the normal to a plane. Let **A** and **B** be any two vectors lying in the plane, or, let any two vectors be brought tail-to-tail to define a plane, as shown in Figure 1.7. The vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is normal to the plane of **A** and **B**. Therefore, $\hat{\mathbf{n}}_{AB} = \mathbf{C}/C$, or

$$\mathbf{n}_{AB} = \frac{\mathbf{A} \times \mathbf{B}}{\|\mathbf{A} \times \mathbf{B}\|} \tag{1.19}$$

Example 1.3

Let $\mathbf{A} = -3\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$ and $\mathbf{B} = 6\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$. Find a unit vector that lies in the plane of **A** and **B** and is perpendicular to **A**.

Solution

The plane of the vectors **A** and **B** is determined by parallel shifting the vectors so that they meet tail to tail. Calculate the vector $\mathbf{D} = \mathbf{A} \times \mathbf{B}$.

$$\mathbf{D} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 7 & 9 \\ 6 & -5 & 8 \end{vmatrix} = 101\hat{\mathbf{i}} + 78\hat{\mathbf{j}} - 27\hat{\mathbf{k}}$$

Note that **A** and **B** are both normal to **D**. We next calculate the vector $\mathbf{C} = \mathbf{D} \times \mathbf{A}$.

$$\mathbf{C} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 101 & 78 & -27 \\ -3 & 7 & 9 \end{vmatrix} = 891\hat{\mathbf{i}} - 828\hat{\mathbf{j}} + 941\hat{\mathbf{k}}$$

C is normal to **D** as well as to **A**. **A**, **B** and **C** are all perpendicular to **D**. Therefore they are coplanar. Thus **C** is not only perpendicular to **A**, it lies in the plane of **A** and **B**. Therefore, the unit vector we are seeking is the unit vector in the direction of **C**, namely,

$$\hat{\mathbf{u}}_C = \frac{\mathbf{C}}{C} = \frac{891\hat{\mathbf{i}} - 828\hat{\mathbf{j}} + 941\hat{\mathbf{k}}}{\sqrt{891^2 + (-828)^2 + 941^2}}$$

$$\boxed{\hat{\mathbf{u}}_C = 0.5794\hat{\mathbf{i}} - 0.5384\hat{\mathbf{j}} + 0.6119\hat{\mathbf{k}}}$$

In the chapters to follow we will often encounter the vector triple product, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. By resolving **A**, **B** and **C** into their Cartesian components, it can easily be shown (see Problem 1.1c) that the vector triple product can be expressed in terms of just the dot products of these vectors as follows:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{1.20}$$

Because of the appearance of the letters on the right-hand side, this is often referred to as the ***bac-cab rule***.

Example 1.4

If $\mathbf{F} = \mathbf{E} \times \{\mathbf{D} \times [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]\}$, use the *bac-cab* rule to reduce this expression to one involving only dot products.

Solution

First we invoke the *bac-cab* rule to obtain

$$\mathbf{F} = \mathbf{E} \times \left\{\mathbf{D} \times [\overbrace{\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})}^{bac-cab\ \text{rule}}]\right\}$$

Expanding and collecting terms leads to

$$\mathbf{F} = (\mathbf{A}\cdot\mathbf{C})[\mathbf{E} \times (\mathbf{D} \times \mathbf{B})] - (\mathbf{A}\cdot\mathbf{B})[\mathbf{E} \times (\mathbf{D} \times \mathbf{C})]$$

We next apply the *bac-cab* rule twice on the right-hand side.

$$\mathbf{F} = (\mathbf{A}\cdot\mathbf{C})\left[\overbrace{\mathbf{D}(\mathbf{E}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{E}\cdot\mathbf{D})}^{bac-cab\ \text{rule}}\right] - (\mathbf{A}\cdot\mathbf{B})\left[\overbrace{\mathbf{D}(\mathbf{E}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{E}\cdot\mathbf{D})}^{bac-cab\ \text{rule}}\right]$$

Expanding and collecting terms yields the sought-for result.

$$\boxed{\mathbf{F} = [(\mathbf{A}\cdot\mathbf{C})(\mathbf{E}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})(\mathbf{E}\cdot\mathbf{C})]\mathbf{D} - (\mathbf{A}\cdot\mathbf{C})(\mathbf{E}\cdot\mathbf{D})\mathbf{B} + (\mathbf{A}\cdot\mathbf{B})(\mathbf{E}\cdot\mathbf{D})\mathbf{C}}$$

Another useful vector identity is the ***interchange of the dot and cross***:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \tag{1.21}$$

It is so-named because interchanging the operations in the expression $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ yields $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$. The parentheses in Equation 1.21 are required to show which operation must be carried out first, according to the rules of vector algebra. (For example, $(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}$, the cross product of a scalar and a vector, is undefined.) It is easy to verify Equation 1.21 by substituting $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$, $\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$ and $\mathbf{C} = C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}} + C_z\hat{\mathbf{k}}$ and observing that both sides of the equal sign reduce to the same expression (Problem 1.1b).

1.3 KINEMATICS

To track the motion of a particle P through Euclidean space we need a frame of reference, consisting of a clock and a Cartesian coordinate system. The clock keeps track of time t and the xyz axes of the Cartesian coordinate system are used to locate the spatial position of the particle. In nonrelativistic mechanics, a single "universal" clock serves for all possible Cartesian coordinate systems. So when we refer to a frame of reference we need think only of the mutually orthogonal axes themselves.

The unit of time used throughout this book is the second (s). The unit of length is the meter (m), but the kilometer (km) will be the length unit of choice when large distances and velocities are involved. Conversion factors between kilometers, miles and nautical miles are listed in Table A.3.

Given a frame of reference, the position of the particle P at a time t is defined by the position vector $\mathbf{r}(t)$ extending from the origin O of the frame out to P itself, as illustrated in Figure 1.8. The components of $\mathbf{r}(t)$ are just the x, y and z coordinates,

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

The distance of P from the origin is the magnitude or length of $\mathbf{r}$, denoted $\|\mathbf{r}\|$ or just r,

$$\|\mathbf{r}\| = r = \sqrt{x^2 + y^2 + z^2}$$

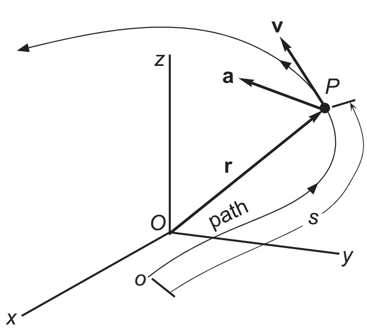


FIGURE 1.8

Position, velocity and acceleration vectors.

As in Equation 1.11, the magnitude of **r** can also be computed by means of the dot product operation,

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$$

The velocity **v** and acceleration **a** of the particle are the first and second time derivatives of the position vector,

$$\mathbf{v}(t) = \frac{dx(t)}{dt}\hat{\mathbf{i}} + \frac{dy(t)}{dt}\hat{\mathbf{j}} + \frac{dy(t)}{dt}\hat{\mathbf{k}} = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}}$$

$$\mathbf{a}(t) = \frac{dv_x(t)}{dt}\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\hat{\mathbf{j}} + \frac{dv_z(t)}{dt}\hat{\mathbf{k}} = a_x(t)\hat{\mathbf{i}} + a_y(t)\hat{\mathbf{j}} + a_z(t)\hat{\mathbf{k}}$$

It is convenient to represent the time derivative by means of an overhead dot. In this shorthand ***overhead dot notation***, if ( ) is any quantity, then

$$\dot{(\ )} \equiv \frac{d(\ )}{dt} \qquad \ddot{(\ )} \equiv \frac{d^2(\ )}{dt^2} \qquad \dddot{(\ )} \equiv \frac{d^3(\ )}{dt^3}, \text{ etc.}$$

Thus, for example,

$$\begin{aligned}
&\mathbf{v} = \dot{\mathbf{r}} \\
&\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} \\
&v_x = \dot{x} \qquad\qquad v_y = \dot{y} \qquad\qquad v_z = \dot{z} \\
&a_x = \dot{v}_x = \ddot{x} \qquad a_y = \dot{v}_y = \ddot{y} \qquad a_z = \dot{v}_z = \ddot{z}
\end{aligned}$$

The locus of points that a particle occupies as it moves through space is called its path or trajectory. If the path is a straight line, then the motion is rectilinear. Otherwise, the path is curved, and the motion is called curvilinear. The velocity vector **v** is tangent to the path. If $\hat{\mathbf{u}}_t$ is the unit vector tangent to the trajectory, then

$$\mathbf{v} = v\hat{\mathbf{u}}_t \tag{1.22}$$

where the speed v is the magnitude of the velocity **v**. The distance ds that P travels along its path in the time interval dt is obtained from the speed by

$$ds = vdt$$

In other words,

$$v = \dot{s}$$

The distance s, measured along the path from some starting point, is what the odometers in our automobiles record. Of course, $\dot{s}$, our speed along the road, is indicated by the dial of the speedometer.

Note carefully that $v \neq \dot{r}$, that is, the magnitude of the derivative of **r** does not equal the derivative of the magnitude of **r**.

Example 1.5

The position vector in meters is given as a function of time in seconds as

$$\mathbf{r} = (8t^2 + 7t + 6)\hat{\mathbf{i}} + (5t^3 + 4)\hat{\mathbf{j}} + (0.3t^4 + 2t^2 + 1)\hat{\mathbf{k}}\ (\text{m}) \tag{a}$$

At $t = 10$ seconds, calculate (a) v (the magnitude of the derivative of **r**) and (b) $\dot{r}$ (the derivative of the magnitude of **r**).

Solution

(a) The velocity **v** is found by differentiating the given position vector with respect to time,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (16t + 7)\hat{\mathbf{i}} + 15t^2\hat{\mathbf{j}} + (1.2t^3 + 4t)\hat{\mathbf{k}}$$

The magnitude of this vector is the square root of the sum of the squares of its components,

$$v = (1.44t^6 + 234.6t^4 + 272t^2 + 224t + 49)^{\frac{1}{2}}$$

Evaluating this at $t = 10\,\text{s}$, we get

$$\boxed{v = 1953.3\ \text{m/s}}$$

(b) Calculating the magnitude of **r** in (a) leads to

$$r = (0.09t^8 + 26.2t^6 + 68.6t^4 + 152t^3 + 149t^2 + 84t + 53)^{\frac{1}{2}}$$

The time derivative of this expression is:

$$\dot{r} = \frac{dr}{dt} = \frac{0.36t^7 + 78.6t^5 + 137.2t^3 + 228t^2 + 149t + 42}{(0.09t^8 + 26.2t^6 + 68.6t^4 + 152t^3 + 149t^2 + 84t + 53)^{\frac{1}{2}}}$$

Substituting $t = 10\,\text{s}$ yields

$$\boxed{\dot{r} = 1935.5\ \text{m/s}}$$

If **v** is given, then we can find the components of the unit tangent $\hat{\mathbf{u}}_t$ in the Cartesian coordinate frame of reference by means of Equation 1.22:

$$\hat{\mathbf{u}}_t = \frac{\mathbf{v}}{v} = \frac{v_x}{v}\hat{\mathbf{i}} + \frac{v_y}{v}\hat{\mathbf{j}} + \frac{v_z}{v}\hat{\mathbf{k}} \qquad \left(v = \sqrt{v_x^{\ 2} + v_y^{\ 2} + v_z^{\ 2}}\right) \tag{1.23}$$

The acceleration may be written

$$\mathbf{a} = a_t\hat{\mathbf{u}}_t + a_n\hat{\mathbf{u}}_n \tag{1.24}$$

where a_t and a_n are the tangential and normal components of acceleration, given by

$$a_t = \dot{v}(= \ddot{s}) \qquad a_n = \frac{v^2}{\rho} \tag{1.25}$$

ρ is the radius of curvature, which is the distance from the particle P to the center of curvature of the path at that point. The unit principal normal $\hat{\mathbf{u}}_n$ is perpendicular to $\hat{\mathbf{u}}_t$ and points towards the center of curvature C, as shown in Figure 1.9. Therefore, the position of C relative to P, denoted $\mathbf{r}_{C/P}$, is

$$\mathbf{r}_{C/P} = \rho\hat{\mathbf{u}}_n \tag{1.26}$$

The orthogonal unit vectors $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ form a plane called the osculating plane. The unit normal to the osculating plane is $\hat{\mathbf{u}}_b$, the binormal, and it is obtained from $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ by taking their cross product:

$$\hat{\mathbf{u}}_b = \hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n \tag{1.27}$$

From Equations 1.22, 1.24 and 1.27 we have

$$\mathbf{v} \times \mathbf{a} = v\hat{\mathbf{u}}_t \times (a_t\hat{\mathbf{u}}_t + a_n\hat{\mathbf{u}}_n) = va_n(\hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n) = va_n\hat{\mathbf{u}}_b = \|\mathbf{v} \times \mathbf{a}\|\hat{\mathbf{u}}_b$$

That is, an alternative to Equation 1.27 for calculating the binormal vector is

$$\hat{\mathbf{u}}_b = \frac{\mathbf{v} \times \mathbf{a}}{\|\mathbf{v} \times \mathbf{a}\|} \tag{1.28}$$

Note that $\hat{\mathbf{u}}_t$, $\hat{\mathbf{u}}_n$ and $\hat{\mathbf{u}}_b$ form a right-handed triad of orthogonal unit vectors. That is,

$$\hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t = \hat{\mathbf{u}}_n \qquad \hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n = \hat{\mathbf{u}}_b \qquad \hat{\mathbf{u}}_n \times \hat{\mathbf{u}}_b = \hat{\mathbf{u}}_t \tag{1.29}$$

The center of curvature lies in the osculating plane. When the particle P moves an incremental distance ds the radial from the center of curvature to the path sweeps out a small angle $d\phi$, measured in the osculating plane. The relationship between this angle and ds is

$$ds = \rho d\phi$$

so that $\dot{s} = \rho\dot{\phi}$, or

$$\dot{\phi} = \frac{v}{\rho} \tag{1.30}$$

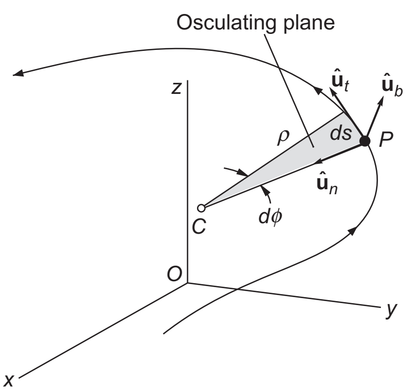


FIGURE 1.9

Orthogonal triad of unit vectors associated with the moving point P.

Example 1.6

Relative to a Cartesian coordinate system, the position, velocity and acceleration of a particle P at a given instant are

$$\mathbf{r} = 250\hat{\mathbf{i}} + 630\hat{\mathbf{j}} + 430\hat{\mathbf{k}}\ (\text{m}) \tag{a}$$

$$\mathbf{v} = 90\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 170\hat{\mathbf{k}}\ (\text{m/s}) \tag{b}$$

$$\mathbf{a} = 16\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 30\hat{\mathbf{k}}\ (\text{m/s})^2 \tag{c}$$

Find the coordinates of the center of curvature at that instant.

Solution

The coordinates of the center of curvature C are the components of its position vector $\mathbf{r}_C$. Consulting Figure 1.9, we observe that

$$\mathbf{r}_C = \mathbf{r} + \rho\hat{\mathbf{u}}_n \tag{d}$$

where $\mathbf{r}$ is the position vector of the point P, ρ is the radius of curvature and $\hat{\mathbf{u}}_n$ is the unit principal normal vector. The position vector $\mathbf{r}$ is given in (a), but ρ and $\hat{\mathbf{u}}_n$ are unknowns at this point. We must use the geometry of Figure 1.9 to find them.

We first seek the value of $\hat{\mathbf{u}}_n$, starting with Equation 1.29_1,

$$\hat{\mathbf{u}}_n = \hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t \tag{e}$$

The unit tangent vector $\hat{\mathbf{u}}_t$ is found at once from the velocity vector in (b) by means of Equation 1.23:

$$\hat{\mathbf{u}}_t = \frac{\mathbf{v}}{v}$$

where

$$v = \sqrt{90^2 + 125^2 + 170^2} = 229.4 \tag{f}$$

Thus

$$\hat{\mathbf{u}}_t = \frac{90\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 170\hat{\mathbf{k}}}{229.4} = 0.39233\hat{\mathbf{i}} + 0.5449\hat{\mathbf{j}} + 0.74106\hat{\mathbf{k}} \tag{g}$$

To find the binormal $\hat{\mathbf{u}}_b$ we insert the given velocity and acceleration vectors into Equation 1.28:

$$\begin{aligned}
\hat{\mathbf{u}}_b &= \frac{\mathbf{v}\times\mathbf{a}}{\|\mathbf{v}\times\mathbf{a}\|} \\
&= \frac{\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 90 & 125 & 170 \\ 16 & 125 & 30 \end{vmatrix}}{\|\mathbf{v}\times\mathbf{a}\|} \\
&= \frac{-17500\hat{\mathbf{i}} + 20\hat{\mathbf{j}} + 9250\hat{\mathbf{k}}}{\sqrt{(-17500)^2 + 20^2 + 9250^2}} \\
&= -0.88409\hat{\mathbf{i}} + 0.0010104\hat{\mathbf{j}} + 0.46731\hat{\mathbf{k}}
\end{aligned} \tag{h}$$

Substituting (g) and (h) back into (e) finally yields the unit principal normal:

$$\hat{\mathbf{u}}_n = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -0.88409 & 0.0010104 & 0.46731 \\ 0.39233 & 0.5449 & 0.74106 \end{vmatrix} = -0.25389\hat{\mathbf{i}} + 0.8385\hat{\mathbf{j}} - 0.48214\hat{\mathbf{k}} \tag{i}$$

The only unknown remaining in (d) is ρ, for which we appeal to Equation 1.25:

$$\rho = \frac{v^2}{a_n} \tag{j}$$

The normal acceleration a_n is calculated by projecting the acceleration vector $\mathbf{a}$ onto the direction of the unit normal $\hat{\mathbf{u}}_n$,

$$a_n = \mathbf{a} \cdot \hat{\mathbf{u}}_n = (16\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 30\hat{\mathbf{k}}) \cdot (-0.25389\hat{\mathbf{i}} + 0.8385\hat{\mathbf{j}} - 0.48214\hat{\mathbf{k}}) = 86.287 \text{ m/s} \tag{k}$$

Putting the values of v and a_n in Equations (f) and (k) into (j) yields the radius of curvature,

$$\rho = \frac{229.4^2}{86.287} = 609.89 \text{ m} \tag{l}$$

Upon substituting (a), (i) and (l) into (d) we obtain the position vector of the center of curvature C:

$$\begin{aligned} \mathbf{r}_C &= (250\hat{\mathbf{i}} + 630\hat{\mathbf{j}} + 430\hat{\mathbf{k}}) + 609.89(-0.25389\hat{\mathbf{i}} + 0.8385\hat{\mathbf{j}} - 0.48214\hat{\mathbf{k}}) \\ &= 95.159\hat{\mathbf{i}} + 1141.4\hat{\mathbf{j}} + 135.95\hat{\mathbf{k}} \text{ (km)} \end{aligned}$$

Therefore, the coordinates of C are

$$\boxed{x = 95.16 \text{ m} \qquad y = 1141 \text{ m} \qquad z = 136.0 \text{ m}}$$

1.4 MASS, FORCE AND NEWTON'S LAW OF GRAVITATION

Mass, like length and time, is a primitive physical concept: it cannot be defined in terms of any other physical concept. Mass is simply the quantity of matter. More practically, mass is a measure of the inertia of a body. Inertia is an object's resistance to changing its state of motion. The larger its inertia (the greater its mass), the more difficult it is to set a body into motion or bring it to rest. The unit of mass is the kilogram (kg).

Force is the action of one physical body on another, either through direct contact or through a distance. Gravity is an example of force acting through a distance, as are magnetism and the force between charged particles. The gravitational force between two masses m_1 and m_2 having a distance r between their centers is

$$F_g = G\frac{m_1 m_2}{r^2} \tag{1.31}$$

This is ***Newton's law of gravity***, in which G, the ***universal gravitational constant***, has the value $G = 6.6742 \times 10^{-11}\ \text{m}^3/(\text{kg} \cdot \text{s}^2)$. Due to the inverse-square dependence on distance, the force of gravity rapidly diminishes with the amount of separation between the two masses. In any case, the force of gravity is minuscule unless at least one of the masses is extremely big.

The force of a large mass (such as the earth) on a mass many orders of magnitude smaller (such as a person) is called weight, W. If the mass of the large object is M and that of the relatively tiny one is m, then the weight of the small body is

$$W = G\frac{Mm}{r^2} = m\left(\frac{GM}{r^2}\right)$$

or

$$W = mg \tag{1.32}$$

where

$$g = \frac{GM}{r^2} \tag{1.33}$$

g has units of acceleration (m/s^2) and is called the acceleration of gravity. If planetary gravity is the only force acting on a body, then the body is said to be in free fall. The force of gravity draws a freely falling object towards the center of attraction (e.g., center of the earth) with an acceleration g. Under ordinary conditions, we sense our own weight by feeling contact forces acting on us in opposition to the force of gravity. In free fall there are, by definition, no contact forces, so there can be no sense of weight. Even though the weight is not zero, a person in free fall experiences weightlessness, or the absence of gravity.

Let us evaluate Equation 1.33 at the surface of the earth, whose radius according to Table A.1 is 6378 km. Letting g_0 represent the standard sea-level value of g, we get

$$g_0 = \frac{GM}{{R_E}^2} \tag{1.34}$$

In SI units,

$$g_0 = 9.807 \text{ m/s}^2 \tag{1.35}$$

Substituting Equation 1.34 into Equation 1.33 and letting z represent the distance above the earth's surface, so that $r = R_E + z$, we obtain

$$g = g_0\frac{{R_E}^2}{(R_E + z)^2} = \frac{g_0}{(1 + z/R_E)^2} \tag{1.36}$$

Commercial airliners cruise at altitudes on the order of ten kilometers (six miles). At that height, Equation 1.36 reveals that g (and hence weight) is only three-tenths of a percent less than its sea level value. Thus, under ordinary conditions, we ignore the variation of g with altitude. A plot of Equation 1.36 out to a height of 1000 km (the upper limit of low-earth orbit operations) is shown in Figure 1.10. The variation of g over that range is significant. Even so, at space station altitude (300 km), weight is only about 10 percent less that it is on the earth's surface. The astronauts experience weightlessness, but they clearly are not weightless.

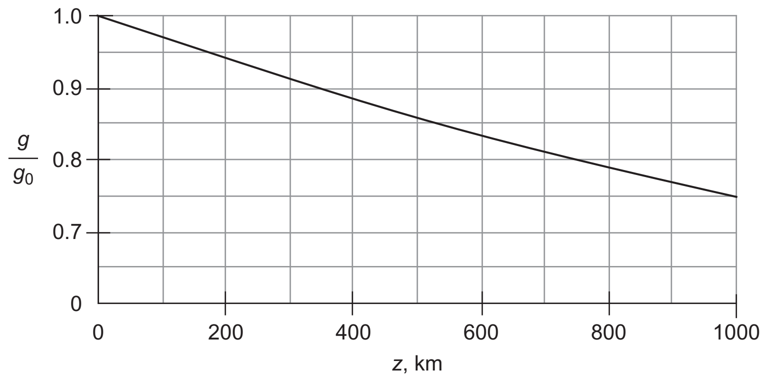


FIGURE 1.10

Variation of the acceleration of gravity with altitude.

Example 1.7

Show that in the absence of an atmosphere, the shape of a low altitude ballistic trajectory is a parabola. Assume the acceleration of gravity g is constant and neglect the earth's curvature.

Solution

Figure 1.11 shows a projectile launched at $t = 0$ with a speed v_0 at a flight path angle γ_0 from the point with coordinates (x_0, y_0). Since the projectile is in free fall after launch, its only acceleration is that of gravity in the negative y-direction:

$$\ddot{x} = 0$$
$$\ddot{y} = -g$$

Integrating with respect to time and applying the initial conditions leads to

$$x = x_0 + (v_0 \cos \gamma_0) \tag{a}$$

$$y = y_0 + (v_0 \sin \gamma_0)t - \frac{1}{2}gt^2 \tag{b}$$

Solving (a) for t and substituting the result into (b) yields

$$\boxed{y = y_0 + (x - x_0)\tan \gamma_0 - \frac{1}{2}\frac{g}{v_0 \cos \gamma_o}(x - x_0)^2} \tag{c}$$

This is the equation of a second-degree curve, a parabola, as sketched in Figure 1.11.

Example 1.8

An airplane flies a parabolic trajectory like that in Figure 1.11 so that the passengers will experience free fall (weightlessness). What is the required variation of the flight path angle γ with speed v? Ignore the curvature of the earth.

Solution

Figure 1.12 reveals that for a "flat" earth, $d\gamma = -d\phi$, i.e.,

$$\dot{\gamma} = -\dot{\phi}$$

It follows from Equation 1.30 that

$$\rho\dot{\gamma} = -v \tag{1.37}$$

The normal acceleration a_n is just the component of the gravitational acceleration g in the direction of the unit principal normal to the curve (from P towards C). From Figure 1.12, then,

$$a_n = g\cos\gamma \tag{a}$$

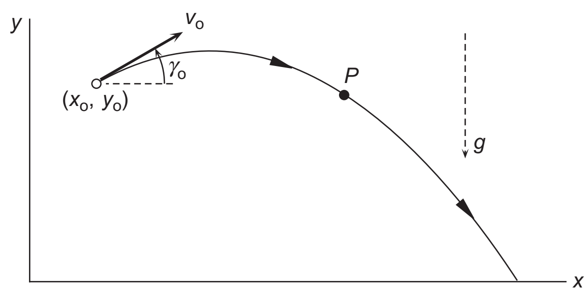


FIGURE 1.11

Flight of a low altitude projectile in free fall (no atmosphere).

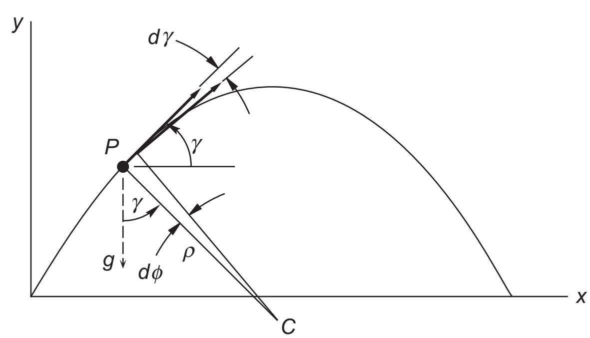


FIGURE 1.12

Relationship between $d\gamma$ and $d\rho$ for a "flat" earth.

Substituting Equation 1.25_2 into (a) and solving for the radius of curvature yields

$$\rho = \frac{v^2}{g\cos\gamma} \tag{b}$$

Combining Equations 1.37 and (b), we find the time rate of change of the flight path angle,

$$\boxed{\dot{\gamma} = -\frac{g\cos\gamma}{v}}$$

1.5 NEWTON'S LAW OF MOTION

Force is not a primitive concept like mass because it is intimately connected with the concepts of motion and inertia. In fact, the only way to alter the motion of a body is to exert a force on it. The degree to which the motion is altered is a measure of the force. ***Newton's second law of motion*** quantifies this. If the resultant or net force on a body of mass m is $\mathbf{F}_{\text{net}}$, then

$$\mathbf{F}_{\text{net}} = m\mathbf{a} \tag{1.38}$$

In this equation, $\mathbf{a}$ is the absolute acceleration of the center of mass. The absolute acceleration is measured in a frame of reference which itself has neither translational nor rotational acceleration relative to the fixed stars. Such a reference is called an absolute or ***inertial frame of reference***.

Force, then, is related to the primitive concepts of mass, length and time by Newton's second law. The unit of force, appropriately, is the newton, which is the force required to impart an acceleration of $1\,\text{m/s}^2$ to a mass of 1 kg. A mass of one kilogram therefore weighs 9.81 newtons at the earth's surface. The kilogram is not a unit of force.

Confusion can arise when mass is expressed in units of force, as frequently occurs in U.S. engineering practice. In common parlance either the pound or the ton (2000 pounds) is more likely to be used to express the mass. The pound of mass is officially defined precisely in terms of the kilogram as shown in Table A.3. Since one pound of mass weighs one pound of force where the standard sea-level acceleration of gravity ($g_0 = 9.80665\,\text{m/s}^2$) exists, we can use Newton's second law to relate the pound of force to the newton:

$$1\text{ lb (force)} = 0.4536\text{ kg} \times 9.807\text{ m/s}^2 = 4.448\text{ N}$$

The slug is the quantity of matter accelerated at one foot per second^2 by a force of one pound. We can again use Newton's second law to relate the slug to the kilogram. Noting the relationship between feet and meters in Table A.3, we find

$$1\text{ slug} = \frac{1\text{ lb}}{1\text{ ft/s}^2} = \frac{4.448\text{ N}}{0.3048\text{ m/s}^2} = 14.59\,\frac{\text{kg}\cdot\text{m/s}^2}{\text{m/s}^2} = 14.59\text{ kg}$$

Example 1.9

On a NASA mission the space shuttle Atlantis orbiter was reported to weigh 239,255 lb just prior to lift off. On orbit 18 at an altitude of about 350 km, the orbiter's weight was reported to be 236,900 lb. (a) What was the mass, in kilograms, of Atlantis on the launch pad and in orbit? (b) If no mass were lost between launch and orbit 18, what would have been the weight of Atlantis, in pounds?

Solution

(a) The given data illustrates the common use of weight in pounds as a measure of mass. The "weights" given are actually the mass in pounds of mass. Therefore, prior to launch

$$m_{\text{launch pad}} = 239{,}255 \text{ lb (mass)} \times \frac{0.4536 \text{ kg}}{1 \text{ lb (mass)}} = \boxed{108{,}500 \text{ kg}}$$

In orbit,

$$m_{\text{orbit 18}} = 236{,}900 \text{ lb (mass)} \times \frac{0.4536 \text{ kg}}{1 \text{ lb (mass)}} = \boxed{107{,}500 \text{ kg}}$$

The decrease in mass is the propellant expended by the orbital maneuvering and reaction control rockets on the orbiter.

(b) Since the space shuttle launch pad at Kennedy Space Center is essentially at sea level, the launch-pad weight of Atlantis in lb (force) is numerically equal to its mass in lb (mass). With no change in mass, the force of gravity at 350 km would be, according to Equation 1.36,

$$W = 239{,}255 \text{ lb (force)} \times \left(\frac{1}{1 + \dfrac{350}{6378}}\right)^2 = \boxed{215{,}000 \text{ lb (force)}}$$

The integral of a force **F** over a time interval is called the ***impulse of the force***, $\mathcal{I}$

$$\mathcal{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt \tag{1.39}$$

Impulse is a vector quantity. From Equation 1.38 it is apparent that if the mass is constant, then

$$\mathcal{I}_{\text{net}} = \int_{t_1}^{t_2} m\frac{d\mathbf{v}}{dt}dt = m\mathbf{v}_2 - m\mathbf{v}_1 \tag{1.40}$$

That is, the net impulse on a body yields a change $m\Delta\mathbf{v}$ in its linear momentum, so that

$$\Delta\mathbf{v} = \frac{\mathcal{I}_{\text{net}}}{m} \tag{1.41}$$

If $\mathbf{F}_{net}$ is constant, then $\mathcal{I}_{net} = \mathbf{F}_{net}\Delta t$, in which case Equation 1.41 becomes

$$\Delta\mathbf{v} = \frac{\mathbf{F}_{net}}{m}\Delta t \qquad (\text{if } \mathbf{F}_{net} \text{ is constant}) \tag{1.42}$$

Let us conclude this section by introducing the concept of angular momentum. The moment of the net force about O in Figure 1.13 is

$$\mathbf{M}_{O\text{net}} = \mathbf{r} \times \mathbf{F}_{net}$$

Substituting Equation 1.38 yields

$$\mathbf{M}_{O\text{net}} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times m\frac{d\mathbf{v}}{dt} \tag{1.43}$$

But, keeping in mind that the mass is constant,

$$\mathbf{r} \times m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) - \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) - (\mathbf{v} \times m\mathbf{v})$$

Since $\mathbf{v} \times m\mathbf{v} = m(\mathbf{v} \times \mathbf{v}) = \mathbf{0}$, it follows that Equation 1.43 can be written

$$\mathbf{M}_{O\text{net}} = \frac{d\mathbf{H}_O}{dt} \tag{1.44}$$

where $\mathbf{H}_O$ is the ***angular momentum*** about O,

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{1.45}$$

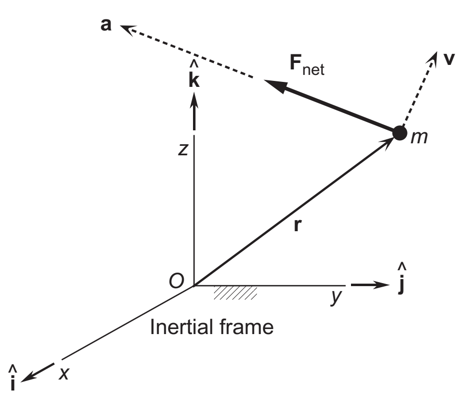


FIGURE 1.13

The absolute acceleration of a particle is in the direction of the net force.

Thus, just as the net force on a particle changes its linear momentum $m\mathbf{v}$, the moment of that force about a fixed point changes the moment of its linear momentum about that point. Integrating Equation 1.44 with respect to time yields

$$\int_{t_1}^{t_2} \mathbf{M}_{O\text{net}} dt = \mathbf{H}_{O2} - \mathbf{H}_{O1} \tag{1.46}$$

The integral on the left is the net ***angular impulse***. This angular impulse–momentum equation is the rotational analog of the linear impulse–momentum relation given in Equation 1.40.

Example 1.10

A particle of mass m is attached to point O by an inextensible string of length l (Figure 1.14). Initially the string is slack when m is moving to the left with a speed v_0 in the position shown. Calculate (a) the speed of m just after the string becomes taut and (b) the average force in the string over the small time interval Δt required to change the direction of the particle's motion.

Solution

(a) Initially, the position and velocity of the particle are

$$\mathbf{r}_1 = c\hat{\mathbf{i}} + d\hat{\mathbf{j}} \qquad \mathbf{v}_1 = -v_o\hat{\mathbf{i}}$$

The angular momentum about O is

$$\mathbf{H}_1 = \mathbf{r}_1 \times m\mathbf{v}_1 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ c & d & 0 \\ -mv_o & 0 & 0 \end{vmatrix} = mv_o d\hat{\mathbf{k}} \tag{a}$$

Just after the string becomes taut

$$\mathbf{r}_2 = -\sqrt{l^2 - d^2}\,\hat{\mathbf{i}} + d\hat{\mathbf{j}} \qquad \mathbf{v}_2 = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} \tag{b}$$

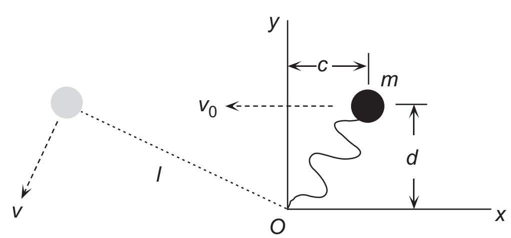


FIGURE 1.14

Particle attached to O by an inextensible string.

and the angular momentum is

$$\mathbf{H}_2 = \mathbf{r}_2 \times m\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sqrt{l^2 - d^2} & d & 0 \\ v_x & v_y & 0 \end{vmatrix} = (-mv_x d - mv_y\sqrt{l^2 - d^2})\hat{\mathbf{k}} \tag{c}$$

Initially the force exerted on m by the slack string is zero. When the string becomes taut, the force exerted on m passes through O. Therefore, the moment of the net force on m about O remains zero. According to Equation 1.46,

$$\mathbf{H}_2 = \mathbf{H}_1$$

Substituting (a) and (c), yields

$$v_x d + \sqrt{l^2 - d^2}\,v_y = -v_o d \tag{d}$$

The string is inextensible, so the component of the velocity of m along the string must be zero:

$$\mathbf{v}_2 \cdot \mathbf{r}_2 = 0$$

Substituting $\mathbf{v}_2$ and $\mathbf{r}_2$ from (b) and solving for v_y, we get

$$v_y = v_x\sqrt{\frac{l^2}{d^2} - 1} \tag{e}$$

Solving (d) and (e) for v_x and v_y leads to

$$v_x = -\frac{d^2}{l^2}v_o \qquad v_y = -\sqrt{1 - \frac{d^2}{l^2}}\,\frac{d}{l}v_o \tag{f}$$

Thus, the speed, $v = \sqrt{v_x^2 + v_y^2}$, after the string becomes taut is

$$\boxed{v = \frac{d}{l}v_o}$$

(b) From Equation 1.40, the impulse on m during the time it takes the string to become taut is

$$\mathcal{I} = m(\mathbf{v}_2 - \mathbf{v}_1) = m\left[\left(-\frac{d^2}{l^2}v_o\hat{\mathbf{i}} - \sqrt{1 - \frac{d^2}{l^2}}\,\frac{d}{l}v_o\hat{\mathbf{j}}\right) - (-v_o\hat{\mathbf{i}})\right] = \left(1 - \frac{d^2}{l^2}\right)mv_o\hat{\mathbf{i}} - \sqrt{1 - \frac{d^2}{l^2}}\,\frac{d}{l}mv_o\hat{\mathbf{j}}$$

The magnitude of this impulse, which is directed along the string, is

$$\mathcal{I} = \sqrt{1 - \frac{d^2}{l^2}}\,mv_o$$

Hence, the average force in the string during the small time interval Δt required to change the direction of the velocity vector turns out to be

$$F_{avg} = \frac{\mathcal{I}}{\Delta t} = \boxed{\sqrt{1 - \frac{d^2}{l^2}}\frac{mv_o}{\Delta t}}$$

1.6 TIME DERIVATIVES OF MOVING VECTORS

Figure 1.15(a) shows a vector **A** inscribed in a rigid body *B* that is in motion relative to an inertial frame of reference (a rigid, Cartesian coordinate system which is fixed relative to the fixed stars). The magnitude of **A** is fixed. The body *B* is shown at two times, separated by the differential time interval *dt*. At time $t + dt$ the orientation of vector **A** differs slightly from that at time *t*, but its magnitude is the same. According to one of the many theorems of the prolific eighteenth century Swiss mathematician Leonhard Euler (1707–1783), there is a unique axis of rotation about which *B*, and therefore, **A** rotates during the differential time interval. If we shift the two vectors $\mathbf{A}(t)$ and $\mathbf{A}(t + dt)$ to the same point on the axis of rotation, so that they are tail-to-tail as shown in Figure 1.15(b), we can assess the difference $d\mathbf{A}$ between them caused by the infinitesimal rotation. Remember that shifting a vector to a parallel line does not change the vector. The rotation of the body *B* is measured in the plane perpendicular to the instantaneous axis of rotation. The amount of rotation is the angle $d\theta$ through which a line element normal to the rotation axis turns in the time interval *dt*. In Figure 1.15(b) that line element is the component of **A** normal to the axis of rotation. We can express the difference $d\mathbf{A}$ between $\mathbf{A}(t)$ and $\mathbf{A}(t + dt)$ as

$$d\mathbf{A} = \overbrace{\left[\left(\|\mathbf{A}\| \cdot \sin\phi\right) d\theta\right]}^{\text{magnitude of } d\mathbf{A}}\hat{\mathbf{n}} \tag{1.47}$$

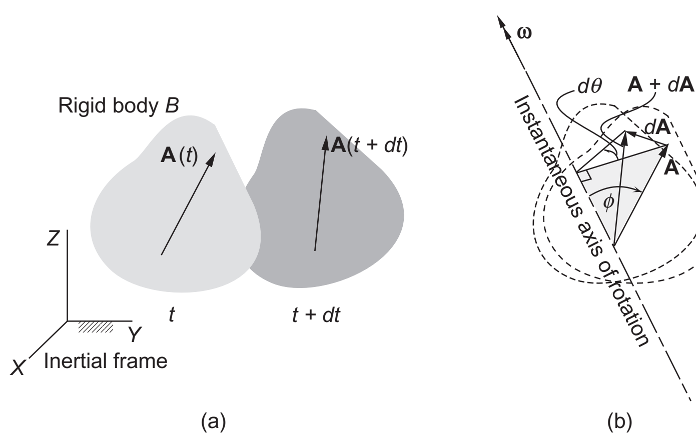


FIGURE 1.15

Displacement of a rigid body.

where $\hat{\mathbf{n}}$ is the unit normal to the plane defined by **A** and the axis of rotation, and it points in the direction of the rotation. The angle ϕ is the inclination of **A** to the rotation axis. By definition,

$$d\theta = \|\boldsymbol{\omega}\|dt \tag{1.48}$$

where $\boldsymbol{\omega}$ is the angular velocity vector, which points along the instantaneous axis of rotation and its direction is given by the right-hand rule. That is, wrapping the right hand around the axis of rotation, with the fingers pointing in the direction of $d\theta$ results in the thumb's defining the direction of $\boldsymbol{\omega}$. This is evident in Figure 1.15(b). It should be pointed out that the time derivative of $\boldsymbol{\omega}$ is the angular acceleration, usually given the symbol $\boldsymbol{\alpha}$. Thus,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} \tag{1.49}$$

Substituting Equation 1.48 into Equation 1.47, we get

$$d\mathbf{A} = \|\mathbf{A}\|\cdot\sin\phi\cdot\|\boldsymbol{\omega}\|dt\cdot\hat{\mathbf{n}} = \left(\|\boldsymbol{\omega}\|\cdot\|\mathbf{A}\|\cdot\sin\phi\right)\hat{\mathbf{n}}dt \tag{1.50}$$

By definition of the cross product, $\boldsymbol{\omega}\times\mathbf{A}$ is the product of the magnitude of $\boldsymbol{\omega}$, the magnitude of **A**, the sine of the angle between $\boldsymbol{\omega}$ and **A** and the unit vector normal to the plane of $\boldsymbol{\omega}$ and **A**, in the rotation direction. That is,

$$\boldsymbol{\omega}\times\mathbf{A} = \|\boldsymbol{\omega}\|\cdot\|\mathbf{A}\|\cdot\sin\phi\cdot\hat{\mathbf{n}} \tag{1.51}$$

Substituting Equation 1.51 into Equation 1.50 yields

$$d\mathbf{A} = \boldsymbol{\omega}\times\mathbf{A}dt$$

Dividing through by dt, we finally obtain

$$\boxed{\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega}\times\mathbf{A}} \quad \left(\text{if } \frac{d}{dt}\|\mathbf{A}\| = 0\right) \tag{1.52}$$

Equation 1.52 is a formula we can use to compute the ***time derivative of a rotating vector of constant magnitude***.

Example 1.11

Calculate the second time derivative of a vector **A** of constant magnitude, expressing the result in terms of $\boldsymbol{\omega}$ and its derivatives and **A**.

Solution

Differentiating Equation 1.52 with respect to time, we get

$$\frac{d^2\mathbf{A}}{dt^2} = \frac{d}{dt}\frac{d\mathbf{A}}{dt} = \frac{d}{dt}(\boldsymbol{\omega}\times\mathbf{A}) = \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{A} + \boldsymbol{\omega}\times\frac{d\mathbf{A}}{dt}$$

Using Equations 1.49 and 1.52, this can be written

$$\boxed{\frac{d^2\mathbf{A}}{dt^2} = \boldsymbol{\alpha}\times\mathbf{A} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})} \tag{1.53}$$

Example 1.12

Calculate the third derivative of a vector $\mathbf{A}$ of constant magnitude, expressing the result in terms of $\boldsymbol{\omega}$ and its derivatives and $\mathbf{A}$.

Solution

$$\begin{aligned}
\frac{d^3\mathbf{A}}{dt^3} &= \frac{d}{dt}\frac{d^2\mathbf{A}}{dt^2} = \frac{d}{dt}[\boldsymbol{\alpha}\times\mathbf{A} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})] \\
&= \frac{d}{dt}(\boldsymbol{\alpha}\times\mathbf{A}) + \frac{d}{dt}[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})] \\
&= \left(\frac{d\boldsymbol{\alpha}}{dt}\times\mathbf{A} + \boldsymbol{\alpha}\times\frac{d\mathbf{A}}{dt}\right) + \left[\frac{d\boldsymbol{\omega}}{dt}\times(\boldsymbol{\omega}\times\mathbf{A}) + \boldsymbol{\omega}\times\frac{d}{dt}(\boldsymbol{\omega}\times\mathbf{A})\right] \\
&= \left[\frac{d\boldsymbol{\alpha}}{dt}\times\mathbf{A} + \boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})\right] + \left[\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A}) + \boldsymbol{\omega}\times\left(\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{A} + \boldsymbol{\omega}\times\frac{d\mathbf{A}}{dt}\right)\right] \\
&= \left[\frac{d\boldsymbol{\alpha}}{dt}\times\mathbf{A} + \boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})\right] + \{\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A}) + \boldsymbol{\omega}\times[\boldsymbol{\alpha}\times\mathbf{A} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]\} \\
&= \frac{d\boldsymbol{\alpha}}{dt}\times\mathbf{A} + \boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A}) + \boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A}) + \boldsymbol{\omega}\times\left(\boldsymbol{\alpha}\times\mathbf{A}\right) + \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})] \\
&= \frac{d\boldsymbol{\alpha}}{dt}\times\mathbf{A} + 2\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A}) + \boldsymbol{\omega}\times(\boldsymbol{\alpha}\times\mathbf{A}) + \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]
\end{aligned}$$

$$\boxed{\frac{d^3\mathbf{A}}{dt^3} = \frac{d\boldsymbol{\alpha}}{dt}\times\mathbf{A} + 2\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A}) + \boldsymbol{\omega}\times[\boldsymbol{\alpha}\times\mathbf{A} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]}$$

Let *XYZ* be a rigid inertial frame of reference and *xyz* a rigid moving frame of reference, as shown in Figure 1.16. The moving frame can be moving (translating and rotating) freely of its own accord, or it can be attached to a physical object, such as a car, an airplane or a spacecraft. Kinematic quantities measured relative to the fixed inertial frame will be called ***absolute*** (e.g., absolute acceleration), and those measured relative to the moving system will be called ***relative*** (e.g., relative acceleration). The unit vectors along the inertial *XYZ* system are $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$, whereas those of the moving *xyz* system are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. The motion of the moving frame is arbitrary, and its absolute angular velocity is $\boldsymbol{\Omega}$. If, however, the moving frame is rigidly attached to an object, so that it not only translates but rotates with it, then the frame is called a body frame and the axes are referred to as body axes. A body frame clearly has the same angular velocity as the body to which it is bound.

Let $\mathbf{B}$ be any time-dependent vector. Resolved into components along the inertial frame of reference, it is expressed analytically as

$$\mathbf{B} = B_X\hat{\mathbf{I}} + B_Y\hat{\mathbf{J}} + B_Z\hat{\mathbf{K}}$$

where B_X, B_Y and B_Z are functions of time. Since $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ are fixed, the time derivative of $\mathbf{B}$ is simply given by

$$\frac{d\mathbf{B}}{dt} = \frac{dB_X}{dt}\hat{\mathbf{I}} + \frac{dB_Y}{dt}\hat{\mathbf{J}} + \frac{dB_Z}{dt}\hat{\mathbf{K}}$$

dB_X/dt, dB_Y/dt and dB_Z/dt are the components of the absolute time derivative of $\mathbf{B}$.

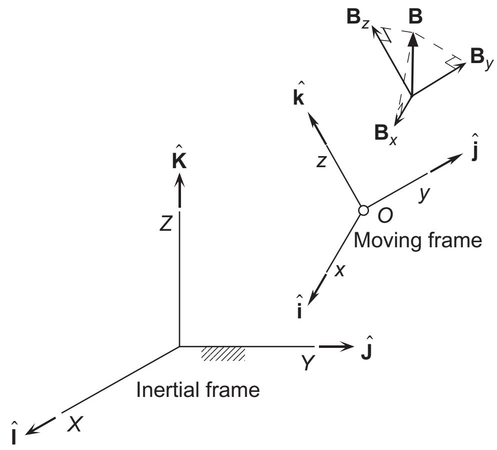


FIGURE 1.16

Fixed (inertial) and moving rigid frames of reference.

B may also be resolved into components along the moving *xyz* frame, so that, at any instant,

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} \tag{1.54}$$

Using this expression to calculate the time derivative of **B** yields

$$\frac{d\mathbf{B}}{dt} = \frac{dB_x}{dt}\hat{\mathbf{i}} + \frac{dB_y}{dt}\hat{\mathbf{j}} + \frac{dB_z}{dt}\hat{\mathbf{k}} + B_x\frac{d\hat{\mathbf{i}}}{dt} + B_y\frac{d\hat{\mathbf{j}}}{dt} + B_z\frac{d\hat{\mathbf{k}}}{dt} \tag{1.55}$$

The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are not fixed in space, but are continuously changing direction; therefore, their time derivatives are not zero. They obviously have a constant magnitude (unity) and, being attached to the *xyz* frame, they all have the angular velocity $\boldsymbol{\Omega}$. It follows from Equation 1.52 that

$$\frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\Omega}\times\hat{\mathbf{i}} \qquad \frac{d\hat{\mathbf{j}}}{dt} = \boldsymbol{\Omega}\times\hat{\mathbf{j}} \qquad \frac{d\hat{\mathbf{k}}}{dt} = \boldsymbol{\Omega}\times\hat{\mathbf{k}}$$

Substituting these on the right-hand side of Equation 1.55 yields

$$\begin{aligned}\frac{d\mathbf{B}}{dt} &= \frac{dB_x}{dt}\hat{\mathbf{i}} + \frac{dB_y}{dt}\hat{\mathbf{j}} + \frac{dB_z}{dt}\hat{\mathbf{k}} + B_x(\boldsymbol{\Omega}\times\hat{\mathbf{i}}) + B_y(\boldsymbol{\Omega}\times\hat{\mathbf{j}}) + B_z(\boldsymbol{\Omega}\times\hat{\mathbf{k}})\\ &= \frac{dB_x}{dt}\hat{\mathbf{i}} + \frac{dB_y}{dt}\hat{\mathbf{j}} + \frac{dB_z}{dt}\hat{\mathbf{k}} + (\boldsymbol{\Omega}\times B_x\hat{\mathbf{i}}) + (\boldsymbol{\Omega}\times B_y\hat{\mathbf{j}}) + (\boldsymbol{\Omega}\times B_z\hat{\mathbf{k}})\\ &= \frac{dB_x}{dt}\hat{\mathbf{i}} + \frac{dB_y}{dt}\hat{\mathbf{j}} + \frac{dB_z}{dt}\hat{\mathbf{k}} + \boldsymbol{\Omega}\times(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})\end{aligned}$$

In view of Equation 1.54, this can be written

$$\frac{d\mathbf{B}}{dt} = \left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} + \boldsymbol{\Omega}\times\mathbf{B} \tag{1.56}$$

where

$$\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} = \frac{dB_x}{dt}\hat{\mathbf{i}} + \frac{dB_y}{dt}\hat{\mathbf{j}} + \frac{dB_z}{dt}\hat{\mathbf{k}} \tag{1.57}$$

$d\mathbf{B}/dt)_{\text{rel}}$ is the time derivative of **B** relative to the moving frame. Equation 1.56 shows how the ***absolute time derivative*** is obtained from the ***relative time derivative***. Clearly, $d\mathbf{B}/dt = d\mathbf{B}/dt)_{\text{rel}}$ only when the moving frame is in pure translation ($\mathbf{\Omega} = \mathbf{0}$)).

Equation 1.56 can be used recursively to compute higher order time derivatives. Thus, differentiating Equation 1.56 with respect to t, we get

$$\frac{d^2\mathbf{B}}{dt^2} = \frac{d}{dt}\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{B} + \mathbf{\Omega}\times\frac{d\mathbf{B}}{dt}$$

Using Equation 1.56 in the last term yields

$$\frac{d^2\mathbf{B}}{dt^2} = \frac{d}{dt}\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{B} + \mathbf{\Omega}\times\left[\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} + \mathbf{\Omega}\times\mathbf{B}\right] \tag{1.58}$$

Equation 1.56 also implies that

$$\left.\frac{d}{dt}\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} = \left.\frac{d^2\mathbf{B}}{dt^2}\right)_{\text{rel}} + \mathbf{\Omega}\times\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} \tag{1.59}$$

where

$$\left.\frac{d^2\mathbf{B}}{dt^2}\right)_{\text{rel}} = \frac{d^2B_x}{dt^2}\hat{\mathbf{i}} + \frac{d^2B_y}{dt^2}\hat{\mathbf{j}} + \frac{d^2B_z}{dt^2}\hat{\mathbf{k}}$$

Substituting Equation 1.59 into Equation 1.58 yields

$$\frac{d^2\mathbf{B}}{dt^2} = \left[\left.\frac{d^2\mathbf{B}}{dt^2}\right)_{\text{rel}} + \mathbf{\Omega}\times\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}}\right] + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{B} + \mathbf{\Omega}\times\left[\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} + \mathbf{\Omega}\times\mathbf{B}\right]$$

Collecting terms, this becomes

$$\frac{d^2\mathbf{B}}{dt^2} = \left.\frac{d^2\mathbf{B}}{dt^2}\right)_{\text{rel}} + \dot{\mathbf{\Omega}}\times\mathbf{B} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{B}) + 2\mathbf{\Omega}\times\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} \tag{1.60}$$

where $\dot{\mathbf{\Omega}} \equiv d\mathbf{\Omega}/dt$ is the absolute angular acceleration of the xyz frame.

Formulas for higher order time derivatives are found in a similar fashion.

1.7 RELATIVE MOTION

Let P be a particle in arbitrary motion. The absolute position vector of P is $\mathbf{r}$ and the position of P relative to the moving frame is $\mathbf{r}_{rel}$. If $\mathbf{r}_O$ is the absolute position of the origin of the moving frame, then it is clear from Figure 1.17 that

$$\mathbf{r} = \mathbf{r}_O + \mathbf{r}_{rel} \tag{1.61}$$

Since $\mathbf{r}_{rel}$ is measured in the moving frame,

$$\mathbf{r}_{rel} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{1.62}$$

where x, y and z are the coordinates of P relative to the moving reference.

The absolute velocity $\mathbf{v}$ of P is $d\mathbf{r}/dt$, so that from Equation 1.61 we have

$$\mathbf{v} = \mathbf{v}_O + \frac{d\mathbf{r}_{rel}}{dt} \tag{1.63}$$

where $\mathbf{v}_O = d\mathbf{r}_O/dt$ is the (absolute) velocity of the origin of the xyz frame. From Equation 1.56, we can write

$$\frac{d\mathbf{r}_{rel}}{dt} = \mathbf{v}_{rel} + \boldsymbol{\Omega} \times \mathbf{r}_{rel} \tag{1.64}$$

where $\mathbf{v}_{rel}$ is the velocity of P relative to the xyz frame:

$$\mathbf{v}_{rel} = \left.\frac{d\mathbf{r}_{rel}}{dt}\right)_{rel} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \tag{1.65}$$

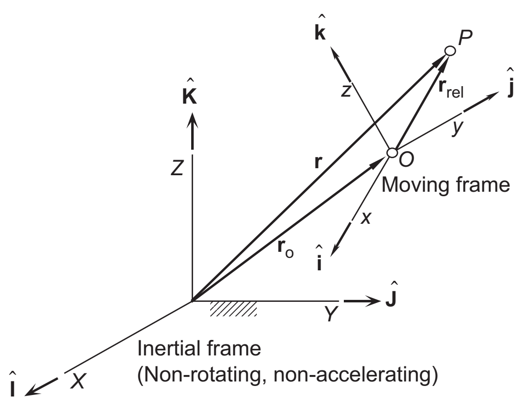


FIGURE 1.17

Absolute and relative position vectors.

Substituting Equation 1.64 into Equation 1.63 yields ***the relative velocity formula***

$$\mathbf{v} = \mathbf{v}_O + \boldsymbol{\Omega} \times \mathbf{r}_{rel} + \mathbf{v}_{rel} \tag{1.66}$$

The absolute acceleration $\mathbf{a}$ of P is $d\mathbf{v}/dt$, so that from Equation 1.63 we have

$$\mathbf{a} = \mathbf{a}_O + \frac{d^2\mathbf{r}_{rel}}{dt^2} \tag{1.67}$$

where $\mathbf{a}_O = d\mathbf{v}_O/dt$ is the absolute acceleration of the origin of the xyz frame. We evaluate the second term on the right using Equation 1.60.

$$\frac{d^2\mathbf{r}_{rel}}{dt^2} = \left.\frac{d^2\mathbf{r}_{rel}}{dt^2}\right)_{rel} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{rel} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\Omega} \times \left.\frac{d\mathbf{r}_{rel}}{dt}\right)_{rel} \tag{1.68}$$

Since $\mathbf{v}_{rel} = d\mathbf{r}_{rel}/dt)_{rel}$ and $\mathbf{a}_{rel} = d^2\mathbf{r}_{rel}/dt^2)_{rel}$, this can be written

$$\frac{d^2\mathbf{r}_{rel}}{dt^2} = \mathbf{a}_{rel} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{rel} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{rel} \tag{1.69}$$

Upon substituting this result into Equation 1.67, we obtain ***the relative acceleration formula***

$$\mathbf{a} = \mathbf{a}_O + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{rel} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel} \tag{1.70}$$

The cross product $2\boldsymbol{\Omega} \times \mathbf{v}_{rel}$ is called the ***Coriolis acceleration*** after Gustave Gaspard de Coriolis (1792–1843), the French mathematician who introduced this term (Coriolis, 1835). Because of the number of terms on the right, Equation 1.70 is sometimes referred to as the five-term acceleration formula.

Example 1.13

At a given instant, the absolute position, velocity and acceleration of the origin O of a moving frame are

$$\begin{aligned} \mathbf{r}_O &= 100\hat{\mathbf{I}} + 200\hat{\mathbf{J}} + 300\hat{\mathbf{K}}\ (\text{m}) \\ \mathbf{v}_O &= -50\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 10\hat{\mathbf{K}}\ (\text{m/s}) \qquad (\text{given}) \\ \mathbf{a}_O &= -15\hat{\mathbf{I}} + 40\hat{\mathbf{J}} + 25\hat{\mathbf{K}}\ (\text{m/s}^2) \end{aligned} \tag{a}$$

The angular velocity and acceleration of the moving frame are:

$$\begin{aligned} \boldsymbol{\Omega} &= 1.0\hat{\mathbf{I}} - 0.4\hat{\mathbf{J}} + 0.6\hat{\mathbf{K}}\ (\text{rad/s}) \\ \dot{\boldsymbol{\Omega}} &= -1.0\hat{\mathbf{I}} + 0.3\hat{\mathbf{J}} - 0.4\hat{\mathbf{K}}\ (\text{rad/s}^2) \qquad (\text{given}) \end{aligned} \tag{b}$$

The unit vectors of the moving frame are:

$$\begin{aligned} \hat{\mathbf{i}} &= 0.5571\hat{\mathbf{I}} + 0.7428\hat{\mathbf{J}} + 0.3714\hat{\mathbf{K}} \\ \hat{\mathbf{j}} &= -0.06331\hat{\mathbf{I}} + 0.4839\hat{\mathbf{J}} - 0.8728\hat{\mathbf{K}} \qquad (\text{given}) \\ \hat{\mathbf{k}} &= -0.8280\hat{\mathbf{I}} + 0.4627\hat{\mathbf{J}} + 0.3166\hat{\mathbf{K}} \end{aligned} \tag{c}$$

The absolute position, velocity and acceleration of P are:

$$\begin{aligned}\mathbf{r} &= 300\hat{\mathbf{I}} - 100\hat{\mathbf{J}} + 150\hat{\mathbf{K}}\ (\text{m})\\ \mathbf{v} &= 70\hat{\mathbf{I}} + 25\hat{\mathbf{J}} - 20\hat{\mathbf{K}}\ (\text{m/s}) \qquad \text{(given)}\\ \mathbf{a} &= 7.5\hat{\mathbf{I}} - 8.5\hat{\mathbf{J}} + 6.0\hat{\mathbf{K}}\ (\text{m/s}^2)\end{aligned} \tag{d}$$

Find (a) the velocity $\mathbf{v}_{\text{rel}}$ and (b) the acceleration $\mathbf{a}_{\text{rel}}$ of P relative to the moving frame.

Solution

Let us first use Equations (c) to solve for $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ (three equations in three unknowns):

$$\begin{aligned}\hat{\mathbf{I}} &= 0.5571\hat{\mathbf{i}} - 0.06331\hat{\mathbf{j}} - 0.8280\hat{\mathbf{k}}\\ \hat{\mathbf{J}} &= 0.7428\hat{\mathbf{i}} + 0.4839\hat{\mathbf{j}} + 0.4627\hat{\mathbf{k}}\\ \hat{\mathbf{K}} &= 0.3714\hat{\mathbf{i}} - 0.8728\hat{\mathbf{j}} + 0.3166\hat{\mathbf{k}}\end{aligned} \tag{e}$$

(a) The relative position vector is

$$\mathbf{r}_{\text{rel}} = \mathbf{r} - \mathbf{r}_O = (300\hat{\mathbf{I}} - 100\hat{\mathbf{J}} + 150\hat{\mathbf{K}}) - (100\hat{\mathbf{I}} + 200\hat{\mathbf{J}} + 300\hat{\mathbf{K}}) = 200\hat{\mathbf{I}} - 300\hat{\mathbf{J}} - 150\hat{\mathbf{K}}\ (\text{m}) \tag{f}$$

From Equation 1.66, the relative velocity vector is

$$\begin{aligned}\mathbf{v}_{\text{rel}} &= \mathbf{v} - \mathbf{v}_O - \boldsymbol{\Omega}\times\mathbf{r}_{\text{rel}}\\ &= (70\hat{\mathbf{I}} + 25\hat{\mathbf{J}} - 20\hat{\mathbf{K}}) - (-50\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 10\hat{\mathbf{K}}) - \begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 1.0 & -0.4 & 0.6\\ 200 & -300 & -150\end{vmatrix}\\ &= (70\hat{\mathbf{I}} + 25\hat{\mathbf{J}} - 20\hat{\mathbf{K}}) - (-50\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 10\hat{\mathbf{K}}) - (240\hat{\mathbf{I}} + 270\hat{\mathbf{J}} - 220\hat{\mathbf{K}})\end{aligned}$$

or

$$\mathbf{v}_{\text{rel}} = -120\hat{\mathbf{I}} - 275\hat{\mathbf{J}} + 210\hat{\mathbf{K}}\ (\text{m/s}) \tag{g}$$

To obtain the components of the relative velocity along the axes of the moving frame, substitute Equations (e) into Equation (g).

$$\begin{aligned}\mathbf{v}_{\text{rel}} &= -120(0.5571\hat{\mathbf{i}} - 0.06331\hat{\mathbf{j}} - 0.8280\hat{\mathbf{k}})\\ &\quad - 275(0.7428\hat{\mathbf{i}} + 0.4839\hat{\mathbf{j}} + 0.4627\hat{\mathbf{k}}) + 210(0.3714\hat{\mathbf{i}} - 0.8728\hat{\mathbf{j}} + 0.3166\hat{\mathbf{k}})\end{aligned}$$

so that

$$\boxed{\mathbf{v}_{\text{rel}} = -193.1\hat{\mathbf{i}} - 308.8\hat{\mathbf{j}} + 38.60\hat{\mathbf{k}}\ (\text{m/s})} \tag{h}$$

Alternatively, in terms of the unit vector $\hat{\mathbf{u}}_v$ in the direction of $\mathbf{v}_{\text{rel}}$

$$\boxed{\mathbf{v}_{\text{rel}} = 366.2\hat{\mathbf{u}}_v\ (\text{m/s}),\ \text{where}\ \hat{\mathbf{u}}_v = -0.5272\hat{\mathbf{i}} - 0.8432\hat{\mathbf{j}} + 0.1005\hat{\mathbf{k}}} \tag{i}$$

(b) To find the relative acceleration, we use the five-term acceleration formula, Equation 1.70:

$$\mathbf{a}_{\text{rel}} = \mathbf{a} - \mathbf{a}_O - \dot{\mathbf{\Omega}} \times \mathbf{r}_{\text{rel}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{\text{rel}}) - 2(\mathbf{\Omega} \times \mathbf{v}_{\text{rel}})$$

$$= \mathbf{a} - \mathbf{a}_O - \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -1.0 & 0.3 & -0.4 \\ 200 & -300 & -150 \end{vmatrix} - \mathbf{\Omega} \times \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 1.0 & -0.4 & 0.6 \\ 200 & -300 & -150 \end{vmatrix} - 2\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 1.0 & -0.4 & 0.6 \\ -120 & -275 & 210 \end{vmatrix}$$

$$= \mathbf{a} - \mathbf{a}_O - (-165\hat{\mathbf{I}} - 230\hat{\mathbf{J}} + 240\hat{\mathbf{K}}) - \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 1.0 & -0.4 & 0.6 \\ 240 & 270 & -220 \end{vmatrix} - (162\hat{\mathbf{I}} - 564\hat{\mathbf{J}} - 646\hat{\mathbf{K}})$$

$$= (7.5\hat{\mathbf{I}} - 8.5\hat{\mathbf{J}} + 6\hat{\mathbf{K}}) - (-15\hat{\mathbf{I}} + 40\hat{\mathbf{J}} + 25\hat{\mathbf{K}}) - (-165\hat{\mathbf{I}} - 230\hat{\mathbf{J}} + 240\hat{\mathbf{K}})$$
$$- (-74\hat{\mathbf{I}} + 364\hat{\mathbf{J}} + 366\hat{\mathbf{K}}) - (162\hat{\mathbf{I}} - 564\hat{\mathbf{J}} - 646\hat{\mathbf{K}})$$

$$\mathbf{a}_{\text{rel}} = 99.5\hat{\mathbf{I}} + 381.5\hat{\mathbf{J}} + 21.0\hat{\mathbf{K}} \ (\text{m/s}^2) \tag{j}$$

The components of the relative acceleration along the axes of the moving frame are found by substituting Equations (e) into Equation (j):

$$\mathbf{a}_{\text{rel}} = 99.5(0.5571\hat{\mathbf{i}} - 0.06331\hat{\mathbf{j}} - 0.8280\hat{\mathbf{k}}) + 381.5(0.7428\hat{\mathbf{i}} + 0.4839\hat{\mathbf{j}} + 0.4627\hat{\mathbf{k}})$$
$$+ 21.0(0.3714\hat{\mathbf{i}} - 0.8728\hat{\mathbf{j}} + 0.3166\hat{\mathbf{k}})$$

$$\boxed{\mathbf{a}_{\text{rel}} = 346.6\mathbf{i} + 160.0\mathbf{j} + 100.8\mathbf{k} \ (\text{m/s}^2)} \tag{k}$$

Or, in terms of the unit vector $\hat{\mathbf{u}}_a$ in the direction of $\mathbf{a}_{\text{rel}}$

$$\boxed{\mathbf{a}_{\text{rel}} = 394.8\hat{\mathbf{u}}_a \ (\text{m/s}^2), \quad \text{where } \hat{\mathbf{u}}_a = 0.8778\hat{\mathbf{i}} + 0.4052\hat{\mathbf{j}} + 0.2553\hat{\mathbf{k}}} \tag{l}$$

Figure 1.18 shows the nonrotating inertial frame of reference *XYZ* with its origin at the center *C* of the earth, which we shall assume to be a sphere. That assumption will be relaxed in Chapter 5. Embedded in the earth and rotating with it is the orthogonal $x'y'z'$ frame, also centered at *C*, with the z' axis parallel to Z, the earth's axis of rotation. The x' axis intersects the equator at the prime meridian (zero degrees longitude), which passes through Greenwich in London, England. The angle between *X* and x' is θ_G, and the rate of increase of θ_G is just the angular velocity Ω of the earth. *P* is a particle (e.g., an airplane, spacecraft, etc.), which is moving in an arbitrary fashion above the surface of the earth. $\mathbf{r}_{\text{rel}}$ is the position vector of *P* relative to *C* in the rotating $x'y'z'$ system. At a given instant, *P* is directly over point *O*, which lies on the earth's surface at longitude Λ and latitude ϕ. Point *O* coincides instantaneously with the origin of what is known as a topocentric-horizon coordinate system *xyz*. For our purposes *x* and *y* are measured positive eastward and northward along the local latitude and meridian, respectively, through *O*. The tangent plane to the earth's surface at *O* is the local horizon. The *z*-axis is the local vertical (straight up), and it is directed radially outward from the center of the earth. The unit vectors of the *xyz* frame are $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$, as indicated in Figure 1.18. Keep in mind that *O* remains directly below *P*, so that as *P* moves, so do the *xyz* axes. Thus, the $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ triad, which comprises the unit vectors of a spherical coordinate system, varies in direction as *P* changes location, thereby accounting for the curvature of the earth.

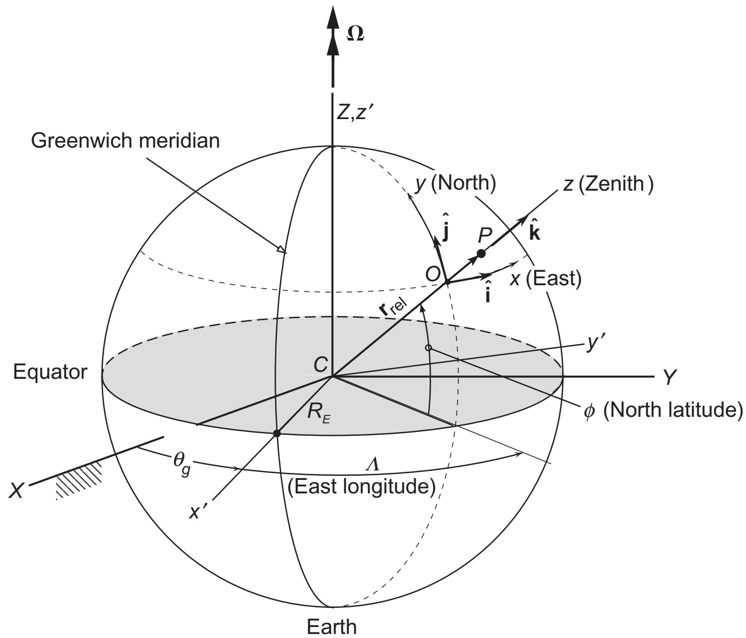


FIGURE 1.18

Earth-centered inertial frame (*XYZ*); earth-centered noninertial $x'y'z'$ frame embedded in and rotating with the earth; and a noninertial, topocentric-horizon frame *xyz* attached to a point *O* on the earth's surface.

Let us find the absolute velocity and acceleration of *P*. It is convenient to first obtain the velocity and acceleration of *P* relative to the nonrotating earth, and then use Equations 1.66 and 1.70 to calculate their inertial values.

The relative position vector can be written

$$\mathbf{r}_{\text{rel}} = (R_E + z)\hat{\mathbf{k}} \tag{1.71}$$

where R_E is the radius of the earth and z is the height of *P* above the earth (i.e., its altitude). The time derivative of $\mathbf{r}_{\text{rel}}$ is the velocity $\mathbf{v}_{\text{rel}}$ relative to the nonrotating earth,

$$\mathbf{v}_{\text{rel}} = \frac{d\mathbf{r}_{\text{rel}}}{dt} = \dot{z}\hat{\mathbf{k}} + (R_E + z)\frac{d\hat{\mathbf{k}}}{dt} \tag{1.72}$$

To calculate $d\hat{\mathbf{k}}/dt$, we must use Equation 1.52. The angular velocity $\boldsymbol{\omega}$ of the *xyz* frame relative to the nonrotating earth is found in terms of the rates of change of latitude ϕ and longitude Λ,

$$\boldsymbol{\omega} = -\dot{\phi}\hat{\mathbf{i}} + \dot{\Lambda}\cos\phi\hat{\mathbf{j}} + \dot{\Lambda}\sin\phi\hat{\mathbf{k}} \tag{1.73}$$

Thus,

$$\frac{d\hat{\mathbf{k}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{k}} = \dot{\Lambda}\cos\phi\hat{\mathbf{i}} + \dot{\phi}\hat{\mathbf{j}} \tag{1.74}$$

Let us also record the following for future use:

$$\frac{d\hat{\mathbf{j}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{j}} = -\dot{\Lambda}\sin\phi\hat{\mathbf{i}} - \dot{\phi}\hat{\mathbf{k}} \tag{1.75}$$

$$\frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{i}} = \dot{\Lambda}\sin\phi\hat{\mathbf{j}} - \dot{\Lambda}\cos\phi\hat{\mathbf{k}} \tag{1.76}$$

Substituting Equation 1.74 into Equation 1.72 yields

$$\mathbf{v}_{\text{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{1.77}$$

where

$$\dot{x} = (R_E + z)\dot{\Lambda}\cos\phi \qquad \dot{y} = (R_E + z)\dot{\phi} \tag{1.77b}$$

It is convenient to use these results to express the rates of change of latitude and longitude in terms of the components of relative velocity over the earth's surface,

$$\dot{\phi} = \frac{\dot{y}}{R_E + z} \qquad \dot{\Lambda} = \frac{\dot{x}}{(R_E + z)\cos\phi} \tag{1.78}$$

The time derivatives of these two expressions are

$$\ddot{\phi} = \frac{(R_E + z)\ddot{y} - \dot{y}\dot{z}}{(R_E + z)^2} \qquad \ddot{\Lambda} = \frac{(R_E + z)\ddot{x}\cos\phi - (\dot{z}\cos\phi - \dot{y}\sin\phi)\dot{x}}{(R_E + z)^2\cos^2\phi} \tag{1.79}$$

The acceleration of P relative to the nonrotating earth is found by taking the time derivative of $\mathbf{v}_{\text{rel}}$. From Equation 1.77 we thereby obtain

$$\begin{aligned}\mathbf{a}_{\text{rel}} &= \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} + \dot{x}\frac{d\hat{\mathbf{i}}}{dt} + \dot{y}\frac{d\hat{\mathbf{j}}}{dt} + \dot{z}\frac{d\hat{\mathbf{k}}}{dt}\\ &= [\dot{z}\dot{\Lambda}\cos\phi + (R_E + z)\ddot{\Lambda}\cos\phi - (R_E + z)\dot{\phi}\dot{\Lambda}\sin\phi]\hat{\mathbf{i}} + [\dot{z}\dot{\phi} + (R_E + z)\ddot{\phi}]\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}\\ &\quad + (R_E + z)\dot{\Lambda}\cos\phi(\boldsymbol{\omega} \times \hat{\mathbf{i}}) + (R_E + z)\dot{\phi}(\boldsymbol{\omega} \times \hat{\mathbf{j}}) + \dot{z}(\boldsymbol{\omega} \times \hat{\mathbf{k}})\end{aligned}$$

Substituting Equations 1.74 through 1.76 together with 1.78 and 1.79 into this expression yields, upon simplification,

$$\mathbf{a}_{\text{rel}} = \left[\ddot{x} + \frac{\dot{x}(\dot{z} - \dot{y}\tan\phi)}{R_E + z}\right]\mathbf{i} + \left(\ddot{y} + \frac{\dot{y}\dot{z} + \dot{x}^2\tan\phi}{R_E + z}\right)\hat{\mathbf{j}} + \left(\ddot{z} - \frac{\dot{x}^2 + \dot{y}^2}{R_E + z}\right)\hat{\mathbf{k}} \tag{1.80}$$

Observe that the curvature of the earth's surface is neglected by letting $R_E + z$ become infinitely large, in which case

$$\left.\mathbf{a}_{\text{rel}}\right)_{\text{neglecting earth's curvature}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}$$

That is, for a "flat earth," the components of the relative acceleration vector are just the derivatives of the components of the relative velocity vector.

For the absolute velocity we have, according to Equation 1.66,

$$\mathbf{v} = \mathbf{v}_C + \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}} \tag{1.81}$$

From Figure 1.18 it can be seen that $\hat{\mathbf{K}} = \cos\phi\hat{\mathbf{j}} + \sin\phi\hat{\mathbf{k}}$, which means the angular velocity of the earth is

$$\boldsymbol{\Omega} = \Omega\hat{\mathbf{K}} = \Omega\cos\phi\hat{\mathbf{j}} + \Omega\sin\phi\hat{\mathbf{k}} \tag{1.82}$$

Substituting this, together with Equations 1.71 and 1.77a and the fact that $\mathbf{v}_C = \mathbf{0}$, into Equation 1.81 yields

$$\mathbf{v} = [\dot{x} + \Omega(R_E + z)\cos\phi]\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{1.83}$$

From Equation 1.70 the absolute acceleration of P is

$$\mathbf{a} = \mathbf{a}_C + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{\text{rel}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

Since $\mathbf{a}_C = \dot{\boldsymbol{\Omega}} = \mathbf{0}$, we find, upon substituting Equations 1.71, 1.77a, 1.80 and 1.82, that

$$\begin{aligned}\mathbf{a} = &\left[\ddot{x} + \frac{\dot{x}(\dot{z} - \dot{y}\tan\phi)}{R_E + z} + 2\Omega(\dot{z}\cos\phi - \dot{y}\sin\phi)\right]\hat{\mathbf{i}} \\ &+ \left\{\ddot{y} + \frac{\dot{y}\dot{z} + \dot{x}^2\tan\phi}{R_E + z} + \Omega\sin\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{j}} \\ &+ \left\{\ddot{z} - \frac{\dot{x}^2 + \dot{y}^2}{R_E + z} - \Omega\cos\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{k}}\end{aligned} \tag{1.84}$$

Some special cases of Equations 1.83 and 1.84 follow.

Straight and level, unaccelerated flight: $\dot{z} = \ddot{z} = \ddot{x} = \ddot{y} = 0$

$$\mathbf{v} = [\dot{x} + \Omega(R_E + z)\cos\phi]\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} \tag{1.85a}$$

$$\begin{aligned}\mathbf{a} = &-\left[\frac{\dot{x}\dot{y}\tan\phi}{R_E + z} + 2\Omega\dot{y}\sin\phi\right]\hat{\mathbf{i}} + \left\{\frac{\dot{x}^2\tan\phi}{R_E + z} + \Omega\sin\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{j}} \\ &-\left\{\frac{\dot{x}^2 + \dot{y}^2}{R_E + z} + \Omega\cos\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{k}}\end{aligned} \tag{1.85b}$$

Flight due north (y) at constant speed and altitude: $\dot{z} = \ddot{z} = \dot{x} = \ddot{x} = \ddot{y} = 0$

$$\mathbf{v} = \Omega(R_E + z)\cos\phi\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} \tag{1.86a}$$

$$\mathbf{a} = -2\Omega\dot{y}\sin\phi\hat{\mathbf{i}} + \Omega^2(R_E + z)\sin\phi\cos\phi\hat{\mathbf{j}} - \left[\frac{\dot{y}^2}{R_E + z} + \Omega^2(R_E + z)\cos^2\phi\right]\hat{\mathbf{k}} \tag{1.86b}$$

Flight due east (x) at constant speed and altitude: $\dot{z} = \ddot{z} = \ddot{x} = \dot{y} = \ddot{y} = 0$

$$\mathbf{v} = \left[\dot{x} + \Omega(R_E + z)\cos\phi\right]\hat{\mathbf{i}} \tag{1.87a}$$

$$\mathbf{a} = \left\{\frac{\dot{x}^2\tan\phi}{R_E + z} + \Omega\sin\phi\left[\Omega(R_E + z)\cos\phi + 2\dot{x}\right]\right\}\hat{\mathbf{j}} - \left\{\frac{\dot{x}^2}{R_E + z} + \Omega\cos\phi\left[\Omega(R_E + z)\cos\phi + 2\dot{x}\right]\right\}\hat{\mathbf{k}} \tag{1.87b}$$

Flight straight up (z): $\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = 0$

$$\mathbf{v} = \Omega(R_E + z)\cos\phi\hat{\mathbf{i}} + \dot{z}\hat{\mathbf{k}} \tag{1.88a}$$

$$\mathbf{a} = 2\Omega(\dot{z}\cos\phi)\hat{\mathbf{i}} + \Omega^2(R_E + z)\sin\phi\cos\phi\hat{\mathbf{j}} + \left[\ddot{z} - \Omega^2(R_E + z)\cos^2\phi\right]\hat{\mathbf{k}} \tag{1.88b}$$

Stationary: $\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0$

$$\mathbf{v} = \Omega(R_E + z)\cos\phi\hat{\mathbf{i}} \tag{1.89a}$$

$$\mathbf{a} = \Omega^2(R_E + z)\sin\phi\cos\phi\hat{\mathbf{j}} - \Omega^2(R_E + z)\cos^2\phi\hat{\mathbf{k}} \tag{1.89b}$$

Example 1.14

An airplane of mass 70,000 kg is traveling due north at latitude 30° north, at an altitude of 10 km (32,800 ft) with a speed of 300 m/s (671 mph). Calculate (a) the components of the absolute velocity and acceleration along the axes of the topocentric-horizon reference frame, and (b) the net force on the airplane.

Solution

(a) First, using the sidereal rotation period of the earth in Table A.1, we note that the earth's angular velocity is

$$\Omega = \frac{2\pi \text{ radians}}{\text{sidereal day}} = \frac{2\pi \text{ radians}}{23.93 \text{ hr}} = \frac{2\pi \text{ radians}}{86\,160 \text{ s}} = 7.292 \times 10^{-5} \text{ radians/s}$$

From Equation 1.86a, the absolute velocity is

$$\mathbf{v} = \Omega(R_E + z)\cos\phi\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} = \left[(7.292 \times 10^{-5}) \cdot (6378 + 10) \cdot 10^3 \cos 30°\right]\hat{\mathbf{i}} + 300\hat{\mathbf{j}}$$

or

$$\boxed{\mathbf{v} = 403.4\hat{\mathbf{i}} + 300\hat{\mathbf{j}} \text{ (m/s)}}$$

The 403.4 m/s (901 mph) component of velocity to the east (*x*-direction) is due entirely to the Earth's rotation.

From Equation 1.86b, the absolute acceleration is

$$\mathbf{a} = -2\Omega\dot{y}\sin\phi\hat{\mathbf{i}} + \Omega^2(R_E + z)\sin\phi\cos\phi\hat{\mathbf{j}} - \left[\frac{\dot{y}^2}{R_E + z} + \Omega^2(R_E + z)\cos^2\phi\right]\hat{\mathbf{k}}$$

$$= -2(7.292 \times 10^{-5}) \cdot 300 \cdot \sin 30°\hat{\mathbf{i}} + (7.292 \times 10^{-5})^2 \cdot (6378 + 10) \cdot 10^3 \cdot \sin 30° \cdot \cos 30°\hat{\mathbf{j}}$$

$$- \left[\frac{300^2}{(6378 + 10) \cdot 10^3} + (7.292 \times 10^{-5})^2 \cdot (6378 + 10) \cdot 10^3 \cdot \cos^2 30°\right]\hat{\mathbf{k}}$$

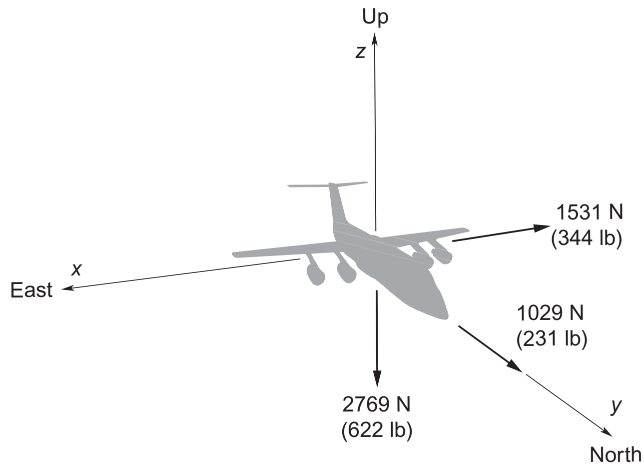


FIGURE 1.19

Components of the net force on the airplane.

or

$$\boxed{\mathbf{a} = -0.02187\hat{\mathbf{i}} + 0.01471\hat{\mathbf{j}} - 0.03956\hat{\mathbf{k}}\ (\mathrm{m/s^2})}$$

The westward (negative x) acceleration of 0.02187 m/s^2 is the Coriolis acceleration.
(b) Since the acceleration in part (a) is the absolute acceleration, we can use it in Newton's law to calculate the net force on the airplane,

$$\mathbf{F}_{\text{net}} = m\mathbf{a} = 70{,}000(-0.02187\hat{\mathbf{i}} + 0.01471\hat{\mathbf{j}} - 0.03956\hat{\mathbf{k}}) = \boxed{-1531\hat{\mathbf{i}} + 1029\hat{\mathbf{j}} - 2769\hat{\mathbf{k}}\ (\mathrm{N})}$$

Figure 1.19 shows the components of this relatively small force. The forward (y) and downward (negative z) forces are in the directions of the airplane's centripetal acceleration, caused by the earth's rotation and, in the case of the downward force, by the earth's curvature as well. The westward force is in the direction of the Coriolis acceleration, which is due the combined effects of the earth's rotation and the motion of the airplane. These net external forces must exist if the airplane is to fly the prescribed path.

In the vertical direction, the net force is that of the upward lift L of the wings plus the downward weight W of the aircraft, so that

$$\left.F_{\text{net}}\right)_z = L - W = -2769 \Rightarrow L = W - 2769\ (\mathrm{N})$$

Thus, the effect of the earth's rotation and curvature is to apparently produce an outward *centrifugal force*, reducing the weight of the airplane a bit, in this case by about 0.4 percent. The fictitious centrifugal force also increases the apparent drag in the flight direction by 1029 N. That is, in the flight direction

$$\left.F_{\text{net}}\right)_y = T - D = 1029\ \mathrm{N}$$

where T is the thrust and D is the drag. Hence

$$T = D + 1029\ (\mathrm{N})$$

The 1531 N force to the left, produced by crabbing the airplane very slightly in that direction, is required to balance the fictitious Coriolis force which would otherwise cause the airplane to deviate to the right of its flight path.

1.8 NUMERICAL INTEGRATION

Analysis of the motion of a spacecraft leads to ordinary differential equations with time as the independent variable. It is often impractical if not impossible to solve them exactly. Therefore, the ability to solve differential equations numerically is important. In this section we will take a look at a few common numerical integration schemes and investigate their accuracy and stability by applying them to some problems, which do have an analytical solution.

Particle mechanics is based on Newton's second law, Equation 1.38, which may be written

$$\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} \tag{1.90}$$

This is a second order, ordinary differential equation for the position vector $\mathbf{r}$ as a function of time. Depending on the complexity of the force function $\mathbf{F}$, there may or may not be a closed form, analytical solution of Equation 1.90. In the most trivial case, the force vector $\mathbf{F}$ and the mass m are constant, which means we can use elementary calculus to integrate Equation 1.90 twice to get

$$\mathbf{r} = \frac{\mathbf{F}}{2m}t^2 + \mathbf{C}_1 t + \mathbf{C}_2 \qquad (\mathbf{F} \text{ and } m \text{ constant}) \tag{1.91}$$

$\mathbf{C}_1$ and $\mathbf{C}_2$ are the two vector constants of integration. Since each vector has three components, there are a total of six scalar constants of integration. If the position and velocity are both specified at time $t = 0$ to be $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, respectively, then we have an initial value problem. Applying the initial conditions to Equation 1.91, we find $\mathbf{C}_1 = \dot{\mathbf{r}}_0$ and $\mathbf{C}_2 = \mathbf{r}_0$, which means

$$\mathbf{r} = \frac{\mathbf{F}}{2m}t^2 + \dot{\mathbf{r}}_0 t + \mathbf{r}_0 \qquad (\mathbf{F} \text{ and } m \text{ constant})$$

On the other hand, we may know the position $\mathbf{r}_0$ at $t = 0$ and the velocity $\dot{\mathbf{r}}_f$ at a later time $t = t_f$. These are boundary conditions and this is an example of a boundary value problem. Applying the boundary conditions to Equation 1.91 yields $\mathbf{C}_1 = \dot{\mathbf{r}}_f - F/m\,t_f$ and $\mathbf{C}_2 = \mathbf{r}_0$, which means

$$\mathbf{r} = \frac{\mathbf{F}}{2m}t^2 + \left(\dot{\mathbf{r}}_f - \frac{F}{m}t_f\right)t + \mathbf{r}_0 \qquad (\mathbf{F} \text{ and } m \text{ constant})$$

For the remainder of this section we will focus on the numerical solution of initial value problems only.

In general the function $\mathbf{F}$ in Equation 1.90 is not constant but is instead a function of time t, position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$. That is, $\mathbf{F} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$. Let us resolve the vector $\mathbf{r}$ and its derivatives as well as the force F into their Cartesian components in three-dimensional space:

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \qquad \ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \qquad \mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$$

The three components of Equation 1.90 are

$$\ddot{x} = \frac{F_x(t, \mathbf{r}, \dot{\mathbf{r}})}{m} \qquad \ddot{y} = \frac{F_y(t, \mathbf{r}, \dot{\mathbf{r}})}{m} \qquad \ddot{z} = \frac{F_z(t, \mathbf{r}, \dot{\mathbf{r}})}{m} \tag{1.92}$$

These are three uncoupled second order differential equations. For the purpose of numerical solution they must be reduced to six first order differential equations. This is accomplished by introducing six auxiliary variables y_1 through y_6, defined as follows:

$$\begin{aligned} y_1 &= x & y_2 &= y & y_3 &= z \\ y_4 &= \dot{x} & y_5 &= \dot{y} & y_6 &= \dot{z} \end{aligned} \tag{1.93}$$

In terms of these auxiliary variables, the position and velocity vectors are

$$\mathbf{r} = y_1\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + y_3\hat{\mathbf{k}} \qquad \dot{\mathbf{r}} = y_4\hat{\mathbf{i}} + y_5\hat{\mathbf{j}} + y_6\hat{\mathbf{k}}$$

Taking the derivative *d/dt* of each of the six expressions in Equation 1.93 yields

$$\begin{aligned} \frac{dy_1}{dt} &= \dot{x} & \frac{dy_2}{dt} &= \dot{y} & \frac{dy_3}{dt} &= \dot{z} \\ \frac{dy_4}{dt} &= \ddot{x} & \frac{dy_5}{dt} &= \ddot{y} & \frac{dy_6}{dt} &= \ddot{z} \end{aligned}$$

Upon substituting Equations 1.92 and 1.93, we arrive at the six first order differential equations

$$\begin{aligned} \dot{y}_1 &= y_4 \\ \dot{y}_2 &= y_5 \\ \dot{y}_3 &= y_6 \\ \dot{y}_4 &= \frac{F_x(t, y_1, y_2, y_3, y_4, y_5, y_6)}{m} \\ \dot{y}_5 &= \frac{F_y(t, y_1, y_2, y_3, y_4, y_5, y_6)}{m} \\ \dot{y}_6 &= \frac{F_z(t, y_1, y_2, y_3, y_4, y_5, y_6)}{m} \end{aligned} \tag{1.94}$$

These equations are coupled because the right side of each one contains variables that belong to other equations as well. The ***system of first order differential equations*** 1.94 can be written more compactly in vector notation as

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{1.95}$$

where the column vectors $\mathbf{y}$, $\dot{\mathbf{y}}$ and $\mathbf{f}$ are

$$\mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} \qquad \dot{\mathbf{y}} = \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \dot{y}_6 \end{Bmatrix} \qquad \mathbf{f} = \begin{Bmatrix} y_2 \\ y_4 \\ y_6 \\ F_x(t, \mathbf{y})/\mathrm{m} \\ F_y(t, \mathbf{y})/\mathrm{m} \\ F_z(t, \mathbf{y})/\mathrm{m} \end{Bmatrix} \tag{1.96}$$

Note that in this case $\mathbf{f}(t, \mathbf{y})$ is shorthand for $\mathbf{f}(t, y_1, y_2, y_3, y_4, y_5, y_6)$. Any set of one or more ordinary differential equations of any order can be cast in the form of Equation 1.95.

Example 1.15

Write the nonlinear differential equation

$$\dddot{x} - x\ddot{x} + \dot{x}^2 = 0 \tag{a}$$

as three first order differential equations.

Solution

Introducing the three auxiliary variables

$$y_1 = x \qquad y_2 = \dot{x} \qquad y_3 = \ddot{x} \tag{b}$$

we take the derivative of each one to get

$$\begin{aligned} dy_1/dt &= dx/dt = \dot{x} \\ dy_2/dt &= d\dot{x}/dt = \ddot{x} \\ dy_3/dt &= d\ddot{x}/dt = \dddot{x} \overbrace{=}^{\text{From (a)}} x\ddot{x} - \dot{x}^2 \end{aligned}$$

Substituting (b) on the right of these expressions yields

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= y_1 y_3 - {y_2}^2 \end{aligned} \tag{c}$$

This is a system of three first order, coupled ordinary differential equations. It is an autonomous system, since time t does not appear explicitly on the right side. The three equations can therefore be written compactly as $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$.

Before discussing some numerical integration schemes, it will be helpful to review the concept of the ***Taylor series***, named for the English mathematician Brook Taylor (1685–1731). Recall from calculus that if we know the value of a function $g(t)$ at time t and wish to approximate its value at a neighboring time $t + h$, we can use the Taylor series to express $g(t + h)$ as an infinite power series in h,

$$g(t + h) = g(t) + c_1 h + c_2 h^2 + c_3 h^3 + \cdots + c_n h^n + O(h^{n+1}) \tag{1.97}$$

The coefficients c_m are found by taking successively higher order derivatives of $g(t)$ according to the formula

$$c_m = \frac{1}{m!}\frac{d^m g(t)}{dt^m} \tag{1.98}$$

$O(h^{n+1})$ ("order of h to the $n+1$") means that the remaining terms of this infinite series all have h^{n+1} as a factor. In other words,

$$\lim_{h\to 0}\frac{O(h^{n+1})}{h^{n+1}} = c_{n+1}$$

$O(h^{n+1})$ is the truncation error due to retaining only terms up to h^n. The order of a Taylor series expansion is the highest power of h retained. The more terms of the Taylor series that we keep, the more accurate will be the representation of the function $g(t + h)$ in the neighborhood of t. Reducing h lowers the truncation error. For example, if we reduce h to $h/2$, then $O(h^n)$ goes down by a factor of $(1/2)^n$.

Example 1.16

Expand the function $\sin(t + h)$ in a Taylor series about $t = 1$. Plot the Taylor series of order 1, 2, 3 and 4 and compare them with $\sin(1 + h)$ for $-2 < h < 2$.

Solution

The nth order Taylor power series expansion of $\sin(t + h)$ is written

$$\sin(t + h) = p_n(h)$$

where, according to Equations 1.97 and 1.98, the polynomial p_n is given by

$$p_n(h) = \sum_{m=0}^{n} \frac{h^m}{m!} \frac{d^m \sin t}{dt^m}$$

Thus, the zero through fourth order Taylor series polynomials in h are

$$\begin{aligned}
p_0 &= \frac{h^0}{0!}\frac{d^0 \sin t}{dt^0} = \sin t \\
p_1 &= p_0 + \frac{h}{1!}\frac{d \sin t}{dt} = \sin t + h \cos t \\
p_2 &= p_1 + \frac{h^2}{2!}\frac{d^2 \sin t}{dt^2} = \sin t + h\cos t - \frac{h^2}{2}\sin t \\
p_3 &= p_2 + \frac{h^3}{3!}\frac{d^3 \sin t}{dt^3} = \sin t + h\cos t - \frac{h^2}{2}\sin t - \frac{h^3}{6}\cos t \\
p_4 &= p_3 + \frac{h^4}{4!}\frac{d^4 \sin t}{dt^4} = \sin t + h\cos t - \frac{h^2}{2}\sin t - \frac{h^3}{6}\cos t + \frac{h^4}{24}\sin t
\end{aligned}$$

For $t = 1$, p_1 through p_4 as well as $\sin(t + h)$ are plotted in Figure 1.20. As expected, we see that the higher degree Taylor polynomials for $\sin(1 + h)$ lie closer to $\sin(1 + h)$ over a wider range of h.

The numerical integration schemes that we shall examine are designed to solve first order ordinary differential equations of the form shown in Equation 1.95. To obtain a numerical solution of $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ over the time interval t_0 to t_f, we divide or "mesh" the interval into N discrete times $t_1, t_2, t_3, \cdots, t_N$, where $t_1 = t_0$ and $t_N = t_f$. The step size h is the difference between two adjacent times on the mesh, that is, $h = t_{i+1} - t_i$. h may be constant for all steps across the entire time span t_0 to t_f. Recent methods have adaptive step size control in which h varies from step to step to provide better accuracy and efficiency.

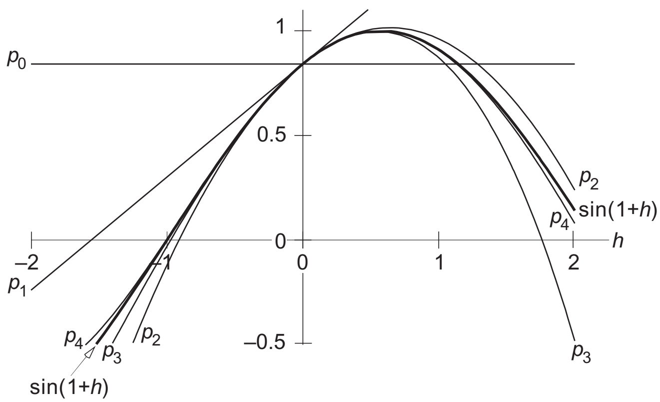


FIGURE 1.20

Plots of zero to fourth order Taylor series expansions of $\sin(1 + h)$.

Let us denote the values of $\mathbf{y}$ and $\dot{\mathbf{y}}$ at time t_i as $\mathbf{y}_i$ and $\mathbf{f}_i$, respectively, where $\mathbf{f}_i = \mathbf{f}(t_i, \mathbf{y}_i)$. In an initial value problem the values of all components of $\mathbf{y}$ at the initial time t_0 together with Equation 1.95 provide the information needed to determine $\mathbf{y}$ at the subsequent discrete times.

1.8.1 Runge-Kutta Methods

The ***Runge-Kutta (RK) methods*** were originally developed by the German mathematicians Carle Runge (1856–1927) and Martin Kutta (1867–1944). In the explicit, single-step *RK* methods, $\mathbf{y}_{i+1}$ at $t_i + h$ is obtained from $\mathbf{y}_i$ at t_i by the formula

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\phi(t_i, \mathbf{y}_i, h) \tag{1.99}$$

The increment function ϕ is an average of the derivative $d\mathbf{y}/dt$ over the time interval t_i to $t_i + h$. This average is obtained by evaluating the derivative $\mathbf{f}(t,\mathbf{y})$ at several points or "stages" within the time interval. The order of an *RK* method reflects the accuracy to which ϕ is computed, compared to a Taylor series expansion. A Runge-Kutta method of order p is called an *RKp* method. An *RKp* method is as accurate in computing $\mathbf{y}_i$ from Equation 1.99 as is the pth order Taylor series

$$\mathbf{y}(t_i + h) = \mathbf{y}_i + \mathbf{c}_1 h + \mathbf{c}_2 h^2 + \cdots \mathbf{c}_p h^p \tag{1.100}$$

An attractive feature of the *RK* schemes is that only the first derivative $\mathbf{f}(t,\mathbf{y})$ is required, and it is available from the differential equation itself (Equation 1.95). By contrast, the pth order Taylor series expansion in Equation 1.100 requires computing all derivatives of $\mathbf{y}$ through order p.

The higher the Runge-Kutta order, the more stages there are and the more accurate is ϕ. The number of stages equals the order of the *RK* method if the order is less than 5. If the number of stages is s, then there are s times $\tilde{t}$ within the interval t_i to $t_i + h$ at which we evaluate the derivatives $\mathbf{f}(t,\mathbf{y})$. These times are given by specifying numerical values of the nodes a_m in the expression

$$\tilde{t}_m = t_i + a_m h \qquad m = 1, 2, \cdots, s$$

At each of these times the value of $\tilde{\mathbf{y}}$ is obtained by providing numerical values for the coupling coefficients β_{mn} in the formula

$$\tilde{\mathbf{y}}_m = \mathbf{y}_i + h\sum_{n=1}^{m-1} b_{mn}\tilde{\mathbf{f}}_n \qquad m = 1,2,\cdots,s \tag{1.101}$$

The vector of derivatives $\tilde{\mathbf{f}}_m$ is evaluated at stage m by substituting $\tilde{t}_m$ and $\tilde{\mathbf{y}}_m$ into Equation 1.95,

$$\tilde{\mathbf{f}}_m = \mathbf{f}(\tilde{t}_m, \tilde{\mathbf{y}}_m) \qquad m = 1,2,\cdots,s \tag{1.102}$$

The increment function ϕ is a weighted sum of the derivatives $\tilde{\mathbf{f}}_m$ over the s stages within the time interval t_i to $t_i + h$,

$$\phi = \sum_{m=1}^{s} c_m \tilde{\mathbf{f}}_m \tag{1.103}$$

The coefficients c_m are known as the weights. Substituting Equation 1.103 into Equation 1.99 yields

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\sum_{m=1}^{s} c_m \tilde{\mathbf{f}}_m \tag{1.104}$$

The numerical values of the coefficients a_m, b_{mn} and c_m depend on which *RK* method is being used. It is convenient to write these coefficients as arrays, so that

$$\{\mathbf{a}\} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{Bmatrix} \qquad [\mathbf{b}] = \begin{bmatrix} b_{11} & & & \\ b_{21} & b_{22} & & \\ \vdots & \vdots & \cdots & \\ b_{s1} & b_{s2} & \cdots & b_{s,s-1} \end{bmatrix} \qquad \{\mathbf{c}\} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{Bmatrix} \quad (s = \text{the number of stages}) \tag{1.105}$$

$[\mathbf{b}]$ is undefined if $s = 1$. The nodes $\{\mathbf{a}\}$, coupling coefficients $[\mathbf{b}]$ and weights $\{\mathbf{c}\}$ for a given *RK* method are not necessarily unique, although research favors the choice of some sets over others. Details surrounding the derivation of these coefficients as well as in-depth discussions of not only *RK* but also the numerous other common numerical integration techniques may be found in numerical analysis textbooks, such as Butcher (2001).

For Runge-Kutta orders 1 through 4 we list below the commonly used values of the coefficients (Equation 1.105), the resulting formula for the derivatives $\tilde{\mathbf{f}}$ at each stage (Equation 1.102), and the formula for $\mathbf{y}_{i+1} - \mathbf{y}_i$ (Equation 1.104). These *RK* schemes all use a uniform step size h.

RK1 (Euler's method)

$$\begin{aligned} &\{\mathbf{a}\} = \{0\} \qquad \{\mathbf{c}\} = \{1\} \\ &\tilde{\mathbf{f}}_1 = \mathbf{f}(t_i, \mathbf{y}_i) \\ &\mathbf{y}_{i+1} = \mathbf{y}_i + h\tilde{\mathbf{f}}_1 \end{aligned} \tag{1.106}$$

RK2 (Heun's method)

$$\{\mathbf{a}\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad [\mathbf{b}] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \{\mathbf{c}\} = \begin{Bmatrix} 1/2 \\ 1/2 \end{Bmatrix}$$
$$\tilde{\mathbf{f}}_1 = \mathbf{f}(t_i, \mathbf{y}_i) \qquad \tilde{\mathbf{f}}_2 = \mathbf{f}(t_i + h, \mathbf{y}_i + h\tilde{\mathbf{f}}_1) \tag{1.107}$$
$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\left(\frac{1}{2}\tilde{\mathbf{f}}_1 + \frac{1}{2}\tilde{\mathbf{f}}_2\right)$$

RK3

$$\{\mathbf{a}\} = \begin{Bmatrix} 0 \\ 1/2 \\ 1 \end{Bmatrix} \qquad [\mathbf{b}] = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \\ -1 & 2 \end{bmatrix} \qquad \{\mathbf{c}\} = \begin{Bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{Bmatrix}$$
$$\tilde{\mathbf{f}}_1 = \mathbf{f}(t_i, \mathbf{y}_i) \qquad \tilde{\mathbf{f}}_2 = \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{y}_i + \frac{1}{2}h\tilde{\mathbf{f}}_1\right) \qquad \tilde{\mathbf{f}}_3 = \mathbf{f}(t_i + h, \mathbf{y}_i + h[-\tilde{\mathbf{f}}_1 + 2\tilde{\mathbf{f}}_2]) \tag{1.108}$$
$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\left(\frac{1}{6}\tilde{\mathbf{f}}_1 + \frac{2}{3}\tilde{\mathbf{f}}_2 + \frac{1}{6}\tilde{\mathbf{f}}_3\right)$$

RK4

$$\{\mathbf{a}\} = \begin{Bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1 \end{Bmatrix} \qquad [\mathbf{b}] = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{\mathbf{c}\} = \begin{Bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{Bmatrix}$$
$$\tilde{\mathbf{f}}_1 = \mathbf{f}(t_i, \mathbf{y}_i) \quad \tilde{\mathbf{f}}_2 = \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{y}_i + \frac{1}{2}h\tilde{\mathbf{f}}_1\right) \quad \tilde{\mathbf{f}}_3 = \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{y}_i + \frac{1}{2}h\tilde{\mathbf{f}}_2\right) \quad \tilde{\mathbf{f}}_4 = \mathbf{f}(t_i + h, \mathbf{y}_i + h\tilde{\mathbf{f}}_3) \tag{1.109}$$
$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\left(\frac{1}{6}\tilde{\mathbf{f}}_1 + \frac{1}{3}\tilde{\mathbf{f}}_2 + \frac{1}{3}\tilde{\mathbf{f}}_3 + \frac{1}{6}\tilde{\mathbf{f}}_4\right)$$

Observe that in each of the four cases the sum of the components of $\{\mathbf{c}\}$ is 1 and the sum of each row of $[\mathbf{b}]$ equals the value in that row of $\{\mathbf{a}\}$. This is characteristic of the *RK* methods.

Algorithm 1.1 Given the vector **y** at time *t*, the derivatives $\mathbf{f}(t,\mathbf{y})$ and the step size *h*, use one of the methods *RK1* through *RK4* to find **y** at time $t + h$. See Appendix D.2 for a MATLAB® software implementation of this algorithm in the form of the function *rk1_4.m*. *rk1_4.m* that executes any of the four *RK* methods according to whether the variable *rk* passed to the function has the value 1, 2, 3 or 4.

1. Evaluate the derivatives $\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2, \ldots, \tilde{\mathbf{f}}_s$ at stages 1 through *s* by means of Equation 1.102.
2. Use Equation 1.104 to compute $\mathbf{y}(t + h) = \mathbf{y}(t) + h\sum_{m=1}^{s} \mathbf{c}_m \tilde{\mathbf{f}}_m$.

Repeat these steps to obtain **y** at subsequent times $t + 2h$, $t + 3h$, etc.

Let us employ the Runge-Kutta methods and Algorithm 1.1 to solve for the motion of the well-known viscously damped spring-mass system pictured in Figure 1.21. The spring has an unstretched length l_0 and a spring constant k. The viscous damper coefficient is c and the mass of the block, which slides on a frictionless surface, is m. A forcing function $F(t)$ is applied to the mass. From the free body diagram in part (c) of the figure we obtain the equation of motion of this one-dimensional system in the x-direction.

$$-F_s - F_d + F(t) = m\ddot{x} \tag{1.110}$$

where F_s and F_d are the forces of the spring and dashpot, respectively. Since $F_s = kx$ and $F_d = c\dot{x}$, Equation 1.110, after dividing through by the mass, can be rewritten as

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m} \tag{1.111}$$

The spring rate k and the mass m determine the natural circular frequency of vibration of the system, $\omega_n = \sqrt{k/m}$ (radians/s). Furthermore, the damping coefficient c may be expressed as $c = 2\zeta m\omega_n$, where ζ is the dimensionless damping factor ($\zeta \geq 0$). Making these substitutions in Equation 1.111, we get the standard form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = \frac{F(t)}{m} \tag{1.112}$$

If the forcing function is sinusoidal with amplitude F_0 and circular frequency ω, then Equation 1.112 becomes

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = \frac{F_0}{m}\sin\omega t \tag{1.113}$$

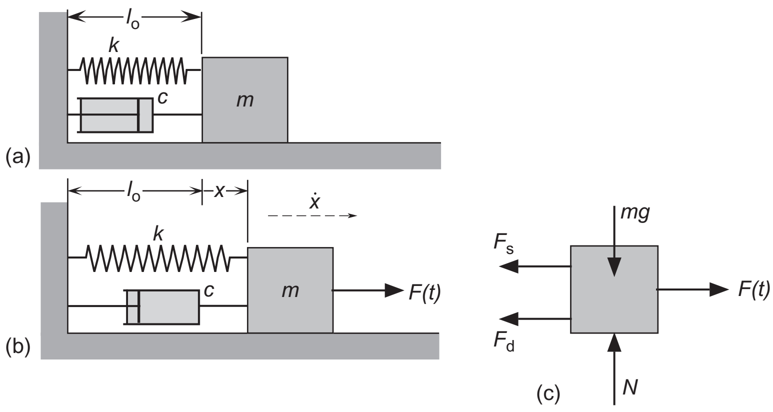


FIGURE 1.21

A damped spring-mass system with a forcing function applied to the mass. (a) At rest. (b) In motion under the action of the applied force $F(t)$. (c) Free body diagram at any instant.

This second order ordinary differential equation has a closed form solution, which is found using procedures taught in a differential equations course. If the system is underdamped, which means $\zeta < 1$, then it can be verified by substitution (see Problem 1.16) that the solution of Equation 1.113 is

$$\begin{aligned} x &= e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t) \\ &+ \frac{F_0/m}{(\omega_n{}^2 - \omega^2)^2 + (2\omega\omega_n\zeta)^2}[(\omega_n{}^2 - \omega^2)\sin\omega t - 2\omega\omega_n\zeta\cos\omega t] \end{aligned} \tag{1.114a}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. The initial conditions determine the values of the coefficients A and B. If at $t = 0$, $x = x_0$ and $\dot{x} = \dot{x}_0$, it turns out (see Problem 1.17) that

$$\begin{aligned} A &= \zeta\frac{\omega_n}{\omega_d}x_0 + \frac{\dot{x}_0}{\omega_d} + \frac{\omega^2 + (2\zeta^2 - 1)\omega_n{}^2}{(\omega_n{}^2 - \omega^2)^2 + (2\omega\omega_n\zeta)^2}\frac{\omega}{\omega_d}\frac{F_0}{m} \\ B &= x_0 + \frac{2\omega\omega_n\zeta}{(\omega_n{}^2 - \omega^2)^2 + (2\omega\omega_n\zeta)^2}\frac{F_0}{m} \end{aligned} \tag{1.114b}$$

The transient term with the exponential factor in Equation 1.114a dies out eventually, leaving only the steady-state solution, which persists as long as the forcing function acts.

Example 1.17

Plot Equation 1.114 from $t = 0$ to $t = 110$ seconds if $m = 1\,\text{kg}$, $\omega_n = 1\,\text{rad/s}$, $\zeta = 0.03$, $F_0 = 1\,\text{N}$, $\omega = 0.4\,\text{rad/s}$ and the initial conditions are $x = \dot{x} = 0$.

Solution

Substituting the given values into Equations 1.114 yields

$$\boxed{x = e^{-0.03t}[0.03399\cos(0.9995t) - 0.4750\sin(0.9995t)] + [1.190\sin(0.4t) - 0.03399\cos(0.4t)]} \tag{1.115}$$

This function is plotted over the time span 0 to 110 s in Figure 1.22. Observe that after about 80 seconds the transient has damped out and the system vibrates at the same frequency as the forcing function (although slightly out of phase due to the small viscosity).

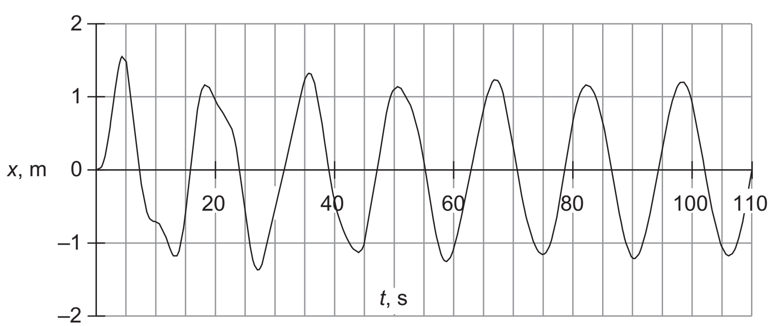


FIGURE 1.22

Over time only the steady state solution of Equation 1.113 remains.

Example 1.18

Solve Equation 1.113 numerically, using the Runge-Kutta method and the data of Example 1.17. Compare the *RK* solution with the exact one, given by Equation 1.115.

Solution

We must first reduce Equation 1.113 to two first order differential equations by introducing the two auxiliary variables

$$y_1 = x(t) \tag{a}$$

$$y_2 = \dot{x}(t) \tag{b}$$

Differentiating (a) we find

$$\dot{y}_1 = \dot{x}(t) = y_2(t) \tag{c}$$

Differentiating (b) and using Equation 1.113 yields

$$\dot{y}_2 = \ddot{x}(t) = \frac{F_0}{m}\sin\omega t - \omega_n{}^2 y_1(t) - 2\zeta\omega_n y_2(t) \tag{d}$$

The system (c) and (d) can be written compactly in the standard vector notation as

$$\dot{\mathbf{y}} = \mathbf{f}(t,\mathbf{y}) \tag{e}$$

where

$$\mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \qquad \dot{\mathbf{y}} = \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} \qquad \mathbf{f}(t,\mathbf{y}) = \begin{Bmatrix} y_2(t) \\ \dfrac{F_0}{m}\sin\omega t - \omega_n{}^2 y_1(t) - 2\zeta\omega_n y_2(t) \end{Bmatrix} \tag{1.116}$$

Equations 1.116 are what we need to implement Algorithm 1.1 for this problem.

We will use the two MATLAB functions listed in Appendix D.2, namely, *Example_1_18.m* and *rk1_4.m. Example_1_18.m* passes the data of Example 1.17 to the function *rk1_4.m*, which executes Algorithm 1.1 for *RK1*, *RK2*, *RK3* and *RK4* over the time interval from 0 to 110 seconds. In each case the problem is solved for two different values of the time step *h*. The subfunction *rates* within *Example_1_18.m* calculates the derivatives **f**(t,**y**) given in Equation 1.116. The exact solution (Equation 1.115) along with the four *RK* solutions is nondimensionalized and plotted at each time step in Figure 1.23.

We see that all of the *RK* solutions agree closely with the analytical one for a sufficiently small step size. The figure shows, as expected, that to obtain accuracy, the uniform step size *h* must be reduced as the order of the *RK* method is reduced. Likewise, the figure suggests that a step size which yields inaccurate results for one *RK* order may work just fine for the next higher order procedure.

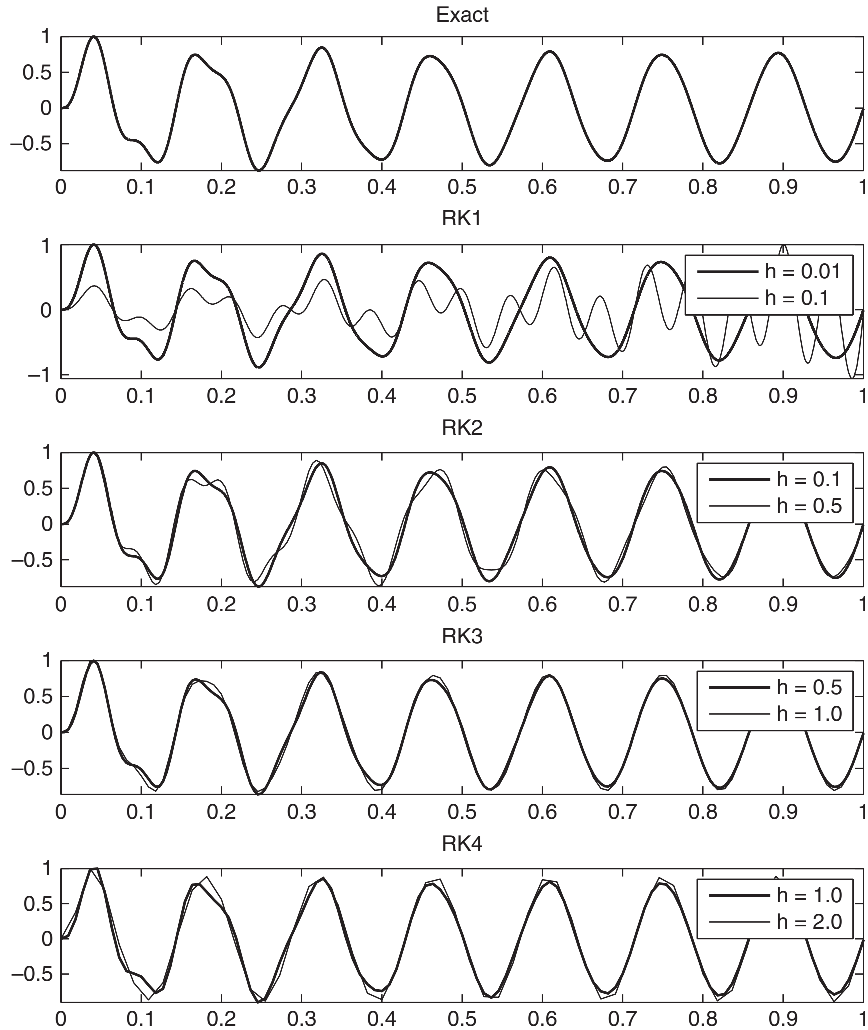


FIGURE 1.23

x/x_{max} vs. t/t_{max} for the *RK1* through *RK4* solutions of Equation 1.113 using the data of Example 1.17. The exact solution is at the top.

1.8.2 Heun's Predictor-Corrector Method

As we have seen, the *RK1* method (Equation 1.106) uses just $\tilde{\mathbf{f}}_1$, the derivative of **y** at the beginning of the time interval, to approximate the value of **y** at the end of the interval. The use of Equation 1.106 for approximate numerical integration of nonlinear functions was introduced by Leonhard Euler in 1768 and is therefore known as Euler's method. *RK2* (Equation 1.107) improves the accuracy by using the average of the derivatives $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$ at each end of the time interval. The ***predictor-corrector method*** due originally to the German mathematician Karl Heun (1859–1929) employs this idea.

First we use *RK1* to estimate the value of $\mathbf{y}$ at t_{i+1}, labeling that approximation $\mathbf{y}^*_{i+1}$:

$$\mathbf{y}^*_{i+1} = \mathbf{y}_i + h\mathbf{f}(t_i, \mathbf{y}_i) \qquad \text{(predictor)} \tag{1.117a}$$

$\mathbf{y}^*_{i+1}$ is then used to compute the derivative $\mathbf{f}$ at $t_i + h$, whereupon the average of the two derivatives is used to correct the estimate

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\frac{\mathbf{f}(t_i, \mathbf{y}_i) + \mathbf{f}(t_i + h, \mathbf{y}^*_{i+1})}{2} \qquad \text{(corrector)} \tag{1.117b}$$

We can iteratively improve the estimate of $\mathbf{y}_{i+1}$ by making the substitution $\mathbf{y}^*_{i+1} \leftarrow \mathbf{y}_{i+1}$ (where $\leftarrow$ means "is replaced by") and computing a new value of $\mathbf{y}_{i+1}$ from Equation 1.117b. That process is repeated until the difference between $\mathbf{y}_{i+1}$ and $\mathbf{y}^*_{i+1}$ becomes acceptably small.

Algorithm 1.2 Given the vector $\mathbf{y}$ at time t and the derivatives $\mathbf{f}(t,\mathbf{y})$, use Heun's method to find $\mathbf{y}$ at time $t + h$. See Appendix D.3 for a MATLAB implementation of this algorithm (*heun.m*).

1. Evaluate the vector of derivatives $\mathbf{f}(t,\mathbf{y})$.
2. Compute the predictor $\mathbf{y}^*(t + h) = \mathbf{y}(t) + \mathbf{f}(t,\mathbf{y})h$.
3. Compute the corrector $\mathbf{y}(t + h) = \mathbf{y}(t) + h\{\mathbf{f}[t, \mathbf{y}] + \mathbf{f}[t + h, \mathbf{y}^*(t + h)]\}/2$.
4. Make the substitution $\mathbf{y}^*(t + h) \leftarrow \mathbf{y}(t + h)$ and use Step 3 to recompute $\mathbf{y}(t + h)$.
5. Repeat Step 4 until $\mathbf{y}(t + h) \approx \mathbf{y}^*(t + h)$ to within a given tolerance.

Repeat these steps to obtain $\mathbf{y}$ at subsequent times $t + 2h$, $t + 3h$, etc.

Example 1.19

Employ Heun's method to solve Equation 1.113 using the data provided in Example 1.17. Use two different time steps, $h = 1$ s and $h = 0.1$ s, and compare the results.

Solution

We use the MATLAB functions *Example_1_19.m* and *heun.m* listed in Appendix D.3. The function *Example_1_19.m* passes the given data to the function *heun.m*, which uses the subfunction *rates* within *Example_1_19.m* to compute the derivatives $\mathbf{f}(t,\mathbf{y})$ in Equation 1.116. *heun.m* executes Algorithm 1.2 over the time interval from 0 to 110 seconds, once for $h = 1$ s and again for $h = 0.1$ s, and plots the output in each case, as illustrated in Figure 1.24.

The graph shows that for $h = 0.1$ s, Heun's method yields a curve identical to the exact solution (whereas the *RK1* method diverged for this time step in Figure 1.23). Even for the rather large time step $h = 1$ s, the Heun solution, though it starts out a bit ragged, proceeds after 60 seconds (about the time the transient dies out) to settle down and coincide thereafter very well with the exact solution. For this problem, Heun's method is a decidedly better choice than *RK1* and competes with *RK2* and *RK3*.

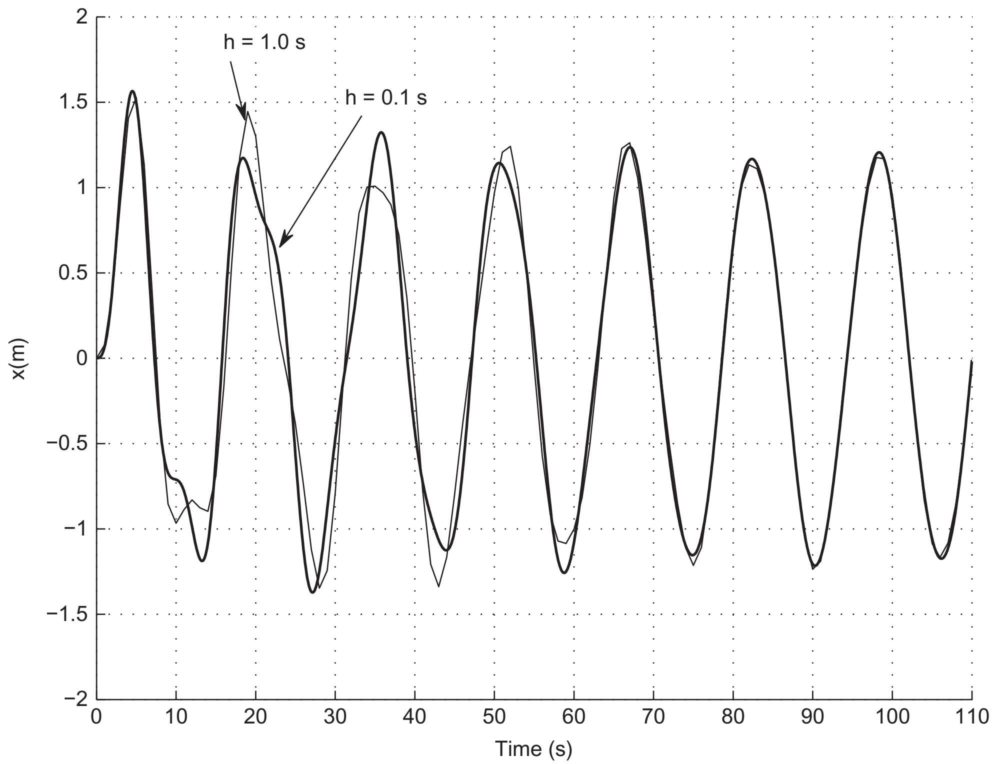


FIGURE 1.24

Numerical solution of Equation 1.113 using Heun's method with two different step sizes.

1.8.3 Runge-Kutta with Variable Step Size

Using a constant step size to integrate a differential equation can be inefficient. The value of h in those regions where the solution varies slowly should be larger than in regions where the variation is more rapid, which requires h to be small in order to maintain accuracy. Methods for automatically adjusting the step size have been developed. They involve combining two adjacent-order *RK* methods into one and using the difference between the higher and lower order solution to estimate the truncation error in the lower order solution. The step size h is adjusted to keep the truncation error in bounds.

A common example is the embedding of *RK4* into *RK5* to produce the *RKF4(5)* method. The *F* is added in recognition of E. Fehlberg's contribution to this extension of the Runge-Kutta method. The ***Runge-Kutta-Fehlberg*** procedure has six stages, and the Fehlberg coefficients are (Fehlberg, 1969)

$$\{\mathbf{a}\} = \begin{Bmatrix} 0 \\ 1/4 \\ 3/8 \\ 12/13 \\ 1 \\ 1/2 \end{Bmatrix} \qquad [\mathbf{b}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 0 \\ 3/32 & 9/32 & 0 & 0 & 0 \\ 1932/2197 & -7200/2197 & 7296/2197 & 0 & 0 \\ 439/216 & -8 & 3680/513 & -845/4104 & 0 \\ -8/27 & 2 & -3544/2565 & 1859/4104 & -11/40 \end{bmatrix} \tag{1.118}$$

$$\{\mathbf{c}\,*\} = \begin{Bmatrix} 25/216 \\ 0 \\ 1408/2565 \\ 2197/4104 \\ -1/5 \\ 0 \end{Bmatrix} \qquad \{\mathbf{c}\} = \begin{Bmatrix} 16/135 \\ 0 \\ 6656/12825 \\ 28561/56430 \\ -9/50 \\ 2/55 \end{Bmatrix} \tag{1.119}$$

Using asterisks to indicate that *RK4* is the lower order of the two, we have from Equations 1.104

$$\mathbf{y}\,*_{i+1} = \mathbf{y}_i + h(c\,*_1\,\tilde{\mathbf{f}}_1 + c\,*_2\,\tilde{\mathbf{f}}_2 + c\,*_3\,\tilde{\mathbf{f}}_3 + c\,*_4\,\tilde{\mathbf{f}}_4 + c\,*_5\,\tilde{\mathbf{f}}_5 + c\,*_6\,\tilde{\mathbf{f}}_6) \quad \text{Low order solution} \ (RK4) \tag{1.120}$$

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h(c_1\tilde{\mathbf{f}}_1 + c_2\tilde{\mathbf{f}}_2 + c_3\tilde{\mathbf{f}}_3 + c_4\tilde{\mathbf{f}}_4 + c_5\tilde{\mathbf{f}}_5 + c_6\tilde{\mathbf{f}}_6) \qquad \text{High order solution} \ (RK5) \tag{1.121}$$

where, from Equations 1.100, 1.101 and 1.102, the derivatives at the six stages are

$$\begin{aligned} \tilde{\mathbf{f}}_1 &= \mathbf{f}(t_i, \mathbf{y}_i) \\ \tilde{\mathbf{f}}_2 &= \mathbf{f}(t_i + a_2h, \mathbf{y}_i + hb_{21}\tilde{\mathbf{f}}_1) \\ \tilde{\mathbf{f}}_3 &= \mathbf{f}(t_i + a_3h, \mathbf{y}_i + h[b_{31}\tilde{\mathbf{f}}_1 + b_{32}\tilde{\mathbf{f}}_2]) \\ \tilde{\mathbf{f}}_4 &= \mathbf{f}(t_i + a_4h, \mathbf{y}_i + h[b_{41}\tilde{\mathbf{f}}_1 + b_{42}\tilde{\mathbf{f}}_2 + b_{43}\tilde{\mathbf{f}}_3]) \\ \tilde{\mathbf{f}}_5 &= \mathbf{f}(t_i + a_5h, \mathbf{y}_i + h[b_{51}\tilde{\mathbf{f}}_1 + b_{52}\tilde{\mathbf{f}}_2 + b_{53}\tilde{\mathbf{f}}_3 + b_{54}\tilde{\mathbf{f}}_4]) \\ \tilde{\mathbf{f}}_6 &= \mathbf{f}(t_i + a_6h, \mathbf{y}_i + h[b_{61}\tilde{\mathbf{f}}_1 + b_{62}\tilde{\mathbf{f}}_2 + b_{63}\tilde{\mathbf{f}}_3 + b_{64}\tilde{\mathbf{f}}_4 + b_{65}\tilde{\mathbf{f}}_5]) \end{aligned} \tag{1.122}$$

Observe that, although the low and high order solutions have different weights ($\{\mathbf{c}^*\}$ and $\{\mathbf{c}\}$, respectively), they share the same nodes $\{\mathbf{a}\}$ and coupling coefficients $[\mathbf{b}]$ and, hence, the same values of the derivatives $\tilde{\mathbf{f}}$. This is another convenient feature of the *RKF* method.

The truncation vector $\mathbf{e}$ is the difference between the higher order solution $\mathbf{y}_{i+1}$ and the lower order solution $\mathbf{y}^*_{i+1}$

$$\begin{aligned} \mathbf{e} &= \mathbf{y}_{i+1} - \mathbf{y}\,*_{i+1} \\ &= h[(c_1 - c\,*_1)\tilde{\mathbf{f}}_1 + (c_2 - c\,*_2)\tilde{\mathbf{f}}_2 + (c_3 - c\,*_3)\tilde{\mathbf{f}}_3 \\ &\quad + (c_4 - c\,*_4)\tilde{\mathbf{f}}_4 + (c_5 - c\,*_5)\tilde{\mathbf{f}}_5 + (c_6 - c\,*_6)\tilde{\mathbf{f}}_6] \end{aligned} \tag{1.123}$$

The number of components of $\mathbf{e}$ equals N, the number of first order differential equations in the system (*e.g.*, three in Example 1.15 and two in Example 1.18). The scalar truncation error e is the largest of the absolute values of the components of $\mathbf{e}$,

$$e = \text{maximum of the set } (|e_1|, |e_2|, |e_3|, \cdots, |e_N|) \tag{1.124}$$

We set up a tolerance *tol*, which the truncation error cannot exceed. Instead of using the same h for every step of the numerical integration process, we can adjust the step size so as to keep the error e from exceeding *tol*. A simple strategy for ***adaptive step size control*** is to update h after each time step using a formula, which is derived in, for example, Bond and Allman (1996),

$$h_{new} = h_{old}\left(\frac{tol}{e}\right)^{\frac{1}{p+1}} \tag{1.125}$$

where p is the lower of the two orders in an *RKFp(p + 1)* method. For *RKF4(5)*, $p = 4$. According to Equation 1.125, if $e > tol$, then $h_{\text{new}} < h_{\text{old}}$, whereas if $e < tol$, then $h_{\text{new}} > h_{\text{old}}$. A factor β is commonly added so that

$$h_{new} = h_{old}\beta\left(\frac{tol}{e}\right)^{\frac{1}{p+1}} \tag{1.126}$$

where β may be 0.8 or 0.9, depending on the computer program.

Algorithm 1.3 Given the vector $\mathbf{y}_i$ at time t_i, the derivative functions $\mathbf{f}(t, \mathbf{y})$, the time step h, and the tolerance *tol*, use the *RKF4(5)* method with adaptive step size control to find $\mathbf{y}_{i+1}$ at time t_{i+1}. See Appendix D.4 for *rkf45.m*, a MATLAB implementation of this algorithm.

1. Evaluate the derivatives $\tilde{\mathbf{f}}_1$ through $\tilde{\mathbf{f}}_6$ using Equations 1.122.
2. Calculate the truncation vector using Equation 1.123.
3. Compute the scalar truncation error e using Equation 1.124.
4. If $e > tol$ then replace h by $h\beta(tol/e)^{1/5}$ and return to Step 1.
5. Replace t by $t + h$ and calculate $\mathbf{y}_{i+1}$ using Equation 1.121.
6. Replace h by $h\beta(tol/e)^{1/5}$.

Repeat these steps to obtain $\mathbf{y}_{i+2}$, $\mathbf{y}_{i+3}$, etc.

Example 1.20

A spacecraft S of mass m travels in a straight line away from the center C of the earth, as illustrated in Figure 1.25. If at a distance of 6500 km from C its outbound velocity is 7.8 km/s, what will be its position and velocity 70 minutes later?

Solution

Solving this problem requires writing down and then integrating the equations of motion. Starting with the free body diagram of S shown in Figure 1.25, we find that Newton's second law (Equation 1.38) for the spacecraft is

$$-F_g = m\ddot{x} \tag{a}$$

The variable force of gravity F_g on the spacecraft is its mass m times the local acceleration of gravity, given by Equation 1.8. That is,

$$F_g = mg = m\frac{g_0 R_E{}^2}{x^2} \tag{b}$$

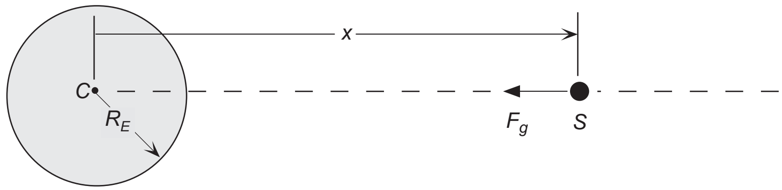


FIGURE 1.25

Spacecraft *S* in rectilinear motion relative to the earth.

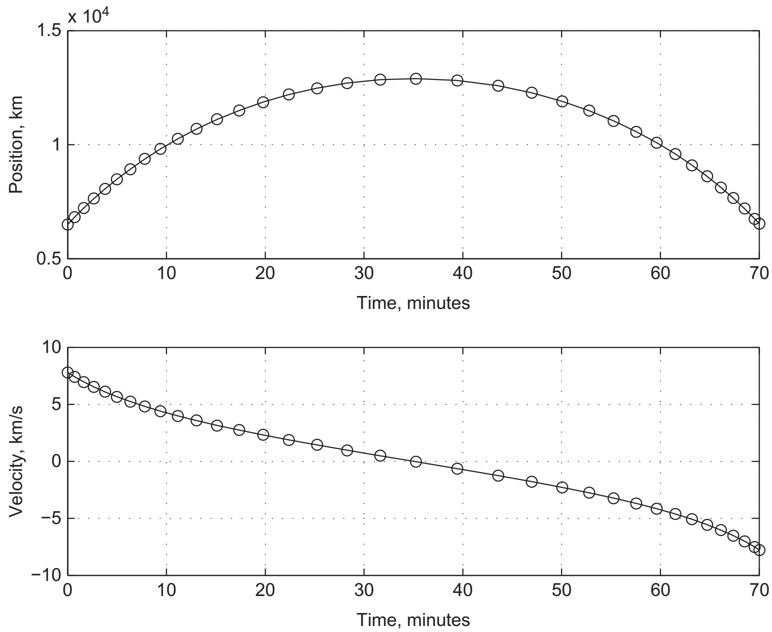


FIGURE 1.26

Position and velocity versus time. The solution points are circled.

R_E is the earth's radius (6378 km) and g_0 is the sea-level acceleration of gravity (9.807 m/s^2). Combining Equations (a) and (b) yields

$$\ddot{x} + \frac{g_0 {R_E}^2}{x^2} = 0 \tag{1.127}$$

This differential equation for the rectilinear motion of the spacecraft has an analytical solution, which we shall not go into here. Instead, we will solve it numerically using Algorithm 1.3 and the given initial conditions. For that we must as usual introduce the auxiliary variables $y_1 = x$ and $y_2 = \dot{x}$ to obtain the two differential equations

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -\frac{g_0 {R_E}^2}{{y_1}^2} \end{aligned} \tag{c}$$

The initial conditions in this case are

$$y_1(0) = 6500 \text{ km} \qquad y_2(0) = 7.8 \text{ km/s}^2 \tag{d}$$

The MATLAB programs *Example_1_20.m* and *rkf45.m*, both in Appendix D.4, were used to produce Figure 1.26, which shows the position and velocity of the spacecraft over the requested time span. *Example_1_20.m* passes the initial conditions and time span to *rkf4.m*, which uses the subroutine *rates* within *Example_1_20.m* to compute the derivatives $\dot{x}$ and $\ddot{x}$.

Figure 1.26 reveals that the spacecraft takes 35 minutes to coast out to twice its original 6500 km distance from *C* before reversing direction and returning 35 minutes later to where it started with a speed of 7.8 km/s. The nonuniform spacing between the solution points shows how *rkf4.m* controlled the step size so that *h* was smaller during rapid variations of the solution but larger elsewhere.

PROBLEMS

Section 1.2

1.1 Given the three vectors $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$, $\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$ and $\mathbf{C} = C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}} + C_z\hat{\mathbf{k}}$, show analytically that

(a) $\mathbf{A}\cdot\mathbf{A} = A^2$

(b) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ (interchangeability of the "dot" and "cross")

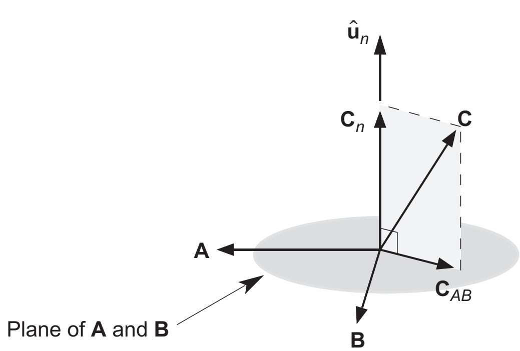
(c) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ (the *bac-cab* rule)

(Hint: Simply compute the expressions on each side of the = signs and demonstrate conclusively that they are the same. Do *not* substitute numbers to "prove" your point. Use Equations 1.9 and 1.16.)

1.2 Use just the vector identities in Exercise 1.1 to show that

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

1.3 Let $\mathbf{A} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$, $\mathbf{B} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{C} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$. Calculate the (scalar) projection C_{AB} of C onto the plane of A and B. See illustration below. (Hint: $C^2 = C_n{}^2 + C_{AB}{}^2$.)
{Ans.: $C_{AB} = 8.952$}



Section 1.3

1.4 Since $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ are perpendicular and $\hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n = \hat{\mathbf{u}}_b$, use the *bac-cab* rule to show that $\hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t = \hat{\mathbf{u}}_n$ and $\hat{\mathbf{u}}_n \times \hat{\mathbf{u}}_b = \hat{\mathbf{u}}_t$, thereby verifying Equation 1.29.

1.5 The x, y and z coordinates (in meters) of a particle P as a function of time (in seconds) are $x = \sin t$, $y = \sin 2t$ and $z = \sin 3t$. At $t = 3\,\text{s}$, determine:

(a) The velocity $\mathbf{v}$, in Cartesian coordinates.
(b) The speed v.
(c) The angles θ_x, θ_y and θ_z which $\mathbf{v}$ makes with the x, y and z axes.
(d) The unit tangent vector $\hat{\mathbf{u}}_t$.
(e) The acceleration a in Cartesian coordinates.
(f) The unit binormal vector $\hat{\mathbf{u}}_b$.
(g) The unit normal vector $\hat{\mathbf{u}}_n$.
(h) The angles ϕ_x, ϕ_y and ϕ_z which a makes with the x, y and z-axes.
(i) The tangential component a_t of the acceleration.
(j) The normal component a_n of acceleration.
(k) The radius of curvature of the path of P.
(l) The Cartesian coordinates of the center of curvature of the path.

{Partial Ans.: (b) 3.484 m/s ; (c) $\theta_x = 106.5°$; (j) $a_n = 1.520\,\text{m/s}^2$; (l) $x_C = 4.724\,\text{m}$}

Section 1.4

1.6 An 80 kg man and 50 kg woman stand 0.5 meter from each other. What is the force of gravitational attraction between the couple?
{Ans.: 1.07 μN}

1.7 If a person's weight is W on the surface of the earth, calculate the earth's gravitational pull on that person at a distance equal to the moon's orbit.
{Ans.: $275 \times 10^6\ W$}

1.8 If a person's weight is W on the surface of the earth, calculate what it would be, in terms of W, at the surface of (a) the moon; (b) Mars; (c) Jupiter.
{Partial Ans.: (c) $2.53W$}

Section 1.5

1.9 A satellite of mass m is in a circular orbit around the earth, whose mass is M. The orbital radius from the center of the earth is r. Use Newton's Second Law of Motion, together with Equations 1.25 and 1.31, to calculate the speed v of the satellite in terms of M, r and the gravitational constant G.
{Ans.: $v = \sqrt{GM/r}$ }

1.10 If the earth takes 365.25 days to complete its circular orbit of radius $149.6 \times 10^6\,\text{km}$ around the sun, use the result of Problem 1.9 to calculate the mass of the sun.
{Ans.: $1.988 \times 10^{30}\,\text{kg}$}

Section 1.6

1.11 $\mathbf{F}$ is a force vector of fixed magnitude embedded on a rigid body in plane motion (in the *xy* plane). At a given instant, $\boldsymbol{\omega} = 3\hat{\mathbf{k}}$ rad/s, $\dot{\boldsymbol{\omega}} = -2\hat{\mathbf{k}}$ rad/s^2, $\ddot{\boldsymbol{\omega}} = \mathbf{0}$ and $\mathbf{F} = 10\hat{\mathbf{i}}$ N. At that instant, calculate $\ddot{\mathbf{F}}$.
{Ans.: $\ddot{\mathbf{F}} = 180\hat{\mathbf{i}} - 270\hat{\mathbf{j}}$ N/s^3}

Section 1.7

1.12 The absolute position, velocity and acceleration of *O* are

$$\begin{aligned}\mathbf{r}_0 &= 300\hat{\mathbf{I}} + 200\hat{\mathbf{J}} + 100\hat{\mathbf{K}}\ \text{(m)}\\ \mathbf{v}_0 &= -10\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 50\hat{\mathbf{K}}\ \text{(m/s)}\\ \mathbf{a}_0 &= 25\hat{\mathbf{I}} + 40\hat{\mathbf{J}} - 15\hat{\mathbf{K}}\ (\text{m/s}^2)\end{aligned}$$

The angular velocity and acceleration of the moving frame are

$$\boldsymbol{\Omega} = 0.6\hat{\mathbf{I}} - 0.4\hat{\mathbf{J}} + 1.0\hat{\mathbf{K}}\ \text{(rad/s)} \qquad \dot{\boldsymbol{\Omega}} = -0.4\hat{\mathbf{I}} + 0.3\hat{\mathbf{J}} - 1.0\hat{\mathbf{K}}\ (\text{rad/s}^2)$$

The unit vectors of the moving frame are

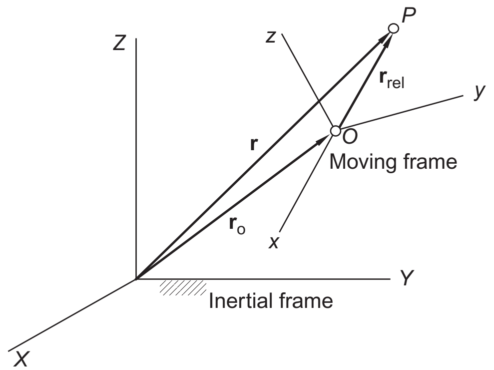
$$\begin{aligned}\hat{\mathbf{i}} &= 0.57735\hat{\mathbf{I}} + 0.57735\hat{\mathbf{J}} + 0.57735\hat{\mathbf{K}}\\ \hat{\mathbf{j}} &= -0.74296\hat{\mathbf{I}} + 0.66475\hat{\mathbf{J}} + 0.078206\hat{\mathbf{K}}\\ \hat{\mathbf{k}} &= -0.33864\hat{\mathbf{I}} - 0.47410\hat{\mathbf{J}} + 0.81274\hat{\mathbf{K}}\end{aligned}$$

The absolute position of *P* is

$$\mathbf{r} = 150\hat{\mathbf{I}} - 200\hat{\mathbf{J}} + 300\hat{\mathbf{K}}\ \text{(m)}$$

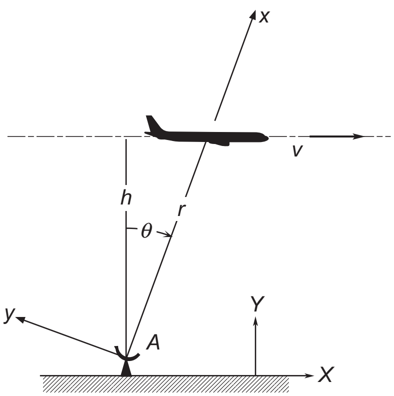
The velocity and acceleration of *P* relative to the moving frame are

$$\mathbf{v}_{rel} = -20\hat{\mathbf{i}} + 25\hat{\mathbf{j}} + 70\hat{\mathbf{k}}\ \text{(m/s)} \qquad \mathbf{a}_{rel} = 7.5\hat{\mathbf{i}} - 8.5\hat{\mathbf{j}} + 6.0\hat{\mathbf{k}}\ (\text{m/s}^2)$$



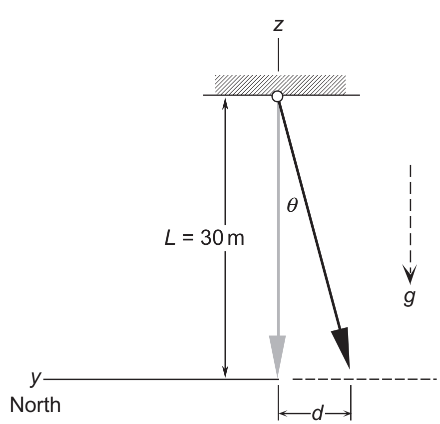
Calculate the absolute velocity $\mathbf{v}_P$ and acceleration $\mathbf{a}_P$ of P.
{Ans.: $\mathbf{v}_P = 478.7\hat{\mathbf{u}}_v$ (m/s), $\hat{\mathbf{u}}_v = 0.5352\hat{\mathbf{I}} - 0.5601\hat{\mathbf{J}} - 0.6324\hat{\mathbf{K}}$;
$\mathbf{a}_P = 616.3\hat{\mathbf{u}}_a$ (m/s^2), $\hat{\mathbf{u}}_a = 0.1655\hat{\mathbf{I}} + 0.9759\hat{\mathbf{J}} + 0.1424\hat{\mathbf{K}}$}

1.13 An airplane in level flight at an altitude h and a uniform speed v passes directly over a radar tracking station A. Calculate the angular velocity $\dot{\theta}$ and angular acceleration of the radar antenna $\ddot{\theta}$ as well as the rate $\dot{r}$ at which the airplane is moving away from the antenna. Use the equations of this chapter (rather than polar coordinates, which you can use to check your work). Attach the inertial frame of reference to the ground and assume a nonrotating earth. Attach the moving frame to the antenna, with the x-axis pointing always from the antenna towards the airplane.
{Ans.: (a) $\dot{\theta} = v\cos^2\theta/h$ (b) $\ddot{\theta} = -2v^2\cos^3\theta\sin\theta/h^2$ (c) $v_{\mathrm{rel}} = v\sin\theta$}



1.14 At 30 degrees north latitude, a 1000 kg (2205 lb) car travels due north at a constant speed of 100 km/hr (62 mph) on a level road at sea level. Taking into account the earth's rotation, calculate the lateral (sideways) force of the road on the car, and the normal force of the road on the car.
{Ans.: $F_{\mathrm{lateral}} = 2.026$ N, to the left (west); $N = 9784$ N}

1.15 At 29 degrees north latitude, what is the deviation d from the vertical of a plumb bob at the end of a 30 m string, due to the earth's rotation?
{Ans.: 44.1 mm to the south.}



Section 1.8

1.16 Verify by substitution that Equation 1.114a is the solution of Equation 1.113.

1.17 Verify that Equation 1.114b is valid.

1.18 Numerically solve the fourth order differential equation

$$\ddddot{y} + 2\ddot{y} + y = 0$$

for y at $t = 20$, if the initial conditions are $y = 1,\ \dot{y} = \ddot{y} = \dddot{y} = 0$ at $t = 0$.
{Ans.: $y(20) = 9.54$}

1.19 Numerically solve the differential equation

$$\dddot{y} + 3\ddot{y} - 4\dot{y} - 12y = te^{2t}$$

for y at $t = 3$ if, at $t = 0$, $y = \dot{y} = \ddot{y} = 0$.
{Ans.: $y(3) = 66.6$}

1.20 Numerically solve the differential equation

$$t\ddot{y} + t^2\dot{y} - 2y = 0$$

to obtain y at $t = 4$ if the initial conditions are $y = 0$ and $\dot{y} = 1$ at $t = 1$.
{Ans.: $y(4) = 1.29$}

1.21 Numerically solve the system

$$\begin{aligned}
\dot{x} + \frac{1}{2}y - \frac{1}{2}z &= 0 \\
-\frac{1}{2}x + \dot{y} + \frac{1}{\sqrt{2}}z &= 0 \\
\frac{1}{2}x - \frac{1}{\sqrt{2}}y + \dot{z} &= 0
\end{aligned}$$

to obtain x, y and z at $t = 20$. The initial conditions are $x = 1$ and $y = z = 0$ at $t = 0$.
{Ans.: $x(20) = 0.703$, $y(20) = 0.666$, $z(20) = -0.247$}

1.22 Use one of the numerical methods discussed in this section to solve Equation 1.127 for the time required for the moon to fall to the earth if it were somehow stopped in its orbit while the earth remained fixed in space. (This will require a trial and error procedure known formally as a shooting method. It is not necessary for this problem to code the procedure. Simply guess a time and let the solver compute the final radius. On the basis of the deviation of that result from the earth's radius (6378 km), revise your time estimate and re-run the problem to compute a new final radius. Repeat this process in a logical fashion until your time estimate yields a final radius that is accurate to at least three significant figures.) Compare your answer with the analytical solution,

$$t = \sqrt{\frac{r_0}{2g_0 R_E^{\,2}}}\left[\frac{\pi}{4}r_0 + \sqrt{r(r_0 - r)} + \frac{r_0}{2}\sin^{-1}\left(\frac{r_0 - 2r}{r_0}\right)\right]$$

where t is the time, r_0 is the initial radius, r is the final radius ($r < r_0$), g_0 is the sea-level acceleration of earth's gravity and R_E is the radius of the earth.

1.23 Use a Runge-Kutta solver such as MATLAB's *ode45* to solve the nonlinear Lorenz equations, due to the American meteorologist and mathematician E. N. Lorenz (1917–2008):

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

Start off by using the values Lorenz used in his paper (Lorenz, 1963): $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$ and the initial conditions $x = 0$, $y = 1$ and $z = 0$ at $t = 0$. Let t range to a value of 20 or higher. Plot the phase trajectory $x = x(t)$, $y = y(t)$, $z = z(t)$ in 3D to see the now-famous "Lorenz attractor." The Lorenz equations are a simplified model of the two-dimensional convective motion within a fluid layer due to a temperature difference ΔT between the upper and lower surfaces. The equations are chaotic in nature. For one thing, this means that the solutions are extremely sensitive to the initial conditions. A minute change yields a completely different solution in the long run. Check this out yourself. (x represents the intensity of the convective motion of the fluid, y is proportional to the temperature difference between rising and falling fluid, and z represents the nonlinearity of the temperature profile across the depth. σ is a fluid property (the Prandtl number), ρ is proportional to ΔT, β is a geometrical parameter and t is a nondimensional time.)

List of Key Terms

absolute time derivative
adaptive step size control
angular impulse
angular momentum
bac-cab rule
Cartesian coordinate system
Coriolis acceleration
cross product
dot product
impulse of a force
inertial frame of reference
interchange of the dot and cross
Newton's law of gravity
Newton's second law of motion
overhead dot for time derivative

predictor-corrector method
relative acceleration formula
relative time derivative
relative velocity formula
resultant of two vectors
Runge-Kutta-Fehlberg method
Runge-Kutta methods
system of first order differential equations
Taylor series
time derivative of a rotating vector of constant magnitude
universal gravitational constant

CHAPTER

The two-body problem

2

Chapter outline

2.1	Introduction	61
2.2	Equations of motion in an inertial frame	62
2.3	Equations of relative motion	70
2.4	Angular momentum and the orbit formulas	74
2.5	The energy law	82
2.6	Circular orbits ($e = 0$)	83
2.7	Elliptical orbits ($0 < e < 1$)	89
2.8	Parabolic trajectories ($e = 1$)	100
2.9	Hyperbolic trajectories ($e > 1$)	104
2.10	Perifocal frame	113
2.11	The Lagrange coefficients	117
2.12	Restricted three-body problem	129

2.1 INTRODUCTION

This chapter presents the vector-based approach to the classical problem of determining the motion of two bodies due solely to their own mutual gravitational attraction. We show that the path of one of the masses relative to the other is a conic section (circle, ellipse, parabola, or hyperbola) whose shape is determined by the eccentricity. Several fundamental properties of the different types of orbits are developed with the aid of the laws of conservation of angular momentum and energy. These properties include the period of elliptical orbits, the escape velocity associated with parabolic paths and the characteristic energy of hyperbolic trajectories. Following the presentation of the four types of orbits, the perifocal frame is introduced. This frame of reference is used to describe orbits in three dimensions, which is the subject of Chapter 4. In this chapter, the perifocal frame provides the backdrop for developing the Lagrange f and g coefficients. By means of the Lagrange f and g coefficients, the position and velocity on a trajectory can be found in terms of the position and velocity at an initial time. These functions are needed in the orbit determination algorithms of Lambert and Gauss presented in Chapter 5.

The chapter concludes with a discussion of the restricted three-body problem in order to provide a basis for understanding the concepts of Lagrange points and the Jacobi constant. This material is optional.

In studying this chapter it would be well from time to time to review the road map provided in Appendix B.

2.2 EQUATIONS OF MOTION IN AN INERTIAL FRAME

Figure 2.1 shows two point masses acted upon only by the mutual force of gravity between them. The positions $\mathbf{R}_1$ and $\mathbf{R}_2$ of their centers of mass are shown relative to an inertial frame of reference *XYZ*. In terms of the coordinates of the two points

$$\begin{aligned}\mathbf{R}_1 &= X_1\hat{\mathbf{I}} + Y_1\hat{\mathbf{J}} + Z_1\hat{\mathbf{K}}\\ \mathbf{R}_2 &= X_2\hat{\mathbf{I}} + Y_2\hat{\mathbf{J}} + Z_2\hat{\mathbf{K}}\end{aligned} \tag{2.1}$$

The origin O of the inertial frame may move with constant velocity (relative to the fixed stars), but the axes do not rotate. Each of the two bodies is acted upon by the gravitational attraction of the other. $\mathbf{F}_{12}$ is the force exerted on m_1 by m_2, and $\mathbf{F}_{21}$ is the force exerted on m_2 by m_1.

The position vector $\mathbf{R}_G$ of the center of mass G of the system in Figure 2.1a is defined by the formula:

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2} \tag{2.2}$$

Therefore, the absolute velocity and the absolute acceleration of G are:

$$\mathbf{v}_G = \dot{\mathbf{R}}_G = \frac{m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2}{m_1 + m_2} \tag{2.3}$$

$$\mathbf{a}_G = \ddot{\mathbf{R}}_G = \frac{m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2}{m_1 + m_2} \tag{2.4}$$

The adjective "absolute" means that the quantities are measured relative to an inertial frame of reference.

Let $\mathbf{r}$ be the position vector of m_2 relative to m_1. Then,

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \tag{2.5}$$

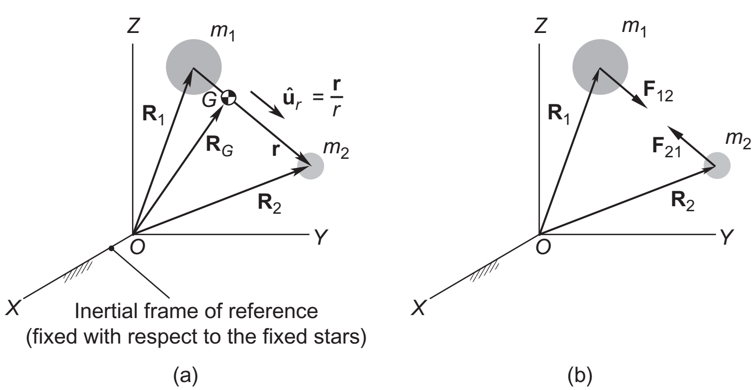


FIGURE 2.1

(a) Two masses located in an inertial frame. (b) Free-body diagrams.

Or, using Equations 2.1,

$$\mathbf{r} = (X_2 - X_1)\hat{\mathbf{I}} + (Y_2 - Y_1)\hat{\mathbf{J}} + (Z_2 - Z_1)\hat{\mathbf{K}} \tag{2.6}$$

Furthermore, let $\hat{\mathbf{u}}_r$ be the unit vector pointing from m_1 towards m_2, so that

$$\hat{\mathbf{u}}_r = \frac{\mathbf{r}}{r} \tag{2.7}$$

where r is the magnitude of $\mathbf{r}$,

$$r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \tag{2.8}$$

The body m_1 is acted upon only by the force of gravitational attraction towards m_2. The force of gravitational attraction, F_g, which acts along the line joining the centers of mass of m_1 and m_2, is given by Equation 1.31. Therefore, the force exerted on m_1 by m_2 is:

$$\mathbf{F}_{12} = \frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \tag{2.9}$$

where $\hat{\mathbf{u}}_r$ accounts for the fact that the force vector $\mathbf{F}_{12}$ is directed from m_1 towards m_2. (Do not confuse the symbol G, used in this context to represent the universal gravitational constant, with its use elsewhere in the book to denote the center of mass.) By Newton's third law (the action-reaction principle), the force $\mathbf{F}_{21}$ exerted on m_2 by m_1 is $-\mathbf{F}_{12}$, so that:

$$\mathbf{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \tag{2.10}$$

Newton's second law of motion as applied to body m_1 is $\mathbf{F}_{12} = m_1\ddot{\mathbf{R}}_1$, where $\ddot{\mathbf{R}}_1$ is the absolute acceleration of m_1. Combining this with Newton's law of gravitation (Equation 2.9) yields:

$$m_1\ddot{\mathbf{R}}_1 = \frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \tag{2.11}$$

Likewise, by substituting $\mathbf{F}_{21} = m_2\ddot{\mathbf{R}}_2$ into Equation 2.10 we get:

$$m_2\ddot{\mathbf{R}}_2 = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \tag{2.12}$$

It is apparent from forming the sum of Equations 2.11 and 2.12 that $m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2 = \mathbf{0}$. According to Equation 2.4, that means the acceleration of the center of mass G of the system of two bodies m_1 and m_2 is zero. Therefore, as is true for any system that is free of external forces, G moves in a straight line through space with a constant velocity $\mathbf{v}_G$. Its position vector relative to XYZ is given by:

$$\mathbf{R}_G = \mathbf{R}_{G_0} + \mathbf{v}_G t \tag{2.13}$$

where $\mathbf{R}_{G_0}$ is the position of G at time $t = 0$. The non-accelerating center of mass of a two-body system may serve as the origin of an inertial frame.

Example 2.1

Use the two-body equations of motion to show why orbiting astronauts experience weightlessness.

Solution

We sense weight by feeling the contact forces that develop wherever our body is supported. Consider an astronaut of mass m_A strapped into a spacecraft of mass m_S, in orbit about the earth. The distance between the center of the earth and the spacecraft is r, and the mass of the earth is M_E. Since the only external force is that of gravity, $\left.\mathbf{F}_S\right)_g$, the equation of motion of the spacecraft is:

$$\left.\mathbf{F}_S\right)_g = m_S \mathbf{a}_S \tag{a}$$

where $\mathbf{a}_S$ is measured in an inertial frame. According to Equation 2.10,

$$\left.\mathbf{F}_S\right)_g = -\frac{GM_E m_S}{r^2}\hat{\mathbf{u}}_r \tag{b}$$

in which $\hat{\mathbf{u}}_r$ is the unit vector pointing outwards from the earth towards the orbiting spacecraft. Thus, (a) and (b) imply that the absolute acceleration of the spacecraft is:

$$\mathbf{a}_S = -\frac{GM_E}{r^2}\hat{\mathbf{u}}_r \tag{c}$$

The equation of motion of the astronaut is:

$$\left.\mathbf{F}_A\right)_g + \mathbf{C}_A = m_A \mathbf{a}_A \tag{d}$$

In this expression $\left.\mathbf{F}_A\right)_g$ is the force of gravity on (i.e., the weight of) the astronaut, $\mathbf{C}_A$ is the net contact force on the astronaut from restraints (e.g., seat, seat belt), and $\mathbf{a}_A$ is the astronaut's absolute acceleration. According to Equation 2.10,

$$\left.\mathbf{F}_A\right)_g = -\frac{GM_E m_A}{r^2}\hat{\mathbf{u}}_r \tag{e}$$

Since the astronaut is moving with the spacecraft we have, noting (c),

$$\mathbf{a}_A = \mathbf{a}_S = -\frac{GM_E}{r^2}\hat{\mathbf{u}}_r \tag{f}$$

Substituting (e) and (f) into (d) yields:

$$-\frac{GM_E m_A}{r^2}\hat{\mathbf{u}}_r + \mathbf{C}_A = m_A\left(-\frac{GM_E}{r^2}\hat{\mathbf{u}}_r\right)$$

from which it is clear that:

$$\boxed{\mathbf{C}_A = \mathbf{0}}$$

The net contact force on the astronaut is zero. With no reaction to the force of gravity exerted on the body, there is no sensation of weight.

The ***gravitational potential energy*** V of the force of attraction $\mathbf{F}$ between two point masses m_1 and m_2 separated by a distance r is given by:

$$V = -\frac{Gm_1m_2}{r} \tag{2.14}$$

A conservative force like gravity can be obtained from its scalar potential energy function V by means of the gradient operator,

$$\mathbf{F} = -\nabla V \tag{2.15}$$

where, in Cartesian coordinates,

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \tag{2.16}$$

For the two-body system in Figure 2.1 we have, by combining Equations 2.8 and 2.14,

$$V = -\frac{Gm_1m_2}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}} \tag{2.17}$$

The attractive forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ in Equations 2.9 and 2.10 are derived from Equation 2.17 as follows

$$\mathbf{F}_{12} = -\left(\frac{\partial V}{\partial X_2}\hat{\mathbf{I}} + \frac{\partial V}{\partial Y_2}\hat{\mathbf{J}} + \frac{\partial V}{\partial Z_2}\hat{\mathbf{K}}\right)$$
$$\mathbf{F}_{21} = -\left(\frac{\partial V}{\partial X_1}\hat{\mathbf{I}} + \frac{\partial V}{\partial Y_1}\hat{\mathbf{J}} + \frac{\partial V}{\partial Z_1}\hat{\mathbf{K}}\right)$$

In Appendix E it is shown that the gravitational potential V, and hence the gravitational force, outside of a sphere with a spherically symmetric mass distribution M is the same as that of a point mass M located at the center of the sphere. Therefore, the two-body problem applies not just to point masses but also to spherical bodies (as long, of course, as they do not come into contact!).

Let us return to Equations 2.11 and 2.12, the equations of motion of the two-body system relative to the XYZ inertial frame. We can divide m_1 out of Equation 2.11 and m_2 out of Equation 2.12 and then substitute Equation 2.7 into both results to obtain

$$\ddot{\mathbf{R}}_1 = Gm_2\frac{\mathbf{r}}{r^3} \tag{2.18a}$$

$$\ddot{\mathbf{R}}_2 = -Gm_1\frac{\mathbf{r}}{r^3} \tag{2.18b}$$

These are the final forms of the equations of motion of the two bodies in inertial space. With the aid of Equations 2.1, 2.6 and 2.8 we can express these equations in terms of the components of the position and acceleration vectors in the inertial *XYZ* frame:

$$\ddot{X}_1 = Gm_2 \frac{X_2 - X_1}{r^3} \quad \ddot{Y}_1 = Gm_2 \frac{Y_2 - Y_1}{r^3} \quad \ddot{Z}_1 = Gm_2 \frac{Z_2 - Z_1}{r^3} \tag{2.19a}$$

$$\ddot{X}_2 = Gm_1 \frac{X_1 - X_2}{r^3} \quad \ddot{Y}_2 = Gm_1 \frac{Y_1 - Y_2}{r^3} \quad \ddot{Z}_2 = Gm_1 \frac{Z_1 - Z_2}{r^3} \tag{2.19b}$$

where $r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)}$.

The position vector **R** and velocity vector **V** of a particle are referred to collectively as its ***state vector***. The fundamental problem before us is to find the state vectors of both particles of the two-body system at a given time given the state vectors at an initial time. The numerical solution procedure is outlined in Algorithm 2.1.

Algorithm 2.1 Numerically compute the state vectors $\mathbf{R}_1$, $\mathbf{V}_1$ and $\mathbf{R}_2$, $\mathbf{V}_2$ of the two body system as a function of time, given their initial values $\mathbf{R}_1^0, \mathbf{V}_1^0$ and $\mathbf{R}_2^0, \mathbf{V}_2^0$. This algorithm is implemented in MATLAB® as the function *twobody3d.m*, which is listed in Appendix D.5.

1. Form the vector consisting of the components of the state vectors at time t_0,

$$\mathbf{y}_0 = \left\lfloor X_1^0 \quad Y_1^0 \quad Z_1^0 \quad X_2^0 \quad Y_2^0 \quad Z_2^0 \quad \dot{X}_1^0 \quad \dot{Y}_1^0 \quad \dot{Z}_1^0 \quad \dot{X}_2^0 \quad \dot{Y}_2^0 \quad \dot{Z}_2^0 \right\rfloor$$

2. Provide $\mathbf{y}_0$ and the final time t_f to Algorithm 1.1, 1.2, or 1.3, along with the vector that comprises the components of the state vector derivatives :

$$\mathbf{f}(t,\mathbf{y}) = \left\lfloor \dot{X}_1 \quad \dot{Y}_1 \quad \dot{Z}_1 \quad \dot{X}_2 \quad \dot{Y}_2 \quad \dot{Z}_2 \quad \ddot{X}_1 \quad \ddot{Y}_1 \quad \ddot{Z}_1 \quad \ddot{X}_2 \quad \ddot{Y}_2 \quad \ddot{Z}_2 \right\rfloor$$

   where the last six components, the accelerations, are given by Equation 2.19.

3. The algorithm selected in Step 2 solves the system $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ for the system state vector:

$$\mathbf{y} = \left\lfloor X_1 \quad Y_1 \quad Z_1 \quad X_2 \quad Y_2 \quad Z_2 \quad \dot{X}_1 \quad \dot{Y}_1 \quad \dot{Z}_1 \quad \dot{X}_2 \quad \dot{Y}_2 \quad \dot{Z}_2 \right\rfloor$$

   at n discrete times t_n from t_0 through t_f.

4. The state vectors of m_1 and m_2 at the discrete times are:

$$\begin{aligned} \mathbf{R}_1 &= X_1\hat{\mathbf{I}} + Y_1\hat{\mathbf{J}} + Z_1\hat{\mathbf{K}} \quad & \mathbf{V}_1 &= \dot{X}_1\hat{\mathbf{I}} + \dot{Y}_1\hat{\mathbf{J}} + \dot{Z}_1\hat{\mathbf{K}} \\ \mathbf{R}_2 &= X_2\hat{\mathbf{I}} + Y_2\hat{\mathbf{J}} + Z_2\hat{\mathbf{K}} \quad & \mathbf{V}_2 &= \dot{X}_2\hat{\mathbf{I}} + \dot{Y}_2\hat{\mathbf{J}} + \dot{Z}_2\hat{\mathbf{K}} \end{aligned}$$

Example 2.2

A system consists of two massive bodies m_1 and m_2 each having a mass of 10^{26} kg. At time $t = 0$ the state vectors of the two particles in an inertial frame are

$$\mathbf{R}_1^{(0)} = \mathbf{0} \qquad \mathbf{V}_1^{(0)} = 10\hat{\mathbf{I}} + 20\hat{\mathbf{J}} + 30\hat{\mathbf{K}} \text{ (km/s)}$$
$$\mathbf{R}_2^{(0)} = 3000\hat{\mathbf{I}} \text{ (km)} \qquad \mathbf{V}_2^{(0)} = 40\hat{\mathbf{J}} \text{ (km/s)}$$

Use Algorithm 2.1 and the *RKF4(5)* method (Algorithm 1.3) to numerically determine the motion of the two masses due solely to their mutual gravitational attraction from $t = 0$ to $t = 480$ seconds.

(a) Plot the motion of m_1 and m_2 relative to the inertial frame.
(b) Plot the motion of m_2 and G relative to m_1.
(c) Plot the motion of m_1 and m_2 relative to the center of mass G of the system.

Solution

The MATLAB function *twobody3d.m* in Appendix D.5 contains within it the data for this problem. Embedded in the program is the subfunction *rates*, which computes the accelerations given by Equation 2.19. *twobody3d.m* uses the solution vector from *rkf45.m* to plot Figures 2.2 and 2.3, which summarize the results requested in the problem statement.

In answer to part (a), Figure 2.2 shows the motion of the two-body system relative to the inertial frame. m_1 and m_2 are soon established in a periodic helical motion around the straight-line trajectory of the center of mass G through space. This pattern continues indefinitely.

Figure 2.3a relates to part (b) of the problem. The very same motion appears rather less complex when viewed from m_1. In fact we see that $\mathbf{R}_2(t) - \mathbf{R}_1(t)$, the trajectory of m_2 relative to m_1, appears to be an elliptical path. So does $\mathbf{R}_G(t) - \mathbf{R}_1(t)$, the path of the center of mass around m_1.

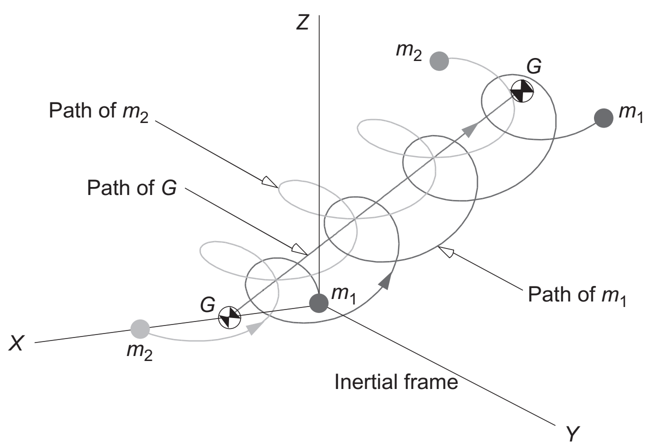


FIGURE 2.2

The motion of two identical bodies acted on only by their mutual gravitational attraction, as viewed from the inertial frame of reference.

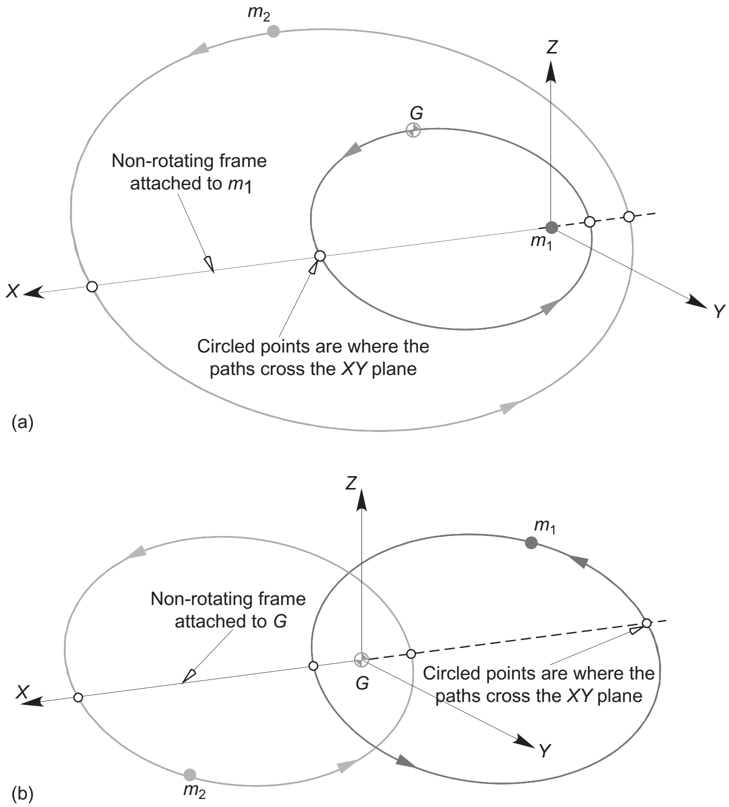


FIGURE 2.3

The motion in Figure 2.2, (a) as viewed relative to m_1 (or m_2); (b) as viewed from the center of mass.

Finally, for part (c) of the problem, Figure 2.3b reveals that both m_1 and m_2 follow apparently elliptical paths around the center of mass.

One may wonder what the motion looks like if there are more than two bodies moving under the influence only of their mutual gravitational attraction. The n-body problem with $n > 2$ has no closed form solution, which is complex and chaotic in nature. The three-body problem is briefly addressed in Appendix C, where the equations of motion of the system are presented. Appendix C lists the MATLAB program *threebody.m* that is used to solve the equations of motion for given initial conditions. Figure 2.4 shows the results for three particles of equal mass, equally spaced initially along the X-axis of an inertial frame. The central mass has an initial velocity in the XY plane, while the other two are at rest. As time progresses, we see no periodic behavior as was evident in the two-body motion in Figure 2.2. The chaos is more obvious if the motion is viewed from the center of mass of the three-body system, as shown in Figure 2.5. The computer simulation reveals that the masses all eventually collide.

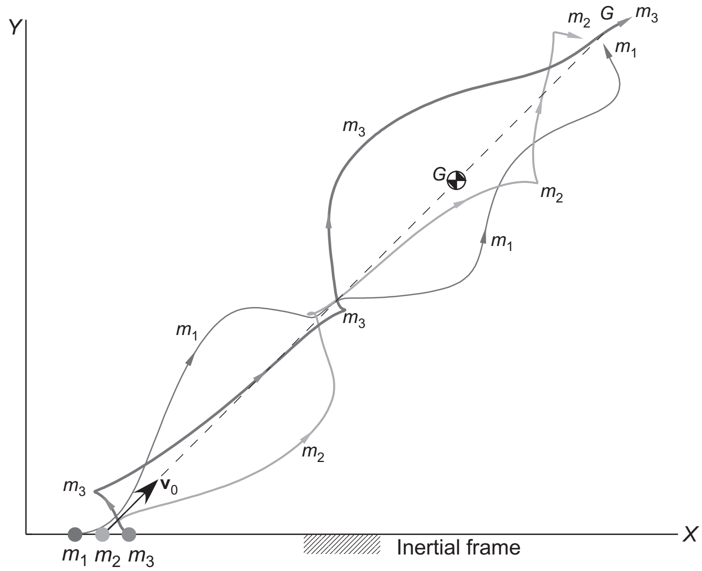


FIGURE 2.4

The motion of three identical masses as seen from the inertial frame in which m_1 and m_3 are initially at rest, while m_2 has an initial velocity $\mathbf{v}_0$ directed upwards and to the right, as shown.

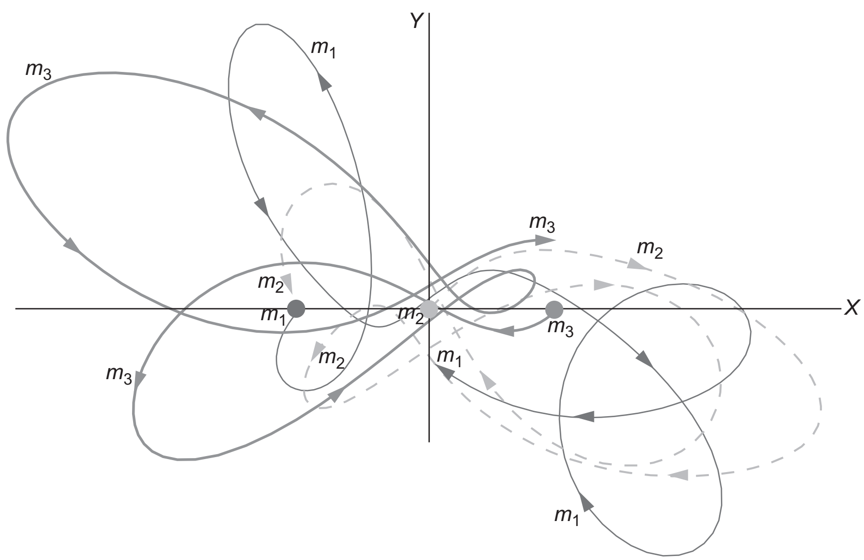


FIGURE 2.5

The same motion as Figure 2.4, as viewed from the inertial frame attached to the center of mass G.

2.3 EQUATIONS OF RELATIVE MOTION

Let us differentiate Equation 2.5 twice with respect to time in order to obtain the relative acceleration vector,

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1$$

Substituting Equations 2.18 into the right side of this expression yields:

$$\ddot{\mathbf{r}} = -\frac{G(m_1 + m_z)}{r^3}\mathbf{r} \tag{2.20}$$

The ***gravitational parameter*** μ is defined as:

$$\mu = G(m_1 + m_2) \tag{2.21}$$

The units of μ are $\mathrm{km}^3\ \mathrm{s}^{-2}$. Using Equation 2.21 we can write Equation 2.20 as:

$$\boxed{\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}} \tag{2.22}$$

This ***fundamental equation of relative two-body motion*** is a nonlinear second order differential equation that governs the motion of m_2 relative to m_1. It has two vector constants of integration, each having three scalar components. Therefore, Equation 2.22 has six constants of integration. Note that interchanging the roles of m_1 and m_2 amounts to simply multiplying Equation 2.22 through by -1, which, of course, changes nothing. Thus, the motion of m_2 as seen from m_1 is precisely the same as the motion of m_1 as seen from m_2. The motion of the moon as observed from earth appears the same as that of the earth as viewed from the moon.

The relative position vector $\mathbf{r}$ in Equation 2.22 was originally defined in the inertial frame (Equation 2.6). It is convenient, however, to measure the components of $\mathbf{r}$ in a frame of reference attached to and moving with m_1. In a co-moving reference frame, such as the *xyz* system illustrated in Figure 2.6, $\mathbf{r}$ has the expression:

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

The relative velocity $\dot{\mathbf{r}}_{\mathrm{rel}}$ and acceleration $\ddot{\mathbf{r}}_{\mathrm{rel}}$ in the co-moving frame are found by simply taking the derivatives of the coefficients of the unit vectors, which themselves are fixed in the moving *xyz* system. Thus,

$$\dot{\mathbf{r}}_{\mathrm{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \qquad \ddot{\mathbf{r}}_{\mathrm{rel}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}$$

From Equation 1.68 we know that the relationship between absolute acceleration $\ddot{\mathbf{r}}$ and relative acceleration $\ddot{\mathbf{r}}_{\mathrm{rel}}$ is:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{\mathrm{rel}} + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}}_{\mathrm{rel}}$$

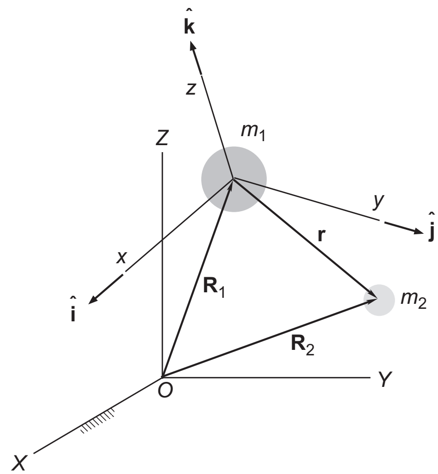


FIGURE 2.6

Moving reference frame *xyz* attached to the center of mass of m_1.

where $\mathbf{\Omega}$ and $\dot{\mathbf{\Omega}}$ are the absolute angular velocity and angular acceleration of the moving frame of reference. Thus $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{\text{rel}}$ only if $\mathbf{\Omega} = \dot{\mathbf{\Omega}} = \mathbf{0}$. That is to say, the relative acceleration may be used on the left of Equation 2.22 as long as the co-moving frame in which it is measured is not rotating.

In the remainder of this chapter and those that follow, the analytical solution of the two-body equation of relative motion (Equation 2.22) will be presented and applied to a variety of practical problems in orbital mechanics. Pending an analytical solution, we can solve Equation 2.22 numerically in a manner similar to Algorithm 2.1.

To begin, we imagine a nonrotating Cartesian coordinate system attached to m_1, as illustrated in Figure 2.6. Resolve $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$ into components in this moving frame of reference to obtain the relative acceleration components:

$$\ddot{x} = -\frac{\mu}{r^3}x \qquad \ddot{y} = -\frac{\mu}{r^3}y \qquad \ddot{z} = -\frac{\mu}{r^3}z \tag{2.23}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The components of the state vector ($\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, $\mathbf{v} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$) are listed in the vector $\mathbf{y}$,

$$\mathbf{y} = \lfloor x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z} \rfloor$$

The time derivative of this vector comprises the state vector rates,

$$\dot{\mathbf{y}} = \lfloor \dot{x} \quad \dot{y} \quad \dot{z} \quad \ddot{x} \quad \ddot{y} \quad \ddot{z} \rfloor$$

where the last three components, the accelerations, are given by Equation 2.23.

Algorithm 2.2 Numerically compute the state vector **r**, **v** of m_1 relative to m_2 as a function of time, given the initial values $\mathbf{r}_0$, $\mathbf{v}_0$. This algorithm is implemented in MATLAB as the function *orbit.m*, which is listed in Appendix D.6.

1. Form the vector comprising the components of the state vector at time t_0,

$$\mathbf{y}_0 = \lfloor x_0 \quad y_0 \quad z_0 \quad \dot{x}_0 \quad \dot{y}_0 \quad \dot{z}_0 \rfloor$$

2. Provide the state vector derivatives:

$$\mathbf{f}(t, \mathbf{y}) = \left\lfloor \dot{x} \quad \dot{y} \quad \dot{z} \quad -\frac{\mu}{r^3}x \quad -\frac{\mu}{r^3}y \quad -\frac{\mu}{r^3}z \right\rfloor$$

together with $\mathbf{y}_0$ and the final time t_f to Algorithm 1.1, 1.2 or 1.3.

3. The algorithm selected in Step 2 solves the system $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ for the state vector:

$$\mathbf{y} = \lfloor x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z} \rfloor$$

at n discrete times t_n from t_0 through t_f.

4. The position and velocity at the discrete times are:

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad \mathbf{v} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$$

Example 2.3

Relative to a nonrotating frame of reference with origin at the center of the earth, a 1000 kg satellite's initial position vector is $\mathbf{r} = 8000\hat{\mathbf{i}} + 6000\hat{\mathbf{k}}$ (km) and its initial velocity vector is $\mathbf{v} = 7\hat{\mathbf{j}}$ (km/s). Use Algorithm 2.2 and the *RKF4(5)* method to solve for the path of the spacecraft over the next four hours. Determine its minimum and maximum distance from the earth's surface during that time.

Solution

The MATLAB function *orbit.m* in Appendix D.6 solves this problem. The initial value of the vector **y** is:

$$\mathbf{y}_0 = \lfloor 8000\,\text{km} \quad 0 \quad 6000\,\text{km} \quad 0 \quad 5\ \text{km/s} \quad 5\ \text{km/s} \rfloor$$

The program provides these initial conditions to the function *rkf45* (Appendix D.4), which integrates the system $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$. *rkf45* uses the function *rates* embedded in *orbit.m* to calculate **f**(t,**y**) at each time step. The command window output of *orbit.m* in Appendix D.6 shows that:

```
The minimum altitude is 3622 km, and the speed at that point is 7 km/s.
The maximum altitude is 9560 km, and the speed at that point is 4.39 km/s.
```

The minimum altitude in this case is at the starting point of the orbit. The maximum altitude occurs two hours later on the opposite side of the earth.

orbit.m also uses some MATLAB plotting features to generate Figure 2.7. Observe that the orbit is inclined to the equatorial plane and has an apparently elliptical shape. The satellite moves eastwardly in the same direction as the earth's rotation.

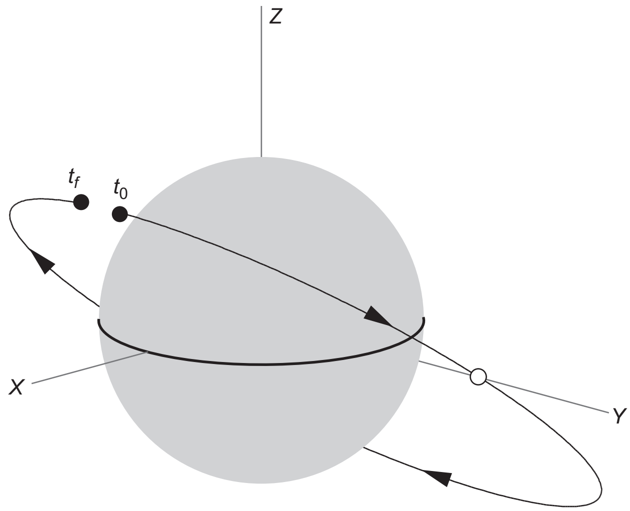


FIGURE 2.7

The computed earth orbit. The beginning of the path is marked by t_0, and t_f marks the end of the path 4 hours later.

As pointed out earlier, since the center of mass G has zero acceleration, we can use it as the origin of an inertial reference frame. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of m_1 and m_2, respectively, relative to the center of mass G in Figure 2.1. The equation of motion of m_2 relative to the center of mass is:

$$-G\frac{m_1m_2}{r^2}\hat{\mathbf{u}}_r = m_2\ddot{\mathbf{r}}_2 \tag{2.24}$$

where, as before, r is the magnitude of r, the position vector of m_2 relative to m_1. In terms of $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \tag{2.25}$$

Since the position vector of the center of mass relative to itself is zero, it follows from Equation 2.2 that:

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{0}$$

Therefore,

$$\mathbf{r}_1 = -\frac{m_2}{m_1}\mathbf{r}_2 \tag{2.26}$$

Substituting 2.26 into 2.25 yields:

$$\mathbf{r} = \frac{m_1 + m_2}{m_1}\mathbf{r}_2$$

Substituting this back into Equation 2.24 and using the fact that $\hat{\mathbf{u}}_r = \mathbf{r}_2/r_2$, we get:

$$-G\frac{m_1^{\,3}m_2}{(m_1 + m_2)^2 r_2^{\,3}}\mathbf{r}_2 = m_2\ddot{\mathbf{r}}_2$$

Upon simplification this becomes:

$$-\left(\frac{m_1}{m_1+m_2}\right)^3 \frac{\mu}{r_2^3}\mathbf{r}_2 = \ddot{\mathbf{r}}_2 \tag{2.27}$$

where μ is the gravitational parameter given by Equation 2.21. If we let:

$$\mu' = \left(\frac{m_1}{m_1+m_2}\right)^3 \mu$$

then Equation 2.27 reduces to:

$$\ddot{\mathbf{r}}_2 = -\frac{\mu'}{r_2^3}\mathbf{r}_2$$

which is identical in form to Equation 2.22.

In a similar fashion, the equation of motion of m_1 relative to the center of mass is found to be:

$$\ddot{\mathbf{r}}_1 = -\frac{\mu''}{r_1^3}\mathbf{r}_1$$

in which,

$$\mu'' = \left(\frac{m_2}{m_1+m_2}\right)^3 \mu$$

Since the equations of motion of either particle relative to the center of mass have the same form as the equations of motion relative to either one of the bodies, m_1 or m_2, it follows that the relative motion as viewed from these different perspectives must be similar, as illustrated in Figure 2.3.

2.4 ANGULAR MOMENTUM AND THE ORBIT FORMULAS

The angular momentum of body m_2 relative to m_1 is the moment of m_2's relative linear momentum $m_2\dot{\mathbf{r}}$ (cf. Equation 1.45),

$$\mathbf{H}_{2/1} = \mathbf{r} \times m_2\dot{\mathbf{r}}$$

where $\dot{\mathbf{r}} = \mathbf{v}$ is the velocity of m_2 relative to m_1. Let us divide this equation through by m_2 and let $\mathbf{h} = \mathbf{H}_{2/1}/m_2$, so that

$$\boxed{\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}} \tag{2.28}$$

$\mathbf{h}$ is the relative angular momentum of m_2 per unit mass, that is, the ***specific relative angular momentum***. The units of $\mathbf{h}$ are km^2s^{-1}.

Taking the time derivative of **h** yields:

$$\frac{d\mathbf{h}}{dt} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}$$

But $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$. Furthermore, $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$, according to Equation 2.22, so that:

$$\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \left(-\frac{\mu}{r^3}\mathbf{r}\right) = -\frac{\mu}{r^3}(\mathbf{r} \times \mathbf{r}) = 0$$

Therefore, we have the ***conservation of angular momentum***,

$$\frac{d\mathbf{h}}{dt} = 0 \qquad (\text{or } \mathbf{r} \times \dot{\mathbf{r}} = \text{constant}) \tag{2.29}$$

If the position vector **r** and the velocity vector $\dot{\mathbf{r}}$ are parallel, then it follows from Equation 2.28 that the angular momentum is zero and, according to Equation 2.29, it remains zero at all points of the trajectory. Zero angular momentum characterizes ***rectilinear trajectories*** whereon m_2 moves towards or away from m_1 in a straight line (see Example 1.20).

At any point of a curvilinear trajectory the position vector **r** and the velocity vector $\dot{\mathbf{r}}$ lie in the same plane, as illustrated in Figure 2.8. Their cross product $\mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to that plane. Since $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$, the unit vector normal to the plane is:

$$\hat{\mathbf{h}} = \frac{\mathbf{h}}{h} \tag{2.30}$$

By the conservation of angular momentum (Equation 2.29), this unit vector is constant. Thus, the path of m_2 around m_1 lies in a single plane.

Since the orbit of m_2 around m_1 forms a plane, it is convenient to orient oneself above that plane and look down upon the path, as shown in Figure 2.9. Let us resolve the relative velocity vector $\dot{\mathbf{r}}$ into components $\mathbf{v}_r = v_r\hat{\mathbf{u}}_r$ and $\mathbf{v}_\perp = v_\perp\hat{\mathbf{u}}_\perp$ along the outward radial from m_1 and perpendicular to it, respectively, where $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\perp$ are the radial and perpendicular (azimuthal) unit vectors. Then we can write Equation 2.28 as:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\hat{\mathbf{u}}_r \times (v_r\hat{\mathbf{u}}_r + v_\perp\hat{\mathbf{u}}_\perp) = rv_\perp\hat{\mathbf{h}}$$

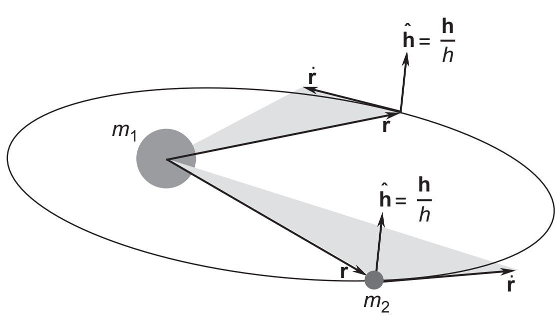


FIGURE 2.8

The path of m_2 around m_1 lies in a plane whose normal is defined by **h**.

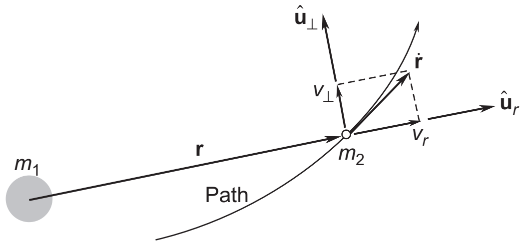


FIGURE 2.9

Components of the velocity of m_2, viewed above the plane of the orbit.

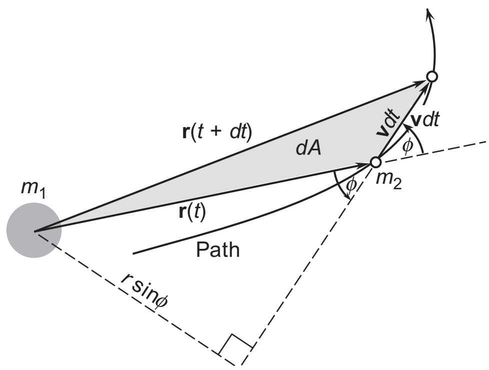


FIGURE 2.10

Differential area *dA* swept out by the relative position vector **r** during time interval *dt*.

That is,

$$\boxed{h = rv_{\perp}} \tag{2.31}$$

Clearly, the angular momentum depends only on the azimuth component of the relative velocity.

During the differential time interval *dt* the position vector **r** sweeps out an area *dA*, as shown in Figure 2.10. From the figure it is clear that the triangular area *dA* is given by:

$$dA = \frac{1}{2} \times \text{base} \times \text{altitude} = \frac{1}{2} \times v\,dt \times r\sin\phi = \frac{1}{2} r(v\sin\phi)dt = \frac{1}{2} rv_{\perp}dt$$

Therefore, using Equation 2.31 we have:

$$\frac{dA}{dt} = \frac{h}{2} \tag{2.32}$$

dA/*dt* is called the areal velocity, and according to Equation 2.32 it is constant. Named for the German astronomer Johannes Kepler (1571–1630), this result is known as ***Kepler's second law***: equal areas are swept out in equal times.

Before proceeding with an effort to integrate Equation 2.22, recall the *bac-cab* rule (Equation 1.20):

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{2.33}$$

Recall as well from Equation 1.11 that

$$\mathbf{r} \cdot \mathbf{r} = r^2 \tag{2.34}$$

so that

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = 2r\frac{dr}{dt}$$

But

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}$$

Thus, we obtain the important identity:

$$\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r} \tag{2.35a}$$

Since $\dot{\mathbf{r}} = \mathbf{v}$ and $r = \|\mathbf{r}\|$, this can be written alternatively as:

$$\mathbf{r} \cdot \mathbf{v} = \|\mathbf{r}\|\frac{d\|\mathbf{r}\|}{dt} \tag{2.35b}$$

Now let us take the cross product of both sides of Equation 2.22 [$\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$] with the specific angular momentum $\mathbf{h}$:

$$\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} \tag{2.36}$$

Since $d/dt\ (\dot{\mathbf{r}} \times \mathbf{h}) = \ddot{\mathbf{r}} \times \mathbf{h} + \dot{\mathbf{r}} \times \dot{\mathbf{h}}$, the left-hand side can be written:

$$\ddot{\mathbf{r}} \times \mathbf{h} = \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) - \dot{\mathbf{r}} \times \dot{\mathbf{h}}$$

But according to Equation 2.29, the angular momentum is constant ($\dot{\mathbf{h}} = 0$), so this reduces to:

$$\ddot{\mathbf{r}} \times \mathbf{h} = \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) \tag{2.37}$$

The right-hand side of Equation 2.36 can be transformed by the following sequence of substitutions:

$$\begin{aligned}
\frac{1}{r^3}\mathbf{r} \times \mathbf{h} &= \frac{1}{r^3}[\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})] && \text{(Equation 2.28 } [\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}]\text{)} \\
&= \frac{1}{r^3}[\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - \dot{\mathbf{r}}(\mathbf{r} \cdot \mathbf{r})] && \text{(Equation 2.33 [}\textit{bac - cab}\text{ rule])} \\
&= \frac{1}{r^3}[\mathbf{r}(r\dot{r}) - \dot{\mathbf{r}}r^2] && \text{(Equations 2.34 and 2.35)} \\
&= \frac{\mathbf{r}\dot{r} - \dot{\mathbf{r}}r}{r^2}
\end{aligned}$$

But

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{r\dot{\mathbf{r}} - \mathbf{r}\dot{r}}{r^2} = -\frac{\mathbf{r}\dot{r} - r\dot{\mathbf{r}}}{r^2}$$

Therefore,

$$\frac{1}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \tag{2.38}$$

Substituting Equations 2.37 and 2.38 into Equation 2.36, we get:

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \frac{d}{dt}\left(\mu\frac{\mathbf{r}}{r}\right)$$

or

$$\frac{d}{dt}\left(\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r}\right) = 0$$

That is,

$$\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r} = \mathbf{C} \tag{2.39}$$

where the vector **C**, called the Laplace vector after the French mathematician Pierre-Simon Laplace (1749–1827), is a constant having the dimensions of μ. Equation 2.39 is the first integral of the equation of motion, $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$. Taking the dot product of both sides of Equation 2.39 with the vector **h** yields:

$$(\dot{\mathbf{r}} \times \mathbf{h}) \cdot \mathbf{h} - \mu\frac{\mathbf{r} \cdot \mathbf{h}}{r} = \mathbf{C} \cdot \mathbf{h}$$

Since $\dot{\mathbf{r}} \times \mathbf{h}$ is perpendicular to both $\dot{\mathbf{r}}$ and **h**, it follows that $(\dot{\mathbf{r}} \times \mathbf{h}) \cdot \mathbf{h} = 0$. Likewise, since $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to both **r** and $\dot{\mathbf{r}}$, it is true that $\mathbf{r} \cdot \mathbf{h} = 0$. Therefore, we have $\mathbf{C} \cdot \mathbf{h} = 0$, i.e., **C** is perpendicular to **h**, which is normal to the orbital plane. That of course means that the Laplace vector must lie in the orbital plane.

Let us rearrange Equation 2.39 and write it as:

$$\frac{\mathbf{r}}{r} + \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} \tag{2.40}$$

where $\mathbf{e} = \mathbf{C}/\mu$. The dimensionless vector **e** is called the ***eccentricity vector***. The line defined by the vector **e** is commonly called the ***apse line***. In order to obtain a scalar equation, let us take the dot product of both sides of Equation 2.40 with the position vector **r**:

$$\frac{\mathbf{r} \cdot \mathbf{r}}{r} + \mathbf{r} \cdot \mathbf{e} = \frac{\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h})}{\mu} \tag{2.41}$$

We can simplify the right-hand side by employing the vector identity presented in Equation 1.21:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \tag{2.42}$$

from which we obtain:

$$\mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{h}) = (\mathbf{r}\times\dot{\mathbf{r}})\cdot\mathbf{h} = \mathbf{h}\cdot\mathbf{h} = h^2 \tag{2.43}$$

Substituting this expression into the right-hand side of Equation 2.41, and substituting $\mathbf{r}\cdot\mathbf{r} = r^2$ on the left yields:

$$r + \mathbf{r}\cdot\mathbf{e} = \frac{h^2}{\mu} \tag{2.44}$$

Observe that by following the steps leading from Equation 2.40 to 2.44, we have lost track of the variable time. This occurred at Equation 2.43, because h is constant. Finally, from the definition of the dot product we have:

$$\mathbf{r}\cdot\mathbf{e} = re\cos\theta$$

in which e is the ***eccentricity*** (the magnitude of the eccentricity vector **e**) and θ is the ***true anomaly***. θ is the angle between the fixed vector **e** and the variable position vector **r**, as illustrated in Figure 2.11. (Other symbols used to represent true anomaly include v, f, ν and ϕ.) In terms of the eccentricity and the true anomaly, we may therefore write Equation 2.44 as

$$r + re\cos\theta = \frac{h^2}{\mu}$$

or

$$\boxed{r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}} \tag{2.45}$$

This is the ***orbit equation***, and it defines the path of the body m_2 around m_1, relative to m_1. Remember that μ, h, and e are constants. Observe as well that there is no significance to negative values of eccentricity; that is, $e \geq 0$. Since the orbit equation describes conic sections, including ellipses, it is a mathematical statement of ***Kepler's first law***, namely, that the planets follow elliptical paths around the sun. Two-body orbits are often referred to as ***Keplerian orbits***.

In Section 2.3 it was pointed out that integration of the equation of relative motion, Equation 2.22, leads to six constants of integration. In this section it would seem that we have arrived at those constants, namely

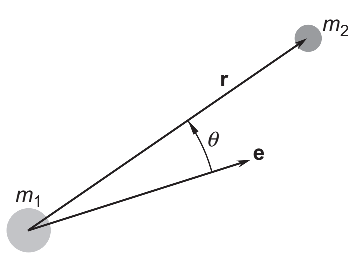


FIGURE 2.11

The true anomaly θ is the angle between the eccentricity vector **e** and the position vector **r**.

the three components of the angular momentum **h** and the three components of the eccentricity vector **e**. However, we showed that **h** is perpendicular to **e**. This places a condition, namely $\mathbf{h}\cdot\mathbf{e} = 0$, on the components of **h** and **e**, so that we really have just five independent constants of integration. The sixth constant of the motion will arise when we work time back into the picture in the next chapter.

The angular velocity of the position vector **r** is $\dot{\theta}$, the rate of change of the true anomaly. The component of velocity normal to the position vector is found in terms of the angular velocity by the formula

$$v_\perp = r\dot{\theta} \tag{2.46}$$

Substituting this into Equation 2.31 ($h = rv_\perp$) yields the specific angular momentum in terms of the angular velocity,

$$h = r^2\dot{\theta} \tag{2.47}$$

It is convenient to have formulas for computing the radial and azimuth components of velocity shown in Figure 2.12. From $h = rv_\perp$ we obtain the ***azimuth component of velocity***:

$$v_\perp = \frac{h}{r}$$

Substituting r from Equation 2.45 readily yields:

$$v_\perp = \frac{\mu}{h}(1 + e\cos\theta) \tag{2.48}$$

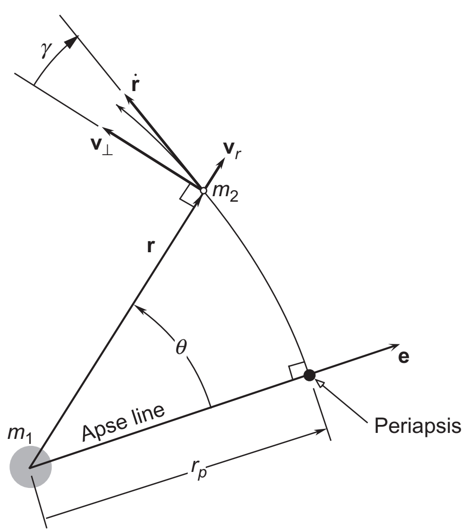


FIGURE 2.12

Position and velocity of m_2 in polar coordinates centered at m_1, with the eccentricity vector being the reference for true anomaly (polar angle) θ. γ is the flight path angle.

Since $v_r = \dot{r}$, we take the derivative of Equation 2.45 to get:

$$\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left[\frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\right] = \frac{h^2}{\mu}\left[-\frac{e(-\dot{\theta}\sin\theta)}{(1+e\cos\theta)^2}\right] = \frac{h^2}{\mu}\frac{e\sin\theta}{(1+e\cos\theta)^2}\frac{h}{r^2}$$

where we made use of the fact that $\dot{\theta} = h/r^2$, from Equation 2.47. Substituting Equation 2.45 once again and simplifying, finally yields the ***radial component of velocity***,

$$\boxed{v_r = \frac{\mu}{h}e\sin\theta} \tag{2.49}$$

We see from Equation 2.45 that m_2 comes closest to m_1 (r is smallest) when $\theta = 0$ (unless $e = 0$, in which case the distance between m_1 and m_2 is constant). The point of closest approach lies on the apse line and is called ***periapsis***. The distance r_p to periapsis, as shown in Figure 2.12, is obtained by setting the true anomaly equal to zero,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} \tag{2.50}$$

From Equation 2.49 it is clear that the radial component of velocity is zero at periapsis. For $0 < \theta < 180°$, v_r is positive, which means m_2 is moving *away* from periapsis. On the other hand Equation 2.49 shows that if $180° < \theta < 360°$, then v_r is negative, which means m_2 is moving *towards* periapsis.

The flight path angle γ is illustrated in Figure 2.12. It is the angle that the velocity vector $\mathbf{v} = \dot{\mathbf{r}}$ makes with the normal to the position vector. The normal to the position vector points in the direction of $\mathbf{v}_\perp$, and it is called the ***local horizon***. From Figure 2.12 it is clear that:

$$\tan\gamma = \frac{v_r}{v_\perp} \tag{2.51}$$

Substituting Equations 2.48 and 2.49 leads at once to the expression:

$$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta} \tag{2.52}$$

The flight path angle, like v_r, is positive (velocity vector directed above the local horizon) when the spacecraft is moving away from periapsis and is negative (velocity vector directed below the local horizon) when the spacecraft is moving towards periapsis.

Since $\cos(-\theta) = \cos\theta$, the trajectory described by the orbit equation is symmetric about the apse line, as illustrated in Figure 2.13, which also shows a chord, the straight line connecting any two points on the orbit. The latus rectum is the chord through the center of attraction perpendicular to the apse line. By symmetry, the center of attraction divides the latus rectum into two equal parts, each of length p, known historically as the ***semi-latus rectum***. In modern parlance, p is called the ***parameter*** of the orbit. From Equation 2.45 it is apparent that

$$p = \frac{h^2}{\mu} \tag{2.53}$$

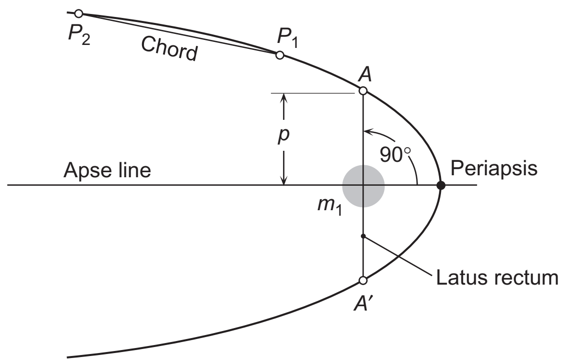


FIGURE 2.13

Illustration of latus rectum, semi-latus rectum p, and the chord between any two points on an orbit.

Since the curvilinear path of m_2 around m_1 lies in a plane, for the time being we will for simplicity continue to view the trajectory from above the plane. Unless there is reason to do otherwise, we will assume that the eccentricity vector points to the right and that m_2 moves counterclockwise around m_1, which means that the true anomaly is measured positive counterclockwise, consistent with the usual polar coordinate sign convention.

2.5 THE ENERGY LAW

By taking the cross product of Equation 2.22, $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$ (Newton's second law of motion), with the relative *angular* momentum per unit mass $\mathbf{h}$, we were led to the vector Equation 2.39, and from that we obtained the orbit formula, Equation 2.45. Now let us see what results from taking the *dot* product of Equation 2.22 with the relative *linear* momentum per unit mass. The relative linear momentum per unit mass is just the relative velocity,

$$\frac{m_2\dot{\mathbf{r}}}{m_2} = \dot{\mathbf{r}}$$

Thus, carrying out the dot product in Equation 2.22 yields:

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -\mu\frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r^3} \tag{2.54}$$

For the left-hand side of this equation we observe that:

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \frac{1}{2}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}\frac{d}{dt}(v^2) = \frac{d}{dt}\left(\frac{v^2}{2}\right) \tag{2.55}$$

For the right-hand side of Equation 2.54 we have, recalling that $\mathbf{r}\cdot\mathbf{r} = r^2$ and that $d(1/r)/dt = (-1/r^2)(dr/dt)$,

$$\mu\frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r^3} = \mu\frac{r\dot{r}}{r^3} = \mu\frac{\dot{r}}{r^2} = -\frac{d}{dt}\left(\frac{\mu}{r}\right) \tag{2.56}$$

Substituting Equations 2.55 and 2.56 into Equation 2.54 yields:

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0$$

or

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \text{ (constant)} \tag{2.57}$$

where ε is a constant. $v^2/2$ is the relative kinetic energy per unit mass. $(-\mu/r)$ is the potential energy per unit mass of the body m_2 in the gravitational field of m_1. The total mechanical energy per unit mass ε is the sum of the kinetic and potential energies per unit mass. Equation 2.57 is a statement of the ***conservation of energy***, namely, that the specific mechanical energy is the same at all points of the trajectory. Equation 2.57 is also known as the ***vis-viva*** ("living force") equation. It is valid for any trajectory, including rectilinear ones.

For curvilinear trajectories, we can evaluate the constant ε at periapsis ($\theta = 0$),

$$\varepsilon = \varepsilon_p = \frac{v_p{}^2}{2} - \frac{\mu}{r_p} \tag{2.58}$$

where r_p and v_p are the position and speed at periapsis. Since $v_r = 0$ at periapsis, the only component of velocity is $v_\perp$, which means $v_p = v_\perp = h/r_p$. Thus,

$$\varepsilon = \frac{1}{2}\frac{h^2}{r_p{}^2} - \frac{\mu}{r_p} \tag{2.59}$$

Substituting the formula for periapse radius (Equation 2.50) into Equation 2.59 yields an expression for the orbital specific energy in terms of the orbital constants h and e,

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2) \tag{2.60}$$

Clearly, the orbital energy is not an independent orbital parameter.

Note that the energy $\mathscr{E}$ of a satellite of mass m is obtained from the specific energy ε by the formula:

$$\mathscr{E} = m\varepsilon \tag{2.61}$$

2.6 CIRCULAR ORBITS ($e = 0$)

Setting $e = 0$ in the orbital equation $r = (h^2/\mu)/(1 + e\cos\theta)$, yields:

$$r = \frac{h^2}{\mu} \tag{2.62}$$

That is, $r =$ constant, which means the orbit of m_2 around m_1 is a circle. Since the radial velocity $\dot{r}$ is zero, it follows that $v = v_\perp$ so that the angular momentum formula $h = rv_\perp$ becomes simply $h = rv$ for a circular orbit. Substituting this expression for h into Equation 2.62 and solving for v yields the velocity of a circular orbit,

$$v_{\text{circular}} = \sqrt{\frac{\mu}{r}} \tag{2.63}$$

The time T required for one orbit is known as the period. Because the speed is constant, the ***period of a circular orbit*** is easy to compute.

$$T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{\sqrt{\mu/r}}$$

so that,

$$T_{\text{circular}} = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} \tag{2.64}$$

The ***specific energy of a circular orbit*** is found by setting $e = 0$ in Equation 2.60,

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}$$

Employing Equation 2.62 yields

$$\varepsilon_{\text{circular}} = -\frac{\mu}{2r} \tag{2.65}$$

Obviously, the energy of a circular orbit is negative. As the radius goes up, the energy becomes less negative, that is, it increases. In other words, the larger the orbit is, the greater is its energy.

To launch a satellite from the surface of the earth into a circular orbit requires increasing its specific energy ε. This energy comes from the rocket motors of the launch vehicle. Since the energy of a satellite of mass m is $\mathcal{E} = m\varepsilon$, a propulsion system that can place a large mass in a low earth orbit can place a smaller mass in a higher earth orbit.

The space shuttle orbiters are the largest man-made satellites so far placed in orbit with a single launch vehicle. For example, on NASA mission STS-82 in February 1997, the orbiter Discovery rendezvoused with the Hubble space telescope to repair and refurbish it. The altitude of the nearly circular orbit was 580 km (360 miles). Discovery's orbital mass early in the mission was 106,000 kg (117 tons). That was only six per cent of the total mass of the shuttle prior to launch (comprising the orbiter's dry mass, plus that of its payload and fuel, plus the two solid rocket boosters, plus the external fuel tank filled with liquid hydrogen and oxygen). This mass of about two million kilograms (2200 tons) was lifted off the launch pad by a total thrust in the vicinity of 35,000 kN (7.8 million pounds). Eighty-five per cent of the thrust was furnished by the solid rocket boosters (SRBs), which were depleted and jettisoned about two minutes into the flight. The remaining thrust came from the three liquid rockets (space shuttle main engines, or SSMEs) on the orbiter. These were fueled by the external tank, which was jettisoned just after the SSMEs were shut down at MECO (main engine cut off), about eight and a half minutes after lift-off.

Manned orbital spacecraft and a host of unmanned remote sensing, imaging and navigation satellites occupy nominally circular, ***low earth orbits (LEOs)***. A low earth orbit (LEO) is one whose altitude lies between about 150 km (100 miles) and about 1000 km (600 miles). An LEO is well above the nominal outer limits of the drag-producing atmosphere (about 80 km or 50 miles), and well below the hazardous Van Allen radiation belts, the innermost of which begins at about 2400 km (1500 miles).

Nearly all of our applications of the orbital equations will be to the analysis of man-made spacecraft, all of which have a mass that is insignificant compared to the sun and planets. For example, since the earth is nearly twenty orders of magnitude more massive than the largest conceivable artificial satellite, the center of mass of the two-body system lies at the center of the earth, and the constant μ in Equation 2.21 becomes:

$$\mu = G\left(m_{\text{earth}} + \overbrace{m_{\text{satellite}}}^{\text{neglect}}\right) = Gm_{\text{earth}}$$

The value of the earth's gravitational parameter to be used throughout this book is found in Table A.2,

$$\mu_{\text{earth}} = 398{,}600 \text{ km}^3/\text{s}^2 \tag{2.66}$$

Example 2.4

Plot the speed v and period T of a satellite in circular LEO as a function of altitude z.

Solution

Equations 2.63 and 2.64 give the speed and period, respectively, of the satellite:

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{R_E + z}} = \sqrt{\frac{398{,}600}{6378 + z}} \qquad T = \frac{2\pi}{\sqrt{\mu}}r^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398{,}600}}(6378 + z)^{\frac{3}{2}}$$

These relations are graphed in Figure 2.14.

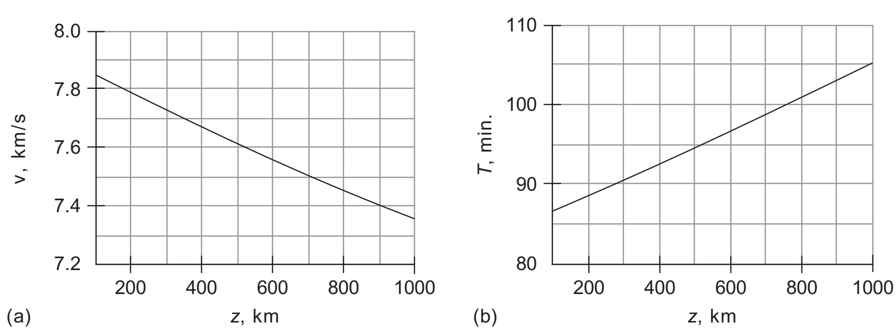


FIGURE 2.14

Circular orbital speed (a) and period (b) as a function of altitude.

If a satellite remains always above the same point on the earth's equator, then it is in a circular, ***geostationary equatorial orbit*** or *GEO*. For GEO, the radial from the center of the earth to the satellite must have the same angular velocity as the earth itself, namely, 2π radians per sidereal day. The ***sidereal day*** is the time it takes the earth to complete one rotation relative to inertial space (the fixed stars). The ordinary 24-hour day, or ***synodic day***, is the time it takes the sun to apparently rotate once around the earth, from high noon one day to high noon the next. The synodic and sidereal days would be identical if the earth stood still in space. However, while the earth makes one absolute rotation around its axis, it advances 2π/365.26 radians along its solar orbit. Therefore, earth's inertial angular velocity ω_E is [(2π + 2π/365.26) radians]/(24 hours); that is,

$$\boxed{\omega_E = 72.9217 \times 10^{-6} \text{ rad/s}} \tag{2.67}$$

Communications satellites and global weather satellites are placed in geostationary orbit because of the large portion of the earth's surface visible from that altitude and the fact that ground stations do not have to track the satellite, which appears motionless in the sky.

Example 2.5

Calculate the altitude z_{GEO} and speed v_{GEO} of a geostationary earth satellite.

Solution

From Equation 2.63, the speed of the satellite in its circular GEO of radius r_{GEO} is:

$$v_{GEO} = \sqrt{\frac{\mu}{r_{GEO}}} \tag{a}$$

On the other hand, the speed v_{GEO} along its circular path is related to the absolute angular velocity ω_E of the earth by the kinematics formula:

$$v_{GEO} = \omega_E r_{GEO}$$

Equating these two expressions and solving for r_{GEO} yields:

$$r_{GEO} = \sqrt[3]{\frac{\mu}{\omega_E{}^2}}$$

Substituting Equations 2.66 and 2.67, we get:

$$r_{GEO} = \sqrt[3]{\frac{398{,}600}{(72.9217 \times 10^{-6})^2}} = 42{,}164 \text{ km} \tag{2.68}$$

Therefore, ***GEO altitude,*** the distance of the satellite above the earth's surface is

$$z_{GEO} = r_{GEO} - R_E = 42{,}164 - 6378$$

$$\boxed{z_{GEO} = 35{,}786 \text{ km} \quad (22{,}241 \text{ mi})}$$

Substituting Equation 2.68 into (a) yields the ***GEO speed***,

$$v_{GEO} = \sqrt{\frac{398{,}600}{42{,}164}} = \boxed{3.075 \text{ km/s}} \tag{2.69}$$

Example 2.6

Calculate the maximum latitude and the percentage of the earth's surface visible from GEO.

Solution

To find the maximum viewable latitude ϕ, use Figure 2.15, from which it is apparent that:

$$\phi = \cos^{-1}\frac{R_E}{r} \tag{a}$$

where $R_E = 6378$ km and, according to Equation 2.68, $r = 42{,}164$ km. Therefore,

$$\phi = \cos^{-1}\frac{6378}{42{,}164}$$

$$\boxed{\phi = 81.30^\circ} \quad \text{Maximum visible north or south latitude} \tag{b}$$

The surface area S visible from GEO is the shaded region illustrated in Figure 2.16. It can be shown that the area S is given by

$$S = 2\pi R_E^2(1 - \cos\phi)$$

where $2\pi {R_E}^2$ is the area of the hemisphere. Therefore, the percentage of the hemisphere visible from GEO is:

$$\frac{S}{2\pi R_E^2} \times 100 = (1 - \cos 81.30^\circ) \times 100 = 84.9\%$$

which of course means that $\boxed{\text{42.4 percent}}$ of the total surface of the earth can be seen from GEO.

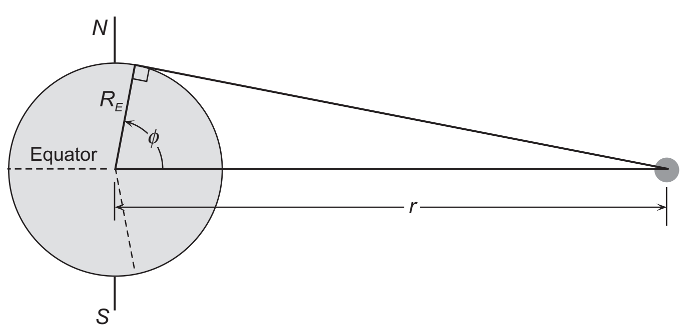


FIGURE 2.15

Satellite in GEO.

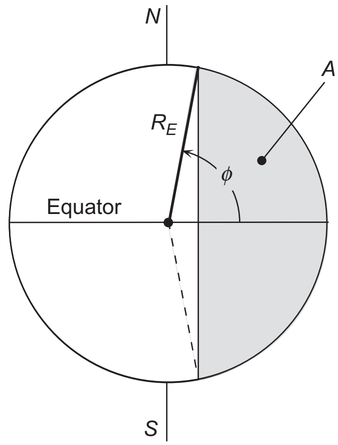


FIGURE 2.16

Surface area *S* visible from GEO.

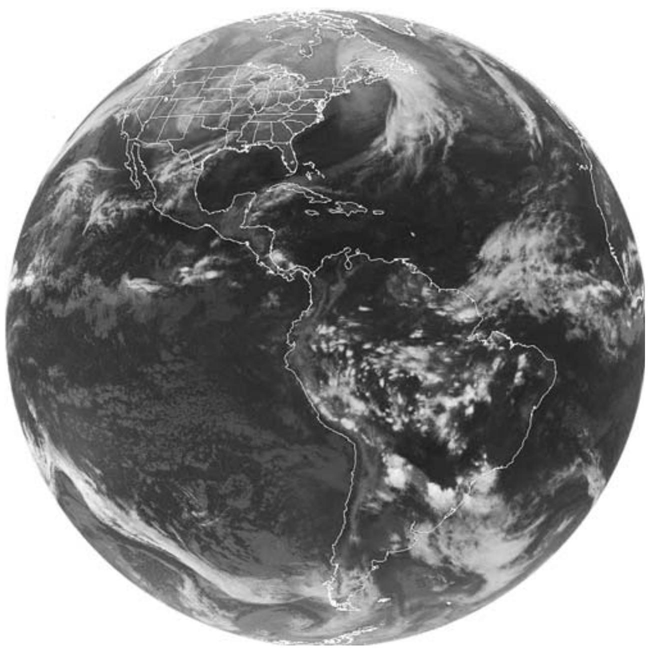


FIGURE 2.17

The view from GEO. *NASA-Goddard Space Flight Center, data from NOAA GOES*.

Figure 2.17 is a photograph taken from geosynchronous equatorial orbit by one of the National Oceanic and Atmospheric Administration's (NOAA), and Geostationary Operational Environmental Satellites (GOES).

2.7 ELLIPTICAL ORBITS ($0 < e < 1$)

If $0 < e < 1$, then the denominator of Equation 2.45 varies with the true anomaly θ, but it remains positive, never becoming zero. Therefore, the relative position vector remains bounded, having its smallest magnitude at periapsis r_p, given by Equation 2.50. The maximum value of r is reached when the denominator of $r = (h^2/\mu)/(1 + e\cos\theta)$ obtains its minimum value, which occurs at $\theta = 180°$. That point is called the ***apoapsis***, and its radial coordinate, denoted r_a, is:

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e} \tag{2.70}$$

The curve defined by Equation 2.45 in this case is an ellipse.

Let $2a$ be the distance measured along the apse line from periapsis P to apoapsis A, as illustrated in Figure 2.18. Then,

$$2a = r_p + r_a$$

Substituting Equations 2.50 and 2.70 into this expression we get:

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} \tag{2.71}$$

a is the ***semimajor axis*** of the ellipse. Solving Equation 2.71 for h^2/μ and putting the result into Equation 2.45 yields an alternative form of the orbit equation,

$$r = a\frac{1-e^2}{1+e\cos\theta} \tag{2.72}$$

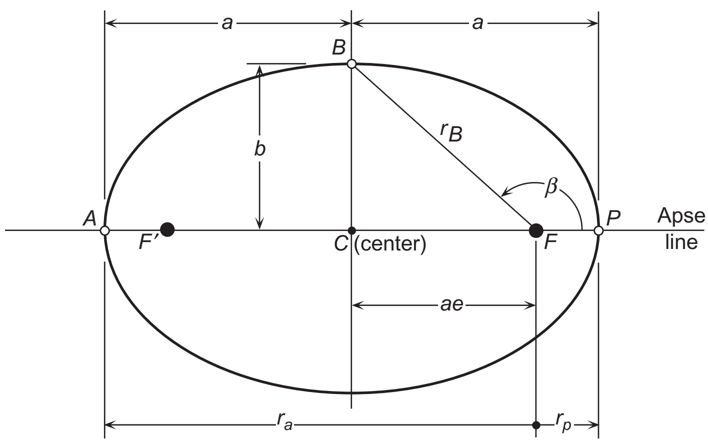


FIGURE 2.18

Elliptical orbit. m_1 is at the focus F. F' is the unoccupied empty focus.

In Figure 2.18, let F denote the location of the body m_1, which is the origin of the r,θ polar coordinate system. The center C of the ellipse is the point lying midway between the apoapsis and periapsis. The distance CF from the center C to the focus F is:

$$CF = a - FP = a - r_p$$

But from Equation 2.72, evaluated at $\theta = 0$,

$$r_p = a(1 - e) \tag{2.73}$$

Therefore, $CF = ae$, as indicated in Figure 2.18.

Let B be the point on the orbit that lies directly above C, on the perpendicular bisector of the major axis AP. The distance b from C to B is the semiminor axis. If the true anomaly of point B is β, then according to Equation 2.72, the radial coordinate of B is

$$r_B = a\frac{1 - e^2}{1 + e\cos\beta} \tag{2.74}$$

The projection of r_B onto the apse line is ae; that is,

$$ae = r_B\cos(180 - \beta) = -r_B\cos\beta = -\left(a\frac{1 - e^2}{1 + e\cos\beta}\right)\cos\beta$$

Solving this expression for e, we obtain:

$$e = -\cos\beta \tag{2.75}$$

Substituting this result into Equation 2.74 reveals the interesting fact that:

$$r_B = a$$

According to the Pythagorean theorem,

$$b^2 = r_B{}^2 - (ae)^2 = a^2 - a^2e^2$$

which means that the ***semiminor axis*** is found in terms of the semimajor axis and the eccentricity of the ellipse as:

$$\boxed{b = a\sqrt{1 - e^2}} \tag{2.76}$$

Let an xy Cartesian coordinate system be centered at C, as shown in Figure 2.19. In terms of r and θ, we see from the figure that the x-coordinate of a point on the orbit is:

$$x = ae + r\cos\theta = ae + \left(a\frac{1 - e^2}{1 + e\cos\theta}\right)\cos\theta = a\frac{e + \cos\theta}{1 + e\cos\theta}$$

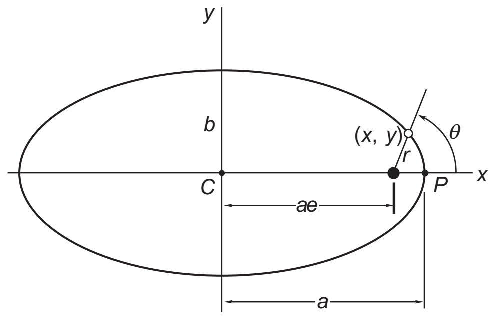


FIGURE 2.19

Cartesian coordinate description of the orbit.

From this we have:

$$\frac{x}{a} = \frac{e + \cos\theta}{1 + e\cos\theta} \tag{2.77}$$

For the *y*-coordinate we make use of Equation 2.76 to obtain:

$$y = r\sin\theta = \left(a\frac{1-e^2}{1+e\cos\theta}\right)\sin\theta = b\frac{\sqrt{1-e^2}}{1+e\cos\theta}\sin\theta$$

Therefore,

$$\frac{y}{b} = \frac{\sqrt{1-e^2}}{1+e\cos\theta}\sin\theta \tag{2.78}$$

Using Equations 2.77 and 2.78, we find:

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{1}{(1+e\cos\theta)^2}[(e+\cos\theta)^2 + (1-e^2)\sin^2\theta] \\
&= \frac{1}{(1+e\cos\theta)^2}[e^2 + 2e\cos\theta + \cos^2\theta + \sin^2\theta - e^2\sin^2\theta] \\
&= \frac{1}{(1+e\cos\theta)^2}[e^2 + 2e\cos\theta + 1 - e^2\sin^2\theta] \\
&= \frac{1}{(1+e\cos\theta)^2}[e^2(1-\sin^2\theta) + 2e\cos\theta + 1] \\
&= \frac{1}{(1+e\cos\theta)^2}[e^2\cos^2\theta + 2e\cos\theta + 1] \\
&= \frac{1}{(1+e\cos\theta)^2}(1+e\cos\theta)^2
\end{aligned}$$

That is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.79}$$

This is the familiar Cartesian coordinate formula for an ellipse centered at the origin, with x-intercepts at $\pm a$ and y-intercepts at $\pm b$. If $a = b$, Equation 2.79 describes a circle, which is really an ellipse whose eccentricity is zero.

The specific energy of an elliptical orbit is negative, and it is found by substituting the angular momentum and eccentricity into Equation 2.60,

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2)$$

According to Equation 2.71, $h^2 = \mu a(1 - e^2)$, so that the ***specific energy of an ellipse*** is

$$\boxed{\varepsilon = -\frac{\mu}{2a}} \tag{2.80}$$

This shows that the specific energy is independent of the eccentricity and depends only on the semimajor axis of the ellipse. For an elliptical orbit, the conservation of energy (Equation 2.57) may therefore be written:

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{2.81}$$

The area of an ellipse is found in terms of its semimajor and semiminor axes by the formula $A = \pi ab$ (which reduces to the formula for the area of a circle if $a = b$). To find the period T of the elliptical orbit, we employ Kepler's second law, $dA/dt = h/2$, to obtain:

$$\Delta A = \frac{h}{2}\Delta t$$

For one complete revolution, $\Delta A = \pi ab$ and $\Delta t = T$. Thus, $\pi ab = (h/2)T$, or

$$T = \frac{2\pi ab}{h}$$

Substituting Equations 2.71 and 2.76, we get:

$$T = \frac{2\pi}{h}a^2\sqrt{1 - e^2} = \frac{2\pi}{h}\left(\frac{h^2}{\mu}\frac{1}{1 - e^2}\right)^2\sqrt{1 - e^2}$$

so that the formula for the ***period of an elliptical orbit***, in terms of the orbital parameters h and e, becomes

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1 - e^2}}\right)^3 \tag{2.82}$$

We can once again appeal to Equation 2.71 to substitute $h = \sqrt{\mu a(1 - e^2)}$ into this equation, thereby obtaining an alternative expression for the period,

$$\boxed{T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}} \tag{2.83}$$

This expression, which is identical to that of a circular orbit of radius a (Equation 2.64), reveals that, like the energy, the period of an elliptical orbit is independent of the eccentricity (see Figure 2.20). Equation 2.83 embodies ***Kepler's third law***: the period of a planet is proportional to the three-halves power of its semimajor axis.

Finally, observe that dividing Equation 2.50 by Equation 2.70 yields:

$$\frac{r_p}{r_a} = \frac{1 - e}{1 + e}$$

Solving this for e results in a useful formula for calculating the eccentricity of an elliptical orbit, namely,

$$\boxed{e = \frac{r_a - r_p}{r_a + r_p}} \tag{2.84}$$

From Figure 2.18 it is apparent that $r_a - r_p = \overline{F'F}$, the distance between the foci. As previously noted, $r_a + r_p = 2a$. Thus, Equation 2.84 has the geometrical interpretation,

$$\textit{eccentricity} = \frac{\text{distance between the foci}}{\text{length of the major axis}}$$

A ***rectilinear ellipse*** is characterized as having zero angular momentum and an eccentricity of 1. That is, the distance between the foci equals the finite length of the major axis, along which the relative motion occurs. Since only the length of the semimajor axis determines the orbital specific energy, Equation 2.80 applies to rectilinear ellipses as well.

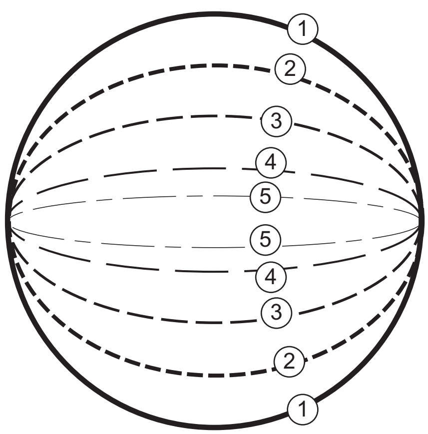


FIGURE 2.20

Since all five ellipses have the same major axis, their periods and energies are identical.

What is the average distance of m_2 from m_1 in the course of one complete orbit? To answer this question, we divide the range of the true anomaly (2π) into n equal segments $\Delta\theta$, so that:

$$n = \frac{2\pi}{\Delta\theta}$$

We then use the orbit formula $r = (h^2/\mu)/(1 + e\cos\theta)$ to evaluate $r(\theta)$ at the n equally spaced values of true anomaly, starting at periapsis:

$$\theta_1 = 0,\ \theta_2 = \Delta\theta,\ \theta_3 = 2\Delta\theta,\ \cdots,\theta_n = (n-1)\Delta\theta$$

The average of this set of n values of r is given by:

$$\bar{r}_\theta = \frac{1}{n}\sum_{i=1}^{n} r(\theta_i) = \frac{\Delta\theta}{2\pi}\sum_{i=1}^{n} r(\theta_i) = \frac{1}{2\pi}\sum_{i=1}^{n} r(\theta_i)\Delta\theta \tag{2.85}$$

Now let n become very large, so that $\Delta\theta$ becomes very small. In the limit as $n \to \infty$, Equation 2.85 becomes:

$$\bar{r}_\theta = \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\, d\theta \tag{2.86}$$

Substituting Equation 2.72 into the integrand yields:

$$\bar{r}_\theta = \frac{1}{2\pi} a(1-e^2)\int_0^{2\pi} \frac{d\theta}{1 + e\cos\theta}$$

The integral in this expression can be found in integral tables (e.g., Beyer, 1991), from which we obtain:

$$\bar{r}_\theta = \frac{1}{2\pi} a(1-e^2)\left(\frac{2\pi}{\sqrt{1-e^2}}\right) = a\sqrt{1-e^2} \tag{2.87}$$

Comparing this result with Equation 2.76, we see that the true-anomaly-averaged orbital radius equals the length of the semiminor axis b of the ellipse. Thus, the semimajor axis, which is the average of the maximum and minimum distances from the focus, is not the mean distance. Since, from Equation 2.72, $r_p = a(1 - e)$ and $r_a = a(1 + e)$, Equation 2.87 also implies that:

$$\bar{r}_\theta = \sqrt{r_p r_a} \tag{2.88}$$

The mean distance is the one-half power of the product of the maximum and minimum distances from the focus and not one-half their sum.

Example 2.7

An earth satellite is in an orbit with perigee altitude $z_p = 400\,\text{km}$ and apogee altitude $z_a = 4000\,\text{km}$. Find each of the following quantities:

(a) Eccentricity, e;
(b) Angular momentum, h;
(c) Perigee velocity, v_p;
(d) Apogee velocity, v_a;
(e) Semimajor axis, a;
(f) Period of the orbit, T;
(g) True-anomaly-averaged radius $\bar{r}_\theta$;
(h) True anomaly when $r = \bar{r}_\theta$;
(i) Satellite speed when $r = \bar{r}_\theta$;
(j) Flight path angle γ when $r = \bar{r}_\theta$;
(k) Maximum flight path angle γ_{max} and the true anomaly at which it occurs.

Recall from Equation 2.66 that $\mu = 398{,}600\,\text{km}^3/\text{s}^2$ and also that R_E, the radius of the earth, is 6378 km.

Solution

The strategy is always to seek the primary orbital parameters (eccentricity e and angular momentum h) first. All of the other orbital parameters are obtained from these two.

(a) A formula that involves the unknown eccentricity e as well as the given perigee and apogee data is Equation 2.84. We must not forget to convert the given altitudes to radii:

$$r_p = R_E + z_p = 6378 + 400 = 6778\,\text{km}$$

$$r_a = R_E + z_a = 6378 + 4000 = 10{,}378\,\text{km}$$

Then,

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{10{,}378 - 6778}{10{,}378 + 6778}$$

$$\boxed{e = 0.2098}$$

(b) Now that we have the eccentricity, we need an expression containing it and the unknown angular momentum h and any other given data. That would be Equation 2.50, the orbit formula evaluated at perigee ($\theta = 0$),

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e}$$

We use this to compute the angular momentum

$$6778 = \frac{h^2}{398{,}600}\frac{1}{1 + 0.2098}$$

$$\boxed{h = 57{,}172\ \text{km}^2/s}$$

(c) The angular momentum h and the perigee radius r_p can be substituted into the angular momentum formula, Equation 2.31, to find the perigee velocity v_p,

$$v_p = v_\perp\big)_{\text{perigee}} = \frac{h}{r_p} = \frac{57{,}172}{6778}$$

$$\boxed{v_p = 8.435 \text{ km/s}}$$

(d) Since h is constant, the angular momentum formula can also be employed to obtain the apogee speed v_a,

$$v_a = \frac{h}{r_a} = \frac{57{,}172}{10{,}378}$$

$$\boxed{v_a = 5.509 \text{ km/s}}$$

(e) The semimajor axis is the average of the perigee and apogee radii (Figure 2.18),

$$a = \frac{r_p + r_a}{2} = \frac{6778 + 10{,}378}{2}$$

$$\boxed{a = 8578 \text{ km}}$$

(f) Since the semimajor axis a has been found, we can use Equation 2.83 to calculate the period T of the orbit:

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398{,}600}} 8578^{\frac{3}{2}} = 7907 \text{ s}$$

$$\boxed{T = 2.196 \text{ h}}$$

Alternatively, we could have used Equation 2.82 for T, since both h and e were calculated above.

(g) Either Equation 2.87 or 2.88 may be used at this point to find the true-anomaly-averaged radius. Choosing the latter, we get:

$$\bar{r}_\theta = \sqrt{r_p r_a} = \sqrt{6778 \cdot 10{,}378}$$

$$\boxed{\bar{r}_\theta = 8387 \text{ km}}$$

(h) To find the true anomaly when $r = \bar{r}_\theta$, we have only one choice, namely, the orbit formula (Equation 2.45):

$$\bar{r}_\theta = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta}$$

Substituting h and e, the primary orbital parameters found above, together with $\bar{r}_\theta$, we get:

$$8387 = \frac{57{,}172^2}{398{,}600} \frac{1}{1 + 0.2098\cos\theta}$$

from which,

$$\cos\theta = -0.1061$$

This means that the true-anomaly-averaged radius occurs at:

$\boxed{\theta = 96.09°}$ where the satellite passes through $\bar{r}_\theta$ on its way *from* perigee.

and at,

$\boxed{\theta = 263.9°}$ where the satellite passes through $\bar{r}_\theta$ on its way *towards* perigee.

(i) To find the speed of the satellite when $r = \bar{r}_\theta$, it is simplest to use the energy equation for the ellipse (Equation 2.81),

$$\frac{v^2}{2} - \frac{\mu}{\bar{r}_\theta} = -\frac{\mu}{2a}$$

$$\frac{v^2}{2} - \frac{398{,}600}{8387} = -\frac{398{,}600}{2 \cdot 8578}$$

$$\boxed{v = 6.970 \text{ km/s}}$$

(j) Equation 2.52 gives the flight path angle in terms of the true anomaly of the average radius $\bar{r}_\theta$. Substituting the smaller of the two angles found in part (h) above yields:

$$\tan\gamma = \frac{e\sin\theta}{1 + e\cos\theta} = \frac{0.2098\sin 96.09°}{1 + 0.2098\cos 96.09°} = 0.2134$$

This means that:

$\boxed{\gamma = 12.05°}$ when the satellite passes through $\bar{r}_\theta$ on its way *from* perigee.

(k) To find where γ is a maximum, we must take the derivative of:

$$\gamma = \tan^{-1}\frac{e\sin\theta}{1 + e\cos\theta} \tag{a}$$

with respect to θ and set the result equal to zero. Using the rules of calculus,

$$\frac{d\gamma}{d\theta} = \frac{1}{1 + \left(\dfrac{e\sin\theta}{1 + e\cos\theta}\right)^2}\frac{d}{d\theta}\left(\frac{e\sin\theta}{1 + e\cos\theta}\right) = \frac{e(e + \cos\theta)}{(1 + e\cos\theta)^2 + e^2\sin^2\theta}$$

For $e < 1$, the denominator is nonzero for all values of θ. Therefore, $d\gamma/d\theta = 0$ only if the numerator vanishes, that is, if $\cos\theta = -e$. Recall from Equation 2.75 that this true anomaly locates the end-point of the minor axis of the ellipse. The maximum positive flight path angle therefore occurs at the true anomaly,

$$\theta = \cos^{-1}(-0.2098)$$

$$\boxed{\theta = 102.1°}$$

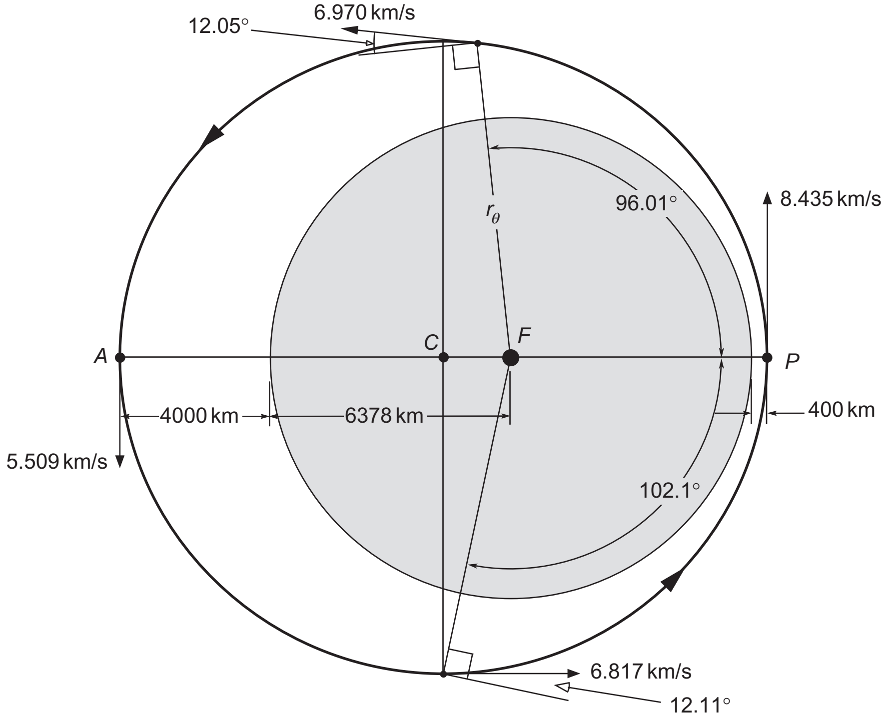


FIGURE 2.21

The orbit of Example 2.7.

Substituting this into (a), we find the maximum value of the flight path angle to be:

$$\gamma_{max} = \tan^{-1}\frac{0.2098\sin 102.1°}{1+0.2098\cos 102.1°}$$

$$\boxed{\gamma_{max} = 12.11°}$$

After attaining this greatest magnitude, the flight path angle starts to decrease steadily towards its value of zero at apogee.

Example 2.8

At two points on a geocentric orbit the altitude and true anomaly are $z_1 = 1545$ km, $\theta_1 = 126°$ and $z_2 = 852$ km, $\theta_2 = 58°$, respectively. Find (a) the eccentricity, (b) the altitude of perigee, (c) the semimajor axis, and (d) the period.

Solution

The first objective is to find the primary orbital parameters e and h, since all other orbital data can be deduced from them.

(a) Before proceeding, we must remember to add the earth's radius to the given altitudes so that we are dealing with orbital radii. The radii of the two points are:

$$r_1 = R_E + z_1 = 6378 + 1545 = 7923 \text{ km}$$
$$r_2 = R_E + z_2 = 6378 + 852 = 7230 \text{ km}$$

The only formula we have that relates orbital position to the orbital parameters e and h is the orbit formula, Equation 2.45. Writing that equation down for each of the two given points on the orbit yields two equations for e and h. For point 1 we obtain:

$$r_1 = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta_1}$$

$$7923 = \frac{h^2}{398{,}600}\frac{1}{1 + e\cos 126°}$$

$$h^2 = 3.158 \times 10^9 - 1.856 \times 10^9 e \qquad \text{(a)}$$

For point 2,

$$r_2 = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta_2}$$

$$7230 = \frac{h^2}{398{,}600}\frac{1}{1 + e\cos 58°}$$

$$h^2 = 2.882 \times 10^9 + 1.527 \times 10^9 e \qquad \text{(b)}$$

Equating (a) and (b), the two expressions for h^2, yields a single equation for the eccentricity e,

$$3.158 \times 10^9 - 1.856 \times 10^9 e = 2.882 \times 10^9 + 1.527 \times 10^9 e$$

or

$$3.384 \times 10^9 e = 276.2 \times 10^6$$

Therefore,

$$\boxed{e = 0.08164} \quad \text{(an ellipse)} \qquad \text{(c)}$$

By substituting the eccentricity back into (a) [or (b)] we find the angular momentum,

$$h^2 = 3.158 \times 10^9 - 1.856 \times 10^9 \cdot 0.08164 \Rightarrow h = 54{,}830 \text{ km}^2/\text{s} \qquad \text{(d)}$$

(b) With the eccentricity and angular momentum available, we can use the orbit equation to obtain the perigee radius (Equation 2.50),

$$r_p = \frac{h^2}{\mu}\frac{1}{1 + e} = \frac{54{,}830^2}{398{,}600}\frac{1}{1 + 0.08164} = 6974 \text{ km} \qquad \text{(e)}$$

From that we find the perigee altitude,

$$z_p = r_p - R_E = 6974 - 6378$$

$$\boxed{z_p = 595.5 \text{ km}}$$

(c) The semimajor axis is the average of the perigee and apogee radii. We just found the perigee radius above in (e). Thus, we need only to compute the apogee radius and that is accomplished by using Equation 2.70, which is the orbit formula evaluated at apogee.

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e} = \frac{54{,}830^2}{398{,}600}\frac{1}{1-0.08164} = 8213 \text{ km} \tag{f}$$

From (e) and (f) it follows that:

$$a = \frac{r_p + r_a}{2} = \frac{8213 + 6974}{2}$$

$$\boxed{a = 7593 \text{ km}}$$

(d) Since the semimajor axis has been determined, it is convenient to use Equation 2.83 to find the period.

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398{,}600}} 7593^{\frac{3}{2}} = 6585 \text{ s}$$

$$\boxed{T = 1.829 \text{ hr}}$$

2.8 PARABOLIC TRAJECTORIES ($e = 1$)

If the eccentricity equals 1, then the orbit equation (Equation 2.45) becomes:

$$r = \frac{h^2}{\mu}\frac{1}{1+\cos\theta} \tag{2.89}$$

As the true anomaly θ approaches 180°, the denominator approaches zero, so that r tends towards infinity. According to Equation 2.60, the energy of a trajectory for which $e = 1$ is zero, so that for a parabolic trajectory the conservation of energy, Equation 2.57, is:

$$\frac{v^2}{2} - \frac{\mu}{r} = 0$$

In other words, the speed anywhere on a parabolic path is:

$$v = \sqrt{\frac{2\mu}{r}} \tag{2.90}$$

If the body m_2 is launched on a parabolic trajectory, it will coast to infinity, arriving there with zero velocity relative to m_1. It will not return. Parabolic paths are therefore called escape trajectories. At a given distance r from m_1, the ***escape velocity*** is given by Equation 2.90,

$$v_{\text{esc}} = \sqrt{\frac{2\mu}{r}} \tag{2.91}$$

Let v_c be the speed of a satellite in a circular orbit of radius r. Then from Equations 2.63 and 2.91 we have:

$$v_{\text{esc}} = \sqrt{2}v_c \tag{2.92}$$

That is, to escape from a circular orbit requires a velocity boost of 41.4%. However, remember our assumption is that m_1 and m_2 are the only objects in the universe. A spacecraft launched from earth with velocity v_{esc} (relative to the earth) will not coast to infinity (i.e., leave the solar system) because it will eventually succumb to the gravitational influence of the sun and, in fact, end up in the same orbit as earth. This will be discussed in more detail in Chapter 8.

For the parabola, Equation 2.52 for the flight path angle takes the form:

$$\tan\gamma = \frac{\sin\theta}{1+\cos\theta}$$

Using the trigonometric identities:

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1$$

we can write,

$$\tan\gamma = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

It follows that:

$$\gamma = \frac{\theta}{2} \tag{2.93}$$

That is, on parabolic trajectories the flight path angle is always one-half the true anomaly.

Equation 2.53 gives the parameter p of an orbit. Let us substitute that expression into Equation 2.89 and then plot $r = p/(1 + \cos\theta)$ in a Cartesian coordinate system centered at the focus, as illustrated in Figure 2.23. From the figure it is clear that:

$$x = r\cos\theta = p\frac{\cos\theta}{1+\cos\theta} \tag{2.94a}$$
$$y = r\sin\theta = p\frac{\sin\theta}{1+\cos\theta} \tag{2.94b}$$

Therefore,

$$\frac{x}{p/2} + \left(\frac{y}{p}\right)^2 = 2\frac{\cos\theta}{1+\cos\theta} + \frac{\sin^2\theta}{(1+\cos\theta)^2}$$

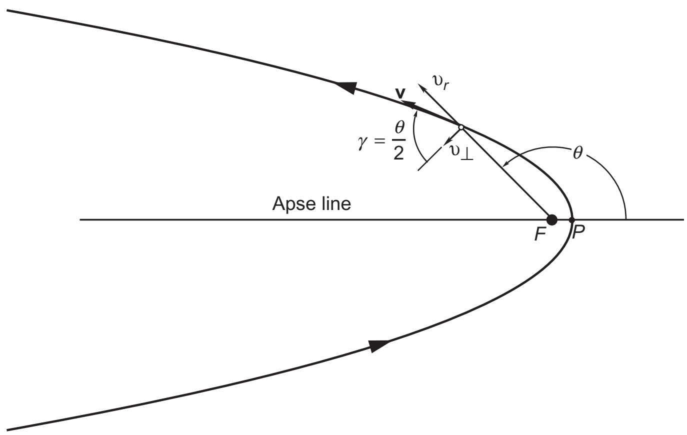


FIGURE 2.22

Parabolic trajectory around the focus F.

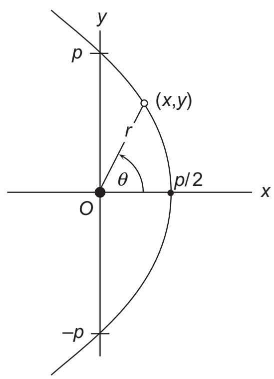


FIGURE 2.23

Parabola with focus at the origin of the Cartesian coordinate system.

Working to simplify the right-hand side, we get:

$$\frac{x}{p/2} + \left(\frac{y}{p}\right)^2 = \frac{2\cos\theta(1+\cos\theta) + \sin^2\theta}{(1+\cos\theta)^2} = \frac{2\cos\theta + 2\cos^2\theta + (1-\cos^2\theta)}{(1+\cos\theta)^2}$$
$$= \frac{1 + 2\cos\theta + \cos^2\theta}{(1+\cos\theta)^2} = \frac{(1+\cos\theta)^2}{(1+\cos\theta)^2} = 1$$

It follows that:

$$x = \frac{p}{2} - \frac{y^2}{2p} \tag{2.95}$$

This is the equation of a parabola in a Cartesian coordinate system whose origin serves as the focus.

Example 2.9

The perigee of a satellite in a parabolic geocentric trajectory is 7000 km. Find the distance d between points P_1 and P_2 on the orbit which are 8000 km and 16,000 km, respectively, from the center of the earth.

Solution

This would be a simple trigonometry problem if we knew the angle $\Delta\theta$ between the radials to P_1 and P_2. We can find that angle by first determining the true anomalies of the two points. The true anomalies are obtained from the orbit formula, Equation 2.89, once we have determined the angular momentum h.

We calculate the angular momentum of the satellite by evaluating the orbit equation at perigee,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+\cos(0)} = \frac{h^2}{2\mu}$$

from which

$$h = \sqrt{2\mu r_p} = \sqrt{2 \cdot 398{,}600 \cdot 7000} = 74{,}700 \text{ km}^2/\text{s} \tag{a}$$

Substituting the radii and the true anomalies of points P_1 and P_2 into Equation 2.89, we get:

$$8000 = \frac{74{,}700^2}{398{,}600}\frac{1}{1+\cos\theta_1} \Rightarrow \cos\theta_1 = 0.75 \Rightarrow \theta_1 = 41.41°$$

$$16{,}000 = \frac{74{,}700^2}{398{,}600}\frac{1}{1+\cos\theta_2} \Rightarrow \cos\theta_2 = -0.125 \Rightarrow \theta_2 = 97.18°$$

The difference between the two angles θ_1 and θ_2 is $\Delta\theta = 97.18° - 41.41° = 55.78°$.

The length of the chord $\overline{P_1P_2}$ can now be found by using the law of cosines from trigonometry,

$$d^2 = 8000^2 + 16{,}000^2 - 2 \cdot 8000 \cdot 16{,}000 \cos\Delta\theta$$

$$\boxed{d = 13{,}270 \text{ km}}$$

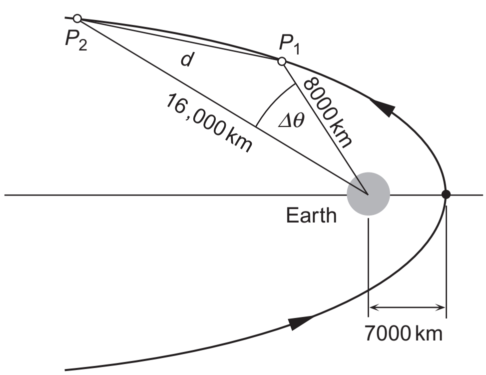


FIGURE 2.24

Parabolic geocentric trajectory.

2.9 HYPERBOLIC TRAJECTORIES ($e > 1$)

If $e > 1$, the orbit formula,

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta} \tag{2.96}$$

describes the geometry of the hyperbola shown in Figure 2.25. The system consists of two symmetric curves. The orbiting body occupies one of them. The other one is its empty mathematical image. Clearly, the denominator of Equation 2.96 goes to zero when $\cos\theta = -1/e$. We denote this value of true anomaly

$$\theta_\infty = \cos^{-1}(-1/e) \tag{2.97}$$

since the radial distance approaches infinity as the true anomaly approaches θ_∞. θ_∞ is known as the ***true anomaly of the asymptote***. Observe that θ_∞ lies between 90° and 180°. From the trig identity $\sin^2\theta_\infty + \cos^2\theta_\infty = 1$ it follows that:

$$\sin\theta_\infty = \frac{\sqrt{e^2 - 1}}{e} \tag{2.98}$$

For $-\theta_\infty < \theta < \theta_\infty$, the physical trajectory is the occupied hyperbola *I* shown on the left in Figure 2.25. For $\theta_\infty < \theta < (360° - \theta_\infty)$, hyperbola *II*—the vacant orbit around the empty focus *F'*is traced out. (The vacant orbit is physically impossible, because it would require a repulsive gravitational force.) Periapsis *P* lies on the apse line on the physical hyperbola *I*, whereas apoapsis *A* lies on the apse line on the vacant orbit. The point halfway between periapsis and apoapsis is the center *C* of the hyperbola. The asymptotes of the hyperbola are the straight lines towards which the curves tend as they approach infinity. The asymptotes

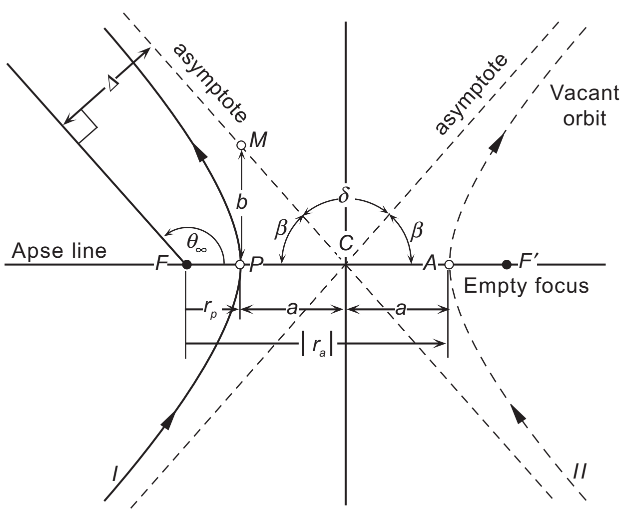


FIGURE 2.25

Hyperbolic trajectory.

intersect at C, making an acute angle β with the apse line, where $\beta = 180° - \theta_\infty$. Therefore, $\cos\beta = -\cos\theta_\infty$, which means:

$$\beta = \cos^{-1}(1/e) \tag{2.99}$$

The angle δ between the asymptotes is called the ***turn angle***. This is the angle through which the velocity vector of the orbiting body is rotated as it rounds the attracting body at F and heads back towards infinity. From the figure we see that $\delta = 180° - 2\beta$, so that:

$$\sin\frac{\delta}{2} = \sin\left(\frac{180° - 2\beta}{2}\right) = \sin(90° - \beta) = \cos\beta \overset{\text{Eq. 2.99}}{=} \frac{1}{e}$$

or

$$\delta = 2\sin^{-1}(1/e) \tag{2.100}$$

Equation 2.50 gives the distance r_p from the focus F to the periapsis,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} \tag{2.101}$$

Just as for an ellipse, the radial coordinate r_a of apoapsis is found by setting $\theta = 180°$ in Equation 2.45:

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e} \tag{2.102}$$

Observe that r_a is negative, since $e > 1$ for the hyperbola. That means the apoapsis lies to the right of the focus F. From Figure 2.25 we see that the distance $2a$ from periapsis P to apoapsis A is:

$$2a = |r_a| - r_p = -r_a - r_p$$

Substituting Equations 2.101 and 2.102 yields:

$$2a = -\frac{h^2}{\mu}\left(\frac{1}{1-e} + \frac{1}{1+e}\right)$$

From this it follows that a, the semimajor axis of the hyperbola, is given by an expression which is nearly identical to that for an ellipse (Equation 2.71):

$$a = \frac{h^2}{\mu}\frac{1}{e^2 - 1} \tag{2.103}$$

Therefore, Equation 2.96 may be written for the hyperbola:

$$r = a\frac{e^2 - 1}{1 + e\cos\theta} \tag{2.104}$$

This formula is analogous to Equation 2.72 for the elliptical orbit. Furthermore, from Equation 2.104 it follows that:

$$r_p = a(e - 1) \tag{2.105a}$$

$$r_a = -a(e + 1) \tag{2.105b}$$

The distance b from periapsis to an asymptote, measured perpendicular to the apse line, is the semiminor axis of the hyperbola. From Figure 2.25, we see that the length b of the semiminor axis $\overline{PM}$ is:

$$b = a\tan\beta = a\frac{\sin\beta}{\cos\beta} = a\frac{\sin(180 - \theta_\infty)}{\cos(180 - \theta_\infty)} = a\frac{\sin\theta_\infty}{-\cos\theta_\infty} \overset{\text{Equations 2.97 \& 2.98}}{=} a\frac{\sqrt{e^2 - 1}/e}{-(-1/e)}$$

so that for the hyperbola,

$$b = a\sqrt{e^2 - 1} \tag{2.106}$$

This relation is analogous to Equation 2.76 for the semiminor axis of an ellipse.

The distance Δ between the asymptote and a parallel line through the focus is called the ***aiming radius***, which is illustrated in Figure 2.25. From that figure we see that:

$$\begin{aligned}
\Delta &= (r_p + a)\sin\beta \\
&= ae\sin\beta && \text{(Equation 2.105a)} \\
&= ae\frac{\sqrt{e^2 - 1}}{e} && \text{(Equation 2.99)} \\
&= ae\sin\theta_\infty && \text{(Equation 2.98)} \\
&= ae\sqrt{1 - \cos^2\theta_\infty} && \text{(trig identity)} \\
&= ae\sqrt{1 - \frac{1}{e^2}} && \text{(Equation 2.97)}
\end{aligned}$$

or

$$\Delta = a\sqrt{e^2 - 1} \tag{2.107}$$

Comparing this result with Equation 2.106, it is clear that the aiming radius equals the length of the semiminor axis of the hyperbola.

As with the ellipse and the parabola, we can express the polar form of the equation of the hyperbola in a Cartesian coordinate system whose origin is in this case midway between the two foci, as illustrated in Figure 2.26. From the figure it is apparent that:

$$x = -a - r_p + r\cos\theta \tag{2.108a}$$

$$y = r\sin\theta \tag{2.108b}$$

Using Equations 2.104 and 2.105a in 2.108a, we obtain:

$$x = -a - a(e-1) + a\frac{e^2-1}{1+e\cos\theta}\cos\theta = -a\frac{e+\cos\theta}{1+e\cos\theta}$$

Substituting Equations 2.104 and 2.106 into 2.108b yields:

$$y = \frac{b}{\sqrt{e^2-1}}\frac{e^2-1}{1+e\cos\theta}\sin\theta = b\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta}$$

It follows that:

$$\begin{aligned}\frac{x^2}{a^2} - \frac{y^2}{b^2} &= \left(\frac{e+\cos\theta}{1+e\cos\theta}\right)^2 - \left(\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta}\right)^2 \\ &= \frac{e^2 + 2e\cos\theta + \cos^2\theta - (e^2-1)(1-\cos^2\theta)}{(1+e\cos\theta)^2} \\ &= \frac{1 + 2e\cos\theta + e^2\cos^2\theta}{(1+e\cos\theta)^2} = \frac{(1+e\cos\theta)^2}{(1+e\cos\theta)^2}\end{aligned}$$

That is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2.109}$$

This is the familiar equation of a hyperbola which is symmetric about the x and y axes, with intercepts on the x-axis.

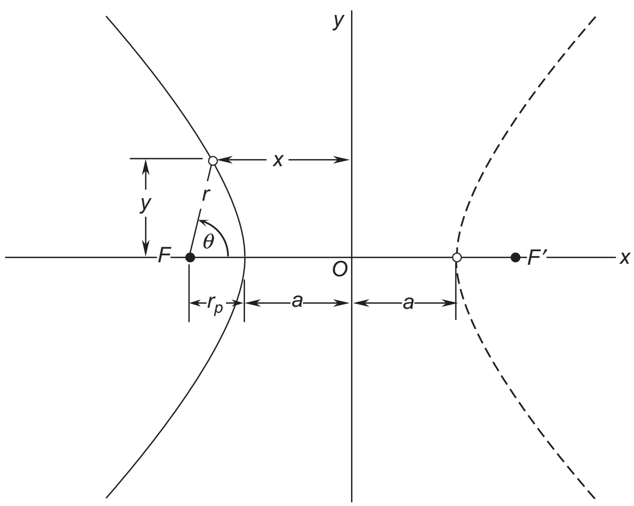


FIGURE 2.26

Plot of Equation 2.104 in a Cartesian coordinate system with origin O midway between the two foci.

Equation 2.60 gives the specific energy of the hyperbolic trajectory. Substituting Equation 2.103 into that expression yields:

$$\varepsilon = \frac{\mu}{2a} \tag{2.110}$$

The specific energy of a hyperbolic orbit is clearly positive and independent of the eccentricity. The conservation of energy for a hyperbolic trajectory is:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \tag{2.111}$$

Let v_∞ denote the speed at which a body on a hyperbolic path arrives at infinity. According to Equation 2.111:

$$v_\infty = \sqrt{\frac{\mu}{a}} \tag{2.112}$$

v_∞ is called the ***hyperbolic excess speed***. In terms of v_∞ we may write Equation 2.111 as:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{v_\infty^{\,2}}{2}$$

Substituting the expression for escape speed, $v_{\text{esc}} = \sqrt{2\mu/r}$ (Equation 2.91), we obtain for a hyperbolic trajectory

$$v^2 = v_{\text{esc}}^{\,2} + v_\infty^{\,2} \tag{2.113}$$

This equation clearly shows that the hyperbolic excess speed v_∞ represents the excess kinetic energy over that which is required to simply escape from the center of attraction. The square of v_∞ is denoted C_3, and is known as the ***characteristic energy***,

$$C_3 = v_\infty^{\,2} \tag{2.114}$$

C_3 is a measure of the energy required for an interplanetary mission and C_3 is also a measure of the maximum energy a launch vehicle can impart to a spacecraft of a given mass. Obviously, to match a launch vehicle with a mission, $C_3)_{\text{launch vehicle}} > C_3)_{\text{mission}}$.

Note that the hyperbolic excess speed can also be obtained from Equations 2.49 and 2.98,

$$v_\infty = \frac{\mu}{h} e \sin\theta_\infty = \frac{\mu}{h}\sqrt{e^2 - 1} \tag{2.115}$$

Finally, for purposes of comparison, Figure 2.27 shows a range of trajectories, from a circle through hyperbolas, all having a common focus and periapsis. The parabola is the demarcation between the closed, negative energy orbits (ellipses) and open, positive energy orbits (hyperbolas).

At this point the reader may be understandably overwhelmed by the number of equations for Keplerian orbits (conic sections) that have been presented thus far in this chapter. As summarized in the Road Map in Appendix B, there is just a small set of equations from which all of the others are derived.

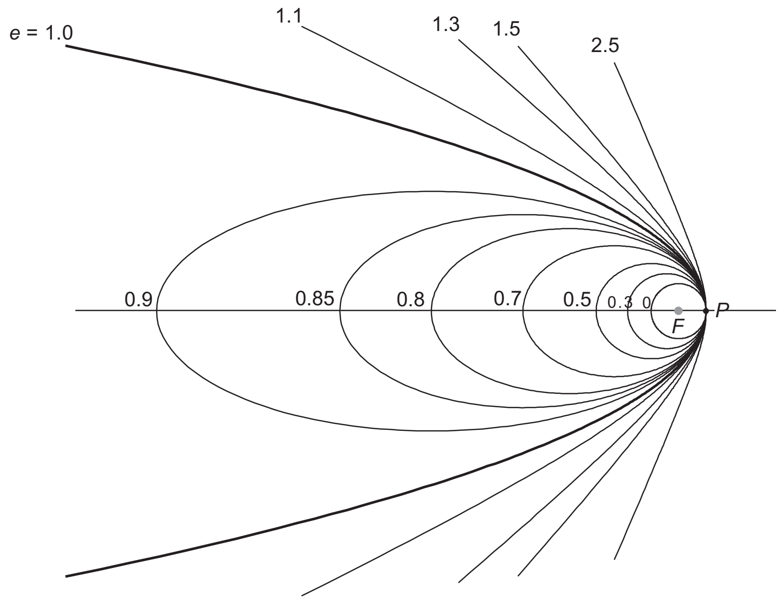


FIGURE 2.27

Orbits of various eccentricities, having a common focus *F* and periapsis *P*.

Here is a "tool box" of the only equations necessary for solving two-dimensional curvilinear orbital problems that do not involve time, which is the subject of Chapter 3.

All orbits:

$$h = rv_{\perp} \tag{2.31}$$

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta} \tag{2.45}$$

$$v_r = \frac{\mu}{h} e\sin\theta \tag{2.49}$$

$$\tan\gamma = \frac{v_r}{v_{\perp}} \tag{2.51}$$

$$v = \sqrt{{v_r}^2 + {v_{\perp}}^2}$$

Ellipses ($0 \le e < 1$):

$$a = \frac{r_p + r_a}{2} = \frac{h^2}{\mu}\frac{1}{1 - e^2} \tag{2.71}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{2.81}$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \tag{2.83}$$

$$e = \frac{r_a - r_p}{r_a + r_p} \tag{2.84}$$

Parabolas ($e = 1$):

$$\frac{v^2}{2} - \frac{\mu}{r} = 0 \tag{2.90}$$

Hyperbolas ($e > 1$):

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right) \tag{2.97}$$

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) \tag{2.100}$$

$$a = \frac{h^2}{\mu}\frac{1}{e^2 - 1} \tag{2.103}$$

$$\Delta = a\sqrt{e^2 - 1} \tag{2.107}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \tag{2.111}$$

Notice that we can rewrite Equations 2.103 and 2.111 as follows (where a is positive),

$$-a = \frac{h^2}{\mu}\frac{1}{1 - e^2} \qquad \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2(-a)}$$

That is, if we assume that the semimajor axis of a hyperbola has a negative value, then the semimajor axis formula and the vis-viva equation become identical for ellipses and hyperbolas. There is no advantage at this point in requiring hyperbolas to have negative semimajor axes. However, doing so will be necessary for the universal variable formulation to presented in the next chapter.

Example 2.10

At a given point of a spacecraft's geocentric trajectory, the radius is 14,600 km, the speed is 8.6 km/s, and the flight path angle is 50°. Show that the path is a hyperbola and calculate the following:

(a) Angular momentum
(b) Eccentricity
(c) True anomaly
(d) Radius of perigee
(e) Semimajor axis
(f) C_3
(g) Turn angle
(h) Aiming radius

This problem is illustrated in Figure 2.28.

Solution

Since both the radius and the speed are given, we can determine the type of trajectory by comparing the speed to the escape speed (of a parabolic trajectory) at the given radius:

$$v_{esc} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{2 \cdot 398{,}600}{14{,}600}} = 7.389 \text{ km/s}$$

The escape speed is less than the spacecraft's speed of 8.6 km/s, which means $\boxed{\text{the path is a hyperbola}}$.

(a) Before embarking on a quest for the required orbital data, remember that everything depends on the primary orbital parameters, angular momentum h and eccentricity e. These are among the list of five unknowns for this problem: h, e, θ, v_r and $v_\perp$. From the "tool box" we have five equations involving these five quantities and the given data:

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta} \tag{a}$$

$$v_r = \frac{\mu}{h} e\sin\theta \tag{b}$$

$$v_\perp = \frac{h}{r} \tag{c}$$

$$v = \sqrt{{v_r}^2 + {v_\perp}^2} \tag{d}$$

$$\tan\gamma = \frac{v_r}{v_\perp} \tag{e}$$

From (e)

$$v_r = v_\perp \tan 50° = 1.1918 v_\perp \tag{f}$$

Substituting this and the given speed into (d) yields:

$$8.6^2 = (1.1918 v_\perp)^2 + {v_\perp}^2 \Rightarrow v_\perp = 5.528 \text{ km/s} \tag{g}$$

The angular momentum may now be found from (c),

$$h = 14{,}600 \cdot 5.528 = \boxed{80{,}708 \text{ km}^2/\text{s}}$$

(b) Substituting $v_\perp$ into (f) we get the radial velocity component,

$$v_r = 1.1918 \cdot 5.528 = 6.588 \text{ km/s}$$

Substituting h and v_r into (b) yields an expression involving the eccentricity and the true anomaly,

$$6.588 = \frac{398{,}600}{80{,}708} e\sin\theta \Rightarrow e\sin\theta = 1.3339 \tag{h}$$

Similarly, substituting h and r into (a) we find:

$$14{,}600 = \frac{80{,}708^2}{398{,}600}\frac{1}{1+e\cos\theta} \Rightarrow e\cos\theta = 0.1193 \tag{i}$$

By squaring the expressions in (h) and (i) and then summing them, we obtain the eccentricity,

$$e^2\overbrace{(\sin^2\theta + \cos^2\theta)}^{=1} = 1.7936$$

$$\boxed{e = 1.3393}$$

(c) To find the true anomaly, substitute the value of e into (i),

$$1.3393\cos\theta = 0.1193 \Rightarrow \theta = 84.889^\circ \ or\ \theta = 275.11^\circ$$

We choose the smaller of the angles because (h) and (i) imply that both $\sin\theta$ and $\cos\theta$ are positive, which means θ lies in the first quadrant ($\theta \le 90^\circ$). Alternatively, we may note that the given flight path angle (50°) is positive, which means the spacecraft is flying away from perigee, so that the true anomaly must be less than 180°. In any case, the true anomaly is given by

$$\boxed{\theta = 84.889^\circ}.$$

(d) The radius of perigee can now be found from the orbit equation (a)

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e\cos(0)} = \frac{80{,}710^2}{398{,}600}\frac{1}{1+1.339} = \boxed{6986\ \text{km}}$$

(e) The semimajor axis of the hyperbola is found in Equation 2.103,

$$a = \frac{h^2}{\mu}\frac{1}{e^2-1} = \frac{80{,}710^2}{398{,}600}\frac{1}{1.339^2-1} = \boxed{20{,}590\ \text{km}}$$

(f) The hyperbolic excess velocity is found using Equation 2.113,

$$v_\infty{}^2 = v^2 - v_{esc}{}^2 = 8.6^2 - 7.389^2 = 19.36\ \text{km}^2/\text{s}^2$$

From Equation 2.114 it follows that:

$$\boxed{C_3 = 19.36\ \text{km}^2/\text{s}^2}$$

(g) The formula for turn angle is Equation 2.100, from which:

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\sin^{-1}\left(\frac{1}{1.339}\right) = \boxed{96.60^\circ}$$

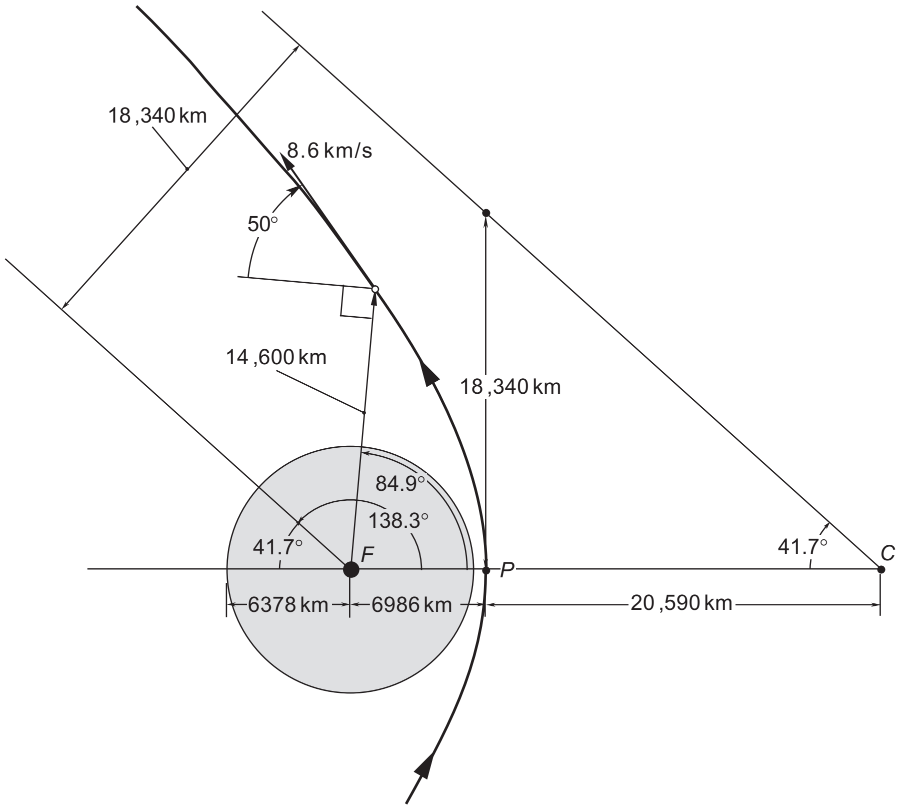


FIGURE 2.28

Solution of Example 2.10.

(h) According to Equation 2.107, the aiming radius is:

$$\Delta = a\sqrt{e^2 - 1} = 20{,}590\sqrt{1.339^2 - 1} = \boxed{18{,}340 \text{ km}}$$

2.10 PERIFOCAL FRAME

The perifocal frame is the "natural frame" for an orbit. It is centered at the focus of the orbit. Its $\bar{x}\bar{y}$ plane is the plane of the orbit, and its $\bar{x}$ axis is directed from the focus through periapsis, as illustrated in Figure 2.29. The unit vector along the $\bar{x}$ axis (the apse line) is denoted $\hat{\mathbf{p}}$. The $\bar{y}$ axis, with unit vector $\hat{\mathbf{q}}$, lies at 90° true anomaly to the $\bar{x}$ axis. The $\bar{z}$ axis is normal to the plane of the orbit in the direction of the angular momentum vector $\mathbf{h}$. The $\bar{z}$ unit vector is $\hat{\mathbf{w}}$,

$$\hat{\mathbf{w}} = \frac{\mathbf{h}}{h} \tag{2.116}$$

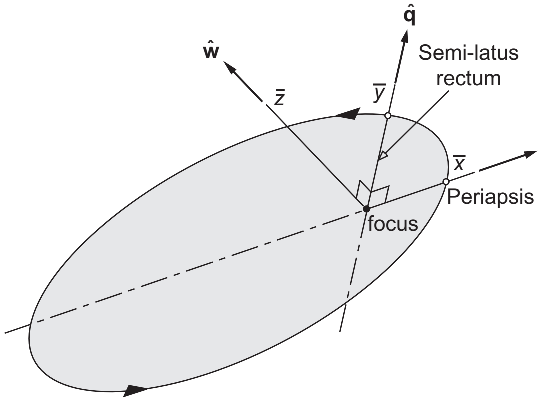


FIGURE 2.29

Perifocal frame $\hat{\mathbf{p}}\hat{\mathbf{q}}\hat{\mathbf{w}}$.

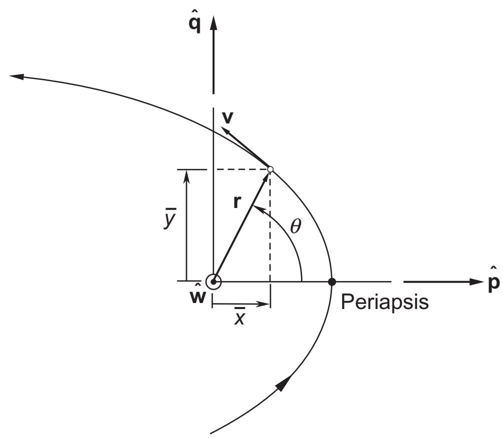


FIGURE 2.30

Position and velocity relative to the perifocal frame.

In the perifocal frame, the position vector **r** is written (see Figure 2.30)

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} \tag{2.117}$$

where,

$$\bar{x} = r\cos\theta \qquad \bar{y} = r\sin\theta \tag{2.118}$$

and r, the magnitude of **r**, is given by the orbit equation, $r = (h^2/\mu)[1/(1 + e\cos\theta)]$. Thus, we may write Equation 2.117 as:

$$\mathbf{r} = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) \tag{2.119}$$

The velocity is found by taking the time derivative of **r**,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} \tag{2.120}$$

From Equations 2.118 we obtain:

$$\begin{aligned} \dot{\bar{x}} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{\bar{y}} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned} \tag{2.121}$$

$\dot{r}$ is the radial component of velocity, v_r. Therefore, according to Equation 2.49,

$$\dot{r} = \frac{\mu}{h}e\sin\theta \tag{2.122}$$

From Equations 2.46 and 2.48 we have:

$$r\dot{\theta} = v_{\perp} = \frac{\mu}{h}(1 + e\cos\theta) \tag{2.123}$$

Substituting Equations 2.122 and 2.123 into 2.121 and simplifying the results yields

$$\begin{aligned} \dot{\bar{x}} &= -\frac{\mu}{h}\sin\theta \\ \dot{\bar{y}} &= \frac{\mu}{h}(e + \cos\theta) \end{aligned} \tag{2.124}$$

Hence, Equation 2.120 becomes

$$\mathbf{v} = \frac{\mu}{h}[-\sin\theta\hat{\mathbf{p}} + (e + \cos\theta)\hat{\mathbf{q}}] \tag{2.125}$$

Formulating the kinematics of orbital motion in the perifocal frame, as we have done here, is a prelude to the study of orbits in three dimensions (Chapter 4). We also need Equations 2.117 and 2.120 in the next section.

Example 2.11

An earth orbit has an eccentricity of 0.3, an angular momentum of 60,000 km^2/s and a true anomaly of 120°. What are the position vector **r** and velocity vector **v** in the perifocal frame of reference?

Solution

From Equation 2.119 we have:

$$\mathbf{r} = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) = \frac{60{,}000^2}{398{,}600}\frac{1}{1 + 0.3\cos 120^\circ}(\cos 120^\circ\hat{\mathbf{p}} + \sin 120^\circ\hat{\mathbf{q}})$$

$$\boxed{\mathbf{r} = -5312.7\hat{\mathbf{p}} + 9201.9\hat{\mathbf{q}}\ (\text{km})}$$

Substituting the given data into Equation 2.125 yields:

$$\mathbf{v} = \frac{\mu}{h}[-\sin\theta\hat{\mathbf{p}} + (e + \cos\theta)\hat{\mathbf{q}}] = \frac{398{,}600}{60{,}000}[-\sin 120°\hat{\mathbf{p}} + (0.3 + \cos 120°)\hat{\mathbf{q}}]$$

$$\boxed{\mathbf{v} = -5.7533\hat{\mathbf{p}} - 1.3287\hat{\mathbf{q}}\ (\text{km/s})}$$

Example 2.12

An earth satellite has the following position and velocity vectors at a given instant:

$$\mathbf{r} = 7000\hat{\mathbf{p}} + 9000\hat{\mathbf{q}}\ (\text{km})$$
$$\mathbf{v} = -5\hat{\mathbf{p}} + 7\hat{\mathbf{q}}\ (\text{km/s})$$

Calculate the specific angular momentum h, the true anomaly θ, and the eccentricity e.

Solution

This problem is obviously the reverse of the situation presented in the previous example. From Equation 2.28 the angular momentum is:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{p}} & \hat{\mathbf{q}} & \hat{\mathbf{w}} \\ 7000 & 9000 & 0 \\ -5 & 7 & 0 \end{vmatrix} = 94{,}000\hat{\mathbf{w}}\ (\text{km}^2/\text{s})$$

Hence the magnitude of the angular momentum is:

$$\boxed{h = 94{,}000\ \text{km}^2/\text{s}}$$

The true anomaly is measured from the positive $\bar{x}$ axis. By definition of the dot product, $\mathbf{r} \cdot \hat{\mathbf{p}} = r\cos\theta$. Thus,

$$\cos\theta = \frac{\mathbf{r}}{r} \cdot \hat{\mathbf{p}} = \frac{7000\hat{\mathbf{p}} + 9000\hat{\mathbf{q}}}{\sqrt{7000^2 + 9000^2}} \cdot \hat{\mathbf{p}} = \frac{7000}{11{,}402} = 0.61394$$

which means $\theta = 52.125°$ or $\theta = -52.125°$. Since the $\bar{y}$ component of $\mathbf{r}$ is positive, the true anomaly must lie between 0 and 180°. It follows that:

$$\boxed{\theta = 52.125°}$$

Finally, the eccentricity may be found from the orbit formula, $r = (h^2/\mu)/(1 + e\cos\theta)$:

$$\sqrt{7000^2 + 9000^2} = \frac{94{,}000^2}{398{,}600}\frac{1}{1 + e\cos 52.125°}$$

$$\boxed{e = 1.538}$$

The trajectory is a hyperbola.

2.11 THE LAGRANGE COEFFICIENTS

In this section we will establish what may seem intuitively obvious: if the position and velocity of an orbiting body are known at a given instant, then the position and velocity at any later time are found in terms of the initial values. Let us start with Equations 2.117 and 2.120,

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} \tag{2.126}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} \tag{2.127}$$

Attach a subscript "zero" to quantities evaluated at time $t = t_0$. Then the expressions for $\mathbf{r}$ and $\mathbf{v}$ evaluated at $t = t_0$ are:

$$\mathbf{r}_0 = \bar{x}_0\hat{\mathbf{p}} + \bar{y}_0\hat{\mathbf{q}} \tag{2.128}$$

$$\mathbf{v}_0 = \dot{\bar{x}}_0\hat{\mathbf{p}} + \dot{\bar{y}}_0\hat{\mathbf{q}} \tag{2.129}$$

The angular momentum $\mathbf{h}$ is constant; so let us calculate it using the initial conditions. Substituting Equations 2.128 and 2.129 into Equation 2.28 yields

$$\mathbf{h} = \mathbf{r}_0 \times \mathbf{v}_0 = \begin{vmatrix} \hat{\mathbf{p}} & \hat{\mathbf{q}} & \hat{\mathbf{w}} \\ \bar{x}_0 & \bar{y}_0 & 0 \\ \dot{\bar{x}}_0 & \dot{\bar{y}}_0 & 0 \end{vmatrix} = \hat{\mathbf{w}}(\bar{x}_0\dot{\bar{y}}_0 - \bar{y}_0\dot{\bar{x}}_0) \tag{2.130}$$

Recall that $\hat{\mathbf{w}}$ is the unit vector in the direction of $\mathbf{h}$ (Equation 2.116). Therefore, the coefficient of $\hat{\mathbf{w}}$ on the right of Equation 2.130 must be the magnitude of the angular momentum. That is,

$$h = \bar{x}_0\dot{\bar{y}}_0 - \bar{y}_0\dot{\bar{x}}_0 \tag{2.131}$$

Now let us solve the two vector equations (2.128) and (2.129) for the unit vectors $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ in terms of $\mathbf{r}_0$ and $\mathbf{v}_0$. From (2.128) we get:

$$\hat{\mathbf{q}} = \frac{1}{\bar{y}_0}\mathbf{r}_0 - \frac{\bar{x}_0}{\bar{y}_0}\hat{\mathbf{p}} \tag{2.132}$$

Substituting this into Equation (2.129), combining terms and using Equation 2.130 yields:

$$\mathbf{v}_0 = \dot{\bar{x}}_0\hat{\mathbf{p}} + \dot{\bar{y}}_0\left(\frac{1}{\bar{y}_0}\mathbf{r}_0 - \frac{\bar{x}_0}{\bar{y}_0}\hat{\mathbf{p}}\right) = \frac{\bar{y}_0\dot{\bar{x}}_0 - \bar{x}_0\dot{\bar{y}}_0}{\bar{y}_0}\hat{\mathbf{p}} + \frac{\dot{\bar{y}}_0}{\bar{y}_0}\mathbf{r}_0 = -\frac{h}{\bar{y}_0}\hat{\mathbf{p}} + \frac{\dot{\bar{y}}_0}{\bar{y}_0}\mathbf{r}_0$$

Solve this for $\hat{\mathbf{p}}$ to obtain:

$$\hat{\mathbf{p}} = \frac{\dot{\bar{y}}_0}{h}\mathbf{r}_0 - \frac{\bar{y}_0}{h}\mathbf{v}_0 \tag{2.133}$$

Putting this result back into Equation 2.132 gives:

$$\hat{\mathbf{q}} = \frac{1}{\bar{y}_0}\mathbf{r}_0 - \frac{\bar{x}_0}{\bar{y}_0}\left(\frac{\dot{\bar{y}}_0}{h}\mathbf{r}_0 - \frac{\bar{y}_0}{h}\mathbf{v}_0\right) = \frac{h - \bar{x}_0\dot{\bar{y}}_0}{\bar{y}_0}\mathbf{r}_0 + \frac{\bar{x}_0}{h}\mathbf{v}_0$$

Upon replacing h by the right-hand side of Equation 2.131 we get:

$$\hat{\mathbf{q}} = -\frac{\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{\bar{x}_0}{h}\mathbf{v}_0 \tag{2.134}$$

Equations 2.133 and 2.134 give $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ in terms of the initial position and velocity. Substituting those two expressions back into Equations 2.126 and 2.127 yields, respectively,

$$\mathbf{r} = \bar{x}\left(\frac{\dot{\bar{y}}_0}{h}\mathbf{r}_0 - \frac{\bar{y}_0}{h}\mathbf{v}_0\right) + \bar{y}\left(-\frac{\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{\bar{x}_0}{h}\mathbf{v}_0\right) = \frac{\bar{x}\,\dot{\bar{y}}_0 - \bar{y}\,\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{-\bar{x}\,\bar{y}_0 + \bar{y}\,\bar{x}_0}{h}\mathbf{v}_0$$

$$\mathbf{v} = \dot{\bar{x}}\left(\frac{\dot{\bar{y}}_0}{h}\mathbf{r}_0 - \frac{\bar{y}_0}{h}\mathbf{v}_0\right) + \dot{\bar{y}}\left(-\frac{\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{\bar{x}_0}{h}\mathbf{v}_0\right) = \frac{\dot{\bar{x}}\,\dot{\bar{y}}_0 - \dot{\bar{y}}\,\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{-\dot{\bar{x}}\,\bar{y}_0 + \dot{\bar{y}}\,\bar{x}_0}{h}\mathbf{v}_0$$

Therefore,

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0 \tag{2.135}$$

$$\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 \tag{2.136}$$

where f and g are given by:

$$f = \frac{\bar{x}\,\dot{\bar{y}}_0 - \bar{y}\,\dot{\bar{x}}_0}{h} \tag{2.137a}$$

$$g = \frac{-\bar{x}\,\bar{y}_0 + \bar{y}\,\bar{x}_0}{h} \tag{2.137b}$$

together with their time derivatives:

$$\dot{f} = \frac{\dot{\bar{x}}\,\dot{\bar{y}}_0 - \dot{\bar{y}}\,\dot{\bar{x}}_0}{h} \tag{2.138a}$$

$$\dot{g} = \frac{-\dot{\bar{x}}\,\bar{y}_0 + \dot{\bar{y}}\,\bar{x}_0}{h} \tag{2.138b}$$

The f and g functions are referred to as the ***Lagrange coefficients*** after Joseph-Louis Lagrange (1736–1813), a French mathematical physicist whose numerous contributions include calculations of planetary motion.

From Equations 2.135 and 2.136 we see that the position and velocity vectors **r** and **v** are indeed linear combinations of the initial position and velocity vectors. The Lagrange coefficients and their time derivatives in these expressions are themselves functions of time and the initial conditions.

Before proceeding, let us show that the conservation of angular momentum **h** imposes a condition on f and g and their time derivatives $\dot{f}$ and $\dot{g}$. Calculate **h** using Equations 2.135 and 2.136,

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = (f\mathbf{r}_0 + g\mathbf{v}_0) \times (\dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0)$$

Expanding the right-hand side yields:

$$\mathbf{h} = (f\mathbf{r}_0 \times \dot{f}\mathbf{r}_0) + (f\mathbf{r}_0 \times \dot{g}\mathbf{v}_0) + (g\mathbf{v}_0 \times \dot{f}\mathbf{r}_0) + (g\mathbf{v}_0 \times \dot{g}\mathbf{v}_0)$$

Factoring out the scalars f, g, $\dot{f}$ and $\dot{g}$, we get:

$$\mathbf{h} = f\dot{f}(\mathbf{r}_0 \times \mathbf{r}_0) + f\dot{g}(\mathbf{r}_0 \times \mathbf{v}_0) + \dot{f}g(\mathbf{v}_0 \times \mathbf{r}_0) + g\dot{g}(\mathbf{v}_0 \times \mathbf{v}_0)$$

But $\mathbf{r}_0 \times \mathbf{r}_0 = \mathbf{v}_0 \times \mathbf{v}_0 = \mathbf{0}$, so

$$\mathbf{h} = f\dot{g}(\mathbf{r}_0 \times \mathbf{v}_0) + \dot{f}g(\mathbf{v}_0 \times \mathbf{r}_0)$$

Since

$$\mathbf{v}_0 \times \mathbf{r}_0 = -(\mathbf{r}_0 \times \mathbf{v}_0)$$

this reduces to:

$$\mathbf{h} = (f\dot{g} - \dot{f}g)(\mathbf{r}_0 \times \mathbf{v}_0)$$

or

$$\mathbf{h} = (f\dot{g} - \dot{f}g)\mathbf{h}_0$$

where $\mathbf{h}_0 = \mathbf{r}_0 \times \mathbf{v}_0$, which is the angular momentum at $t = t_0$. But the angular momentum is constant (recall Equation 2.29), which means $\mathbf{h} = \mathbf{h}_0$, so that

$$\mathbf{h} = (f\dot{g} - \dot{f}g)\mathbf{h}$$

Since $\mathbf{h}$ cannot be zero (unless the body is traveling a straight line towards the center of attraction), it follows that:

$$f\dot{g} - \dot{f}g = 1 \quad \text{(Conservation of angular momentum)} \tag{2.139}$$

Thus, if any three of the functions f, g, $\dot{f}$ and $\dot{g}$ are known, the fourth may be found from Equation 2.139.

Let us use Equations 2.137 and 2.138 to evaluate the Lagrange coefficients and their time derivatives in terms of the true anomaly. First of all, note that evaluating Equations 2.118 at time $t = t_0$ yields:

$$\begin{aligned} \bar{x}_0 &= r_0 \cos\theta_0 \\ \bar{y}_0 &= r_0 \sin\theta_0 \end{aligned} \tag{2.140}$$

Likewise, from Equations 2.124 we get:

$$\begin{aligned} \dot{\bar{x}}_0 &= -\frac{\mu}{h}\sin\theta_0 \\ \dot{\bar{y}}_0 &= \frac{\mu}{h}(e + \cos\theta_0) \end{aligned} \tag{2.141}$$

To evaluate the function f, we substitute Equations 2.118 and 2.141 into Equation 2.137a,

$$\begin{aligned} f &= \frac{\bar{x}\dot{\bar{y}}_0 - \bar{y}\dot{\bar{x}}_0}{h} \\ &= \frac{1}{h}\left\{[r\cos\theta]\left[\frac{\mu}{h}(e + \cos\theta_0)\right] - [r\sin\theta]\left[-\frac{\mu}{h}\sin\theta_0\right]\right\} \\ &= \frac{\mu r}{h^2}[e\cos\theta + (\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0)] \end{aligned} \tag{2.142}$$

If we invoke the trig identity:

$$\cos(\theta - \theta_0) = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0 \tag{2.143}$$

and let $\Delta\theta$ represent the difference between the current and initial true anomalies,

$$\Delta\theta = \theta - \theta_0 \tag{2.144}$$

then Equation 2.142 reduces to:

$$f = \frac{\mu r}{h^2}(e\cos\theta + \cos\Delta\theta) \tag{2.145}$$

Finally, from Equation 2.45, we have:

$$e\cos\theta = \frac{h^2}{\mu r} - 1 \tag{2.146}$$

Substituting this into Equation 2.145 leads to:

$$f = 1 - \frac{\mu r}{h^2}(1 - \cos\Delta\theta) \tag{2.147}$$

We obtain r from the orbit formula, Equation 2.45, in which the true anomaly θ appears, whereas the difference in the true anomalies occurs on the right hand side of Equation 2.147. However, we can express the orbit equation in terms of the difference in true anomalies as follows. From Equation 2.144 we have $\theta = \theta_0 + \Delta\theta$, which means we can write the orbit equation as:

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos(\theta_0 + \Delta\theta)} \tag{2.148}$$

By replacing θ_0 by $-\Delta\theta$ in Equation 2.143, Equation 2.148 becomes:

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta_0\cos\Delta\theta - e\sin\theta_0\sin\Delta\theta} \tag{2.149}$$

To remove θ_0 from this expression, observe first of all that Equation 2.146 implies that, at $t = t_0$,

$$e\cos\theta_0 = \frac{h^2}{\mu r_0} - 1 \tag{2.150}$$

Furthermore, from Equation 2.49 for the radial velocity we obtain:

$$e\sin\theta_0 = \frac{h v_{r0}}{\mu} \tag{2.151}$$

Substituting Equations 2.150 and 2.151 into 2.149 yields:

$$r = \frac{h^2}{\mu}\frac{1}{1 + \left(\dfrac{h^2}{\mu r_0} - 1\right)\cos\Delta\theta - \dfrac{h v_{r0}}{\mu}\sin\Delta\theta} \tag{2.152}$$

Using this form of the orbit equation, we can find r in terms of the initial conditions and the change in the true anomaly. Thus f in Equation 2.147 depends only on $\Delta\theta$.

The Lagrange coefficient g is found by substituting Equations 2.118 and 2.140 into Equation 2.137b:

$$\begin{aligned} g &= \frac{-\overline{x}\,\overline{y}_0 + \overline{y}\,\overline{x}_0}{h} \\ &= \frac{1}{h}[(-r\cos\theta)(r_0\sin\theta_0) + (r\sin\theta)(r\cos\theta_0)] \\ &= \frac{rr_0}{h}(\sin\theta\cos\theta_0 - \cos\theta\sin\theta_0) \end{aligned} \tag{2.153}$$

Making use of the trig identity

$$\sin(\theta - \theta_0) = \sin\theta\cos\theta_0 - \cos\theta\sin\theta_0$$

together with Equation 2.144, we find:

$$g = \frac{rr_0}{h}\sin(\Delta\theta) \tag{2.154}$$

To obtain $\dot{g}$, substitute Equations 2.124 and 2.140 into Equation 2.138b:

$$\begin{aligned} \dot{g} &= \frac{-\dot{\overline{x}}\,\overline{y}_0 + \dot{\overline{y}}\,\overline{x}_0}{h} \\ &= \frac{1}{h}\left\{-\left[-\frac{\mu}{h}\sin\theta\right][r_0\sin\theta_0] + \left[\frac{\mu}{h}(e + \cos\theta)\right](r_0\cos\theta_0)\right\} \\ &= \frac{\mu r_0}{h^2}[e\cos\theta_0 + (\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0)] \end{aligned}$$

With the aid of Equations 2.143 and 2.150, this reduces to:

$$\dot{g} = 1 - \frac{\mu r_0}{h^2}(1 - \cos\Delta\theta) \tag{2.155}$$

$\dot{f}$ can be found using Equation 2.139. Thus,

$$\dot{f} = \frac{1}{g}(f\dot{g} - 1) \tag{2.156}$$

Substituting Equations 2.147, 2.153, and 2.155 results in:

$$\dot{f} = \frac{1}{\dfrac{rr_0}{h}\sin\Delta\theta}\left\{\left[1 - \frac{\mu r}{h^2}(1-\cos\Delta\theta)\right]\left[1 - \frac{\mu r_0}{h^2}(1-\cos\Delta\theta)\right] - 1\right\}$$
$$= \frac{1}{\dfrac{rr_0}{h}\sin\Delta\theta}\frac{h^2 \mu r r_0}{h^4}\left[(1-\cos\Delta\theta)^2\frac{\mu}{h^2} - (1-\cos\Delta\theta)\left(\frac{1}{r_0}+\frac{1}{r}\right)\right]$$

or

$$\dot{f} = \frac{\mu}{h}\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1-\cos\Delta\theta) - \frac{1}{r_0} - \frac{1}{r}\right] \tag{2.157}$$

To summarize, the ***Lagrange coefficients in terms of the change in true anomaly*** are

$$f = 1 - \frac{\mu r}{h^2}(1-\cos\Delta\theta) \tag{2.158a}$$

$$g = \frac{rr_0}{h}\sin\Delta\theta \tag{2.158b}$$

$$\dot{f} = \frac{\mu}{h}\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1-\cos\Delta\theta) - \frac{1}{r_0} - \frac{1}{r}\right] \tag{2.158c}$$

$$\dot{g} = 1 - \frac{\mu r_0}{h^2}(1-\cos\Delta\theta) \tag{2.158d}$$

where r is given by Equation 2.152. The implementation of these four functions in MATLAB is presented in Appendix D.7.

Observe that using the Lagrange coefficients to determine the position and velocity from the initial conditions does not require knowing the type of orbit we are dealing with (ellipse, parabola, hyperbola), since the eccentricity does not appear in Equations 2.152 and 2.158. However, the initial position and velocity give us that information. From $\mathbf{r}_0$ and $\mathbf{v}_0$ we obtain the angular momentum $h = \|\mathbf{r}_0 \times \mathbf{v}_0\|$. The initial radius r_0 is just the magnitude of the vector $\mathbf{r}_0$. The initial radial velocity v_{r0} is the projection of $\mathbf{v}_0$ onto the direction of $\mathbf{r}_0$,

$$v_{r0} = \mathbf{v}_0 \cdot \frac{\mathbf{r}_0}{r_0}$$

From Equations 2.45 and 2.49 we have:

$$r_0 = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_0} \qquad v_{r0} = \frac{\mu}{h}e\sin\theta_0$$

These two equations can be solved for the eccentricity e and for the true anomaly of the initial point θ_0.

Algorithm 2.3 Given $\mathbf{r}_0$ and $\mathbf{v}_0$, find $\mathbf{r}$ and $\mathbf{v}$ after the true anomaly changes by $\Delta\theta$. See Appendix D.8 for an implementation of this procedure in MATLAB.

1. Compute the f and g functions and their derivatives by the following steps:
   (a) Calculate the magnitude of $\mathbf{r}_0$ and $\mathbf{v}_0$:

$$r_0 = \sqrt{\mathbf{r}_0 \cdot \mathbf{r}_0} \qquad v_0 = \sqrt{\mathbf{v}_0 \cdot \mathbf{v}_0}$$

   (b) Calculate the radial component of $\mathbf{v}_0$ by projecting it onto the direction of $\mathbf{r}_0$:

$$v_{r0} = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{r_0}$$

   (c) Calculate the magnitude of the constant angular momentum:

$$h = r_0 v_{\perp 0} = r_0\sqrt{{v_0}^2 - {v_{r0}}^2}$$

   (d) Substitute r_0, v_{r0}, h and $\Delta\theta$ in Equation 2.152 to calculate r.
   (e) Substitute r, r_0, h and $\Delta\theta$ into Equations 2.158 to find f, g, $\dot{f}$ and $\dot{g}$.
2. Use Equations 2.135 and 2.136 to calculate $\mathbf{r}$ and $\mathbf{v}$.

Example 2.13

An earth satellite moves in the xy plane of an inertial frame with origin at the earth's center. Relative to that frame, the position and velocity of the satellite at time t_0 are:

$$\begin{aligned}\mathbf{r}_0 &= 8182.4\hat{\mathbf{i}} - 6865.9\hat{\mathbf{j}}\ \text{(km)}\\ \mathbf{v}_0 &= 0.47572\hat{\mathbf{i}} + 8.8116\hat{\mathbf{j}}\ \text{(km/s)}\end{aligned} \tag{a}$$

Use Algorithm 2.3 to compute the position and velocity vectors after the satellite has traveled through a true anomaly of 120°.

Solution

Step 1:

(a) $r_0 = \sqrt{\mathbf{r}_0 \cdot \mathbf{r}_0} = 10{,}861$ km $\qquad v_0 = \sqrt{\mathbf{v}_0 \cdot \mathbf{v}_0} = 8.8244$ km/s

(b) $$v_{r0} = \mathbf{v}_0 \cdot \frac{\mathbf{r}_0}{r_0} = \frac{(0.47572\hat{\mathbf{i}} + 8.8116\hat{\mathbf{j}}) \cdot (8182.4\hat{\mathbf{i}} - 6865.9\hat{\mathbf{j}})}{10{,}681} = -5.2996\ \text{km/s}$$

(c) $$h = r_0\sqrt{{v_0}^2 - {v_{r0}}^2} = 10{,}861\sqrt{8.8244^2 - (-5.2996)^2} = 75{,}366\ \text{km}^2/\text{s}$$

(d) $$\begin{aligned} r &= \frac{h^2}{\mu}\frac{1}{1 + \left(\dfrac{h^2}{\mu r_0} - 1\right)\cos\Delta\theta - \dfrac{h v_{r0}}{\mu}\sin\Delta\theta}\\ &= \frac{75{,}366^2}{398{,}600}\frac{1}{1 + \left(\dfrac{75{,}366^2}{398{,}600 \cdot 10{,}681} - 1\right)\cos 120^\circ - \dfrac{75{,}366 \cdot (-5.2996)}{398{,}600}\sin 120^\circ}\\ &= 8378.8\ \text{km}\end{aligned}$$

$$\text{(e) } f = 1-\frac{\mu r}{h^2}(1-\cos\Delta\theta) = 1-\frac{398{,}600\cdot 8378.8}{75{,}366^2}(1-\cos 120°) = 0.11802 \text{ (dimensionless)}$$

$$g = \frac{r r_0}{h}\sin(\Delta\theta) = \frac{8378.8\cdot 10{,}681}{75{,}366}\sin(120°) = 1028.4 \text{ s}$$

$$\begin{aligned}\dot{f} &= \frac{\mu}{h}\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1-\cos\Delta\theta)-\frac{1}{r_0}-\frac{1}{r}\right]\\ &= \frac{398{,}600}{75{,}366}\frac{1-\cos 120°}{\sin 120°}\left[\frac{398{,}600}{75{,}366^2}(1-\cos 120°)-\frac{1}{10{,}681}-\frac{1}{8378.9}\right]\\ &= -9.8666\times 10^{-4} \text{ (dimensionless)}\end{aligned}$$

$$\dot{g} = 1-\frac{\mu r_0}{h^2}(1-\cos\Delta\theta) = 1-\frac{398{,}600\cdot 10{,}681}{75{,}366^2}(1-\cos 120°) = -0.12435 \text{ (dimensionless)}$$

Step 2:

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0 = 0.11802(8182.4\hat{\mathbf{i}} - 6865.9\hat{\mathbf{j}}) + 1028.4(0.47572\hat{\mathbf{i}} + 8.8116\hat{\mathbf{j}})$$

$$\mathbf{r} = \boxed{1454.9\hat{\mathbf{i}} + 8251.6\hat{\mathbf{j}} \text{ (km)}}$$

$$\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 = (-9.8666\times 10^{-4})(8182.4\hat{\mathbf{i}} - 6865.9\hat{\mathbf{j}}) + (-0.12435)(0.47572\hat{\mathbf{i}} + 8.8116\hat{\mathbf{j}})$$

$$\mathbf{v} = \boxed{-8.1323\hat{\mathbf{i}} + 5.6785\hat{\mathbf{j}} \text{ (km/s)}}$$

These results are shown in Figure 2.31.

Example 2.14

Find the eccentricity of the orbit in Example 2.13 as well as the true anomaly at the initial time t_0 and, hence, the location of perigee for this orbit.

Solution

In Example 2.13 we found:

$$\begin{aligned} r_0 &= 10{,}861 \text{ km}\\ v_{r0} &= -5.2996 \text{ km/s}\\ h &= 75{,}366 \text{ km}^2/\text{s}\end{aligned} \tag{a}$$

Since v_{r0} is negative, we know that the spacecraft is approaching perigee, which means that:

$$180° < \theta_0 < 360° \tag{b}$$

The orbit formula and the radial velocity formula (Equations 2.45 and 2.49), evaluated at t_0 are:

$$r_0 = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_0} \qquad v_{r0} = \frac{\mu}{h}e\sin\theta_0$$

Substituting the numerical values from (a) into these formulas yields:

$$10{,}861 = \frac{75{,}366^3}{398{,}600}\frac{1}{1 + e\cos\theta_0} \qquad -5.2996 = \frac{398{,}600}{75{,}366}e\sin\theta_0$$

From these we obtain two equations for the two unknowns e and θ_0:

$$e\cos\theta_0 = 0.3341 \qquad e\sin\theta_0 = -1.002 \tag{c}$$

Squaring these two expressions and then summing them gives:

$$e^2(\sin^2\theta_0 + \cos^2\theta_0) = 1.1157$$

Recalling the trig identity $\sin^2\theta_0 + \cos^2\theta_0 = 1$, we get:

$$\boxed{e = 1.0563} \qquad \text{(hyperbola)}$$

The eccentricity may be substituted back into either of the two expressions in (c) in order to find the true anomaly θ_0. Choosing $(c)_1$, we find:

$$\cos\theta_0 = \frac{0.3341}{1.0563} = 0.3163$$

This means either $\theta_0 = 71.56°$ (moving away from perigee) or $\theta_0 = 288.44°$ (moving towards perigee). From (a) we know the motion is towards perigee, so that:

$$\boxed{\theta_0 = 288.44°}$$

Figure 2.31 shows the computed location of perigee relative to the initial and final position vectors.

In order to use the Lagrange coefficients to find the position and velocity as a function of time instead of true anomaly, we need to come up with a relation between $\Delta\theta$ and time. We will deal with that complex problem in the next chapter. Meanwhile, for times t that are close to the initial time t_0, we can obtain polynomial expressions for f and g in which the variable $\Delta\theta$ is replaced by the time interval $\Delta t = t - t_0$.

To do so, we expand the position vector $\mathbf{r}(t)$, considered to be a function of time, in a Taylor series about $t = t_0$. As pointed out previously (Equations 1.97 and 1.98), the Taylor series is given by

$$\mathbf{r}(t) = \sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{r}^{(n)}(t_0)(t - t_0)^n \tag{2.159}$$

where $\mathbf{r}^{(n)}(t_0)$ is the nth time derivative of $\mathbf{r}(t)$, evaluated at t_0,

$$\mathbf{r}^{(n)}(t_0) = \left(\frac{d^n\mathbf{r}}{dt^n}\right)_{t=t_0} \tag{2.160}$$

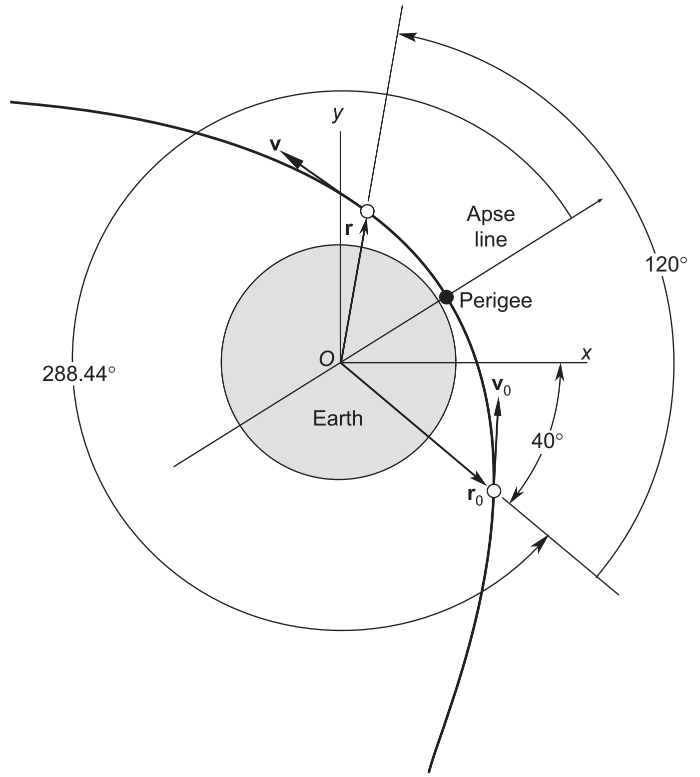


FIGURE 2.31

The initial and final position and velocity vectors and the perigee location for Examples 2.13 and 2.14.

Let us truncate this infinite series at four terms. Then, to that degree of approximation,

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \left(\frac{d\mathbf{r}}{dt}\right)_{t=t_0} \Delta t + \frac{1}{2}\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{t=t_0} \Delta t^2 + \frac{1}{6}\left(\frac{d^3\mathbf{r}}{dt^3}\right)_{t=t_0} \Delta t^3 + \frac{1}{24}\left(\frac{d^4\mathbf{r}}{dt^4}\right)_{t=t_0} \Delta t^4 \tag{2.161}$$

where $\Delta t = t - t_0$. To evaluate the four derivatives, we note first that $(d\mathbf{r}/dt)_{t=t_0}$ is just the velocity $\mathbf{v}_0$ at $t = t_0$,

$$\left(\frac{d\mathbf{r}}{dt}\right)_{t=t_0} = \mathbf{v}_0 \tag{2.162}$$

$(d^2\mathbf{r}/dt^2)_{t=t_0}$ is evaluated using Equation 2.22,

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{2.163}$$

Thus,

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{t=t_0} = -\frac{\mu}{r_0^{\ 3}}\mathbf{r}_0 \tag{2.164}$$

$(d^3\mathbf{r}/dt^3)_{t=t_0}$ is evaluated by differentiating Equation 2.163,

$$\frac{d^3\mathbf{r}}{dt^3} = -\mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r^3}\right) = -\mu\left(\frac{r^3\mathbf{v} - 3\mathbf{r}r^2\dot{r}}{r^6}\right) = -\mu\frac{\mathbf{v}}{r^3} + 3\mu\frac{\dot{r}\mathbf{r}}{r^4} \tag{2.165}$$

From Equation 2.35a we have:

$$\dot{r} = \frac{\mathbf{r}\cdot\mathbf{v}}{r} \tag{2.166}$$

Hence, Equation 2.165, evaluated at $t = t_0$, is

$$\left(\frac{d^3\mathbf{r}}{dt^3}\right)_{t=t_0} = -\mu\frac{\mathbf{v}_0}{r_0^{\ 3}} + 3\mu\frac{\mathbf{r}_0\cdot\mathbf{v}_0}{r_0^{\ 5}}\mathbf{r}_0 \tag{2.167}$$

Finally, $(d^4\mathbf{r}/dt^4)_{t=t_0}$ is found by first differentiating Equation 2.165:

$$\frac{d^4\mathbf{r}}{dt^4} = \frac{d}{dt}\left(-\mu\frac{\dot{\mathbf{r}}}{r^3} + 3\mu\frac{\dot{r}\mathbf{r}}{r^4}\right) = -\mu\left(\frac{r^3\ddot{\mathbf{r}} - 3r^2\dot{r}\dot{\mathbf{r}}}{r^6}\right) + 3\mu\left[\frac{r^4(\ddot{r}\mathbf{r} + \dot{r}\dot{\mathbf{r}}) - 4r^3\dot{r}^2\mathbf{r}}{r^8}\right] \tag{2.168}$$

$\ddot{r}$ is found in terms of **r** and **v** by differentiating Equation 2.166 and making use of Equation 2.163. This leads to the expression:

$$\ddot{r} = \frac{d}{dt}\left(\frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r}\right) = \frac{v^2}{r} - \frac{\mu}{r^2} - \frac{(\mathbf{r}\cdot\mathbf{v})^2}{r^3} \tag{2.169}$$

Substituting Equations 2.163, 2.166, and 2.169 into Equation 2.168, combining terms and evaluating the result at $t = t_0$ yields:

$$\left(\frac{d^4\mathbf{r}}{dt^4}\right)_{t=t_0} = \left[-2\frac{\mu^2}{r_0^{\ 6}} + 3\mu\frac{v_0^{\ 2}}{r_0^{\ 5}} - 15\mu\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)^2}{r_0^{\ 7}}\right]\mathbf{r}_0 + 6\mu\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)}{r_0^{\ 5}}\mathbf{v}_0 \tag{2.170}$$

After substituting Equations 2.162, 2.164, 2.167, and 2.170 into Equation 2.161 and rearranging and collecting terms, we obtain:

$$\begin{aligned}\mathbf{r}(t) = &\left\{1 - \frac{\mu}{2r_0^{\ 3}}\Delta t^2 + \frac{\mu}{2}\frac{\mathbf{r}_0\cdot\mathbf{v}_0}{r_0^{\ 5}}\Delta t^3 + \frac{\mu}{24}\left[-2\frac{\mu}{r_0^{\ 6}} + 3\frac{v_0^{\ 2}}{r_0^{\ 5}} - 15\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)^2}{r_0^{\ 7}}\right]\Delta t^4\right\}\mathbf{r}_0 \\ &+ \left[\Delta t - \frac{1}{6}\frac{\mu}{r_0^{\ 3}}\Delta t^3 + \frac{\mu}{4}\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)}{r_0^{\ 5}}\Delta t^4\right]\mathbf{v}_0\end{aligned} \tag{2.171}$$

Comparing this expression with Equation 2.135, we see that, to the fourth order in Δt,

$$\begin{aligned} f &= 1 - \frac{\mu}{2r_0^3}\Delta t^2 + \frac{\mu}{2}\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{r_0^5}\Delta t^3 + \frac{\mu}{24}\left[-2\frac{\mu}{r_0^6} + 3\frac{v_0^2}{r_0^5} - 15\frac{(\mathbf{r}_0 \cdot \mathbf{v}_0)^2}{r_0^7}\right]\Delta t^4 \\ g &= \Delta t - \frac{1}{6}\frac{\mu}{r_0^3}\Delta t^3 + \frac{\mu}{4}\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{r_0^5}\Delta t^4 \end{aligned} \tag{2.172}$$

For small values of elapsed time Δt these f and g series may be used to calculate the position of an orbiting body from the initial conditions.

Example 2.15

The orbit of an earth satellite has an eccentricity $e = 0.2$ and a perigee radius of 7000 km. Starting at perigee, plot the radial distance as a function of time using the f and g series and compare the curve with the exact solution.

Solution

Since the satellite starts at perigee, $t_0 = 0$ and we have, using the perifocal frame,

$$\mathbf{r}_0 = 7000\hat{\mathbf{p}} \text{ (km)} \tag{a}$$

The orbit equation evaluated at perigee is Equation 2.50, which in the present case becomes:

$$7000 = \frac{h^2}{398{,}600}\frac{1}{1 + 0.2}$$

Solving for the angular momentum, we get $h = 57{,}864\ \text{km}^2/\text{s}$. Then, using the angular momentum formula, Equation 2.31, we find that the speed at perigee is $v_0 = 8.2663$ km/s, so that

$$\mathbf{v}_0 = 8.2663\hat{\mathbf{q}} \text{ (km/s)} \tag{b}$$

Clearly, $\mathbf{r}_0 \cdot \mathbf{v}_0 = 0$. Hence, with $\mu = 398{,}600\ \text{km}^3/\text{s}^2$, the two Lagrange series in Equation 2.172 become (setting $\Delta t = t$):

$$\begin{aligned} f &= 1 - 5.8105(10^{-7})t^2 + 9.0032(10^{-14})t^4 \\ g &= t - 1.9368(10^{-7})t^3 \end{aligned}$$

where the units of t are seconds. Substituting f and g into Equation 2.135 yields:

$$\mathbf{r} = [1 - 5.8105(10^{-7})t^2 + 9.0032(10^{-14})t^4](7000\hat{\mathbf{p}}) + [t - 1.9368(10^{-7})t^3](8.2663\hat{\mathbf{q}})$$

From this we obtain:

$$\boxed{r = \|\mathbf{r}\| = \sqrt{49(10^6) + 11.389t^2 - 1.103(10^{-6})t^4 - 2.5633(10^{-12})t^6 + 3.9718(10^{-19})t^8}} \tag{c}$$

For the exact solution of r versus time we must appeal to the methods presented in the next chapter. The exact solution and the series solution [Equation (c)] are plotted in Figure 2.32. As can be seen, the series solution begins to seriously diverge from the exact solution after about ten minutes.

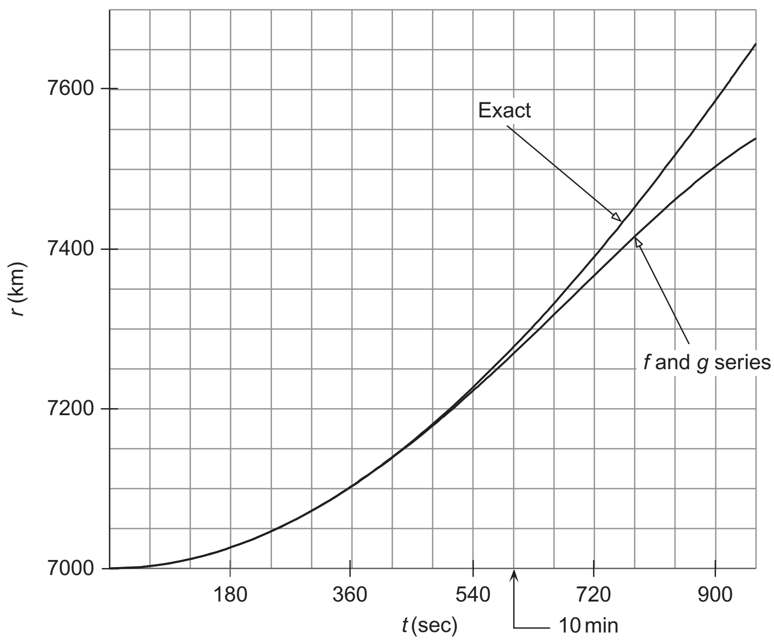


FIGURE 2.32

Exact and series solutions for the radial position of the satellite.

If we include terms of fifth and higher order in the *f* and *g* series, Equations 2.172, then the approximate solution in the above example will agree with the exact solution for a longer time interval than that indicated in Figure 2.32. However, there is a time interval beyond which the series solution will diverge from the exact one no matter how many terms we include. This time interval is called the radius of convergence. According to Bond and Allman (1996), for the elliptical orbit of Example 2.15, the radius of convergence is 1700 seconds (not quite half an hour), which is one fifth of the period of that orbit. This further illustrates the fact that the series forms of the Lagrange coefficients are applicable only over small time intervals. For arbitrary time intervals the closed form of these functions, presented in Chapter 3, must be employed.

2.12 RESTRICTED THREE-BODY PROBLEM

Consider two bodies m_1 and m_2 moving under the action of just their mutual gravitation, and let their orbit around each other be a circle of radius r_{12}. Consider a noninertial, co-moving frame of reference xyz whose origin lies at the center of mass G of the two-body system, with the x-axis directed towards m_2, as shown in Figure 2.33. The y-axis lies in the orbital plane, to which the z-axis is perpendicular. In this rotating frame of reference, m_1 and m_2 appear to be at rest, the force of gravity on each one seemingly balanced by the fictitious centripetal force required to hold it in its circular path around the system center of mass.

The constant, inertial angular velocity $\mathbf{\Omega}$ is given by:

$$\mathbf{\Omega} = \Omega\hat{\mathbf{k}} \tag{2.173}$$

where,

$$\Omega = \frac{2\pi}{T}$$

and T is the period of the orbit (Equation 2.64),

$$T = \frac{2\pi}{\sqrt{\mu}} r_{12}^{\frac{3}{2}}$$

Thus,

$$\Omega = \sqrt{\frac{\mu}{r_{12}^{3}}} \quad (2.174)$$

Recall that if M is the total mass of the system,

$$M = m_1 + m_2 \quad (2.175)$$

then

$$\mu = GM \quad (2.176)$$

m_1 and m_2 lie in the orbital plane, so their y and z coordinates are zero. To determine their locations on the x-axis, we use the definition of the center of mass (Equation 2.2) to write:

$$m_1 x_1 + m_2 x_2 = 0$$

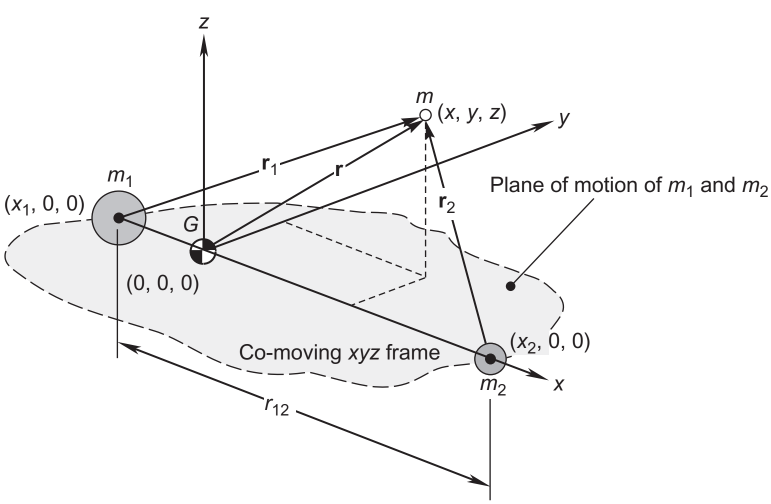


FIGURE 2.33

Primary bodies m_1 and m_2 in circular orbit around each other, plus a secondary mass m.

Since m_2 is at a distance r_{12} from m_1 in the positive x-direction, it is also true that:

$$x_2 = x_1 + r_{12}$$

From these two equations we obtain:

$$x_1 = -\pi_2 r_{12} \tag{2.177a}$$

$$x_2 = \pi_1 r_{12} \tag{2.177b}$$

where the dimensionless mass ratios π_1 and π_2 are given by:

$$\begin{aligned}\pi_1 &= \frac{m_1}{m_1 + m_2} \\ \pi_2 &= \frac{m_2}{m_1 + m_2}\end{aligned} \tag{2.178}$$

Since m_1 and m_2 have the same period in their circular orbits around G, the larger mass (the one closest to G) has the greater orbital speed and hence the greatest centripetal force.

We now introduce a third body of mass m, which is vanishingly small compared to the primary masses m_1 and m_2—like the mass of a spacecraft compared to that of a planet or moon of the solar system. This is called the restricted three-body problem, because the mass m is assumed to be so small that it has no effect on the motion of the primary bodies. We are interested in the motion of m due to the gravitational fields of m_1 and m_2. Unlike the two-body problem, there is no general, closed form solution for this motion. However, we can set up the equations of motion and draw some general conclusions from them.

In the co-moving coordinate system, the position vector of the secondary mass m relative to m_1 is given by:

$$\mathbf{r}_1 = (x - x_1)\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = (x + \pi_2 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{2.179}$$

Relative to m_2 the position of m is:

$$\mathbf{r}_2 = (x - \pi_1 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{2.180}$$

Finally, the position vector of the secondary body relative to center of mass is:

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{2.181}$$

The inertial velocity of m is found by taking the time derivative of Equation 2.181. However, relative to inertial space, the xyz coordinate system is rotating with the angular velocity $\boldsymbol{\Omega}$, so that the time derivatives of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are not zero. To account for the rotating frame, we use Equation 1.66 to obtain:

$$\dot{\mathbf{r}} = \mathbf{v}_G + \boldsymbol{\Omega} \times \mathbf{r} + \mathbf{v}_{rel} \tag{2.182}$$

$\mathbf{v}_G$ is the inertial velocity of the center of mass (the origin of the xyz frame), and $\mathbf{v}_{rel}$ is the velocity of m as measured in the moving xyz frame, namely,

$$\mathbf{v}_{rel} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{2.183}$$

The absolute acceleration of m is found using the "five-term" relative acceleration formula, Equation 1.70,

$$\ddot{\mathbf{r}} = \mathbf{a}_G + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{2.184}$$

Recall from Section 2.2 that the velocity $\mathbf{v}_G$ of the center of mass is constant, so that $\mathbf{a}_G = 0$. Furthermore, $\dot{\boldsymbol{\Omega}} = 0$ since the angular velocity of the circular orbit is constant. Therefore, Equation 2.184 reduces to:

$$\ddot{\mathbf{r}} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{2.185}$$

where,

$$\mathbf{a}_{\text{rel}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{2.186}$$

Substituting Equations 2.173, 2.181, 2.183, and 2.186 into Equation 2.185 yields:

$$\begin{aligned}\ddot{\mathbf{r}} &= (\Omega\mathbf{k}) \times [(\Omega\hat{\mathbf{k}}) \times (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})] + 2(\Omega\hat{\mathbf{k}}) \times (\dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}) + \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \\ &= -\Omega^2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \qquad + 2\Omega\dot{x}\hat{\mathbf{j}} - 2\Omega\dot{y}\hat{\mathbf{i}} \qquad + \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}\end{aligned}$$

Collecting terms, we find:

$$\ddot{\mathbf{r}} = (\ddot{x} - 2\Omega\dot{y} - \Omega^2 x)\hat{\mathbf{i}} + (\ddot{y} + 2\Omega\dot{x} - \Omega^2 y)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{2.187}$$

Now that we have an expression for the inertial acceleration in terms of quantities measured in the rotating frame, let us observe that Newton's second law for the secondary body is

$$m\ddot{\mathbf{r}} = \mathbf{F}_1 + \mathbf{F}_2 \tag{2.188}$$

$\mathbf{F}_1$ and $\mathbf{F}_2$ are the gravitational forces exerted on m by m_1 and m_2, respectively. Recalling Equation 2.10, we have:

$$\begin{aligned}\mathbf{F}_1 &= -\frac{Gm_1 m}{r_1^2}\mathbf{u}_{r_1} = -\frac{\mu_1 m}{r_1^3}\mathbf{r}_1 \\ \mathbf{F}_2 &= -\frac{Gm_2 m}{r_2^2}\mathbf{u}_{r_2} = -\frac{\mu_2 m}{r_2^3}\mathbf{r}_2\end{aligned} \tag{2.189}$$

where,

$$\mu_1 = Gm_1 \qquad \mu_2 = Gm_2 \tag{2.190}$$

Substituting Equations 2.189 into 2.188 and canceling out m yields:

$$\ddot{\mathbf{r}} = -\frac{\mu_1}{r_1^3}\mathbf{r}_1 - \frac{\mu_2}{r_2^3}\mathbf{r}_2 \tag{2.191}$$

Finally, we substitute Equation 2.187 on the left and Equations 2.179 and 2.180 on the right to obtain:

$$\begin{aligned}(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x)\hat{\mathbf{i}} + (\ddot{y} + 2\Omega\dot{x} - \Omega^2 y)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} = &-\frac{\mu_1}{r_1^3}[(x + \pi_2 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \\ &-\frac{\mu_2}{r_2^3}[(x - \pi_1 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}]\end{aligned}$$

Equating the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ on each side of this equation yields the three scalar equations of motion for the restricted three-body problem:

$$\ddot{x} - 2\Omega\dot{y} - \Omega^2 x = -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \tag{2.192a}$$

$$\ddot{y} + 2\Omega\dot{x} - \Omega^2 y = -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \tag{2.192b}$$

$$\ddot{z} = -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z \tag{2.192c}$$

2.12.1 Lagrange Points

Although Equations 2.192 have no closed form analytical solution, we can use them to determine the location of the equilibrium points. These are the locations in space where the secondary mass m would have zero velocity and zero acceleration, that is, where m would appear permanently at rest relative to m_1 and m_2 (and therefore appear to an inertial observer to move in circular orbits around m_1 and m_2). Once placed at an equilibrium point (also called *libration point* or *Lagrange point*), a body will presumably stay there. The equilibrium points are therefore defined by the conditions:

$$\dot{x} = \dot{y} = \dot{z} = 0 \quad \text{and} \quad \ddot{x} = \ddot{y} = \ddot{z} = 0$$

Substituting these conditions into Equations 2.192 yields:

$$-\Omega^2 x = -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \tag{2.193a}$$

$$-\Omega^2 y = -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \tag{2.193b}$$

$$0 = -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z \tag{2.193c}$$

From Equation 2.193c we have:

$$\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right)z = 0 \tag{2.194}$$

Since $\mu_1/r_1^3 > 0$ and $\mu_2/r_2^3 > 0$, it must therefore be true that $z = 0$. That is, the equilibrium points lie in the orbital plane.

From Equations 2.178 it is clear that:

$$\pi_1 = 1 - \pi_2 \tag{2.195}$$

Using this, along with Equation 2.174, and assuming $y \neq 0$, we can write Equations 2.193a and 2.193b as:

$$\begin{aligned} (1-\pi_2)(x+\pi_2 r_{12})\frac{1}{r_1^3} + \pi_2(x+\pi_2 r_{12}-r_{12})\frac{1}{r_2^3} &= \frac{x}{r_{12}^3} \\ (1-\pi_2)\frac{1}{r_1^3} + \pi_2\frac{1}{r_2^3} &= \frac{1}{r_{12}^3} \end{aligned} \tag{2.196}$$

where we made use of the fact that:

$$\pi_1 = \mu_1/\mu \qquad \pi_2 = \mu_2/\mu \tag{2.197}$$

Treating Equations 2.196 as two linear equations in $1/r_1^3$ and $1/r_2^3$, we solve them simultaneously to find that:

$$\frac{1}{r_1^3} = \frac{1}{r_2^3} = \frac{1}{r_{12}^3}$$

or

$$r_1 = r_2 = r_{12} \tag{2.198}$$

Using this result, together with $z = 0$ and Equation 2.195 we obtain from Equations 2.179 and 2.180, respectively,

$$r_{12}^2 = (x + \pi_2 r_{12})^2 + y^2 \tag{2.199}$$

$$r_{12}^2 = (x + \pi_2 r_{12} - r_{12})^2 + y^2 \tag{2.200}$$

Equating the right-hand sides of these two equations leads at once to the conclusion that:

$$x = \frac{r_{12}}{2} - \pi_2 r_{12} \tag{2.201}$$

Substituting this result into Equation 2.199 or 2.200 and solving for y yields:

$$y = \pm\frac{\sqrt{3}}{2} r_{12}$$

We have thus found two of the equilibrium points, the ***Lagrange points*** L_4 ***and*** L_5. As Equation 2.198 shows, these points are the same distance r_{12} from the primary bodies m_1 and m_2 that the primary bodies are from each other, and in the co-moving coordinate system their coordinates are:

$$L_4, L_5 : x = \frac{r_{12}}{2} - \pi_2 r_{12},\ y = \pm\frac{\sqrt{3}}{2} r_{12},\ z = 0 \tag{2.202}$$

Therefore, the two primary bodies and these two Lagrange points lie at the vertices of equilateral triangles, as illustrated in Figure 2.36.

The remaining equilibrium points are found by setting $y = 0$ as well as $z = 0$, which satisfy both Equations 2.193b and 2.193c. For these values, Equations 2.179 and 2.180 become:

$$\mathbf{r}_1 = (x + \pi_2 r_{12})\hat{\mathbf{i}}$$
$$\mathbf{r}_2 = (x - \pi_1 r_{12})\hat{\mathbf{i}} = (x + \pi_2 r_{12} - r_{12})\hat{\mathbf{i}}$$

Therefore,

$$r_1 = |x + \pi_2 r_{12}|$$
$$r_2 = |x + \pi_2 r_{12} - r_{12}|$$

Substituting these expressions together with Equations 2.174, 2.195, and 2.197 into Equation 2.193a yields:

$$\frac{1-\pi_2}{|x + \pi_2 r_{12}|^3}(x + \pi_2 r_{12}) + \frac{\pi_2}{|x + \pi_2 r_{12} - r_{12}|^3}(x + \pi_2 r_{12} - r_{12}) - \frac{1}{r_{12}^{\ 3}}x = 0 \tag{2.203}$$

Further simplification is obtained by nondimensionalizing x,

$$\xi = \frac{x}{r_{12}}$$

In terms of ξ, Equation 2.203 becomes $f(\pi_2,\xi) = 0$, where:

$$f(\pi_2,\xi) = \frac{1-\pi_2}{|\xi + \pi_2|^3}(\xi + \pi_2) + \frac{\pi_2}{|\xi + \pi_2 - 1|^3}(\xi + \pi_2 - 1) - \xi \tag{2.204}$$

Figure 2.34 is a contour plot showing the locus of points (π_2,ξ) at which f is zero. For a given value of the mass ratio π_2 $(0 < \pi_2 < 1)$, the chart shows that there are three values of ξ, corresponding to each of the three colinear ***Lagrange points*** L_1, L_2 ***and*** L_3.

We cannot read these values precisely off the figure, but we can use them as starting points to solve for the roots of the function $f(\pi_2,\xi)$ in Equation 2.204. The bisection method is a simple, though not very efficient, procedure that we can employ here as well as in other problems that require the root of a nonlinear function.

If r is a root of the function $f(x)$, then $f(r) = 0$. To find r by the bisection method, we first select two values of x that we know lie close to and on each side of the root. Label these values x_l and x_u, where $x_l < r$ and $x_u > r$. Since the function f changes sign at a root, it follows that $f(x_l)$ and $f(x_u)$ must be of opposite sign, which means $f(x_l)\cdot f(x_u) < 0$. For the sake of argument, suppose $f(x_l) < 0$ and $f(x_u) > 0$, as in Figure 2.35. Bisect the interval from x_l to x_u by computing $x_m = (x_l + x_u)/2$. If $f(x_m)$ is positive, then the root r lies between x_l and x_m, so (x_l,x_m) becomes our new search interval. If instead $f(x_m)$ is negative, then (x_m,x_u) becomes our search interval. In either case, we bisect the new search interval and repeat the process over and over again, the search interval becoming smaller and smaller, until we eventually converge to r within a desired accuracy E. To achieve that accuracy from the starting values of x_l and x_u requires no more than n iterations, where n is the smallest integer such that (Hahn, 2002):

$$n > \frac{1}{\ln 2}\ln\left(\frac{|x_u - x_l|}{E}\right)$$

Let us summarize the procedure as follows.

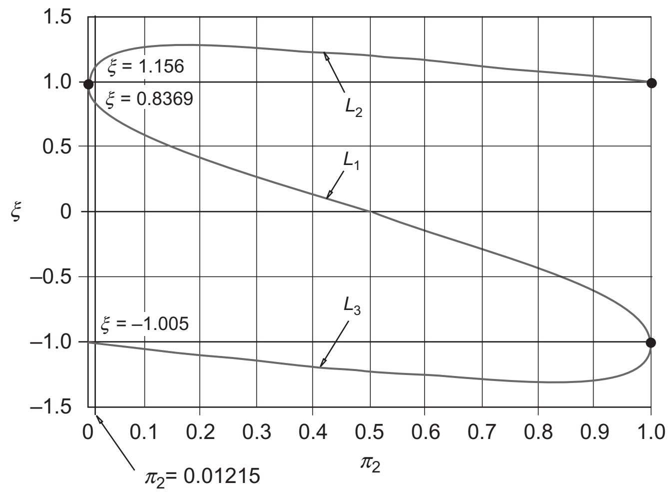


FIGURE 2.34

Contour plot of $f(\pi_2,\xi) = 0$ for the colinear equilibrium points of the restricted three-body problem. $\pi_2 = 0.01215$ for the earth-moon system.

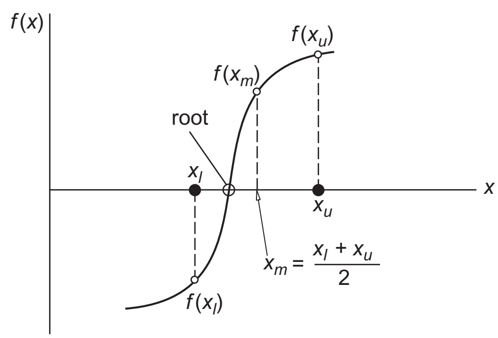


FIGURE 2.35

Determining a root by the bisection method.

Algorithm 2.4 Find a root r of the function $f(x)$ using the bisection method. See Appendix D.9 for a MATLAB implementation of this procedure in the script named *bisect.m*.

1. Select values x_l and x_u, which are known to be fairly close to r and such that $x_l < r$ and $x_u > r$.
2. Choose a tolerance E and determine the number of iterations n from the above formula.
3. Repeat the following steps n times:
   a. Compute $x_m = (x_l + x_u)/2$.
   b. If $f(x_l) \cdot f(x_m) > 0$ then $x_l \leftarrow x_m$; otherwise, $x_u \leftarrow x_m$.
   c. Return to a.
4. $r = x_m$.

Example 2.16

Locate the five Lagrange points for the earth-moon system.

Solution

From Table A.1 we find:

$$\begin{aligned} m_1 &= 5.974 \times 10^{24} \text{ kg} \quad \text{(earth)} \\ m_2 &= 7.348 \times 10^{22} \text{ kg} \quad \text{(moon)} \\ r_{12} &= 3.844 \times 10^{5} \text{ km} \quad \text{(distance between the earth and moon)} \end{aligned} \tag{2.205}$$

We know that Lagrange points L_4 and L_5 lie on the moon's orbit around the earth. L_4 is 60° ahead of the moon and L_5 lies 60° behind the moon, as illustrated in Figure 2.36.

To find L_1, L_2 and L_3 requires finding the roots of Equation 2.204 in which, for the case at hand, the mass ratio is:

$$\pi_2 = \frac{m_2}{m_1 + m_2} = 0.01215$$

Using Algorithm 2.4, we proceed as follows.

Step 1:
For the above value of π_2, Figure 2.34 shows that L_3 lies near $\xi = -1$, whereas L_1 and L_2 lie on the low and high side, respectively, of $\xi = +1$. We cannot read these values precisely off the graph, but we can use them to select the starting values for the bisection method. For L_3, we choose $\xi_l = -1.1$ and $\xi_u = -0.9$.

Step 2:
Choose an error tolerance of $E = 10^{-6}$, which sets the number of iterations,

$$n > \frac{1}{\ln 2}\ln\left(\frac{|\xi_u - \xi_l|}{E}\right) = \frac{1}{\ln 2}\ln\left(\frac{|-0.9 - (-1.1)|}{10^{-6}}\right) = 17.61$$

That is, $n = 18$.

Step 3:
This is summarized in Table 2.1.

We conclude that, to five significant figures, $\xi_3 = -1.0050$.

The values of ξ for the Lagrange points L_1 and L_2 are found the same way using Algorithm 2.4, starting with estimates obtained from Figure 2.34. Rather than repeating the lengthy hand computations, see instead Appendix D.9 for the MATLAB program *Example_2_16.m*, which carries out the calculations of all three roots. It uses the program *bisect.m* to do the iterations, leading to $\xi_1 = 0.8369$ and $\xi_2 = 1.156$, as well as $\xi_3 = -1.005$ computed in Table 2.1.

Multiplying each dimensionless root by r_{12} yields the x coordinates of the colinear Lagrange points in kilometers.

$$\boxed{\begin{aligned} L_1 &: x = 0.8369 r_{12} = 3.217 \times 10^5 \text{ km} \\ L_2 &: x = 1.156 r_{12} = 4.444 \times 10^5 \text{ km} \\ L_3 &: x = -1.005 r_{12} = -3.863 \times 10^5 \text{ km} \end{aligned}} \tag{2.206}$$

Table 2.1 Steps of the Bisection Method Leading to $\xi = 1.0050$ for L_3

n	ξ_l	ξ_u	ξ_m	Sign of $f(\pi_{12},\xi_l)\cdot f(\pi_{12},\xi_u)$
1	−1.1	−0.9	−1	<0
2	−1.1	−1	−1.05	>0
3	−1.05	−1	−1.025	>0
4	−1.025	−1	−1.0125	>0
5	−1.0125	−1	−1.00625	>0
6	−1.00625	−1	−1.003125	<0
7	−1.00625	−1.003125	−1.0046875	<0
8	−1.00625	−1.0046875	−1.00546875	>0
9	−1.00546875	−1.0046875	−1.005078125	>0
10	−1.005078125	−1.0046875	−1.0049882812	<0
11	−1.005078125	−1.0049882812	−1.004980469	<0
12	−1.004980469	−1.0049882812	−1.005029297	>0
13	−1.005029297	−1.0049882812	−1.005004883	>0
14	−1.005004883	−1.0049882812	−1.004992676	<0
15	−1.005004883	−1.004992676	−1.004998779	>0
16	−1.004998779	−1.004992676	−1.004995728	>0
17	−1.004995728	−1.004992676	−1.004994202	<0
18	−1.004995728	−1.004994202	−1.004994965	>0

The locations of the five Lagrange points for the earth-moon system are shown in Figure 2.36. For convenience, all of their positions are shown relative to the center of the earth, instead of the center of mass. As can be seen from Equation 2.177a, the center of mass of the earth-moon system is only 4670 km from the center of the earth. That is, it lies within the earth at 73 percent of its radius. Since the Lagrange points are fixed relative to the earth and moon, they follow circular orbits around the earth with the same period as the moon.

If an equilibrium point is stable, then a small mass occupying that point will tend to return to that point if nudged out of position. The perturbation results in a small oscillation (orbit) about the equilibrium point. Thus, objects can be placed in small orbits (called halo orbits) around stable equilibrium points without requiring much in the way of station keeping. On the other hand, if a body located at an unstable equilibrium point is only slightly perturbed, it will oscillate in a divergent fashion, drifting eventually completely away from that point. It turns out that the Lagrange points L_1, L_2 and L_3 on the apse line are unstable, whereas L_4 and L_5, which lie 60° ahead of m_1 and 60 behind m_1 in its orbit, are stable if the ratio m_2/m_1 exceeds 24.96. For the earth-moon system that ratio is 81.3. However, L_4 and L_5 are destabilized by the influence of the sun's gravity, so that in actuality station keeping would be required to maintain position in the neighborhood of those points of the earth-moon system.

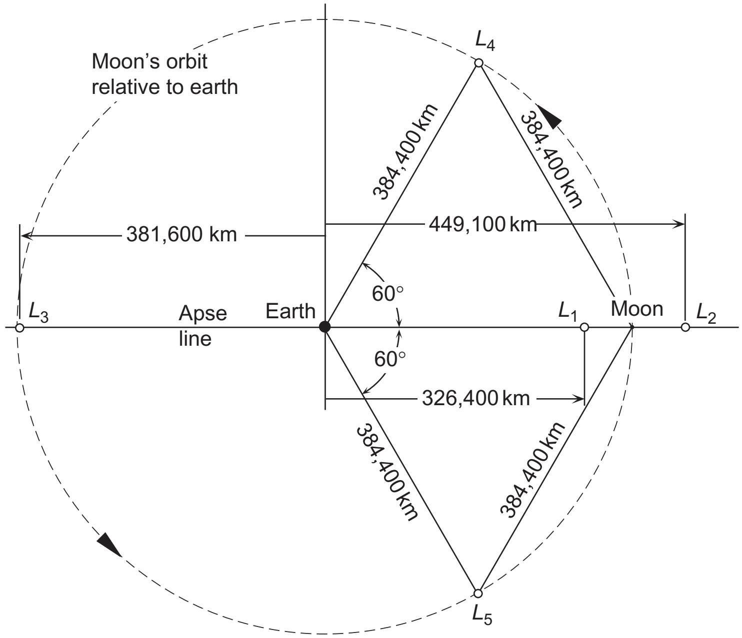


FIGURE 2.36

Location of the five Lagrange points of the earth-moon system. These points orbit the Earth with the same period as the moon.

Solar observation spacecraft have been placed in halo orbits around the L_1 point of the sun-earth system. L_1 lies about 1.5 million kilometers from the earth (1/100 the distance to the sun) and well outside the earth's magnetosphere. Three such missions were the International Sun-Earth Explorer 3 (ISSUE-3) launched in August 1978; the Solar and Heliospheric Observatory (SOHO) launched in December 1995; and the Advanced Composition Explorer (ACE), launched in August 1997.

In June 2001, the 830 kg Wilkinson Microwave Anisotropy Probe (WMAP) was launched aboard a Delta II rocket on a three month journey to sun-earth Lagrange point L_2, which lies 1.5 million kilometers from the earth in the opposite direction from L_1. WMAP's several-year mission was to measure cosmic microwave background radiation. The 6200 kg James Webb Space Telescope (JWST) is scheduled for a 2013 launch aboard an Arianne 5 to an orbit around L_2. This successor to the Hubble Space Telescope, which is in low earth orbit, will use a 6.5-meter mirror to gather data in the infrared spectrum over a period of 5 to 10 years.

2.12.2 Jacobi Constant

Multiply Equation 2.192a by $\dot{x}$, Equation 2.192b by $\dot{y}$ and Equation 2.192c by $\dot{z}$ to obtain:

$$\begin{aligned}
\ddot{x}\dot{x} - 2\Omega\dot{x}\dot{y} - \Omega^2 x\dot{x} &= -\frac{\mu_1}{r_1^3}(x\dot{x} + \pi_2 r_{12}\dot{x}) - \frac{\mu_2}{r_2^3}(x\dot{x} - \pi_1 r_{12}\dot{x}) \\
\ddot{y}\dot{y} + 2\Omega\dot{x}\dot{y} - \Omega^2 y\dot{y} &= -\frac{\mu_1}{r_1^3}y\dot{y} - \frac{\mu_2}{r_2^3}y\dot{y} \\
\ddot{z}\dot{z} &= -\frac{\mu_1}{r_1^3}z\dot{z} - \frac{\mu_2}{r_2^3}z\dot{z}
\end{aligned}$$

Sum the left and right side of these equations to get:

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} - \Omega^2(x\dot{x} + y\dot{y}) = -\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right)(x\dot{x} + y\dot{y} + z\dot{z}) + r_{12}\left(\frac{\pi_1\mu_2}{r_2^3} - \frac{\pi_2\mu_1}{r_1^3}\right)\dot{x}$$

or, rearranging terms,

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} - \Omega^2(x\dot{x} + y\dot{y}) = -\frac{\mu_1}{r_1^3}(x\dot{x} + y\dot{y} + z\dot{z} + \pi_2 r_{12}\dot{x}) - \frac{\mu_2}{r_2^3}(x\dot{x} + y\dot{y} + z\dot{z} - \pi_1 r_{12}\dot{x}) \quad (2.207)$$

Note that

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = \frac{1}{2}\frac{d}{dt}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}v^2 \quad (2.208)$$

where v is the speed of the secondary mass relative to the rotating frame. Similarly,

$$x\dot{x} + y\dot{y} = \frac{1}{2}\frac{d}{dt}(x^2 + y^2) \quad (2.209)$$

From Equation 2.179 we obtain:

$$r_1^2 = (x + \pi_2 r_{12})^2 + y^2 + z^2$$

Therefore,

$$2r_1\frac{dr_1}{dt} = 2(x + \pi_2 r_{12})\dot{x} + 2y\dot{y} + 2z\dot{z}$$

or

$$\frac{dr_1}{dt} = \frac{1}{r_1}(\pi_2 r_{12}\dot{x} + x\dot{x} + y\dot{y} + z\dot{z})$$

It follows that

$$\frac{d}{dt}\frac{1}{r_1} = -\frac{1}{r_1^2}\frac{dr_1}{dt} = -\frac{1}{r_1^3}(x\dot{x} + y\dot{y} + z\dot{z} + \pi_2 r_{12}\dot{x}) \quad (2.210)$$

In a similar fashion, starting with Equation 2.180, we find:

$$\frac{d}{dt}\frac{1}{r_2} = -\frac{1}{r_2^3}(x\dot{x} + y\dot{y} + z\dot{z} - \pi_1 r_{12}\dot{x}) \quad (2.211)$$

Substituting Equations 2.208, 2.209, 2.210 and 2.211 into Equation 2.207 yields:

$$\frac{1}{2}\frac{dv^2}{dt} - \frac{1}{2}\Omega^2\frac{d}{dt}(x^2 + y^2) = \mu_1\frac{d}{dt}\frac{1}{r_1} + \mu_2\frac{d}{dt}\frac{1}{r_2}$$

Alternatively, upon rearranging terms:

$$\frac{d}{dt}\left[\frac{1}{2}v^2 - \frac{1}{2}\Omega^2(x^2 + y^2) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}\right] = 0$$

which means the bracketed expression is a constant:

$$\frac{1}{2}v^2 - \frac{1}{2}\Omega^2(x^2 + y^2) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} = C \tag{2.212}$$

$v^2/2$ is the kinetic energy per unit mass relative to the rotating frame. $-\mu_1/r_1$ and $-\mu_2/r_2$ are the gravitational potential energies of the two primary masses. $-\Omega^2(x^2 + y^2)/2$ may be interpreted as the potential energy of the centrifugal force per unit mass $\Omega^2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$ induced by the rotation of the reference frame. The constant C is known as the Jacobi constant, after the German mathematician Carl Jacobi (1804–1851), who discovered it in 1836. Jacobi's constant may be interpreted as the total energy of the secondary particle relative to the rotating frame. C is a constant of the motion of the secondary mass just like the energy and angular momentum are constants of the relative motion in the two-body problem.

Solving Equation 2.212 for v^2 yields:

$$v^2 = \Omega^2(x^2 + y^2) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C \tag{2.213}$$

If we restrict the motion of the secondary mass to lie in the plane of motion of the primary masses, then:

$$r_1 = \sqrt{(x + \pi_2 r_{12})^2 + y^2} \qquad r_2 = \sqrt{(x - \pi_1 r_{12})^2 + y^2} \tag{2.214}$$

For a given value of the Jacobi constant, Equation 2.213 shows that v^2 is a function only of position in the rotating frame. Since v^2 cannot be negative, it must be true that:

$$\Omega^2(x^2 + y^2) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C \geq 0 \tag{2.215}$$

Trajectories of the secondary body in regions where this inequality is violated are not allowed. The boundaries between forbidden and allowed regions of motion are found by setting $v^2 = 0$, that is:

$$\Omega^2(x^2 + y^2) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C = 0 \tag{2.216}$$

For a given value of the Jacobi constant the curves of zero velocity are determined by this equation. These boundaries cannot be crossed by a secondary mass (spacecraft) moving within an allowed region.

Since the first three terms on the left of Equation 2.216 are all positive, it follows that the zero velocity curves correspond to negative values of the Jacobi constant. Large negative values of C mean that the secondary body is far from the system center of mass ($x^2 + y^2$ is large) or that the body is close to one of the primary bodies (r_1 is small or r_2 is small).

Let us consider again the earth-moon system. From Equations 2.174, 2.175, 2.176, 2.190, and 2.205 we have:

$$\begin{aligned}
\Omega &= \sqrt{\frac{G(m_1 + m_2)}{{r_{12}}^3}} = \sqrt{\frac{6.67259 \times 10^{-20}(6.04748 \times 10^{24})}{384,400^3}} = 2.66538 \times 10^{-6} \text{ rad/s} \\
\mu_1 &= Gm_1 = 6.67259 \times 10^{-20} \cdot 5.9742 \times 10^{24} = 398,620 \text{ km}^3/\text{s}^2 \\
\mu_2 &= Gm_2 = 6.67259 \times 10^{-20} \cdot 7.348 \times 10^{22} = 4903.02 \text{ km}^3/\text{s}^2
\end{aligned} \tag{2.217}$$

Substituting these values into Equation 2.216, we can plot the zero velocity curves for different values of Jacobi's constant. The curves bound regions in which the motion of a spacecraft is not allowed.

For C = $-1.8\,\text{km}^2/\text{s}^2$, the allowable regions are circles surrounding the earth and the moon, as shown in Figure 2.37(a). A spacecraft launched from the earth with this value of C cannot reach the moon, to say nothing of escaping the earth-moon system.

Substituting the coordinates of the Lagrange points L_1, L_2 and L_3 into Equation 2.216, we obtain the successively larger values of the Jacobi constants C_1, C_2 and C_3 which are required to arrive at those points with zero velocity. These are shown along with the allowable regions in Figure 2.37. From part (c) of that figure we see that C_2 represents the minimum energy for a spacecraft to escape the earth-moon system via a narrow corridor around the moon. Increasing C widens that corridor and at C_3 escape becomes possible in the opposite direction from the moon. The last vestiges of the forbidden regions surround L_4 and L_5. Further increase in Jacobi's constant make the entire earth-moon system and beyond accessible to an earth-launched spacecraft.

For a given value of the Jacobi constant, the relative speed at any point within an allowable region can be found using Equation 2.213.

Example 2.17

The earth-orbiting spacecraft in Figure 2.38 has a relative burnout velocity v_{bo} at an altitude of $d = 200$ km on a radial for which $\phi = -90°$. Find the value of v_{bo} for each of the scenarios depicted in Figure 2.37.

Solution

From Equations 2.177 and 2.205 we have

$$\begin{aligned}
\pi_1 &= \frac{m_1}{m_1 + m_2} = \frac{5.974 \times 10^{24}}{6.047 \times 10^{24}} = 0.9878 \qquad \pi_2 = 1 - \pi_1 = 0.1215 \\
x_1 &= -\pi_1 r_{12} = -0.9878 \cdot 384,400 = -4670.6 \text{ km}
\end{aligned}$$

Therefore, the coordinates of the burnout point are

$$x = -4670.6 \text{ km} \qquad y = -6578 \text{ km}$$

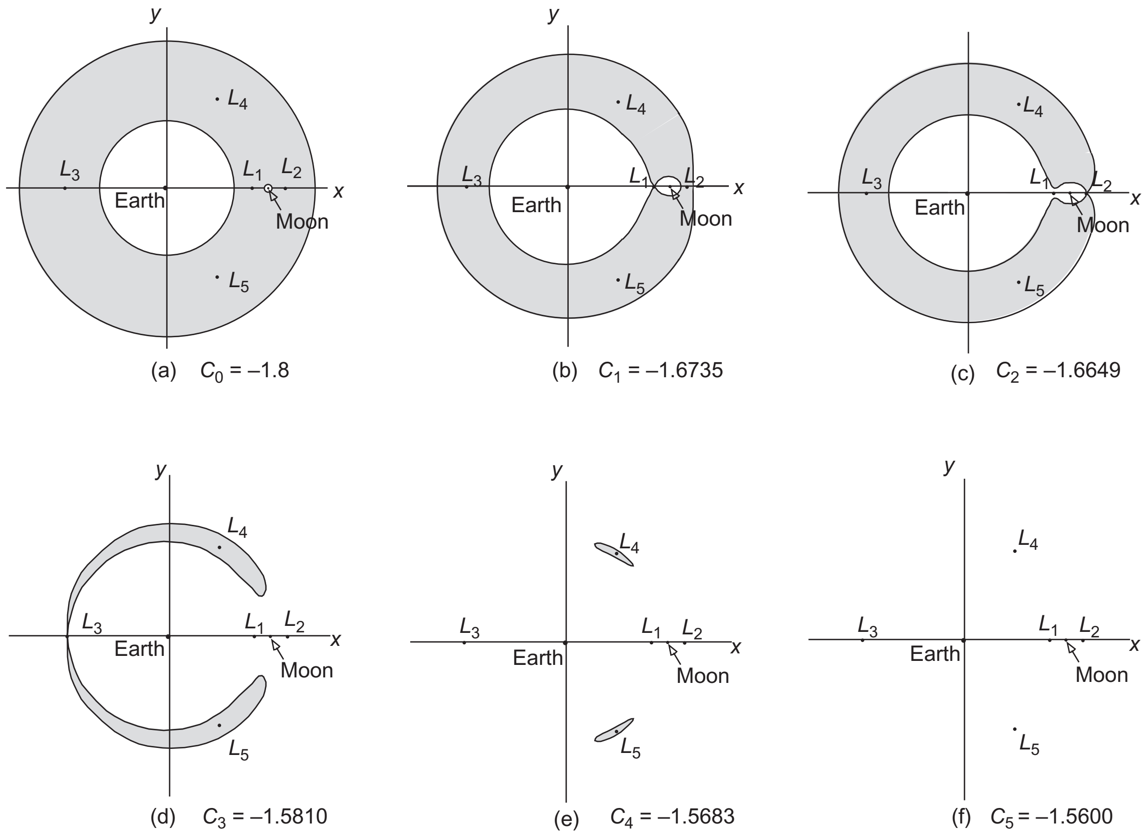


FIGURE 2.37

Forbidden regions (shaded) within the earth-moon system for increasing values of Jacobi's constant (km^2/s^2).

Substituting these values along with the Jacobi constant into Equations 2.213 and 2.214 yields the relative burnout speed v_{bo}. For the six Jacobi constants in Figure 2.38 we obtain:

$$\boxed{\begin{array}{l} C_0 :v_{bo} = 10.84518 \text{ km/s} \\ C_1 :v_{bo} = 10.85683 \text{ km/s} \\ C_2 :v_{bo} = 10.85762 \text{ km/s} \\ C_3 :v_{bo} = 10.86535 \text{ km/s} \\ C_4 :v_{bo} = 10.86652 \text{ km/s} \\ C_5 :v_{bo} = 10.86728 \text{ km/s} \end{array}}$$

These velocities are not substantially different from the escape velocity (Equation 2.91) at 200 km altitude,

$$v_{\text{esc}} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{2 \cdot 398{,}600}{6578}} = 11.01 \text{ km/s}$$

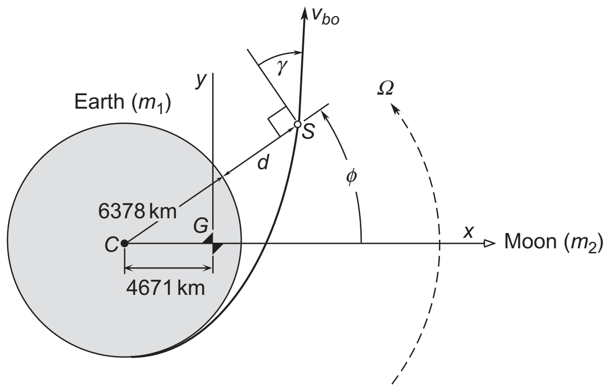


FIGURE 2.38

Spacecraft S burnout position and velocity relative to the rotating earth-moon frame.

Observe that a change in v_{bo} on the order of only 10 m/s or less can have a significant influence on the regions of earth-moon space accessible to the spacecraft.

Example 2.18

For the spacecraft in Figure 2.38 the initial conditions ($t = 0$) are $d = 200$ km, $\phi = -90°$, $\gamma = 20°$ and $v_{bo} = 10.9148$ km/s. Use Equations 2.192, the restricted three body equations of motion, to determine the trajectory and locate its position at $t = 3.16689$ days.

Solution

Since z and $\dot{z}$ are initially zero, Equation 2.192c implies that z remains zero. The motion is therefore confined to the xy plane and is governed by Equations 2.192a and 2.192b. These have no analytical solution, so we must use a numerical approach.

In order to get Equations 2.192a and 2.192b into the standard form for numerical solution (see Section 1.8), we introduce the auxiliary variables

$$y_1 = x \quad y_2 = y \quad y_3 = \dot{x} \quad y_4 = \dot{y} \tag{a}$$

The time derivatives of these variables are:

$$\begin{aligned}
\dot{y}_1 &= y_3 \\
\dot{y}_2 &= y_4 \\
\dot{y}_3 &= 2\Omega y_4 + \Omega^2 y_1 - \frac{\mu_1}{r_1^3}(y_1 + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(y_1 - \pi_1 r_{12}) \qquad \text{(Equation 2.192a)} \\
\dot{y}_4 &= -2\Omega y_3 + \Omega^2 y_2 - \frac{\mu_1}{r_1^3} y_2 - \frac{\mu_2}{r_2^3} y_2 \qquad \text{(Equation 2.192b)}
\end{aligned} \tag{b}$$

where, from Equations 2.179 and 2.180,

$$r_1 = \sqrt{(y_1 + \pi_2 r_{12})^2 + y_2^2} \qquad r_2 = \sqrt{(y_1 - \pi_1 r_{12})^2 + y_2^2} \tag{c}$$

Equations (b) are of the form $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ given by Equation 1.95.

To solve this system let us use the Runge-Kutta-Fehlberg 4(5) method and Algorithm 1.3, which is implemented in MATLAB as the program *rkf45.m* in Appendix D.4. The MATLAB function named *Example_2_18.m* in Appendix D.10 contains the data for this problem, the given initial conditions and the time range. To perform the numerical integration, *Example_2_18.m* calls *rkf45.m*, which uses the subfunction *rates*, which is embedded within *Example_2_18.m,* to compute the derivatives in (b) above. Running *Example_2_18.m* yields the plot of the trajectory shown in Figure 2.39. After coasting 3.16689 days as specified in the problem statement,

the spacecraft arrives at the far side of the moon on the earth-moon line at an altitude of 256 km.

For comparison, the 1969 Apollo 11 translunar trajectory, which differed from this one in many details (including the use of midcourse corrections), required 3.04861 days to arrive at the lunar orbit insertion point.

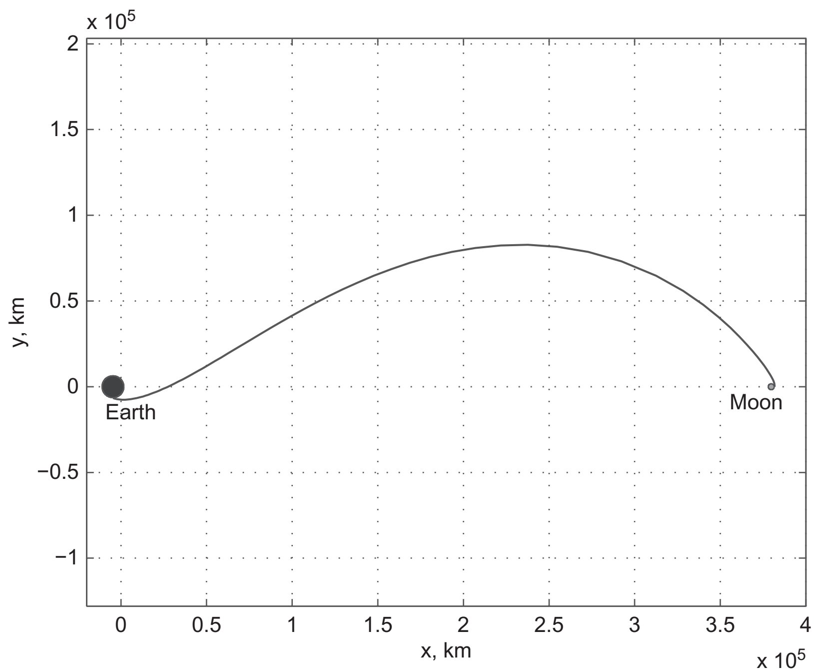


FIGURE 2.39

Translunar coast trajectory computed numerically from the restricted three-body differential equations using the *RKF4(5)* method.

PROBLEMS

For man-made earth satellites use $\mu = 398{,}600\,\text{km}^3/\text{s}^2$. $R_E = 6378\,\text{km}$ (Tables A.1 and A.2).

Section 2.2

2.1 Two particles of identical mass m are acted on only by the gravitational force of one upon the other. If the distance d between the particles is constant, what is the angular velocity of the line joining them? Use Newton's second law with the center of mass of the system as the origin of the inertial frame.
{Ans.: $\omega = \sqrt{2Gm/d^3}$ }

2.2 Three particles of identical mass m are acted on only by their mutual gravitational attraction. They are located at the vertices of an equilateral triangle with sides of length d. Consider the motion of any one of the particles about the system center of mass G and, using G as the origin of the inertial frame, employ Newton's second law to determine the angular velocity ω required for d to remain constant.
{Ans.: $\omega = \sqrt{3Gm/d^3}$ }

Section 2.3

2.3 Consider the two-body problem illustrated in Figure 2.1. If a force $\mathbf{T}$ (such as rocket thrust) acts on m_2 in addition to the mutual force of gravitation $\mathbf{F}_{21}$, show that:

(a) Equation 2.22 becomes:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \frac{\mathbf{T}}{m_2}.$$

(b) If the thrust vector $\mathbf{T}$ has magnitude T and is aligned with the velocity vector $\mathbf{v}$, then:

$$\mathbf{T} = T\frac{\mathbf{v}}{v}.$$

2.4 At a given instant t_0, an earth-orbiting satellite has the inertial position and velocity vectors $\mathbf{r}_0 = 3207\hat{\mathbf{i}} + 5459\hat{\mathbf{j}} + 2714\hat{\mathbf{k}}$ (km) and $\mathbf{v}_0 = -6.532\hat{\mathbf{i}} + 0.7835\hat{\mathbf{j}} + 6.142\hat{\mathbf{k}}$ (km/s). Solve Equation 2.22 numerically to find maximum altitude reached by the satellite and the time at which it occurs.
{Ans.: Using MATLAB's *ode45.m*, maximum altitude = 9680 km at 1.66 hours after t_0.}

2.5 At a given instant, an earth-orbiting satellite has the inertial position and velocity vectors $\mathbf{r}_0 = 6600\hat{\mathbf{i}}$ (km) and $\mathbf{v}_0 = 12\hat{\mathbf{j}}$ (km/s). Solve Equation 2.22 numerically to find the distance of the spacecraft from the center of the earth and its speed 24 hours later.
{Ans.: Using MATLAB's *ode45.m*, distance = 463,300 km, speed = 4.995 km/s.}

Section 2.4

2.6 If r, in meters, is given by $\mathbf{r} = 3t^4\hat{\mathbf{I}} + 2t^3\hat{\mathbf{J}} + 9t^2\hat{\mathbf{K}}$, where t is time in seconds, calculate (a) $\dot{r}$ (where $r = \|\mathbf{r}\|$) and (b) $\|\dot{\mathbf{r}}\|$ at $t = 2$s.
{Ans.: (a) $\dot{r} = 101.3$ m/s; (b) $\|\dot{\mathbf{r}}\| = 105.3$ m/s}

2.7 Starting with Equation 2.35a, prove that $\dot{r} = \mathbf{v} \cdot \hat{\mathbf{u}}_r$ and interpret this result.

2.8 Show that $\hat{\mathbf{u}}_r \cdot d\hat{\mathbf{u}}_r/dt = 0$, where $\hat{\mathbf{u}}_r = \mathbf{r}/r$. Use only the fact that $\hat{\mathbf{u}}_r$ is a unit vector. Interpret this result.

2.9 Starting with Equation 2.38, show that $\hat{\mathbf{u}}_r \cdot d\hat{\mathbf{u}}_r/dt = 0$.

2.10 Show that $v = \frac{\mu}{h}\sqrt{1 + 2e\cos\theta + e^2}$ for any orbit.

2.11 Relative to a nonrotating, earth-centered Cartesian coordinate system, the position and velocity vectors of a spacecraft are $\mathbf{r} = -8900\hat{\mathbf{i}} - 1690\hat{\mathbf{j}} + 5210\hat{\mathbf{k}}$ (km) and $\mathbf{v} = -6\hat{\mathbf{i}} - 4.5\hat{\mathbf{j}} - 1.5\hat{\mathbf{k}}$ (km/s). Calculate the orbit's (a) eccentricity vector and (b) the true anomaly.
{Ans.: (a) $\mathbf{e} = 0.3461\hat{\mathbf{i}} + 0.5142\hat{\mathbf{j}} + 0.4663\hat{\mathbf{k}}$; (b) $\theta = 100.8°$}

2.12 Show that the eccentricity is 1 for rectilinear orbits ($\mathbf{h} = 0$).

2.13 Relative to a nonrotating, earth-centered Cartesian coordinate system, the velocity of a spacecraft is $\mathbf{v} = -8.2\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 1.3\hat{\mathbf{k}}$ (km/s) and the unit vector in the direction of the radius is $\hat{\mathbf{u}}_r = -0.4835\hat{\mathbf{i}} + 0.09667\hat{\mathbf{j}} + 0.8700\hat{\mathbf{k}}$. Calculate (a) the radial component of velocity v_r, (b) the azimuth component of velocity $v_\perp$, and the flight path angle γ.
{Ans.: (a) 5.966 km/s; (b) 10.69 km/s; (c) 29.16°}

Section 2.5

2.14 If the specific energy ε of the two-body problem is negative, show that m_2 cannot move outside a sphere of radius $\mu/|\varepsilon$ centered at m_1.

2.15 Relative to a nonrotating Cartesian coordinate frame with origin at the center of the earth, a spacecraft has the position and velocity vectors $\mathbf{r} = 10{,}000\hat{\mathbf{i}} + 5000\hat{\mathbf{j}} + 15{,}000\hat{\mathbf{k}}$ (km) and $\mathbf{v} = 5\hat{\mathbf{i}} + 2.5\hat{\mathbf{j}} + 7.5\hat{\mathbf{k}}$ (km/s). Later, when the speed is $v = 7$ km/s, what is the position vector?
{Ans.: $\mathbf{r} = 103{,}600\hat{\mathbf{i}} + 51{,}810\hat{\mathbf{j}} + 155{,}400\hat{\mathbf{k}}$ (km)}

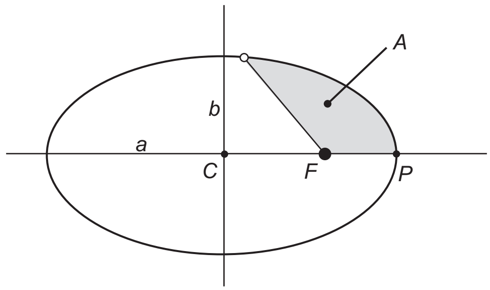
Section 2.6

2.16 A satellite is in a circular, 350 km orbit (i.e., it is 350 km above the earth's surface). Calculate (a) the speed in km/s and (b) the period.
{Ans.: (a) 7.697 km/s; (b) 91 min 32 s}

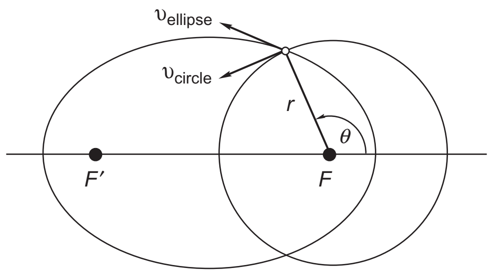
2.17 A spacecraft is in a circular orbit of the moon at an altitude of 80 km. Calculate its speed and its period.
{Ans.: 1.642 km/s; 1 hr 56 min.}

Section 2.7

2.18 Calculate the area A swept out during the time $t = T/3$ since periapsis, where T is the period of the elliptical orbit.
{Ans.: 1.047ab}



2.19 Determine the true anomaly θ of the point(s) on an elliptical orbit at which the speed equals the speed of a circular orbit with the same radius, i.e., $v_{\text{ellipse}} = v_{\text{circle}}$.
{Ans.: $\theta = \cos^{-1}(-e)$, where e is the eccentricity of the ellipse.}



2.20 Calculate the flight path angle at the locations found in Problem 2.19.
{Ans.: $\gamma = \tan^{-1}\left(e/\sqrt{1-e^2}\right)$}

2.21 An unmanned satellite orbits the earth with a perigee radius of 7000 km and an apogee radius of 70,000 km.Calculate:

(a) the eccentricity of the orbit;
(b) the semimajor axis of the orbit (km);
(c) the period of the orbit (hours);
(d) the specific energy of the orbit (km^2/s^2);
(e) the true anomaly at which the altitude is 1000 km (degrees);
(f) v_r and $v_\perp$ at the points found in part (e) (km/s);
(g) the speed at perigee and apogee (km/s).

{Partial Ans.: (c) 20.88 hr; (e) 27.61°; (g) 10.18 km/s, 1.018 km/s}

2.22 A spacecraft is in a 250 km by 300 km low earth orbit. How long (in minutes) does it take to coast from perigee to apogee?
{Ans.: 45.00 min}

2.23 The altitude of a satellite in an elliptical orbit around the earth is 1600 km at apogee and 600 km at perigee. Determine (a) the eccentricity of the orbit; (b) the orbital speeds at perigee and apogee; (c) the period of the orbit.
{Ans.: (a) 0.06686; (b) $v_p = 7.81$ km/s; $v_A = 6.83$ km/s; (c) $T = 107.2$ minutes.}

2.24 A satellite is placed into an earth orbit at perigee at an altitude of 1270 km with a speed of 9 km/s. Calculate the flight path angle γ and the altitude of the satellite at a true anomaly of 100°.
{Ans.: $\gamma = 31.1°$, z = 6774 km}

2.25 A satellite is launched into earth orbit at an altitude of 640 km with a speed of 9.2 km/s and a flight path angle of 10°. Calculate the true anomaly of the launch point and the period of the orbit.
{Ans.: $\theta = 29.8°$; $T = 4.46$ hrs}

2.26 A satellite has perigee and apogee altitudes of 250 km and 42,000 km. Calculate the orbit period, eccentricity, and the maximum speed.
{Ans.: 12 h 36 m, 0.759, 10.3 km/s}

2.27 A satellite is launched parallel to the earth's surface with a speed of 8 km/s at an altitude of 640 km. Calculate the apogee altitude and the period.
{Ans.: 2679 km, 1 h 59 m 30 s}

2.28 A satellite in orbit around the earth has a perigee velocity of 8 km/sec. Its period is 2 hours. Calculate its altitude at perigee.
{Ans.: 648 km}

2.29 A satellite in polar orbit around the earth comes within 150 km of the north pole at its point of closest approach. If the satellite passes over the pole once every 90 minutes, calculate the eccentricity of its orbit.
{Ans.: 0.0187}

2.30 The following position data for an earth orbiter are given:
Altitude = 1700 km at a true anomaly of 130°.
Altitude = 500 km at a true anomaly of 50°.

Calculate:
(a) The eccentricity.
(b) The perigee altitude (km).
(c) The semimajor axis (km)
{Partial ans.: (c) 7547 km}

2.31 An earth satellite has a speed of 7 km/s and a flight path angle of 15° when its radius is 9000 km. Calculate: (a) the true anomaly (degrees), and (b) the eccentricity of the orbit.
{Ans.: (a) 83.35°; (b) 0.2785}

2.32 If, for an earth satellite, the specific angular momentum is 60,000 km^2/s and the specific energy is $-20\ km^2/s^2$, calculate the apogee and perigee altitudes.
{Ans.: 6,637 km and 537.2 km}

2.33 A rocket launched from the surface of the earth has a speed of 8.85 km/s when powered flight ends at an altitude of 550 km.The flight path angle at this time is 6°. Determine (a) the eccentricity of the trajectory; (b) the period of the orbit.
{Ans.: (a) $e = 0.3742$; (b) $T = 187.4$ min.}

2.34 If the perigee velocity is c times the apogee velocity, calculate the eccentricity of the orbit in terms of c.
{Ans.: $e = (c - 1)/(c + 1)$}

Section 2.8

2.35 Find the minimum additional speed required to escape from GEO.
{Ans.: 1.274 km/s}

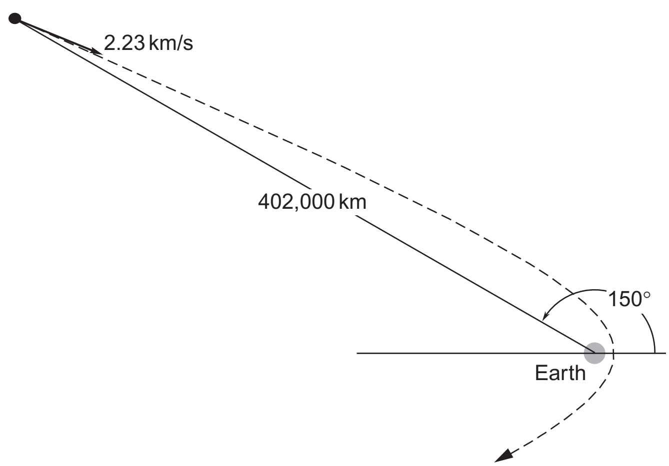
2.36 What velocity, relative to the earth, is required to escape the solar system on a parabolic path from earth's orbit?
{Ans.: 12.34 km/s}

Section 2.9

2.37 A hyperbolic earth departure trajectory has a perigee altitude of 300 km and a perigee speed of 15 km/s.

(a) Calculate the hyperbolic excess speed (km/s).
(b) Find the radius (km) when the true anomaly is 100°. {Ans.: 48,497 km}
(c) Find v_r and $v_\perp$ (km/s) when the true anomaly is 100°.

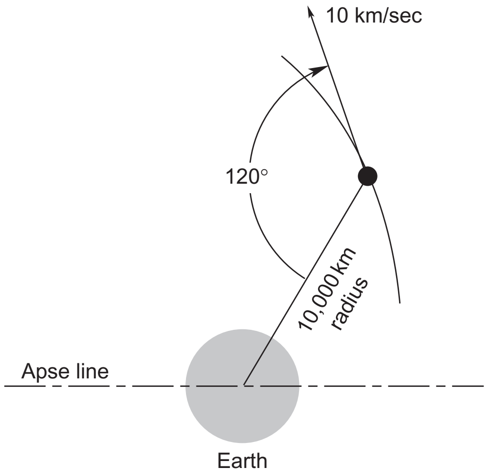
2.38 A meteoroid is first observed approaching the earth when it is 402,000 km from the center of the earth with a true anomaly of 150°. If the speed of the meteoroid at that time is 2.23 km/s, calculate (a) the eccentricity of the trajectory; (b) the altitude at closest approach; (c) the speed at closest approach.
{Ans.: (a) 1.086; (b) 5088 km; (c) 8.516 km/s}



2.39 Calculate the radius r at which the speed on a hyperbolic trajectory is 1.1 times the hyperbolic excess speed. Express your result in terms of the periapsis radius r_p and the eccentricity e.
{Ans.: $r = 9.524r_p/(e-1)$}

2.40 A hyperbolic trajectory has an eccentricity $e = 3.0$ and an angular momentum $h = 105{,}000\ \text{km}^2/\text{s}$. Without using the energy equation, calculate the hyperbolic excess speed. {Ans.: 10.7 km/s}

2.41 A space vehicle has a velocity of 10 km/s in the direction shown when it is 10 000 km from the center of the earth. Calculate its true anomaly.
{Ans.: 51°}



2.42 A space vehicle has a velocity of 10 km/s and a flight path angle of 20° when it is 15 000 km from the center of the earth. Calculate its true anomaly.
{Ans.: 27.5°}

2.43 For a spacecraft trajectory around the earth, $r = 10{,}000$ km when $\theta = 30°$, and $r = 30{,}000$ km when $\theta = 105°$. Calculate the eccentricity.
{Ans.: 1.22}

Section 2.11

2.44 At a given instant, a spacecraft has the position and velocity vectors $\mathbf{r}_0 = 7000\hat{\mathbf{i}}$ (km) and $\mathbf{v}_0 = 7\hat{\mathbf{i}} + 7\hat{\mathbf{j}}$ (km/s) relative to an earth-centered non-rotating frame. (a) What is the position vector after the true anomaly increases by 90°? (b) What is the true anomaly of the initial point?
{Ans.: (a) $\mathbf{r} = 43{,}180\,\hat{\mathbf{j}}$ (km); (b) 99.21°}

2.45 Relative to an earth-centered, non-rotating frame the position and velocity vectors of a spacecraft are $\mathbf{r}_0 = 3450\hat{\mathbf{i}} - 1700\hat{\mathbf{j}} + 7750\hat{\mathbf{k}}$ (km) and $\mathbf{v}_0 = 5.4\hat{\mathbf{i}} - 5.4\hat{\mathbf{j}} + 1.0\hat{\mathbf{k}}$ (km/s), respectively. (a) Find the distance and speed of the spacecraft after the true anomaly changes by 82°. (b) Verify that the specific angular momentum h and total energy ε are conserved.
{Ans.: (a) $r = 19{,}266$ km, $v = 2.925$ km/s.}

2.46 Relative to an earth-centered, non-rotating frame the position and velocity vectors of a spacecraft are $\mathbf{r}_0 = 6320\hat{\mathbf{i}} + 7750\hat{\mathbf{k}}$ (km) and $\mathbf{v}_0 = 11\hat{\mathbf{j}}$ (km/s). (a) Find the position vector ten minutes later. (b) Calculate the change in true anomaly over the ten-minute time span.
{Ans.: (a) $\mathbf{r} = 5905.1 - 6442.2 + 7241.2$ (km); (b) 34.6°

Section 2.12

2.47 For the sun-earth system, find the distance of the L_1, L_2 and L_3 Lagrange points from the center of mass of the system.
{Ans.: $x_1 = 151.101 \times 10^6$ km, $x_2 = 148.108 \times 10^6$ km, $x_3 = -149.600 \times 10^6$ km (opposite side of the sun)}

2.48 Write a program like that for Example 2.18 to compute the trajectory of a spacecraft using the restricted three-body equations of motion. Use the program to design a trajectory from earth to earth-moon Lagrange point L_4, starting at a 200 km altitude burnout point. The path should take the coasting spacecraft to within 500 km of L_4 with a relative speed of no more than 1 km/s.

List of Key Terms

aiming radius
apoapsis
apse line
azimuth component of velocity
characteristic energy
conservation of angular momentum
conservation of energy
earth's inertial angular velocity
eccentricity
eccentricity vector
fundamental equation of relative two-body motion
geostationary equatorial orbit (GEO)
GEO altitude
GEO speed
gravitational parameter
gravitational potential energy
hyperbolic excess speed
Kepler's first law
Kepler's second law
Kepler's third law
Keplerian orbits
Lagrange f and g coefficients
Lagrange f and g coefficients in terms of the true anomaly
Lagrange points L_1, L_2 and L_3
Lagrange points L_4 and L_5
local horizon
low earth orbit (LEO)
orbit equation
parameter
periapsis
period of a circular orbit
period of an elliptical orbit
radial component of velocity

rectilinear trajectories
semimajor axis
sidereal day
specific energy of an ellipse
synodic day
semi-latus rectum
specific relative angular momentum
state vector
total specific energy of a circular orbit
true anomaly
true anomaly of the asymptote
turn angle
vis-viva equation

CHAPTER

Orbital position as a function of time

3

Chapter outline

3.1	Introduction	155
3.2	Time since periapsis	155
3.3	Circular orbits ($e = 0$)	156
3.4	Elliptical orbits ($e < 1$)	157
3.5	Parabolic trajectories ($e = 1$)	172
3.6	Hyperbolic trajectories ($e < 1$)	174
3.7	Universal variables	182

3.1 INTRODUCTION

In Chapter 2 we found the relationship between position and true anomaly for the two-body problem. The only place time appeared explicitly was in the expression for the period of an ellipse. Obtaining position as a function of time is a simple matter for circular orbits. For elliptical, parabolic and hyperbolic paths we are led to the various forms of Kepler's equation relating position to time. These transcendental equations must be solved iteratively using a procedure like Newton's method, which is presented and illustrated in this chapter.

The different forms of Kepler's equation are combined into a single universal Kepler's equation by introducing universal variables. Implementation of this appealing notion is accompanied by the introduction of an unfamiliar class of functions known as Stumpff functions. The universal variable formulation is required for the Lambert and Gauss orbit determination algorithms in Chapter 5.

The road map of Appendix B may aid in grasping how the material presented here depends on that of Chapter 2.

3.2 TIME SINCE PERIAPSIS

The orbit formula, $r = (h^2/\mu)/(1 + e\cos\theta)$, gives the position of body m_2 in its orbit around m_1 as a function of the true anomaly. For many practical reasons we need to be able to determine the position of m_2 as a function of time. For elliptical orbits, we have a formula for the period T (Equation 2.83), but we cannot yet calculate the time required to fly between any two true anomalies. The purpose of this section is to come up with the formulas that allow us to do that calculation.

The one equation we have which relates true anomaly directly to time is Equation 2.47, $h = r^2\dot{\theta}$, which can be written

$$\frac{d\theta}{dt} = \frac{h}{r^2}$$

Substituting $r = (h^2/\mu)/(1 + e\cos\theta)$ we find, after separating variables,

$$\frac{\mu^2}{h^3}dt = \frac{d\theta}{(1 + e\cos\theta)^2}$$

Integrating both sides of this equation yields

$$\frac{\mu^2}{h^3}(t - t_p) = \int_0^\theta \frac{d\vartheta}{(1 + e\cos\vartheta)^2} \tag{3.1}$$

in which the constant of integration t_p is the time at periapsis passage, where by definition $\theta = 0$. t_p is the sixth constant of the motion that was missing in Chapter 2. The origin of time is arbitrary. It is convenient to measure time from periapsis passage, so we will usually set $t_p = 0$. In that case the ***time versus true anomaly*** integral is

$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{d\vartheta}{(1 + e\cos\vartheta)^2} \tag{3.2}$$

The integral on the right may be found in any standard mathematical handbook, such as Beyer (1991), in which we find:

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{1}{(a^2 - b^2)^{3/2}}\left(2a\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2} - \frac{b\sqrt{a^2 - b^2}\sin x}{a + b\cos x}\right) \quad (b < a) \tag{3.3}$$

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{1}{a^2}\left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{6}\tan^3\frac{x}{2}\right) \quad (b = a) \tag{3.4}$$

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{1}{(b^2 - a^2)^{3/2}}\left[\frac{b\sqrt{b^2 - a^2}\sin x}{a + b\cos x} - a\ln\left(\frac{\sqrt{b+a} + \sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a} - \sqrt{b-a}\tan\frac{x}{2}}\right)\right] \quad (b > a) \tag{3.5}$$

3.3 CIRCULAR ORBITS ($e = 0$)

If $e = 0$ the integral in Equation 3.2 is simply $\int_0^\theta d\vartheta$, which means

$$t = \frac{h^3}{\mu^2}\theta$$

Recall that for a circle (Equation 2.62), $r = h^2/\mu$. Therefore $h^3 = r^{3/2}\mu^{3/2}$, so that

$$t = \frac{r^{\frac{3}{2}}}{\sqrt{\mu}}\theta$$

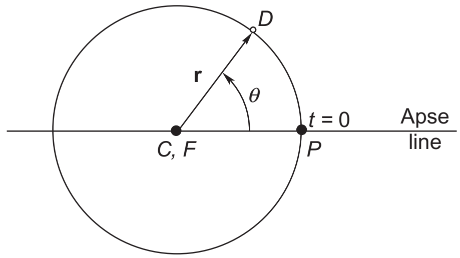


FIGURE 3.1

Time since periapsis is directly proportional to true anomaly in a circular orbit.

Finally, substituting the formula (Equation 2.64) for the period T of a circular orbit, $T = 2\pi r^{\frac{3}{2}}/\sqrt{\mu}$, yields

$$t = \frac{\theta}{2\pi}T$$

or,

$$\theta = \frac{2\pi}{T}t$$

The reason that t is directly proportional to θ in a circular orbit is simply that the angular velocity $2\pi/T$ is constant. Therefore the time Δt to fly through a true anomaly of $\Delta\theta$ is $(\Delta\theta/2\pi)T$.

Because the circle is symmetric about any diameter, the apse line—and therefore the periapsis—can be chosen arbitrarily.

3.4 ELLIPTICAL ORBITS ($e < 1$)

Set $a = 1$ and $b = e$ in Equation 3.3 to obtain

$$\int_0^\theta \frac{d\vartheta}{(1 + e\cos\vartheta)^2} = \frac{1}{(1-e^2)^{\frac{3}{2}}}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

Therefore, Equation 3.2 in this case becomes

$$\frac{\mu^2}{h^3}t = \frac{1}{(1-e^2)^{\frac{3}{2}}}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

or

$$M_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \tag{3.6}$$

where

$$M_e = \frac{\mu^2}{h^3}(1-e^2)^{\frac{3}{2}}t \tag{3.7}$$

M_e is called the *mean anomaly*. The subscript e reminds us this is ***mean anomaly for the ellipse*** and not for parabolas and hyperbolas, which have their own "mean anomaly" formulas. Equation 3.6 is plotted in

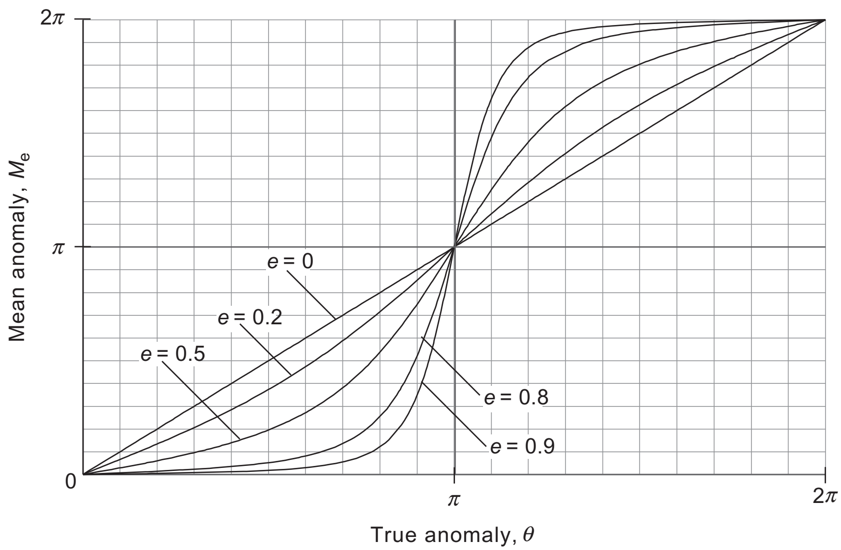


FIGURE 3.2

Mean anomaly versus true anomaly for ellipses of various eccentricities.

Figure 3.2. Observe that for all values of the eccentricity e, M_e is a monotonically increasing function of the true anomaly θ.

From Equation 2.82, the formula for the period T of an elliptical orbit, we have $\mu^2(1-e^2)^{3/2}/h^3 = 2\pi/T$, so that the mean anomaly in Equation 3.7 can be written much more simply as

$$M_e = \frac{2\pi}{T}t \tag{3.8}$$

The angular velocity of the position vector of an elliptical orbit is not constant, but since 2π radians are swept out per period T, the ratio $2\pi/T$ is the average angular velocity, which is given the symbol n and called the ***mean motion***,

$$n = \frac{2\pi}{T} \tag{3.9}$$

In terms of the mean motion, Equation 3.5 can be written simpler still as

$$M_e = nt$$

The mean anomaly is the azimuth position (in radians) of a fictitious body moving around the ellipse at the constant angular speed n. For a circular orbit, the mean anomaly M_e and the true anomaly θ are identical.

It is convenient to simplify Equation 3.6 by introducing an auxiliary angle E called the ***eccentric anomaly***, which is shown in Figure 3.3. This is done by circumscribing the ellipse with a concentric auxiliary circle having a radius equal to the semimajor axis a of the ellipse. Let S be that point on the ellipse whose true anomaly is θ. Through point S we pass a perpendicular to the apse line, intersecting the auxiliary circle at

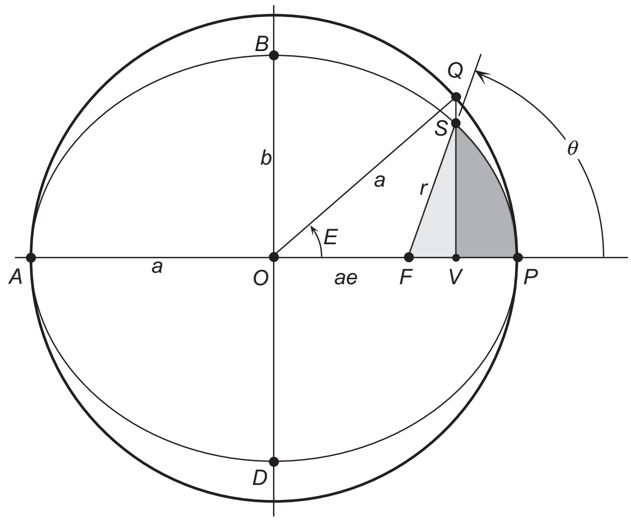


FIGURE 3.3

Ellipse and the circumscribed auxiliary circle.

point Q and the apse line at point V. The angle between the apse line and the radius drawn from the center of the circle to Q on its circumference is the eccentric anomaly E. Observe that E lags θ from periapsis P to apoapsis A ($0 \leq \theta < 180°$) whereas it leads θ from A to P ($180° \leq \theta < 360°$).

To find E as a function of θ, we first observe from Figure 3.3 that, in terms of the eccentric anomaly, $\overline{OV} = a\cos E$ whereas in terms of the true anomaly, $\overline{OV} = ae + r\cos\theta$. Thus,

$$a\cos E = ae + r\cos\theta$$

Using Equation 2.72, $r = a(1 - e^2)/(1 + e\cos\theta)$, we can write this as

$$a\cos E = ae + \frac{a(1 - e^2)\cos\theta}{1 + e\cos\theta}$$

Simplifying the right-hand side, we get

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} \tag{3.10a}$$

Solving this for $\cos\theta$ we obtain the inverse relation

$$\cos\theta = \frac{e - \cos E}{e\cos E - 1} \tag{3.10b}$$

Substituting Equation 3.10a into the trigonometric identity $\sin^2 E + \cos^2 E = 1$ and solving for $\sin E$ yields

$$\sin E = \frac{\sqrt{1 - e^2}\sin\theta}{1 + e\cos\theta} \tag{3.11}$$

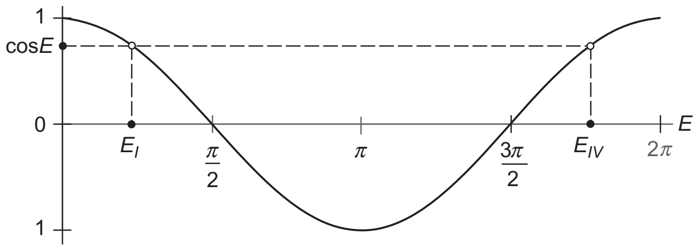


FIGURE 3.4

For $0 < \cos E < 1$, E can lie in the first or fourth quadrant. For $-1 < \cos E < 0$, E can lie in the second or third quadrant.

Equation 3.10a would be fine for obtaining E from θ, except that, given a value of cos E between –1 and 1, there are two values of E between 0 and 360°, as illustrated in Figure 3.4. The same comments hold for Equation 3.11. To resolve this quadrant ambiguity, we use the following trigonometric identity,

$$\tan^2\frac{E}{2} = \frac{\sin^2 E/2}{\cos^2 E/2} = \frac{\dfrac{1-\cos E}{2}}{\dfrac{1+\cos E}{2}} = \frac{1-\cos E}{1+\cos E} \tag{3.12}$$

From Equation 3.10a

$$1-\cos E = \frac{1-\cos\theta}{1+e\cos\theta}(1-e) \qquad \text{and} \qquad 1+\cos E = \frac{1+\cos\theta}{1+e\cos\theta}(1+e)$$

Therefore,

$$\tan^2\frac{E}{2} = \frac{1-e}{1+e}\,\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-e}{1+e}\tan^2\frac{\theta}{2}$$

where the last step required applying the trig identity in Equation 3.12 to the term $(1-\cos\theta)/(1+\cos\theta)$. Finally, therefore, we obtain

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} \tag{3.13a}$$

or

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) \tag{3.13b}$$

Observe from Figure 3.5 that for any value of tan (E/2), there is only one value of E between 0 and 360°. There is no quadrant ambiguity.

Substituting Equations 3.11 and 3.13b into Equation 3.6 yields ***Kepler's equation***,

$$\boxed{M_e = E - e\sin E} \tag{3.14}$$

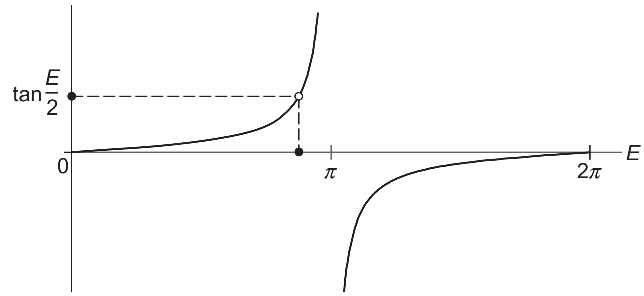


FIGURE 3.5

To any value of tan (E/2) there corresponds a unique value of E in the range 0 to 2π.

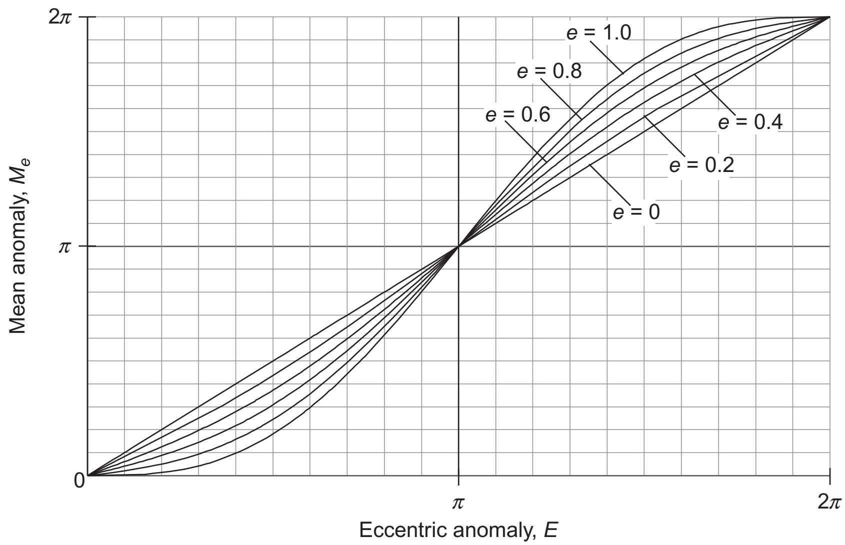


FIGURE 3.6

Plot of Kepler's equation for an elliptical orbit.

This monotonically increasing relationship between mean anomaly and eccentric anomaly is plotted for several values of eccentricity in Figure 3.6.

Given the true anomaly θ, we calculate the eccentric anomaly E using Equations 3.13. Substituting E into Kepler's formula, Equation 3.14, yields the mean anomaly directly. From the mean anomaly and the period T we find the time (since periapsis) from Equation 3.5,

$$t = \frac{M_e}{2\pi}T \tag{3.15}$$

On the other hand, if we are given the time, then Equation 3.15 yields the mean anomaly M_e. Substituting M_e into Kepler's equation, we get the following expression for the eccentric anomaly:

$$E - e\sin E = M_e$$

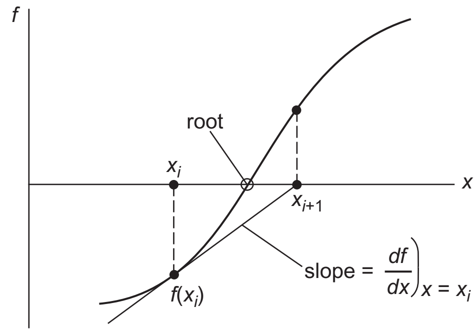


FIGURE 3.7

Newton's method for finding a root of $f(x) = 0$.

We cannot solve this transcendental equation directly for E. A rough value of E might be read off Figure 3.6. However, an accurate solution requires an iterative, "trial and error" procedure.

Newton's method, or one of its variants, is one of the more common and efficient ways of finding the root of a well-behaved function. To find a root of the equation $f(x) = 0$ in Figure 3.7, we estimate it to be x_i, and evaluate the function $f(x)$ and its first derivative $f'(x)$ at that point. We then extend the tangent to the curve at $f(x_i)$ until it intersects the x-axis at x_{i+1}, which becomes our updated estimate of the root. The intercept x_{i+1} is found by setting the slope of the tangent line equal to the slope of the curve at x_i, that is,

$$f'(x_i) = \frac{0 - f(x_i)}{x_{i+1} - x_i}$$

from which we obtain

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{3.16}$$

The process is repeated, using x_{i+1} to estimate x_{i+2}, and so on, until the root has been found to the desired level of precision.

To apply Newton's method to the solution of Kepler's equation, we form the function

$$f(E) = E - e\sin E - M_e$$

and seek the value of eccentric anomaly that makes $f(E) = 0$. Since

$$f'(E) = 1 - e\cos E$$

for this problem Equation 3.16 becomes

$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M_e}{1 - e\cos E_i} \tag{3.17}$$

Algorithm 3.1 Solve Kepler's equation for the eccentric anomaly E given the eccentricity e and the mean anomaly M_e. See Appendix D.11 for the implementation of this algorithm in MATLAB®.

1. Choose an initial estimate of the root E as follows (Prussing and Conway, 1993). If $M_e < \pi$, then $E = M_e + e/2$. If $M_e > \pi$, then $E = M_e - e/2$. Remember that the angles E and M_e are in radians. (When using a hand-held calculator, be sure it is in radian mode.)
2. At any given step, having obtained E_i from the previous step, calculate

$$f(E_i) = E_i - e\sin E_i - M_e \text{ and } f'(E_i) = 1 - e\cos E_i.$$

3. Calculate $\text{ratio}_i = f(E_i)/f'(E_i)$.
4. If $|\text{ratio}_i|$ exceeds the chosen tolerance (e.g., 10^{-8}), then calculate an updated value of E:

$$E_{i+1} = E_i - \text{ratio}_i$$

   Return to step 2.

5. If $|\text{ratio}_i|$ is less than the tolerance, then accept E_i as the solution to within the chosen accuracy.

Example 3.1

A geocentric elliptical orbit has a perigee radius of 9600 km and an apogee radius of 21,000 km. Calculate the time to fly from perigee P to a true anomaly of 120°.

Solution

Before anything else, let us find the primary orbital parameters e and h. The eccentricity is readily obtained from the perigee and apogee radii by means of Equation 2.84,

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{21{,}000 - 9600}{21{,}000 + 9600} = 0.37255 \quad \text{(a)}$$

We find the angular momentum using the orbit equation, evaluated at perigee:

$$9600 = \frac{h^2}{398{,}600}\frac{1}{1 + 0.37255\cos(0)} \Rightarrow h = 72{,}472 \text{ km}^2/\text{s}$$

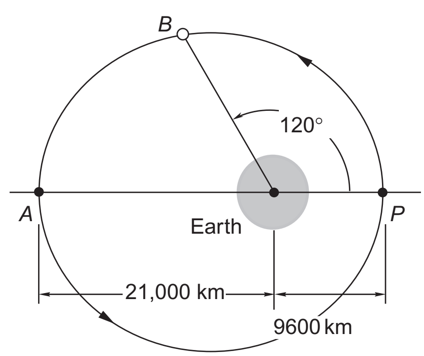


FIGURE 3.8

Geocentric elliptical orbit.

With h and e, the period of the orbit is obtained from Equation 2.82,

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398{,}600^2}\left(\frac{72{,}472}{\sqrt{1-0.37255^2}}\right)^3 = 18{,}834 \text{ s} \tag{b}$$

Equation 3.11a yields the eccentric anomaly from the true anomaly,

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} = \sqrt{\frac{1-0.37255}{1+0.37255}}\tan\frac{120°}{2} = 1.1711 \Rightarrow E = 1.7281 \text{ rad.}$$

Then Kepler's equation, Equation 3.14, is used to find the mean anomaly,

$$M_e = 1.7281 - 0.37255\sin 1.7281 = 1.3601 \text{ rad.}$$

Finally, the time follows from Equation 3.12,

$$t = \frac{M_e}{2\pi}T = \frac{1.3601}{2\pi}18{,}834 = \boxed{4077 \text{ s } (1.132 \text{ hr})}$$

Example 3.2

In the previous example, find the true anomaly at three hours after perigee passage.

Solution

Since the time (10,800 seconds) is greater than one-half the period, the true anomaly must be greater than 180°.

First, we use Equation 3.12 to calculate the mean anomaly for $t = 10{,}800$ s.

$$M_e = 2\pi\frac{t}{T} = 2\pi\frac{10{,}800}{18{,}830} = 3.6029 \text{ rad.} \tag{a}$$

Kepler's equation, $E - e\sin(E) = M_e$ (with all angles in radians) is then employed to find the eccentric anomaly. This transcendental equation will be solved using Algorithm 3.1 with an error tolerance of 10^{-6}. Since $M_e > \pi$, a good starting value for the iteration is $E_0 = M_e - e/2 = 3.4166$. Executing the algorithm yields the following steps:

Step 1:

$$E_0 = 3.4166$$

$$f(E_0) = -0.085124$$

$$f'(E_0) = 1.3585$$

$$\text{ratio} = \frac{-0.085124}{1.3585} = -0.062658$$

$|\text{ratio}| > 10^{-6}$, so repeat.

Step 2:

$$E_1 = 3.4166 - (-0.062658) = 3.4793$$

$$f(E_1) = -0.0002134$$

$$f'(E_1) = 1.3515$$

$$\text{ratio} = \frac{-0.0002134}{1.3515} = -1.5778 \times 10^{-4}$$

$|\text{ratio}| > 10^{-6}$, so repeat.

Step 3:

$$E_2 = 3.4793 - (-1.5778 \times 10^{-4}) = 3.4794$$

$$f(E_2) = -1.5366 \times 10^{-9}$$

$$f'(E_2) = 1.3515$$

$$\text{ratio} = \frac{-1.5366 \times 10^{-9}}{1.3515} - 1.137 \times 10^{-9}$$

$|\text{ratio}| < 10^{-6}$, so accept $E = 3.4794$ as the solution.

Convergence to even more than the desired accuracy occurred after just two iterations. With this value of the eccentric anomaly, the true anomaly is found from Equation 3.13a:

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} = \sqrt{\frac{1+0.37255}{1-0.37255}}\tan\frac{3.4794}{2} = -8.6721 \Rightarrow \boxed{\theta = 193.2^\circ}$$

Example 3.3

Let a satellite be in a 500 km by 5000 km orbit with its apse line parallel to the line from the earth to the sun, as shown in Figure 3.9. Find the time that the satellite is in the earth's shadow if: (a) the apogee is towards the sun; (b) the perigee is towards the sun.

Solution

We start by using the given data to find the primary orbital parameters, e and h. The eccentricity is obtained from Equation 2.84,

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{(6378 + 5000) - (6378 + 500)}{(6378 + 5000) + (6378 + 500)} = 0.24649 \qquad \text{(a)}$$

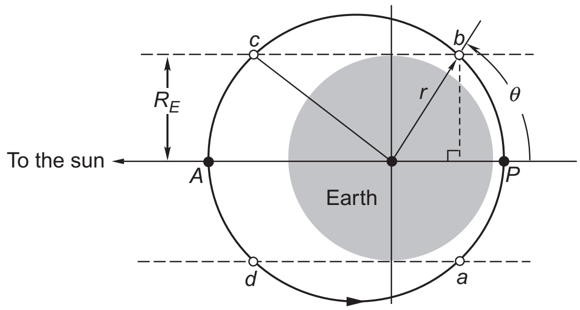


FIGURE 3.9

Satellite passing in and out of the earth's shadow.

The orbit equation can then be used to find the angular momentum

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e\cos(0)} \Rightarrow 6878 = \frac{h^2}{398{,}600}\frac{1}{1+0.24649} \Rightarrow h = 58{,}458 \text{ km}^2/\text{s} \tag{b}$$

The semimajor axis may be found from Equation 2.71,

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} = \frac{58{,}458^2}{398{,}600}\frac{1}{1-0.24649^2} = 9128 \text{ km} \tag{c}$$

or from the fact that $a = (r_p + r_a)/2$. The period of the orbit follows from Equation 2.83

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{398{,}600}}9128^{3/2} = 8679.1 \text{ s } (2.4109 \text{ h}) \tag{d}$$

(a) If the apogee is towards the sun, as in Figure 3.9, then the satellite is in earth's shadow between points a and b on its orbit. These are two of the four points of intersection of the orbit with the lines that are parallel to the earth-sun line and lie at a distance R_E from the center of the earth. The true anomaly of b is therefore given by $\sin\theta = R_E/r$, where r is the radial position of the satellite. It follows that the radius of b is:

$$r = \frac{R_E}{\sin\theta} \tag{e}$$

From Equation 2.72 we also have

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{f}$$

Substituting (e) into (f), collecting terms and simplifying yields an equation in θ,

$$e\cos\theta - (1-e^2)\frac{a}{R_E}\sin\theta + 1 = 0 \tag{g}$$

Substituting (a) and (c) together with $R_E = 6378\,\text{km}$ into (g) yields

$$0.24649\cos\theta - 1.3442\sin\theta = -1 \tag{h}$$

This equation is of the form

$$A\cos\theta + B\sin\theta = C \tag{i}$$

It has two roots, which are given by (see Problem 3.12 at the end of the chapter):

$$\theta = \tan^{-1}\frac{B}{A} \pm \cos^{-1}\left[\frac{C}{A}\cos\left(\tan^{-1}\frac{B}{A}\right)\right] \tag{j}$$

For the case at hand,

$$\begin{aligned}\theta &= \tan^{-1}\frac{-1.3442}{0.24649} \pm \cos^{-1}\left[\frac{-1}{0.24649}\cos\left(\tan^{-1}\frac{-1.3442}{0.24649}\right)\right]\\ &= -79.607^\circ \pm 137.03^\circ\end{aligned}$$

That is,

$$\begin{aligned}\theta_b &= 57.423^\circ\\ \theta_c &= -216.64^\circ \quad (+143.36^\circ)\end{aligned} \tag{k}$$

For apogee towards the sun, the flight from perigee to point b will be in shadow. To find the time of flight from perigee to point b, we first compute the eccentric anomaly of b using Equation 3.13b:

$$E_b = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_b}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.24649}{1+0.24649}}\tan\frac{1.0022}{2}\right) = 0.80521\text{ rad.} \tag{l}$$

From this we find the mean anomaly using Kepler's equation,

$$M_e = E - e\sin E = 0.80521 - 0.24649\sin 0.80521 = 0.62749\text{ rad.} \tag{m}$$

Finally, Equation (3.5) yields the time at b,

$$t_b = \frac{M_e}{2\pi}T = \frac{0.62749}{2\pi}8679.1 = 866.77\text{ s.} \tag{n}$$

The total time in shadow, from a to b, during which the satellite passes through perigee, is

$$\boxed{t = 2t_b = 1734\text{ sec. (28.98 min.)}} \tag{o}$$

(b) If the perigee is towards the sun, then the satellite is in shadow near apogee, from point c ($\theta_c = 143.36^\circ$) to d on the orbit. Following the same procedure as above, we obtain (see Problem 3.13):

$$\begin{aligned}E_c &= 2.3364\text{ rad.}\\ M_c &= 2.1587\text{ rad.}\\ t_c &= 2981.8\text{ sec.}\end{aligned} \tag{p}$$

The total time in shadow, from c to d, is:

$$\boxed{t = T - 2t_c = 8679.1 - 2\cdot 2891.8 = 2716 \text{ sec (45.26 min.)}} \tag{q}$$

The time is longer than that given by (n) since the satellite travels slower near apogee.

We have observed that there is no closed form solution for the eccentric anomaly E in Kepler's equation, $E - e\sin E = M_e$. However, there exist infinite series solutions. One of these, due to Lagrange (Battin, 1999), is a power series in the eccentricity e,

$$E = M_e + \sum_{n=1}^{\infty} a_n e^n \tag{3.18}$$

where the coefficients a_n are given by the somewhat intimidating expression

$$a_n = \frac{1}{2^{n-1}} \sum_{k=0}^{floor(n/2)} (-1)^k \frac{1}{(n-k)!k!}(n-2k)^{n-1} \sin\left[(n-2k)M_e\right] \tag{3.19}$$

Here, *floor* (x) means x rounded to the next lowest integer [e.g., *floor* (0.5) = 0, *floor* (π) = 3]. If e is sufficiently small, then the Lagrange series converges. That means by including enough terms in the summation, we can obtain E to any desired degree of precision. Unfortunately, if e exceeds 0.6627434193, the series diverges, which means taking more and more terms yields worse and worse results for some values of M.

The limiting value for the eccentricity was discovered by the French mathematician Pierre-Simone Laplace (1749–1827) and is called the *Laplace limit*.

In practice, we must truncate the Lagrange series to a finite number of terms N, so that

$$E = M_e + \sum_{n=1}^{N} a_n e^n \tag{3.20}$$

For example, setting $N = 3$ and calculating each a_n by means of Equation 3.19 leads to

$$E = M_e + e\sin M_e + \frac{e^2}{2}\sin 2M_e + \frac{e^3}{8}(3\sin 3M_e - \sin M_e) \tag{3.21}$$

For small values of the eccentricity e, this yields good agreement with the exact solution of Kepler's equation (plotted in Figure 3.6). However, as we approach the Laplace limit, the accuracy degrades unless more terms of the series are included. Figure 3.10 shows that for an eccentricity of 0.65, just below the Laplace limit, Equation 3.21 ($N = 3$) yields a solution that oscillates around the exact solution, but is fairly close to it everywhere. Setting $N = 10$ in Equation 3.20 produces a curve that, at the given scale, is indistinguishable from the exact solution. On the other hand, for an eccentricity of 0.90, far above the Laplace limit, Figure 3.11 reveals that Equation 3.21 is a poor approximation to the exact solution, and using $N = 10$ makes matters even worse.

Another infinite series for E (Battin, 1999) is given by

$$E = M_e + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne)\sin nM_e \tag{3.22}$$

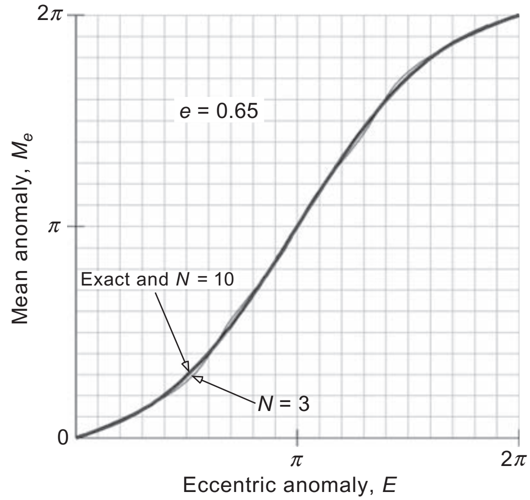


FIGURE 3.10

Comparison of the exact solution of Kepler's equation with the truncated Lagrange series solution ($N = 3$ and $N = 10$) for an eccentricity of 0.65.

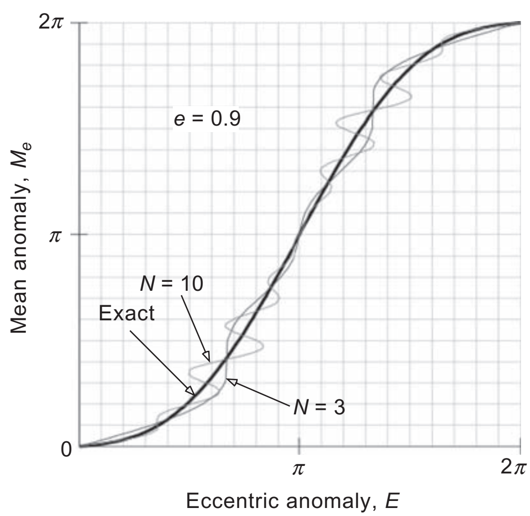


FIGURE 3.11

Comparison of the exact solution of Kepler's equation with the truncated Lagrange series solution ($N = 3$ and $N = 10$) for an eccentricity of 0.90.

where the coefficients J_n are Bessel functions of the first kind, defined by the infinite series

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k} \tag{3.23}$$

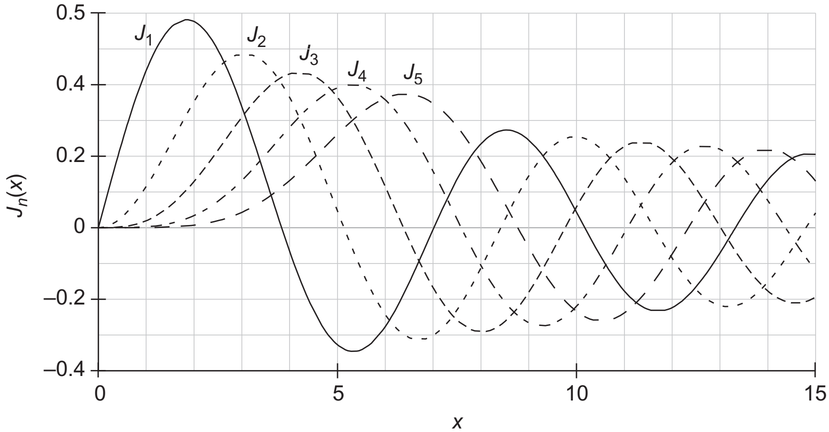


FIGURE 3.12

Bessel functions of the first kind.

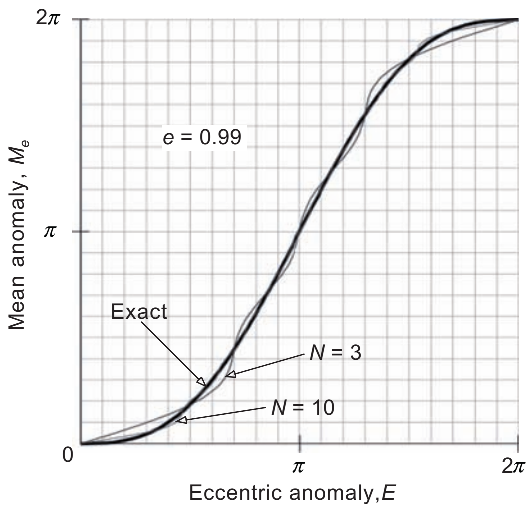


FIGURE 3.13

Comparison of the exact solution of Kepler's equation with the truncated Bessel series solution ($N = 3$ and $N = 10$) for an eccentricity of 0.99.

J_1 through J_5 are plotted in Figure 3.12. Clearly, they are oscillatory in appearance and tend towards zero with increasing x.

It turns out that, unlike the Lagrange series, the Bessel function series solution converges for all values of the eccentricity less than 1. Figure 3.13 shows how the truncated Bessel series solutions

$$E = M_e + \sum_{n=1}^{N} \frac{2}{n} J_n(ne) \sin nM_e \tag{3.24}$$

for $N = 3$ and $N = 10$ compare to the exact solution of Kepler's equation for the very large elliptical eccentricity of $e = 0.99$. It can be seen that the case $N = 3$ yields a poor approximation for all but a few values of M_e. Increasing the number of terms in the series to $N = 10$ obviously improves the approximation, and adding even more terms will make the truncated series solution indistinguishable from the exact solution at the given scale.

Observe that we can combine Equations 3.10 and 2.72 as follows to obtain the ***orbit equation for the ellipse in terms of the eccentric anomaly***

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{a(1-e^2)}{1+e\left(\dfrac{e-\cos E}{e\cos E-1}\right)}$$

From this it is easy to see that:

$$r = a(1 - e\cos E) \tag{3.25}$$

In Equation 2.86 we defined the true-anomaly-averaged radius $\overline{r}_\theta$ of an elliptical orbit. Alternatively, the time-averaged radius $\overline{r}_t$ of an elliptical orbit is defined as

$$\overline{r}_t = \frac{1}{T}\int_0^T r\,dt \tag{3.26}$$

According to Equations 3.14 and 3.15,

$$t = \frac{T}{2\pi}(E - e\sin E)$$

Therefore,

$$dt = \frac{T}{2\pi}(1 - e\cos E)\,dE$$

Upon using this relationship to change the variable of integration from t to E and substituting Equation 3.25, Equation 3.26 becomes

$$\begin{aligned}
\overline{r}_t &= \frac{1}{T}\int_0^{2\pi}[a(1-e\cos E)]\left[\frac{T}{2\pi}(1-e\cos E)\right]dE \\
&= \frac{a}{2\pi}\int_0^{2\pi}(1-e\cos E)^2\,dE \\
&= \frac{a}{2\pi}\int_0^{2\pi}(1-2e\cos E+e^2\cos^2 E)\,dE \\
&= \frac{a}{2\pi}(2\pi - 0 + e^2\pi)
\end{aligned}$$

so that

$$\bar{r}_t = a\left(1 + \frac{e^2}{2}\right) \quad \text{Time-averaged radius of an elliptical orbit.} \tag{3.27}$$

Comparing this result with Equation 2.87 reveals, as we should have expected (Why?), that $\bar{r}_t > \bar{r}_\theta$. In fact, combining Equations 2.87 and 3.27 yields

$$\bar{r}_\theta = a\sqrt{3 - 2\frac{\bar{r}_t}{a}} \tag{3.28}$$

3.5 PARABOLIC TRAJECTORIES ($e = 1$)

For the parabola, Equation 3.2 becomes

$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{d\vartheta}{(1 + \cos\vartheta)^2} \tag{3.29}$$

Setting $a = b = 1$ in Equation 3.4 yields

$$\int_0^\theta \frac{d\vartheta}{(1 + \cos\vartheta)^2} = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}$$

Therefore, Equation 3.29 may be written as

$$\boxed{M_p = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}} \tag{3.30}$$

where

$$M_p = \frac{\mu^2 t}{h^3} \tag{3.31}$$

M_p is dimensionless, and it may be thought of as the mean anomaly for the parabola (***parabolic mean anomaly***). Equation 3.30, which plays the role of Kepler's equation for parabolic trajectories, is also known as ***Barker's equation***. It is plotted in Figure 3.14.

There is no "eccentric anomaly" for the parabola. Given the true anomaly θ, we find the time directly from Equations 3.30 and 3.31. If time is the given variable, then we must solve the cubic equation

$$\frac{1}{6}\left(\tan\frac{\theta}{2}\right)^3 + \frac{1}{2}\tan\frac{\theta}{2} - M_p = 0$$

which has but one real root, namely,

$$\tan\frac{\theta}{2} = \left(3M_p + \sqrt{(3M_p)^2 + 1}\right)^{\frac{1}{3}} - \left(3M_p + \sqrt{(3M_p)^2 + 1}\right)^{-\frac{1}{3}} \tag{3.32}$$

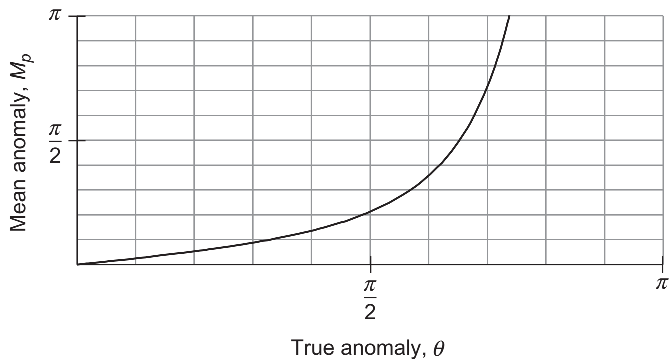


FIGURE 3.14

Graph of Equation 3.30.

Example 3.4

A geocentric parabola has a perigee velocity of 10 km/s. How far is the satellite from the center of the earth six hours after perigee passage?

Solution

The first step is to find the orbital parameters e and h. Of course we know that $e = 1$ for a parabola. To get the angular momentum, we can use the given perigee speed and Equation 2.90 (the energy equation) to find the perigee radius,

$$r_p = \frac{2\mu}{v_p^2} = \frac{2 \cdot 398{,}600}{10^2} = 7972 \text{ km}$$

It follows from Equation 2.31 that the angular momentum is

$$h = r_p v_p = 7972 \cdot 10 = 79{,}720 \text{ km}^2/\text{s}$$

We can now calculate the parabolic mean anomaly by means of Equation 3.31,

$$M_p = \frac{\mu^2 t}{h^3} = \frac{398{,}600^2 \cdot (6 \cdot 3600)}{79{,}720^3} = 6.7737 \text{ rad}$$

Therefore, $3M_p = 20.321$ rad, which, when substituted into Equation 3.32, yields the true anomaly,

$$\tan\frac{\theta}{2} = \left(20.321 + \sqrt{20.321^2 + 1}\right)^{\frac{1}{3}} - \left(20.321 + \sqrt{20.321^2 + 1}\right)^{-\frac{1}{3}} = 3.1481 \Rightarrow \theta = 144.75°$$

Finally, we substitute the true anomaly into the orbit equation to find the radius,

$$r = \frac{79{,}720^2}{398{,}600}\frac{1}{1 + \cos(144.75°)} = \boxed{86{,}899 \text{ km}}$$

3.6 HYPERBOLIC TRAJECTORIES ($e < 1$)

Setting $a = 1$ and $b = e$ in Equation 3.5 yields:

$$\int_0^\theta \frac{d\vartheta}{(1+e\cos\vartheta)^2} = \frac{1}{e^2-1}\left[\frac{e\sin\theta}{1+e\cos\theta} - \frac{1}{\sqrt{e^2-1}}\ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right)\right]$$

Therefore, for the hyperbola Equation 3.1 becomes

$$\frac{\mu^2}{h^3}t = \frac{1}{e^2-1}\frac{e\sin\theta}{1+e\cos\theta} - \frac{1}{(e^2-1)^{\frac{3}{2}}}\ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right)$$

Multiplying both sides by $(e^2-1)^{3/2}$, we get

$$M_h = \frac{e\sqrt{e^2-1}\sin\theta}{1+e\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right) \tag{3.33}$$

where

$$M_h = \frac{\mu^2}{h^3}(e^2-1)^{\frac{3}{2}}t \tag{3.34}$$

M_h is the ***hyperbolic mean anomaly***. Equation 3.33 is plotted in Figure 3.15. Recall that θ cannot exceed θ_∞ (see Equation 2.97).

We can simplify Equation 3.33 by introducing an auxiliary angle analogous to the eccentric anomaly E for the ellipse. Consider a point on a hyperbola whose polar coordinates are r and θ. Referring to Figure 3.16, let x be the horizontal distance of the point from the center C of the hyperbola, and let y be its distance

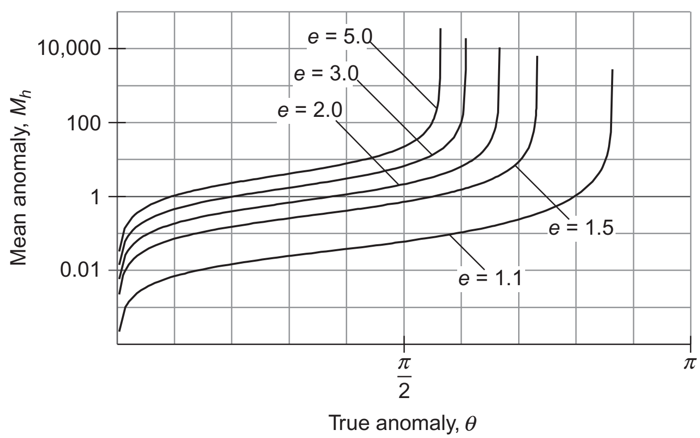


FIGURE 3.15

Plots of Equation 3.33 for several different eccentricities.

above the apse line. The ratio y/b defines the hyperbolic sine of the dimensionless variable F that we will use as the hyperbolic eccentric anomaly. That is, we define F to be such that

$$\sinh F = \frac{y}{b} \tag{3.35}$$

In view of the equation of a hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

it is consistent with the definition of sinh F to define the hyperbolic cosine as

$$\cosh F = \frac{x}{a} \tag{3.36}$$

(It should be recalled that $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$ and, therefore, that $\cosh^2 x - \sinh^2 x = 1$.)

From Figure 3.16 we see that $y = r\sin\theta$. Substituting this into Equation 3.35, along with $r = a(e^2 - 1)/(1 + e\cos\theta)$ (Equation 2.104) and $b = a\sqrt{e^2 - 1}$ (Equation 2.106), we get

$$\sinh F = \frac{1}{b} r\sin\theta = \frac{1}{a\sqrt{e^2 - 1}} \frac{a(e^2 - 1)}{1 + e\cos\theta}\sin\theta$$

so that

$$\sinh F = \frac{\sqrt{e^2 - 1}\sin\theta}{1 + e\cos\theta} \tag{3.37}$$

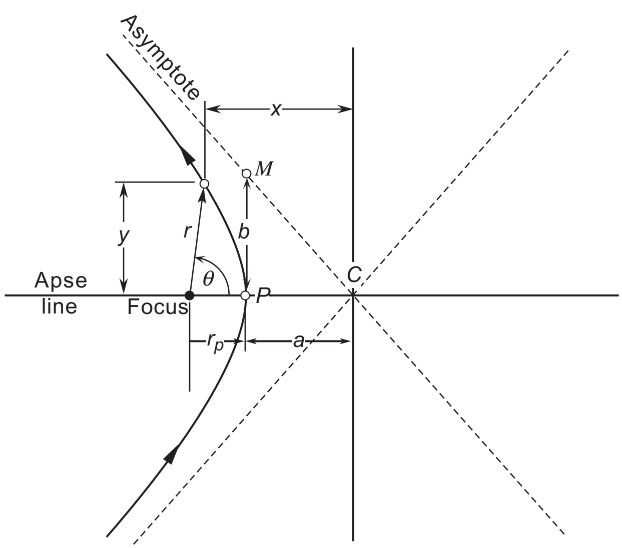


FIGURE 3.16

Hyperbolic parameters.

This can be used to solve for F in terms of the true anomaly,

$$F = \sinh^{-1}\left(\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta}\right) \tag{3.38}$$

Using the formula $\sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$, we can, after simplifying the algebra, write Equation 3.38 as

$$F = \ln\left(\frac{\sin\theta\sqrt{e^2-1}+\cos\theta+e}{1+e\cos\theta}\right)$$

Substituting the trigonometric identities

$$\sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} \qquad \cos\theta = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$$

and doing some more algebra yields

$$F = \ln\left[\frac{1+e+(e-1)\tan^2(\theta/2)+2\tan(\theta/2)\sqrt{e^2-1}}{1+e+(1-e)\tan^2(\theta/2)}\right]$$

Fortunately, but not too obviously, the numerator and the denominator in the brackets have a common factor, so that this expression for the hyperbolic eccentric anomaly reduces to

$$F = \ln\left[\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right] \tag{3.39}$$

Substituting Equations 3.37 and 3.39 into Equation 3.33 yields ***Kepler's equation for the hyperbola***,

$$\boxed{M_h = e\sinh F - F} \tag{3.40}$$

This equation is plotted for several different eccentricities in Figure 3.17.

If we substitute the expression for sinh F, Equation 3.37, into the hyperbolic trig identity $\cosh^2 F - \sinh^2 F = 1$, we get

$$\cosh^2 F = 1 + \left(\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta}\right)^2$$

A few steps of algebra lead to

$$\cosh^2 F = \left(\frac{\cos\theta+e}{1+e\cos\theta}\right)^2$$

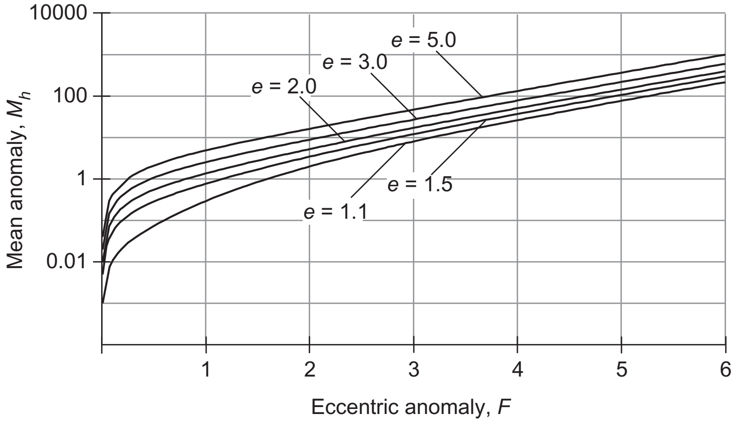


FIGURE 3.17

Plot of Kepler's equation for the hyperbola.

so that

$$\cosh F = \frac{\cos\theta + e}{1 + e\cos\theta} \tag{3.41a}$$

Solving this for cos θ, we obtain the inverse relation,

$$\cos\theta = \frac{\cosh F - e}{1 - e\cosh F} \tag{3.41b}$$

The hyperbolic tangent is found in terms of the hyperbolic sine and cosine by the formula

$$\tanh F = \frac{\sinh F}{\cosh F}$$

In mathematical handbooks we can find the hyperbolic trig identity,

$$\tanh\frac{F}{2} = \frac{\sinh F}{1 + \cosh F} \tag{3.42}$$

Substituting Equations 3.37 and 3.41a into this formula and simplifying yields

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\frac{\sin\theta}{1 + \cos\theta} \tag{3.43}$$

Interestingly enough, Equation 3.42 holds for ordinary trig functions, too; that is,

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}$$

Therefore, Equation 3.43 can be written

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2} \tag{3.44a}$$

This is a somewhat simpler alternative to Equation 3.39 for computing eccentric anomaly from true anomaly, and it is a whole lot simpler to invert:

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} \tag{3.44b}$$

If time is the given quantity, then Equation 3.40—a transcendental equation—must be solved for F by an iterative procedure, as was the case for the ellipse. To apply Newton's procedure to the solution of Kepler's equation for the hyperbola, we form the function

$$f(F) = e\sinh F - F - M_h$$

and seek the value of F that makes $f(F) = 0$. Since

$$f'(F) = e\cosh F - 1$$

Equation 3.16 becomes

$$F_{i+1} = F_i - \frac{e\sinh F_i - F_i - M_h}{e\cosh F_i - 1} \tag{3.45}$$

All quantities in this formula are dimensionless (radians, not degrees).

Algorithm 3.2 Solve Kepler's equation for the hyperbola for the hyperbolic eccentric anomaly F given the eccentricity e and the hyperbolic mean anomaly M_h. See Appendix D.12 for the implementation of this algorithm in MATLAB.

1. Choose an initial estimate of the root F.
   a. For hand computations read a rough value of F_0 (no more than two significant figures) from Figure 3.17 in order to keep the number of iterations to a minimum.
   b. In computer software let $F_0 = M_h$, an inelegant choice which may result in many iterations but will nevertheless rapidly converge on today's high speed desktop and laptop computers.
2. At any given step, having obtained F_i from the previous step, calculate $f(F_i) = e\sinh F_i - F_i - M_h$ and $f'(F_i) = e\cosh F_i - 1$.
3. Calculate $\text{ratio}_i = f(F_i)/f'(F_i)$.
4. If $|\text{ratio}_i|$ exceeds the chosen tolerance (e.g., 10^{-8}), then calculate an updated value of F:

   $$F_{i+1} = F_i - \text{ratio}_i$$

   Return to step 2.
5. If $|\text{ratio}_i|$ is less than the tolerance, then accept F_i as the solution to within the desired accuracy.

Example 3.5

A geocentric trajectory has a perigee velocity of 15 km/s and a perigee altitude of 300 km. (a) Find the radius and the time when the true anomaly is 100°; (b) find the position and speed three hours later.

Solution

We first calculate the primary orbital parameters e and h. The angular momentum is calculated from Equation 2.31 and the given perigee data:

$$h = r_p v_p = (6378 + 300)\cdot 15 = 100{,}170\ \text{km}^2/\text{s}$$

The eccentricity is found by evaluating the orbit equation, $r = (h^2/\mu)[1/(1 + e\cos\theta)]$, at perigee:

$$6378 + 300 = \frac{100{,}170^2}{398{,}600}\frac{1}{1+e} \Rightarrow e = 2.7696$$

(a) Since $e > 1$ the trajectory is a hyperbola. Note that the true anomaly of the asymptote of the hyperbola is, according to Equation 2.97:

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{2.7696}\right) = 111.17°$$

Evaluating the orbit equation at $\theta = 100°$ yields:

$$r = \frac{100{,}170^2}{398{,}600}\frac{1}{1 + 2.7696\cos 100°} = \boxed{48{,}497\ \text{km}}$$

To find the time since perigee passage at $\theta = 100°$, we first use Equation 3.44a to calculate the hyperbolic eccentric anomaly,

$$\tanh\frac{F}{2} = \sqrt{\frac{2.7696 - 1}{2.7696 + 1}}\tan\frac{100°}{2} = 0.81653 \Rightarrow F = 2.2927\ \text{rad.}$$

Kepler's equation for the hyperbola then yields the mean anomaly,

$$M_h = e\sinh F - F = 2.7696\sinh 2.2927 - 2.2927 = 11.279\ \text{rad.}$$

The time since perigee passage is found by means of Equation 3.34,

$$t = \frac{h^3}{\mu^2}\frac{1}{(e^2 - 1)^{3/2}}M_h = \frac{100{,}170^3}{398{,}600^2}\cdot\frac{1}{(2.7696^2 - 1)^{3/2}}\cdot 11.279 = \boxed{4141\ \text{s}}$$

(b) Three hours later the time since perigee passage is

$$t = 4141.4 + 3\cdot 3600 = 14{,}941\ \text{s} \quad (4.15\ \text{hr})$$

The corresponding mean anomaly, from Equation 3.34, is

$$M_h = \frac{398{,}600^2}{100{,}170^3}(2.7696^2 - 1)^{\frac{3}{2}}\ 14{,}941 = 40.690 \text{ rad.}$$

We will use Algorithm 3.2 with an error tolerance of 10^{-6} to find the hyperbolic eccentric anomaly F. Referring to Figure 3.17, we see that for $M_h = 40.69$ and $e = 2.7696$, F lies between 3 and 4. Let us arbitrarily choose $F_0 = 3$ as our initial estimate of F. Executing the algorithm yields the following steps:

$$F_0 = 3$$

Step 1:

$$\begin{aligned} f(F_0) &= -15.944494 \\ f'(F_0) &= 26.883397 \\ \text{ratio} &= -0.59309818 \\ F_1 &= 3 - (-0.59309818) = 3.5930982 \end{aligned}$$

$|\text{ratio}| > 10^{-6}$, so repeat.

Step 2:

$$\begin{aligned} f(F_1) &= 6.0114484 \\ f'(F_1) &= 49.370747 \\ \text{ratio} &= -0.12176134 \\ F_2 &= 3.5930982 - (-0.12176134) = 3.4713368 \end{aligned}$$

$|\text{ratio}| > 10^{-6}$, so repeat.

Step 3:

$$\begin{aligned} f(F_2) &= 0.35812370 \\ f'(F_2) &= 43.605527 \\ \text{ratio} &= 8.2128052 \times 10^{-3} \\ F_3 &= 3.4713368 - (8.2128052 \times 10^{-3}) = 3.4631240 \end{aligned}$$

$|\text{ratio}| > 10^{-6}$, so repeat.

Step 4:

$$\begin{aligned} f(F_3) &= 1.4973128 \times 10^{-3} \\ f'(F_3) &= 43.241398 \\ \text{ratio} &= 3.4626836 \times 10^{-5} \\ F_4 &= 3.4631240 - (3.4626836 \times 10^{-5}) = 3.4630894 \end{aligned}$$

$|\text{ratio}| > 10^{-6}$, so repeat

Step 5:

$$
\begin{aligned}
f(F_4) &= 2.6470781 \times 10^{-3} \\
f'(F_4) &= 43.239869 \\
\text{ratio} &= 6.1218459 \times 10^{-10} \\
F_5 &= 3.4630894 - (6.1218459 \times 10^{-10}) = 3.4630894
\end{aligned}
$$

$|\text{ratio}| < 10^{-6}$, so accept $F = 3.4631$ as the solution.

We substitute this value of F into Equation 3.44b to find the true anomaly,

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} = \sqrt{\frac{2.7696+1}{2.7696-1}}\tanh\frac{3.4631}{2} = 1.3708 \Rightarrow \theta = 107.78°$$

With the true anomaly, the orbital equation yields the radial coordinate at the final time

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} = \frac{100{,}170^2}{398{,}600}\frac{1}{1+2.7696\cos 107.78} = \boxed{163{,}180\ \text{km}}$$

The velocity components are obtained from Equation 2.31,

$$v_\perp = \frac{h}{r} = \frac{100{,}170}{163{,}180} = 0.61386\ \text{km/s}$$

and Equation 2.49,

$$v_r = \frac{\mu}{h}e\sin\theta = \frac{398{,}600}{100{,}170}2.7696\sin 107.78° = 10.494\ \text{km/s}$$

Therefore, the speed of the spacecraft is

$$v = \sqrt{{v_r}^2 + {v_\perp}^2} = \sqrt{10.494^2 + 0.61386^2} = \boxed{10.51\ \text{km/s}}$$

Note that the hyperbolic excess speed for this orbit is

$$v_\infty = \frac{\mu}{h}e\sin\theta_\infty = \frac{398{,}600}{100{,}170}\cdot 2.7696\cdot\sin 111.7° = 10.277\ \text{km/s}$$

The results of this analysis are shown in Figure 3.18.

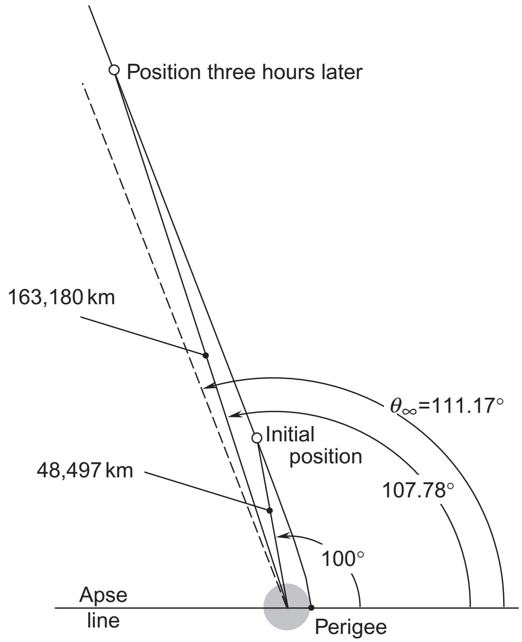


FIGURE 3.18

Given and computed data for Example 3.5.

When determining orbital position as a function of time with the aid of Kepler's equation, it is convenient to have position r as a function of eccentric anomaly F. The ***orbit equation in terms of hyperbolic eccentric anomaly*** is obtained by substituting Equation 3.41b into Equation 2.104,

$$r = \frac{a(e^2 - 1)}{1 + e\cos\theta} = \frac{a(e^2 - 1)}{1 + e\left(\dfrac{\cosh F - e}{1 - e\cos F}\right)}$$

This reduces to

$$r = a(e\cosh F - 1) \tag{3.46}$$

3.7 UNIVERSAL VARIABLES

The equations for elliptical and hyperbolic trajectories are very similar, as can be seen from Table 3.1. Observe, for example, that the hyperbolic mean anomaly is obtained from that of the ellipse as follows:

$$\begin{aligned} M_h &= \frac{\mu^2}{h^3}(e^2 - 1)^{\frac{3}{2}} t \\ &= \frac{\mu^2}{h^3}[(-1)(1 - e^2)]^{\frac{3}{2}} t \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu^2}{h^3}(-1)^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}t \\
&= \frac{\mu^2}{h^3}(-i)(1-e^2)^{\frac{3}{2}}t \\
&= -i\left[\frac{\mu^2}{h^3}(1-e^2)^{\frac{3}{2}}t\right] \\
&= -iM_e
\end{aligned}
$$

In fact, the formulas for the hyperbola can all be obtained from those of the ellipse by replacing the variables in the ellipse equations according to the following scheme, wherein "←" means "replace by":

$$
\begin{aligned}
a &\leftarrow -a \\
b &\leftarrow ib \\
M_e &\leftarrow -iM_h \qquad (i = \sqrt{-1}) \\
E &\leftarrow iF
\end{aligned}
$$

Note in this regard that sin $(iF) = i$ sinh F and cos (iF) = cosh F. Relations among the circular and hyperbolic trig functions are found in mathematics handbooks, such as Beyer (1991).

Table 3.1 Comparison of Some of the Orbital Formulas for the Ellipse and Hyperbola

Equation		Ellipse ($e < 1$)	Hyperbola ($e > 1$)
1.	Orbit equation (2.45)	$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$	same
2.	Conic equation in Cartesian coordinates (2.79), (2.109)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
3.	Semimajor axis (2.71), (2.103)	$a = \frac{h^2}{\mu}\frac{1}{1-e^2}$	$a = \frac{h^2}{\mu}\frac{1}{e^2-1}$
4.	Semiminor axis (2.76), (2.106)	$b = a\sqrt{1-e^2}$	$b = a\sqrt{e^2-1}$
5.	Energy equation (2.81), (2.111)	$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$	$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$
6.	Mean anomaly (3.7), (3.34)	$M_e = \frac{\mu^2}{h^3}(1-e^2)^{\frac{3}{2}}t$	$M_h = \frac{\mu^2}{h^3}(e^2-1)^{\frac{3}{2}}t$
7.	Kepler's equation (3.14), (3.40)	$M_e = E - e\sin E$	$M_h = e\sinh F - F$
8.	Orbit equation in terms of eccentric anomaly (3.25), (3.46)	$r = a(1 - e\cos E)$	$r = a(e\cosh F - 1)$

In the universal variable approach, ***the semimajor axis of the hyperbola is negative***, so that the energy equation (row 5 of Table 3.1) has the same form for any type of orbit, including the parabola, for which $a = \infty$. In this formulation, the semimajor axis of any orbit is found using (row 3)

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} \tag{3.47}$$

If the position r and velocity v are known at a given point on the path, then the energy equation (row 5) is convenient for finding the semimajor axis of any orbit,

$$a = \frac{1}{\dfrac{2}{r} - \dfrac{v^2}{\mu}} \tag{3.48}$$

Kepler's equation may also be written in terms of a universal variable, or ***universal anomaly*** χ, that is valid for all orbits. See, for example, Battin (1999), Bond & Allman (1993), and Prussing & Conway (1993). If t_0 is the time when the universal variable is zero, then the value of χ at time $t_0 + \Delta t$ is found by iterative solution of the ***universal Kepler's equation***

$$\boxed{\sqrt{\mu}\Delta t = \frac{r_0 v_{r0}}{\sqrt{\mu}}\chi^2 C(\alpha\chi^2) + (1-\alpha r_0)\chi^3 S(\alpha\chi^2) + r_0\chi} \tag{3.49}$$

in which r_0 and v_{r0} are the radius and radial velocity at $t = t_0$, and α is the reciprocal of the semimajor axis:

$$\alpha = \frac{1}{a} \tag{3.50}$$

$\alpha < 0$, $\alpha = 0$ and $\alpha > 0$ for hyperbolas, parabolas and ellipses, respectively. The units of χ are $km^{1/2}$ (so $\alpha\chi^2$ is dimensionless). The functions $C(z)$ and $S(z)$ belong to the class known as ***Stumpff functions***, and they are defined by the infinite series,

$$S(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(2k+3)!} = \frac{1}{6} - \frac{z}{120} + \frac{z^2}{5040} - \frac{z^3}{362{,}880} + \frac{z^4}{39{,}916{,}800} - \frac{z^5}{6{,}227{,}020{,}800} + \cdots \tag{3.51a}$$

$$C(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(2k+2)!} = \frac{1}{2} - \frac{z}{24} + \frac{z^2}{720} - \frac{z^3}{40{,}320} + \frac{z^4}{3{,}628{,}800} - \frac{z^5}{479{,}001{,}600} + \cdots \tag{3.51b}$$

$C(z)$ and $S(z)$ are related to the circular and hyperbolic trig functions as follows:

$$S(z) = \begin{cases} \dfrac{\sqrt{z} - \sin\sqrt{z}}{(\sqrt{z})^3} & (z > 0) \\ \dfrac{\sinh\sqrt{-z} - \sqrt{-z}}{(\sqrt{-z})^3} & (z < 0) \\ \dfrac{1}{6} & (z = 0) \end{cases} \qquad (z = \alpha\chi^2) \tag{3.52}$$

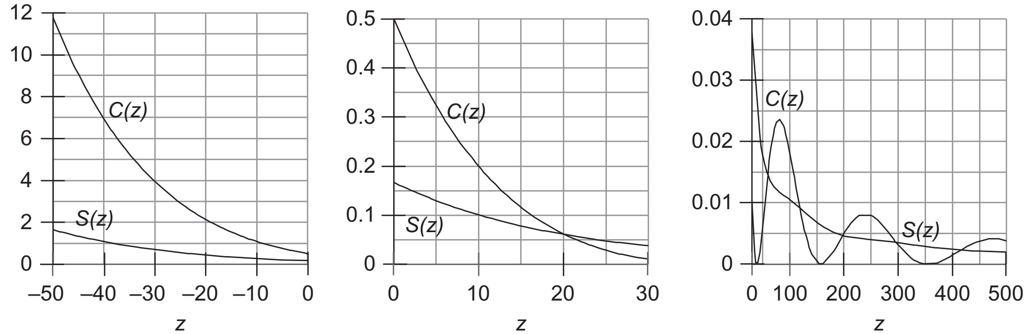


FIGURE 3.19

A plot of the Stumpff functions $C(z)$ and $S(z)$.

$$C(z) = \begin{cases} \dfrac{1-\cos\sqrt{z}}{z} & (z>0) \\ \dfrac{\cosh\sqrt{-z}-1}{-z} & (z<0) \\ \dfrac{1}{2} & (z=0) \end{cases} \qquad (z=\alpha\chi^2) \tag{3.53}$$

Clearly, $z<0$, $z=0$ and $z>0$ for hyperbolas, parabolas and ellipses, respectively. It should be pointed out that if $C(z)$ and $S(z)$ are computed by the series expansions, Equations 3.51a and 3.51b, then the forms of $C(z)$ and $S(z)$, depending on the sign of z, are selected, so to speak, automatically. $C(z)$ and $S(z)$ behave as shown in Figure 3.19. Both $C(z)$ and $S(z)$ are nonnegative functions of z. They increase without bound as z approaches $-\infty$ and tend towards zero for large positive values of z. As can be seen from Equation 3.53_1, for $z>0$, $C(z)=0$ when $\cos\sqrt{z}=1$, that is, when $z=(2\pi)^2, (4\pi)^2, (6\pi)^2, \cdots$.

The price we pay for using the universal variable formulation is having to deal with the relatively unknown Stumpff functions. However, Equations 3.52 and 3.53 are easy to implement in both computer programs and programmable calculators. See Appendix D.13 for the implementation of these expressions in MATLAB.

To gain some insight into how Equation 3.49 represents the Kepler equations for all of the conic sections, let t_0 be the time at periapse passage and let us set $t_0=0$, as we have assumed previously. Then $\Delta t = t$, $v_{r0}=0$ and r_0 equals r_p, the periapsis radius. In that case Equation 3.49 reduces to

$$\sqrt{\mu}t = (1-\alpha r_p)\chi^3 S(\alpha\chi^2) + r_p\chi \quad (t=0 \text{ at periapse passage}) \tag{3.54}$$

Consider first the parabola. In that case $\alpha=0$, and $S=S(0)=1/6$, so that Equation 3.54 becomes a cubic polynomial in χ:

$$\sqrt{\mu}t = \frac{1}{6}\chi^3 + r_p\chi$$

Multiply this equation through by $\left(\sqrt{\mu}/h\right)^3$ to obtain

$$\frac{\mu^2}{h^3}t = \frac{1}{6}\left(\frac{\chi\sqrt{\mu}}{h}\right)^3 + r_p\chi\left(\frac{\sqrt{\mu}}{h}\right)^3$$

Since $r_p = h^2/2\mu$ for a parabola, we can write this as

$$\frac{\mu^2}{h^3}t = \frac{1}{6}\left(\frac{\sqrt{\mu}}{h}\chi\right)^3 + \frac{1}{2}\left(\frac{\sqrt{\mu}}{h}\chi\right) \tag{3.55}$$

Upon setting $\chi = h\tan(\theta/2)/\sqrt{\mu}$, Equation 3.55 becomes identical to Equation 3.30, the time versus true anomaly relation for the parabola.

Kepler's equation for the ellipse can be obtained by multiplying Equation 3.54 through by $\left(\sqrt{\mu(1-e^2)}/h\right)^3$:

$$\frac{\mu^2}{h^3}(1-e^2)^{\frac{3}{2}}t = \left(\chi\frac{\sqrt{\mu}}{h}\sqrt{1-e^2}\right)^3(1-\alpha r_p)S(z) + r_p\chi\left(\frac{\sqrt{\mu}}{h}\sqrt{1-e^2}\right)^3 \quad (z=\alpha\chi^2) \tag{3.56}$$

Recall that for the ellipse, $r_p = h^2/[\mu(1+e)]$ and $\alpha = 1/a = \mu(1-e^2)/h^2$. Using these two expressions in Equation 3.56, along with $S(z) = \left[\sqrt{\alpha}\chi - \sin(\sqrt{\alpha}\chi)\right]/\alpha^{\frac{3}{2}}\chi^3$ (from Equation 3.52_1), and working through the algebra ultimately leads to

$$M_e = \frac{\chi}{\sqrt{a}} - e\sin\left(\frac{\chi}{\sqrt{a}}\right)$$

Comparing this with Kepler's equation for an ellipse (Equation 3.14) reveals that the relationship between the universal variable χ and the eccentric anomaly E is $\chi = \sqrt{a}E$. Similarly, it can be shown for hyperbolic orbits that $\chi = \sqrt{-a}F$. In summary, the ***relations between the universal anomaly and the various eccentric anomalies*** encountered previously are:

$$\chi = \begin{cases} \dfrac{h}{\sqrt{\mu}}\tan\dfrac{\theta}{2} & \text{parabola} \\ \sqrt{a}E & \text{ellipse} \quad (t_0 = 0, \text{ at periapsis}) \\ \sqrt{-a}F & \text{hyperbola} \end{cases} \tag{3.57}$$

When t_0 is the time at a point other than periapsis, so that Equation 3.49 applies, then Equations 3.57 becomes

$$\chi = \begin{cases} \dfrac{h}{\sqrt{\mu}}\left(\tan\dfrac{\theta}{2} - \tan\dfrac{\theta_0}{2}\right) & \text{parabola} \\ \sqrt{a}(E - E_0) & \text{ellipse} \\ \sqrt{-a}(F - F_0) & \text{hyperbola} \end{cases} \tag{3.58}$$

As before, we can use Newton's method to solve Equation 3.49 for the universal anomaly χ, given the time interval Δt. To do so, we form the function

$$f(\chi) = \frac{r_0 v_{r0}}{\sqrt{\mu}}\chi^2 C(z) + (1 - \alpha r_0)\chi^3 S(z) + r_0\chi - \sqrt{\mu}\Delta t \tag{3.59}$$

and its derivative

$$\frac{df(\chi)}{d\chi} = 2\frac{r_0 v_{r0}}{\sqrt{\mu}}\chi C(z) + \frac{r_0 v_{r0}}{\sqrt{\mu}}\chi^2 \frac{dC(z)}{dz}\frac{dz}{d\chi} + 3(1 - \alpha r_0)\chi^2 S(z) + (1 - r_0\alpha)\chi^3 \frac{dS(z)}{dz}\frac{dz}{d\chi} + r_0 \tag{3.60}$$

where it is to be recalled that

$$z = \alpha\chi^2 \tag{3.61}$$

which means of course that

$$\frac{dz}{d\chi} = 2\alpha\chi \tag{3.62}$$

It turns out that

$$\begin{aligned} \frac{dS(z)}{dz} &= \frac{1}{2z}\left[C(z) - 3S(z)\right] \\ \frac{dC(z)}{dz} &= \frac{1}{2z}\left[1 - zS(z) - 2C(z)\right] \end{aligned} \tag{3.63}$$

Substituting Equations 3.61, 3.62 and 3.63 into Equation 3.60 and simplifying the result yields

$$\frac{df(\chi)}{d\chi} = \frac{r_0 v_{r0}}{\sqrt{\mu}}\chi\left[1 - \alpha\chi^2 S(z)\right] + (1 - \alpha r_0)\chi^2 C(z) + r_0 \tag{3.64}$$

With Equations 3.59 and 3.64, Newton's algorithm (Equation 3.16) for the universal Kepler equation becomes

$$\chi_{i+1} = \chi_i - \frac{\dfrac{r_0 v_{r0}}{\sqrt{\mu}}\chi_i^2 C(z_i) + (1 - \alpha r_0)\chi_i^3 S(z_i) + r_0\chi_i - \sqrt{\mu}\Delta t}{\dfrac{r_0 v_{r0}}{\sqrt{\mu}}\chi_i\left[1 - \alpha\chi_i^2 S(z_i)\right] + (1 - \alpha r_0)\chi_i^2 C(z_i) + r_0} \qquad (z_i = \alpha\chi_i^2) \tag{3.65}$$

According to Chobotov (2002), a reasonable estimate for the starting value χ_0 is

$$\chi_0 = \sqrt{\mu}|\alpha|\Delta t \tag{3.66}$$

Algorithm 3.3 Solve the universal Kepler's equation for the universal anomaly χ given Δt, r_0, v_{r0} and α. See Appendix D.14 for an implementation of this procedure in MATLAB.

1. Use Equation 3.66 for an initial estimate of χ_0.
2. At any given step, having obtained χ_i from the previous step, calculate

$$f(\chi_i) = \frac{r_0 v_{r0}}{\sqrt{\mu}}\chi_i^2 C(z_i) + (1 - \alpha r_0)\chi_i^3 S(z_i) + r_0\chi_i - \sqrt{\mu}\Delta t$$

   and

$$f'(\chi_i) = \frac{r_0 v_{r0}}{\sqrt{\mu}}\chi_i\left[1 - \alpha\chi_i^2 S(z_i)\right] + (1 - \alpha r_0)\chi_i^2 C(z_i) + r_0$$

   where $z_i = \alpha\chi_i^2$.
3. Calculate $\text{ratio}_i = f(\chi_i)/f'(\chi_i)$
4. If $|\text{ratio}_i|$ exceeds the chosen tolerance (e.g., 10^{-8}), then calculate an updated value of χ,

$$\chi_{i+1} = \chi_i - \text{ratio}_i$$

   Return to step 2.
5. If $|\text{ratio}_i|$ is less than the tolerance, then accept χ_i as the solution to within the desired accuracy.

Example 3.6

An earth satellite has an initial true anomaly of $\theta_0 = 30°$, a radius of $r_0 = 10{,}000\,\text{km}$, and a speed of $v_0 = 10\,\text{km/s}$. Use the universal Kepler's equation to find the change in universal anomaly χ after one hour and use that information to determine the true anomaly θ at that time.

Solution

Using the initial conditions, let us first determine the angular momentum and the eccentricity of the trajectory. From the orbit formula, Equation 2.45, we have

$$h = \sqrt{\mu r_0(1 + e\cos\theta_0)} = \sqrt{398{,}600\cdot 10{,}000\cdot(1 + e\cos 30°)} = 63{,}135\sqrt{1 + 0.86602e} \qquad \text{(a)}$$

This, together with the angular momentum formula, Equation 2.31, yields

$$v_{\perp 0} = \frac{h}{r_0} = \frac{63{,}135\sqrt{1 + 0.86602e}}{10{,}000} = 6.3135\sqrt{1 + 0.86602e}$$

Using the radial velocity relation, Equation 2.49, we find

$$v_{r0} = \frac{\mu}{h}e\sin\theta_0 = \frac{398{,}600}{63{,}135\sqrt{1 + 0.86602e}}e\sin 30° = 3.1567\frac{e}{\sqrt{1 + 0.86602e}} \qquad \text{(b)}$$

Since $v_{r0}^2 + v_{\perp 0}^2 = v_0^2$, it follows that

$$\left(3.1567\frac{e}{\sqrt{1+0.86602e}}\right)^2 + (6.3135\sqrt{1+0.86602e})^2 = 10^2$$

which simplifies to become $39.86e^2 - 17.563e - 60.14 = 0$. The only positive root of this quadratic equation is

$$e = 1.4682$$

Since e is greater than 1, the orbit is a hyperbola. Substituting this value of the eccentricity back into (a) and (b) yields the angular momentum

$$h = 95{,}154 \text{ km}^2/\text{s}$$

as well as the initial radial speed

$$v_{r0} = 3.0752 \text{ km/s}$$

The hyperbolic eccentric anomaly F_0 for the initial conditions may now be found from Equation 3.44a,

$$\tanh\frac{F_0}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta_0}{2} = \sqrt{\frac{1.4682-1}{1.4682+1}}\tan\frac{30°}{2} = 0.16670$$

Solving for F_0 yields

$$F_0 = 0.23448 \text{ rad.} \tag{c}$$

In the universal variable formulation, we calculate the semimajor axis of the orbit by means of Equation 3.47,

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} = \frac{95{,}154^2}{398{,}600}\frac{1}{1-1.4682^2} = -19{,}655 \text{ km} \tag{d}$$

The negative value is consistent with the fact that the orbit is a hyperbola. From Equation 3.50 we get

$$\alpha = \frac{1}{a} = \frac{1}{-19{,}655} = -5.0878 \times 10^{-5} \text{ km}^{-1}$$

which appears throughout the universal Kepler's equation.

We will use Algorithm 3.3 with an error tolerance of 10^{-6} to find the universal anomaly. From Equation 3.66, our initial estimate is

$$\chi_0 = \sqrt{398{,}600}\cdot\left|-5.0878 \times 10^{-6}\right|\cdot 3600 = 115.6$$

Executing the algorithm yields the following steps:

$\chi_0 = 115.6$

Step 1:

$$f(\chi_0) = -370{,}650.01$$
$$f'(\chi_0) = 26{,}956.300$$
$$\text{ratio} = -13.7500033$$
$$\chi_1 = 115.6 - (-13.750033) = 129.35003$$
$|\text{ratio}| > 10^{-6}$, so repeat.

Step 2:

$$f(\chi_1) = 25{,}729.002$$
$$f'(\chi_1) = 30{,}776.401$$
$$\text{ratio} = 0.83599669$$
$$\chi_2 = 129.35003 - 0.83599669 = 128.51404$$
$|\text{ratio}| > 10^{-6}$, so repeat.

Step 3:

$$f(\chi_2) = 102.83891$$
$$f'(\chi_2) = 30{,}530.672$$
$$\text{ratio} = 3.3683800 \times 10^{-3}$$
$$\chi_3 = 128.51404 - 3.3683800 \times 10^{-3} = 128.51067$$
$|\text{ratio}| > 10^{-6}$, so repeat.

Step 4:

$$f(\chi_3) = 1.6614116 \times 10^{-3}$$
$$f'(\chi_3) = 30{,}529.686$$
$$\text{ratio} = 5.4419545 \times 10^{-8}$$
$$\chi_4 = 128.51067 - 5.4419545 \times 10^{-8} = 128.51067$$
$|\text{ratio}| < 10^{-6}$

So we accept

$$\boxed{\chi = 128.51\ \text{km}^{\frac{1}{2}}}$$

as the solution after four iterations. Substituting this value of χ together with the semimajor axis [Equation (d)] into Equation 3.58_3 yields

$$F - F_0 = \frac{\chi}{\sqrt{-a}} = \frac{128.51}{\sqrt{-(-19{,}655)}} = 0.91664$$

It follows from (c) that the hyperbolic eccentric anomaly after one hour is

$$F = 0.23448 + 0.91664 = 1.1511$$

Finally, we calculate the corresponding true anomaly using Equation 3.44b,

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} = \sqrt{\frac{1.4682+1}{1.4682-1}}\tanh\frac{1.1511}{2} = 1.1926$$

which means that after one hour

$$\boxed{\theta = 100.04^\circ}$$

Recall from Section 2.11 that the position $\mathbf{r}$ and velocity $\mathbf{v}$ on a trajectory at any time t can be found in terms of the position $\mathbf{r}_0$ and velocity $\mathbf{v}_0$ at time t_0 by means of the Lagrange f and g coefficients and their first derivatives,

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0 \tag{3.67}$$

$$\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 \tag{3.68}$$

Equations 2.158 give f, g, $\dot{f}$ and $\dot{g}$ explicitly in terms of the change in true anomaly $\Delta\theta$ over the time interval $\Delta t = t - t_0$. The Lagrange coefficients can also be derived in terms of changes in the eccentric anomaly ΔE for elliptical orbits, ΔF for hyperbolas or $\Delta\tan(\theta/2)$ for parabolas. However, if we take advantage of the universal variable formulation, we can cover all of these cases with the same set of Lagrange coefficients (Bond and Allman, 1996). By means of the Stumpff functions $C(z)$ and $S(z)$, the ***Lagrange f and g coefficients in terms of the universal anomaly*** are:

$$f = 1 - \frac{\chi^2}{r_0}C(\alpha\chi^2) \tag{3.69a}$$

$$g = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(\alpha\chi^2) \tag{3.69b}$$

$$\dot{f} = \frac{\sqrt{\mu}}{r r_0}\left[\alpha\chi^3 S(\alpha\chi^2) - \chi\right] \tag{3.69c}$$

$$\dot{g} = 1 - \frac{\chi^2}{r}C(\alpha\chi^2) \tag{3.69d}$$

The implementation of these four functions in MATLAB is found in Appendix D.15.

Algorithm 3.4 Given $\mathbf{r}_0$ and $\mathbf{v}_0$, find $\mathbf{r}$ and $\mathbf{v}$ at a time Δt later. See Appendix D.16 for an implementation of this procedure in MATLAB.

1. Use the initial conditions to find:
   a. The magnitude of $\mathbf{r}_0$ and $\mathbf{v}_0$,

$$r_0 = \sqrt{\mathbf{r}_0\cdot\mathbf{r}_0} \qquad v_0 = \sqrt{\mathbf{v}_0\cdot\mathbf{v}_0}$$

b. The radial component velocity of v_{r0} by projecting $\mathbf{v}_0$ onto the direction of $\mathbf{r}_0$,

$$v_{r0} = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{r_0}$$

c. The reciprocal α of the semimajor axis, using Equation 3.48,

$$\alpha = \frac{2}{r_0} - \frac{{v_0}^2}{\mu}$$

The sign of α determines whether the trajectory is an ellipse ($\alpha > 0$), parabola ($\alpha = 0$) or hyperbola ($\alpha < 0$).

2. With r_0, v_{r0}, α and Δt, use Algorithm 3.3 to find the universal anomaly χ.
3. Substitute α, r_0, Δt and χ into Equations 3.69a and b to obtain f, and g.
4. Use Equation 3.67 to compute $\mathbf{r}$ and, from that, its magnitude r.
5. Substitute α, r_0, r and χ into Equations 3.69c and d to obtain $\dot{f}$ and $\dot{g}$.
6. Use Equation 3.68 to compute $\mathbf{v}$.

Example 3.7

An earth satellite moves in the xy plane of an inertial frame with origin at the earth's center. Relative to that frame, the position and velocity of the satellite at time t_0 are

$$\begin{aligned} \mathbf{r}_0 &= 7000.0\hat{\mathbf{i}} - 12{,}124\hat{\mathbf{j}}\ (\text{km}) \\ \mathbf{v}_0 &= 2.6679\hat{\mathbf{i}} + 4.6210\hat{\mathbf{j}}\ (\text{km/s}) \end{aligned} \quad \text{(a)}$$

Compute the position and velocity vectors of the satellite 60 minutes later using Algorithm 3.4.

Solution

Step 1.

$$\begin{aligned} r_0 &= \sqrt{7000.0^2 + (-12{,}124)^2} = 14{,}000\ \text{km} \\ v_0 &= \sqrt{2.6679^2 + 4.6210^2} = 5.3359\ \text{km/s} \\ v_{r0} &= \frac{7000.0 \cdot 2.6679 + (-12{,}124) \cdot 4.6210}{14{,}000} = -2.6679\ \text{km/s} \\ \alpha &= \frac{2}{14{,}000} - \frac{5.3359^2}{398{,}600} = 7.1429 \times 10^{-5}\ \text{km}^{-1} \end{aligned}$$

The trajectory is an ellipse, because α is positive.

Step 2.

Using the results of Step 1, Algorithm 3.3 yields

$$\chi = 253.53\ \text{km}^{\frac{1}{2}}$$

which means

$$z = \alpha\chi^2 = 7.1429 \times 10^5 \cdot 253.53^2 = 4.5911$$

Step 3.

Substituting the above values of χ and z into Equations 3.69a and 3.69b we find

$$f = 1 - \frac{\chi^2}{r_0} C(\alpha\chi^2) = 1 - \frac{253.53^2}{14{,}000} \overbrace{C(4.5911)}^{0.3357} = -0.54123$$

$$g = \Delta t - \frac{1}{\sqrt{\mu}} \chi^3 S(\alpha\chi^2) = 3600 - \frac{253.53^3}{\sqrt{398{,}600}} \overbrace{S(4.5911)}^{0.13233} = 184.35 \text{ s}^{-1}$$

Step 4.

$$\begin{aligned} \mathbf{r} &= f\mathbf{r}_0 + g\mathbf{v}_0 \\ &= (-0.54123)(7000.0\hat{\mathbf{i}} - 12.124\hat{\mathbf{j}}) + 184.35(2.6679\hat{\mathbf{i}} + 4.6210\hat{\mathbf{j}}) \\ &= -3296.8\hat{\mathbf{i}} + 7413.9\hat{\mathbf{j}} \text{ (km)} \end{aligned}$$

Therefore, the magnitude of $\mathbf{r}$ is

$$r = \sqrt{(-3296.8)^2 + 7413.9^2} = 8113.9 \text{ km}$$

Step 5.

$$\begin{aligned} \dot{f} &= \frac{\sqrt{\mu}}{rr_0}\left[\alpha\chi^3 S(\alpha\chi^2) - \chi\right] \\ &= \frac{\sqrt{398{,}600}}{8113.9 \cdot 14{,}000}\left[(7.1429 \times 10^5) \cdot 253.53^3 \cdot \overbrace{S(4.5911)}^{0.13233} - 253.53\right] \\ &= -0.00055298 \text{ s}^{-1} \end{aligned}$$

$$\dot{g} = 1 - \frac{\chi^2}{r} C(\alpha\chi^2) = 1 - \frac{253.53^2}{8113.9} \overbrace{C(4.5911)}^{0.3357} = -1.6593$$

Step 6.

$$\begin{aligned} \mathbf{v} &= \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 \\ &= (-0.00055298)(7000.0\hat{\mathbf{i}} - 12.124\hat{\mathbf{j}}) + (-1.6593)\mathbf{v}_0(2.6679\hat{\mathbf{i}} + 4.6210\hat{\mathbf{j}}) \\ &= -8.2977\hat{\mathbf{i}} - 0.96309\hat{\mathbf{j}} \text{ (km/s)} \end{aligned}$$

The initial and final position and velocity vectors, as well as the trajectory, are accurately illustrated in Figure 3.20.

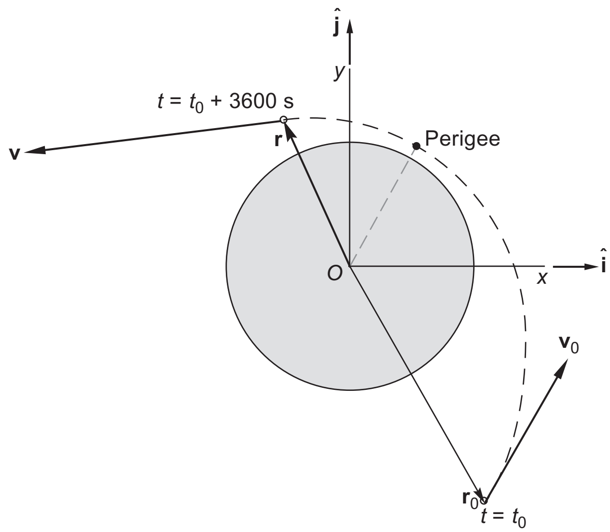


FIGURE 3.20

Initial and final points on the geocentric trajectory of Example 3.7.

PROBLEMS

Section 3.2

3.1 If $f = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{6}\tan^3\frac{x}{2}$, then show that $df/dx = 1/(1 + \cos x)^2$, thereby verifying the integral in Equation 3.4.

Section 3.4

3.2 Find the three positive roots of the equation $10_e^{\sin x} = x^2 - 5x + 4$ to eight significant figures. Use:

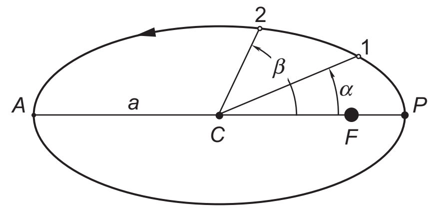
(a) Newton's method.
(b) Bisection method.

3.3 Find the first four non-negative roots of the equation $\tan(x) = \tanh(x)$ to eight significant figures. Use:

(a) Newton's method.
(b) Bisection method.

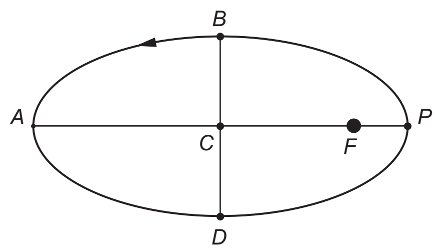
3.4 In terms of the eccentricity e, the period T and the angles α and β (in radians) find the time t required to fly from point 1 to point 2 on the ellipse. C is the center of the ellipse.

$$\left\{\text{Ans.: } t = \frac{T}{2\pi}\left[\beta - \alpha - 2e\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right]\right\}$$

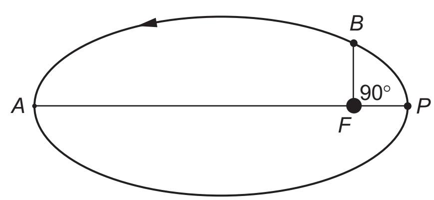


3.5 Calculate the time required to fly from P to B, in terms of the eccentricity e and the period T. B lies on the minor axis.

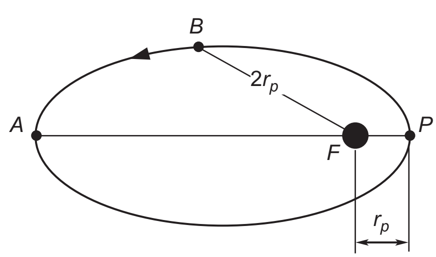
$$\left\{\text{Ans.: } \left(\frac{1}{4} - \frac{e}{2\pi}\right)T\right\}$$



3.6 If the eccentricity of the elliptical orbit is 0.3, calculate, in terms of the period T, the time required to fly from P to B.
{Ans.: $0.157T$}



3.7 If the eccentricity of the elliptical orbit is 0.5, calculate, in terms of the period T, the time required to fly from P to B.
{Ans.: 0.170T}



3.8 A satellite is in earth orbit for which the perigee altitude is 200 km and the apogee altitude is 600 km. Find the time interval during which the satellite remains above an altitude of 400 km.
{Ans.: 47.15 minutes}

3.9 An earth-orbiting satellite has a perigee radius of 7000 km and an apogee radius of 10,000 km. (a) What true anomaly $\Delta\theta$ is swept out between $t = 0.5$ hr and $t = 1.5$ hr after perigee passage? (b) What area is swept out by the position vector during that time interval?
{Ans.: (a) 128.7°; (b) 1.03×10^8 km^2}

3.10 An earth-orbiting satellite has a period of 15.743 hours and a perigee radius of 12,756 km. At time $t = 10$ hours after perigee passage, determine:

(a) The radius.
(b) The speed.
(c) The radial component of the velocity.
{Ans.: (a) 48,290 km; (b) 2.00 km/s; (c) –0.7210 km/s}

3.11 A satellite in earth orbit has perigee and apogee radii of $r_p = 7000$ km and $r_a = 14{,}000$ km, respectively. Find its true anomaly 30 minutes after passing true anomaly of 60°.
{Ans.: 127°}

3.12 Show that the solution to $a\cos\theta + b\sin\theta = c$, where a, b and c are given, is:

$$\theta = \phi \pm \cos^{-1}\left(\frac{c}{a}\cos\phi\right)$$

where $\tan\phi = b/a$.

3.13 Verify the results of part (b) of Example 3.3.

Section 3.5

3.14 Calculate the time required for a spacecraft launched into a parabolic trajectory at a perigee altitude of 500 km to leave the earth's sphere of influence (see Table A.2).
{Ans.: 7d 18h 34m}

3.15 A spacecraft on a parabolic trajectory around the earth has a perigee radius of 7500 km.

(a) How long does it take to coast from $\theta = -90$ degrees to $\theta = +90$ degrees?
(b) How far is the spacecraft from the center of the earth 24 hours after passing through perigee?
{Ans.: (a) 1.078 h; (b) 230,200 km}

Section 3.6

3.16 A spacecraft on a hyperbolic trajectory around the earth has a perigee radius of 7500 km and a perigee speed of $1.1v_{esc}$.

(a) How long does it take to coast from $\theta = -90°$ to $\theta = +90°$?
(b) How far is the spacecraft from the center of the earth 24 hours after passing through perigee?
{Ans.: (a) 1.14 h; (b) 456,000 km}

3.17 A trajectory has a perigee velocity of 11.5 km/s and a perigee altitude of 300 km. If at 6 AM the satellite is traveling towards the earth with a speed of 10 km/s, how far will it be from the earth's surface at 11 AM the same day?
{Ans.: 88,390 km}

3.18 An incoming object is sighted at an altitude of 37,000 km with a speed of 8 km/s and a flight path angle of −65°. (a) Will it impact the earth or fly by? (b) What is the time to impact or closest passage?
{Ans.: (b) 1 h 24 m}

Section 3.7

3.19 At a given instant the radial position of an earth-orbiting satellite is 7200 km, its radial speed is 1 km/s. If the semimajor axis is 10,000 km, use Algorithm 3.3 to find the universal anomaly 60 minutes later. Check your result using Equation 3.58.

3.20 At a given instant a space object has the following position and velocity vectors relative to an earth-centered inertial frame of reference:

$$\mathbf{r}_0 = 20{,}000\hat{\mathbf{i}} - 105{,}000\hat{\mathbf{j}} - 19{,}000\hat{\mathbf{k}} \text{ (km)}$$
$$\mathbf{v}_0 = 0.9000\hat{\mathbf{i}} - 3.4000\hat{\mathbf{j}} - 1.5000\hat{\mathbf{k}} \text{ (km/s)}$$

Find **r** and **v** two hours later.

$\left\{\text{Ans.: } \mathbf{r} = 26{,}338\hat{\mathbf{i}} - 128{,}750\hat{\mathbf{j}} - 29{,}656\hat{\mathbf{k}} \text{ (km)};\ \mathbf{v} = 0.862{,}800\hat{\mathbf{i}} - 3.2116\hat{\mathbf{j}} - 1.4613\hat{\mathbf{k}} \text{ (km/s)}\right\}$

List of Key Terms

Barker's equation
eccentric anomaly
hyperbolic mean anomaly
Kepler's equation
Kepler's equation for the hyperbola
Lagrange f and g coefficients in terms of the universal anomaly
mean anomaly for the ellipse
mean motion
Newton's method
orbit equation in terms of hyperbolic eccentric anomaly
orbit equation in terms of the eccentric anomaly
parabolic mean anomaly
relation between universal anomaly and the different eccentric anomalies
Stumpff functions
time versus true anomaly
universal anomaly
universal Kepler's equation

CHAPTER

Orbits in three dimensions

4

Chapter outline

4.1	Introduction	199
4.2	Geocentric right ascension-declination frame	200
4.3	State vector and the geocentric equatorial frame	203
4.4	Orbital elements and the state vector	208
4.5	Coordinate transformation	216
4.6	Transformation between geocentric equatorial and perifocal frames	229
4.7	Effects of the earth's oblateness	233
4.8	Ground tracks	244

4.1 INTRODUCTION

The discussion of orbital mechanics up to now has been confined to two dimensions, that is, to the plane of the orbits themselves. This chapter explores the means of describing orbits in three-dimensional space, which, of course, is the setting for real missions and orbital maneuvers. Our focus will be on the orbits of earth satellites, but the applications are to any two-body trajectories, including interplanetary missions to be discussed in Chapter 8.

We begin with a discussion of the ancient concept of the celestial sphere and the use of right ascension and declination to define the location of stars, planets and other celestial objects on the sphere. This leads to the establishment of the inertial geocentric equatorial frame of reference and the concept of state vector. The six components of this vector give the instantaneous position and velocity of an object relative to the inertial frame and define the characteristics of the orbit. Following that discussion is a presentation of the six classical orbital elements, which also uniquely define the shape and orientation of an orbit and the location of body on it. We then show how to transform the state vector into orbital elements and vice versa, taking advantage of the perifocal frame introduced in Chapter 2.

We go on to summarize two of the major perturbations of earth orbits due to the earth's nonspherical shape. These perturbations are exploited to place satellites in sun-synchronous and molniya orbits.

The chapter concludes with a discussion of ground tracks and how to compute them.

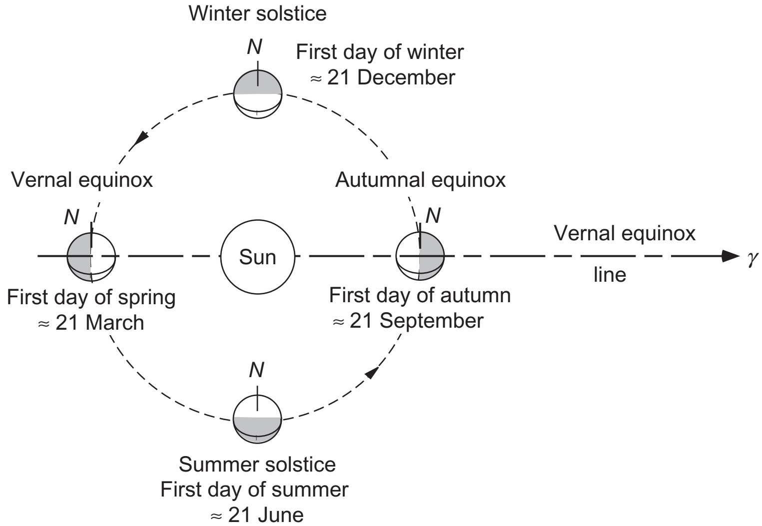


FIGURE 4.1

The earth's orbit around the sun, viewed from above the ecliptic plane, showing the change of seasons in the northern hemisphere.

4.2 GEOCENTRIC RIGHT ASCENSION-DECLINATION FRAME

The coordinate system used to describe earth orbits in three dimensions is defined in terms of earth's equatorial plane, the ***ecliptic plane***, and the earth's axis of rotation. The ecliptic is the plane of the earth's orbit around the sun, as illustrated in Figure 4.1. The earth's axis of rotation, which passes through the north and south poles, is not perpendicular to the ecliptic. It is tilted away by an angle known as the ***obliquity of the ecliptic***, ε. For the earth ε is approximately 23.4 degrees. Therefore, the earth's equatorial plane and the ecliptic intersect along a line, which is known as the ***vernal equinox line***. On the calendar, "vernal equinox" is the first day of spring in the northern hemisphere, when the noontime sun crosses the equator from south to north. The position of the sun at that instant defines the location of a point in the sky called the vernal equinox, for which the symbol γ is used. On the day of the vernal equinox, the number of hours of daylight and darkness are equal; hence, the word equinox. The other equinox occurs precisely one-half year later, when the sun crosses back over the equator from north to south, thereby defining the first day of autumn. The vernal equinox lies today in the constellation Pisces, which is visible in the night sky during the fall. The direction of the vernal equinox line is from the earth towards γ, as shown in Figure 4.1.

For many practical purposes, the vernal equinox line may be considered fixed in space. However, it actually rotates slowly because the earth's tilted spin axis precesses westward around the normal to the ecliptic at the rate of about 1.4° per century. This slow ***precession of the vernal equinox line*** is due primarily to the action of the sun and the moon on the nonspherical distribution of mass within the earth. Due to the centrifugal force of rotation about its own axis, the earth bulges very slightly outward at its equator. This effect is shown highly exaggerated in Figure 4.2. One of the bulging sides is closer to the sun than the other, so the force of the sun's gravity $\mathbf{f}_1$ on its mass is slightly larger than the force $\mathbf{f}_2$ on opposite the side, farthest from the sun. The forces $\mathbf{f}_1$ and $\mathbf{f}_2$, along with the dominant force $\mathbf{F}$ on the spherical mass, comprise the total force

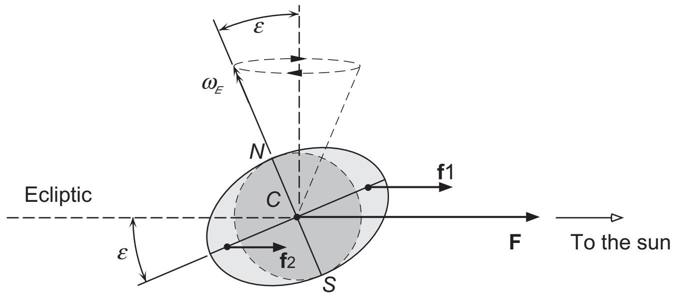


FIGURE 4.2

Secondary (perturbing) gravitational forces on the earth.

of the sun on the earth, holding it in its solar orbit. Taken together, $\mathbf{f}_1$ and $\mathbf{f}_2$ produce a net clockwise moment (a vector into the page) about the center of the earth. That moment would rotate the earth's equator into alignment with the ecliptic if it were not for the fact that the earth has an angular momentum directed along its south-to-north polar axis due to its spin around that axis at an angular velocity ω_E of about 360° per day. The effect of the moment is to rotate the angular momentum vector in the direction of the moment (into the page). The result is that the spin axis is forced to precess in a clockwise direction around the normal to the ecliptic, sweeping out a cone as illustrated in the figure. The moon exerts a torque on the earth for the same reason, and the combined effect of the sun and the moon is a precession of the spin axis, and hence γ, with a period of about 26,000 years. The moon's action also superimposes a small nutation on the precession. This causes the obliquity ε to vary with a maximum amplitude of 0.0025° over a period of 18.6 years.

Four thousand years ago, when the first recorded astronomical observations were being made, γ was located in the constellation Aries, the ram. The Greek letter γ is a descendent of the ancient Babylonian symbol resembling the head of a ram.

To the human eye, objects in the night sky appear as points on a ***celestial sphere*** surrounding the earth, as illustrated in Figure 4.3. The north and south poles of this fixed sphere correspond to those of the earth rotating within it. Coordinates of latitude and longitude are used to locate points on the celestial sphere in much the same way as on the surface of the earth. The projection of the earth's equatorial plane outward onto the celestial sphere defines the celestial equator. The vernal equinox γ, which lies on the celestial equator, is the origin for measurement of longitude, which in astronomical parlance is called ***right ascension***. Right ascension (RA or α) is measured along the celestial equator in degrees east from the vernal equinox. (Astronomers measure right ascension in hours instead of degrees, where 24 hours equals 360°.) Latitude on the celestial sphere is called ***declination***. Declination (Dec or δ) is measured along a meridian in degrees, positive to the north of the equator and negative to the south. Figure 4.4 is a sky chart showing how the heavenly grid appears from a given point on the earth. Notice that the sun is located at the intersection of the equatorial and ecliptic planes, so this must be the first day of spring.

Stars are so far away from the earth that their positions relative to each other appear stationary on the celestial sphere. Planets, comets, satellites, etc., move upon the fixed backdrop of the stars. A table of the coordinates of celestial bodies as a function of time is called an ***ephemeris***, for example, the *Astronomical Almanac* (U.S. Naval Observatory, 2008). Table 4.1 is an abbreviated ephemeris for the Moon and for Venus. An ephemeris depends on the location of the vernal equinox at a given time or epoch, for we know that even the positions of the stars relative to the equinox change slowly with time. For example, Table 4.2 shows the celestial coordinates of the star Regulus at five epochs since AD 1700. Currently, the position of

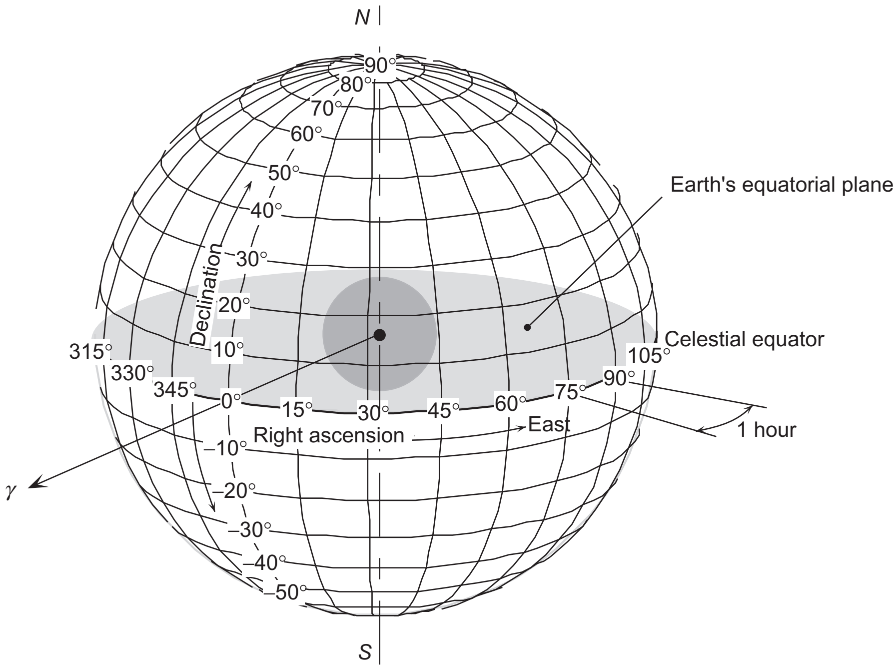


FIGURE 4.3

The celestial sphere, with grid lines of right ascension and declination.

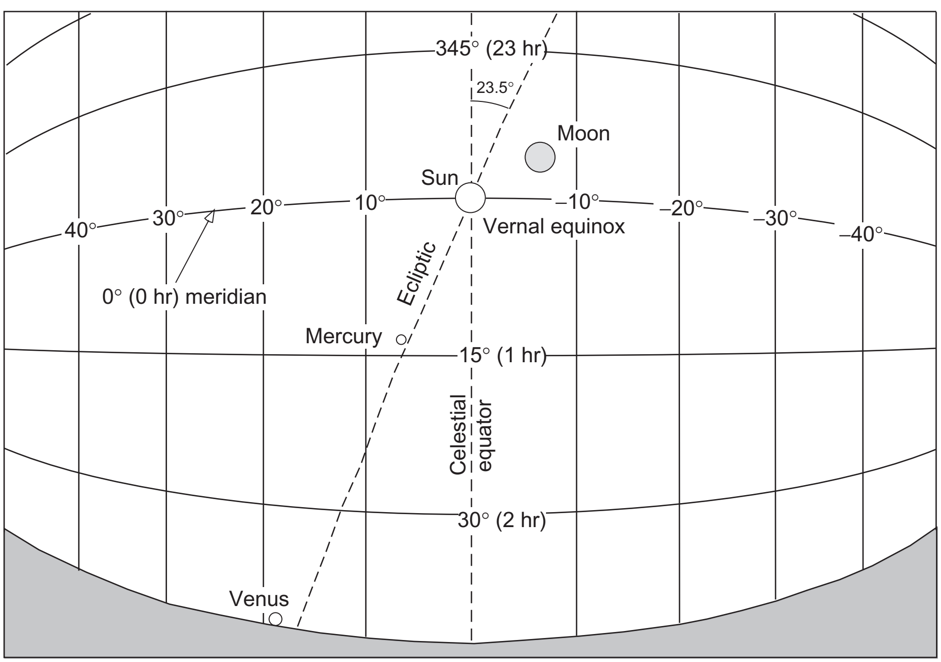


FIGURE 4.4

A view of the sky above the eastern horizon from 0° longitude on the equator at 9 am local time, 20 March, 2004 (Precession epoch AD 2000).

Table 4.1 Venus and Moon Ephemeris for 0 Hours Universal Time. (Precession Epoch: 2000 AD)

	Venus		Moon	
Date	**RA**	**Dec**	**RA**	**Dec**
1 Jan 2004	21 h 05.0 m	−18° 36′	1 h 44.9 m	+8° 47′
1 Feb 2004	23 h 28.0 m	−04° 30′	4 h 37.0 m	+24° 11′
1 Mar 2004	01 h 30.0 m	+10° 26′	6 h 04.0 m	+08° 32′
1 Apr 2004	03 h 37.6 m	+22° 51′	9 h 18.7 m	+21° 08′
1 May 2004	05 h 20.3 m	+27° 44′	11 h 28.8 m	+07° 53′
1 Jun 2004	05 h 25.9 m	+24° 43′	14 h 31.3 m	−14° 48′
1 Jul 2004	04 h 34.5 m	+17° 48′	17 h 09.0 m	−26° 08′
1 Aug 2004	05 h 37.4 m	+19° 04′	21 h 05.9 m	−21° 49′
1 Sep 2004	07 h 40.9 m	+19° 16′	00 h 17.0 m	−00° 56′
1 Oct 2004	09 h 56.5 m	+12° 42′	02 h 20.9 m	+14° 35′
1 Nov 2004	12 h 15.8 m	+00° 01′	05 h 26.7 m	+27° 18′
1 Dec 2004	14 h 34.3 m	−13° 21′	07 h 50.3 m	+26° 14′
1 Jan 2005	17 h 12.9 m	−22° 15′	10 h 49.4 m	+11° 39′

Table 4.2 Variation of the Coordinates of the Star Regulus Due to Precession of the Equinox

Precession Epoch	RA	Dec
1700 AD	9 h 52.2 m (148.05°)	+13° 25′
1800 AD	9 h 57.6 m (149.40°)	+12° 56′
1900 AD	10 h 3.0 m (150.75°)	+12° 27′
1950 AD	10 h 5.7 m (151.42°)	+12° 13′
2000 AD	10 h 8.4 m (152.10°)	+11° 58′

the vernal equinox in the year 2000 is used to define the standard grid of the celestial sphere. In 2025, the position will be updated to that of the year 2050, and so on at twenty-five year intervals. Since observations are made relative to the actual orientation of the earth, these measurements must be transformed into the standardized celestial frame of reference. As Table 4.2 suggests, the adjustments will be small if the current epoch is within 25 years of the standard precession epoch.

4.3 STATE VECTOR AND THE GEOCENTRIC EQUATORIAL FRAME

At any given time, the state vector of a satellite comprises its position **r** and orbital velocity **v**. Orbital mechanics is concerned with specifying or predicting state vectors over intervals of time. From Chapter 2, we know that the equation governing the state vector of a satellite traveling around the earth, is, under the familiar assumptions:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{4.1}$$

$\mathbf{r}$ is the position vector of the satellite relative to the center of the earth. The components of $\mathbf{r}$ and, especially, those of its time derivatives $\dot{\mathbf{r}} = \mathbf{v}$ and $\ddot{\mathbf{r}} = \mathbf{a}$, must be measured in a non-rotating frame attached to the earth. A commonly used non-rotating right-handed Cartesian coordinate system is the ***geocentric equatorial frame*** shown in Figure 4.5. The X-axis points in the vernal equinox direction. The XY plane is the earth's equatorial plane, and the Z-axis coincides with the earth's axis of rotation and points northward. The unit vectors $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ form a right-handed triad. The non-rotating geocentric equatorial frame serves as an inertial frame for the two-body earth satellite problem, as embodied in Equation 4.1. It is not truly an inertial frame, however, since the center of the earth is always accelerating towards a third body, the sun (to say nothing of the moon), a fact which we ignore in the two-body formulation.

In the geocentric equatorial frame the state vector is given in component form by

$$\mathbf{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}} \tag{4.2}$$

$$\mathbf{v} = v_X\hat{\mathbf{I}} + v_Y\hat{\mathbf{J}} + v_Z\hat{\mathbf{K}} \tag{4.3}$$

If r is the magnitude of the position vector, then

$$\mathbf{r} = r\hat{\mathbf{u}}_r \tag{4.4}$$

Figure 4.5 shows that the components of $\hat{\mathbf{u}}_r$ (the direction cosines l, m and n of $\mathbf{r}$) are found in terms of the right ascension α and declination δ as follows:

$$\hat{\mathbf{u}}_r = l\hat{\mathbf{I}} + m\hat{\mathbf{J}} + n\hat{\mathbf{K}} = \cos\delta\cos\alpha\hat{\mathbf{I}} + \cos\delta\sin\alpha\hat{\mathbf{J}} + \sin\delta\hat{\mathbf{K}} \tag{4.5}$$

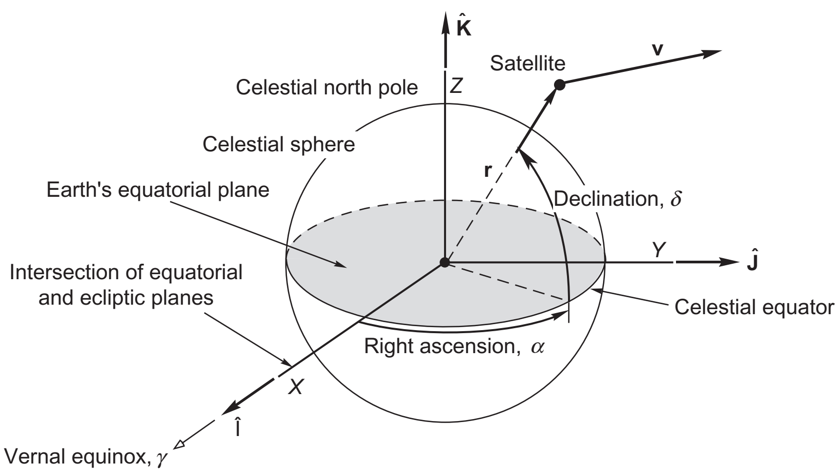


FIGURE 4.5

The geocentric equatorial frame.

From this we see that the declination is obtained as $\delta = \sin^{-1}n$. There is no quadrant ambiguity since, by definition, the declination lies between $-90°$ and $+90°$, which is precisely the range of the principal values of the arcsine function. It follows that $\cos\delta$ cannot be negative. Equation 4.5 also reveals that $l = \cos\delta\cos\alpha$. Hence, we find the right ascension from $\alpha = \cos^{-1}(l/\cos\delta)$, which yields two values of α between 0 and 360°. To determine the correct quadrant for α, we check the sign of the direction cosine $m = \cos\delta\sin\alpha$. Since $\cos\delta$ cannot be negative, the sign of m is the same as the sign of $\sin\alpha$. If $\sin\alpha > 0$ then α lies in the range 0 to 180°, whereas $\sin\alpha < 0$ means that α lies between 180° and 360°.

Algorithm 4.1 Given the position vector $\mathbf{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}}$, calculate the right ascension α and declination δ. This procedure is implemented in MATLAB® as *ra_and_dec_from_r.m*, which appears in Appendix D.17.

1. Calculate the magnitude of **r**:

$$r = \sqrt{X^2 + Y^2 + Z^2}$$

2. Calculate the direction cosines of **r**:

$$l = \frac{X}{r} \qquad m = \frac{Y}{r} \qquad n = \frac{Z}{r}$$

3. Calculate the declination:

$$\delta = \sin^{-1} n$$

4. Calculate the right ascension:

$$\alpha = \begin{cases} \cos^{-1}\left(\dfrac{l}{\cos\delta}\right) & (m > 0) \\ 360° - \cos^{-1}\left(\dfrac{l}{\cos\delta}\right) & (m \le 0) \end{cases}$$

Although the position vector furnishes the right ascension and declination, the right ascension and declination alone do not furnish **r**. For that we need the distance r in order to obtain the position vector from Equation 4.4.

Example 4.1

If the position vector of the International Space Station in the geocentric equatorial frame is

$$\mathbf{r} = -5368\hat{\mathbf{I}} - 1784\hat{\mathbf{J}} + 3691\hat{\mathbf{K}} \text{ (km)}$$

what are its right ascension and declination?

Solution

We employ Algorithm 4.1.

Step 1.

$$r = \sqrt{(-5368)^2 + (-1784)^2 + 3691^2} = 6754\,\text{km}$$

Step 2.

$$l = \frac{-5368}{6754} = -0.7947 \qquad m = \frac{-1784}{6754} = -0.2642 \qquad n = \frac{3691}{6754} = 0.5462$$

Step 3.

$$\delta = \sin^{-1} 0.5464 = \boxed{33.12°}$$

Step 4.

Since the direction cosine m is negative,

$$\alpha = 360° - \cos^{-1}\left(\frac{l}{\cos\delta}\right) = 360° - \cos^{-1}\left(\frac{-0.7947}{\cos 33.12°}\right) = 360° - 161.6° = \boxed{198.4°}$$

From Section 2.11 we know that if we are provided the state vector ($\mathbf{r}_0$,$\mathbf{v}_0$) at a given instant, then we can determine the state vector at any other time in terms of the initial vector by means of the expressions

$$\begin{aligned}\mathbf{r} &= f\mathbf{r}_0 + g\mathbf{v}_0 \\ \mathbf{v} &= \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0\end{aligned} \tag{4.6}$$

where the Lagrange coefficients f and g and their time derivatives are given in Equation 3.69. Specifying the total of six components of $\mathbf{r}_0$ and $\mathbf{v}_0$ therefore completely determines the size, shape and orientation of the orbit.

Example 4.2

At time t_0 the state vector of an earth satellite is

$$\mathbf{r}_0 = 1600\hat{\mathbf{I}} + 5310\hat{\mathbf{J}} + 3800\hat{\mathbf{K}}\ \text{(km)} \tag{a}$$

$$\mathbf{v}_0 = -7.350\hat{\mathbf{I}} + 0.4600\hat{\mathbf{J}} + 2.470\hat{\mathbf{K}}\ \text{(km/s)} \tag{b}$$

Determine the position and velocity 3200 seconds later and plot the orbit in three dimensions.

Solution

We will use the universal variable formulation and Algorithm 3.4, which was illustrated in detail in Example 3.7. Therefore, only the results of each step are presented here.

Step 1. (α here is not to be confused with the right ascension.)

$$\alpha = 1.4613 \times 10^{-4}\ \text{km}^{-1}.\ \text{Since this is positive, the orbit is an ellipse.}$$

Step 2.

$$\chi = 294.42\ \text{km}^{\frac{1}{2}}.$$

Step 3.

$$f = -0.94843 \quad \text{and} \quad g = -354.89\ \text{s}^{-1}.$$

Step 4.

$$\boxed{\mathbf{r} = 1090.9\hat{\mathbf{I}} - 5199.4\hat{\mathbf{J}} - 4480.6\hat{\mathbf{K}}\ (\text{km})} \Rightarrow r = 6949.8\ \text{km}$$

Step 5.

$$\dot{f} = 0.00045324\ \text{s}^{-1}, \dot{g} = -0.88479.$$

Step 6.

$$\boxed{\mathbf{v} = 7.2284\hat{\mathbf{I}} + 1.9997\hat{\mathbf{J}} - 0.46311\hat{\mathbf{K}}\ (\text{km/s})}$$

To plot the elliptical orbit, we observe that one complete revolution means a change in the eccentric anomaly E of 2π radians. According to Equation 3.57_2, the corresponding change in the universal anomaly is:

$$\chi = \sqrt{a}E = \sqrt{\frac{1}{\alpha}}E = \sqrt{\frac{1}{0.00014613}} \cdot 2\pi = 519.77\ \text{km}^{\frac{1}{2}}$$

Letting χ vary from 0 to 519.77 in small increments, we employ the Lagrange coefficient formulation (Equation 3.67 plus 3.69a and 3.69b) to compute:

$$\mathbf{r} = \left[1 - \frac{\chi^2}{r_0}C(\alpha\chi^2)\right]\mathbf{r}_0 + \left[\Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(\alpha\chi^2)\right]\mathbf{v}_0$$

where Δt for a given value of χ is given by Equation 3.49. Using a computer to plot the points obtained in this fashion yields Figure 4.6, which also shows the state vectors at t_0 and $t_0 + 3200$ s.

The previous example illustrates the fact that the six quantities or orbital elements comprising the state vector **r** and **v** completely determine the orbit. Other elements may be chosen. The classical orbital elements are introduced and related to the state vector in the next section.

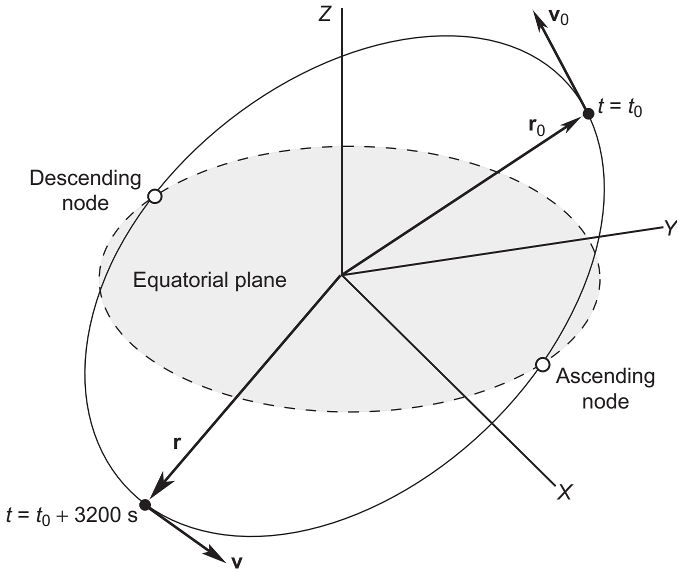


FIGURE 4.6

The orbit corresponding to the initial conditions given in Equations (a) and (b) of Example 4.2.

4.4 ORBITAL ELEMENTS AND THE STATE VECTOR

To define an orbit in the plane requires two parameters: eccentricity and angular momentum. Other parameters, such as the semimajor axis, the specific energy, and (for an ellipse) the period are obtained from these two. To locate a point on the orbit requires a third parameter, the true anomaly, which leads us to the time since perigee. Describing the orientation of an orbit in three dimensions requires three additional parameters, called the *Euler angles*, which are illustrated in Figure 4.7.

First, we locate the intersection of the orbital plane with the equatorial (XY) plane. That line is called the *node line*. The point on the node line where the orbit passes above the equatorial plane from below it is called the *ascending node*. The node line vector **N** extends outward from the origin through the ascending node. At the other end of the node line, where the orbit dives below the equatorial plane, is the descending node. The angle between the positive X-axis and the node line is the first Euler angle Ω, the right ascension of the ascending node. Recall from Section 4.2 that right ascension is a positive number lying between 0° and 360°.

The dihedral angle between the orbital plane and the equatorial plane is the inclination i, measured according to the right-hand rule, that is, counterclockwise around the node line vector from the equator to the orbit. The inclination is also the angle between the positive Z-axis and the normal to the plane of the orbit. The two equivalent means of measuring i are indicated in Figure 4.7. Recall from Chapter 2 that the angular momentum vector **h** is normal to the plane of the orbit. Therefore, the inclination i is the angle between the positive Z-axis and **h**. The inclination is a positive number between 0° and 180°.

It remains to locate the perigee of the orbit. Recall that perigee lies at the intersection of the eccentricity vector **e** with the orbital path. The third Euler angle ω, the argument of perigee, is the angle between the node line vector **N** and the eccentricity vector **e**, measured in the plane of the orbit. The argument of perigee is a positive number between 0° and 360°.

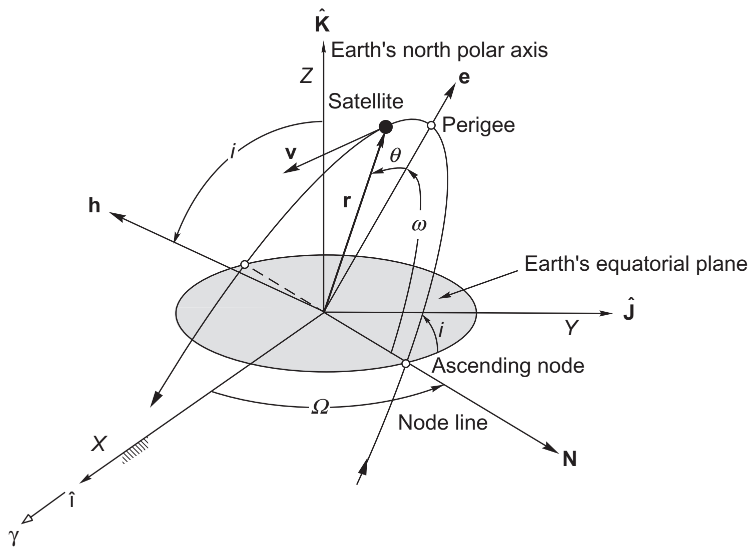


FIGURE 4.7

Geocentric equatorial frame and the orbital elements.

In summary, the six ***orbital elements*** are:

h specific angular momentum.
i inclination.
Ω right ascension (RA) of the ascending node.
e eccentricity.
ω argument of perigee.
θ true anomaly.

The angular momentum h and true anomaly θ are frequently replaced by the semimajor axis a and the mean anomaly M, respectively.

Given the position $\mathbf{r}$ and velocity $\mathbf{v}$ of a spacecraft in the geocentric equatorial frame, how do we obtain the orbital elements? The step-by-step procedure is outlined next in Algorithm 4.2. Note that each step incorporates results obtained in the previous steps. Several steps require resolving the quadrant ambiguity that arises in calculating the arccosine (recall Figure 3.4).

Algorithm 4.2 Obtain orbital elements from the state vector. A MATLAB version of this procedure appears in Appendix D.18. Applying this algorithm to orbits around other planets or the sun amounts to defining the frame of reference and substituting the appropriate gravitational parameter μ.

1. Calculate the distance:

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{X^2 + Y^2 + Z^2}$$

2. Calculate the speed:

$$v = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{{v_X}^2 + {v_Y}^2 + {v_Z}^2}$$

3. Calculate the radial velocity:

$$v_r = \mathbf{r} \cdot \mathbf{v}/r = (Xv_X + Yv_Y + Zv_Z)/r.$$

Note that if $v_r > 0$, the satellite is flying away from perigee. If $v_r < 0$, it is flying towards perigee.

4. Calculate the specific angular momentum:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ X & Y & Z \\ v_X & v_Y & v_Z \end{vmatrix}.$$

5. Calculate the magnitude of the specific angular momentum,

$$h = \sqrt{\mathbf{h} \cdot \mathbf{h}}$$

the first orbital element.

6. Calculate the inclination:

$$i = \cos^{-1}\left(\frac{h_Z}{h}\right) \tag{4.7}$$

This is the second orbital element. Recall that i must lie between 0° and 180°, which is precisely the range (principal values) of the arccosine function. Hence, there is no quadrant ambiguity to contend with here. If $90° < i \leq 180°$, the angular momentum **h** points in a southerly direction. In that case the orbit is retrograde, which means that the motion of the satellite around the earth is opposite to the earth's rotation.

7. Calculate:

$$\mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ h_X & h_Y & h_Z \end{vmatrix} \tag{4.8}$$

This vector defines the node line.

8. Calculate the magnitude of **N**:

$$N = \sqrt{\mathbf{N} \cdot \mathbf{N}}.$$

9. Calculate the right ascension of the ascending node:

$$\Omega = \cos^{-1}(N_X/N)$$

the third orbital element. If $(N_X/N) > 0$, then Ω lies in either the first or fourth quadrant. If $(N_X/N) < 0$, then Ω lies in either the second or third quadrant. To place Ω in the proper quadrant, observe that the ascending node lies on the positive side of the vertical XZ plane ($0 \le \Omega < 180°$) if $N_Y > 0$. On the other hand, the ascending node lies on the negative side of the XZ plane ($180° \le \Omega < 360°$) if $N_Y < 0$. Therefore, $N_Y > 0$ implies that $0 < \Omega < 180°$, whereas $N_Y < 0$ implies that $180° < \Omega < 360°$. In summary,

$$\Omega = \begin{cases} \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y \ge 0) \\ 360° - \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y < 0) \end{cases} \tag{4.9}$$

10. Calculate the eccentricity vector. Starting with Equation 2.40,

$$\mathbf{e} = \frac{1}{\mu}\left[\mathbf{v} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\mathbf{v} \times (\mathbf{r} \times \mathbf{v}) - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\overbrace{\mathbf{r}v^2 - \mathbf{v}(\mathbf{r}\cdot\mathbf{v})}^{bac-cab\ \text{rule}} - \mu\frac{\mathbf{r}}{r}\right]$$

so that

$$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - rv_r\mathbf{v}\right] \tag{4.10}$$

11. Calculate the eccentricity,

$$e = \sqrt{\mathbf{e}\cdot\mathbf{e}}$$

the fourth orbital element. Substituting Equation 4.10 leads to a form depending only on the scalars obtained thus far:

$$e = \sqrt{1 + \frac{h^2}{\mu^2}\left(v^2 - \frac{2\mu}{r}\right)^2} \tag{4.11}$$

12. Calculate the argument of perigee,

$$\omega = \cos^{-1}\left(\frac{\mathbf{N}}{N}\cdot\frac{\mathbf{e}}{e}\right)$$

the fifth orbital element. If $\mathbf{N}\cdot\mathbf{e} > 0$, then ω lies in either the first or fourth quadrant. If $\mathbf{N}\cdot\mathbf{e} < 0$, then ω lies in either the second or third quadrant. To place ω in the proper quadrant, observe that perigee lies above the equatorial plane ($0 \le \omega < 180°$) if $\mathbf{e}$ points up (in the positive Z direction), and perigee lies below the plane ($180° \leq \omega < 360°$) if **e** points down. Therefore, $e_Z \geq 0$ implies that $0 < \omega < 180°$, whereas $e_Z < 0$ implies that $180° < \omega < 360°$. To summarize:

$$\omega = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z < 0) \end{cases} \tag{4.12}$$

13. Calculate the true anomaly,

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}}{e}\cdot\frac{\mathbf{r}}{r}\right)$$

the sixth and final orbital element. If $\mathbf{e}\cdot\mathbf{r} > 0$, then θ lies in the first or fourth quadrant. If $\mathbf{e}\cdot\mathbf{r} < 0$, then θ lies in the second or third quadrant. To place θ in the proper quadrant, note that if the satellite is flying away from perigee ($\mathbf{r}\cdot\mathbf{v} \geq 0$), then $0 \leq \theta < 180°$, whereas if the satellite is flying towards perigee ($\mathbf{r}\cdot\mathbf{v} < 0$), then $180° \leq \theta < 360°$. Therefore, using the results of step 3 above:

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{e}}{e}\cdot\dfrac{\mathbf{r}}{r}\right) & (v_r \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{e}}{e}\cdot\dfrac{\mathbf{r}}{r}\right) & (v_r < 0) \end{cases} \tag{4.13a}$$

Substituting Equation 4.10 yields an alternative form of this expression,

$$\theta = \begin{cases} \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r \geq 0) \\ 360° - \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r < 0) \end{cases} \tag{4.13b}$$

The procedure described above for calculating the orbital elements is not unique.

Example 4.3

Given the state vector:

$$\mathbf{r} = -6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}\ \text{(km)}$$
$$\mathbf{v} = -3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}}\ \text{(km/s)}$$

find the orbital elements h, i, Ω, e, ω and θ using Algorithm 4.2.

Solution

Step 1.

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{(-6045)^2 + (-3490)^2 + 2500^2} = 7414 \text{ km} \tag{a}$$

Step 2.

$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(-3.457)^2 + 6.618^2 + 2.533^2} = 7.884 \text{ km/s} \tag{b}$$

Step 3.

$$v_r = \frac{\mathbf{v}\cdot\mathbf{r}}{r} = \frac{(-3.457)\cdot(-6045) + 6.618\cdot(-3490) + 2.533\cdot 2500}{7414} = 0.5575 \text{ km/s} \tag{c}$$

Since $v_r > 0$, the satellite is flying away from perigee.

Step 4.

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -6045 & -3490 & 2500 \\ -3.457 & 6.618 & 2.533 \end{vmatrix} = -25{,}380\hat{\mathbf{I}} + 6670\hat{\mathbf{J}} - 52{,}070\hat{\mathbf{K}} \ (\text{km}^2/\text{s}) \tag{d}$$

Step 5.

$$h = \sqrt{\mathbf{h}\cdot\mathbf{h}} = \sqrt{(-25{,}380)^2 + 6670^2 + (-52{,}070)^2} = \boxed{58{,}310 \text{ km}^2/\text{s}} \tag{e}$$

Step 6.

$$i = \cos^{-1}\frac{h_Z}{h} = \cos^{-1}\left(\frac{-52{,}070}{58{,}310}\right) = \boxed{153.2^\circ} \tag{f}$$

Since i is greater than 90°, this is a retrograde orbit.

Step 7.

$$\mathbf{N} = \hat{\mathbf{K}}\times\mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ -25{,}380 & 6670 & -52{,}070 \end{vmatrix} = -6670\hat{\mathbf{I}} - 25{,}380\hat{\mathbf{J}} \quad (\text{km}^2/\text{s}) \tag{g}$$

Step 8.

$$N = \sqrt{\mathbf{N}\cdot\mathbf{N}} = \sqrt{(-6670)^2 + (-25{,}380)^2} = 26{,}250 \text{ km}^2/\text{s} \tag{h}$$

Step 9.

$$\Omega = \cos^{-1}\frac{N_X}{N} = \cos^{-1}\left(\frac{-6670}{26{,}250}\right) = 104.7^\circ \text{ or } 255.3^\circ$$

From (g) we know that $N_Y < 0$; therefore, Ω must lie in the third quadrant,

$$\boxed{\Omega = 255.3°} \tag{i}$$

Step 10.

$$\begin{aligned}\mathbf{e} &= \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - rv_r\mathbf{v}\right] \\ &= \frac{1}{398{,}600}\left[\left(7.884^2 - \frac{398{,}600}{7414}\right)(-6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}) - (7414)(0.5575)\right. \\ &\qquad \left.(-3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}})\right] \\ \mathbf{e} &= -0.09160\hat{\mathbf{I}} - 0.1422\hat{\mathbf{J}} + 0.02644\hat{\mathbf{K}}\end{aligned} \tag{j}$$

Step 11.

$$e = \sqrt{\mathbf{e}\cdot\mathbf{e}} = \sqrt{(-0.09160)^2 + (-0.1422)^2 + (0.02644)^2} = \boxed{0.1712} \tag{k}$$

Clearly, the orbit is an ellipse.
Step 12.

$$\begin{aligned}\omega &= \cos^{-1}\frac{\mathbf{N}\cdot\mathbf{e}}{Ne} = \cos^{-1}\left[\frac{(-6670)(-0.09160) + (-25{,}380)(-0.1422) + (0)(0.02644)}{(26{,}250)(0.1712)}\right] \\ &= 20.07° \text{ or } 339.9°\end{aligned}$$

ω lies in the first quadrant if $e_Z > 0$, which true in this case, as we see from (j). Therefore,

$$\boxed{\omega = 20.07°} \tag{l}$$

Step 13.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = \cos^{-1}\left[\frac{(-0.09160)(-6045) + (-0.1422)\cdot(-3490) + (0.02644)(2500)}{(0.1712)(7414)}\right] \\ &= 28.45° \text{ or } 331.6°\end{aligned}$$

From (c) we know that $v_r > 0$, which means $0 \le \theta < 180°$. Therefore,

$$\boxed{\theta = 28.45°}$$

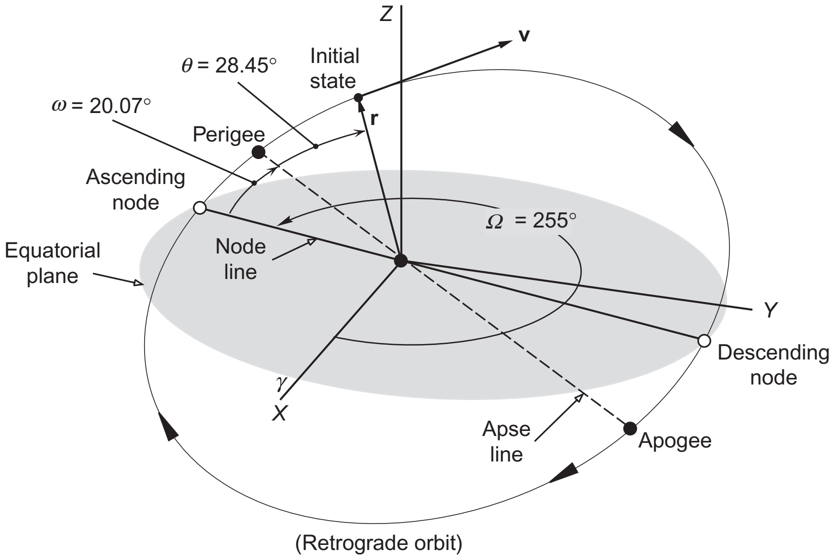


FIGURE 4.8

A plot of the orbit identified in Example 4.3.

Having found the orbital elements, we can go on to compute other parameters. The perigee and apogee radii are:

$$r_p = \frac{h^2}{\mu}\frac{1}{1 + e\cos(0)} = \frac{58{,}310^2}{398{,}600}\frac{1}{1 + 0.1712} = 7284 \text{ km}$$

$$r_a = \frac{h^2}{\mu}\frac{1}{1 + e\cos(180°)} = \frac{58{,}310^2}{398{,}600}\frac{1}{1 - 0.1712} = 10{,}290 \text{ km}$$

From these it follows that the semimajor axis of the ellipse is:

$$a = \frac{1}{2}(r_p + r_a) = 8788 \text{ km}$$

This leads to the period:

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = 2.278 \text{ hr}$$

The orbit is illustrated in Figure 4.8.

We have seen how to obtain the orbital elements from the state vector. To arrive at the state vector, given the orbital elements, requires performing coordinate transformations, which are discussed in the next section.

4.5 COORDINATE TRANSFORMATION

The Cartesian coordinate system was introduced in Section 1.2. Figure 4.9 shows two such coordinate systems: the unprimed system with axes xyz, and the primed system with axes $x'y'z'$. The orthogonal unit basis vectors for the unprimed system are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. The fact they are unit vectors means

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1 \tag{4.14}$$

Since they are orthogonal,

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0 \tag{4.15}$$

The orthonormal basis vectors $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ of the primed system share these same properties. That is,

$$\hat{\mathbf{i}}'\cdot\hat{\mathbf{i}}' = \hat{\mathbf{j}}'\cdot\hat{\mathbf{j}}' = \hat{\mathbf{k}}'\cdot\hat{\mathbf{k}}' = 1 \tag{4.16}$$

and

$$\hat{\mathbf{i}}'\cdot\hat{\mathbf{j}}' = \hat{\mathbf{i}}'\cdot\hat{\mathbf{k}}' = \hat{\mathbf{j}}'\cdot\hat{\mathbf{k}}' = 0 \tag{4.17}$$

We can express the unit vectors of the primed system in terms of their components in the unprimed system as follows:

$$\begin{aligned}\hat{\mathbf{i}}' &= Q_{11}\hat{\mathbf{i}} + Q_{12}\hat{\mathbf{j}} + Q_{13}\hat{\mathbf{k}}\\ \hat{\mathbf{j}}' &= Q_{21}\hat{\mathbf{i}} + Q_{22}\hat{\mathbf{j}} + Q_{23}\hat{\mathbf{k}}\\ \hat{\mathbf{k}}' &= Q_{31}\hat{\mathbf{i}} + Q_{32}\hat{\mathbf{j}} + Q_{33}\hat{\mathbf{k}}\end{aligned} \tag{4.18}$$

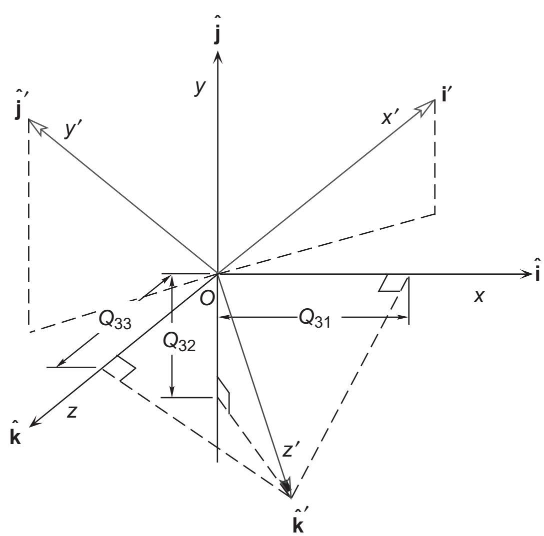


FIGURE 4.9

Two sets of Cartesian reference axes, xyz and $x'y'z'$.

The Q's in these expressions are just the direction cosines of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$. Figure 4.9 illustrates the components of $\hat{\mathbf{k}}'$, which are, of course, the projections of $\hat{\mathbf{k}}'$ onto the x, y and z axes. The unprimed unit vectors may be resolved into components along the primed system to obtain a set of equations similar to Equation 4.18.

$$\begin{aligned}\hat{\mathbf{i}} &= Q'_{11}\hat{\mathbf{i}}' + Q'_{12}\hat{\mathbf{j}}' + Q'_{13}\hat{\mathbf{k}}' \\ \hat{\mathbf{j}} &= Q'_{21}\hat{\mathbf{i}}' + Q'_{22}\hat{\mathbf{j}}' + Q'_{23}\hat{\mathbf{k}}' \\ \hat{\mathbf{k}} &= Q'_{31}\hat{\mathbf{i}}' + Q'_{32}\hat{\mathbf{j}}' + Q'_{33}\hat{\mathbf{k}}'\end{aligned} \tag{4.19}$$

However, $\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{i}}'$, so that, from Equations 4.18_1 and 4.19_1, we find $Q_{11} = Q'_{11}$. Likewise, $\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{i}}'$, which, according to Equations 4.18_1 and 4.19_2, means $Q_{12} = Q'_{21}$. Proceeding in this fashion, it is clear that the direction cosines in Equation 4.19 may be expressed in terms of those in Equation 4.18. That is, Equation 4.19 may be written:

$$\begin{aligned}\hat{\mathbf{i}} &= Q_{11}\hat{\mathbf{i}}' + Q_{21}\hat{\mathbf{j}}' + Q_{31}\hat{\mathbf{k}}' \\ \hat{\mathbf{j}} &= Q_{12}\hat{\mathbf{i}}' + Q_{22}\hat{\mathbf{j}}' + Q_{32}\hat{\mathbf{k}}' \\ \hat{\mathbf{k}} &= Q_{13}\hat{\mathbf{i}}' + Q_{23}\hat{\mathbf{j}}' + Q_{33}\hat{\mathbf{k}}'\end{aligned} \tag{4.20}$$

Substituting Equation 4.20 into Equation 4.14 and making use of Equations 4.16 and 4.17, we get the three relations

$$\begin{aligned}\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = 1 \quad &\Rightarrow \quad Q_{11}^2 + Q_{21}^2 + Q_{31}^2 = 1 \\ \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1 \quad &\Rightarrow \quad Q_{12}^2 + Q_{22}^2 + Q_{32}^2 = 1 \\ \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \quad &\Rightarrow \quad Q_{13}^2 + Q_{23}^2 + Q_{33}^2 = 1\end{aligned} \tag{4.21}$$

Substituting Equation 4.20 into Equation 4.15 and, again, making use of Equation 4.16 and 4.17, we obtain the three equations

$$\begin{aligned}\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0 \quad &\Rightarrow \quad Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} = 0 \\ \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = 0 \quad &\Rightarrow \quad Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} = 0 \\ \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \quad &\Rightarrow \quad Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} = 0\end{aligned} \tag{4.22}$$

Let $[\mathbf{Q}]$ represent the matrix of direction cosines of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ relative to $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, as given by Equation 4.18. $[\mathbf{Q}]$ is referred to as the ***direction cosine matrix*** or *DCM*.

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{j}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{k}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}} \end{bmatrix} \tag{4.23}$$

The transpose of the matrix $[\mathbf{Q}]$, denoted $[\mathbf{Q}]^T$, is obtained by interchanging the rows and columns of $[\mathbf{Q}]$. Thus,

$$[\mathbf{Q}]^T = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{i}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{i}} \cdot \hat{\mathbf{k}}' \\ \hat{\mathbf{j}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{j}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{j}} \cdot \hat{\mathbf{k}}' \\ \hat{\mathbf{k}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{k}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' \end{bmatrix} \tag{4.24}$$

Forming the product $[\mathbf{Q}]^T[\mathbf{Q}]$, we get:

$$[\mathbf{Q}]^T[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$
$$= \begin{bmatrix} Q_{11}^2 + Q_{21}^2 + Q_{31}^2 & Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} & Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} \\ Q_{12}Q_{11} + Q_{22}Q_{21} + Q_{32}Q_{31} & Q_{12}^2 + Q_{22}^2 + Q_{32}^2 & Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} \\ Q_{13}Q_{11} + Q_{23}Q_{21} + Q_{33}Q_{31} & Q_{13}Q_{12} + Q_{23}Q_{22} + Q_{33}Q_{32} & Q_{13}^2 + Q_{23}^2 + Q_{33}^2 \end{bmatrix}$$

From this we obtain, with the aid of Equations 4.21 and 4.22,

$$[\mathbf{Q}]^T[\mathbf{Q}] = [\mathbf{1}] \tag{4.25}$$

where

$$[\mathbf{1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[\mathbf{1}]$ stands for the identity matrix or unit matrix.

In a similar fashion, we can substitute Equation 4.18 into Equations 4.16 and 4.17 and make use of Equations 4.14 and 4.15 to finally obtain:

$$[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{1}] \tag{4.26}$$

Since $[\mathbf{Q}]$ satisfies Equations 4.25 and 4.26, it is called an ***orthogonal matrix***.

Let $\mathbf{v}$ be a vector. It can be expressed in terms of its components along the unprimed system:

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

or along the primed system

$$\mathbf{v} = v_x'\hat{\mathbf{i}}' + v_y'\hat{\mathbf{j}}' + v_z'\hat{\mathbf{k}}'$$

These two expressions for $\mathbf{v}$ are equivalent ($\mathbf{v} = \mathbf{v}$) since a vector is independent of the coordinate system used to describe it. Thus,

$$v_x'\hat{\mathbf{i}}' + v_y'\hat{\mathbf{j}}' + v_z'\hat{\mathbf{k}}' = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \tag{4.27}$$

Substituting Equation 4.20 into the right-hand side of Equation 4.27 yields

$$v_x'\hat{\mathbf{i}}' + v_y'\hat{\mathbf{j}}' + v_z'\hat{\mathbf{k}}' = v_x(Q_{11}\hat{\mathbf{i}}' + Q_{21}\hat{\mathbf{j}}' + Q_{31}\hat{\mathbf{k}}') + v_y(Q_{12}\hat{\mathbf{i}}' + Q_{22}\hat{\mathbf{j}}' + Q_{32}\hat{\mathbf{k}}') + v_z(Q_{13}\hat{\mathbf{i}}' + Q_{23}\hat{\mathbf{j}}' + Q_{33}\hat{\mathbf{k}}')$$

Upon collecting terms on the right, we get

$$v'_x\hat{\mathbf{i}}' + v'_y\hat{\mathbf{j}}' + v'_z\hat{\mathbf{k}}' = (Q_{11}v_x + Q_{12}v_y + Q_{13}v_z)\hat{\mathbf{i}}' + (Q_{21}v_x + Q_{22}v_y + Q_{23}v_z)\hat{\mathbf{j}}' + (Q_{31}v_x + Q_{32}v_y + Q_{33}v_z)\hat{\mathbf{k}}'$$

Equating the components of like unit vectors on each side of the equals sign yields

$$\begin{aligned} v'_x &= Q_{11}v_x + Q_{12}v_y + Q_{13}v_z \\ v'_y &= Q_{21}v_x + Q_{22}v_y + Q_{23}v_z \\ v'_z &= Q_{31}v_x + Q_{32}v_y + Q_{33}v_z \end{aligned} \tag{4.28}$$

In matrix notation, this may be written

$$\{\mathbf{v}'\} = [\mathbf{Q}]\{\mathbf{v}\} \tag{4.29}$$

where

$$\{\mathbf{v}'\} = \begin{Bmatrix} v'_x \\ v'_y \\ v'_z \end{Bmatrix} \qquad \{\mathbf{v}\} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} \tag{4.30}$$

and $[\mathbf{Q}]$ is given by Equation 4.23. Equation 4.28 (or Equation 4.29) shows how to transform the components of the vector $\mathbf{v}$ in the unprimed system into its components in the primed system. The inverse transformation, from primed to unprimed, is found by multiplying Equation 4.29 through by $[\mathbf{Q}]^T$:

$$[\mathbf{Q}]^T\{\mathbf{v}'\} = [\mathbf{Q}]^T[\mathbf{Q}]\{\mathbf{v}\}$$

But, according to Equation 4.25, $[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{1}]$, so that

$$[\mathbf{Q}]^T\{\mathbf{v}'\} = [\mathbf{1}]\{\mathbf{v}\}$$

Since $[\mathbf{1}]\{\mathbf{v}\} = \{\mathbf{v}\}$, we obtain

$$\{\mathbf{v}\} = [\mathbf{Q}]^T\{\mathbf{v}'\} \tag{4.31}$$

Therefore, to go from the unprimed system to the primed system we use $[\mathbf{Q}]$, and in the reverse direction—from primed to unprimed—we use $[\mathbf{Q}]^T$.

Example 4.4

In Figure 4.10, the x' axis is defined by the line segment $O'P$. The $x'y'$ plane is defined by the intersecting line segments $O'P$ and $O'Q$. The z' axis is normal to the plane of $O'P$ and $O'Q$ and obtained by rotating $O'P$ towards $O'Q$ and using the right-hand rule.

(a) Find the direction cosine matrix $[\mathbf{Q}]$.
(b) If $\{\mathbf{v}\} = \lfloor 2\ 4\ 6 \rfloor^T$, find $\{\mathbf{v}'\}$.
(c) If $\{\mathbf{v}'\} = \lfloor 2\ 4\ 6 \rfloor^T$, find $\{\mathbf{v}\}$.

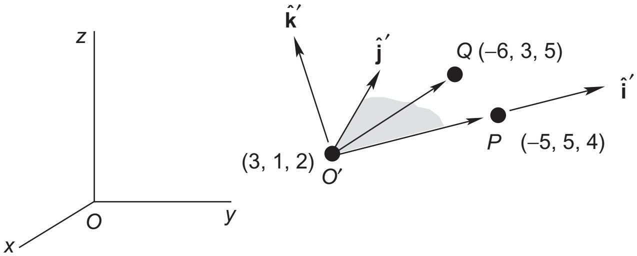


FIGURE 4.10

Defining a unit triad from the coordinates of three noncollinear points, O', P and Q.

Solution

(a) Resolve the directed line segments $\overrightarrow{O'P}$ and $\overrightarrow{O'Q}$ into components along the unprimed system:

$$\overrightarrow{O'P} = (-5-3)\hat{\mathbf{i}} + (5-1)\hat{\mathbf{j}} + (4-2)\hat{\mathbf{k}} = -8\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$
$$\overrightarrow{O'Q} = (-6-3)\hat{\mathbf{i}} + (3-1)\hat{\mathbf{j}} + (5-2)\hat{\mathbf{k}} = -9\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

Taking the cross product of $\overrightarrow{O'P}$ into $\overrightarrow{O'Q}$ yields a vector $\mathbf{Z}'$, which lies in the direction of the desired positive z' axis:

$$\mathbf{Z}' = \overrightarrow{O'P} \times \overrightarrow{O'Q} = 8\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 20\hat{\mathbf{k}}$$

Taking the cross product of $\mathbf{Z}'$ into $\overrightarrow{O'P}$ then yields a vector $\mathbf{Y}'$ which points in the positive y' direction:

$$\mathbf{Y}' = \mathbf{Z} \times \overrightarrow{O'P} = -68\hat{\mathbf{i}} - 176\hat{\mathbf{j}} + 80\hat{\mathbf{k}}$$

Normalizing the vectors $\overrightarrow{O'P}$, $\mathbf{Y}'$ and $\mathbf{Z}'$ produces the $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ unit vectors, respectively. Thus,

$$\hat{\mathbf{i}}' = \frac{\overrightarrow{O'P}}{\left\|\overrightarrow{O'P}\right\|} = -0.8729\hat{\mathbf{i}} + 0.4364\hat{\mathbf{j}} + 0.2182\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}}' = \frac{\mathbf{Y}'}{\|\mathbf{Y}'\|} = -0.3318\hat{\mathbf{i}} - 0.8588\hat{\mathbf{j}} + 0.3904\hat{\mathbf{k}}$$

and,

$$\hat{\mathbf{k}}' = \frac{\mathbf{Z}'}{\|\mathbf{Z}'\|} = 0.3578\hat{\mathbf{i}} + 0.2683\hat{\mathbf{j}} + 0.8944\hat{\mathbf{k}}$$

The components of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ are the rows of the direction cosine matrix [**Q**]. Thus,

$$[\mathbf{Q}] = \begin{bmatrix} -0.8729 & 0.4364 & 0.2182 \\ -0.3318 & -0.8588 & 0.3904 \\ 0.3578 & 0.2683 & 0.8944 \end{bmatrix}$$

(b)

$$\{\mathbf{v}'\} = [\mathbf{Q}]\{\mathbf{v}\} = \begin{bmatrix} -0.8729 & 0.4364 & 0.2182 \\ -0.3318 & -0.8588 & 0.3904 \\ 0.3578 & 0.2683 & 0.8944 \end{bmatrix} \begin{Bmatrix} 2 \\ 4 \\ 6 \end{Bmatrix} = \begin{Bmatrix} 1.309 \\ -1.756 \\ 7.155 \end{Bmatrix}$$

(c)

$$\{\mathbf{v}\} = [\mathbf{Q}]^T\{\mathbf{v}'\} = \begin{bmatrix} -0.8729 & -0.3318 & 0.3578 \\ 0.4364 & -0.8588 & 0.2683 \\ 0.2182 & 0.3904 & 0.8944 \end{bmatrix} \begin{Bmatrix} 2 \\ 4 \\ 6 \end{Bmatrix} = \begin{Bmatrix} -0.9263 \\ -0.9523 \\ 7.364 \end{Bmatrix}$$

Let us consider the special case in which the coordinate transformation involves a rotation about only one of the coordinate axes, as shown in Figure 4.11. If the rotation is about the x axis, then according to Equations 4.18 and 4.23,

$$\begin{aligned} \hat{\mathbf{i}}' &= \hat{\mathbf{i}} \\ \hat{\mathbf{j}}' &= (\hat{\mathbf{j}}' \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{j}}' \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{j}} + \cos(90 - \phi)\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{j}} + \sin(\phi)\hat{\mathbf{k}} \\ \hat{\mathbf{k}}' &= (\hat{\mathbf{k}}' \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \cos(90^\circ + \phi)\hat{\mathbf{j}} + \cos\phi\hat{\mathbf{k}} = -\sin\phi\hat{\mathbf{j}} + \cos\phi\hat{\mathbf{k}} \end{aligned}$$

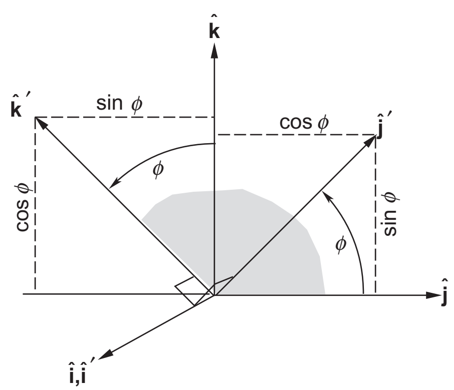


FIGURE 4.11

Rotation about the x-axis.

or

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

The transformation from the xyz coordinate system to the $xy'z'$ system having a common x-axis is given by the direction cosine matrix on the right. Since this is a rotation through the angle ϕ about the x-axis, we denote this matrix by $[\mathbf{R}_1(\phi)]$, in which the subscript 1 stands for axis 1 (the x-axis). Thus,

$$[\mathbf{R}_1(\phi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{4.32}$$

If the rotation is about the y-axis, as shown in Figure 4.12, then Equation 4.18 yields

$$\begin{aligned} \hat{\mathbf{i}}' &= (\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{i}}' \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{i}} + \cos(\phi + 90°)\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{i}} - \sin\phi\hat{\mathbf{k}} \\ \hat{\mathbf{j}}' &= \hat{\mathbf{j}} \\ \hat{\mathbf{k}}' &= (\hat{\mathbf{k}}' \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \cos(90° - \phi)\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{k}} = \sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{k}} \end{aligned}$$

or, more compactly,

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

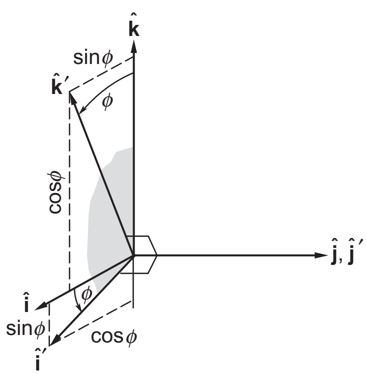


FIGURE 4.12

Rotation about the y-axis.

We represent this transformation between two Cartesian coordinate systems having a common y-axis (axis 2) as $[\mathbf{R}_2(\phi)]$. Therefore,

$$[\mathbf{R}_2(\phi)] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \tag{4.33}$$

Finally, if the rotation is about the z-axis, as shown in Figure 4.13, then we have from Equation 4.18 that

$$\begin{aligned} \hat{\mathbf{i}}' &= (\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} = \cos\phi\hat{\mathbf{i}} + \cos(90^\circ - \phi)\hat{\mathbf{j}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \\ \hat{\mathbf{j}}' &= (\hat{\mathbf{j}}' \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} = \cos(90^\circ + \phi)\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \\ \hat{\mathbf{k}}' &= \hat{\mathbf{k}} \end{aligned}$$

or

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

In this case the rotation is around axis 3, the z-axis, so

$$[\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.34}$$

The single transformation between the xyz and $x'y'z'$ Cartesian coordinate frames in Figure 4.9 can be viewed as a sequence of three coordinate transformations, starting with xyz:

$$xyz \overbrace{\longrightarrow}^{\alpha} x_1y_1z_1 \overbrace{\longrightarrow}^{\beta} x_2y_2z_2 \overbrace{\longrightarrow}^{\gamma} x'y'z'$$

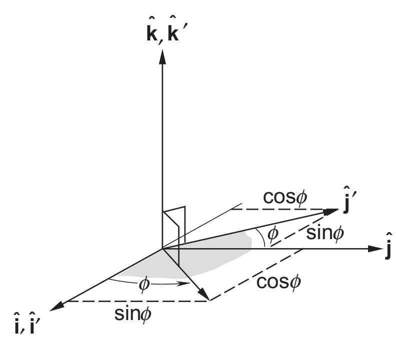


FIGURE 4.13

Rotation about the z-axis.

Each coordinate system is obtained from the previous one by means of an elementary rotation about one of the axes of the previous frame. Two successive rotations cannot be about the same axis. The first rotation angle is α, the second one is β and the final one is γ. In specific applications, the Greek letters that are traditionally used to represent the three rotations are not α, β, and γ. For those new to the subject, however, it might initially be easier to remember that the first, second and third rotation angles are represented by the first, second and third letters of the Greek alphabet ($\alpha\beta\gamma$). Each one of the three transformations has the direction cosine matrix $[\mathbf{R}_i(\phi)]$, where $i = 1$, 2 or 3 and $\phi = \alpha$, β or γ. The sequence of three such elementary rotations relating two different Cartesian frames of reference is called an ***Euler angle sequence***. Each of the twelve possible Euler angle sequences has a direction cosine matrix $[\mathbf{Q}]$, which is the product of three elementary rotation matrices. The six symmetric Euler sequences are those that begin and end with rotation about the same axis:

$$\begin{array}{ll} [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_1(\alpha)] & [\mathbf{R}_1(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_1(\alpha)] \\ [\mathbf{R}_2(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_2(\alpha)] & [\mathbf{R}_2(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_2(\alpha)] \\ [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] & [\mathbf{R}_3(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] \end{array} \tag{4.35}$$

The asymmetric Euler sequences involve rotations about all three axes:

$$\begin{array}{ll} [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] & [\mathbf{R}_1(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_2(\alpha)] \\ [\mathbf{R}_2(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_1(\alpha)] & [\mathbf{R}_2(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] \\ [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_2(\alpha)] & [\mathbf{R}_3(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_1(\alpha)] \end{array} \tag{4.36}$$

One of the symmetric sequences which has frequent application in space mechanics is the ***classical Euler angle sequence***,

$$[\mathbf{Q}] = [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] \qquad (0 \le \alpha < 360^\circ \quad 0 \le \beta \le 180^\circ \quad 0 \le \gamma < 360^\circ) \tag{4.37}$$

which is illustrated in Figure 4.14. The first rotation is around the z-axis, through the angle α. It rotates the x and y axes into the x_1 and y_1 directions. Viewed down the z-axis, this rotation appears as shown in the insert at the top of the figure. The direction cosine matrix associated with this rotation is $[\mathbf{R}_3(\alpha)]$. The subscript means that the rotation is around the current '3' direction, which was the z axis (and is now the z_1 axis). The second rotation, represented by $[\mathbf{R}_1(\beta)]$, is around the x_1 axis through the angle β required to rotate the z_1 axis into the z_2 axis, which coincides with the target z' axis. Simultaneously, y_1 rotates into y_2. The insert in the lower right of Figure 4.14 shows how this rotation appears when viewed from the x_1 direction. $[\mathbf{R}_3(\gamma)]$ represents the third and final rotation, which rotates the x_2 axis (formerly the x_1 axis) and the y_2 axis through the angle γ around the z' axis so that they become aligned with the target x' and y' axes, respectively. This rotation appears from the z'-direction as shown in the insert on the left of Figure 4.14.

Applying the transformation in Equation 4.37 to the xyz components $\{\mathbf{b}\}_x$ of the vector $\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$ yields the components of the same vector in the $x'y'z'$ frame

$$\{\mathbf{b}\}_{x'} = [\mathbf{Q}]\{\mathbf{b}\}_x \qquad (\mathbf{b} = b_{x'}\hat{\mathbf{i}}' + b_{y'}\hat{\mathbf{j}}' + b_z{}'\hat{\mathbf{k}}')$$

That is,

$$[\mathbf{Q}]\{\mathbf{b}\}_x = [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)]\overbrace{[\mathbf{R}_3(\alpha)]\{\mathbf{b}\}_x}^{\{\mathbf{b}\}_{x_1}} = [\mathbf{R}_3(\gamma)]\overbrace{[\mathbf{R}_1(\beta)]\{\mathbf{b}\}_{x_1}}^{\{\mathbf{b}\}_{x_2}} = \overbrace{[\mathbf{R}_3(\gamma)]\{\mathbf{b}\}_{x_2}}^{\{\mathbf{b}\}_{x'}}$$

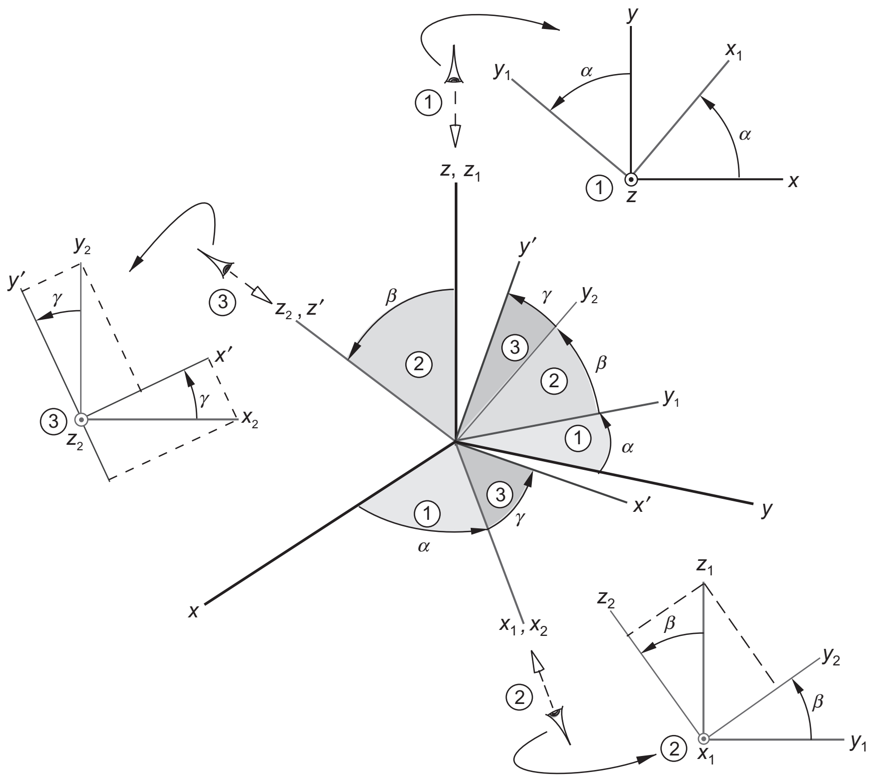


FIGURE 4.14

Classical Euler sequence of three rotations transforming *xyz* into $x'y'z'$. The "eye" viewing down an axis sees the illustrated rotation about that axis.

The column vector $\{\mathbf{b}\}_{x_1}$ contains the components of the vector $\mathbf{b}$ $(\mathbf{b} = b_{x_1}\hat{\mathbf{i}}_1 + b_{y_1}\hat{\mathbf{j}}_1 + b_{z_1}\hat{\mathbf{k}}_1)$ in the first intermediate frame $x_1y_1z_1$. The column vector $\{\mathbf{b}\}_{x_2}$ contains the components of the vector $\mathbf{b}$ $(\mathbf{b} = b_{x_2}\hat{\mathbf{i}}_2 + b_{y_2}\hat{\mathbf{j}}_2 + b_{z_2}\hat{\mathbf{k}}_2)$ in the second intermediate frame $x_2y_2z_2$. Finally, the column vector $\{\mathbf{b}\}_{x'}$ contains the components in the target $x'y'z'$ frame.

Substituting Equations 4.32, and 4.34 into Equation 4.37 yields the direction cosine matrix of the classical Euler sequence $[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$,

$$[\mathbf{Q}] = \begin{bmatrix} -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{bmatrix} \tag{4.38}$$

From this we can see that, given a direction cosine matrix $[\mathbf{Q}]$, the angles of the classical Euler sequence may be found as follows:

$$\tan\alpha = \frac{Q_{31}}{-Q_{32}} \qquad \cos\beta = Q_{33} \qquad \tan\gamma = \frac{Q_{13}}{Q_{23}} \qquad \text{Classical Euler angle sequence} \tag{4.39}$$

We see that $\beta = \cos^{-1} Q_{33}$. There is no quadrant uncertainty because the principal values of the arccosine function coincide with the range of the angle β given in Equation 4.37 (0 to 180°). Finding α and γ involves computing the inverse tangent (arctan), whose principal values lie in the range $-90°$ to $+90°$, whereas the range of both α and γ is 0 to 360°. Placing $\tan^{-1}(y/x)$ in the correct quadrant is accomplished by taking into considering the signs of x and y. The MATLAB function *atan2d_0_360.m* in Appendix D.19 does just that.

Algorithm 4.3 Given the direction cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

find the angles $\alpha\beta\gamma$ of the classical Euler rotation sequence. This algorithm is implemented by the MATLAB function *dcm_to_euler.m* in Appendix D.20.

1. $\alpha = \tan^{-1}\left(\dfrac{Q_{31}}{-Q_{32}}\right) \qquad (0 \le \alpha < 360°)$
2. $\beta = \cos^{-1} Q_{33} \qquad (0 \le \beta \le 180°)$
3. $\gamma = \tan^{-1}\dfrac{Q_{13}}{Q_{23}} \qquad (0 \le \gamma < 360°)$

Example 4.5

If the direction cosine matrix for the transformation from xyz to $x'y'z'$ is:

$$[\mathbf{Q}] = \begin{bmatrix} 0.64050 & 0.75319 & -0.15038 \\ 0.76736 & -0.63531 & 0.086824 \\ -0.030154 & -0.17101 & -0.98481 \end{bmatrix}$$

find the angles α, β and γ of the classical Euler sequence.

Solution

Use Algorithm 4.3.
Step 1.

$$\alpha = \tan^{-1}\left(\frac{Q_{31}}{-Q_{32}}\right) = \tan^{-1}\left(\frac{-0.030154}{-[-0.17101]}\right)$$

Since the numerator is negative and the denominator is positive, α must lie in the fourth quadrant. Thus,

$$\tan^{-1}\left(\frac{-0.030154}{-[-0.17101]}\right) = \tan^{-1}(-0.17633) = -10° \Rightarrow \boxed{\alpha = 350°}$$

Step 2.

$$\beta = \cos^{-1} Q_{33} = \cos^{-1}(-0.98481) = \boxed{170.0°}$$

Step 3.

$$\gamma = \tan^{-1}\frac{Q_{13}}{Q_{23}} = \tan^{-1}\left(\frac{-0.15038}{0.086824}\right)$$

The numerator is negative and the denominator is positive, so γ lies in the fourth quadrant:

$$\tan^{-1}\left(\frac{-0.15038}{0.086824}\right) = \tan^{-1}(-0.17320) = -60° \Rightarrow \boxed{\gamma = 300°}$$

Another commonly used set of Euler angles for rotating xyz into alignment with $x'y'z'$ is the asymmetric ***yaw, pitch, and roll sequence*** found in Equation 4.36:

$$[\mathbf{Q}] = [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] \qquad (0 \le \alpha < 360° \quad -90° < \beta < 90° \quad 0 \le \gamma < 360°) \tag{4.40}$$

It is illustrated in Figure 4.15.

The first rotation $[\mathbf{R}_3(\alpha)]$ is about the z-axis through the angle α. It carries the y-axis into the y_1 axis normal to the plane of z and x' while rotating the x-axis into x_1. This rotation appears as shown in the insert at the top right of Figure 4.15. The second rotation $[\mathbf{R}_3(\beta)]$, shown in the auxiliary view at the bottom right of the figure, is a pitch around y_1 through the angle β. This carries the x_1 axis into x_2, lined up with the target x' direction, and rotates the original z axis (now z_1) into z_2. The final rotation $[\mathbf{R}_1(\gamma)]$ is a roll through the angle y around the x_2 axis so as to carry y_2 (originally y_1) and z_2 into alignment with the target y' and z' axes.

Substituting Equations 4.32, 4.33 and 4.34 into Equation 4.40 yields the direction cosine matrix for the yaw, pitch and roll sequence,

$$[\mathbf{Q}] = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\beta\sin\gamma \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\beta\cos\gamma \end{bmatrix} \tag{4.41}$$

From this it is apparent that

$$\tan\alpha = \frac{Q_{12}}{Q_{11}} \qquad \sin\beta = -Q_{13} \qquad \tan\gamma = \frac{Q_{23}}{Q_{33}} \qquad \text{Yaw-pitch-roll sequence} \tag{4.42}$$

For β we simply compute $\sin^{-1}(-Q_{13})$. There is no quadrant uncertainty because the principal values of the arcsine function coincide with the range of the pitch angle ($-90° < \beta < 90°$). Finding α and γ involves computing the inverse tangent, so we must once again be careful to place the results of these calculations in the range 0 to 360°. As pointed out previously, the MATLAB function *atan2d_0_360.m* in Appendix D.19 takes care of that.

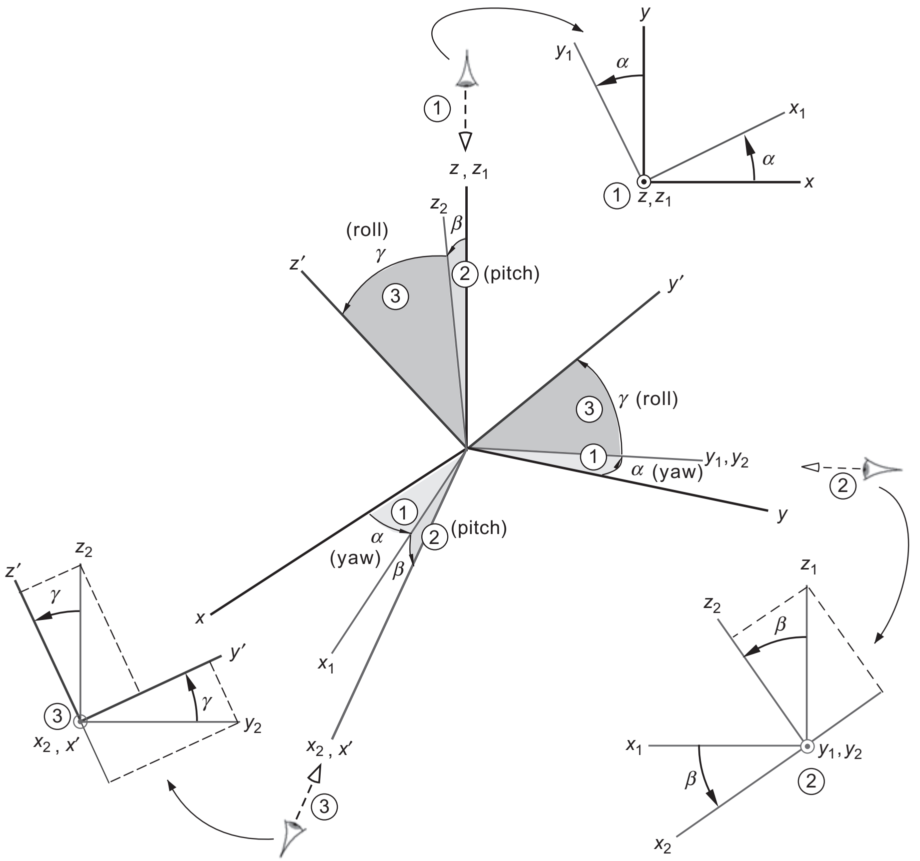


FIGURE 4.15

Yaw, pitch, and roll sequence transforming *xyz* into *x′y′z′*.

Algorithm 4.4 Given the direction cosine matrix:

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

Find the angles $\alpha\beta\gamma$ of the yaw, pitch and roll sequence. This algorithm is implemented by the MATLAB function *dcm_to_ypr.m* in Appendix D.21.

1. $\alpha = \tan^{-1}\dfrac{Q_{12}}{Q_{11}} \qquad (0 \le \alpha < 360°)$
2. $\beta = \sin^{-1}(-Q_{13}) \qquad (-90° < \beta < 90°)$
3. $\gamma = \tan^{-1}\dfrac{Q_{23}}{Q_{33}} \qquad (0 \le \gamma < 360°)$

Example 4.6

If the direction cosine matrix for the transformation from xyz to $x'y'z'$ is the same as it was in Example 4.5:

$$[\mathbf{Q}] = \begin{bmatrix} 0.64050 & 0.75319 & -0.15038 \\ 0.76736 & -0.63531 & 0.086824 \\ -0.030154 & -0.17101 & -0.98481 \end{bmatrix}$$

find the angles α, β and γ of the yaw, pitch, roll sequence.

Solution

Use Algorithm 4.4.

Step 1.

$$\alpha = \tan^{-1}\frac{Q_{12}}{Q_{11}} = \tan^{-1}\left(\frac{0.75319}{0.64050}\right)$$

Since both the numerator and the denominator are positive, α must lie in the first quadrant. Thus,

$$\tan^{-1}\left(\frac{0.75319}{0.64050}\right) = \tan^{-1}1.1759 = \boxed{49.62^\circ}$$

Step 2.

$$\beta = \sin^{-1}(-Q_{13}) = \sin^{-1}\left[-(-0.15038)\right] = \sin^{-1}(0.15038) = \boxed{8.649^\circ}$$

Step 3.

$$\gamma = \tan^{-1}\frac{Q_{23}}{Q_{33}} = \tan^{-1}\left(\frac{0.086824}{-0.98481}\right)$$

The numerator is positive and the denominator is negative, so γ lies in the second quadrant:

$$\tan^{-1}\left(\frac{0.086824}{-0.98481}\right) = \tan^{-1}(-0.088163) = -5.0383^\circ \Rightarrow \boxed{\gamma = 174.96^\circ}$$

4.6 TRANSFORMATION BETWEEN GEOCENTRIC EQUATORIAL AND PERIFOCAL FRAMES

The perifocal frame of reference for a given orbit was introduced in Section 2.10. Figure 4.16 illustrates the relationship between the perifocal and geocentric equatorial frames. Since the orbit lies in the $\bar{x}\bar{y}$ plane, the

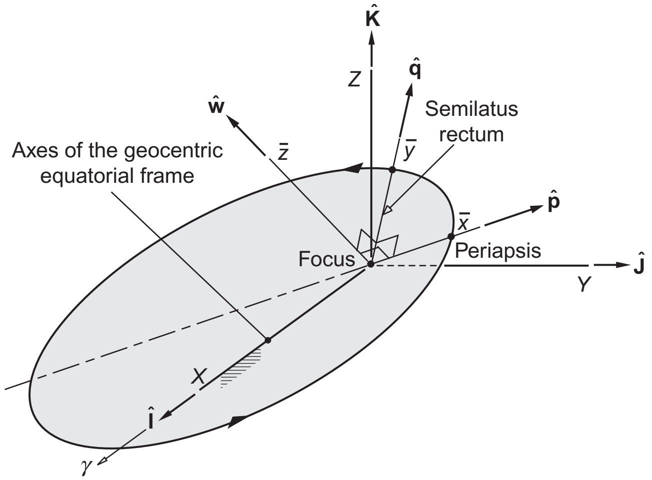


FIGURE 4.16

Perifocal ($\bar{x}\bar{y}\bar{z}$) and geocentric equatorial (*XYZ*) frames.

components of the state vector of a body relative to its perifocal reference are, according to Equations 2.119 and 2.125,

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) \tag{4.43}$$

$$\mathbf{v} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} = \frac{\mu}{h}\left[-\sin\theta\hat{\mathbf{p}} + (e+\cos\theta)\,\hat{\mathbf{q}}\right] \tag{4.44}$$

In matrix notation these may be written

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} \tag{4.45}$$

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} \tag{4.46}$$

The subscript $\bar{x}$ is shorthand for "the $\bar{x}\bar{y}\bar{z}$ coordinate system" and is used to indicate that the components of these vectors are given in the perifocal frame, as opposed to, say, the geocentric equatorial frame (Equations 4.2 and 4.3).

The transformation from the geocentric equatorial frame into the perifocal frame may be accomplished by the classical Euler angle sequence $[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$ in Equation 4.37. Refer to Figure 4.7. In this case the first rotation angle is Ω, the right ascension of the ascending node. The second rotation is i, the orbital inclination angle, and the third rotation angle is ω, the argument of perigee. Ω is measured around the Z-axis of the geocentric equatorial frame, i is measured around the node line, and ω is measured around the $\bar{z}$ axis of the perifocal frame. Therefore the direction cosine matrix $[\mathbf{Q}]_{X\bar{x}}$ of the transformation from XYZ to $\bar{x}\bar{y}\bar{z}$ is

$$[\mathbf{Q}]_{X\bar{x}} = [\mathbf{R}_3(\omega)][\mathbf{R}_1(i)][\mathbf{R}_3(\Omega)] \tag{4.47}$$

From Equation 4.38 we get

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \sin i\sin\omega \\ -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \tag{4.48}$$

Remember that this is an orthogonal matrix, which means that the inverse transformation $[\mathbf{Q}]_{\bar{x}X}$, from $\bar{x}\bar{y}\bar{z}$ to XYZ is given by $[\mathbf{Q}]_{\bar{x}X} = ([\mathbf{Q}]_{X\bar{x}})^T$, or

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \sin\Omega\sin i \\ \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix} \tag{4.49}$$

If the components of the state vector are given in the geocentric equatorial frame

$$\{\mathbf{r}\}_X = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \qquad \{\mathbf{v}\}_X = \begin{Bmatrix} v_X \\ v_Y \\ v_Z \end{Bmatrix}$$

then the components in the perifocal frame are found by carrying out the matrix multiplications

$$\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} \bar{x} \\ \bar{y} \\ 0 \end{Bmatrix} = [\mathbf{Q}]_{X\bar{x}}\{\mathbf{r}\}_X \qquad \{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ 0 \end{Bmatrix} = [\mathbf{Q}]_{X\bar{x}}\{\mathbf{v}\}_X \tag{4.50}$$

Likewise, the transformation from perifocal to geocentric equatorial components is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} \qquad \{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} \tag{4.51}$$

Algorithm 4.5 Given the orbital elements h, e, i, Ω, ω and θ, compute the state vectors $\mathbf{r}$ and $\mathbf{v}$ in the geocentric equatorial frame of reference. A MATLAB implementation of this procedure is listed in Appendix D.22. This algorithm can be applied to orbits around other planets or the sun.

1. Calculate position vector $\{\mathbf{r}\}_{\bar{x}}$ in perifocal coordinates using Equation 4.45.
2. Calculate velocity vector $\{\mathbf{v}\}_{\bar{x}}$ in perifocal coordinates using Equation 4.46.
3. Calculate the matrix $[\mathbf{Q}]_{\bar{x}X}$ of the transformation from perifocal to geocentric equatorial coordinates using Equation 4.49.
4. Transform $\{\mathbf{r}\}_{\bar{x}}$ and $\{\mathbf{v}\}_{\bar{x}}$ into the geocentric frame by means of Equation 4.51.

Example 4.7

For a given earth orbit, the elements are $h = 80{,}000\,\text{km}^2/\text{s}$, $e = 1.4$, $i = 30°$, $\Omega = 40°$, $\omega = 60°$ and $\theta = 30°$. Using Algorithm 4.5 find the state vectors **r** and **v** in the geocentric equatorial frame.

Step 1.

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{80{,}000^2}{398{,}600}\frac{1}{1+1.4\cos 30°}\begin{Bmatrix}\cos 30°\\ \sin 30°\\ 0\end{Bmatrix} = \begin{Bmatrix}6285.0\\ 3628.6\\ 0\end{Bmatrix}\text{ km}$$

Step 2.

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398{,}600}{80{,}000}\begin{Bmatrix}-\sin 30°\\ 1.4+\cos 30°\\ 0\end{Bmatrix} = \begin{Bmatrix}-2.4913\\ 11.290\\ 0\end{Bmatrix}\text{ km/s}$$

Step 3.

$$\begin{aligned}[\mathbf{Q}]_{X\bar{x}} &= \begin{bmatrix}\cos\omega & \sin\omega & 0\\ -\sin\omega & \cos\omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & \sin i\\ 0 & -\sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos\Omega & \sin\Omega & 0\\ -\sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}\cos 60° & \sin 60° & 0\\ -\sin 60° & \cos 60° & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos 30° & \sin 30°\\ 0 & -\sin 30° & \cos 30°\end{bmatrix}\begin{bmatrix}\cos 40° & \sin 40° & 0\\ -\sin 40° & \cos 40° & 0\\ 0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}-0.099068 & 0.89593 & 0.43301\\ -0.94175 & -0.22496 & 0.25\\ 0.32139 & -0.38302 & 0.86603\end{bmatrix}\end{aligned}$$

This is the direction cosine matrix for $XYZ \rightarrow \bar{x}\bar{y}\bar{z}$. The transformation matrix for $\bar{x}\bar{y}\bar{z} \rightarrow XYZ$ is the transpose,

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix}-0.099068 & -0.94175 & 0.32139\\ 0.89593 & -0.22496 & -0.38302\\ 0.43301 & 0.25 & 0.86603\end{bmatrix}$$

Step 4.
The geocentric equatorial position vector is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{bmatrix}-0.099068 & -0.94175 & 0.32139\\ 0.89593 & -0.22496 & -0.38302\\ 0.43301 & 0.25 & 0.86603\end{bmatrix}\begin{Bmatrix}6285.0\\ 3628.6\\ 0\end{Bmatrix} = \boxed{\begin{Bmatrix}-4040\\ 4815\\ 3629\end{Bmatrix}\text{ (km)}} \tag{a}$$

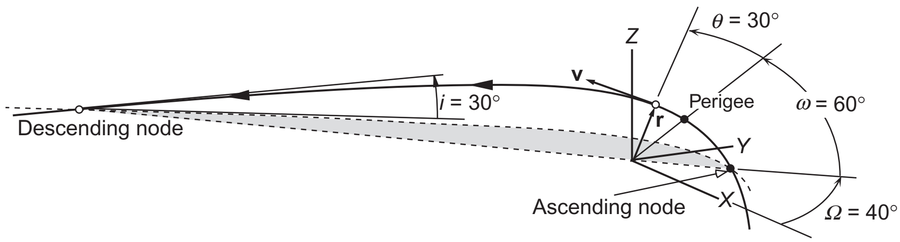


FIGURE 4.17

A portion of the hyperbolic trajectory of Example 4.7.

whereas the geocentric equatorial velocity vector is

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{bmatrix} -0.099068 & -0.94175 & 0.32139 \\ 0.89593 & -0.22496 & -0.38302 \\ 0.43301 & 0.25 & 0.86603 \end{bmatrix} \begin{Bmatrix} -2.4913 \\ 11.290 \\ 0 \end{Bmatrix} = \boxed{\begin{Bmatrix} -10.39 \\ -4.772 \\ 1.744 \end{Bmatrix} (\text{km/s})}$$

The state vectors **r** and **v** are shown in Figure 4.17. By holding all of the orbital parameters except the true anomaly fixed and allowing θ to take on a range of values, we generate a sequence of position vectors $\mathbf{r}_{\bar{x}}$ from Equation 4.37. Each of these is projected into the geocentric equatorial frame as in (a), using repeatedly the same transformation matrix $[\mathbf{Q}]_{\bar{x}X}$. By connecting the end points of all of the position vectors $\mathbf{r}_X$, we trace out the trajectory illustrated in Figure 4.17.

4.7 EFFECTS OF THE EARTH'S OBLATENESS

The earth, like all of the planets with comparable or higher rotational rates, bulges out at the equator because of centrifugal force. The earth's equatorial radius is 21 km (13 miles) larger than the polar radius. This flattening at the poles is called ***oblateness***, which is defined as follows:

$$\text{Oblateness} = \frac{\text{Equatorial radius} - \text{Polar radius}}{\text{Equatorial radius}}$$

The earth is an oblate spheroid, lacking the perfect symmetry of a sphere. (A basketball can be made an oblate spheroid by sitting on it.) This lack of symmetry means that the force of gravity on an orbiting body is not directed towards the center of the earth. Whereas the gravitational field of a perfectly spherical planet depends only on the distance from its center, oblateness causes a variation also with latitude, that is, the angular distance from the equator (or pole). This is called a *zonal variation*. The dimensionless parameter which quantifies the major effects of oblateness on orbits is J_2, the ***second zonal harmonic*** J_2 is not a universal constant. Each planet has its own value, as illustrated in Table 4.3, which lists variations of J_2 as well as oblateness.

The gravitational acceleration (force per unit mass) arising from an oblate planet is given by

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\hat{\mathbf{u}}_r + \mathbf{p}$$

Table 4.3 Oblateness and Second Zonal Harmonics

Planet	Oblateness	J_2
Mercury	0.000	60×10^{-6}
Venus	0.000	4.458×10^{-6}
Earth	0.003353	1.08263×10^{-3}
Mars	0.00648	1.96045×10^{-3}
Jupiter	0.06487	14.736×10^{-3}
Saturn	0.09796	16.298×10^{-3}
Uranus	0.02293	3.34343×10^{-3}
Neptune	0.01708	3.411×10^{-3}
(Moon)	0.0012	202.7×10^{-6}

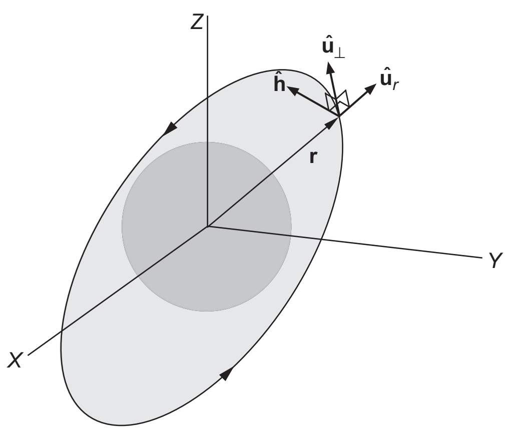


FIGURE 4.18

Unit vectors attached to an orbiting body.

The first term is the familiar one (Equation 4.1) due to a spherical planet. The second term, **p**, which is several orders of magnitude smaller than μ/r^2 is a disturbing acceleration due to the oblateness. This perturbing acceleration can be resolved into components,

$$\mathbf{p} = p_r\hat{\mathbf{u}}_r + p_\perp\hat{\mathbf{u}}_\perp + p_h\hat{\mathbf{h}}$$

where $\hat{\mathbf{u}}_r$, $\hat{\mathbf{u}}_\perp$ and $\hat{\mathbf{h}}$ are the radial, transverse and normal unit vectors attached to the satellite, as illustrated in Figure 4.18. $\hat{\mathbf{u}}_r$ points in the direction of the radial position vector **r**, $\hat{\mathbf{h}}$ is the unit vector normal to the plane of the orbit and $\hat{\mathbf{u}}_\perp$ is perpendicular to **r**, lying in the orbital plane and pointing in the direction of the motion.

The perturbation components p_r, $p_\perp$ and p_h are all directly proportional to J_2 and are functions of otherwise familiar orbital parameters as well as the planet radius R,

$$\begin{aligned} p_r &= -\frac{\mu}{r^2}\frac{3}{2}J_2\left(\frac{R}{r}\right)^2\left[1-3\sin^2 i\sin^2(\omega+\theta)\right] \\ p_\perp &= -\frac{\mu}{r^2}\frac{3}{2}J_2\left(\frac{R}{r}\right)^2 \sin^2 i\sin\left[2(\omega+\theta)\right] \\ p_h &= -\frac{\mu}{r^2}\frac{3}{2}J_2\left(\frac{R}{r}\right)^2 \sin 2i\sin(\omega+\theta) \end{aligned}$$

These relations are derived by Prussing and Conway (1993), who also show how p_r, $p_\perp$ and p_h induce time rates of change in all of the orbital parameters. For example,

$$\begin{aligned} \dot{\Omega} &= \frac{h}{\mu}\frac{\sin(\omega+\theta)}{\sin i(1+e\cos\theta)}p_h \\ \dot{\omega} &= -\frac{r\cos\theta}{eh}p_r + \frac{(2+e\cos\theta)\sin\theta}{eh}p_\perp - \frac{r\sin(\omega+\theta)}{h\tan i}p_h \end{aligned}$$

Clearly, the time variation of the right ascension Ω depends only on the component of the perturbing force normal to the (instantaneous) orbital plane, whereas the rate of change of the argument of perigee is influenced by all three perturbation components.

Integrating $\dot{\Omega}$ over one complete orbit yields the average rate of change,

$$\dot{\Omega}_{avg} = \frac{1}{T}\int_0^T \dot{\Omega}dt$$

where T is the period. Carrying out the mathematical details leads to an expression for the average rate of precession of the node line, and hence, the orbital plane,

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{\left(1-e^2\right)^2 a^{7/2}}\right]\cos i \tag{4.52}$$

where we have dropped the subscript *avg*. R and μ are the radius and gravitational parameter of the planet, a and e are the semimajor axis and eccentricity of the orbit, and i is the orbit's inclination. Observe that if $0 \le i < 90°$, then $\dot{\Omega} < 0$. That is, for prograde orbits, the node line drifts westward. Therefore, since the right ascension of the node continuously decreases, this phenomenon is called ***regression of the nodes***. If $90° < i \le 180°$, we see that $\dot{\Omega} > 0$. The node line of retrograde orbits therefore advances eastward. For polar orbits ($i = 90°$), the node line is stationary.

In a similar fashion, the time rate of change of the argument of perigee is found to be

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{\left(1-e^2\right)^2 a^{7/2}}\right]\left(\frac{5}{2}\sin^2 i - 2\right) \tag{4.53}$$

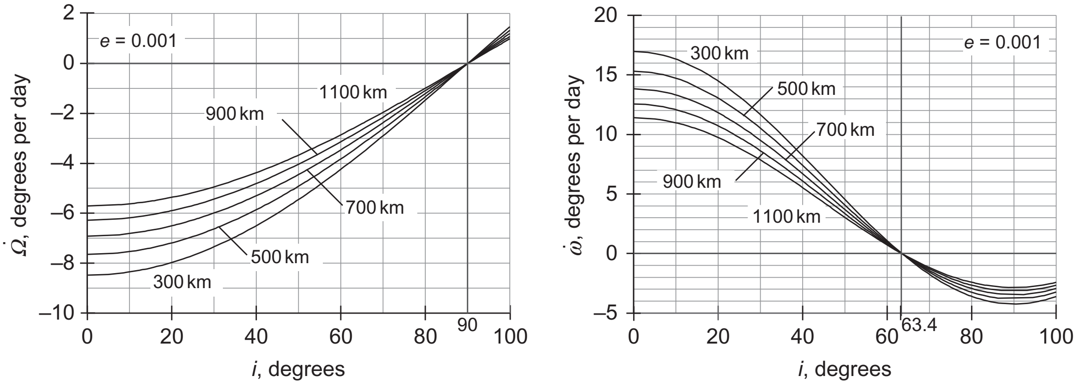


FIGURE 4.19

Regression of the node and advance of perigee for nearly circular orbits of altitudes 300 to 1100 km.

This expression shows that if $0° \leq i < 63.4°$ or $116.6° < i \leq 180°$ then $\dot{\omega}$ is positive, which means the perigee advances in the direction of the motion of the satellite (hence, the name ***advance of perigee*** for this phenomenon). If $63.4° < i \leq 116.6°$, the perigee regresses, moving opposite to the direction of motion. $i = 63.4°$ and $i = 116.6°$ are the critical inclinations at which the apse line does not move.

Observe that the coefficient of the trigonometric terms in Equations 4.52 and 4.53 are identical, so that

$$\dot{\omega} = \dot{\Omega}\frac{(5/2)\sin^2 i - 2}{\cos i} \tag{4.54}$$

Figure 4.19 is a plot of Equations 4.52 and 4.53 for several circular low earth orbits. The effect of oblateness on both $\dot{\Omega}$ and $\dot{\omega}$ is greatest at low inclinations, for which the orbit is near the equatorial bulge for longer portions of each revolution. The effect decreases with increasing semimajor axis because the satellite becomes further from the bulge and its gravitational influence. Obviously, $\dot{\Omega} = \dot{\omega} = 0$ if $J_2 = 0$ (no equatorial bulge).

The time-averaged rates of change for the inclination, eccentricity and semimajor axis are zero.

Example 4.8

The space shuttle is in a 280 km by 400 km orbit with an inclination of 51.43°. Find the rates of node regression and perigee advance.

Solution

The perigee and apogee radii are

$$r_p = 6378 + 280 = 6658 \text{ km} \qquad r_a = 6378 + 400 = 6778 \text{ km}$$

Therefore the eccentricity and semimajor axis are

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.008931$$

$$a = \frac{1}{2}(r_a + r_p) = 6718 \text{ km}$$

From Equation 4.52 we obtain the rate of node line regression

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{398,600}\cdot 0.0010826\cdot 6378^2}{(1-0.0089312^2)^2\cdot 6718^{7/2}}\right]\cos 51.43^\circ = -1.0465\times 10^{-6}\,\text{rad/s}$$

or

$$\boxed{\dot{\Omega} = 5.181 \text{ degrees per day to the west}}$$

From Equation 4.54,

$$\dot{\omega} = -1.0465\times 10^{-6}\cdot\left(\frac{5}{2}\sin^2 51.43^\circ - 2\right) = +7.9193\times 10^{-7}\,\text{rad/s}$$

or

$$\boxed{\dot{\omega} = 3.920 \text{ degrees per day in the flight direction}}$$

The effect of orbit inclination on node regression and advance of perigee is taken advantage of for two very important types of orbits. ***Sun-synchronous orbits*** are those whose orbital plane makes a constant angle α with the radial from the sun to the earth, as illustrated in Figure 4.20. For that to occur, the orbital plane must rotate in inertial space with the angular velocity of the earth in its orbit around the sun, which is 360° per 365.26 days, or 0.9856° per day. With the orbital plane precessing eastward at this rate, the

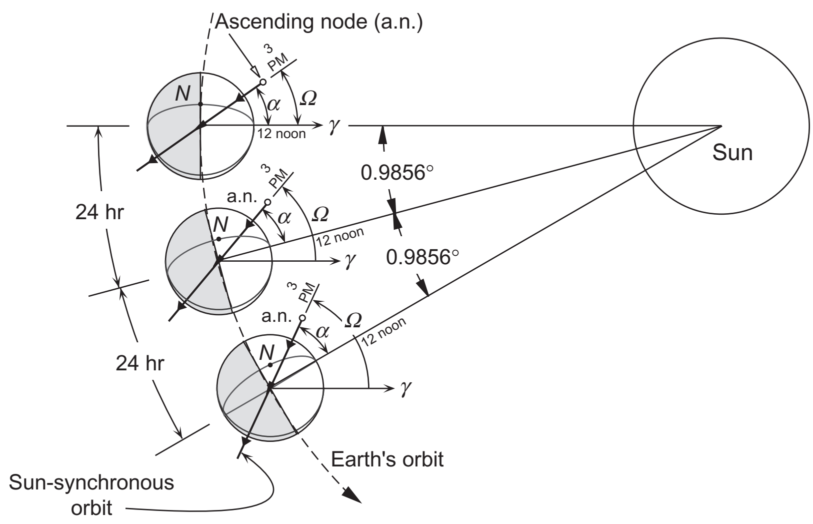


FIGURE 4.20

Sun-synchronous orbit.

ascending node will lie at a fixed local time. In the illustration it happens to be 3 p.m. During every orbit, the satellite sees any given swath of the planet under nearly the same conditions of daylight or darkness day after day. The satellite also has a constant perspective on the sun. Sun-synchronous satellites, like the NOAA Polar-orbiting Environmental Satellites (NOAA/POES) and those of the Defense Meteorological Satellite Program (DMSP) are used for global weather coverage, while Landsat and the French SPOT series are intended for high-resolution earth observation.

Example 4.9

A satellite is to be launched into a sun-synchronous circular orbit with period of 100 minutes. Determine the required altitude and inclination of its orbit.

Solution

We find the altitude z from the period relation for a circular orbit, Equation 2.64:

$$T = \frac{2\pi}{\sqrt{\mu}}(R_E + z)^{3/2} \Rightarrow 100 \cdot 60 = \frac{2\pi}{\sqrt{398{,}600}}(6378 + z)^{3/2} \Rightarrow \boxed{z = 758.63 \text{ km}}$$

For a sun-synchronous orbit, the ascending node must advance at the rate

$$\dot{\Omega} = \frac{2\pi \text{ rad}}{365.26 \cdot 24 \cdot 3600 \text{ s}} = 1.991 \times 10^{-7} \text{ rad/s}$$

Substituting this and the altitude into Equation 4.47, we obtain

$$1.991 \times 10^{-7} = -\left[\frac{3}{2}\frac{\sqrt{398{,}600} \cdot 0.00108263 \cdot 6378^2}{(1-0^2)^2(6378 + 758.63)^{7/2}}\right]\cos i \Rightarrow \cos i = -0.14658$$

Thus, the inclination of the orbit is

$$\boxed{i = \cos^{-1}(-0.14658) = 98.43°}$$

This illustrates the fact that sun-synchronous orbits are very nearly polar orbits ($i = 90°$).

If a satellite is launched into an orbit with an inclination of 63.4° (prograde) or 116.6° (retrograde), then Equation 4.53 shows that the apse line will remain stationary. The Russian space program made this a key element in the design of the system of Molniya ("lightning") communications satellites. All of the Russian launch sites are above 45° latitude, the northernmost, Plesetsk, being located at 62.8°N. As we shall see in Chapter 6, launching a satellite into a geostationary orbit would involve a costly plane change maneuver. Furthermore, recall from Example 2.6 that a geostationary satellite cannot view effectively the far northern latitudes into which Russian territory extends.

The Molniya telecommunications satellites are launched from Plesetsk into 63° inclination orbits having a period of twelve hours. From Equation 2.83 we conclude that the major axis of these orbits is 53,000 km long. Perigee (typically 500 km altitude) lies in the southern hemisphere, while apogee is at an altitude of 40,000 km (25,000 miles) above the northern latitudes, farther out than the geostationary satellites. Figure 4.21 illustrates a typical ***Molniya orbit***. A Molniya "constellation" consists of eight satellites in planes separated by 45°. Each satellite is above 30° north latitude for over eight hours, coasting towards and away from apogee.

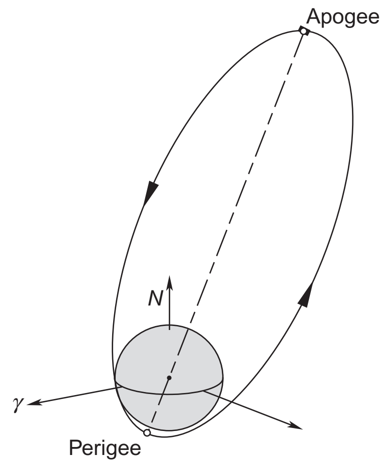


FIGURE 4.21

A typical Molniya orbit (to scale).

Example 4.10

Determine the perigee and apogee for an earth satellite whose orbit satisfies all of the following conditions: it is sun-synchronous, its argument of perigee is constant, and its period is three hours.

Solution

The period determines the semimajor axis,

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \Rightarrow 3 \cdot 3600 = \frac{2\pi}{\sqrt{398{,}600}} a^{3/2} \Rightarrow a = 10{,}560 \text{ km}$$

For the apse line to be stationary we know from Equation 4.53 that $i = 64.435°$ or $i = 116.57°$. However, an inclination of less than 90° causes a westward regression of the node, whereas a sun-synchronous orbit requires an eastward advance, which $i = 116.57°$ provides. Substituting this, the semimajor axis and the $\dot{\Omega}$ in radians per second for a sun-synchronous orbit (cf. Example 4.9) into Equation 4.52, we get

$$1.991 \times 10^{-7} = -\frac{3}{2} \frac{\sqrt{398{,}600} \cdot 0.0010826 \cdot 6378^2}{(1 - e^2)^2 \cdot 10{,}560^{7/2}} \cos 116.57° \Rightarrow e = 0.3466$$

Now we can find the angular momentum from the period expression (Equation 2.82)

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1 - e^2}}\right)^3 \Rightarrow 3 \cdot 3600 = \frac{2\pi}{398{,}600^2}\left(\frac{h}{\sqrt{1 - 0.34655^2}}\right)^3 \Rightarrow h = 60{,}850 \text{ km}^2/\text{s}$$

Finally, to obtain the perigee and apogee radii, we use the orbit formula.

$$z_p + 6378 = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{60{,}860^2}{398{,}600}\frac{1}{1+0.34655} \Rightarrow \boxed{z_p = 522.6 \text{ km}}$$

$$z_a + 6378 = \frac{h^2}{\mu}\frac{1}{1-e} \Rightarrow \boxed{z_a = 7842 \text{ km}}$$

Example 4.11

Given the following state vector of a satellite in geocentric equatorial coordinates:

$$\begin{aligned}\mathbf{r} &= -3670\hat{\mathbf{I}} - 3870\hat{\mathbf{J}} + 4400\hat{\mathbf{K}} \text{ km}\\ \mathbf{v} &= 4.7\hat{\mathbf{I}} - 7.4\hat{\mathbf{J}} + 1\hat{\mathbf{K}} \text{ km/s}\end{aligned}$$

find the state vector after four days (96 hours) of coasting flight, assuming that there are no perturbations other than the influence of the earth's oblateness on Ω and ω.

Solution

Four days is a long enough time interval that we need to take into consideration not only the change in true anomaly but also the regression of the ascending node and the advance of perigee. First, we must determine the orbital elements at the initial time using Algorithm 4.2, which yields:

$$\begin{aligned}h &= 58{,}930 \text{ km}^2/\text{s}\\ i &= 39.687°\\ e &= 0.42607 \qquad \text{(The orbit is an ellipse.)}\\ \Omega_0 &= 130.32°\\ \omega_0 &= 42.373°\\ \theta_0 &= 52.404°\end{aligned}$$

We use Equation 2.71 to determine the semimajor axis,

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} = \frac{58{,}930^2}{398{,}600}\frac{1}{1-0.4261^2} = 10{,}640 \text{ km}$$

so that, according to Equation 2.83, the period is

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = 10{,}928 \text{ s}$$

From this we obtain the mean motion

$$n = \frac{2\pi}{T} = 0.00057495 \text{ rad/s}$$

The initial value E_0 of eccentric anomaly is found from the true anomaly θ_0 using Equation 3.13a,

$$\tan\frac{E_0}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_0}{2} = \sqrt{\frac{1-0.42607}{1+0.42607}}\tan\frac{52.404°}{2} \Rightarrow E_0 = 0.60520 \text{ rad}$$

With E_0, we use Kepler's equation to calculate the time t_0 since perigee at the initial epoch,

$$nt_0 = E_0 - e\sin E_0 \Rightarrow 0.00057495t_0 = 0.60520 - 0.42607\sin 0.60520 \Rightarrow t_1 = 631.00 \text{ s}$$

Now we advance the time to t_f, that of the final epoch, given as 96 hours later. That is, $\Delta t = 345{,}600$ s, so that

$$t_f = t_1 + \Delta t = 631.00 + 345{,}600 = 346{,}230 \text{ s}$$

The number of periods n_p since passing perigee in the first orbit is

$$n_P = \frac{t_f}{T} = \frac{346{,}230}{10{,}928} = 31.682$$

From this we see that the final epoch occurs in the 32nd orbit, whereas t_0 was in orbit 1. Time since passing perigee in the 32nd orbit, which we will denote t_{32}, is

$$t_{32} = (31.682 - 31)T \Rightarrow t_{32} = 7455.7 \text{ s}$$

The mean anomaly corresponding to that time in the 32nd orbit is

$$M_{32} = nt_{32} = 0.00057495 \cdot 7455.7 = 4.2866 \text{ rad}$$

Kepler's equation yields the eccentric anomaly

$$\begin{aligned}
&E_{32} - e\sin E_{32} = M_{32}\\
&E_{32} - 0.42607\sin E_{32} = 4.2866\\
&\therefore E_{32} = 3.9721 \text{ rad} \qquad \text{(Algorithm 3.1)}
\end{aligned}$$

The true anomaly follows in the usual way,

$$\tan\frac{\theta_{32}}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_{32}}{2} \Rightarrow \theta_{32} = 211.25°$$

At this point, we use the newly found true anomaly to calculate the state vector of the satellite in perifocal coordinates. Thus, from Equation 4.43

$$\mathbf{r} = r\cos\theta_{32}\hat{\mathbf{p}} + r\sin\theta_{32}\hat{\mathbf{q}} = -11{,}714\hat{\mathbf{p}} - 7108.8\hat{\mathbf{q}} \text{ (km)}$$

or, in matrix notation,

$$\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} -11\,714 \\ -7108.8 \\ 0 \end{Bmatrix} \text{(km)}$$

Likewise, from Equation 4.44,

$$\mathbf{v} = -\frac{\mu}{h}\sin\theta_{32}\hat{\mathbf{p}} + \frac{\mu}{h}(e + \cos\theta_{32})\hat{\mathbf{q}} = 3.509\,3\hat{\mathbf{p}} - 2.9007\hat{\mathbf{q}} \text{ (km/s)}$$

or

$$\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} 3.5093 \\ -2.9007 \\ 0 \end{Bmatrix} \text{(km/s)}$$

Before we can project **r** and **v** into the geocentric equatorial frame, we must update the right ascension of the node and the argument of perigee. The regression rate of the ascending node is

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{\left(1-e^2\right)^2 a^{7/2}}\right]\cos i = -\frac{3}{2}\frac{\sqrt{398{,}600}\cdot 0.00108263\cdot 6378^2}{\left(1-0.42607^2\right)^2\cdot 10{,}644^{7/2}}\cos 39.69° = -3.8514\times 10^{-7} \text{ rad/s}$$

or

$$\dot{\Omega} = -2.2067\times 10^{-5}\ °/\text{s}$$

Therefore, right ascension at epoch in the 32nd orbit is

$$\Omega_{32} = \Omega_0 + \dot{\Omega}\Delta t = 130.32 + (-2.2067\times 10^{-5})\cdot 345{,}600 = 122.70°$$

Likewise, the perigee advance rate is

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1-e^2)^2a^{7/2}}\right]\left(\frac{5}{2}\sin^2 i - 2\right) = 4.9072\times 10^{-7}\text{ rad/s} = 2.8116\times 10^{-5}\ °/\text{s}$$

which means the argument of perigee at epoch in the 32nd orbit is

$$\omega_{32} = \omega_0 + \dot{\omega}\Delta t = 42.373 + 2.8116\times 10^{-5}\cdot 345{,}600 = 52.090°$$

Substituting the updated values of Ω and ω, together with the inclination i, into Equation 4.47 yields the updated transformation matrix from geocentric equatorial to the perifocal frame,

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} \cos\omega_{32} & \sin\omega_{32} & 0 \\ -\sin\omega_{32} & \cos\omega_{32} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega_{32} & \sin\Omega_{32} & 0 \\ -\sin\Omega_{32} & \cos\Omega_{32} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 52.09^\circ & \sin 52.09^\circ & 0 \\ -\sin 52.09^\circ & \cos 52.09^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 39.687^\circ & \sin 39.687^\circ \\ 0 & -\sin 39.687^\circ & \cos 39.687^\circ \end{bmatrix} \begin{bmatrix} \cos 122.70^\circ & \sin 122.70^\circ & 0 \\ -\sin 122.70^\circ & \cos 122.70^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -0.84285 & 0.18910 & 0.50383 \\ 0.028276 & -0.91937 & 0.39237 \\ 0.53741 & 0.34495 & 0.76955 \end{bmatrix}$$

For the inverse transformation, from perifocal to geocentric equatorial, we need the transpose of this matrix,

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -0.84285 & 0.18910 & 0.50383 \\ 0.028276 & -0.91937 & 0.39237 \\ 0.53741 & 0.34495 & 0.76955 \end{bmatrix}^T = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix}$$

Thus, according to Equation 4.51, the final state vector in the geocentric equatorial frame is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix} \begin{Bmatrix} -11\,714 \\ -7108.8 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 9672 \\ 4320 \\ -8691 \end{Bmatrix} \text{(km)}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix} \begin{Bmatrix} 3.5093 \\ -2.9007 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -3.040 \\ 3.330 \\ 0.6299 \end{Bmatrix} \text{(km/s)}$$

or, in vector notation,

$$\boxed{\begin{aligned} \mathbf{r} &= 9672\hat{\mathbf{I}} + 4320\hat{\mathbf{J}} - 8691\hat{\mathbf{K}}\ (\text{km}) \\ \mathbf{v} &= -3.040\hat{\mathbf{I}} + 3.330\hat{\mathbf{J}} + 0.6299\hat{\mathbf{K}}\ (\text{km/s}) \end{aligned}}$$

The two orbits are plotted in Figure 4.22.

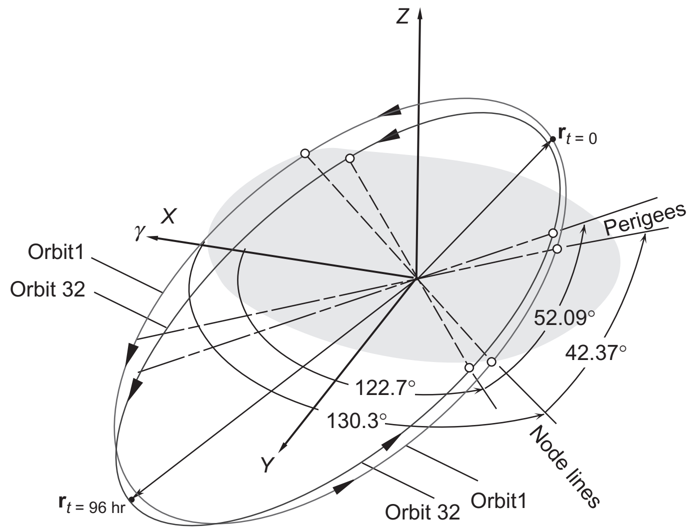


FIGURE 4.22

The initial and final position vectors.

4.8 GROUND TRACKS

The projection of a satellite's orbit onto the earth's surface is called its ***ground track***. At a given instant one can imagine a radial line drawn outward from the center of the earth to the satellite. Where this line pierces the earth's spherical surface is a point on the ground track. We locate this point by giving its latitude and longitude relative to the earth. As the satellite moves around the earth, the trace of these points is its ground track.

Because the satellite reaches a maximum and minimum latitude ("amplitude") during each orbit while passing over the equator twice, on a Mercator projection the ground track of a satellite in low earth orbit often resembles a sine curve. If the earth did not rotate, there would be just one sinusoid-like track, traced repeatedly as the satellite orbits the earth. However, the earth rotates eastward beneath the satellite orbit at 15.04° per hour, so the ground track advances westward at that rate. Figure 4.23 shows about two and a half orbits of a satellite, with the beginning and end of this portion of the ground track labeled. The distance between two successive crossings of the equator is measured to be 23.2°, which is the amount of earth rotation in one orbit of the spacecraft. Therefore, the ground track reveals that the period of the satellite is

$$T = \frac{23.2\text{ degrees}}{15.04\,\text{degrees/hr}} = 1.54\text{ hr} = 92.6\text{ min}$$

This is a typical low earth orbital period.

Given a satellite's position vector $\mathbf{r}$, we can use Algorithm 4.1 to find its right ascension and declination relative to the geocentric equatorial XYZ frame, which is fixed in space. The earth rotates at an angular velocity ω_E relative to this system. Let us attach an $x'y'z'$ Cartesian coordinate system to the earth with its origin located at the earth's center, as illustrated in Figure 1.18. The $x'y'$ axes lie in the equatorial plane and the z' axis points north. (In Figure 1.18 the x' axis is directed towards the prime meridian, which passes

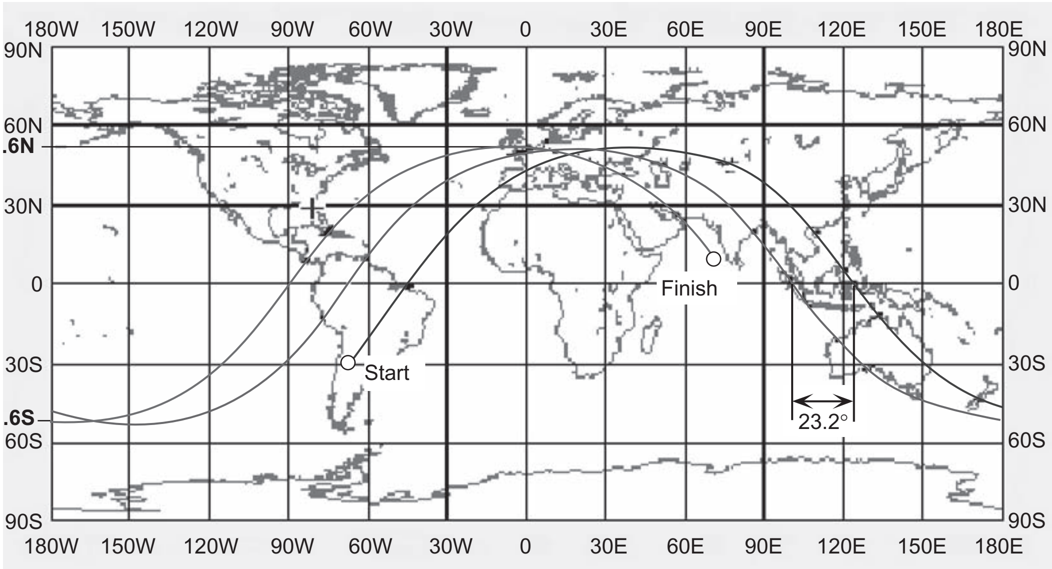


FIGURE 4.23

Ground track of a satellite.

through Greenwich, England.) The *XYZ* and $x'y'z'$ differ only by the angle θ between the stationary *X* axis and the rotating x′ axis. If the *X* and x' axes line up at time t_0, then at any time t thereafter the angle θ will be given by $\omega_E\,(t–t_0)$. The transformation from *XYZ* to $x'y'z'$ is represented by the elementary rotation matrix (recall Equation 4.34),

$$[\mathbf{R}_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \theta = \omega_E(t - t_0) \tag{4.55}$$

Thus, if the components of the position vector **r** in the inertial *XYZ* frame are given by $\{\mathbf{r}\}_X$, its components $\{\mathbf{r}\}_{x'}$ in the rotating, earth-fixed $x'y'z'$ frame are:

$$\{\mathbf{r}\}_{x'} = \left[\mathbf{R}_3(\theta)\right]\{\mathbf{r}\}_X \tag{4.56}$$

Knowing $\{\mathbf{r}\}_{x'}$, we use Algorithm 4.1 to determine the right ascension (longitude east of x') and declination (latitude) in the earth-fixed system. These points are usually plotted on a rectangular Mercator projection of the earth's surface, as in Figure 4.23.

Algorithm 4.6 Given the initial orbital elements (h, e, a, T, i, ω_0, Ω_0, and θ_0) of a satellite relative to the geocentric equatorial frame, compute the right ascension α and declination δ relative to the rotating earth after a time interval Δt. This algorithm is implemented in MATLAB as the script *ground_track.m* in Appendix D.23.

1. Compute $\dot{\Omega}$ and $\dot{\omega}$ from Equations 4.52 and 4.53.
2. Calculate the initial time t_0 (time since perigee passage):
   a. Find the eccentric anomaly E_0 from Equation 3.13b.
   b. Find the mean anomaly M_0 from Equation 3.14.
   c. Find t_0 from Equation 3.15.
3. At time $t = t_0 + \Delta t$, calculate α and δ.
   a. Calculate the true anomaly:
      i. Find M from Equation 3.8
      ii. Find E from Equation 3.14 using Algorithm 3.1.
      iii. Find θ from Equation 3.13a.
   b. Update Ω and ω:

$$\Omega = \Omega_0 + \dot{\Omega}\Delta t \qquad \omega = \omega_0 + \dot{\omega}\Delta t$$

   c. Find $\{\mathbf{r}\}_x$ using Algorithm 4.5.
   d. Find $\{\mathbf{r}\}_{x'}$ using Equations 4.55 and 4.56.
   e. Use Algorithm 4.1 to obtain compute α and δ from $\{\mathbf{r}\}_{x'}$.
4. Repeat Step 3 for additional times ($t = t_0 + 2\Delta t$, $t = t_0 + 3\Delta t$, etc.).

Example 4.12

An earth satellite has the following orbital parameters:

$r_p = 6700\,\text{km}$	Perigee
$r_a = 10{,}000\,\text{km}$	Apogee
$\theta_0 = 230°$	True anomaly
$\Omega_0 = 270°$	Right ascension of the ascending node
$i_0 = 60°$	Inclination
$\omega_0 = 45°$	Argument of perigee

Calculate the right ascension (longitude east of x') and declination (latitude) relative to the rotating earth 45 minutes later.

Solution

First, we compute the semimajor axis a, eccentricity e, the angular momentum h, the semimajor axis a, and the period T. For the semimajor axis we recall that

$$a = \frac{r_p + r_a}{2} = \frac{6700 + 10{,}000}{2} = 8350\,\text{km}$$

From Equation 2.84 we get

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{10{,}000 - 6700}{10{,}000 + 6700} = 0.19760$$

Equation 2.50 yields

$$h = \sqrt{\mu r_p (1 + e)} = \sqrt{398{,}600 \cdot 6700 \cdot (1 + 0.19760)} = 56{,}554\ \text{km}^2/\text{s}$$

Finally, we obtain the period from Equation 2.83:

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} 8350^{3/2} = 7593.5\,\text{s}$$

Now we can proceed with Algorithm 4.6.

Step 1.

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2 R_{\text{earth}}^2}{\left(1-e^2\right)a^{7/2}}\right]\cos i = -\left[\frac{3}{2}\frac{\sqrt{398{,}600}\cdot 0.0010836 \cdot 6378^2}{(1-0.19760^2)\,8350^{7/2}}\right]\cos 60^\circ = -2.3394\times 10^{-7}\,{}^\circ/\text{s}$$

$$\dot{\omega} = \dot{\Omega}\frac{5/2\sin^2 i - 2}{\cos i} = -2.3394\times 10^{-5}\left(\frac{5/2\sin^2 60^\circ - 2}{\cos 60^\circ}\right) = 5.8484\times 10^{-6}\,{}^\circ/\text{s}$$

Step 2.

a. $$E = 2\tan^{-1}\left(\tan\frac{\theta}{2}\sqrt{\frac{1-e}{1+e}}\right) = 2\tan^{-1}\left(\tan\frac{230^\circ}{2}\sqrt{\frac{1-0.19760}{1+0.19760}}\right) = -2.1059\,\text{rad}$$

b. $$M = E - e\sin E = -2.1059 - 0.19760\sin(-2.1059) = -1.9360\,\text{rad}$$

c. $$t_0 = \frac{M}{2\pi}T = \frac{-1.9360}{2\pi}\cdot 7593.5 = -2339.7\text{ s} \qquad (2339.7\text{ s } \textit{until} \text{ perigee})$$

Step 3. $t = t_0 + 45\text{ min} = -2339.7 + 45\cdot 60 = 360.33\text{ s}$ (360.33 s *after perigee*)

a. $$M = 2\pi\frac{t}{T} = 2\pi\frac{360.33}{7593.5} = 0.29815\,\text{rad}$$

$$E - 0.19760\sin E = 0.29815 \quad \overset{\text{Algorithm 3.1}}{\Rightarrow} \quad E = 0.36952\,\text{rad}$$

$$\theta = 2\tan^{-1}\left(\tan\frac{E}{2}\sqrt{\frac{1+e}{1-e}}\right) = 2\tan^{-1}\left(\tan\frac{0.36952}{2}\sqrt{\frac{1+0.19760}{1-0.19760}}\right) = 25.723^\circ$$

b. $$\Omega = \Omega_0 + \dot{\Omega}\Delta t = 270^\circ + (-2.3394\times 10^{-5}\ {}^\circ/\text{s})(2700\text{ s}) = 269.94^\circ$$

$$\omega = \omega_0 + \dot{\omega}\Delta t = 45^\circ + (5.8484\times 10^{-6}\ {}^\circ/\text{s})(2700\text{ s}) = 45.016^\circ$$

c. $\{\mathbf{r}\}_X \overset{\text{Algorithm 4.5}}{\overbrace{=}} \begin{Bmatrix} 3212.6 \\ -2250.5 \\ 5568.6 \end{Bmatrix} \text{ (km)}$

d. $\theta = \omega_E \Delta t = \dfrac{360^\circ\left(1 + \dfrac{1}{365.26}\right)}{24 \cdot 3600\text{s}} \cdot 2700\text{s} = 11.281^\circ$

$$[\mathbf{R}_3(\theta)] = \begin{bmatrix} \cos 11.281^\circ & \sin 11.281^\circ & 0 \\ -\sin 11.281^\circ & \cos 11.281^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.98068 & 0.19562 & 0 \\ -0.19562 & 0.98068 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{\mathbf{r}\}_{x'} = [\mathbf{R}_3(\theta)]\text{e}\,\{\mathbf{r}\}_X = \begin{bmatrix} 0.98068 & 0.19562 & 0 \\ -0.19562 & 0.98068 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 3212.6 \\ -2250.5 \\ 5568.6 \end{Bmatrix} = \begin{Bmatrix} 2710.3 \\ -2835.4 \\ 5568.6 \end{Bmatrix} \text{ (km)}$$

e. $\mathbf{r} = 2710.3\hat{\mathbf{i}}' - 2835.4\hat{\mathbf{j}}' + 5568.6\hat{\mathbf{k}}' \overset{\text{Algorithm 4.1}}{\overbrace{\Rightarrow}} \boxed{\alpha = 313.7^\circ \quad \delta = 54.84^\circ}$

The script *ground_track.m* in Appendix D.23 can be used to plot ground tracks. For the data of Example 4.12 the ground track for 3.25 periods appears in Figure 4.24. The ground track for one orbit of a Molniya satellite is featured more elegantly in Figure 4.25.

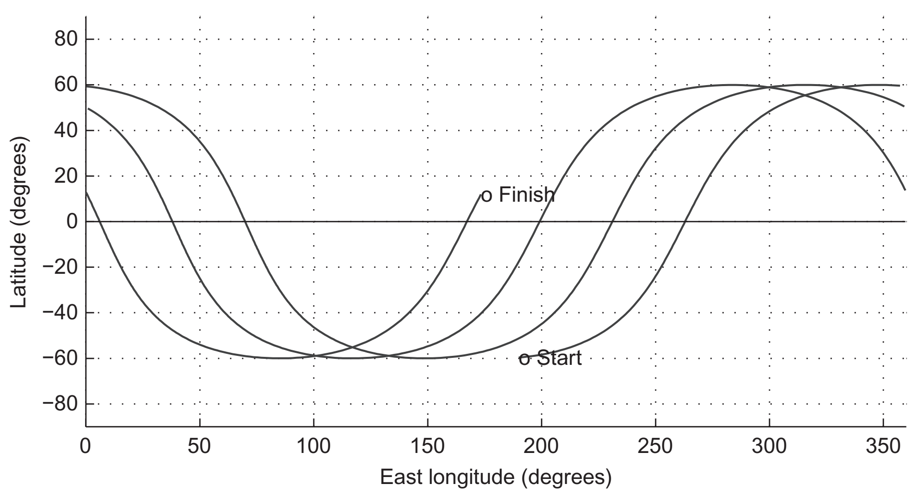


FIGURE 4.24

Ground track for 3.25 orbits of the satellite in Example 4.6.

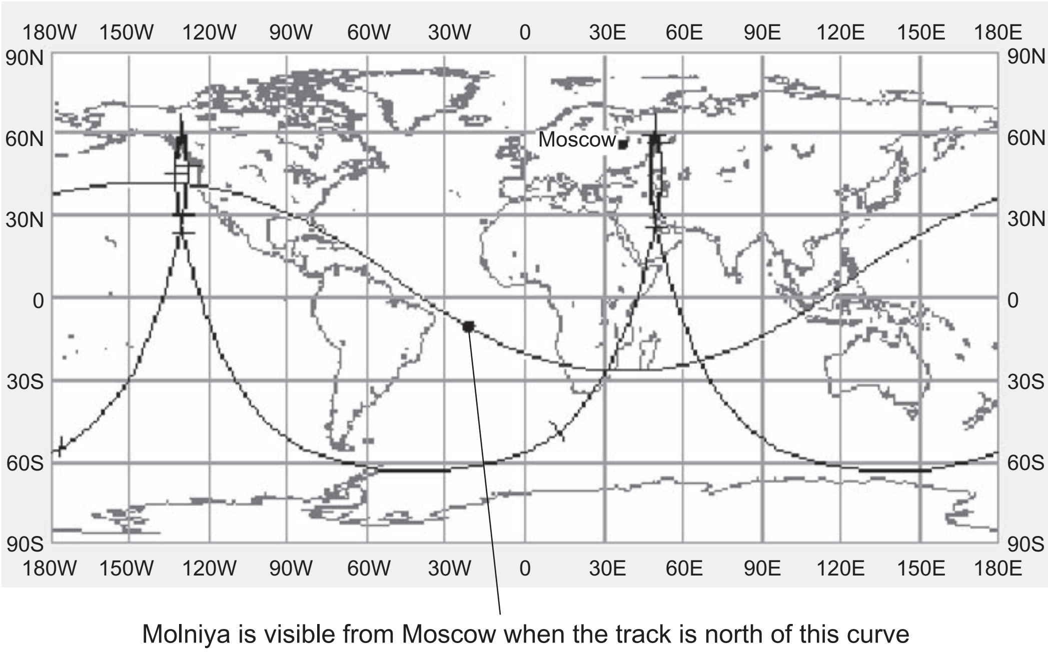


FIGURE 4.25

Ground track for two orbits of a Molniya satellite with a 12-hour period. Tick marks are one hour apart.

PROBLEMS

Section 4.3

4.1 For each of the following geocentric equatorial position vectors (in kilometers) find the right ascension and declination.

(a) $\mathbf{r} = 9000\hat{\mathbf{I}} + 6000\hat{\mathbf{J}} + 3000\hat{\mathbf{K}}$
(b) $\mathbf{r} = -3000\hat{\mathbf{I}} - 6000\hat{\mathbf{J}} - 9000\hat{\mathbf{K}}$
(c) $\mathbf{r} = -9000\hat{\mathbf{I}} - 3000\hat{\mathbf{J}} + 6000\hat{\mathbf{K}}$
(d) $\mathbf{r} = 6000\hat{\mathbf{I}} - 9000\hat{\mathbf{J}} - 3000\hat{\mathbf{K}}$

{Ans.: (b) $\alpha = 24.4°$, $\delta = 53.30°$}

4.2 At a given instant, a spacecraft is 1000 km above the earth, with a right ascension of 150° and declination of 20° relative to the geocentric equatorial frame. Its velocity is 10 km/s directly north, normal to the equatorial plane. Find its right ascension and declination 15 minutes later.
{Ans.: $\alpha = 150°$, $\delta = 63.37°$}

Section 4.4

4.3 Find the orbital elements of a geocentric satellite whose inertial position and velocity vectors in a geocentric equatorial frame are

$$\mathbf{r} = 2615\hat{\mathbf{I}} + 15{,}881\hat{\mathbf{J}} + 3980\hat{\mathbf{K}} \text{ (km)}$$
$$\mathbf{v} = -2.767\hat{\mathbf{I}} - 0.7905\hat{\mathbf{J}} + 4.980\hat{\mathbf{K}} \text{ (km/s)}$$

{Ans.: $e = 0.3760$, $h = 95{,}360\,\text{km}^2/\text{s}$, $i = 63.95°$, $\Omega = 73.71°$, $\omega = 15.43°$, $\theta = 0.06764°$}

4.4 At a given instant the position **r** and velocity **v** of a satellite in the geocentric equatorial frame are $\mathbf{r} = 12670\hat{\mathbf{K}}$ (km) and $\mathbf{v} = -3.874\hat{\mathbf{J}} - 0.7905\hat{\mathbf{K}}$ (km). Find the orbital elements.
{Ans.: $h = 49{,}080\,\text{km}^2/\text{s}$, $e = 0.5319$, $\Omega = 90°$, $\omega = 259.50°$, $\theta = 190.50°$, $i = 90°$.}

4.5 At time t_0 the position **r** and velocity **v** of a satellite in the geocentric equatorial frame are $\mathbf{r} = 6472.7\hat{\mathbf{I}} - 7470.8\hat{\mathbf{J}} - 2469.8\hat{\mathbf{K}}$ (km) and $\mathbf{v} = 3.9914\hat{\mathbf{I}} + 2.7916\hat{\mathbf{J}} - 3.2948\hat{\mathbf{K}}$ (km/s). Find the orbital elements.
{Ans.: $h = 58{,}461\,\text{km}^2/\text{s}$, $e = 0.2465$, $\Omega = 110°$, $\omega = 75°$, $\theta = 130°$, $i = 35°$}

4.6 Given that, with respect to the geocentric equatorial frame, $\mathbf{r} = -6634.2\hat{\mathbf{I}} - 1261.8\hat{\mathbf{J}} - 5230.9\hat{\mathbf{K}}$ (km), $\mathbf{v} = 5.7644\hat{\mathbf{I}} - 7.2005\hat{\mathbf{J}} - 1.8106\hat{\mathbf{K}}$ (km/s) and the eccentricity vector is $\mathbf{e} = -0.40907\hat{\mathbf{I}} - 0.48751\hat{\mathbf{J}} - 0.63640\hat{\mathbf{K}}$, calculate the true anomaly θ of the earth-orbiting satellite.
{Ans.: 330°}

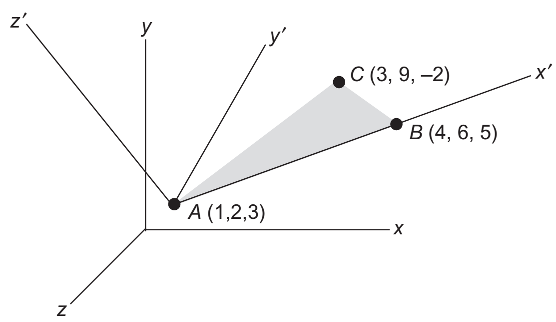
4.7 Given that, relative to the geocentric equatorial frame $\mathbf{r} = -6634.2\hat{\mathbf{I}} - 1261.8\hat{\mathbf{J}} - 5230.9\hat{\mathbf{K}}$ (km), the eccentricity vector is $\mathbf{e} = -0.40907\hat{\mathbf{I}} - 0.48751\hat{\mathbf{J}} - 0.63640\hat{\mathbf{K}}$, and the satellite is flying towards perigee, calculate the inclination of the orbit.
{Ans.: 45°}

Section 4.5

4.8 The right-handed, primed *xyz* system is defined by the three points *A*, *B* and *C*. The $x'y'$ plane is defined by the plane *ABC*. The x' axis runs from *A* through *B*. The z' axis is defined by the cross product of $\overrightarrow{AB}$ into $\overrightarrow{AC}$, so that the $+y'$ axis lies on the same side of the x' axis as point C.

(a) Find the direction cosine matrix [Q] relating the two coordinate bases.
(b) If the components of a vector **v** in the primed system are $\lfloor 2 \quad -1 \quad 3 \rfloor^T$, find the components of **v** in the unprimed system.

{Ans.: $\lfloor -1.307 \quad 2.390 \quad 2.565 \rfloor^T$}



4.9 The unit vectors in a *uvw* Cartesian coordinate frame have the following components in the *xyz* frame:

$$\hat{\mathbf{u}} = 0.26726\hat{\mathbf{i}} + 0.53452\hat{\mathbf{j}} + 0.80178\hat{\mathbf{k}}$$

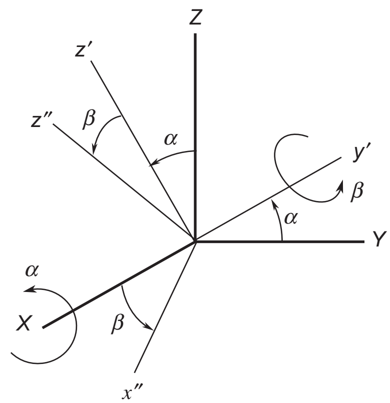
$$\hat{\mathbf{v}} = -0.44376\hat{\mathbf{i}} + 0.80684\hat{\mathbf{j}} - 0.38997\hat{\mathbf{k}}$$

$$\hat{\mathbf{w}} = -0.85536\hat{\mathbf{i}} - 0.25158\hat{\mathbf{j}} + 0.45284\hat{\mathbf{k}}$$

If, in the *xyz* frame $\mathbf{V} = -50\hat{\mathbf{i}} + 100\hat{\mathbf{j}} + 75\hat{\mathbf{k}}$, find the components of the vector **V** in the *uvw* frame.
{Ans.: $\mathbf{V} = 100.2\hat{\mathbf{u}} + 73.62\hat{\mathbf{v}} + 51.57\hat{\mathbf{w}}$}

4.10 Calculate the direction cosine matrix [Q] for the sequence of two rotations: $\alpha = 40°$ about the positive *X* axis, followed by $\beta = 25°$ about the positive y' axis. The result is that the *XYZ* axes are rotated into the $x''y'z''$ axes.

{Partial ans.: $Q_{11} = 0.9063 \quad Q_{12} = 0.2716 \quad Q_{13} = -0.3237$}



4.11 For the direction cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} 0.086824 & -0.77768 & 0.62264 \\ -0.49240 & -0.57682 & -0.65178 \\ 0.86603 & -0.25000 & -0.43301 \end{bmatrix}$$

calculate,
(a) the classical Euler angle sequence;
{Ans.: $\alpha = 73.90°$, $\beta = 115.7°$, $\gamma = 136.3°$}

(b) the yaw, pitch and roll angle sequence.
{Ans.: $\alpha = 276.4°$, $\beta = 38.51°$, $\gamma = 236.4°$}

4.12 What yaw, pitch and roll sequence yields the same DCM as the classical Euler sequence $\alpha = 350°$, $\beta = 170°$, $\gamma = 300°$?
{Ans.: $\alpha = 49.62°$, $\beta = 8.649°$, $\gamma = 175.0°$}

4.13 What classical Euler angle sequence yields the same DCM as the yaw-pitch-roll sequence $\alpha = 300°$, $\beta = 80°$, $\gamma = 30°$?
{Ans.: $\alpha = 240.4°$, $\beta = 81.35°$, $\gamma = 84.96°$}

Section 4.6

4.14 At time t_0 the position $\mathbf{r}$ and velocity $\mathbf{v}$ of a satellite in the geocentric equatorial frame are:

$$\mathbf{r} = -5102\hat{\mathbf{I}} - 8228\hat{\mathbf{J}} - 2105\hat{\mathbf{K}}\ \text{(km)}$$
$$\mathbf{v} = -4.348\hat{\mathbf{I}} + 3.478\hat{\mathbf{J}} - 2.846\hat{\mathbf{K}}\ \text{(km/s)}$$

Find $\mathbf{r}$ and $\mathbf{v}$ at time $t_0 + 50$ minutes. ($t_0 \neq 0$!)
{Ans.: $\mathbf{r} = -4198\hat{\mathbf{I}} + 7856\hat{\mathbf{J}} - 3199\hat{\mathbf{K}}$ (km); $\mathbf{v} = 4.952\hat{\mathbf{I}} + 3.482\hat{\mathbf{J}} + 2.495\hat{\mathbf{K}}$ (km/s) c}

4.15 For a spacecraft, the following orbital parameters are given: $e = 1.5$; perigee altitude = 300 km; $i = 35°$; $\Omega = 130°$; $\omega = 115°$. Calculate $\mathbf{r}$ and $\mathbf{v}$ at perigee relative to (a) the perifocal reference frame and (b) the geocentric equatorial frame.
{Ans.: (a) $\mathbf{r} = 6678\hat{\mathbf{p}}$ (km), $\mathbf{v} = 12.22\hat{\mathbf{q}}$ (km/s) .
(b) $\mathbf{r} = -1984\hat{\mathbf{I}} - 5348\hat{\mathbf{J}} + 3471\hat{\mathbf{K}}$ (km), $\mathbf{v} = 10.36\hat{\mathbf{I}} - 5.763\hat{\mathbf{J}} - 2.961\hat{\mathbf{K}}$ (km/s)}

4.16 For the spacecraft of Problem 4.15 calculate $\mathbf{r}$ and $\mathbf{v}$ at two hours past perigee relative to (a) the perifocal reference frame, and; (b) the geocentric equatorial frame.
{Ans.: (a) $\mathbf{r} = -25{,}010\hat{\mathbf{p}} + 48{,}090\hat{\mathbf{q}}$ (km), $\mathbf{v} = -4.335\hat{\mathbf{p}} + 5.075\hat{\mathbf{q}}$ (km/s) ;
(b) $\mathbf{r} = 48{,}200\hat{\mathbf{I}} - 2658\hat{\mathbf{J}} - 24{,}660\hat{\mathbf{K}}$ E(km), $\mathbf{v} = 5.590\hat{\mathbf{I}} + 1.078\hat{\mathbf{J}} - 3.484\hat{\mathbf{K}}$ (km/s)}

4.17 Calculate $\mathbf{r}$ and $\mathbf{v}$ for the satellite in Problem 4.15 at time $t_0 + 50$ minutes. ($t_0 \neq 0$!)
{Ans.: $\mathbf{r} = 6862\hat{\mathbf{I}} + 5920\hat{\mathbf{J}} - 5933\hat{\mathbf{K}}$ (km), $\mathbf{v} = -3.565\hat{\mathbf{I}} + 3.904\hat{\mathbf{J}} + 1.411\hat{\mathbf{K}}$ (km/s)}

4.18 For a spacecraft, the following orbital parameters are given: $e = 1.2$; perigee altitude = 200 km; $i = 50°$; $\Omega = 75°$; $\omega = 80°$. Calculate $\mathbf{r}$ and $\mathbf{v}$ at perigee relative to (a) the perifocal reference frame and (b) the geocentric equatorial frame.
{Ans. (a) $\mathbf{r} = 6578\hat{\mathbf{p}}$ (km), $\mathbf{v} = 11.55\hat{\mathbf{q}}$ (km/s);
(b) $\mathbf{r} = -3726\hat{\mathbf{I}} + 2181\hat{\mathbf{J}} + 4962\hat{\mathbf{K}}$ (km), $\mathbf{v} = -4.188\hat{\mathbf{I}} - 10.65\hat{\mathbf{J}} + 1.536\hat{\mathbf{K}}$ (km/s)}

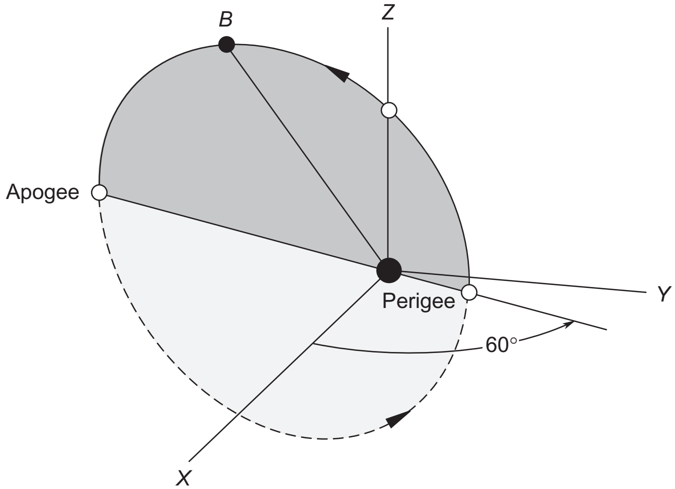
4.19 For the spacecraft of Problem 4.18 calculate $\mathbf{r}$ and $\mathbf{v}$ at two hours past perigee relative to (a) the perifocal reference frame and (b) the geocentric equatorial frame.
{Ans.: (a) $\mathbf{r} = -26{,}340\mathbf{p} + 37{,}810\mathbf{q}$ (km), $\mathbf{v} = -4.306\mathbf{p} + 3.298\mathbf{q}$ (km/s)
(b) $\mathbf{r} = 1207\hat{\mathbf{I}} - 43{,}600\hat{\mathbf{J}} - 14{,}840\hat{\mathbf{K}}$ (km), $\mathbf{v} = 1.243\hat{\mathbf{I}} - 4.4700\hat{\mathbf{J}} - 2.810\hat{\mathbf{K}}$ (km/s)}

4.20 Given that $e = 0.7$, $h = 75{,}000\ \text{km}^2/\text{s}$, and $\theta = 25°$, calculate the components of velocity in the geocentric equatorial frame if:

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -.83204 & -.13114 & 0.53899 \\ 0.02741 & -.98019 & -.19617 \\ 0.55403 & -.14845 & 0.81915 \end{bmatrix}$$

{Ans.: $\mathbf{v} = 2.103\hat{\mathbf{I}} - 8.073\hat{\mathbf{J}} - 2.885\hat{\mathbf{K}}$ (km/s)}

4.21 The apse line of the elliptical orbit lies in the *XY* plane of the geocentric equatorial frame, whose *Z*-axis lies in the plane of the orbit. At *B* (for which $\theta = 140°$) the perifocal velocity vector is $\{\mathbf{v}\}_{\bar{x}} = \lfloor -3.208 \quad -0.8288 \quad 0 \rfloor^T$ (km/s). Calculate the geocentric-equatorial components of the velocity at *B*.
{Ans.: $\{\mathbf{v}\}_X = \lfloor -1.604 \quad -2.778 \quad -0.8288 \rfloor^T$ (km/s)}



4.22 A satellite in earth orbit has the following orbital parameters: $a = 7016$ km, $e = 0.05$, $i = 45°$, $\Omega = 0°$, $\omega = 20°$ and $\theta = 10°$. Find the position vector in the geocentric-equatorial frame.
{Ans.: $\mathbf{r} = 5776.4\hat{\mathbf{I}} + 2358.2\hat{\mathbf{J}} + 2358.2\hat{\mathbf{K}}$ E (km)}

Section 4.7

4.23 Calculate the orbital inclination required to place an earth satellite in a 500 km by 1000 km sun-synchronous orbit.
{Ans.: 98.37°}.

4.24 A satellite in a circular, sun-synchronous low earth orbit passes over the same point on the equator once each day, at 12 o'clock noon. Calculate the inclination, altitude and period of the orbit.
{This problem has more than one solution.}

4.25 The orbit of a satellite around an unspecified planet has an inclination of 40°, and its perigee advances at the rate of 7° per day. At what rate does the node line regress?
{Ans.: $\dot{\Omega} = 5.545$ deg/day}

4.26 At a given time, the position and velocity of an earth satellite in the geocentric equatorial frame are $\mathbf{r} = -2429.1\hat{\mathbf{I}} + 4555.1\hat{\mathbf{J}} + 4577.0\hat{\mathbf{K}}$ (km) and $\mathbf{v} = -4.7689\hat{\mathbf{I}} - 5.6113\hat{\mathbf{J}} + 3.0535\hat{\mathbf{K}}$ (km/s). Find **r** and **v** precisely 72 hours later, taking into consideration the node line regression and the advance of perigee.
{Ans.: $\mathbf{r} = 4596\hat{\mathbf{I}} + 5759\hat{\mathbf{J}} - 1266\hat{\mathbf{K}}$ (km), $\mathbf{v} = -3.601\hat{\mathbf{I}} + 3.179\hat{\mathbf{J}} + 5.617\hat{\mathbf{K}}$ (km/s)}

Section 4.8

4.27 The space shuttle is in a circular orbit of 180 km altitude and inclination 30°. What is the spacing, in kilometers, between successive ground tracks at the equator, including the effect of earth's oblateness?
{Ans.: 2511 km}

List of Key Terms

advance of perigee
celestial sphere
classical Euler angle sequence
declination
direction cosine matrix (DCM)
ecliptic plane
ephemeris
Euler angle sequence
geocentric equatorial frame
ground tracks
Molniya orbit
oblateness
obliquity of the ecliptic
orbital elements
orthogonal matrix
precession of the vernal equinox line
regression of the node
right ascension
second zonal harmonic (J_2)
sun-synchronous orbit
vernal equinox line
yaw, pitch and roll sequence

CHAPTER

Preliminary orbit determination

5

Chapter outline

5.1	Introduction	255
5.2	Gibbs method of orbit determination from three position vectors	256
5.3	Lambert's problem	263
5.4	Sidereal time	275
5.5	Topocentric coordinate system	280
5.6	Topocentric equatorial coordinate system	283
5.7	Topocentric horizon coordinate system	284
5.8	Orbit determination from angle and range measurements	289
5.9	Angles only preliminary orbit determination	297
5.10	Gauss method of preliminary orbit determination	297

5.1 INTRODUCTION

In this chapter we will consider some (by no means all) of the classical ways in which the orbit of a satellite can be determined from earth-bound observations. All of the methods presented here are based on the two-body equations of motion. As such, they must be considered preliminary orbit determination techniques because the actual orbit is influenced over time by other phenomena (perturbations), such as the gravitational force of the moon and sun, atmospheric drag, solar wind and the nonspherical shape and nonuniform mass distribution of the earth. We took a brief look at the dominant effects of the earth's oblateness in Section 4.7. To accurately propagate an orbit into the future from a set of initial observations requires taking the various perturbations, as well as instrumentation errors themselves, into account. More detailed considerations, including the means of updating the orbit based on additional observations, are beyond our scope. Introductory discussions may be found elsewhere. See Bate, Mueller and White (1971), Boulet (1991), Prussing and Conway (1993) and Wiesel (1997), to name but a few.

We begin with the Gibbs method of predicting an orbit using three geocentric position vectors. This is followed by a presentation of Lambert's problem, in which an orbit is determined from two position vectors and the time between them. Both the Gibbs and Lambert procedures are based on the fact that two-body orbits lie in a plane. The Lambert problem is more complex and requires using the Lagrange f and g functions introduced in Chapter 2 as well as the universal variable formulation introduced in Chapter 3. The Lambert algorithm is employed in Chapter 8 to analyze interplanetary missions.

In preparation for explaining how satellites are tracked, the Julian day numbering scheme is introduced along with the notion of sidereal time. This is followed by a description of the topocentric coordinate systems and the relationships among topocentric right ascension/declension angles and azimuth/elevation angles. We then describe how orbits are determined from measuring the range and angular orientation of the line of sight together with their rates. The chapter concludes with a presentation of the Gauss method of angles-only orbit determination.

5.2 GIBBS METHOD OF ORBIT DETERMINATION FROM THREE POSITION VECTORS

Suppose that from observations of a space object at the three successive times t_1, t_2 and t_3 ($t_1 < t_2 < t_3$) we have obtained the geocentric position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. The problem is to determine the velocities $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ at t_1, t_2 and t_3 assuming that the object is in a two-body orbit. The solution using purely vector analysis is due to J. W. Gibbs (1839–1903), an American scholar who is known primarily for his contributions to thermodynamics. Our explanation is based on that in Bate, Mueller and White (1971).

We know that the conservation of angular momentum requires that the position vectors of an orbiting body must lie in the same plane. In other words, the unit vector normal to the plane of $\mathbf{r}_2$ and $\mathbf{r}_3$ must be perpendicular to the unit vector in the direction of $\mathbf{r}_1$. Thus, if $\hat{\mathbf{u}}_{r_1} = \mathbf{r}_1/r_1$ and $\hat{\mathbf{C}}_{23} = (\mathbf{r}_2 \times \mathbf{r}_3)/\|\mathbf{r}_2 \times \mathbf{r}_3\|$, then the dot product of these two unit vectors must vanish,

$$\hat{\mathbf{u}}_{r_1} \cdot \hat{\mathbf{C}}_{23} = 0$$

Furthermore, as illustrated in Figure 5.1, the fact that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ lie in the same plane means we can apply scalar factors c_1 and c_3 to $\mathbf{r}_1$ and $\mathbf{r}_3$ so that $\mathbf{r}_2$ is the vector sum of $c_1\mathbf{r}_1$ and $c_3\mathbf{r}_3$

$$\mathbf{r}_2 = c_1\mathbf{r}_1 + c_3\mathbf{r}_3 \tag{5.1}$$

The coefficients c_1 and c_3 are readily obtained from $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ as we shall see in Section 5.10 (Equations 5.89 and 5.90).

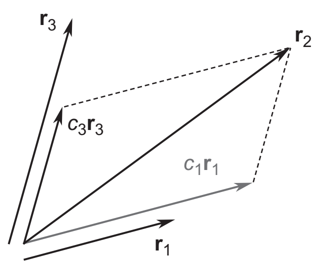


FIGURE 5.1

Any one of a set of three coplanar vectors ($\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$) can be expressed as the vector sum of the other two.

To find the velocity **v** corresponding to any of the three given position vectors **r**, we start with Equation 2.40, which may be written

$$\mathbf{v} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right)$$

where **h** is the angular momentum and **e** is the eccentricity vector. To isolate the velocity, take the cross product of this equation with the angular momentum,

$$\mathbf{h} \times (\mathbf{v} \times \mathbf{h}) = \mu\left(\frac{\mathbf{h} \times \mathbf{r}}{r} + \mathbf{h} \times \mathbf{e}\right) \tag{5.2}$$

By means of the *bac-cab* rule (Equation 2.33), the left side becomes

$$\mathbf{h} \times (\mathbf{v} \times \mathbf{h}) = \mathbf{v}(\mathbf{h} \cdot \mathbf{h}) - \mathbf{h}(\mathbf{h} \cdot \mathbf{v})$$

But $\mathbf{h} \cdot \mathbf{h} = h^2$, and $\mathbf{v} \cdot \mathbf{h} = 0$, since **v** is perpendicular to **h**. Therefore,

$$\mathbf{h} \times (\mathbf{v} \times \mathbf{h}) = h^2\mathbf{v}$$

which means Equation 5.2 may be written

$$\mathbf{v} = \frac{\mu}{h^2}\left(\frac{\mathbf{h} \times \mathbf{r}}{r} + \mathbf{h} \times \mathbf{e}\right) \tag{5.3}$$

In Section 2.10 we introduced the perifocal coordinate system, in which the unit vector $\hat{\mathbf{p}}$ lies in the direction of the eccentricity vector **e** and $\hat{\mathbf{w}}$ is the unit vector normal to the orbital plane, in the direction of the angular momentum vector **h**. Thus, we can write

$$\mathbf{e} = e\hat{\mathbf{p}} \tag{5.4a}$$

$$\mathbf{h} = h\hat{\mathbf{w}} \tag{5.4b}$$

so that Equation 5.3 becomes

$$\mathbf{v} = \frac{\mu}{h^2}\left(\frac{h\hat{\mathbf{w}} \times \mathbf{r}}{r} + h\hat{\mathbf{w}} \times e\hat{\mathbf{p}}\right) = \frac{\mu}{h}\left[\frac{\hat{\mathbf{w}} \times \mathbf{r}}{r} + e(\hat{\mathbf{w}} \times \hat{\mathbf{p}})\right] \tag{5.5}$$

Since $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$ and $\hat{\mathbf{w}}$ form a right-handed triad of unit vectors, it follows that $\hat{\mathbf{p}} \times \hat{\mathbf{q}} = \hat{\mathbf{w}}$, $\hat{\mathbf{q}} \times \hat{\mathbf{w}} = \hat{\mathbf{p}}$ and

$$\hat{\mathbf{w}} \times \hat{\mathbf{p}} = \hat{\mathbf{q}} \tag{5.6}$$

Therefore, Equation 5.5 reduces to

$$\mathbf{v} = \frac{\mu}{h}\left(\frac{\hat{\mathbf{w}} \times \mathbf{r}}{r} + e\hat{\mathbf{q}}\right) \tag{5.7}$$

This is an important result, because if we can somehow use the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ to calculate $\hat{\mathbf{q}}$, $\hat{\mathbf{w}}$, h and e, then the velocities $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ will each be determined by this formula.

So far the only condition we have imposed on the three position vectors is that they are coplanar (Equation 5.1). To bring in the fact that they describe an orbit, let us take the dot product of Equation 5.1 with the eccentricity vector $\mathbf{e}$ to obtain the scalar equation

$$\mathbf{r}_2 \cdot \mathbf{e} = c_1 \mathbf{r}_1 \cdot \mathbf{e} + c_3 \mathbf{r}_3 \cdot \mathbf{e} \tag{5.8}$$

According to Equation 2.44—the orbit equation—we have the following relations among h, e and each of the position vectors,

$$\mathbf{r}_1 \cdot \mathbf{e} = \frac{h^2}{\mu} - r_1 \qquad \mathbf{r}_2 \cdot \mathbf{e} = \frac{h^2}{\mu} - r_2 \qquad \mathbf{r}_3 \cdot \mathbf{e} = \frac{h^2}{\mu} - r_3 \tag{5.9}$$

Substituting these equations into equation 5.8 yields

$$\frac{h^2}{\mu} - r_2 = c_1\left(\frac{h^2}{\mu} - r_1\right) + c_3\left(\frac{h^2}{\mu} - r_3\right) \tag{5.10}$$

To eliminate the unknown coefficients c_1 and c_2 from this expression, let us take the cross product of Equation 5.1 first with $\mathbf{r}_1$ and then with $\mathbf{r}_3$. This results in two equations, both having $\mathbf{r}_3 \times \mathbf{r}_1$ on the right,

$$\mathbf{r}_2 \times \mathbf{r}_1 = c_3(\mathbf{r}_3 \times \mathbf{r}_1) \qquad \mathbf{r}_2 \times \mathbf{r}_3 = -c_1(\mathbf{r}_3 \times \mathbf{r}_1) \tag{5.11}$$

Now multiply Equation 5.10 through by the vector $\mathbf{r}_3 \times \mathbf{r}_1$ to obtain

$$\frac{h^2}{\mu}(\mathbf{r}_3 \times \mathbf{r}_1) - r_2(\mathbf{r}_3 \times \mathbf{r}_1) = c_1(\mathbf{r}_3 \times \mathbf{r}_1)\left(\frac{h^2}{\mu} - r_1\right) + c_3(\mathbf{r}_3 \times \mathbf{r}_1)\left(\frac{h^2}{\mu} - r_3\right)$$

Using Equations 5.11, this becomes

$$\frac{h^2}{\mu}(\mathbf{r}_3 \times \mathbf{r}_1) - r_2(\mathbf{r}_3 \times \mathbf{r}_1) = -(\mathbf{r}_2 \times \mathbf{r}_3)\left(\frac{h^2}{\mu} - r_1\right) + (\mathbf{r}_2 \times \mathbf{r}_1)\left(\frac{h^2}{\mu} - r_3\right)$$

Observe that c_1 and c_2 have been eliminated. Rearranging terms, we get

$$\frac{h^2}{\mu}(\mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1) = r_1(\mathbf{r}_2 \times \mathbf{r}_3) + r_2(\mathbf{r}_3 \times \mathbf{r}_1) + r_3(\mathbf{r}_1 \times \mathbf{r}_2) \tag{5.12}$$

This is an equation involving the given position vectors and the unknown angular momentum h. Let us introduce the following notation for the vectors on each side of Equation 5.12,

$$\mathbf{N} = r_1(\mathbf{r}_2 \times \mathbf{r}_3) + r_2(\mathbf{r}_3 \times \mathbf{r}_1) + r_3(\mathbf{r}_1 \times \mathbf{r}_2) \tag{5.13}$$

and

$$\mathbf{D} = \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1 \tag{5.14}$$

Then Equation 5.12 may be written more simply as

$$\mathbf{N} = \frac{h^2}{\mu}\mathbf{D}$$

from which we obtain

$$N = \frac{h^2}{\mu}D \tag{5.15}$$

where $N = \|\mathbf{N}\|$ and $D = \|\mathbf{D}\|$. It follows from Equation 5.15 that the angular momentum h is determined from $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ by the formula

$$h = \sqrt{\mu\frac{N}{D}} \tag{5.16}$$

Since $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are coplanar, all of the cross products $\mathbf{r}_1 \times \mathbf{r}_2$, $\mathbf{r}_2 \times \mathbf{r}_3$ and $\mathbf{r}_3 \times \mathbf{r}_1$ lie in the same direction, namely, normal to the orbital plane. Therefore, it is clear from Equation 5.14 that $\mathbf{D}$ must be normal to the orbital plane. In the context of the perifocal frame, we use $\hat{\mathbf{w}}$ to denote the orbit unit normal. Therefore,

$$\hat{\mathbf{w}} = \frac{\mathbf{D}}{D} \tag{5.17}$$

So far we have found h and $\hat{\mathbf{w}}$ in terms of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. We need likewise to find an expression for $\hat{\mathbf{q}}$ to use in Equation 5.7. From Equations 5.4a, 5.6, and 5.17 it follows that

$$\hat{\mathbf{q}} = \hat{\mathbf{w}} \times \hat{\mathbf{p}} = \frac{1}{De}(\mathbf{D} \times \mathbf{e}) \tag{5.18}$$

Substituting Equation 5.14, we get

$$\hat{\mathbf{q}} = \frac{1}{De}\left[(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} + (\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} + (\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e}\right] \tag{5.19}$$

We can apply the *bac-cab* rule (Equation 1.20) to the right side by noting

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$$

Using this vector identity we obtain

$$\begin{aligned}
(\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} &= \mathbf{r}_3(\mathbf{r}_2 \cdot \mathbf{e}) - \mathbf{r}_2(\mathbf{r}_3 \cdot \mathbf{e}) \\
(\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e} &= \mathbf{r}_1(\mathbf{r}_3 \cdot \mathbf{e}) - \mathbf{r}_3(\mathbf{r}_1 \cdot \mathbf{e}) \\
(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} &= \mathbf{r}_2(\mathbf{r}_1 \cdot \mathbf{e}) - \mathbf{r}_1(\mathbf{r}_2 \cdot \mathbf{e})
\end{aligned}$$

Once again employing Equations 5.9, these become

$$(\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} = \mathbf{r}_3\left(\frac{h^2}{\mu} - r_2\right) - \mathbf{r}_2\left(\frac{h^2}{\mu} - r_3\right) = \frac{h^2}{\mu}(\mathbf{r}_3 - \mathbf{r}_2) + r_3\mathbf{r}_2 - r_2\mathbf{r}_3$$
$$(\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e} = \mathbf{r}_1\left(\frac{h^2}{\mu} - r_3\right) - \mathbf{r}_3\left(\frac{h^2}{\mu} - r_1\right) = \frac{h^2}{\mu}(\mathbf{r}_1 - \mathbf{r}_3) + r_1\mathbf{r}_3 - r_3\mathbf{r}_1$$
$$(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = \mathbf{r}_2\left(\frac{h^2}{\mu} - r_1\right) - \mathbf{r}_1\left(\frac{h^2}{\mu} - r_2\right) = \frac{h^2}{\mu}(\mathbf{r}_2 - \mathbf{r}_1) + r_2\mathbf{r}_1 - r_1\mathbf{r}_2$$

Summing these three equations, collecting terms and substituting the result into Equation 5.19 yields

$$\hat{\mathbf{q}} = \frac{1}{De}\mathbf{S} \tag{5.20}$$

where

$$\mathbf{S} = \mathbf{r}_1(r_2 - r_3) + \mathbf{r}_2(r_3 - r_1) + \mathbf{r}_3(r_1 - r_2) \tag{5.21}$$

Finally, we substitute Equations 5.16, 5.17 and 5.20 into Equation 5.7 to obtain

$$\mathbf{v} = \frac{\mu}{h}\left(\frac{\hat{\mathbf{w}} \times \mathbf{r}}{r} + e\hat{\mathbf{q}}\right) = \frac{\mu}{\sqrt{\mu\dfrac{N}{D}}}\left[\frac{\dfrac{\mathbf{D}}{D} \times \mathbf{r}}{r} + e\left(\frac{1}{De}\mathbf{S}\right)\right]$$

Simplifying this expression for the velocity yields

$$\mathbf{v} = \sqrt{\frac{\mu}{ND}}\left(\frac{\mathbf{D} \times \mathbf{r}}{r} + \mathbf{S}\right) \tag{5.22}$$

All of the terms on the right depend only on the given position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$.

The Gibbs procedure may be summarized in the following algorithm.

Algorithm 5.1 Gibbs method of preliminary orbit determination. A MATLAB® implementation of this procedure is found in Appendix D.24.

Given $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, the steps are as follows:

1. Calculate r_1, r_2 and r_3.
2. Calculate $\mathbf{C}_{12} = \mathbf{r}_1 \times \mathbf{r}_2$, $\mathbf{C}_{23} = \mathbf{r}_2 \times \mathbf{r}_3$ and $\mathbf{C}_{31} = \mathbf{r}_3 \times \mathbf{r}_1$.
3. Verify that $\hat{\mathbf{u}}_{r_1} \cdot \hat{\mathbf{C}}_{23} = 0$.
4. Calculate $\mathbf{N}$, $\mathbf{D}$ and $\mathbf{S}$ using Equations 5.13, 5.14 and 5.21, respectively.
5. Calculate $\mathbf{v}_2$ using Equation 5.22.
6. Use $\mathbf{r}_2$ and $\mathbf{v}_2$ to compute the orbital elements by means of Algorithm 4.2.

Example 5.1

The geocentric position vectors of a space object at three successive times are

$$\begin{aligned}\mathbf{r}_1 &= -294.32\hat{\mathbf{I}} + 4265.1\hat{\mathbf{J}} + 5986.7\hat{\mathbf{K}} \text{ (km)}\\ \mathbf{r}_2 &= -1365.5\hat{\mathbf{I}} + 3637.6\hat{\mathbf{J}} + 6346.8\hat{\mathbf{K}} \text{ (km)}\\ \mathbf{r}_3 &= -2940.3\hat{\mathbf{I}} + 2473.7\hat{\mathbf{J}} + 6555.8\hat{\mathbf{K}} \text{ (km)}\end{aligned}$$

Determine the classical orbital elements using Gibbs' procedure.

Solution

We employ Algorithm 5.1.

Step 1.

$$\begin{aligned}r_1 &= \sqrt{(-294.32)^2 + 4265.1^2 + 5986.7^2} = 7356.5 \text{ km}\\ r_2 &= \sqrt{(-1365.5)^2 + 3637.6^2 + 6346.8^2} = 7441.7 \text{ km}\\ r_3 &= \sqrt{(-2940.3)^2 + 2473.7^2 + 6555.8^2} = 7598.9 \text{ km}\end{aligned}$$

Step 2.

$$\mathbf{C}_{12} = \begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ -294.32 & 4265.1 & 5986.7\\ -1365.5 & 3637.6 & 6346.8\end{vmatrix} = (5.2925\hat{\mathbf{I}} - 6.3068\hat{\mathbf{J}} + 4.7534\hat{\mathbf{K}}) \times 10^6 \text{ (km}^2\text{)}$$

$$\mathbf{C}_{23} = \begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ -1365.5 & 3637.6 & 6346.8\\ -2940.3 & 2473.7 & 6555.8\end{vmatrix} = (8.1473\hat{\mathbf{I}} - 9.7096\hat{\mathbf{J}} + 7.3178\hat{\mathbf{K}}) \times 10^6 \text{ (km}^2\text{)}$$

$$\mathbf{C}_{31} = \begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ -2940.3 & 2473.7 & 6555.8\\ -294.32 & 4265.1 & 5986.7\end{vmatrix} = (-1.3152\hat{\mathbf{I}} + 1.5673\hat{\mathbf{J}} - 1.1813\hat{\mathbf{K}}) \times 10^7 \text{ (km}^2\text{)}$$

Step 3.

$$\hat{\mathbf{C}}_{23} = \frac{\mathbf{C}_{23}}{\|\mathbf{C}_{23}\|} = \frac{8.1473\hat{\mathbf{I}} - 9.7096\hat{\mathbf{J}} + 7.3178\hat{\mathbf{K}}}{\sqrt{8.1473^2 + (-9.7096)^2 + 7.3178^2}} = 0.55667\hat{\mathbf{I}} - 0.66342\hat{\mathbf{J}} + 0.5000\hat{\mathbf{K}}$$

Therefore,

$$\hat{\mathbf{u}}_{r_1} \cdot \hat{\mathbf{C}}_{23} = \frac{-294.32\hat{\mathbf{I}} + 4265.1\hat{\mathbf{J}} + 5986.7\hat{\mathbf{K}}}{7356.5} \cdot (0.55667\hat{\mathbf{I}} - 0.66342\hat{\mathbf{J}} + 0.5000\hat{\mathbf{K}}) = -6.1181 \times 10^{-6}$$

This is close enough to zero for our purposes. The three vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are coplanar.

Step 4.

$$\begin{aligned}\mathbf{N} &= r_1\mathbf{C}_{23} + r_2\mathbf{C}_{31} + r_3\mathbf{C}_{12} \\ &= 7356.5[(8.1473\hat{\mathbf{I}} - 9.7096\hat{\mathbf{J}} + 7.3178\hat{\mathbf{K}}) \times 10^6] + 7441.7[(-1.3152\hat{\mathbf{I}} + 1.5673\hat{\mathbf{J}} - 1.1813\hat{\mathbf{K}}) \times 10^6] \\ &\quad + 7598.9[(5.2925\hat{\mathbf{I}} - 6.3068\hat{\mathbf{J}} + 4.7534\hat{\mathbf{K}}) \times 10^6]\end{aligned}$$

or

$$\mathbf{N} = (2.2811\hat{\mathbf{I}} - 2.7186\hat{\mathbf{J}} + 2.0481\hat{\mathbf{K}}) \times 10^9 \ (\text{km}^3)$$

so that

$$N = \sqrt{[2.2811^2 + (-2.7186)^2 + 2.0481^2] \times 10^{18}} = 4.0975 \times 10^9 \ (\text{km}^3)$$

$$\begin{aligned}\mathbf{D} &= \mathbf{C}_{12} + \mathbf{C}_{23} + \mathbf{C}_{31} \\ &= [(5.295\hat{\mathbf{I}} - 6.3068\hat{\mathbf{J}} + 4.7534\hat{\mathbf{K}}) \times 10^6] + [(8.1473\hat{\mathbf{I}} - 9.7096\hat{\mathbf{J}} + 7.3178\hat{\mathbf{K}}) \times 10^6] \\ &\quad + [(-1.3152\hat{\mathbf{I}} + 1.5673\hat{\mathbf{J}} - 1.1813\hat{\mathbf{K}}) \times 10^6]\end{aligned}$$

or

$$\mathbf{D} = (2.8797\hat{\mathbf{I}} - 3.4321\hat{\mathbf{J}} + 2.5856\hat{\mathbf{K}}) \times 10^5 \ (\text{km}^2)$$

so that

$$D = \sqrt{[2.8797^2 + (-3.4321)^2 + 2.5856^2] \times 10^{10}} = 5.1728 \times 10^5 \ (\text{km}^2)$$

Lastly,

$$\begin{aligned}\mathbf{S} &= \mathbf{r}_1(r_2 - r_3) + \mathbf{r}_2(r_3 - r_1) + \mathbf{r}_3(r_1 - r_2) \\ &= (-294.32\hat{\mathbf{I}} + 4265.1\hat{\mathbf{J}} + 5986.7\hat{\mathbf{K}})(7441.7 - 7598.9) + (-1365.5\hat{\mathbf{I}} + 3637.6\hat{\mathbf{J}} \\ &\quad + 6346.8\hat{\mathbf{K}})(7598.9 - 7356.5) + (-2940.3\hat{\mathbf{I}} + 2473.7\hat{\mathbf{J}} + 6555.8\hat{\mathbf{K}})(7356.5 - 7441.7)\end{aligned}$$

or

$$\mathbf{S} = -34{,}276\hat{\mathbf{I}} + 478.58\hat{\mathbf{J}} + 38{,}810\hat{\mathbf{K}} \ (\text{km}^2)$$

Step 5.

$$\begin{aligned}\mathbf{v}_2 &= \sqrt{\frac{\mu}{ND}}\left(\frac{\mathbf{D} \times \mathbf{r}_2}{r_2} + \mathbf{S}\right) \\ &= \sqrt{\frac{398{,}600}{(4.0971 \times 10^9)(5.1728 \times 10^3)}} \\ &\quad \times \left[\frac{\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 2.8797 \times 10^6 & -3.4321 \times 10^6 & 2.5856 \times 10^6 \\ -1365.5 & 3637.6 & 6346.8 \end{vmatrix}}{7441.7} + (-34.276\hat{\mathbf{I}} + 478.57\hat{\mathbf{J}} + 38.810\hat{\mathbf{K}})\right]\end{aligned}$$

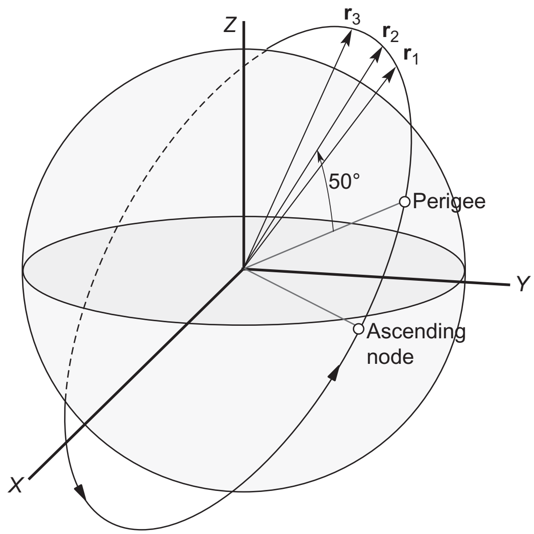


FIGURE 5.2

Sketch of the orbit of Example 5.1.

or

$$\mathbf{v}_2 = -6.2174\hat{\mathbf{I}} - 4.0122\hat{\mathbf{J}} + 1.5990\hat{\mathbf{K}} \text{ (km/s)}$$

Step 6.

Using $\mathbf{r}_2$ and $\mathbf{v}_2$, Algorithm 4.2 yields the orbital elements:

$$\boxed{\begin{aligned} a &= 8000 \text{ km} \\ e &= 0.1 \\ i &= 60^\circ \\ \Omega &= 40^\circ \\ \omega &= 30^\circ \\ \theta &= 50^\circ \text{ (for position vector } \mathbf{r}_2\text{)} \end{aligned}}$$

The orbit is sketched in Figure 5.2.

5.3 LAMBERT'S PROBLEM

Suppose we know the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of two points P_1 and P_2 on the path of mass m around mass M, as illustrated in Figure 5.3. $\mathbf{r}_1$ and $\mathbf{r}_2$ determine the change in the true anomaly $\Delta\theta$, since

$$\cos\Delta\theta = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2} \tag{5.23}$$

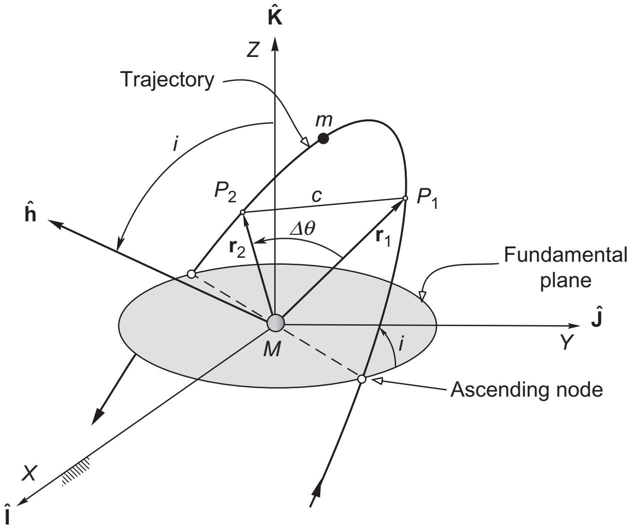


FIGURE 5.3

Lambert's problem.

where

$$r_1 = \sqrt{\mathbf{r}_1 \cdot \mathbf{r}_1} \qquad r_2 = \sqrt{\mathbf{r}_2 \cdot \mathbf{r}_2} \tag{5.24}$$

However, if $\cos \Delta\theta > 0$, then $\Delta\theta$ lies in either the first or fourth quadrant, whereas if $\cos \Delta\theta < 0$, then $\Delta\theta$ lies in the second or third quadrant. (Recall Figure 3.4.) The first step in resolving this quadrant ambiguity is to calculate the Z component of $\mathbf{r}_1 \times \mathbf{r}_2$,

$$(\mathbf{r}_1 \times \mathbf{r}_2)_Z = \hat{\mathbf{K}} \cdot (\mathbf{r}_1 \times \mathbf{r}_2) = \hat{\mathbf{K}} \cdot (r_1 r_2 \sin \Delta\theta \hat{\mathbf{h}}) = r_1 r_2 \sin \Delta\theta (\hat{\mathbf{K}} \cdot \hat{\mathbf{h}})$$

where $\hat{\mathbf{h}}$ is the unit normal to the orbital plane. Therefore, $\hat{\mathbf{K}} \cdot \hat{\mathbf{w}} = \cos i$, where i is the inclination of the orbit, so that

$$(\mathbf{r}_1 \times \mathbf{r}_2)_Z = r_1 r_2 \sin \Delta\theta \cos i \tag{5.25}$$

We use the sign of the scalar $(\mathbf{r}_1 \times \mathbf{r}_2)_Z$ to determine the correct quadrant for $\Delta\theta$.

There are two cases to consider: prograde trajectories ($0 < i < 90°$), and retrograde trajectories ($90° < i < 180°$).

For prograde trajectories (like the one illustrated in Figure 5.3), $\cos i > 0$, so that if $(\mathbf{r}_1 \times \mathbf{r}_2)_Z > 0$, then Equation 5.25 implies that $\sin \Delta\theta > 0$, which means $0° < \Delta\theta < 180°$. Since $\Delta\theta$ therefore lies in the first or second quadrant, it follows that $\Delta\theta$ is given by $\cos^{-1}(\mathbf{r}_1 \cdot \mathbf{r}_2 / r_1 r_2)$. On the other hand, if $(\mathbf{r}_1 \times \mathbf{r}_2)_Z < 0$, Equation 5.25 implies that $\sin \Delta\theta < 0$, which means $180° < \Delta\theta < 360°$. In this case $\Delta\theta$ lies in the third or fourth quadrant and is given by $360° - \cos^{-1}(\mathbf{r}_1 \cdot \mathbf{r}_2 / r_1 r_2)$. For retrograde trajectories, $\cos i < 0$. Thus, if $(\mathbf{r}_1 \times \mathbf{r}_2)_Z > 0$ then $\sin \Delta\theta < 0$, which places $\Delta\theta$ in the third or fourth quadrant. Similarly, if $(\mathbf{r}_1 \times \mathbf{r}_2)_Z > 0$, $\Delta\theta$ must lie in the first or second quadrant.

This logic can be expressed more concisely as follows:

$$\Delta\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) & \text{if } (\mathbf{r}_1 \times \mathbf{r}_2)_Z \geq 0 \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) & \text{if } (\mathbf{r}_1 \times \mathbf{r}_2)_Z < 0 \end{cases} \quad \textit{prograde trajectory}$$

$$\Delta\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) & \text{if } (\mathbf{r}_1 \times \mathbf{r}_2)_Z < 0 \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) & \text{if } (\mathbf{r}_1 \times \mathbf{r}_2)_Z \geq 0 \end{cases} \quad \textit{retrograde trajectory} \tag{5.26}$$

J. H. Lambert (1728–1777) was a French-born German astronomer, physicist and mathematician, According to a theorem of his, the transfer time Δt from P_1 to P_2 is independent of the orbit's eccentricity and depends only on the sum $r_1 + r_2$ of the magnitudes of the position vectors, the semimajor axis a and the length c of the chord joining P_1 and P_2. It is noteworthy that the period (of an ellipse) and the specific mechanical energy are also independent of the eccentricity (Equations 2.83, 2.80 and 2.110).

If we know the time of flight Δt from P_1 to P_2, then Lambert's problem is to find the trajectory joining P_1 and P_2. The trajectory is determined once we find $\mathbf{v}_1$, because, according to Equations 2.135 and 2.136, the position and velocity of any point on the path are determined by $\mathbf{r}_1$ and $\mathbf{v}_1$. That is, in terms of the notation in Figure 5.3,

$$\mathbf{r}_2 = f\mathbf{r}_1 + g\mathbf{v}_1 \tag{5.27a}$$

$$\mathbf{v}_2 = \dot{f}\mathbf{r}_1 + \dot{g}\mathbf{v}_1 \tag{5.27b}$$

Solving the first of these for $\mathbf{v}_1$ yields

$$\mathbf{v}_1 = \frac{1}{g}(\mathbf{r}_2 - f\mathbf{r}_1) \tag{5.28}$$

Substitute this result into Equation 5.27b to get

$$\mathbf{v}_2 = \dot{f}\mathbf{r}_1 + \frac{\dot{g}}{g}(\mathbf{r}_2 - f\mathbf{r}_1) = \frac{\dot{g}}{g}\mathbf{r}_2 - \frac{f\dot{g} - \dot{f}g}{g}\mathbf{r}_1$$

However, according to Equation 2.139, $f\dot{g} - \dot{f}g = 1$. Hence,

$$\mathbf{v}_2 = \frac{1}{g}(\dot{g}\mathbf{r}_2 - \mathbf{r}_1) \tag{5.29}$$

By means of Algorithm 4.2 we can find the orbital elements from either $\mathbf{r}_1$ and $\mathbf{v}_1$ or $\mathbf{r}_2$ and $\mathbf{v}_2$. Clearly, Lambert's problem is solved once we determine the Lagrange coefficients f, g and $\dot{g}$.

The Lagrange f and g coefficients and their time derivatives are listed as functions of the change in true anomaly $\Delta\theta$ in Equations 2.158,

$$f = 1 - \frac{\mu r_2}{h^2}(1 - \cos\Delta\theta) \qquad g = \frac{r_1 r_2}{h}\sin\Delta\theta \tag{5.30a}$$

$$\dot{f} = \frac{\mu}{h}\frac{1 - \cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1 - \cos\Delta\theta) - \frac{1}{r_0} - \frac{1}{r}\right] \qquad \dot{g} = 1 - \frac{\mu r_1}{h^2}(1 - \cos\Delta\theta) \tag{5.30b}$$

Equations 3.69 express these quantities in terms of the universal anomaly χ,

$$f = 1 - \frac{\chi^2}{r_1}C(z) \qquad g = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(z) \tag{5.31a}$$

$$\dot{f} = \frac{\sqrt{\mu}}{r_1 r_2}\chi[zS(z) - 1] \qquad \dot{g} = 1 - \frac{\chi^2}{r_2}C(z) \tag{5.31b}$$

where $z = \alpha\chi^2$. The f and g functions do not depend on the eccentricity, which would seem to make them an obvious choice for the solution of Lambert's problem.

The unknowns on the right of the above sets of equations are h, χ and z, whereas $\Delta\theta$, Δt, r_1 and r_2 are given. Equating the four pairs of expressions for f, g, $\dot{f}$ and $\dot{g}$ in Equations 5.30 and 5.31 yields four equations in the three unknowns h, χ and z. However, because of the fact that $f\dot{g} - \dot{f}g = 1$ (Equation 2.139), only three of these equations are independent. We must solve them for h, χ and z in order to evaluate the Lagrange coefficients and thereby obtain the solution to Lambert's problem. We will follow the procedure presented by Bate, Mueller and White (1971) and Bond and Allman (1996).

While $\Delta\theta$ appears throughout Equations 5.30, the time interval Δt does not. However, Δt does appear in Equation 5.31a. A relationship between $\Delta\theta$ and Δt can therefore be found by equating the two expressions for g,

$$\frac{r_1 r_2}{h}\sin\Delta\theta = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(z) \tag{5.32}$$

To eliminate the unknown angular momentum h, equate the expressions for f in Equations 5.30a and 5.31a,

$$1 - \frac{\mu r_2}{h^2}(1 - \cos\Delta\theta) = 1 - \frac{\chi^2}{r_1}C(z)$$

Upon solving this for h we obtain

$$h = \sqrt{\frac{\mu r_1 r_2(1 - \cos\Delta\theta)}{\chi^2 C(z)}} \tag{5.33}$$

(Equating the two expressions for $\dot{g}$ leads to the same result.) Substituting Equation 5.33 into 5.32, simplifying and rearranging terms yields

$$\sqrt{\mu}\Delta t = \chi^3 S(z) + \chi\sqrt{C(z)}\left(\sin\Delta\theta\sqrt{\frac{r_1 r_2}{1 - \cos\Delta\theta}}\right) \tag{5.34}$$

The term in parentheses on the right is a constant comprised solely of the given data. Let us assign it the symbol A,

$$A = \sin \Delta\theta \sqrt{\frac{r_1 r_2}{1 - \cos \Delta\theta}} \tag{5.35}$$

Then Equation 5.34 assumes the simpler form

$$\sqrt{\mu}\Delta t = \chi^3 S(z) + A\chi\sqrt{C(z)} \tag{5.36}$$

The right side of this equation contains both of the unknown variables χ and z. We cannot use the fact that $z = \alpha\chi^2$ to reduce the unknowns to one since α is the reciprocal of the semimajor axis of the unknown orbit.

In order to find a relationship between z and χ which does not involve orbital parameters, we equate the expressions for $\dot{f}$ (Equations 5.30b and 5.31b) to obtain

$$\frac{\mu}{h}\frac{1 - \cos \Delta\theta}{\sin \Delta\theta}\left[\frac{\mu}{h^2}(1 - \cos \Delta\theta) - \frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{\sqrt{\mu}}{r_1 r_2}\chi[zS(z) - 1]$$

Multiplying through by $r_1 r_2$ and substituting for the angular momentum using Equation 5.33 yields

$$\frac{\mu}{\sqrt{\dfrac{\mu r_1 r_2(1 - \cos \Delta\theta)}{\chi^2 C(z)}}}\frac{1 - \cos \Delta\theta}{\sin \Delta\theta}\left[\frac{\mu}{\dfrac{\mu r_1 r_2(1 - \cos \Delta\theta)}{\chi^2 C(z)}}(1 - \cos \Delta\theta) - r_1 - r_2\right] = \sqrt{\mu}\chi[zS(z) - 1]$$

Simplifying and dividing out common factors leads to

$$\frac{\sqrt{1 - \cos \Delta\theta}}{\sqrt{r_1 r_2}\sin \Delta\theta}\sqrt{C(z)}[\chi^2 C(z) - r_1 - r_2] = zS(z) - 1$$

We recognize the reciprocal of A on the left, so we can rearrange this expression to read as follows,

$$\chi^2 C(z) = r_1 + r_2 + A\frac{zS(z) - 1}{\sqrt{C(z)}}$$

The right-hand side depends exclusively on z. Let us call that function $y(z)$, so that

$$\chi = \sqrt{\frac{y(z)}{C(z)}} \tag{5.37}$$

where

$$y(z) = r_1 + r_2 + A\frac{zS(z) - 1}{\sqrt{C(z)}} \tag{5.38}$$

Equation 5.37 is the relation between χ and z that we were seeking. Substituting it back into Equation 5.36 yields

$$\sqrt{\mu}\Delta t = \left[\frac{y(z)}{C(z)}\right]^{3/2} S(z) + A\sqrt{y(z)} \tag{5.39}$$

We can use this equation to solve for z, given the time interval Δt. It must be done iteratively.

Using Newton's method, we form the function

$$F(z) = \left[\frac{y(z)}{C(z)}\right]^{3/2} S(z) + A\sqrt{y(z)} - \sqrt{\mu}\Delta t \tag{5.40}$$

and its derivative

$$F'(z) = \frac{1}{2\sqrt{y(z)C^5(z)}}\left\{[2C(z)S'(z) - 3C'(z)S(z)]y^2(z) + \left[AC^{3/2}(z) + 3C(z)S(z)y(z)\right]y'(z)\right\} \tag{5.41}$$

in which $C'(z)$ and $S'(z)$ are the derivatives of the Stumpff functions, which are given by Equations 3.63. $y'(z)$ is obtained by differentiating $y(z)$ in Equation 5.38,

$$y'(z) = \frac{A}{2C(z)^{3/2}}\{[1 - zS(z)]C'(z) + 2[S(z) + zS'(z)]C(z)\}$$

If we substitute Equations 3.63 into this expression a much simpler form is obtained, namely

$$y'(z) = \frac{A}{4}\sqrt{C(z)} \tag{5.42}$$

This result can be worked out by using Equations 3.52 and 3.53 to express $C(z)$ and $S(z)$ in terms of the more familiar trig functions. Substituting Equation 5.42 along with Equations 3.63 into Equation 5.41 yields

$$F'(z) = \begin{cases} \left[\frac{y(z)}{C(z)}\right]^{3/2}\left\{\frac{1}{2z}\left[C(z) - \frac{3}{2}\frac{S(z)}{C(z)}\right] + \frac{3}{4}\frac{S(z)^2}{C(z)}\right\} + \frac{A}{8}\left[3\frac{S(z)}{C(z)}\sqrt{y(z)} + A\sqrt{\frac{C(z)}{y(z)}}\right] & (z \neq 0) \\ \frac{\sqrt{2}}{40}y(0)^{3/2} + \frac{A}{8}\left[\sqrt{y(0)} + A\sqrt{\frac{1}{2y(0)}}\right] & (z = 0) \end{cases} \tag{5.43}$$

Evaluating $F'(z)$ at $z = 0$ must be done carefully (and is therefore shown as a special case), because of the z in the denominator within the curly brackets. To handle $z = 0$, we assume that z is very small (almost, but not quite zero) so that we can retain just the first two terms in the series expansions of $C(z)$ and $S(z)$ (Equations 3.51),

$$C(z) = \frac{1}{2} - \frac{z}{24} + \ldots \qquad S(z) = \frac{1}{6} - \frac{z}{120} + \ldots$$

Then we evaluate the term within the curly brackets as follows.

$$\begin{aligned}\frac{1}{2z}\left[C(z)-\frac{3}{2}\frac{S(z)}{C(z)}\right] &\approx \frac{1}{2z}\left[\left(\frac{1}{2}-\frac{z}{24}\right)-\frac{3}{2}\frac{\left(\frac{1}{6}-\frac{z}{120}\right)}{\left(\frac{1}{2}-\frac{z}{24}\right)}\right]\\ &= \frac{1}{2z}\left[\left(\frac{1}{2}-\frac{z}{24}\right)-3\left(\frac{1}{6}-\frac{z}{120}\right)\left(1-\frac{z}{12}\right)^{-1}\right]\\ &\approx \frac{1}{2z}\left[\left(\frac{1}{2}-\frac{z}{24}\right)-3\left(\frac{1}{6}-\frac{z}{120}\right)\left(1+\frac{z}{12}\right)\right]\\ &= \frac{1}{2z}\left(-\frac{7z}{120}+\frac{z^2}{480}\right)\\ &= -\frac{7}{240}+\frac{z}{960}\end{aligned}$$

In the third step we used the familiar binomial expansion theorem,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots \tag{5.44}$$

to set $(1 - z/12)^{-1} \approx 1 + z/12$, which is true if z is close to zero. Thus, when z is actually zero,

$$\frac{1}{2z}\left[C(z)-\frac{3}{2}\frac{S(z)}{C(z)}\right] = -\frac{7}{240}$$

Evaluating the other terms in $F'(z)$ presents no difficulties.

$F(z)$ in Equation 5.40 and $F'(z)$ in Equation 5.43 are used in Newton's formula, Equation 3.16, for the iterative procedure,

$$z_{i+1} = z_i - \frac{F(z_i)}{F'(z_i)} \tag{5.45}$$

For choice of a starting value for z, recall that $z = (1/a)\chi^2$. According to Equation 3.57, $z = E^2$ for an ellipse and $z = -F^2$ for a hyperbola. Since we do not know what the orbit is, setting $z_0 = 0$ seems a reasonable, simple choice. Alternatively, one can plot or tabulate $F(z)$ and choose z_0 to be a point near where $F(z)$ changes sign.

Substituting Equation 5.37 and 5.39 into Equations 5.31 yields the Lagrange coefficients as functions of z alone.

$$f = 1 - \frac{\left[\sqrt{\frac{y(z)}{C(z)}}\right]^2}{r_1}C(z) = 1 - \frac{y(z)}{r_1} \tag{5.46a}$$

$$g = \frac{1}{\sqrt{\mu}}\left\{\left[\frac{y(z)}{C(z)}\right]^{3/2} S(z) + A\sqrt{y(z)}\right\} - \frac{1}{\sqrt{\mu}}\left[\frac{y(z)}{C(z)}\right]^{3/2} S(z) = A\sqrt{\frac{y(z)}{\mu}} \tag{5.46b}$$

$$\dot{f} = \frac{\sqrt{\mu}}{r_1 r_2}\sqrt{\frac{y(z)}{C(z)}}[zS(z) - 1] \tag{5.46c}$$

$$\dot{g} = 1 - \frac{\left[\sqrt{\frac{y(z)}{C(z)}}\right]^2}{r_2} C(z) = 1 - \frac{y(z)}{r_2} \tag{5.46d}$$

We are now in a position to present the solution of Lambert's problem in universal variables, following Bond and Allman (1996).

Algorithm 5.2 Solve Lambert's problem. A MATLAB implementation appears in Appendix D.25.
Given $\mathbf{r}_1$, $\mathbf{r}_2$ and Δt, the steps are as follows:

1. Calculate r_1 and r_2 using Equation 5.24.
2. Choose either a prograde or a retrograde trajectory and calculate $\Delta\theta$ using Equation 5.26.
3. Calculate A in Equation 5.35.
4. By iteration, using Equations 5.40, 5.43 and 5.45, solve Equation 5.39 for z. The sign of z tells us whether the orbit is a hyperbola ($z < 0$), parabola ($z = 0$) or ellipse ($z > 0$).
5. Calculate y using Equation 5.38.
6. Calculate the Lagrange f, g and $\dot{g}$ functions using Equations 5.46.
7. Calculate $\mathbf{v}_1$ and $\mathbf{v}_2$ from Equations 5.28 and 5.29.
8. Use $\mathbf{r}_1$ and $\mathbf{v}_1$ (or $\mathbf{r}_2$ and $\mathbf{v}_2$) in Algorithm 4.2 to obtain the orbital elements.

Example 5.2

The position of an earth satellite is first determined to be $\mathbf{r}_1 = 5000\hat{\mathbf{I}} + 10{,}000\hat{\mathbf{J}} + 2100\hat{\mathbf{K}}$ (km). After one hour the position vector is $\mathbf{r}_2 = -14{,}600\hat{\mathbf{I}} + 2500\hat{\mathbf{J}} + 7000\mathbf{K}$ (km). Determine the orbital elements and find the perigee altitude and the time since perigee passage of the first sighting.

Solution

We first must execute the steps of Algorithm 5.2 in order to find $\mathbf{v}_1$ and $\mathbf{v}_2$.
Step 1.

$$r_1 = \sqrt{5000^2 + 10{,}000^2 + 2100^2} = 11{,}375 \text{ km}$$
$$r_2 = \sqrt{(-14{,}600)^2 + 2500^2 + 7000^2} = 16{,}383 \text{ km}$$

Step 2.
Assume a prograde trajectory.

$$\mathbf{r}_1 \times \mathbf{r}_2 = (64.75\hat{\mathbf{I}} - 65.66\hat{\mathbf{J}} + 158.5\hat{\mathbf{K}}) \times 10^6$$

$$\cos^{-1}\frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2} = 100.29°$$

Since the trajectory is prograde and the z component of $\mathbf{r}_1 \times \mathbf{r}_2$ is positive, it follows from Equation 5.26 that

$$\Delta\theta = 100.29°$$

Step 3.

$$A = \sin\Delta\theta\sqrt{\frac{r_1 r_2}{1-\cos\Delta\theta}} = \sin 100.29°\sqrt{\frac{11{,}375 \cdot 16{,}383}{1-\cos 100.29°}} = 12{,}372 \text{ km}$$

Step 4.

Using this value of A and $\Delta t = 3600$ s, we can evaluate the functions $F(z)$ and $F'(z)$ given by Equations 5.40 and 5.43, respectively. Let us first plot $F(z)$ to estimate where it crosses the z axis. As can be seen from Figure 5.4, $F(z) = 0$ near $z = 1.5$. With $z_0 = 1.5$ as our initial estimate, we execute Newton's procedure, Equation 5.45,

$$z_{i+1} = z_i - \frac{F(z_i)}{F'(z_i)}$$

$$z_1 = 1.5 - \frac{-14{,}476.4}{362{,}642} = 1.53991$$

$$z_2 = 1.53991 - \frac{23.6274}{363{,}828} = 1.53985$$

$$z_3 = 1.53985 - \frac{6.29457 \times 10^{-5}}{363{,}826} = 1.53985$$

Thus, to five significant figures $z = 1.5398$. The fact that z is positive means the orbit is an ellipse.

Step 5.

$$y = r_1 + r_2 + A\frac{zS(z)-1}{\sqrt{C(z)}} = 11{,}375 + 16{,}383 + 12{,}372\frac{1.5398S(1.5398)}{\sqrt{C(1.5398)}} = 13{,}523 \text{ km}$$

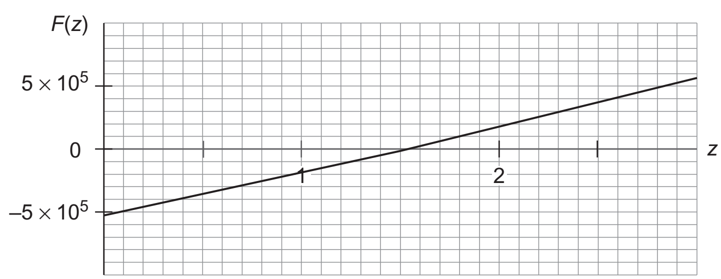


FIGURE 5.4

Graph of $F(z)$.

Step 6.
Equations 5.46 yield the Lagrange functions

$$f = 1 - \frac{y}{r_1} = 1 - \frac{13{,}523}{11{,}375} = -0.188{,}77$$
$$g = A\sqrt{\frac{y}{\mu}} = 12{,}372\sqrt{\frac{13{,}523}{398{,}600}} = 2278.9 \text{ s}$$
$$\dot{g} = 1 - \frac{y}{r_2} = 1 - \frac{13{,}523}{16{,}383} = 0.174{,}57$$

Step 7.

$$\mathbf{v}_1 = \frac{1}{g}(\mathbf{r}_2 - f\mathbf{r}_1) = \frac{1}{2278.9}[(-14{,}600\hat{\mathbf{I}} + 2500\hat{\mathbf{J}} + 7000\hat{\mathbf{K}}) - (-0.188{,}77)(5000\hat{\mathbf{I}} + 10{,}000\hat{\mathbf{J}} + 2100\hat{\mathbf{K}})]$$
$$\mathbf{v}_1 = -5.9925\hat{\mathbf{I}} + 1.9254\hat{\mathbf{J}} + 3.2456\hat{\mathbf{K}} \text{ (km)}$$
$$\mathbf{v}_2 = \frac{1}{g}(\dot{g}\mathbf{r}_2 - \mathbf{r}_1) = \frac{1}{2278.9}[(0.174{,}57)(-14{,}600\hat{\mathbf{I}} + 2500\hat{\mathbf{J}} + 7000\hat{\mathbf{K}}) - (5000\hat{\mathbf{I}} + 10\,000\hat{\mathbf{J}} + 2100\hat{\mathbf{K}})]$$
$$\mathbf{v}_2 = -3.3125\hat{\mathbf{I}} - 4.1966\hat{\mathbf{J}} - 0.385{,}29\hat{\mathbf{K}} \text{ (km)}$$

Step 8.
Using $\mathbf{r}_1$ and $\mathbf{v}_1$ Algorithm 4.2 yields the orbital elements:

$$\boxed{\begin{aligned} h &= 80{,}470 \text{ km}^2/\text{s} \\ a &= 20{,}000 \text{ km} \\ e &= 0.4335 \\ \Omega &= 44.60^\circ \\ i &= 30.19^\circ \\ \omega &= 30.71^\circ \\ \theta_1 &= 350.8^\circ \end{aligned}}$$

This elliptical orbit is plotted in Figure 5.5. The perigee of the orbit is

$$r_p = \frac{h^2}{\mu}\frac{1}{1 + e\cos(0)} = \frac{80{,}470^2}{398{,}600}\frac{1}{1 + 0.4335} = 11{,}330 \text{ k}$$

Therefore the perigee altitude is $11330 - 6378 = \boxed{4952\,\text{km.}}$

To find the time of the first sighting, we first calculate the eccentric anomaly by means of Equation 3.13b,

$$E_1 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.4335}{1+0.4335}}\tan\frac{350.8^\circ}{2}\right) = 2\tan^{-1}(-0.050{,}41) = -0.1007 \text{ rad.}$$

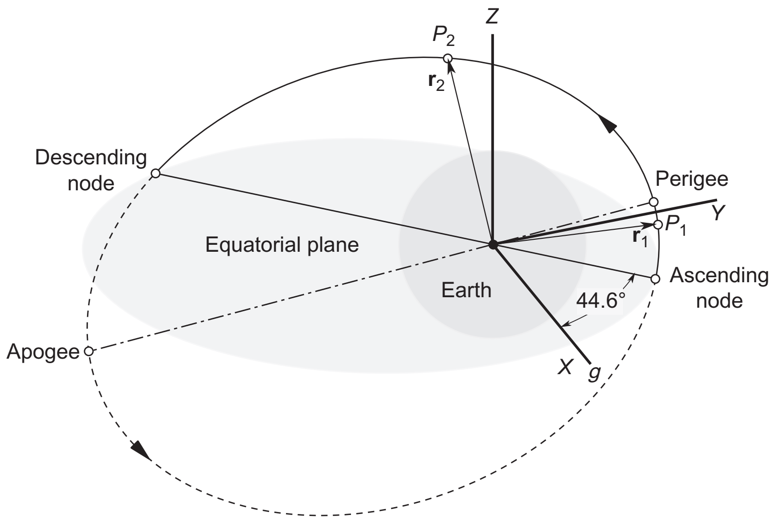


FIGURE 5.5

The solution of Example 5.2 (Lambert's problem).

Then using Kepler's equation for the ellipse (Equation 3.14), the mean anomaly is found to be

$$M_{e_1} = E_1 - e\sin E_1 = -0.1007 - 0.4335\sin(-0.1007) = -0.05715 \text{ rad.}$$

so that from Equation 3.7, the time since perigee passage is

$$t_1 = \frac{h^3}{\mu^2}\frac{1}{(1-e^2)^{3/2}}M_{e_1} = \frac{80{,}470^3}{398{,}600^2}\frac{1}{(1-.4335^2)^{3/2}}(-0.05715) = \boxed{-256.1 \text{ s}}$$

The minus sign means there are 256.1 seconds until perigee encounter after the initial sighting.

Example 5.3

A meteoroid is sighted at an altitude of 267,000 km. 13.5 hours later, after a change in true anomaly of 5°, the altitude is observed to be 140,000 km. Calculate the perigee altitude and the time to perigee after the second sighting.

Solution

We have

$$\begin{aligned} P_1 : r_1 &= 6378 + 267{,}000 = 273{,}378 \text{ km} \\ P_2 : r_2 &= 6378 + 140{,}000 = 146{,}378 \text{ km} \\ \Delta t &= 13.5 \cdot 3600 = 48{,}600 \text{ s} \\ \Delta\theta &= 5 \text{ degrees} \end{aligned}$$

Since r_1, r_2 and $\Delta\theta$ are given, we can skip to Step 3 of Algorithm 5.2 and compute

$$A = 2.8263 \times 10^5 \text{ km}$$

Then, solving for z as in the previous example we obtain

$$z = -0.17344$$

Since z is negative, the path of the meteoroid is a hyperbola.

With z available, we evaluate the Lagrange functions,

$$\begin{aligned} f &= 0.95846 \\ g &= 47{,}708 \text{ s} \\ \dot{g} &= 0.92241 \end{aligned} \tag{a}$$

Step 7 requires the initial and final position vectors. Therefore, for purposes of this problem let us define a geocentric coordinate system with the x axis aligned with $\mathbf{r}_1$ and the y axis at 90° thereto in the direction of the motion (see Figure 5.6). The z axis is therefore normal to the plane of the orbit. Then

$$\begin{aligned} \mathbf{r}_1 &= r_1\hat{\mathbf{i}} = 273{,}378\hat{\mathbf{i}} \text{ (km)} \\ \mathbf{r}_2 &= r_2\cos\Delta\theta\hat{\mathbf{i}} + r_2\sin\Delta\theta\hat{\mathbf{j}} = 145{,}820\hat{\mathbf{i}} + 12{,}758\hat{\mathbf{j}} \text{ (km)} \end{aligned} \tag{b}$$

With (a) and (b) we obtain the velocity at P_1,

$$\begin{aligned} \mathbf{v}_1 &= \frac{1}{g}(\mathbf{r}_2 - f\mathbf{r}_1) \\ &= \frac{1}{47{,}708}[(145{,}820\hat{\mathbf{i}} + 12{,}758\hat{\mathbf{j}}) - 0.95846(273{,}378\hat{\mathbf{i}})] \\ &= -2.4356\hat{\mathbf{i}} - 0.26741\hat{\mathbf{j}} \text{ (km/s)} \end{aligned}$$

Using $\mathbf{r}_1$ and $\mathbf{v}_1$, Algorithm 4.2 yields

$$\begin{aligned} h &= 73{,}105 \text{ km}^2/\text{s} \\ e &= 1.0506 \\ \theta_1 &= 205.16° \end{aligned}$$

The orbit is now determined except for its orientation in space, for which no information was provided. In the plane of the orbit, the trajectory is as shown in Figure 5.6.

The perigee radius is

$$r_p = \frac{h^2}{\mu}\frac{1}{1 + e\cos(0)} = 6538.2 \text{ km}$$

which means the perigee altitude is dangerously low for a large meteoroid,

$$z_p = 6538.2 - 6378 = \boxed{160.2 \text{ km (100 miles)}}$$

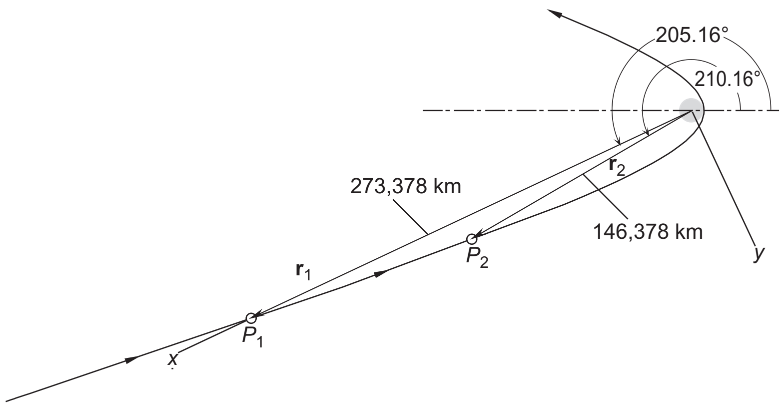


FIGURE 5.6

Solution of Example 5.3 (Lambert's problem).

To find the time of flight from P_2 to perigee, we note that the true anomaly of P_2 is

$$\theta_2 = \theta_1 + 5° = 210.16°$$

The hyperbolic eccentric anomaly F_2 follows from Equation 3.44a,

$$F_2 = 2\tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\frac{\theta_2}{2}\right) = -1.3347 \text{ rad.}$$

From this we appeal to Kepler's equation (Equation 3.40) for the mean anomaly M_h,

$$M_{h_2} = e\sinh(F_2) - F_2 = -0.52265 \text{ rad.}$$

Finally, Equation 3.34 yields the time

$$t_2 = \frac{M_{h_2}h^3}{\mu^2(e^2-1)^{3/2}} = -38{,}396 \text{ s}$$

The minus sign means that 38,396 seconds (a scant 10.6 hours) remain until the meteoroid passes through perigee.

5.4 SIDEREAL TIME

To deduce the orbit of a satellite or celestial body from observations requires, among other things, recording the time of each observation. The time we use in everyday life, the time we set our clocks by, is ***solar time***. It is reckoned by the motion of the sun across the sky. A solar day is the time required for the sun to return to the same position overhead, that is, to lie on same meridian. A solar day—from high noon to high noon—comprises 24 hours. ***Universal time*** (UT) is determined by the sun's passage across the Greenwich meridian, which is 0° terrestrial longitude. See Figure 1.18. At noon UT the sun lies on the Greenwich meridian. Local standard time, or civil time, is obtained from universal time by adding one hour for each time zone between Greenwich and the site, measured westward.

Sidereal time is measured by the rotation of the earth relative to the fixed stars (i.e., the celestial sphere, Figure 4.3). The time it takes for a distant star to return to its same position overhead, i.e, to lie on the same meridian, is one sidereal day (24 sidereal hours). As illustrated in Figure 4.20, the earth's orbit around the sun results in the sidereal day being slightly shorter than the solar day. One sidereal day is 23 hours and 56 minutes. To put it another way, the earth rotates 360° in one sidereal day whereas it rotates 360.986° in a solar day.

Local sidereal time θ of a site is the time elapsed since the local meridian of the site passed through the vernal equinox. The number of degrees (measured eastward) between the vernal equinox and the local meridian is the sidereal time multiplied by 15. To know the location of a point on the earth at any given instant relative to the geocentric equatorial frame requires knowing its local sidereal time. The local sidereal time of a site is found by first determining the Greenwich sidereal time θ_G (the sidereal time of the Greenwich meridian), and then adding the east longitude (or subtracting the west longitude) of the site. Algorithms for determining sidereal time rely on the notion of ***Julian day (JD)***.

The Julian day number is the number of days since noon UT on January 1, 4713 BC. The origin of this time scale is placed in antiquity so that, except for prehistoric events, we do not have to deal with positive and negative dates. The Julian day count is uniform and continuous and does not involve leap years or different numbers of days in different months. The number of days between two events is found by simply subtracting the Julian day of one from that of the other. The Julian day begins at noon rather than at midnight so that astronomers observing the heavens at night would not have to deal with a change of date during their watch.

The Julian day numbering system is not to be confused with the Julian calendar, which the Roman emperor Julius Caesar introduced in 46 BC. The Gregorian calendar, introduced in 1583, has largely supplanted the Julian calendar and is in common civil use today throughout much of the world.

J_0 is the symbol for the Julian day number at 0 hr UT (which is half way into the Julian day). At any other UT, the Julian day is given by

$$JD = J_0 + \frac{UT}{24} \tag{5.47}$$

Algorithms and tables for obtaining J_0 from the ordinary year (y), month (m) and day (d) exist in the literature and on the World Wide Web. One of the simplest formulas is found in Boulet (1991),

$$J_0 = 367y - \text{INT}\left\{\frac{7\left[y + \text{INT}\left(\frac{m+9}{12}\right)\right]}{4}\right\} + \text{INT}\left(\frac{275m}{9}\right) + d + 1{,}721{,}013.5 \tag{5.48}$$

where y, m and d are integers lying in the following ranges

$$\begin{aligned} 1901 \le\ & y \le 2099 \\ 1 \le\ & m \le 12 \\ 1 \le\ & d \le 31 \end{aligned}$$

INT(x) means to retain only the integer portion of x, without rounding (or, in other words, round towards zero); that is, INT(−3.9) = −3 and INT(3.9) = 3. Appendix D.26 lists a MATLAB implementation of Equation 5.48.

Example 5.4

What is the Julian day number for May 12, 2004 at 14:45:30 UT?

Solution

In this case $y = 2004$, $m = 5$ and $d = 12$. Therefore, Equation 5.48 yields the Julian day number at 0hr UT,

$$\begin{aligned} J_0 &= 367 \cdot 2004 - \text{INT}\left\{\frac{7\left[2004 + \text{INT}\left(\frac{5+9}{12}\right)\right]}{4}\right\} + \text{INT}\left(\frac{275 \cdot 5}{9}\right) + 12 + 1{,}721{,}013.5 \\ &= 735{,}468 - \text{INT}\left\{\frac{7[2004+1]}{4}\right\} + 152 + 12 + 1{,}721{,}013.5 \\ &= 735{,}468 - 3508 + 152 + 12 + 1{,}721{,}013.5 \end{aligned}$$

or

$$J_0 = 2{,}453{,}137.5 \text{ days}$$

The universal time, in hours, is

$$UT = 14 + \frac{45}{60} + \frac{30}{3600} = 14.758 \text{ hr}$$

Therefore, from Equation 5.47 we obtain the Julian day number at the desired UT,

$$JD = 2{,}453{,}137.5 + \frac{14.758}{24} = \boxed{2{,}453{,}138.115 \text{ days}}$$

Example 5.5

Find the elapsed time between October 4, 1957 UT 19:26:24 and the date of the previous example.

Solution

Proceeding as in Example 5.4 we find that the Julian day number of the given event (the launch of the first man-made satellite, Sputnik I) is

$$JD_1 = 2{,}436{,}116.3100 \text{ days}$$

The Julian day of the previous example is

$$JD_2 = 2{,}453{,}138.1149 \text{ days}$$

Hence, the elapsed time is

$$\Delta JD = 2{,}453{,}138.1149 - 2{,}436{,}116.3100 = \boxed{17{,}021{,}805 \text{ days}} \text{ (46 years, 220 days)}$$

The current Julian epoch is defined to have been noon on January 1, 2000. This epoch is denoted J2000 and has the exact Julian day number 2,451,545.0. Since there are 365.25 days in a Julian year, a Julian century has 36,525 days. It follows that the time T_0 in Julian centuries between the Julian day J_0 and J2000 is

$$T_0 = \frac{J_0 - 2{,}451{,}545}{36{,}525} \tag{5.49}$$

The Greenwich sidereal time θ_{G0} at 0 hr UT may be found in terms of this dimensionless time (Seidelmann, 1992, Section 2.24). θ_{G0} in degrees is given by the series

$$\theta_{G0} = 100.4606184 + 36{,}000.77004T_0 + 0.000387933T_0^2 - 2.583(10^{-8})T_0^3 \quad \text{(degrees)} \tag{5.50}$$

This formula can yield a value outside of the range $0 \le \theta_{G0} \le 360°$. If so, then the appropriate integer multiple of 360° must be added or subtracted to bring θ_{G0} into that range.

Once θ_{G0} has been determined, the Greenwich sidereal time θ_G at any other universal time is found using the relation

$$\theta_G = \theta_{G0} + 360.98564724\frac{UT}{24} \tag{5.51}$$

where UT is in hours. The coefficient of the second term on the right is the number of degrees the earth rotates in 24 hours (solar time).

Finally, the local sidereal time θ of a site is obtained by adding its east longitude Λ to the Greenwich sidereal time,

$$\theta = \theta_G + \Lambda \tag{5.52}$$

Here again it is possible for the computed value of θ to exceed 360°. If so, it must be reduced to within that limit by subtracting the appropriate integer multiple of 360°. Figure 5.7 illustrates the relationship among θ_{G0}, θ_G, Λ and θ.

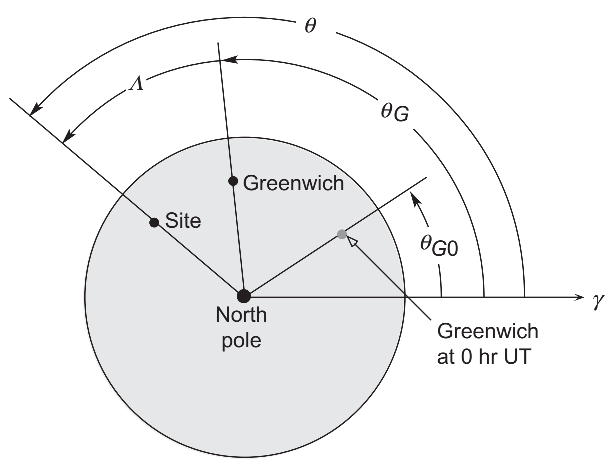


FIGURE 5.7

Schematic of the relationship among θ_{G0}, θ_G, Λ and θ.

Algorithm 5.3 Calculate the local sidereal time, given the date, the local time and the east longitude of the site. This is implemented in MATLAB in Appendix D.27.

1. Using the year, month and day, calculate J_0 using Equation 5.48.
2. Calculate T_0 by means of Equation 5.49.
3. Compute θ_{G0} from Equation 5.50. If θ_{G0} lies outside the range $0° \leq \theta_{G0} \leq 360°$, then subtract the multiple of 360° required to place θ_{G0} in that range.
4. Calculate θ_G using Equation 5.51.
5. Calculate the local sidereal time θ by means of Equation 5.52, adjusting the final value so it lies between 0 and 360°.

Example 5.6

Use Algorithm 5.3 to find the local sidereal time (in degrees) of Tokyo, Japan, on March 3, 2004 at 4:30:00 UT. The east longitude of Tokyo is 139.80°. (This places Tokyo nine time zones ahead of Greenwich, so the local time is 1:30 in the afternoon.)

Step 1.

$$\begin{aligned} J_0 &= 367 \cdot 2004 - \text{INT}\left\{\frac{7\left[2004 + \text{INT}\left(\frac{3+9}{12}\right)\right]}{4}\right\} + \text{INT}\left(\frac{275 \cdot 3}{9}\right) + 3 + 1{,}721{,}013.5 \\ &= 2{,}453{,}067.5 \text{ days} \end{aligned}$$

Recall that the .5 means that we are halfway into the Julian day, which began at noon UT of the previous day.

Step 2.

$$T_0 = \frac{2{,}453{,}067.5 - 2{,}451{,}545}{36{,}525} = 0.041683778$$

Step 3.

$$\begin{aligned} \theta_{G0} &= 100.4606184 + 36{,}000.77004(0.041683778) + 0.000387933(0.041683778)^2 \\ &\quad - 2.583(10^{-8})(0.041683778)^3 \\ &= 1601.1087° \end{aligned}$$

The right-hand side is too large. We must reduce θ_{G0} to an angle that does not exceed 360°. To that end observe that

$$\text{INT}(1601.1087/360) = 4$$

Hence,

$$\theta_{G0} = 1601.1087 - 4 \cdot 360 = 161.10873° \tag{a}$$

Step 4.

The universal time of interest in this problem is

$$UT = 4 + \frac{30}{60} + \frac{0}{3600} = 4.5 \text{ hr}$$

Substitute this and (a) into Equation 5.51 to get the Greenwich sidereal time.

$$\theta_G = 161.10873 + 360.98564724 \frac{4.5}{24} = 228.79354°$$

Step 5.

Add the east longitude of Tokyo to this value to obtain the local sidereal time,

$$\theta = 228.79354 + 139.80 = 368.59°$$

To reduce this result into the range $0 \leq \theta \leq 360°$ we must subtract 360° to get

$$\theta = 368.59 - 360 = \boxed{8.59° \text{ (0.573 hr)}}$$

Observe that the right ascension of a celestial body lying on Tokyo's meridian is 8.59°.

5.5 TOPOCENTRIC COORDINATE SYSTEM

A topocentric coordinate system is one that is centered at the observer's location on the surface of the earth. Consider an object *B*—a satellite or celestial body—and an observer *O* on the earth's surface, as illustrated in Figure 5.8. **r** is the position of the body *B* relative to the center of attraction *C*; **R** is the position vector of the observer relative to C; and **ρ** is the position of the body *B* relative to the observer. **r**, **R** and **ρ** comprise the fundamental vector triangle. The relationship among these three vectors is

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho} \tag{5.53}$$

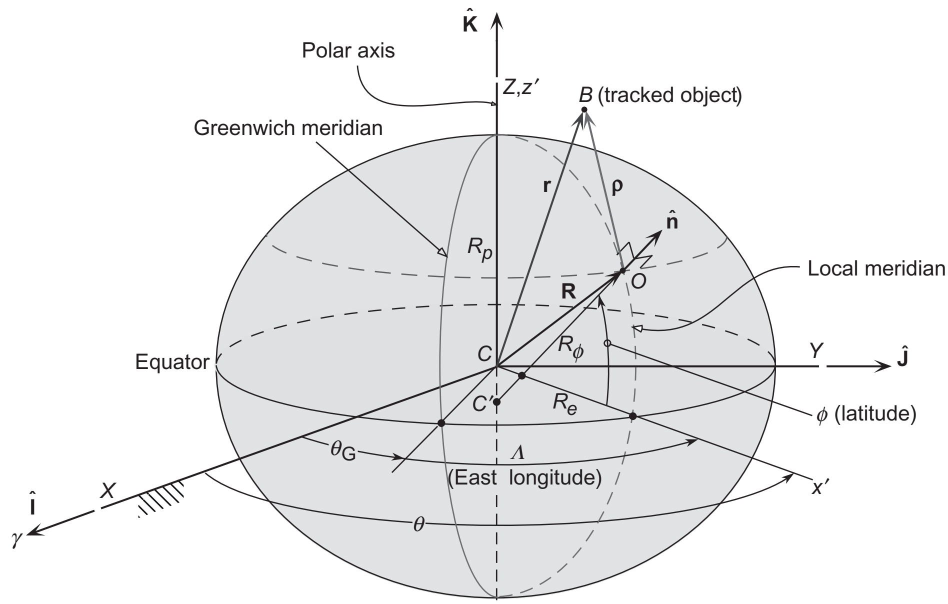


FIGURE 5.8

Oblate spheroidal earth (exaggerated).

As we know, the earth is not a sphere, but a slightly oblate spheroid. This ellipsoidal shape is exaggerated in Figure 5.8. The location of the observation site O is determined by specifying its east longitude Λ and latitude ϕ. East longitude Λ is measured positive eastward from the Greenwich meridian to the meridian through O. The angle between the vernal equinox direction (XZ plane) and the meridian of O is the local sidereal time θ. Likewise, θ_G is the Greenwich sidereal time. Once we know θ_G, then the local sidereal time is given by Equation 5.52.

Latitude ϕ is the angle between the equator and the normal $\hat{\mathbf{n}}$ to the earth's surface at O. Since the earth is not a perfect sphere, the position vector $\mathbf{R}$, directed from the center C of the earth to O, does not point in the direction of the normal except at the equator and the poles.

The oblateness, or flattening f, was defined in Section 4.7,

$$f = \frac{R_e - R_p}{R_e}$$

where R_e is the equatorial radius and R_p is the polar radius. (Review from Table 4.3 that $f = 0.000335$ for the earth.) Figure 5.9 shows the ellipse of the meridian through O. Obviously, R_e and R_p are, respectively, the semimajor and semiminor axes of the ellipse. According to Equation 2.76,

$$R_p = R_e\sqrt{1 - e^2}$$

It is easy to show from the above two relations that flattening and eccentricity are related as follows

$$e = \sqrt{2f - f^2} \qquad f = 1 - \sqrt{1 - e^2}$$

As illustrated in Figure 5.8 and again in Figure 5.9, the normal to the earth's surface at O intersects the polar axis at a point C' that lies below the center C of the earth (if O is in the northern hemisphere). The angle ϕ between the normal and the equator is called the ***geodetic latitude***, as opposed to ***geocentric latitude*** ϕ', which is the angle between the equatorial plane and line joining O to the center of the earth.

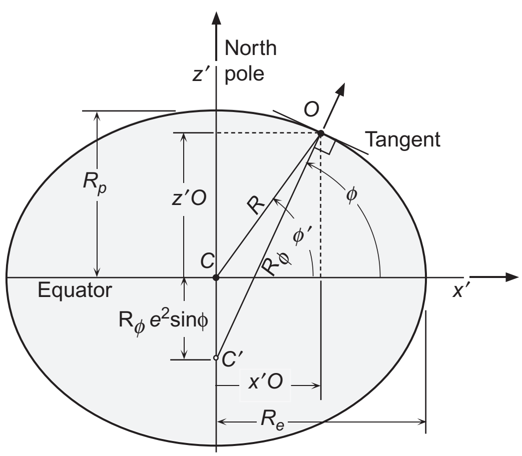


FIGURE 5.9

The relationship between geocentric latitude (ϕ') and geodetic latitude (ϕ).

The distance from C to C' is $R_\phi e^2 \sin^2\phi$, where R_ϕ, the distance from C' to O, is a function of latitude (Seidelmann, 1991, Section 4.22)

$$R_\phi = \frac{R_e}{\sqrt{1 - e^2\sin^2\phi}} = \frac{R_e}{\sqrt{1 - (2f - f^2)\sin^2\phi}} \tag{5.54}$$

Thus, the meridional coordinates of O are

$$\begin{aligned} x'_O &= R_\phi \cos\phi \\ z'_O &= (1 - e^2)R_\phi \sin\phi = (1 - f)^2 R_\phi \sin\phi \end{aligned}$$

If the observation point O is at an elevation H above the ellipsoidal surface, then we must add $H \cos\phi$ to x'_O and $H \sin\phi$ to z'_O to obtain

$$x'_O = R_c \cos\phi \qquad z'_O = R_s \sin\phi \tag{5.55a}$$

where

$$R_c = R_\phi + H \qquad R_s = (1 - f)^2 R_\phi + H \tag{5.55b}$$

Observe that whereas R_c is the distance of O from point C′ on the earth's axis, R_s is the distance from O to the intersection of the line OC' with the equatorial plane.

The geocentric equatorial coordinates of O are

$$X = x'_O \cos\theta \qquad Y = x'_O \sin\theta \qquad Z = z'_O$$

where θ is the local sidereal time given in Equation 5.52. Hence, the position vector **R** shown in Figure 5.8 is

$$\mathbf{R} = R_c \cos\phi\cos\theta\hat{\mathbf{I}} + R_c \cos\phi\sin\theta\hat{\mathbf{J}} + R_s \sin\phi\hat{\mathbf{K}}$$

Substituting Equation 5.54 and Equations 5.55b yields

$$\begin{aligned} \mathbf{R} = &\left[\frac{R_e}{\sqrt{1 - (2f - f^2)\sin^2\phi}} + H\right]\cos\phi(\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}}) \\ &+ \left[\frac{R_e(1 - f)^2}{\sqrt{1 - (2f - f^2)\sin^2\phi}} + H\right]\sin\phi\hat{\mathbf{K}} \end{aligned} \tag{5.56}$$

In terms of the geocentric latitude ϕ'

$$\mathbf{R} = R_e \cos\phi'\cos\theta\hat{\mathbf{I}} + R_e \cos\phi'\sin\theta\hat{\mathbf{J}} + R_e \sin\phi'\hat{\mathbf{K}}$$

By equating these two expressions for **R** and setting $H = 0$ it is easy to show that at sea level geodetic latitude is related to geocentric latitude ϕ' as follows,

$$\tan\phi' = (1 - f)^2 \tan\phi$$

5.6 TOPOCENTRIC EQUATORIAL COORDINATE SYSTEM

The topocentric equatorial coordinate system with origin at point O on the surface of the earth uses a non-rotating set of xyz axes through O which coincide with the XYZ axes of the geocentric equatorial frame, as illustrated in Figure 5.10. As can be inferred from the figure, the relative position vector $\boldsymbol{\rho}$ in terms of the ***topocentric right ascension and declination*** is

$$\boldsymbol{\rho} = \rho\cos\delta\cos\alpha\hat{\mathbf{I}} + \rho\cos\delta\sin\alpha\hat{\mathbf{J}} + \rho\sin\delta\hat{\mathbf{K}}$$

since at all times, $\hat{\mathbf{i}} = \hat{\mathbf{I}}$, $\hat{\mathbf{j}} = \hat{\mathbf{J}}$ and $\hat{\mathbf{k}} = \hat{\mathbf{K}}$ for this frame of reference. We can write $\boldsymbol{\rho}$ as

$$\boldsymbol{\rho} = \rho\hat{\boldsymbol{\rho}}$$

where ρ is the ***slant range*** and $\hat{\boldsymbol{\rho}}$ is the unit vector in the direction of the position vector $\boldsymbol{\rho}$,

$$\hat{\boldsymbol{\rho}} = \cos\delta\cos\alpha\hat{\mathbf{I}} + \cos\delta\sin\alpha\hat{\mathbf{J}} + \sin\delta\hat{\mathbf{K}} \tag{5.57}$$

Since the origins of the geocentric and topocentric systems do not coincide, the direction cosines of the position vectors $\mathbf{r}$ and $\boldsymbol{\rho}$ will in general differ. In particular the topocentric right ascension and declination of an earth-orbiting body B will not be the same as the geocentric right ascension and declination. This is an example of parallax. On the other hand, if $\|\mathbf{r}\| >> \|\mathbf{R}\|$ then the difference between the geocentric and topocentric position vectors, and hence the right ascension and declination, is negligible. This is true for the distant planets and stars.

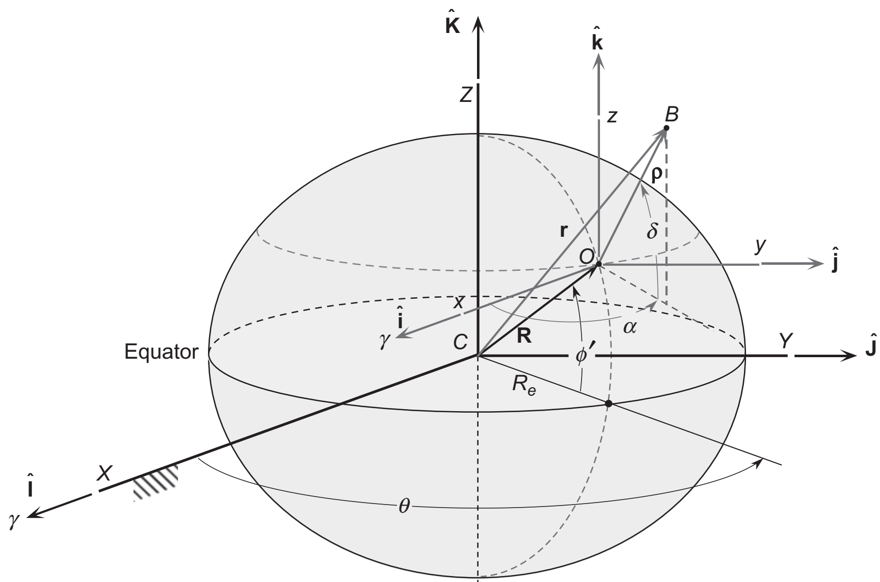


FIGURE 5.10

Topocentric equatorial coordinate system.

Example 5.7

At the instant when the Greenwich sidereal time is $\theta_G = 126.7°$, the geocentric equatorial position vector of the International Space Station is

$$\mathbf{r} = -5368\hat{\mathbf{I}} - 1784\hat{\mathbf{J}} + 3691\hat{\mathbf{K}} \text{ (km)}$$

Find its topocentric right ascension and declination at sea level ($H = 0$), latitude $\phi = 20°$ and east longitude $\Lambda = 60°$.

Solution

According to Equation 5.52, the local sidereal time at the observation site is

$$\theta = \theta_G + \Lambda = 126.7 + 60 = 186.7°$$

Substituting $R_e = 6378$ km, $f = 0.003353$ (Table 4.3), $\theta = 189.7°$ and $\phi = 20°$ into Equation 5.56 yields the geocentric position vector of the site.

$$\mathbf{R} = -5955\hat{\mathbf{I}} - 699.5\hat{\mathbf{J}} + 2168\hat{\mathbf{K}} \text{ (km)}$$

Having found $\mathbf{R}$, we obtain the position vector of the space station relative to the site from Equation 5.53.

$$\begin{aligned}\boldsymbol{\rho} &= \mathbf{r} - \mathbf{R} \\ &= (-5368\hat{\mathbf{I}} - 1784\hat{\mathbf{J}} + 3691\hat{\mathbf{K}}) - (-5955\hat{\mathbf{I}} - 699.5\hat{\mathbf{J}} + 2168\hat{\mathbf{K}}) \\ &= 586.8\hat{\mathbf{I}} - 1084\hat{\mathbf{J}} + 1523\hat{\mathbf{K}} \text{ (km)}\end{aligned}$$

Applying Algorithm 4.1 to this vector yields

$$\boxed{\alpha = 298.4°} \qquad \boxed{\delta = 51.01°}$$

Compare these with the geocentric right ascension α_0 and declination δ_0, which were computed in Example 4.1,

$$\alpha_0 = 198.4° \qquad \delta_0 = 33.12°$$

5.7 TOPOCENTRIC HORIZON COORDINATE SYSTEM

The topocentric horizon system was introduced in Section 1.7 and is illustrated again in Figure 5.11. It is centered at the observation point O whose position vector is $\mathbf{R}$. The xy plane is the local horizon, which is the plane tangent to the ellipsoid at point O. The z axis is normal to this plane directed outward towards the zenith. The x axis is directed eastward and the y axis points north. Because the x axis points east, this may be referred to as an *ENZ* (East-North-Zenith) frame. In the *SEZ* topocentric reference frame the x axis points towards the south and the y axis towards the east. The *SEZ* frame is obtained from *ENZ* by a 90° clockwise rotation around the zenith. Therefore, the matrix of the transformation from *NEZ* to *SEZ* is $[\mathbf{R}_3(-90°)]$, where $[\mathbf{R}_3(\phi)]$ is found in Equation 4.34.

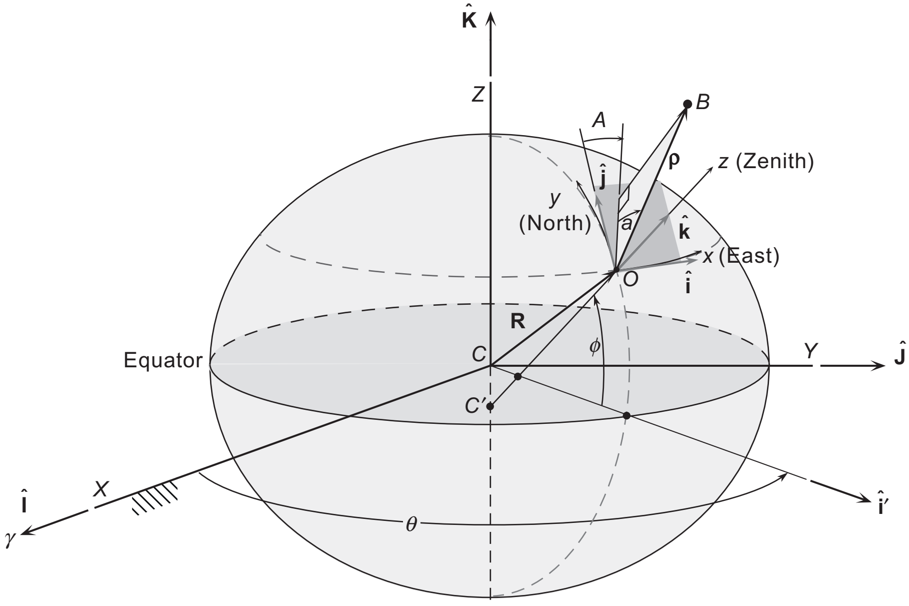


FIGURE 5.11

Topocentric horizon (*xyz*) coordinate system on the surface of the oblate earth.

The position vector $\boldsymbol{\rho}$ of a body B relative to the topocentric horizon system in Figure 5.11 is

$$\boldsymbol{\rho} = \rho \cos a \sin A \hat{\mathbf{i}} + \rho \cos a \cos A \hat{\mathbf{j}} + \rho \sin a \hat{\mathbf{k}}$$

in which ρ is the range; A is the azimuth measured positive clockwise from due north ($0 \leq A \leq 360°$); and a is the elevation angle or altitude measured from the horizontal to the line of sight of the body B ($-90° \leq a \leq 90$). The unit vector $\hat{\boldsymbol{\rho}}$ in the line of sight direction is

$$\hat{\boldsymbol{\rho}} = \cos a \sin A \hat{\mathbf{i}} + \cos a \cos A \hat{\mathbf{j}} + \sin a \hat{\mathbf{k}} \tag{5.58}$$

The transformation between geocentric equatorial and topocentric horizon systems is found by first determining the projections of the topocentric base vectors $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ onto those of the geocentric equatorial frame. From Figure 5.11 it is apparent that

$$\hat{\mathbf{k}} = \cos\phi \hat{\mathbf{i}}' + \sin\phi \hat{\mathbf{K}}$$

and

$$\hat{\mathbf{i}}' = \cos\theta \hat{\mathbf{I}} + \sin\theta \hat{\mathbf{J}}$$

where $\hat{\mathbf{i}}'$ lies in the local meridional plane and is normal to the Z axis. Hence

$$\hat{\mathbf{k}} = \cos\phi \cos\theta \hat{\mathbf{I}} + \cos\phi \sin\theta \hat{\mathbf{J}} + \sin\phi \hat{\mathbf{K}} \tag{5.59}$$

The eastward-directed unit vector $\hat{\mathbf{i}}$ may be found by taking the cross product of $\hat{\mathbf{K}}$ into the unit normal $\hat{\mathbf{k}}$,

$$\hat{\mathbf{i}} = \frac{\hat{\mathbf{K}} \times \hat{\mathbf{k}}}{\|\hat{\mathbf{K}} \times \hat{\mathbf{k}}\|} = \frac{-\cos\phi\sin\theta\hat{\mathbf{I}} + \cos\phi\cos\theta\hat{\mathbf{J}}}{\sqrt{\cos^2\phi(\sin^2\theta + \cos^2\theta)}} = -\sin\theta\hat{\mathbf{I}} + \cos\theta\hat{\mathbf{J}} \tag{5.60}$$

Finally, crossing $\hat{\mathbf{k}}$ into $\hat{\mathbf{i}}$ yields $\hat{\mathbf{j}}$,

$$\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = -\sin\phi\cos\theta\hat{\mathbf{I}} - \sin\phi\sin\theta\hat{\mathbf{J}} + \cos\phi\hat{\mathbf{K}} \tag{5.61}$$

Let us denote the matrix of the transformation from geocentric equatorial to topocentric horizon as $[\mathbf{Q}]_{Xx}$. Recall from Section 4.5 that the rows of this matrix comprise the direction cosines of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively. It follows from Equations 5.59 through 5.61 that

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \end{bmatrix} \tag{5.62a}$$

The reverse transformation, from topocentric horizon to geocentric equatorial, is represented by the transpose of this matrix,

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\theta & -\sin\phi\cos\theta & \cos\phi\cos\theta \\ \cos\theta & -\sin\phi\sin\theta & \cos\phi\sin\theta \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \tag{5.62b}$$

Observe that these matrices also represent the transformation between topocentric horizon and topocentric equatorial frames, because the unit basis vectors of the latter coincide with those of the geocentric equatorial coordinate system.

Example 5.8

The east longitude and latitude of an observer near San Francisco are $\Lambda = 238°$ and $\rho = 38°$, respectively. The local sidereal time, in degrees, is $\theta = 215.1°$ (14h 20m). At that time the planet Jupiter is observed by means of a telescope to be located at azimuth $A = 214.3°$ and angular elevation $a = 43°$. What are Jupiter's right ascension and declination in the topocentric equatorial system?

Solution

The given information allows us to formulate the matrix of the transformation from topocentric horizon to topocentric equatorial using Equation 5.62b,

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin 215.1° & -\sin 38°\cos 215.1° & \cos 38°\cos 215.1° \\ \cos 215.1° & -\sin 38°\sin 215.1° & \cos 38°\sin 215.1° \\ 0 & \cos 38° & \sin 38° \end{bmatrix} = \begin{bmatrix} 0.5750 & 0.5037 & -0.6447 \\ -0.8182 & 0.3540 & -0.4531 \\ 0 & 0.7880 & 0.6157 \end{bmatrix}$$

From Equation 5.58 we have

$$\begin{aligned}\hat{\boldsymbol{\rho}} &= \cos a \sin A\hat{\mathbf{i}} + \cos a \cos A\hat{\mathbf{j}} + \sin a\hat{\mathbf{k}} \\ &= \cos 43^\circ \sin 214.3^\circ\hat{\mathbf{i}} + \cos 43^\circ \cos 214.3^\circ\hat{\mathbf{j}} + \sin 43^\circ\hat{\mathbf{k}} \\ &= -0.4121\hat{\mathbf{i}} - 0.6042\hat{\mathbf{j}} + 0.6820\hat{\mathbf{k}}\end{aligned}$$

Therefore, in matrix notation the topocentric horizon components of $\boldsymbol{\rho}$ are

$$\{\hat{\boldsymbol{\rho}}\}_x = \begin{Bmatrix} -0.4121 \\ -0.6042 \\ 0.6820 \end{Bmatrix}$$

We obtain the topocentric equatorial components $\{\hat{\boldsymbol{\rho}}\}_X$ by the matrix operation

$$\{\hat{\boldsymbol{\rho}}\}_X = [\mathbf{Q}]_{xX}\{\hat{\boldsymbol{\rho}}\}_x = \begin{bmatrix} 0.5750 & 0.5037 & -0.6447 \\ -0.8182 & 0.3540 & -0.4531 \\ 0 & 0.7880 & 0.6157 \end{bmatrix}\begin{Bmatrix} -0.4121 \\ -0.6042 \\ 0.6820 \end{Bmatrix} = \begin{Bmatrix} -0.9810 \\ -0.1857 \\ -0.05621 \end{Bmatrix}$$

so that the topocentric equatorial line of sight unit vector is

$$\hat{\boldsymbol{\rho}} = -0.9810\hat{\mathbf{I}} - 0.1857\hat{\mathbf{J}} - 0.05621\hat{\mathbf{K}} \tag{b}$$

Using this vector in Algorithm 4.1 yields the topocentric equatorial right ascension and declination,

$$\boxed{\alpha = 190.7^\circ \qquad \delta = -3.222^\circ}$$

Jupiter is sufficiently far away that we can ignore the radius of the earth in Equation 5.53. That is, to our level of precision, there is no distinction between the topocentric equatorial and geocentric equatorial systems:

$$\mathbf{r} \approx \boldsymbol{\rho}$$

Therefore the topocentric right ascension and declination computed above are the same as the geocentric equatorial values.

Example 5.9

At a given time, the geocentric equatorial position vector of the International Space Station is

$$\mathbf{r} = -2032.4\hat{\mathbf{I}} + 4591.2\hat{\mathbf{J}} - 4544.8\hat{\mathbf{K}} \text{ (km)}$$

Determine the azimuth and elevation angle relative to a sea-level ($H = 0$) observer whose latitude is $\phi = -40^\circ$ and local sidereal time is $\theta = 110^\circ$.

Solution

Using Equation 5.56 we find the position vector of the observer to be

$$\mathbf{R} = -1673\hat{\mathbf{I}} + 4598\hat{\mathbf{J}} - 4078\hat{\mathbf{K}} \text{ (km)}$$

For the position vector of the space station relative to the observer we have (Equation 5.53)

$$\begin{aligned}\boldsymbol{\rho} &= \mathbf{r} - \mathbf{R}\\ &= (-2032\hat{\mathbf{I}} + 4591\hat{\mathbf{J}} - 4545\hat{\mathbf{K}}) - (-1673\hat{\mathbf{I}} + 4598\hat{\mathbf{J}} - 4078\hat{\mathbf{K}})\\ &= -359.0\hat{\mathbf{I}} - 6.342\hat{\mathbf{J}} - 466.9\hat{\mathbf{K}} \text{ (km)}\end{aligned}$$

or, in matrix notation,

$$\{\boldsymbol{\rho}\}_X = \begin{Bmatrix} -359.0 \\ -6.342 \\ -466.9 \end{Bmatrix} \text{ (km)}$$

To transform these geocentric equatorial components into the topocentric horizon system we need the direction cosine matrix $[\mathbf{Q}]_{Xx}$, which is given by Equation 5.62a,

$$\begin{aligned}[\mathbf{Q}]_{Xx} &= \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \end{bmatrix}\\ &= \begin{bmatrix} -\sin 110° & \cos 110° & 0 \\ -\sin(-40°)\cos 110° & -\sin(-40°)\sin 110° & \cos(-40°) \\ \cos(-40°)\cos 110° & \cos(-40°)\sin 110° & \sin(-40°) \end{bmatrix}\end{aligned}$$

Thus,

$$\{\boldsymbol{\rho}\}_x = [\mathbf{Q}]_{Xx}\{\boldsymbol{\rho}\}_X = \begin{bmatrix} -0.9397 & -0.3420 & 0 \\ -0.2198 & 0.6040 & 0.7660 \\ -0.2620 & 0.7198 & -0.6428 \end{bmatrix}\begin{Bmatrix} -359.0 \\ -6.342 \\ -466.9 \end{Bmatrix} = \begin{Bmatrix} 339.5 \\ -282.6 \\ 389.6 \end{Bmatrix} \text{ (km)}$$

or, reverting to vector notation,

$$\boldsymbol{\rho} = 339.5\hat{\mathbf{i}} - 282.6\hat{\mathbf{j}} + 389.6\hat{\mathbf{k}} \text{ (km)}$$

The magnitude of this vector is $\rho = 589.0$ km. Hence, the unit vector in the direction of $\boldsymbol{\rho}$ is

$$\hat{\boldsymbol{\rho}} = \frac{\boldsymbol{\rho}}{\rho} = 0.5765\hat{\mathbf{i}} - 0.4787\hat{\mathbf{j}} + 0.6615\hat{\mathbf{k}}$$

Comparing this with Equation 5.58, we see that $\sin a = 0.6615$, so that the angular elevation is

$$a = \sin^{-1} 0.6615 = \boxed{41.41°}$$

Furthermore

$$\sin A = \frac{0.5765}{\cos a} = 0.7687$$

$$\cos A = \frac{-0.4787}{\cos a} = -0.6397$$

It follows that

$$A = \cos^{-1}(-0.6397) = 129.8° \text{ (second quadrant) or } 230.2° \text{ (third quadrant)}$$

A must lie in the second quadrant because $\sin A > 0$. Thus, the azimuth is

$$\boxed{A = 129.8°}$$

5.8 ORBIT DETERMINATION FROM ANGLE AND RANGE MEASUREMENTS

We know that an orbit around the earth is determined once the state vectors **r** and **v** in the inertial geocentric equatorial frame are provided at a given instant of time (epoch). Satellites are of course observed from the earth's surface and not from its center. Let us briefly consider how the state vector is determined from measurements by an earth-based tracking station.

The fundamental vector triangle formed by the topocentric position vector **ρ** of a satellite relative to a tracking station, the position vector **R** of the station relative to the center of attraction *C* and the geocentric position vector **r** was illustrated in Figure 5.8 and is shown again schematically in Figure 5.12. The relationship among these three vectors is given by Equation 5.53, which can be written

$$\mathbf{r} = \mathbf{R} + \rho\hat{\boldsymbol{\rho}} \tag{5.63}$$

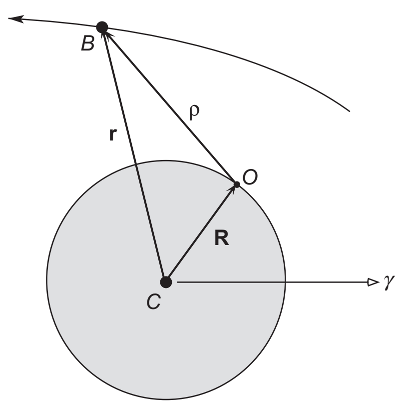


FIGURE 5.12

Earth-orbiting body *B* tracked by an observer *O*.

where the range ρ is the distance of the body B from the tracking site and $\hat{\boldsymbol{\rho}}$ is the unit vector containing the directional information about B. By differentiating Equation 5.63 with respect to time we obtain the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\hat{\boldsymbol{\rho}}} \tag{5.64}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \ddot{\rho}\hat{\boldsymbol{\rho}} + 2\dot{\rho}\dot{\hat{\boldsymbol{\rho}}} + \rho\ddot{\hat{\boldsymbol{\rho}}} \tag{5.65}$$

The vectors in these equations must all be expressed in the common basis ($\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$) of the inertial (nonrotating) geocentric equatorial frame.

Since $\mathbf{R}$ is a vector fixed in the earth, whose constant angular velocity is $\boldsymbol{\Omega} = \omega_E\hat{\mathbf{K}}$ (see Equation 2.67), it follows from Equations 1.52 and 1.53 that

$$\dot{\mathbf{R}} = \boldsymbol{\Omega} \times \mathbf{R} \tag{5.66}$$

$$\ddot{\mathbf{R}} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) \tag{5.67}$$

If L_X, L_Y and L_Z are the topocentric equatorial direction cosines, then the direction cosine vector $\hat{\boldsymbol{\rho}}$ is

$$\hat{\boldsymbol{\rho}} = L_X\hat{\mathbf{I}} + L_Y\hat{\mathbf{J}} + L_Z\hat{\mathbf{K}} \tag{5.68}$$

and its first and second derivatives are

$$\dot{\hat{\boldsymbol{\rho}}} = \dot{L}_X\hat{\mathbf{I}} + \dot{L}_Y\hat{\mathbf{J}} + \dot{L}_Z\hat{\mathbf{K}} \tag{5.69}$$

and

$$\ddot{\hat{\boldsymbol{\rho}}} = \ddot{L}_X\hat{\mathbf{I}} + \ddot{L}_Y\hat{\mathbf{J}} + \ddot{L}_Z\hat{\mathbf{K}} \tag{5.70}$$

Comparing Equations 5.57 and 5.68 reveals that the topocentric equatorial direction cosines in terms of the topocentric right ascension α and declension δ are

$$\begin{Bmatrix} L_X \\ L_Y \\ L_Z \end{Bmatrix} = \begin{Bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{Bmatrix} \tag{5.71}$$

Differentiating this equation twice yields

$$\begin{Bmatrix} \dot{L}_X \\ \dot{L}_Y \\ \dot{L}_Z \end{Bmatrix} = \begin{Bmatrix} -\dot{\alpha}\sin\alpha\cos\delta - \dot{\delta}\cos\alpha\sin\delta \\ \dot{\alpha}\cos\alpha\cos\delta - \dot{\delta}\sin\alpha\sin\delta \\ \dot{\delta}\cos\delta \end{Bmatrix} \tag{5.72}$$

and

$$\begin{Bmatrix} \ddot{L}_X \\ \ddot{L}_Y \\ \ddot{L}_Z \end{Bmatrix} = \begin{Bmatrix} -\ddot{\alpha}\sin\alpha\cos\delta - \ddot{\delta}\cos\alpha\sin\delta - (\dot{\alpha}^2 + \dot{\delta}^2)\cos\alpha\cos\delta + 2\dot{\alpha}\dot{\delta}\sin\alpha\sin\delta \\ \ddot{\alpha}\cos\alpha\cos\delta - \ddot{\delta}\sin\alpha\sin\delta - (\dot{\alpha}^2 + \dot{\delta}^2)\sin\alpha\cos\delta - 2\dot{\alpha}\dot{\delta}\cos\alpha\sin\delta \\ \ddot{\delta}\cos\delta - \dot{\delta}^2\sin\delta \end{Bmatrix} \tag{5.73}$$

Equations 5.71 through 5.73 show how the direction cosines and their rates are obtained from the right ascension and declination and their rates.

In the topocentric horizon system, the relative position vector is written

$$\hat{\boldsymbol{\rho}} = l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}} + l_z\hat{\mathbf{k}} \tag{5.74}$$

where, according to Equation 5.58, the direction cosines l_x, l_y and l_z are found in terms of the azimuth A and elevation a as

$$\begin{Bmatrix} l_x \\ l_y \\ l_z \end{Bmatrix} = \begin{Bmatrix} \sin A \cos a \\ \cos A \cos a \\ \sin a \end{Bmatrix} \tag{5.75}$$

L_X, L_Y and L_Z are obtained from l_x, l_y and l_z by the coordinate transformation

$$\begin{Bmatrix} L_X \\ L_Y \\ L_Z \end{Bmatrix} = [\mathbf{Q}]_{xX} \begin{Bmatrix} l_x \\ l_y \\ l_z \end{Bmatrix} \tag{5.76}$$

where $[\mathbf{Q}]_{xX}$ is given by Equation 5.62b. Thus,

$$\begin{Bmatrix} L_X \\ L_Y \\ L_Z \end{Bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta & -\sin\theta\sin\phi & \sin\theta\cos\phi \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{Bmatrix} \sin A \cos a \\ \cos A \cos a \\ \sin a \end{Bmatrix} \tag{5.77}$$

Substituting Equation 5.71 we see that topocentric right ascension/declination and azimuth/elevation are related by

$$\begin{Bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{Bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta & -\sin\theta\sin\phi & \sin\theta\cos\phi \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{Bmatrix} \sin A \cos a \\ \cos A \cos a \\ \sin a \end{Bmatrix}$$

Expanding the right-hand side and solving for $\sin\delta$, $\sin\alpha$ and $\cos\alpha$ we get

$$\sin\delta = \cos\phi\cos A\cos a + \sin\phi\sin a \tag{5.78a}$$

$$\sin\alpha = \frac{(\cos\phi\sin a - \cos A\cos a\sin\phi)\sin\theta + \cos\theta\sin A\cos a}{\cos\delta} \tag{5.78b}$$

$$\cos\alpha = \frac{(\cos\phi\sin a - \cos A\cos a\sin\phi)\cos\theta - \sin\theta\sin A\cos a}{\cos\delta} \tag{5.78c}$$

We can simplify Equations 5.78b and c by introducing the hour angle h,

$$h = \theta - \alpha \tag{5.79}$$

h is the angular distance between the object and the local meridian. If h is positive, the object is west of the meridian; if h is negative, the object is east of the meridian.
Using well-known trig identities we have

$$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha \tag{5.80a}$$

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha \tag{5.80b}$$

Substituting Equations 5.78b and c on the right of 5.80a and simplifying yields

$$\sin(h) = -\frac{\sin A\cos a}{\cos\delta} \tag{5.81}$$

Likewise, Equation 5.80b leads to

$$\cos(h) = \frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta} \tag{5.82}$$

We calculate h from this equation, resolving quadrant ambiguity by checking the sign of $\sin(h)$. That is,

$$h = \cos^{-1}\left(\frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right)$$

if $\sin(h)$ is positive. Otherwise, we must subtract h from 360°. Since both the elevation angle a and the declination δ lie between $-90°$ and $+90°$, neither $\cos a$ nor $\cos\delta$ can be negative. It follows from Equation 5.81 that the sign of $\sin(h)$ depends only on that of $\sin A$.

To summarize, given the topocentric azimuth A and altitude a of the target together with the sidereal time θ and latitude ϕ of the tracking station, we compute the topocentric declension δ and right ascension α as follows,

$$\delta = \sin^{-1}(\cos\phi\cos A\cos a + \sin\phi\sin a) \tag{5.83a}$$

$$h = \begin{cases} 360° - \cos^{-1}\left(\dfrac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right) & 0° < A < 180° \\ \cos^{-1}\left(\dfrac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right) & 180° \le A \le 360° \end{cases} \tag{5.83b}$$

$$\alpha = \theta - h \tag{5.83c}$$

If A and a are provided as a functions of time, then α and δ are found as functions of time by means of Equations 5.83. The rates $\dot{\alpha}$, $\ddot{\alpha}$, $\dot{\delta}$ and $\ddot{\delta}$ are determined by differentiating $\alpha(t)$ and $\delta(t)$ and substituting the results into Equations 5.68 through 5.73 to calculate the direction cosine vector $\hat{\boldsymbol{\rho}}$ and its rates $\dot{\hat{\boldsymbol{\rho}}}$ and $\ddot{\hat{\boldsymbol{\rho}}}$.

It is a relatively simple matter to find $\dot{\alpha}$ and $\dot{\delta}$ in terms of $\dot{A}$ and $\dot{a}$. Differentiating Equation 5.78a with respect to time yields

$$\dot{\delta} = \frac{1}{\cos\delta}[-\dot{A}\cos\phi\sin A\cos a + \dot{a}(\sin\phi\cos a - \cos\phi\cos A\sin a)] \tag{5.84}$$

Differentiating Equation 5.81, we get

$$\dot{h}\cos(h) = -\frac{1}{\cos^2\delta}[(\dot{A}\cos A\cos a - \dot{a}\sin A\sin a)\cos\delta + \dot{\delta}\sin A\cos a\sin\delta]$$

Substituting Equation 5.82 and simplifying leads to

$$\dot{h} = -\frac{\dot{A}\cos A\cos a - \dot{a}\sin A\sin a + \dot{\delta}\sin A\cos a\tan\delta}{\cos\phi\sin a - \sin\phi\cos A\cos a}$$

But $\dot{h} = \dot{\theta} - \dot{\alpha} = \omega_E - \dot{\alpha}$, so that, finally,

$$\dot{\alpha} = \omega_E + \frac{\dot{A}\cos A\cos a - \dot{a}\sin A\sin a + \dot{\delta}\sin A\cos a\tan\delta}{\cos\phi\sin a - \sin\phi\cos A\cos a} \tag{5.85}$$

Algorithm 5.4 Given the range ρ, azimuth A, angular elevation a together with the rates $\dot{\rho}$, $\dot{A}$ and $\dot{a}$ relative to an earth-based tracking station (for which the altitude H, latitude ϕ and local sidereal time are known), calculate the state vectors **r** and **v** in the geocentric equatorial frame. A MATLAB script of this procedure appears in Appendix D.28.

1. Using the altitude H, latitude ϕ and local sidereal time θ of the site, calculate its geocentric position vector **R** from Equation 5.56.

$$\mathbf{R} = \left[\frac{R_e}{\sqrt{1-(2f-f^2)\sin^2\phi}} + H\right]\cos\phi(\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}}) + \left[\frac{R_e(1-f)^2}{\sqrt{1-(2f-f^2)\sin^2\phi}} + H\right]\sin\phi\hat{\mathbf{K}}$$

where f is the earth's flattening factor.

2. Calculate the topocentric declination δ using Equation 5.83a.
3. Calculate the topocentric right ascension α from Equations 5.83b and 5.83c.
4. Calculate the direction cosine unit vector $\hat{\boldsymbol{\rho}}$ from Equations 5.68 and 5.71,

$$\hat{\boldsymbol{\rho}} = \cos\delta(\cos\alpha\hat{\mathbf{I}} + \sin\alpha\hat{\mathbf{J}}) + \sin\delta\hat{\mathbf{K}}$$

5. Calculate the geocentric position vector **r** from Equation 5.63,

$$\mathbf{r} = \mathbf{R} + \rho\hat{\boldsymbol{\rho}}$$

6. Calculate the inertial velocity $\dot{\mathbf{R}}$ of the site from Equation 5.66.
7. Calculate the declination rate $\dot{\delta}$ using Equation 5.84.
8. Calculate the right ascension rate $\dot{\alpha}$ by means of Equation 5.85.
9. Calculate the direction cosine rate vector $\dot{\hat{\boldsymbol{\rho}}}$ from Equations 5.69 and 5.72,

$$\dot{\hat{\boldsymbol{\rho}}} = (-\dot{\alpha}\sin\alpha\cos\delta - \dot{\delta}\cos\alpha\sin\delta)\hat{\mathbf{I}} + (\dot{\alpha}\cos\alpha\cos\delta - \dot{\delta}\sin\alpha\sin\delta)\hat{\mathbf{J}} + \dot{\delta}\cos\delta\hat{\mathbf{K}}$$

10. Calculate the geocentric velocity vector **v** from Equation 5.64.

$$\mathbf{v} = \dot{\mathbf{R}} + \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\hat{\boldsymbol{\rho}}}$$

Example 5.10

At $\theta = 300°$ local sidereal time a sea-level ($H = 0$) tracking station at latitude $\phi = 60°$ detects a space object and obtains the following data:

Slant range:	$\rho = 2551$ km
Azimuth:	$A = 90°$
Elevation:	$a = 30°$
Range rate:	$\dot{\rho} = 0$
Azimuth rate:	$\dot{A} = 1.973 \times 10^{-3}$ rad/s (0.1130 deg/s)
Elevation rate:	$\dot{a} = 9.864 \times 10^{-4}$ rad/s (0.05651 deg/s)

What are the orbital elements of the object?

Solution

We must first employ Algorithm 5.4 to obtain the state vectors **r** and **v** in order to compute the orbital elements by means of Algorithm 4.2.

Step 1.

The equatorial radius of the earth is $R_e = 6378$ km and the flattening factor is $f = 0.003353$. It follows from Equation 5.56 that the position vector of the observer is

$$\mathbf{R} = 1598\hat{\mathbf{I}} - 2769\hat{\mathbf{J}} + 5500\hat{\mathbf{K}}\,(\text{km})$$

Step 2.

$$\begin{aligned}\delta &= \sin^{-1}(\cos\phi\cos A\cos a + \sin\phi\sin a)\\ &= \sin^{-1}(\cos 60°\cos 90°\cos 30° + \sin 60°\sin 30°)\\ &= 25.66°\end{aligned}$$

Step 3.

Since the given azimuth lies between 0° and 180°, Equation 5.83b yields

$$\begin{aligned}h &= 360° - \cos^{-1}\left(\frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right)\\ &= 360° - \cos^{-1}\left(\frac{\cos 60°\sin 30° - \sin 60°\cos 90°\cos 30°}{\cos 25.66°}\right)\\ &= 360° - 73.90° = 286.1°\end{aligned}$$

Therefore, the right ascension is

$$\alpha = \theta - h = 300° - 286.1° = 13.90°$$

Step 4.

$$\begin{aligned}\hat{\boldsymbol{\rho}} &= \cos\delta(\cos\alpha\hat{\mathbf{I}} + \sin\alpha\hat{\mathbf{J}}) + \sin\delta\hat{\mathbf{K}}a\\ &= \cos 25.66°(\cos 13.90°\hat{\mathbf{I}} + \sin 13.90°\hat{\mathbf{J}}) + \sin 25.66°\hat{\mathbf{K}}\\ &= 0.8750\hat{\mathbf{I}} + 0.2165\hat{\mathbf{J}} + 0.4330\hat{\mathbf{K}}\end{aligned}$$

Step 5.

$$\begin{aligned}\mathbf{r} &= \mathbf{R} + \rho\hat{\boldsymbol{\rho}}\\ &= (1598\hat{\mathbf{I}} - 2769\hat{\mathbf{J}} + 5500\hat{\mathbf{K}}) + 2551(0.8750\hat{\mathbf{I}} + 0.2165\hat{\mathbf{J}} + 0.4330\hat{\mathbf{K}})\\ &= 3831\hat{\mathbf{I}} - 2216\hat{\mathbf{J}} + 6605\hat{\mathbf{K}}\ (\text{km})\end{aligned}$$

Step 6.

Recalling from Equation 2.67 that the angular velocity ω_E of the earth is 72.92×10^{-6} rad/s,

$$\begin{aligned}\dot{\mathbf{R}} &= \boldsymbol{\Omega} \times \mathbf{R}\\ &= (72.92 \times 10^{-6}\hat{\mathbf{K}}) \times (1598\hat{\mathbf{I}} - 2769\hat{\mathbf{J}} + 5500\hat{\mathbf{K}})\\ &= 0.2019\hat{\mathbf{I}} + 0.1166\hat{\mathbf{J}}\ (\text{km/s})\end{aligned}$$

Step 7.

$$\begin{aligned}\dot{\delta} &= \frac{1}{\cos\delta}[-\dot{A}\cos\phi\sin A\cos a + \dot{a}(\sin\phi\cos a - \cos\phi\cos A\sin a)]\\ &= \frac{1}{\cos 25.66^\circ}[-1.973 \times 10^{-3} \cdot \cos 60^\circ \sin 90^\circ \cos 30^\circ + 9.864\\ &\qquad \times 10^{-4}(\sin 60^\circ \cos 30^\circ - \cos 60^\circ \cos 90^\circ \sin 30^\circ)]\\ &= -1.2696 \times 10^{-4}\ (\text{rad/s})\end{aligned}$$

Step 8.

$$\begin{aligned}\dot{\alpha} - \omega_E &= \frac{\dot{A}\cos A\cos a - \dot{a}\sin A\sin a + \dot{\delta}\sin A\cos a\tan\delta}{\cos\phi\sin a - \sin\phi\cos A\cos a}\\ &= \frac{\begin{array}{l}\{1.973 \times 10^{-3}\cos 90^\circ \cos 30^\circ - 9.864 \times 10^{-4}\sin 90^\circ \sin 30^\circ\\ + (-1.2696 \times 10^{-4})\sin 90^\circ \cos 30^\circ \tan 25.66^\circ\}\end{array}}{\cos 60^\circ \sin 30^\circ - \sin 60^\circ \cos 90^\circ \cos 30^\circ}\\ &= -0.002184\end{aligned}$$

$$\dot{\alpha} = 72.92 \times 10^{-6} - 0.002184 = -0.002111\ (\text{rad/s})$$

Step 9.

$$\begin{aligned}\dot{\hat{\boldsymbol{\rho}}} &= (-\dot{\alpha}\sin\alpha\cos\delta - \dot{\delta}\cos\alpha\sin\delta)\hat{\mathbf{I}} + (\dot{\alpha}\cos\alpha\cos\delta - \dot{\delta}\sin\alpha\sin\delta)\hat{\mathbf{J}} + \dot{\delta}\cos\delta\hat{\mathbf{K}}\\ &= [-(-0.002111)\sin 13.90^\circ \cos 25.66^\circ - (-0.1270)\cos 13.90^\circ \sin 25.66^\circ]\hat{\mathbf{I}}\\ &\quad + [(-0.002111)\cos 13.90^\circ \cos 25.66^\circ - (-0.1270)\sin 13.90^\circ \sin 25.66^\circ]\hat{\mathbf{J}}\\ &\quad + [-0.1270\cos 25.66^\circ]\hat{\mathbf{K}}\end{aligned}$$

$$\dot{\hat{\boldsymbol{\rho}}} = (0.5104\hat{\mathbf{I}} - 1.834\hat{\mathbf{J}} - 0.1144\hat{\mathbf{K}})(10^{-3}) \qquad (\text{rad/s})$$

Step 10.

$$\begin{aligned}\mathbf{v} &= \dot{\mathbf{R}} + \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\hat{\boldsymbol{\rho}}} \\ &= (0.2019\hat{\mathbf{I}} + 0.1166\hat{\mathbf{J}}) + 0\cdot(0.8750\hat{\mathbf{I}} + 0.2165\hat{\mathbf{J}} + 0.4330\hat{\mathbf{K}}) \\ &\quad + 2551(0.5104\times 10^{-3}\hat{\mathbf{I}} - 1.834\times 10^{-3}\hat{\mathbf{J}} - 0.1144\times 10^{-3}\hat{\mathbf{K}})\end{aligned}$$

$$\mathbf{v} = 1.504\hat{\mathbf{I}} - 4.562\hat{\mathbf{J}} - 0.2920\hat{\mathbf{K}}\ \text{(km/s)}$$

Using the position and velocity vectors from Steps 5 and 10, the reader can verify that Algorithm 4.2 yields the following orbital elements of the tracked object

$$\boxed{\begin{aligned} a &= 5170\ \text{km} \\ i &= 113.4^\circ \\ \Omega &= 109.8^\circ \\ e &= 0.6195 \\ \omega &= 309.8^\circ \\ \theta &= 165.3^\circ \end{aligned}}$$

This is a highly elliptical orbit with a semimajor axis less than the earth's radius, so the object will impact the earth (at a true anomaly of 216°).

For objects orbiting the sun (planets, asteroids, comets and man-made interplanetary probes), the fundamental vector triangle is as illustrated in Figure 5.13. The tracking station is on the earth but, of course, the sun rather than the earth is the center of attraction. The procedure for finding the heliocentric state vector **r** and **v** is similar to that outlined above. Because of the vast distances involved, the observer can usually be imagined to reside at the center of the earth. Dealing with **R** is different in this case. The daily position of the sun relative to the earth (−**R** in Figure 5.13) may be found in ephemerides, such as Astronomical Almanac (U.S. Naval Observatory, 2008). A discussion of interplanetary trajectories appears in Chapter 8 of this text.

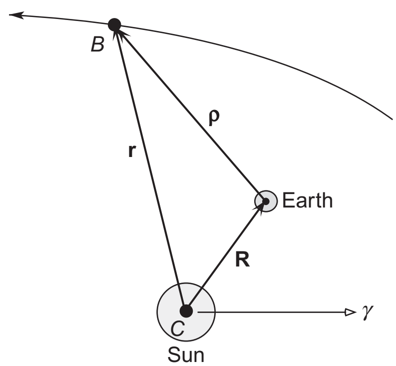


FIGURE 5.13

An object B orbiting the sun and tracked from earth.

5.9 ANGLES ONLY PRELIMINARY ORBIT DETERMINATION

To determine an orbit requires specifying six independent quantities. These can be the six classical orbital elements or the total of six components of the state vector, **r** and **v**, at a given instant. To determine an orbit solely from observations therefore requires six independent measurements. In the previous section we assumed the tracking station was able to measure simultaneously the six quantities range and range rate; azimuth and azimuth rate; plus elevation and elevation rate. This data leads directly to the state vector and, hence, to a complete determination of the orbit. In the absence of range and range rate measuring capability, as with a telescope, we must rely on measurements of just the two angles, azimuth and elevation, to determine the orbit. A minimum of three observations of azimuth and elevation is therefore required to accumulate the six quantities we need to predict the orbit. We shall henceforth assume that the angular measurements are converted to topocentric right ascension α and declination δ, as described in the previous section.

We shall consider the classical method of angles-only orbit determination due to Carl Friedrich Gauss (1777–1855), a German mathematician who many consider was one of the greatest mathematicians ever. This method requires gathering angular information over closely spaced intervals of time and yields a preliminary orbit determination based on those initial observations.

5.10 GAUSS METHOD OF PRELIMINARY ORBIT DETERMINATION

Suppose we have three observations of an orbiting body at times t_1, t_2 and t_3, as shown in Figure 5.14. At each time the geocentric position vector **r** is related to the observer's position vector **R**, the slant range ρ and the topocentric direction cosine vector $\hat{\boldsymbol{\rho}}$ by Equation 5.63,

$$\mathbf{r}_1 = \mathbf{R}_1 + \rho_1\hat{\boldsymbol{\rho}}_1 \tag{5.86a}$$

$$\mathbf{r}_2 = \mathbf{R}_2 + \rho_2\hat{\boldsymbol{\rho}}_2 \tag{5.86b}$$

$$\mathbf{r}_3 = \mathbf{R}_3 + \rho_3\hat{\boldsymbol{\rho}}_3 \tag{5.86c}$$

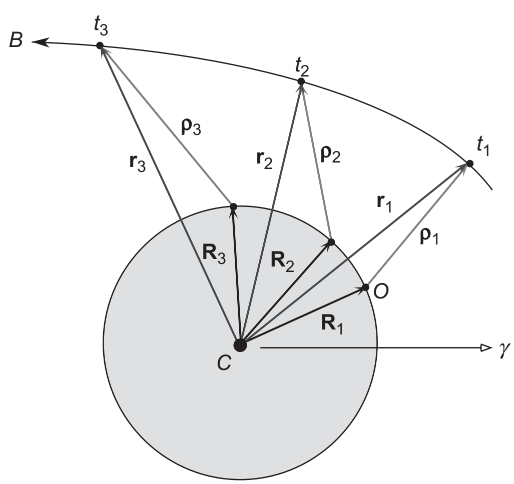


FIGURE 5.14

Center of attraction *C*, observer *O* and tracked body *B*.

The positions $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ of the observer O are known from the location of the tracking station and the time of the observations. $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_3$ are obtained by measuring the right ascension α and declination δ of the body at each of the three times (recall Equation 5.57). Equations 5.86 are three vector equations, and therefore nine scalar equations, in twelve unknowns: the three components of each of the three vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, plus the three slant ranges ρ_1, ρ_2 and ρ_3.

An additional three equations are obtained by recalling from Chapter 2 that the conservation of angular momentum requires the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ to lie in the same plane. As in our discussion of the Gibbs method in Section 5.2, that means $\mathbf{r}_2$ is a linear combination $\mathbf{r}_1$ and $\mathbf{r}_3$.

$$\mathbf{r}_2 = c_1\mathbf{r}_1 + c_3\mathbf{r}_3 \tag{5.87}$$

Adding this equation to those in (5.86) introduces two new unknowns, c_1 and c_3. At this point we therefore have 12 scalar equations in 14 unknowns.

Another consequence of the two-body equation of motion (Equation 2.22) is that the state vectors $\mathbf{r}$ and $\mathbf{v}$ of the orbiting body can be expressed in terms of the state vectors at any given time by means of the Lagrange coefficients, Equations 2.135 and 2.136. For the case at hand, that means we can express the position vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ in terms of the position $\mathbf{r}_2$ and velocity $\mathbf{v}_2$ at the intermediate time t_2 as follows,

$$\mathbf{r}_1 = f_1\mathbf{r}_2 + g_1\mathbf{v}_2 \tag{5.88a}$$

$$\mathbf{r}_3 = f_3\mathbf{r}_2 + g_3\mathbf{v}_2 \tag{5.88b}$$

where f_1 and g_1 are the Lagrange coefficients evaluated at t_1 while f_3 and g_3 are those same functions evaluated at time t_3. If the time intervals between the three observations are sufficiently small then Equations 2.172 reveal that f and g depend approximately only on the distance from the center of attraction at the initial time. For the case at hand that means the coefficients in Equations 5.88 depend only on r_2. Hence, Equations 5.88 add six scalar equations to our previous list of 12 while adding to the list of 14 unknowns only four: the three components of $\mathbf{v}_2$ and the radius r_2. We have arrived at 18 equations in 18 unknowns, so the problem is well posed and we can proceed with the solution. The ultimate objective is to determine the state vector $\mathbf{r}_2$, $\mathbf{v}_2$ at the intermediate time t_2.

Let us start out by solving for c_1 and c_3 in Equation 5.87. First, take the cross product of each term in that equation with $\mathbf{r}_3$,

$$\mathbf{r}_2 \times \mathbf{r}_3 = c_1(\mathbf{r}_1 \times \mathbf{r}_3) + c_3(\mathbf{r}_3 \times \mathbf{r}_3)$$

Since $\mathbf{r}_3 \times \mathbf{r}_3 = 0$, this reduces to

$$\mathbf{r}_2 \times \mathbf{r}_3 = c_1(\mathbf{r}_1 \times \mathbf{r}_3)$$

Taking the dot product of this result with $\mathbf{r}_1 \times \mathbf{r}_3$ and solving for c_1 yields

$$c_1 = \frac{(\mathbf{r}_2 \times \mathbf{r}_3)\cdot(\mathbf{r}_1 \times \mathbf{r}_3)}{\|\mathbf{r}_1 \times \mathbf{r}_3\|^2} \tag{5.89}$$

In a similar fashion, by forming the dot product of Equation 5.87 with $\mathbf{r}_1$, we are led to

$$c_3 = \frac{(\mathbf{r}_2 \times \mathbf{r}_1)\cdot(\mathbf{r}_3 \times \mathbf{r}_1)}{\|\mathbf{r}_1 \times \mathbf{r}_3\|^2} \tag{5.90}$$

Let us next use Equations 5.88 to eliminate $\mathbf{r}_1$ and $\mathbf{r}_3$ from the expressions for c_1 and c_3. First of all,

$$\mathbf{r}_1 \times \mathbf{r}_3 = (f_1\mathbf{r}_2 + g_1\mathbf{v}_2) \times (f_3\mathbf{r}_2 + g_3\mathbf{v}_2) = f_1 g_3(\mathbf{r}_2 \times \mathbf{v}_2) + f_3 g_1(\mathbf{v}_2 \times \mathbf{r}_2)$$

But $\mathbf{r}_2 \times \mathbf{v}_2 = \mathbf{h}$, where $\mathbf{h}$ is the constant angular momentum of the orbit (Equation 2.28). It follows that

$$\mathbf{r}_1 \times \mathbf{r}_3 = (f_1 g_3 - f_3 g_1)\mathbf{h} \tag{5.91}$$

and, of course,

$$\mathbf{r}_3 \times \mathbf{r}_1 = -(f_1 g_3 - f_3 g_1)\mathbf{h} \tag{5.92}$$

Therefore,

$$\left\|\mathbf{r}_1 \times \mathbf{r}_3\right\|^2 = (f_1 g_3 - f_3 g_1)^2 h^2 \tag{5.93}$$

Similarly

$$\mathbf{r}_2 \times \mathbf{r}_3 = \mathbf{r}_2 \times (f_3\mathbf{r}_2 + g_3\mathbf{v}_2) = g_3\mathbf{h} \tag{5.94}$$

and

$$\mathbf{r}_2 \times \mathbf{r}_1 = \mathbf{r}_2 \times (f_1\mathbf{r}_2 + g_1\mathbf{v}_2) = g_1\mathbf{h} \tag{5.95}$$

Substituting Equations 5.91, 5.93 and 5.94 into Equation 5.89 yields

$$c_1 = \frac{g_3\mathbf{h} \cdot (f_1 g_3 - f_3 g_1)\mathbf{h}}{(f_1 g_3 - f_3 g_1)^2 h^2} = \frac{g_3(f_1 g_3 - f_3 g_1)h^2}{(f_1 g_3 - f_3 g_1)^2 h^2}$$

or

$$c_1 = \frac{g_3}{f_1 g_3 - f_3 g_1} \tag{5.96}$$

Likewise, substituting Equations 5.92, 5.93 and 5.95 into Equation 5.90 leads to

$$c_3 = -\frac{g_1}{f_1 g_3 - f_3 g_1} \tag{5.97}$$

The coefficients in Equation 5.87 are now expressed solely in terms of the Lagrange functions, and so far no approximations have been made. However, we will have to make some approximations in order to proceed.

We must approximate c_1 and c_2 under the assumption that the times between observations of the orbiting body are small. To that end, let us introduce the notation

$$\begin{aligned} \tau_1 &= t_1 - t_2 \\ \tau_3 &= t_3 - t_2 \end{aligned} \tag{5.98}$$

τ_1 and τ_3 are the time intervals between the successive measurements of $\hat{\boldsymbol{\rho}}_1$, $\hat{\boldsymbol{\rho}}_2$ and $\hat{\boldsymbol{\rho}}_3$. If the time intervals τ_1 and τ_3 are small enough, we can retain just the first two terms of the series expressions for the Lagrange coefficients f and g in Equations 2.172, thereby obtaining the approximations

$$f_1 \approx 1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_1^2 \tag{5.99a}$$

$$f_3 \approx 1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_3^2 \tag{5.99b}$$

and

$$g_1 \approx \tau_1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_1^3 \tag{5.100a}$$

$$g_3 \approx \tau_3 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^3 \tag{5.100b}$$

We want to exclude all terms in f and g beyond the first two so that only the unknown r_2 appears in Equations 5.99 and 5.100. One can see from Equations 2.172 that the higher order terms include the unknown $\mathbf{v}_2$ as well.

Using Equations 5.99 and 5.100 we can calculate the denominator in Equations 5.96 and 5.97,

$$f_1 g_3 - f_3 g_1 = \left(1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_1^2\right)\left(\tau_3 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^3\right) - \left(1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_3^2\right)\left(\tau_1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_1^3\right)$$

Expanding the right side and collecting terms yields

$$f_1 g_3 - f_3 g_1 = (\tau_3 - \tau_1) - \frac{1}{6}\frac{\mu}{r_2^3}(\tau_3 - \tau_1)^3 + \frac{1}{12}\frac{\mu^2}{r_2^6}(\tau_1^2\tau_3^3 - \tau_1^3\tau_3^2)$$

Retaining terms of at most third order in the time intervals τ_1 and τ_3, and setting

$$\tau = \tau_3 - \tau_1 \tag{5.101}$$

reduces this expression to

$$f_1 g_3 - f_3 g_1 \approx \tau - \frac{1}{6}\frac{\mu}{r_2^3}\tau^3 \tag{5.102}$$

From Equation 5.98 observe that τ is just the time interval between the first and last observations. Substituting Equations 5.100b and 5.102 into Equation 5.96, we get

$$c_1 \approx \frac{\tau_3 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^3}{\tau - \frac{1}{6}\frac{\mu}{r_2^3}\tau^3} = \frac{\tau_3}{\tau}\left(1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^2\right)\cdot\left(\frac{1}{1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau^2}\right)^{-1} \tag{5.103}$$

We can use the binomial theorem to simplify (linearize) the last term on the right. Setting $a = 1$, $b = -\frac{1}{6}\frac{\mu}{r_2^3}\tau^2$ and $n = -1$ in Equation 5.44, and neglecting terms of higher order than 2 in τ, yields

$$\left(1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau^2\right)^{-1} \approx 1 + \frac{1}{6}\frac{\mu}{r_2^3}\tau^2$$

Hence Equation 5.103 becomes

$$c_1 \approx \frac{\tau_3}{\tau}\left[1 + \frac{1}{6}\frac{\mu}{r_2^3}(\tau^2 - \tau_3^2)\right] \tag{5.104}$$

where only second order terms in the time have been retained. In precisely the same way it can be shown that

$$c_3 \approx -\frac{\tau_1}{\tau}\left[1 + \frac{1}{6}\frac{\mu}{r_2^3}(\tau^2 - \tau_1^2)\right] \tag{5.105}$$

Finally, we have managed to obtain approximate formulas for the coefficients in Equation 5.87 in terms of just the time intervals between observations and the as yet unknown distance r_2 from the center of attraction at the central time t_2.

The next stage of the solution is to seek formulas for the slant ranges ρ_1, ρ_2 and ρ_3 in terms of c_1 and c_2. To that end, substitute Equations 5.86 into Equation 5.87 to get

$$\mathbf{R}_2 + \rho_2\hat{\boldsymbol{\rho}}_2 = c_1(\mathbf{R}_1 + \rho_1\hat{\boldsymbol{\rho}}_1) + c_3(\mathbf{R}_3 + \rho_3\hat{\boldsymbol{\rho}}_3)$$

which we rearrange into the form

$$c_1\rho_1\hat{\boldsymbol{\rho}}_1 - \rho_2\hat{\boldsymbol{\rho}}_2 + c_3\rho_3\hat{\boldsymbol{\rho}}_3 = -c_1\mathbf{R}_1 + \mathbf{R}_2 - c_3\mathbf{R}_3 \tag{5.106}$$

Let us isolate the slant ranges ρ_1, ρ_2 and ρ_3 in turn by taking the dot product of this equation with appropriate vectors. To isolate ρ_1 take the dot product of each term in this equation with $\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3$, which gives

$$\begin{aligned} &c_1\rho_1\hat{\boldsymbol{\rho}}_1 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) - \rho_2\hat{\boldsymbol{\rho}}_2 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) + c_3\rho_3\hat{\boldsymbol{\rho}}_3 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \\ &= -c_1\mathbf{R}_1 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) + \mathbf{R}_2 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) - c_3\mathbf{R}_3 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \end{aligned}$$

Since $\hat{\boldsymbol{\rho}}_2 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) = \hat{\boldsymbol{\rho}}_3 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) = 0$, this reduces to

$$c_1\rho_1\hat{\boldsymbol{\rho}}_1 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) = (-c_1\mathbf{R}_1 + \mathbf{R}_2 - c_3\mathbf{R}_3) \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \tag{5.107}$$

Let D_0 represent the scalar triple product of $\hat{\boldsymbol{\rho}}_1$, $\hat{\boldsymbol{\rho}}_2$ and $\hat{\boldsymbol{\rho}}_3$,

$$D_0 = \hat{\boldsymbol{\rho}}_1 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \tag{5.108}$$

We will assume that D_0 is not zero, which means that $\hat{\boldsymbol{\rho}}_1$, $\hat{\boldsymbol{\rho}}_2$ and $\hat{\boldsymbol{\rho}}_3$ do not lie in the same plane. Then we can solve Equation 5.107 for ρ_1 to get

$$\rho_1 = \frac{1}{D_0}\left(-D_{11} + \frac{1}{c_1}D_{21} - \frac{c_3}{c_1}D_{31}\right) \tag{5.109a}$$

where the D's stand for the scalar triple products

$$D_{11} = \mathbf{R}_1 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \qquad D_{21} = \mathbf{R}_2 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \qquad D_{31} = \mathbf{R}_3 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \tag{5.109b}$$

In a similar fashion, by taking the dot product of Equation 5.106 with $\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3$ and then $\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2$ we obtain ρ_2 and ρ_3,

$$\rho_2 = \frac{1}{D_0}(-c_1 D_{12} + D_{22} - c_3 D_{32}) \tag{5.110a}$$

where

$$D_{12} = \mathbf{R}_1 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3) \qquad D_{22} = \mathbf{R}_2 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3) \qquad D_{32} = \mathbf{R}_3 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3) \tag{5.110b}$$

and

$$\rho_3 = \frac{1}{D_0}\left(-\frac{c_1}{c_3} D_{13} + \frac{1}{c_3} D_{23} - D_{33}\right) \tag{5.111a}$$

where

$$D_{13} = \mathbf{R}_1 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2) \qquad D_{23} = \mathbf{R}_2 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2) \qquad D_{33} = \mathbf{R}_3 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2) \tag{5.111b}$$

To obtain these results we used the fact that $\hat{\boldsymbol{\rho}}_2 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3) = -D_0$ and $\hat{\boldsymbol{\rho}}_3 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2) = D_0$ (Equation 2.42).

Substituting Equations 5.104 and 5.105 into Equation 5.110a yields the slant range ρ_2,

$$\rho_2 = A + \frac{\mu B}{r_2^{\,3}} \tag{5.112a}$$

where

$$A = \frac{1}{D_0}\left(-D_{12}\frac{\tau_3}{\tau} + D_{22} + D_{32}\frac{\tau_1}{\tau}\right) \tag{5.112b}$$

$$B = \frac{1}{6D_0}\left[D_{12}(\tau_3^{\,2} - \tau^2)\frac{\tau_3}{\tau} + D_{32}(\tau^2 - \tau_1^{\,2})\frac{\tau_1}{\tau}\right] \tag{5.112c}$$

On the other hand, making the same substitutions into Equations 5.109a and 5.111a leads to the following expressions for the slant ranges ρ_1 and ρ_3,

$$\rho_1 = \frac{1}{D_0}\left[\frac{6\left(D_{31}\dfrac{\tau_1}{\tau_3} + D_{21}\dfrac{\tau}{\tau_3}\right)r_2^{\,3} + \mu D_{31}(\tau^2 - \tau_1^{\,2})\dfrac{\tau_1}{\tau_3}}{6r_2^{\,3} + \mu(\tau^2 - \tau_3^{\,2})} - D_{11}\right] \tag{5.113}$$

$$\rho_3 = \frac{1}{D_0}\left[\frac{6\left(D_{13}\dfrac{\tau_3}{\tau_1} - D_{23}\dfrac{\tau}{\tau_1}\right)r_2^{\,3} + \mu D_{13}(\tau^2 - \tau_3^{\,2})\dfrac{\tau_3}{\tau_1}}{6r_2^{\,3} + \mu(\tau^2 - \tau_1^{\,2})} - D_{33}\right] \tag{5.114}$$

Equation 5.112a is a relation between the slant range ρ_2 and the geocentric radius r_2. Another expression relating these two variables is obtained from Equation 5.86b,

$$\mathbf{r}_2 \cdot \mathbf{r}_2 = (\mathbf{R}_2 + \rho_2\hat{\boldsymbol{\rho}}_2)\cdot(\mathbf{R}_2 + \rho_2\hat{\boldsymbol{\rho}}_2)$$

or

$$r_2^{\,2} = \rho_2^{\,2} + 2E\rho_2 + R_2^{\,2} \tag{5.115a}$$

where

$$E = \mathbf{R}_2 \cdot \hat{\boldsymbol{\rho}}_2 \tag{5.115b}$$

Substituting Equation 5.112a into 5.115a gives

$$r_2^{\,2} = \left(A + \frac{\mu B}{r_2^{\,3}}\right)^2 + 2C\left(A + \frac{\mu B}{r_2^{\,3}}\right) + R_2^{\,2}$$

Expanding and rearranging terms leads to an eighth degree polynomial,

$$x^8 + ax^6 + bx^3 + c = 0 \tag{5.116}$$

where $x = r_2$ and the coefficients are

$$a = -(A^2 + 2AE + R_2^{\,2}) \qquad b = -2\mu B(A + E) \qquad c = -\mu^2 B^2 \tag{5.117}$$

We solve Equation 5.116 for r_2 and substitute the result into Equations 5.112 through 5.114 to obtain the slant ranges ρ_1, ρ_2 and ρ_3. Then Equations 5.86 yield the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. Recall that finding $\mathbf{r}_2$ was one of our objectives.

To attain the other objective, the velocity $\mathbf{v}_2$, we first solve Equation 5.88a for $\mathbf{r}_2$

$$\mathbf{r}_2 = \frac{1}{f_1}\mathbf{r}_1 - \frac{g_1}{f_1}\mathbf{v}_2$$

Substitute this result into Equation 5.88b to get

$$\mathbf{r}_3 = \frac{f_3}{f_1}\mathbf{r}_1 + \left(\frac{f_1 g_3 - f_3 g_1}{f_1}\right)\mathbf{v}_2$$

Solving this for $\mathbf{v}_2$ yields

$$\mathbf{v}_2 = \frac{1}{f_1 g_3 - f_3 g_1}(-f_3\mathbf{r}_1 + f_1\mathbf{r}_3) \tag{5.118}$$

in which the approximate Lagrange functions appearing in Equations 5.99 and 5.100 are used.

The approximate values we have found for $\mathbf{r}_2$ and $\mathbf{v}_2$ are used as the starting point for iteratively improving the accuracy of the computed $\mathbf{r}_2$ and $\mathbf{v}_2$ until convergence is achieved. The entire step by step procedure is summarized in Algorithms 5.5 and 5.6 presented below. See also Appendix D.29.

Algorithm 5.5 Gauss method of preliminary orbit determination. Given the direction cosine vectors $\hat{\boldsymbol{\rho}}_1$, $\hat{\boldsymbol{\rho}}_2$ and $\hat{\boldsymbol{\rho}}_3$ and the observer's position vectors $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ at the times t_1, t_2 and t_3, compute the orbital elements.

1. Calculate the time intervals τ_1, τ_3 and τ using Equations 5.98 and 5.101.
2. Calculate the cross products $\mathbf{p}_1 = \hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3$, $\mathbf{p}_2 = \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3$ and $\mathbf{p}_3 = \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2$.
3. Calculate $D_0 = \hat{\boldsymbol{\rho}}_1 \cdot \mathbf{p}_1$ (Equation 5.108).
4. From Equations 5.109b, 5.110b and 5.111b compute the six scalar quantities

$$\begin{aligned} D_{11} &= \mathbf{R}_1 \cdot \mathbf{p}_1 & D_{12} &= \mathbf{R}_1 \cdot \mathbf{p}_2 & D_{13} &= \mathbf{R}_1 \cdot \mathbf{p}_3 \\ D_{21} &= \mathbf{R}_2 \cdot \mathbf{p}_1 & D_{22} &= \mathbf{R}_2 \cdot \mathbf{p}_2 & D_{23} &= \mathbf{R}_2 \cdot \mathbf{p}_3 \\ D_{31} &= \mathbf{R}_3 \cdot \mathbf{p}_1 & D_{32} &= \mathbf{R}_3 \cdot \mathbf{p}_2 & D_{33} &= \mathbf{R}_3 \cdot \mathbf{p}_3 \end{aligned}$$

5. Calculate A and B using Equations 5.112b and 5.112c.
6. Calculate E, using Equation 5.115b, and $R_2^{\,2} = \mathbf{R}_2 \cdot \mathbf{R}_2$.
7. Calculate a, b and c from Equation 5.117.
8. Find the roots of Equation 5.116 and select the most reasonable one as r_2. Newton's method can be used, in which case Equation 3.16 becomes

$$x_{i+1} = x_i - \frac{x_i^{\,8} + a x_i^{\,6} + b x_i^{\,3} + c}{8x_i^{\,7} + 6a x_i^{\,5} + 3b x_i^{\,2}} \tag{5.119}$$

   One must first print or graph the function $F = x^8 + ax^6 + bx^3 + c$ for $x > 0$ and choose as an initial estimate a value of x near the point where F changes sign. If there is more than one physically reasonable root, then each one must be used and the resulting orbit checked against knowledge that may already be available about the general nature of the orbit. Alternatively, the analysis can be repeated using additional sets of observations.
9. Calculate ρ_1, ρ_2 and ρ_3 using Equations 5.113, 5.112a and 5.114.
10. Use Equations 5.86 to calculate $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$.
11. Calculate the Lagrange coefficients f_1, g_1, f_3 and g_3 from Equations 5.99 and 5.100.
12. Calculate $\mathbf{v}_2$ using Equation 5.118.
13. (a) Use $\mathbf{r}_2$ and $\mathbf{v}_2$ from Steps 10 and 12 to obtain the orbital elements from Algorithm 4.2.
    (b) Alternatively, proceed to Algorithm 5.6 to improve the preliminary estimate of the orbit.

Algorithm 5.6 Iterative improvement of the orbit determined by Algorithm 5.5.
Use the values of $\mathbf{r}_2$ and $\mathbf{v}_2$ obtained from Algorithm 5.5 to compute the "exact" values of the f and g functions from their universal formulation, as follows:

1. Calculate the magnitude of $\mathbf{r}_2$ ($r_2 = \sqrt{\mathbf{r}_2 \cdot \mathbf{r}_2}$) and $\mathbf{v}_2$ ($v_2 = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2}$).
2. Calculate α, the reciprocal of the semimajor axis: $\alpha = 2/r_2 - v_2^2/\mu$.
3. Calculate the radial component of $\mathbf{v}_2$, $v_{r_2} = \mathbf{v}_2 \cdot \mathbf{r}_2 / r_2$.

4. Use Algorithm 3.3 to solve the universal Kepler's equation (Equation 3.49) for the universal variables χ_1 and χ_3 at times t_1 and t_3, respectively:

$$\sqrt{\mu}\tau_1 = \frac{r_2 v_{r2}}{\sqrt{\mu}}\chi_1{}^2 C(\alpha\chi_1{}^2) + (1 - \alpha r_2)\chi_1{}^3 S(\alpha\chi_1{}^2) + r_2\chi_1$$

$$\sqrt{\mu}\tau_1 = \frac{r_2 v_{r2}}{\sqrt{\mu}}\chi_3{}^2 C(\alpha\chi_3{}^2) + (1 - \alpha r_2)\chi_3{}^3 S(\alpha\chi_3{}^2) + r_2\chi_3$$

5. Use χ_1 and χ_3 to calculate f_1, g_1, f_3 and g_3 from Equations 3.69:

$$f_1 = 1 - \frac{\chi_1{}^2}{r_2}C(\alpha\chi_1{}^2) \qquad g_1 = \tau_1 - \frac{1}{\sqrt{\mu}}\chi_1{}^3 S(\alpha\chi_1{}^2)$$

$$f_3 = 1 - \frac{\chi_3{}^2}{r_2}C(\alpha\chi_3{}^2) \qquad g_3 = \tau_3 - \frac{1}{\sqrt{\mu}}\chi_3{}^3 S(\alpha\chi_3{}^2)$$

6. Use these values of f_1, g_1, f_3 and g_3 to calculate c_1 and c_3 from Equations 5.96 and 5.97.
7. Use c_1 and c_3 to calculate updated values of ρ_1, ρ_2 and ρ_3 from Equations 5.109 through 5.111.
8. Calculate updated $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ from Equations 5.86.
9. Calculate updated $\mathbf{v}_2$ using Equation 5.118 and the f and g values computed in Step 5.
10. Go back to Step 1 and repeat until, to the desired degree of precision, there is no further change in ρ_1, ρ_2 and ρ_3.
11. Use $\mathbf{r}_2$ and $\mathbf{v}_2$ to compute the orbital elements by means of Algorithm 4.2.

Example 5.11

A tracking station is located at $\phi = 40°$ north latitude at an altitude of $H = 1$ km. Three observations of an earth satellite yield the values for the topocentric right ascension and declination listed in the following table, which also shows the local sidereal time θ of the observation site.

Use the Gauss Algorithm 5.5 to estimate the state vector at the second observation time. Recall that $\mu = 398{,}600\,\text{km}^3/\text{s}^2$.

Table 5.1 Data for Example 5.11

Observation	Time (seconds)	Right ascension, α (degrees)	Declension, δ (degrees)	Local sidereal time, θ (degrees)
1	0	43.537	−8.7833	44.506
2	118.10	54.420	−12.074	45.000
3	237.58	64.318	−15.105	45.499

Solution

Recalling that the equatorial radius of the earth is $R_e = 6378$ km and the flattening factor is $f = 0.003353$, we substitute $\phi = 40°$, $H = 1$ km and the given values of θ into Equation 5.56 to obtain the inertial position vector of the tracking station at each of the three observation times.

$$\begin{aligned}\mathbf{R}_1 &= 3489.8\hat{\mathbf{I}} + 3430.2\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}} \text{ (km)}\\ \mathbf{R}_2 &= 3460.1\hat{\mathbf{I}} + 3460.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}} \text{ (km)}\\ \mathbf{R}_3 &= 3429.9\hat{\mathbf{I}} + 3490.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}} \text{ (km)}\end{aligned}$$

Using Equation 5.57 we compute the direction cosine vectors at each of the three observation times from the right ascension and declination data

$$\begin{aligned}\hat{\boldsymbol{\rho}}_1 &= \cos(-8.7833°)\cos 43.537°\hat{\mathbf{I}} + \cos(-8.7833°)\sin 43.537°\hat{\mathbf{J}} + \sin(-8.7833°)\hat{\mathbf{K}}\\ &= 0.71643\hat{\mathbf{I}} + 0.68074\hat{\mathbf{J}} - 0.15270\hat{\mathbf{K}}\\ \hat{\boldsymbol{\rho}}_2 &= \cos(-12.074°)\cos 54.420°\hat{\mathbf{I}} + \cos(-12.074°)\sin 54.420°\hat{\mathbf{J}} + \sin(-12.074°)\hat{\mathbf{K}}\\ &= 0.56897\hat{\mathbf{I}} + 0.79531\hat{\mathbf{J}} - 0.20917\hat{\mathbf{K}}\\ \hat{\boldsymbol{\rho}}_3 &= \cos(-15.105°)\cos 64.318°\hat{\mathbf{I}} + \cos(-15.105°)\sin 64.318°\hat{\mathbf{J}} + \sin(-15.105°)\hat{\mathbf{K}}\\ &= 0.41841\hat{\mathbf{I}} + 0.87007\hat{\mathbf{J}} - 0.260{,}59\hat{\mathbf{K}}\end{aligned} \tag{b}$$

We can now proceed with Algorithm 5.5.

Step 1.

$$\begin{aligned}\tau_1 &= 0 - 118.10 = -118.10 \text{ s}\\ \tau_3 &= 237.58 - 118.10 = 119.47 \text{ s}\\ \tau &= 119.47 - (-118.1) = 237.58 \text{ s}\end{aligned}$$

Step 2.

$$\begin{aligned}\mathbf{p}_1 &= \hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3 = -0.025258\hat{\mathbf{I}} + 0.060753\hat{\mathbf{J}} + 0.16229\\ \mathbf{p}_2 &= \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 = -0.044538\hat{\mathbf{I}} + 0.12281\hat{\mathbf{J}} + 0.33853\hat{\mathbf{K}}\\ \mathbf{p}_3 &= \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2 = -0.020950\hat{\mathbf{I}} + 0.062977\hat{\mathbf{J}} + 0.18246\end{aligned}$$

Step 3.

$$D_0 = \hat{\boldsymbol{\rho}}_1 \cdot \mathbf{p}_1 = -0.0015198$$

Step 4.

$$\begin{aligned}D_{11} &= \mathbf{R}_1 \cdot \mathbf{p}_1 = 782.15 \text{ km} & D_{12} &= \mathbf{R}_1 \cdot \mathbf{p}_2 = 1646.5 \text{ km} & D_{13} &= \mathbf{R}_1 \cdot \mathbf{p}_3 = 887.10 \text{ km}\\ D_{21} &= \mathbf{R}_2 \cdot \mathbf{p}_1 = 784.72 \text{ km} & D_{22} &= \mathbf{R}_2 \cdot \mathbf{p}_2 = 1651.5 \text{ km} & D_{23} &= \mathbf{R}_2 \cdot \mathbf{p}_3 = 889.60 \text{ km}\\ D_{31} &= \mathbf{R}_3 \cdot \mathbf{p}_1 = 787.31 \text{ km} & D_{32} &= \mathbf{R}_3 \cdot \mathbf{p}_2 = 1656.6 \text{ km} & D_{33} &= \mathbf{R}_3 \cdot \mathbf{p}_3 = 892.13 \text{ km}\end{aligned}$$

Step 5:

$$A = \frac{1}{-0.0015198}\left[-1646.5\frac{119.47}{237.58} + 1651.5 + 1656.6\frac{(-118.10)}{237.58}\right] = -6.6858 \text{ km}$$

$$B = \frac{1}{6(-0.0015198)}\left\{1646.5(119.47^2 - 237.58^2)\frac{119.47}{237.58} + 1656.6\left[237.58^2 - (-118.10)^2\right]\frac{(-118.10)}{237.58}\right\}$$
$$= 7.6667 \times 10^9 \text{ km-s}^2$$

Step 6.

$$E = \mathbf{R}_2\hat{\boldsymbol{\rho}}_2 = 3867.5 \text{ km}$$
$$R_2{}^2 = \mathbf{R}_2 \cdot \mathbf{R}_2 = 4.058 \times 10^7 \text{ km}^2$$

Step 7.

$$a = -\left[(-6.6858)^2 + 2(-6.6858)(3875.8) + 4.058 \times 10^7\right] = -4.0528 \times 10^7 \text{ km}^2$$
$$b = -2(389{,}600)(7.6667 \times 10^9)(-6.6858 + 3875.8) = -2.3597 \times 10^{19} \text{ km}^5$$
$$c = -(398{,}600)^2(7.6667 \times 10^9)^2 = -9.3387 \times 10^{30} \text{ km}^8$$

Step 8.

$$F(x) = x^8 - 4.0528 \times 10^7 x^6 - 2.3597 \times 10^{19} x^3 - 9.3387 \times 10^{30} = 0$$

The graph of $F(x)$ in Figure 5.15 shows that it changes sign near $x = 9000$ km. Let us use that as the starting value in Newton's method for finding the roots of $F(x)$. For the case at hand, Equation 5.119 is

$$x_{i+1} = x_i - \frac{x_i{}^8 - 4.0528 \times 10^7 x_i{}^6 - 2.3622 \times 10^{19} x_i{}^3 - 9.3186 \times 10^{30}}{8x_i{}^7 - 2.4317 \times 10^8 x_i{}^5 - 7.0866 \times 10^{19} x_i{}^2}$$

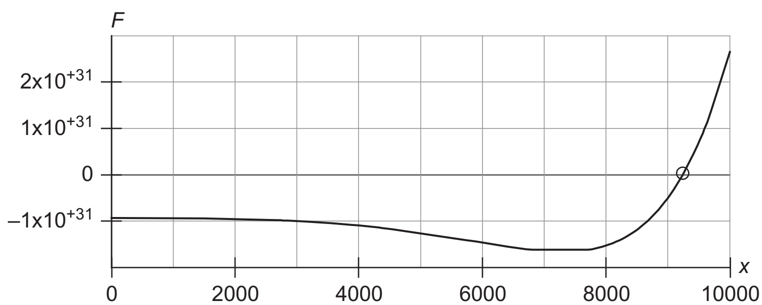


FIGURE 5.15

Graph of the polynomial $F(x)$ in Step 8.

Stepping through Newton's iterative procedure yields

$$\begin{aligned} x_0 &= 9000 \\ x_1 &= 9000 - (-276.93) = 9276.9 \\ x_2 &= 9276.9 - 34.526 = 9242.4 \\ x_3 &= 9242.4 - 0.63428 = 9241.8 \\ x_4 &= 9241.8 - 0.00021048 = 9241.8 \end{aligned}$$

Thus, after four steps we converge to

$$r_2 = 9241.8 \text{ km}$$

The other roots are either negative or complex and are therefore physically unacceptable.

Step 9.

$$\rho_1 = \frac{1}{-0.0015198} \times \left\{ \frac{6\left[787.31\dfrac{(-118.10)}{119.47} + 784.72\dfrac{237.58}{119.47}\right]9241.8^3 + 398{,}600 \cdot 787.31[237.58^2 - (-118.10)^2]\dfrac{-118.10}{119.47}}{6 \cdot 9241.8^3 + 398{,}600(237.58^2 - 119.47^2)} - 782.15 \right\}$$
$$= 3639.1 \text{ km}$$

$$\rho_2 = -6.6858 + \frac{398{,}600 \cdot 7.6667 \times 10^9}{9241.8^3} = 3864.8 \text{ km}$$

$$\rho_3 = \frac{1}{-0.0015198} \times \left[\frac{6\left(887.10\dfrac{119.47}{-118.10} - 889.60\dfrac{237.58}{-118.10}\right)9241.8^3 + 398{,}600 \cdot 887.10(237.58^2 - 119.47^2)\dfrac{119.47}{-118.10}}{6 \cdot 9241.8^3 + 398{,}600\left[237.58^2 - (-118.10)^2\right]} - 892.13 \right]$$
$$= 4172.8 \text{ km}$$

Step 10.

$$\begin{aligned} \mathbf{r}_1 &= (3489.8\hat{\mathbf{I}} + 3430.2\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}) + 3639.1(0.71643\hat{\mathbf{I}} + 0.68074\hat{\mathbf{J}} - 0.15270\hat{\mathbf{K}}) \\ &= 6096.9\hat{\mathbf{I}} + 5907.5\hat{\mathbf{J}} + 3522.9\hat{\mathbf{K}} \text{ (km)} \\ \mathbf{r}_2 &= (3460.1\hat{\mathbf{I}} + 3460.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}) + 3864.8(0.56897\hat{\mathbf{I}} + 0.79531\hat{\mathbf{J}} - 0.20917\hat{\mathbf{K}}) \\ &= 5659.1\hat{\mathbf{I}} + 6533.8\hat{\mathbf{J}} + 3270.1\hat{\mathbf{K}} \text{ (km)} \\ \mathbf{r}_3 &= (3429.9\hat{\mathbf{I}} + 3490.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}) + 4172.8(0.41841\hat{\mathbf{I}} + 0.87007\hat{\mathbf{J}} - 0.26059\hat{\mathbf{K}}) \\ &= 5175.8\hat{\mathbf{I}} + 7120.8\hat{\mathbf{J}} + 2991.1\hat{\mathbf{K}} \text{ (km)} \end{aligned}$$

Step 11.

$$f_1 \approx 1 - \frac{1}{2}\frac{398{,}600}{9241.8^3}(-118.10)^2 = 0.99648$$

$$f_3 \approx 1 - \frac{1}{2}\frac{398{,}600}{9241.8^3}(119.47)^2 = 0.99640$$

$$g_1 \approx -118.10 - \frac{1}{6}\frac{398{,}600}{9241.8^3}(-118.10)^3 = -117.97$$

$$g_3 \approx 119.47 - \frac{1}{6}\frac{398{,}600}{9241.8^3}(119.47)^3 = 119.33$$

Step 12.

$$\begin{aligned}\mathbf{v}_2 &= \frac{-0.99640(6096.9\hat{\mathbf{I}} + 5907.5\hat{\mathbf{J}} + 3522.9\hat{\mathbf{K}}) + 0.99648(5175.8\hat{\mathbf{I}} + 7120.8\hat{\mathbf{J}} + 2991.1\hat{\mathbf{K}})}{0.99648 \cdot 119.33 - 0.99640(-117.97)} \\ &= -3.8800\hat{\mathbf{I}} + 5.1156\hat{\mathbf{J}} - 2.2397\hat{\mathbf{K}}\ \text{(km/s)}\end{aligned}$$

In summary, the state vector at time t_2 is, approximately,

$$\boxed{\begin{aligned}\mathbf{r}_2 &= 5659.1\hat{\mathbf{I}} + 6533.8\hat{\mathbf{J}} + 3270.1\hat{\mathbf{K}}\ \text{(km)} \\ \mathbf{v}_2 &= -3.8800\hat{\mathbf{I}} + 5.1156\hat{\mathbf{J}} - 2.2397\hat{\mathbf{K}}\ \text{(km/s)}\end{aligned}}$$

Example 5.12

Starting with the state vector determined in Example 5.11, use Algorithm 5.6 to improve the vector to five significant figures.

Step 1.

$$r_2 = \|\mathbf{r}_2\| = \sqrt{5659.1^2 + 6533.8^2 + 3270.1^2} = 9241.8\ \text{km}$$

$$v_2 = \|\mathbf{v}_2\| = \sqrt{(-3.8800)^2 + 5.1156^2 + (-2.2397)^2} = 6.7999\ \text{km/s}$$

Step 2.

$$\alpha = \frac{2}{r_2} - \frac{{v_2}^2}{\mu} = \frac{2}{9241.8} - \frac{6.7999^2}{398{,}600} = 1.0040 \times 10^{-4}\ \text{km}^{-1}$$

Step 3.

$$v_{r2} = \frac{\mathbf{v}_2 \cdot \mathbf{r}_2}{r_2} = \frac{(-3.8800) \cdot 5659.1 + 5.1156 \cdot 6533.8 + (-2.2397) \cdot 3270.1}{9241.8} = 0.44829\ \text{km/s}$$

Step 4.

The universal Kepler's equation at times t_1 and t_3, respectively, becomes

$$\begin{aligned}\sqrt{398{,}600}\tau_1 &= \frac{9241.8\cdot 0.44829}{\sqrt{398{,}600}}{\chi_1}^2 C(1.0040\times 10^{-4}{\chi_1}^2)\\ &\quad + (1-1.0040\times 10^{-4}\cdot 9241.8){\chi_1}^3 S(1.0040\times 10^{-4}{\chi_1}^2) + 9241.8\chi_1\\ \sqrt{398{,}600}\tau_3 &= \frac{9241.8\cdot 0.44829}{\sqrt{398{,}600}}{\chi_3}^2 C(1.0040\times 10^{-4}{\chi_3}^2)\\ &\quad + (1-1.0040\times 10^{-4}\cdot 9241.8){\chi_3}^3 S(1.0040\times 10^{-4}{\chi_3}^2) + 9241.8\chi_3\end{aligned}$$

or

$$\begin{aligned}631.35\tau_1 &= 6.5622{\chi_1}^2 C(1.0040\times 10^{-4}{\chi_1}^2) + 0.072085{\chi_1}^3 S(1.0040\times 10^{-4}{\chi_1}^2) + 9241.8\chi_1\\ 631.35\tau_3 &= 6.5622{\chi_3}^2 C(1.0040\times 10^{-4}{\chi_3}^2) + 0.072085{\chi_1}^3 S(1.0040\times 10^{-4}{\chi_3}^2) + 9241.8\chi_3\end{aligned}$$

Applying Algorithm 3.3 to each of these equations yields

$$\begin{aligned}\chi_1 &= -8.0908\ \sqrt{\text{km}}\\ \chi_3 &= 8.1375\ \sqrt{\text{km}}\end{aligned}$$

Step 5.

$$\begin{aligned}f_1 &= 1-\frac{{\chi_1}^2}{r_2}C(\alpha{\chi_1}^2) = 1-\frac{(-8.0908)^2}{9241.8}\cdot\overbrace{C(1.0040\times 10^{-4}[-8.0908]^2)}^{0.49973} = 0.99646\\ g_1 &= \tau_1-\frac{1}{\sqrt{\mu}}{\chi_1}^3 S(\alpha{\chi_1}^2) = -118.1-\frac{1}{\sqrt{398{,}600}}(-8.0908)^3\cdot\overbrace{S(1.0040\times 10^{-4}[-8.0908]^2)}^{0.16661} = -117.96\ \text{s}\end{aligned}$$

and

$$\begin{aligned}f_3 &= 1-\frac{{\chi_3}^2}{r_2}C(\alpha{\chi_3}^2) = 1-\frac{8.1375^2}{9241.8}\cdot\overbrace{C(1.0040\times 10^{-4}\cdot 8.1375^2)}^{0.49972} = 0.99642\\ g_3 &= \tau_3-\frac{1}{\sqrt{\mu}}{\chi_3}^3 S(\alpha{\chi_3}^2) = -118.1-\frac{1}{\sqrt{398{,}600}}8.1375^3\cdot\overbrace{S(1.0040\times 10^{-4}\cdot 8.1375^2)}^{0.16661} = 119.33\end{aligned}$$

It turns out that the procedure converges more rapidly if the Lagrange coefficients are set equal to the average of those computed for the current step and those computed for the previous step. Thus, we set

$$f_1 = \frac{0.99648 + 0.99646}{2} = 0.99647$$
$$g_1 = \frac{-117.97 + (-117.96)}{2} = -117.96 \text{ s}$$
$$f_3 = \frac{0.99642 + 0.99641}{2} = 0.99641$$
$$g_3 = \frac{119.33 + 119.33}{2} = 119.34 \text{ s}$$

Step 6.

$$c_1 = \frac{119.33}{(0.99647)(119.33) - (0.99641)(-117.96 \text{ s})} = 0.50467$$
$$c_3 = -\frac{-117.96}{(0.99647)(119.33) - (0.99641)(-117.96)} = 0.49890$$

Step 7.

$$\rho_1 = \frac{1}{-0.0015198}\left(-782.15 + \frac{1}{0.50467}784.72 - \frac{0.49890}{0.50467}787.31\right) = 3650.6 \text{ km}$$
$$\rho_2 = \frac{1}{-0.0015198}(-0.50467 \cdot 1646.5 + 1651.5 - 0.49890 \cdot 1656.6) = 3877.2 \text{ km}$$
$$\rho_3 = \frac{1}{-0.0015198}\left(-\frac{0.50467}{0.49890}887.10 + \frac{1}{0.49890}889.60 - 892.13\right) = 4186.2 \text{ km}$$

Step 8.

$$\begin{aligned}
\mathbf{r}_1 &= (3489.8\hat{\mathbf{I}} + 3430.2\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}) + 3650.6(0.71643\hat{\mathbf{I}} + 0.68074\hat{\mathbf{J}} - 0.15270\hat{\mathbf{K}}) \\
&= 6105.2\hat{\mathbf{I}} + 5915.3\hat{\mathbf{J}} + 3521.1\hat{\mathbf{K}} \text{ (km)} \\
\mathbf{r}_2 &= (3460.1\hat{\mathbf{I}} + 3460.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}) + 3877.2(0.568{,}97\hat{\mathbf{I}} + 0.79531\hat{\mathbf{J}} - 0.209{,}17\hat{\mathbf{K}}) \\
&= 5666.6\hat{\mathbf{I}} + 6543.7\hat{\mathbf{J}} + 3267.5\hat{\mathbf{K}} \text{ (km)} \\
\mathbf{r}_3 &= (3429.9\hat{\mathbf{I}} + 3490.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}) + 4186.2(0.418{,}41\hat{\mathbf{I}} + 0.87007\hat{\mathbf{J}} - 0.260{,}59\hat{\mathbf{K}}) \\
&= 5181.4\hat{\mathbf{I}} + 7132.4\hat{\mathbf{J}} + 2987.6\hat{\mathbf{K}} \text{ (km)}
\end{aligned}$$

Step 9.

$$\begin{aligned}
\mathbf{v}_2 &= \frac{1}{0.99647 \cdot 119.33 - 0.99641(-117.96)} \times \\
&\quad [-0.99641(6105.2\hat{\mathbf{I}} + 5915.3\hat{\mathbf{J}} + 3521.1\hat{\mathbf{K}}) + 0.99647(5181.4\hat{\mathbf{I}} + 7132.4\hat{\mathbf{J}} + 2987.6\hat{\mathbf{K}})] \\
&= -3.8856\hat{\mathbf{I}} + 5.1214\hat{\mathbf{J}} - 2.2434\hat{\mathbf{K}} \text{ (km/s)}
\end{aligned}$$

Table 5.2 Key Results at Each Step of the Iterative Procedure

Step	χ_1	χ_3	f_1	g_1	f_3	g_3	ρ_1	ρ_2	ρ_3
1	−8.0908	8.1375	0.99647	−117.97	0.99641	119.33	3650.6	3877.2	4186.2
2	−8.0818	8.1282	0.99647	−117.96	0.99642	119.33	3643.8	3869.9	4178.3
3	−8.0871	8.1337	0.99647	−117.96	0.99642	119.33	3644.0	3870.1	4178.6
4	−8.0869	8.1336	0.99647	−117.96	0.99642	119.33	3644.0	3870.1	4178.6

This completes the first iteration.

The updated position $\mathbf{r}_2$ and velocity $\mathbf{v}_2$ are used to repeat the procedure beginning at Step 1. The results of the first and subsequent iterations are shown in Table 5.2. Convergence to five significant figures in the slant ranges ρ_1, ρ_2 and ρ_3 occurs in four steps, at which point the state vector is

$$\boxed{\begin{aligned}\mathbf{r}_2 &= 5662.1\hat{\mathbf{I}} + 6538.0\hat{\mathbf{J}} + 3269.0\hat{\mathbf{K}}\ \text{(km)}\\ \mathbf{v}_2 &= -3.8856\hat{\mathbf{I}} + 5.1214\hat{\mathbf{J}} - 2.2433\hat{\mathbf{K}}\ \text{(km/s)}\end{aligned}}$$

Using $\mathbf{r}_2$ and $\mathbf{v}_2$ in Algorithm 4.2 we find that the orbital elements are

$$\begin{aligned}a &= 10{,}000\ \text{km} \quad (h = 62{,}818\ \text{km}^2/\text{s})\\ e &= 0.1000\\ i &= 30^\circ\\ \Omega &= 270^\circ\\ \omega &= 90^\circ\\ \theta &= 45.01^\circ\end{aligned}$$

PROBLEMS

Section 5.2

5.1 The geocentric equatorial position vectors of a satellite at three separate times are

$$\begin{aligned}\mathbf{r}_1 &= 5887\hat{\mathbf{I}} - 3520\hat{\mathbf{J}} - 1204\hat{\mathbf{K}}\ \text{(km)}\\ \mathbf{r}_2 &= 5572\hat{\mathbf{I}} - 3457\hat{\mathbf{J}} - 2376\hat{\mathbf{K}}\ \text{(km)}\\ \mathbf{r}_3 &= 5088\hat{\mathbf{I}} - 3289\hat{\mathbf{J}} - 3480\hat{\mathbf{K}}\ \text{(km)}\end{aligned}$$

Use Gibbs' method to find $\mathbf{v}_2$. {Partial ans.: $v_2 = 7.59\,\text{km/s}$}

5.2 Calculate the orbital elements and perigee altitude of the space object in the previous problem. {Partial ans.: $z_p = 567\,\text{km}$.}

Section 5.3

5.3 At a given instant the altitude of an earth satellite is 600 km. Fifteen minutes later the altitude is 300 km and the true anomaly has increased by 60°. Find the perigee altitude.
{Ans.: $z_p = 298$ km.}

5.4 At a given instant, the geocentric equatorial position vector of an earth satellite is

$$\mathbf{r}_1 = -3600\hat{\mathbf{I}} + 3600\hat{\mathbf{J}} + 5100\hat{\mathbf{K}} \text{ (km)}.$$

Thirty minutes later the position is

$$\mathbf{r}_2 = -5500\hat{\mathbf{I}} - 6240\hat{\mathbf{J}} - 520\hat{\mathbf{K}} \text{ (km)}.$$

Calculate $\mathbf{v}_1$ and $\mathbf{v}_2$. {Partial ans.: $v_1 = 7.711$ km/s, $v_2 = 6.670$ km/s}

5.5 Compute the orbital elements and perigee altitude for the previous problem.
{Partial ans.: $z_p = 648$ km}

5.6 At a given instant, the geocentric equatorial position vector of an earth satellite is

$$\mathbf{r}_1 = 5644\hat{\mathbf{I}} - 2830\hat{\mathbf{J}} - 4170\hat{\mathbf{K}} \text{ (km)}.$$

Twenty minutes later the position is

$$\mathbf{r}_2 = -2240\hat{\mathbf{I}} + 7320\hat{\mathbf{J}} - 4980\hat{\mathbf{K}} \text{ (km)}.$$

Calculate $\mathbf{v}_1$ and $\mathbf{v}_2$. {Partial ans.: $v_1 = 10.84$ km/s, $v_2 = 9.970$ km/s.}

5.7 Compute the orbital elements and perigee altitude for the previous problem.
{Partial ans.: $z_p = 224$ km.}

Section 5.4

5.8 Calculate the Julian day (JD) number for the following epochs:

(a) 5:30 UT on August 14, 1914.
(b) 14:00 UT on April 18, 1946.
(c) 0:00 UT on September 1, 2010.
(d) 12:00 UT on October 16, 2007.
(e) Noon today, your local time.
{Ans.: (a) 2,420,358.729, (b) 2,431,929.083, (c) 2,455,440.500, (d) 2,454,390.000}

5.9 Calculate the number of days from 12:00 UT on your date of birth to 12:00 UT on today's date.

5.10 Calculate the local sidereal time (in degrees) at:

(a) Stockholm, Sweden (east longitude 18°03′) at 12:00 UT on January 1, 2008.
(b) Melbourne, Australia (east longitude 144°58′) at 10:00 UT on December 21, 2007.
(c) Los Angeles, California (west longitude 118°15′) at 20:00 UT on July 4, 2005.
(d) Rio de Janeiro, Brazil (west longitude 43°06′) at 3:00 UT on February 15, 2006.
(e) Vladivostok, Russia (east longitude 131°56′) at 8:00 UT on March 21, 2006.
(f) At noon today, your local time and place.

{Ans.: (a) 298.6°, (b) 24.6°, (c) 104.7°, (d) 146.9°, (e) 70.6°.}

Section 5.8

5.11 Relative to a tracking station whose local sidereal time is 117° and latitude is +51°, the azimuth and elevation angle of a satellite are 28° and 68°, respectively. Calculate the topocentric right ascension and declination of the satellite.
{Ans.: $\alpha = 145.3°$, $\delta = 68.24°$}

5.12 A sea-level tracking station at whose local sidereal time is 40° and latitude is 35° makes the following observations of a space object:

Azimuth:	36.0°
Azimuth rate:	0.590°
Elevation:	36.6°
Elevation rate:	−0.263°
Range:	988 km
Range rate:	4.86 km/s

What is the state vector of the object? {Partial ans.: $r = 7003.3$ km, $v = 10.922$ km/s}

5.13 Calculate the orbital elements of the satellite in the previous problem.
{Partial ans.: $e = 1.1$, $i = 40°$}

5.14 A tracking station at latitude −20° and elevation 500 m makes the following observations of a satellite at the given times.

Time (min)	Local sidereal time (degrees)	Azimuth (degrees)	Elevation angle (degrees)	Range (km)
0	60.0	165.932	8.81952	1212.48
2	60.5014	145.970	44.2734	410.596
4	61.0027	2.40973	20.7594	726.464

Use the Gibbs method to calculate the state vector of the satellite at the central observation time. {Partial ans.: $r_2 = 6684$ km, $v_2 = 7.7239$ km/s}

5.15 Calculate the orbital elements of the satellite in the previous problem.
{Partial ans.: $e = 0.001$, $i = 95°$}

Section 5.10

5.16 A sea-level tracking station at latitude +29° makes the following observations of a satellite at the given times.

Time (min)	Local sidereal time (degrees)	Topocentric Right ascension (degrees)	Topocentric Declination (degrees)
0.0	0	0	51.5110
1.0	0.250684	65.9279	27.9911
2.0	0.501369	79.8500	14.6609

Use the Gauss method without iterative improvement to estimate the state vector of the satellite at the middle observation time. {Partial ans.: $r = 6700.9$ km $v = 8.0757$ km/s}

5.17 Refine the estimate in the previous problem using iterative improvement.
{Partial ans.: $r = 6701.5$ km $v = 8.0881$ km/s}

5.18 Calculate the orbital elements from the state vector obtained in the previous problem.
{Partial ans.: $e = 0.10$, $i = 30°$}

5.19 A sea-level tracking station at latitude +29° makes the following observations of a satellite at the given times.

Time (min)	Local sidereal time (degrees)	Topocentric Right ascension (degrees)	Topocentric Declination (degrees)
0.0	90	15.0394	20.7487
1.0	90.2507	25.7539	30.1410
2.0	90.5014	48.6055	43.8910

Use the Gauss method without iterative improvement to estimate the state vector of the satellite. {Partial ans.: $r = 6999.1$ km, $v = 7.5541$ km/s}

5.20 Refine the estimate in the previous problem using iterative improvement.
{Partial ans.: $r = 7000.0$ km $v = 7.5638$ km/s}

5.21 Calculate the orbital elements from the state vector obtained in the previous problem.
{Partial ans.: e = 0.0048, $i = 31°$}

5.22 The position vector **R** of a tracking station and the direction cosine vector $\hat{\boldsymbol{\rho}}$ of a satellite relative to the tracking station at three times are as follows:

$$\begin{aligned} t_1 &= 0 \text{ min} \\ \mathbf{R}_1 &= -1825.96\hat{\mathbf{I}} + 3583.66\hat{\mathbf{J}} + 4933.54\hat{\mathbf{K}} \text{ (km)} \\ \hat{\boldsymbol{\rho}}_1 &= -0.301687\hat{\mathbf{I}} + 0.200673\hat{\mathbf{J}} + 0.932049\hat{\mathbf{K}} \end{aligned}$$

$$\begin{aligned} t_2 &= 1 \text{ min} \\ \mathbf{R}_2 &= -1816.30\hat{\mathbf{I}} + 3575.63\hat{\mathbf{J}} + 4933.54\hat{\mathbf{K}} \text{ (km)} \\ \hat{\boldsymbol{\rho}}_2 &= -0.793090\hat{\mathbf{I}} - 0.210324\hat{\mathbf{J}} + 0.571640\hat{\mathbf{K}} \end{aligned}$$

$$\begin{aligned} t_3 &= 2 \text{ min} \\ \mathbf{R}_3 &= -1857.25\hat{\mathbf{I}} + 3567.54\hat{\mathbf{J}} + 4933.54\hat{\mathbf{K}} \text{ (km)} \\ \hat{\boldsymbol{\rho}}_3 &= -0.873085\hat{\mathbf{I}} - 0.362969\hat{\mathbf{J}} + 0.325539\hat{\mathbf{K}} \end{aligned}$$

Use the Gauss method without iterative improvement to estimate the state vector of the satellite at the central observation time. {Partial ans.: $r = 6742.3$ km $v = 7.6799$ km/s}

5.23 Refine the estimate in the previous problem using iterative improvement.
{Partial ans.: $r = 6743.0$ km, $v = 7.6922$ km/s}

5.24 Calculate the orbital elements from the state vector obtained in the previous problem.
{Partial ans.: $e = 0.001$, $i = 52°$}

5.25 A tracking station at latitude 60°N and 500 m elevation obtains the following data:

Time (min)	Local sidereal time (degrees)	Topocentric Right ascension (degrees)	Topocentric Declination (degrees)
0.0	150	157.783	20.2403
5.0	151.253	159.221	27.2993
10.0	152.507	160.526	29.8982

Use the Gauss method without iterative improvement to estimate the state vector of the satellite.
{Partial ans.: $r = 25132$ km, $v = 6.0588$ km/s}

5.26 Refine the estimate in the previous problem using iterative improvement.
{Partial ans.: $r = 25169$ km, $v = 6.0671$ km/s}

5.27 Calculate the orbital elements from the state vector obtained in the previous problem.
{Partial ans.: $e = 1.09$, $i = 63°$}

5.28 The position vector **R** of a tracking station and the direction cosine vector $\hat{\boldsymbol{\rho}}$ of a satellite relative to the tracking station at three times are as follows:

$$
\begin{aligned}
t_1 &= 0 \text{ min} \\
\mathbf{R}_1 &= 5582.84\hat{\mathbf{I}} + 3073.90\hat{\mathbf{K}} \text{ (km)} \\
\hat{\boldsymbol{\rho}}_1 &= 0.846{,}428\hat{\mathbf{I}} + 0.532{,}504\hat{\mathbf{K}}
\end{aligned}
$$

$$
\begin{aligned}
t_2 &= 5 \text{ min} \\
\mathbf{R}_2 &= 5581.50\hat{\mathbf{I}} + 122.122\hat{\mathbf{J}} + 3073.90\hat{\mathbf{K}} \text{ (km)} \\
\hat{\boldsymbol{\rho}}_2 &= 0.749{,}290\hat{\mathbf{I}} + 0.463{,}023\hat{\mathbf{J}} + 0.473{,}470\hat{\mathbf{K}}
\end{aligned}
$$

$$
\begin{aligned}
t_3 &= 10 \text{ min} \\
\mathbf{R}_3 &= 5577.50\hat{\mathbf{I}} + 244.186\hat{\mathbf{J}} + 3073.90\hat{\mathbf{K}} \text{ (km)} \\
\hat{\boldsymbol{\rho}}_3 &= 0.529{,}447\hat{\mathbf{I}} + 0.777{,}163\hat{\mathbf{J}} + 0.340{,}152\hat{\mathbf{K}}
\end{aligned}
$$

Use the Gauss method without iterative improvement to estimate the state vector of the satellite.
{Partial ans.: $r = 9729.6$ km, $v = 6.0234$ km/s}

5.29 Refine the estimate in the previous problem using iterative improvement.
{Partial ans.: $r = 9759.8$ km, $v = 6.0713$ km/s}

5.30 Calculate the orbital elements from the state vector obtained in the previous problem.
{Partial ans.: $e = 0.1$, $i = 30°$}

List of Key Terms

geocentric latitude
geodetic latitude
Julian day
sidereal time
slant range
solar time
topcentric right ascension and declination
universal time

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CHAPTER

Orbital maneuvers

6

Chapter outline

6.1 Introduction 319
6.2 Impulsive maneuvers 320
6.3 Hohmann transfer 321
6.4 Bi-elliptic Hohmann transfer 328
6.5 Phasing maneuvers 332
6.6 Non-Hohmann transfers with a common apse line 338
6.7 Apse line rotation 343
6.8 Chase maneuvers 350
6.9 Plane change maneuvers 355
6.10 Nonimpulsive orbital maneuvers 368

6.1 INTRODUCTION

Orbital maneuvers transfer a spacecraft from one orbit to another. Orbital changes can be dramatic, such as the transfer from a low-earth parking orbit to an interplanetary trajectory. They can also be quite small, as in the final stages of the rendezvous of one spacecraft with another. Changing orbits requires the firing of onboard rocket engines. We will be concerned primarily with impulsive maneuvers in which the rockets fire in relatively short bursts to produce the required velocity change (delta-v).

We start with the classical, energy-efficient Hohmann transfer maneuver, and generalize it to the bi-elliptic Hohmann transfer to see if even more efficiency can be obtained. The phasing maneuver, a form of Hohmann transfer, is considered next. This is followed by a study of non-Hohmann transfer maneuvers with and without rotation of the apse line. We then analyze chase maneuvers, which requires solving Lambert's problem as explained in Chapter 5. The energy-demanding chase maneuvers may be impractical for low earth orbits, but they are necessary for interplanetary missions, as we shall see in Chapter 8. After having focused on impulsive transfers between coplanar orbits, we finally turn our attention to plane change maneuvers and their delta-v requirements, which can be very large.

The chapter concludes with a brief consideration of some orbital transfers in which the propulsion system delivers the impulse during a finite (perhaps very long) time interval instead of instantaneously. This makes it difficult to obtain closed-form solutions, so we illustrate the use of the numerical integration techniques presented in Chapter 1 as an alternative.

6.2 IMPULSIVE MANEUVERS

Impulsive maneuvers are those in which brief firings of on-board rocket motors change the magnitude and direction of the velocity vector instantaneously. During an impulsive maneuver, the position of the spacecraft is considered to be fixed; only the velocity changes. The impulsive maneuver is an idealization by means of which we can avoid having to solve the equations of motion (Equation 2.22) with the rocket thrust included. The idealization is satisfactory for those cases in which the position of the spacecraft changes only slightly during the time that the maneuvering rockets fire. This is true for high-thrust rockets with burn times short compared with the coasting time of the vehicle.

Each impulsive maneuver results in a change $\Delta\mathbf{v}$ in the velocity of the spacecraft. $\Delta\mathbf{v}$ can represent a change in the magnitude ("pumping maneuver") or the direction ("cranking maneuver") of the velocity vector, or both. The magnitude Δv of the velocity increment is related to Δm, the mass of propellant consumed, by the ideal rocket equation (see Equation 11.30).

$$\frac{\Delta m}{m} = 1 - e^{-\frac{\Delta v}{I_{sp} g_o}} \tag{6.1}$$

where m is the mass of the spacecraft before the burn, g_o is the sea-level standard acceleration of gravity, and I_{sp} is the specific impulse of the propellants. Specific impulse is defined as follows:

$$I_{sp} = \frac{\text{thrust}}{\text{sea-level weight rate of fuel consumption}}$$

Specific impulse has units of seconds, and it is a measure of the performance of a rocket propulsion system. I_{sp} for some common propellant combinations are shown in Table 6.1. Figure 6.1 is a graph of Equation 6.1 for a range of specific impulses. Note that for Δv's on the order of 1 km/s or higher, the required propellant exceeds 25% of the spacecraft mass prior to the burn.

There are no refueling stations in space, so a mission's delta-v schedule must be carefully planned to minimize the propellant mass carried aloft in favor of payload.

Table 6.1 Some Typical Specific Impulses

Propellant	I_{sp} (seconds)
Cold gas	50
Monopropellant hydrazine	230
Solid propellant	290
Nitric acid/monomethylhydrazine	310
Liquid oxygen/liquid hydrogen	455
Ion propulsion	>3000

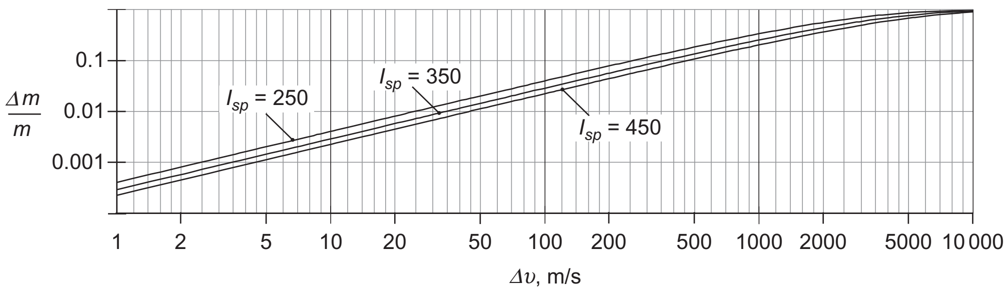


FIGURE 6.1

Propellant mass fraction versus Δv for typical specific impulses.

6.3 HOHMANN TRANSFER

The Hohmann transfer (Hohmann, 1925) is the most energy efficient two-impulse maneuver for transferring between two coplanar circular orbits sharing a common focus. The Hohmann transfer is an elliptical orbit tangent to both circles on its apse line, as illustrated in Figure 6.2. The periapsis and apoapsis of the transfer ellipse are the radii of the inner and outer circles, respectively. Obviously, only one-half of the ellipse is flown during the maneuver, which can occur in either direction, from the inner to the outer circle, or vice-versa.

It may be helpful in sorting out orbit transfer strategies to use the fact that the energy of an orbit depends only on its semimajor axis a. Recall that for an ellipse (Equation 2.80), the specific energy is negative,

$$\varepsilon = -\frac{\mu}{2a}$$

Increasing the energy requires reducing its magnitude, in order to make ε less negative. Therefore, the larger the semimajor axis is, the more energy the orbit has. In Figure 6.2, the energies increase as we move from the inner to the outer circle.

Starting at A on inner circle, a velocity increment Δv_A in the direction of flight is required to boost the vehicle onto the higher-energy elliptical trajectory. After coasting from A to B, another forward velocity increment Δv_B places the vehicle on the still higher-energy, outer circular orbit. Without the latter delta-v burn, the spacecraft would, of course, remain on the Hohmann transfer ellipse and return to A. The total energy expenditure is reflected in the total delta-v requirement, $\Delta v_{\text{total}} = \Delta v_A + \Delta v_B$.

The same total delta-v is required if the transfer begins at B on the outer circular orbit. Since moving to the lower-energy inner circle requires lowering the energy of the spacecraft, the Δv's must be accomplished by retrofires. That is, the thrust of the maneuvering rocket is directed opposite to the flight direction in order to act as a brake on the motion. Since Δv represents the same propellant expenditure regardless of the direction the thruster is aimed, when summing up Δv's, we are concerned only with their magnitudes.

Recall that the eccentricity of an elliptical orbit is found from its radius to perisapsis r_p and its radius to apoapsis r_a by means of Equation 2.84,

$$e = \frac{r_a - r_p}{r_a + r_p}$$

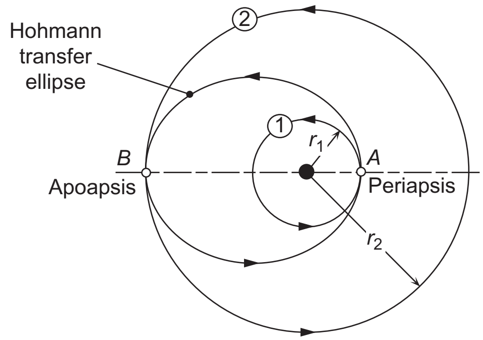


FIGURE 6.2

Hohmann transfer.

The radius to periapsis is given by Equation 2.50,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e}$$

Combining these last two expressions yields

$$r_p = \frac{h^2}{\mu}\frac{1}{1+\dfrac{r_a - r_p}{r_a + r_p}} = \frac{h^2}{\mu}\frac{r_a + r_p}{2r_a}$$

Solving for h, we get the ***angular momentum of an elliptical orbit***,

$$h = \sqrt{2\mu}\sqrt{\frac{r_a r_p}{r_a + r_p}} \tag{6.2}$$

This is a useful formula for analyzing Hohmann transfers, because knowing h we can find the apsidal velocities from Equation 2.31. Note that for circular orbits ($r_a = r_p$) Equation 6.2 yields

$$h = \sqrt{\mu r} \quad \text{(Circular orbit)}$$

Alternatively, one may prefer to compute the velocities by means of the energy equation (Equation 2.81) in the form

$$v = \sqrt{2\mu}\sqrt{\frac{1}{r} - \frac{1}{2a}} \tag{6.3}$$

This of course yields Equation 2.63 for circular orbits.

Example 6.1

A 2000 kg spacecraft is in a 480 km by 800 km earth orbit (orbit 1 in Figure 6.3). Find

(a) The Δv required at perigee A to place the spacecraft in a 480 km by 16,000 km transfer ellipse (orbit 2).
(b) The Δv (apogee kick) required at B of the transfer orbit to establish a circular orbit of 16,000 km altitude (orbit 3).
(c) The total required propellant if the specific impulse is 300 s.

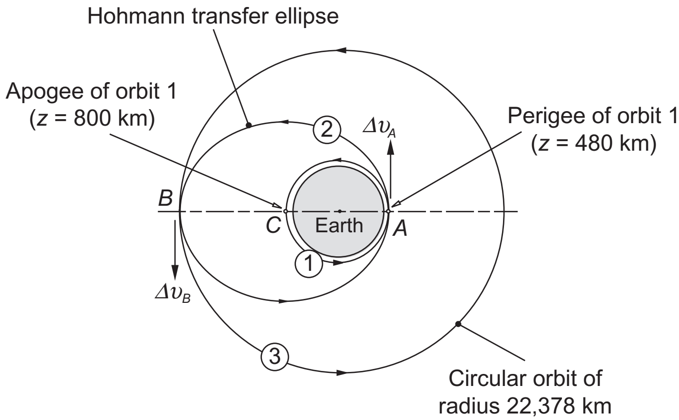


FIGURE 6.3

Hohmann transfer between two earth orbits.

Solution

Since we know the perigee and apogee of all three of the orbits, let us first use Equation 6.2 to calculate their angular momenta.

Orbit 1: $r_p = 6378 + 480 = 6858$ km $\quad r_a = 6378 + 800 = 7178$ km

$$\therefore \quad h_1 = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{7178 \cdot 6858}{7178 + 6858}} = 52{,}876.5 \text{ km/s}^2 \tag{a}$$

Orbit 2: $r_p = 6378 + 480 = 6858$ km $\quad r_a = 6378 + 16{,}000 = 22{,}378$ km

$$\therefore \quad h_2 = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{22{,}378 \cdot 6858}{22{,}378 + 6858}} = 64{,}689.5 \text{ km/s}^2 \tag{b}$$

Orbit 3: $r_a = r_p = 22{,}378$ km

$$\therefore \quad h_3 = \sqrt{398{,}600 \cdot 22{,}378} = 94{,}445.1 \text{ km/s}^2 \tag{c}$$

(a) The speed on orbit 1 at point A is

$$v_A)_1 = \frac{h_1}{r_A} = \frac{52{,}876}{6858} = 7.71019 \text{ km/s}$$

The speed on orbit 2 at point A is

$$v_A)_2 = \frac{h_2}{r_A} = \frac{64{,}689.5}{6858} = 9.43271 \text{ km/s}$$

Therefore, the delta-v required at point A is

$$\Delta v_A = v_A)_2 - v_A)_1 = \boxed{1.7225 \text{ km/s}}$$

(b) The speed on orbit 2 at point B is

$$v_B)_2 = \frac{h_2}{r_B} = \frac{64{,}689.5}{22{,}378} = 2.89076 \text{ km/s}$$

The speed on orbit 3 at point B is

$$v_B)_3 = \frac{h_3}{r_B} = \frac{94{,}445.1}{22{,}378} = 4.22044 \text{ km/s}$$

Hence, the apogee kick required at point B is

$$\Delta v_B = v_B)_3 - v_B)_2 = \boxed{1.3297 \text{ km/s}}$$

(c) The total delta-v requirement for this Hohmann transfer is

$$\Delta v_{\text{total}} = |\Delta v_A| + |\Delta v_B| = 1.7225 + 1.3297 = 3.0522 \text{ km/s}$$

According to Equation 6.1 (converting velocity to m/s),

$$\frac{\Delta m}{m} = 1 - e^{-\frac{3052.2}{300 \cdot 9.807}} = 0.64563$$

Therefore, the mass of propellant expended is

$$\boxed{\Delta m = 0.64563 \cdot 2000 = 1291.3 \text{ kg}}$$

In the previous example the initial orbit of the Hohmann transfer sequence was an ellipse, rather than a circle. Since no real orbit is perfectly circular, we must generalize the notion of a Hohmann transfer to include two-impulsive transfers between elliptical orbits that are coaxial, i.e., share the same apse line, as shown in Figure 6.4. The transfer ellipse must be tangent to both the initial and target ellipses 1 and 2. As can be seen, there are two such transfer orbits, 3 and 3'. It is not immediately obvious which of the two requires the lowest energy expenditure.

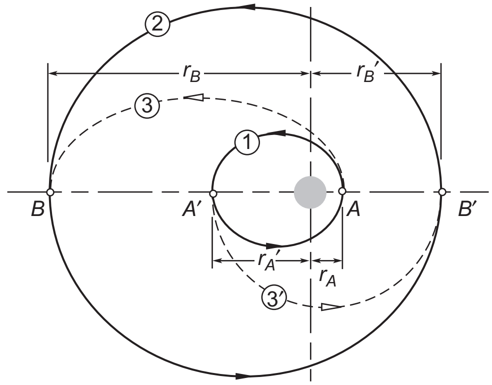


FIGURE 6.4

Hohmann transfers between coaxial elliptical orbits. In this illustration, $r_{A'}/r_o = 3$, $r_B/r_o = 8$ and $r_B/r_o = 4$.

To find out which is the best transfer orbit in general, we must calculate the individual total delta-v requirement for orbits 3 and 3'. This requires finding the velocities at A, A', B and B' for each pair of orbits having those points in common. We employ Equation 6.2 to evaluate the angular momentum of each of the four orbits in Figure 6.4.

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_A r_{A'}}{r_A + r_{A'}}} \qquad h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_{B'}}{r_B + r_{B'}}} \qquad h_3 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} \qquad h_{3'} = \sqrt{2\mu}\sqrt{\frac{r_{A'} r_{B'}}{r_{A'} + r_{B'}}}$$

From these we obtain the velocities,

$$\begin{aligned} v_A)_1 &= \frac{h_1}{r_A} & v_A)_3 &= \frac{h_3}{r_A} \\ v_B)_2 &= \frac{h_2}{r_B} & v_B)_3 &= \frac{h_3}{r_B} \\ v_{A'})_1 &= \frac{h_1}{r_{A'}} & v_{A'})_{3'} &= \frac{h_{3'}}{r_{A'}} \\ v_{B'})_2 &= \frac{h_2}{r_{B'}} & v_{B'})_{3'} &= \frac{h_{3'}}{r_{B'}} \end{aligned}$$

These lead to the delta-vs

$$\begin{aligned} \Delta v_A &= \left|v_A)_3 - v_A)_1\right| & \Delta v_B &= \left|v_B)_2 - v_B)_3\right| \\ \Delta v_{A'} &= \left|v_{A'})_{3'} - v_{A'})_1\right| & \Delta v_{B'} &= \left|v_{B'})_2 - v_{B'})_{3'}\right| \end{aligned}$$

and, finally, to the total delta-v requirement for the two possible transfer trajectories,

$$\Delta v_{\text{total}})_3 = \Delta v_A + \Delta v_B \qquad \Delta v_{\text{total}})_{3'} = \Delta v_{A'} + \Delta v_{B'}$$

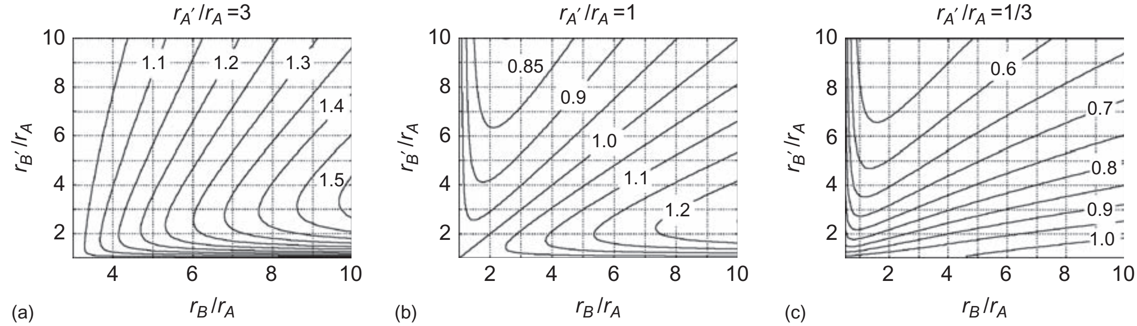


FIGURE 6.5

Contour plots of $\Delta v_{\text{total}})_{3'}/\Delta v_{\text{total}})_3$ for different relative sizes of the ellipses in Figure 6.4. Note that $r_B > r_{A'}$ and $r_{B'} > r_A$.

If $\Delta v_{\text{total}})_{3'}/\Delta v_{\text{total}})_3 > 1$, then orbit 3 is the most efficient. On the other hand, if $\Delta v_{\text{total}})_{3'}/\Delta v_{\text{total}})_3 < 1$ then orbit 3′ is more efficient than orbit 3.

Three contour plots of $\Delta v_{\text{total}})_{3'}/\Delta v_{\text{total}})_3$ are shown in Figure 6.5, for three different shapes of the inner orbit 1 of Figure 6.4. Figure 6.5(a) is for $r_A = 3$, which is the situation represented in Figure 6.4, in which point A is the periapsis of the initial ellipse. In Figure 6.5(b) $r_{A'}/r_A = 1$, which means the starting ellipse is a circle. Finally, in Figure 6.5(c)$r_{A'}/r_A = 1/3$, which corresponds to an initial orbit of the same shape as orbit 1 in Figure 6.4, but with point A being the apoapsis instead of periapsis.

Figure 6.5(a), for which $r_{A'} > r_A$, implies that if point A is the periapsis of orbit 1, then transfer orbit 3 is the most efficient. Figure 6.5(c), for which $r_{A'} < r_A$, shows that if point A' is the periapsis of orbit 1, then transfer orbit 3′ is the most efficient. Together, these results lead us to conclude that it is most efficient for the transfer orbit to begin at the periapsis on the inner orbit 1, where its kinetic energy is greatest, regardless of shape of the outer target orbit. If the starting orbit is a circle, then Figure 6.5(b) shows that transfer orbit 3′ is most efficient if $r_{B'} > r_B$. That is, from an inner circular orbit, the transfer ellipse should terminate at apoapsis of the outer target ellipse, where the speed is slowest.

If the Hohmann transfer is in the reverse direction, that is, to a lower-energy inner orbit, the above analysis still applies, since the same total delta-v is required whether the Hohmann transfer runs forwards or backwards. Thus, from an outer circle or ellipse to an inner ellipse, the most energy-efficient transfer ellipse terminates at periapsis of the inner target orbit. If the inner orbit is a circle, the transfer ellipse should start at apoapsis of the outer ellipse.

We close this section with an illustration of the careful planning required for one spacecraft to rendezvous with another at the end of a Hohmann transfer.

Example 6.2

A spacecraft returning from a lunar mission approaches earth on a hyperbolic trajectory. At its closest approach A it is at an altitude of 5000 km, traveling at 10 km/s. At A retrorockets are fired to lower the spacecraft into a 500 km altitude circular orbit, where it is to rendezvous with a space station. Find the location of the space station at retrofire so that rendezvous will occur at B.

Solution

The time of flight from A to B is one-half the period T_2 of the elliptical transfer orbit 2. While the spacecraft coasts from A to B, the space station coasts through the angle ϕ_{CB} from C to B. Hence, this mission has to be carefully planned and executed, going all the way back to lunar departure, so that the two vehicles meet at B.

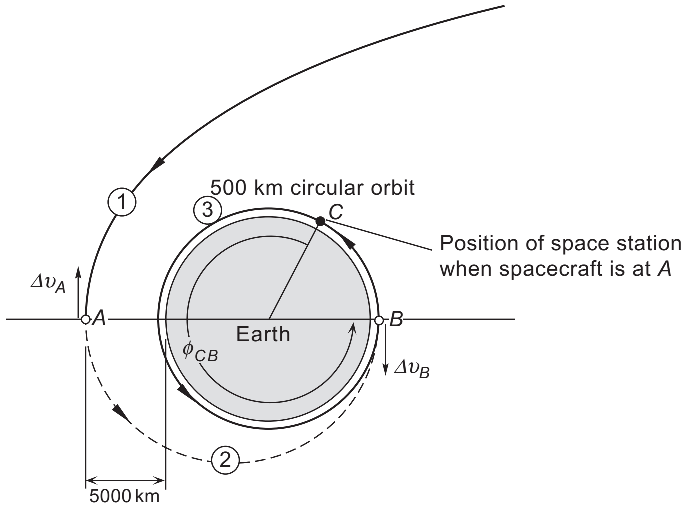


FIGURE 6.6

Relative position of spacecraft and space station at beginning of the transfer ellipse.

According to Equation 2.83, to find the period T_2 we need only determine the semimajor axis of orbit 2. The apogee and perigee of orbit 2 are

$$r_A = 5000 + 6378 = 11{,}378 \text{ km}$$
$$r_B = 500 + 6378 = 6878 \text{ km}$$

Therefore, the semimajor axis is

$$a = \frac{1}{2}(r_A + r_B) = 9128 \text{ km}$$

From this we obtain

$$T_2 = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} 9128^{3/2} = 8679.1 \text{ s} \tag{a}$$

The period of circular orbit 3 is

$$T_3 = \frac{2\pi}{\sqrt{\mu}} r_B^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} 6878^{3/2} = 5676.8 \text{ s} \tag{b}$$

The time of flight from C to B on orbit 3 must equal the time of flight from A to B on orbit 2.

$$t_{CB} = \frac{1}{2} T_2 = \frac{1}{2} \cdot 8679.1 = 4339.5 \text{ s}$$

Since orbit 3 is a circle, its angular velocity, unlike an ellipse, is constant. Therefore, we can write

$$\frac{\phi_{CB}}{t_{CB}} = \frac{360°}{T_3} \Rightarrow \phi_{CB} = \frac{4339.5}{5676.8} \cdot 360 = \boxed{275.2 \text{ degrees}}$$

(The reader should verify that the total delta-v required to lower the spacecraft from the hyperbola into the parking orbit is 5.749 km/s. According to Equation 6.1, that means over 85% of the spacecraft mass must be expended as propellant.)

6.4 BI-ELLIPTIC HOHMANN TRANSFER

A Hohmann transfer from circular orbit 1 to circular orbit 4 in Figure 6.7 is the dotted ellipse lying inside the outer circle, outside the inner circle, and tangent to both. The bi-elliptic Hohmann transfer uses two coaxial semi-ellipses, 2 and 3, which extend beyond the outer target orbit. Each of the two ellipses is tangent to one of the circular orbits, and they are tangent to each other at B, which is the apoapsis of both. The idea is to place B sufficiently far from the focus so that the Δv_B will be very small. In fact, as r_B approaches infinity, Δv_B approaches zero. For the bi-elliptic scheme to be more energy efficient than the Hohmann transfer, it must be true that

$$\Delta v_{\text{total}})_{\text{bi-elliptical}} < \Delta v_{\text{total}})_{\text{Hohmann}}$$

Let v_o be the speed in the circular inner orbit 1,

$$v_o = \sqrt{\frac{\mu}{r_A}}$$

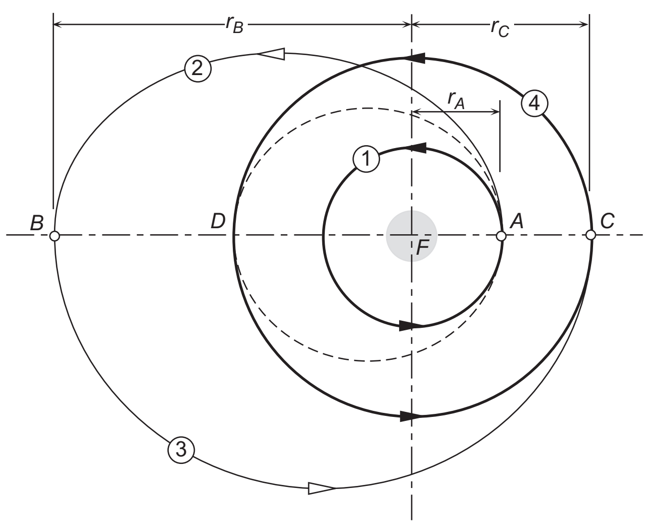


FIGURE 6.7

Bi-elliptic transfer from inner orbit 1 to outer orbit 4.

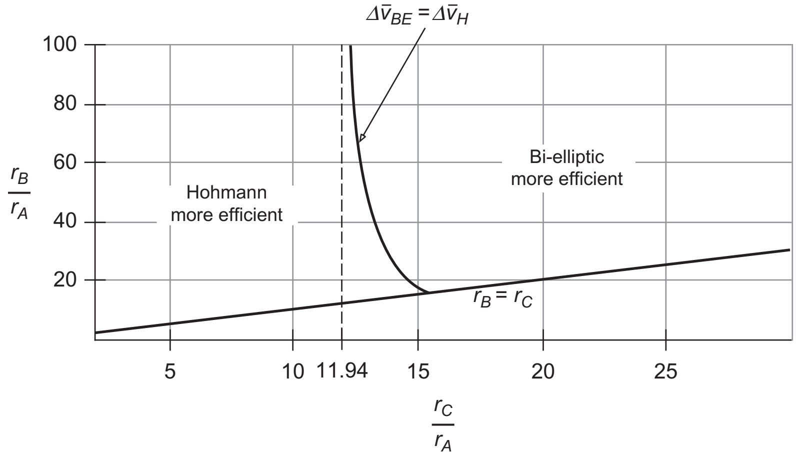


FIGURE 6.8

Orbits for which the bi-elliptic transfer is either less efficient or more efficient than the Hohmann transfer.

Then calculating the total delta-v requirements of the Hohmann and bi-elliptic transfers leads to the following two expressions, respectively,

$$\begin{aligned}\Delta\bar{v}_H &= \frac{1}{\sqrt{\alpha}} - \frac{\sqrt{2}(1-\alpha)}{\sqrt{\alpha(1+\alpha)}} - 1 \\ \Delta\bar{v}_{BE} &= \sqrt{\frac{2(\alpha+\beta)}{\alpha\beta}} - \frac{1+\sqrt{\alpha}}{\sqrt{\alpha}} - \sqrt{\frac{2}{\beta(1+\beta)}}(1-\beta)\end{aligned} \tag{6.4a}$$

where the nondimensional terms are

$$\Delta\bar{v}_H = \left.\frac{\Delta v_{\text{total}}}{v_o}\right)_{\text{Hohmann}} \qquad \Delta\bar{v}_{BE} = \left.\frac{\Delta v_{\text{total}}}{v_o}\right)_{\text{bi-elliptical}} \qquad \alpha = \frac{r_C}{r_A} \quad \beta = \frac{r_B}{r_A} \tag{6.4b}$$

Plotting the difference between $\Delta\bar{v}_H$ and $\Delta\bar{v}_{BE}$ as a function of α and β reveals the regions in which the difference is positive, negative and zero. These are shown in Figure 6.8.

From the figure we see that if the radius of the outer circular target orbit (r_C) is less than 11.94 times that of the inner one (r_A), then the standard Hohmann maneuver is the more energy efficient. If the ratio exceeds 15.58, then the bi-elliptic strategy is better in that regard. Between those two ratios, large values of the apoapsis radius r_B favor the bi-elliptic transfer, while smaller values favor the Hohmann transfer.

Small gains in energy efficiency may be more than offset by the much longer flight times around the bi-elliptic trajectories as compared with the time of flight on the single semi-ellipse of the Hohmann transfer.

Example 6.3

Find the total delta-v requirement for a bi-elliptic Hohmann transfer from a geocentric circular orbit of 7000 km radius to one of 105,000 km radius. Let the apogee of the first ellipse be 210,000 km. Compare the delta-v schedule and total flight time with that for an ordinary single Hohmann transfer ellipse. See Figure 6.9.

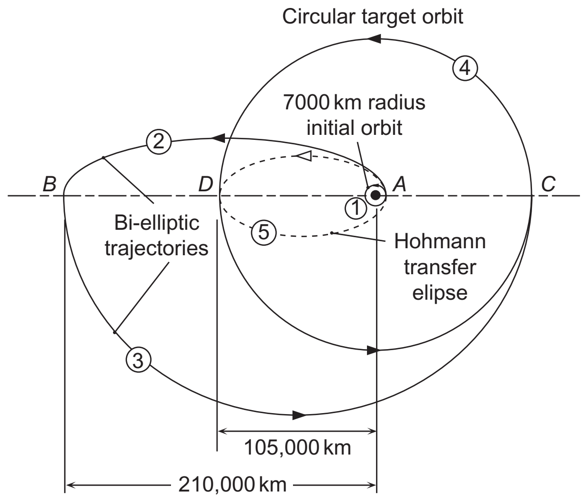


FIGURE 6.9

Bi-elliptic transfer.

Solution

Since

$$r_A = 7000 \text{ km} \qquad r_B = 210{,}000 \text{ km} \qquad r_C = r_D = 105{,}000 \text{ km}$$

we have $r_B/r_A = 30$ and $r_C/r_A = 15$, so that from Figure 6.8 it is apparent right away that the bi-elliptic transfer will be the more energy efficient.

To do the delta-v analysis requires analyzing each of the five orbits.

Orbit 1:

Since this is a circular orbit, we have, simply,

$$v_A)_1 = \sqrt{\frac{\mu}{r_A}} = \sqrt{\frac{398{,}600}{7000}} = 7.546 \text{ km/s} \tag{a}$$

Orbit 2:

For this transfer ellipse, Equation 6.2 yields

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{7000 \cdot 210{,}000}{7000 + 210{,}000}} = 73{,}487 \text{ km}^2\text{/s}$$

Therefore,

$$v_A)_2 = \frac{h_2}{r_A} = \frac{73{,}487}{7000} = 10.498 \text{ km/s} \tag{b}$$

$$v_B)_2 = \frac{h_2}{r_B} = \frac{73{,}487}{210{,}000} = 0.34994 \text{ km/s} \tag{c}$$

Orbit 3:

For the second transfer ellipse, we have

$$h_3 = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{105{,}000 \cdot 210{,}000}{105{,}000 + 210{,}000}} = 236{,}230 \text{ km}^2/\text{s}$$

From this we obtain

$$v_B)_3 = \frac{h_3}{r_B} = \frac{236{,}230}{210{,}000} = 1.1249 \text{ km/s} \tag{d}$$

$$v_C)_3 = \frac{h_3}{r_C} = \frac{236{,}230}{105{,}000} = 2.2498 \text{ km/s} \tag{e}$$

Orbit 4:

The target orbit, like orbit 1, is a circle, which means

$$v_C)_4 = v_D)_4 = \sqrt{\frac{398{,}600}{105{,}000}} = 1.9484 \text{ km/s} \tag{f}$$

For the bi-elliptic maneuver, the total delta-v is, therefore,

$$\begin{aligned} \Delta v_{\text{total}})_{\text{bi-elliptical}} &= \Delta v_A + \Delta v_B + \Delta v_C \\ &= \left|v_A)_2 - v_A)_1\right| + \left|v_B)_3 - v_B)_2\right| + \left|v_C)_4 - v_C)_3\right| \\ &= \left|10.498 - 7.546\right| + \left|1.1249 - 0.34994\right| + \left|1.9484 - 2.2498\right| \end{aligned}$$

or,

$$\boxed{\Delta v_{\text{total}})_{\text{bi-elliptical}} = 4.0285 \text{ km/s}} \tag{g}$$

The semimajor axes of transfer orbits 2 and 3 are

$$\begin{aligned} a_2 &= \frac{1}{2}\left(7000 + 210{,}000\right) = 108{,}500 \text{ km} \\ a_3 &= \frac{1}{2}\left(105{,}000 + 210{,}000\right) = 157{,}500 \text{ km} \end{aligned}$$

With this information and the period formula, Equation 2.83, the time of flight for the two semi-ellipses of the bi-elliptic transfer is found to be

$$t_{\text{bi-elliptical}} = \frac{1}{2}\left(\frac{2\pi}{\sqrt{\mu}}a_2^{3/2} + \frac{2\pi}{\sqrt{\mu}}a_3^{3/2}\right) = 488{,}870 \text{ s} = \boxed{5.66 \text{ days}} \tag{h}$$

Orbit 5 (Hohmann transfer ellipse):

$$h_5 = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{7000 \cdot 105{,}000}{7000 + 105{,}000}} = 72{,}330 \text{ km}^2/\text{s}$$

Hence,

$$v_A)_5 = \frac{h_5}{r_A} = \frac{72{,}330}{7000} = 10.333 \text{ km/s} \tag{i}$$

$$v_D)_5 = \frac{h_5}{r_D} = \frac{72{,}330}{105{,}000} = 0.68886 \text{ km/s} \tag{j}$$

It follows that

$$\begin{aligned}\Delta v_{\text{total}})_{\text{Hohmann}} &= \left|v_A)_5 - v_A)_1\right| + \left|v_D)_5 - v_D)_1\right| \\ &= (10.333 - 7.546) + (1.9484 - 0.68886) \\ &= 2.7868 + 1.2595\end{aligned}$$

or

$$\boxed{\Delta v_{\text{total}})_{\text{Hohmann}} = 4.0463 \text{ km/s}} \tag{k}$$

This is only slightly (0.44%) larger than that of the bi-elliptic transfer.

Since the semimajor axis of the Hohmann semiellipse is

$$a_5 = \frac{1}{2}(7000 + 105{,}000) = 56{,}000 \text{ km}$$

the time of flight from A to D is

$$t_{\text{Hohmann}} = \frac{1}{2}\left(\frac{2\pi}{\sqrt{\mu}} a_5{}^{3/2}\right) = 65{,}942 \text{ s} = \boxed{0.763 \text{ days}} \tag{l}$$

The time of flight of the bi-elliptic maneuver is over seven times longer than that of the Hohmann transfer.

6.5 PHASING MANEUVERS

A phasing maneuver is a two-impulse Hohmann transfer from and back to the same orbit, as illustrated in Figure 6.10. The Hohmann transfer ellipse is the phasing orbit with a period selected to return the spacecraft to the main orbit within a specified time. Phasing maneuvers are used to change the position of a spacecraft in its orbit. If two spacecraft, destined to rendezvous, are at different locations in the same orbit, then one of them may perform a phasing maneuver in order to catch the other one. Communications and weather satellites in geostationary earth orbit use phasing maneuvers to move to new locations above the

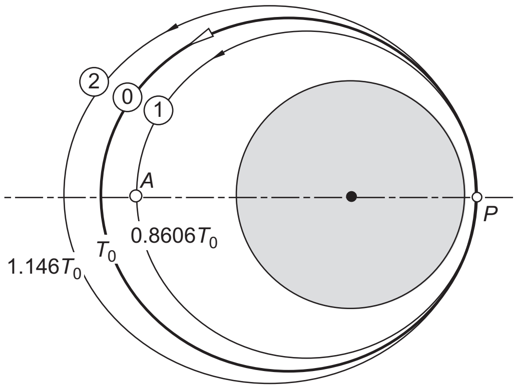


FIGURE 6.10

Main orbit (0) and two phasing orbits, faster (1) and slower (2). T_0 is the period of the main orbit.

equator. In that case, the rendezvous is with an empty point in space rather than with a physical target. In Figure 6.10, phasing orbit 1 might be used to return to P in less than one period of the main orbit. This would be appropriate if the target is ahead of the chasing vehicle. Note that a retrofire is required to enter orbit 1 at P. That is, it is necessary to slow the spacecraft down in order to speed it up, relative to the main orbit. If the chaser is ahead of the target, then phasing orbit 2 with its longer period might be appropriate. A forward fire of the thruster boosts the spacecraft's speed in order to slow it down.

Once the period T of the phasing orbit is established, then Equation 2.83 should be used to determine the semimajor axis of the phasing ellipse,

$$a = \left(\frac{T\sqrt{\mu}}{2\pi}\right)^{\frac{2}{3}} \tag{6.5}$$

With the semimajor axis established, the radius of point A opposite to P is obtained from the fact that $2a = r_P + r_A$. Equation 6.2 may then be used to obtain the angular momentum.

Example 6.4

Spacecraft at A and B are in the same orbit (1). At the instant shown in Figure 6.11 the chaser vehicle at A executes a phasing maneuver so as to catch the target spacecraft back at A after just one revolution of the chaser's phasing orbit (2). What is the required total delta-v?

Solution

We must find the angular momenta of orbits 1 and 2 so that we can use Equation 2.31 to find the velocities on orbits 1 and 2 at point A. (We can alternatively use energy, Equation 2.81, to find the speeds at A.) These velocities furnish the delta-v required to leave orbit 1 for orbit 2 at the beginning of the phasing maneuver and to return to orbit 1 at the end.

Angular momentum of orbit 1

From Figure 6.11 we observe that perigee and apogee radii of orbit 1 are, respectively,

$$r_A = 6800 \text{ km} \qquad r_C = 13{,}600 \text{ km}$$

It follows from Equation 6.2 that the orbit's angular momentum is

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_A r_C}{r_A + r_C}} = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{6800 \cdot 13{,}600}{6800 + 13{,}600}} = 60{,}116 \text{ km/s}^2$$

Angular momentum of orbit 2

The phasing orbit must have a period T_2 equal to the time it takes the target vehicle at B to coast around to point A on orbit 1. That flight time equals the period of orbit 1 minus the flight time t_{AB} from A to B. That is,

$$T_2 = T_1 - t_{AB} \tag{a}$$

The period of orbit 1 is found by computing its semimajor axis,

$$a_1 = \frac{1}{2}(r_A + r_C) = 10{,}200 \text{ km}$$

and substituting that result into Equation 2.83,

$$T_1 = \frac{2\pi}{\sqrt{\mu}} a_1^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} 10{,}200^{3/2} = 10{,}252 \text{ s} \tag{b}$$

The flight time from the perigee A of orbit 1 to point B is obtained from Kepler's equation (Equations 3.8 and 3.14),

$$t_{AB} = \frac{T_1}{2\pi}(E_B - e_1 \sin E_B) \tag{c}$$

Since the eccentricity of orbit 1 is

$$e_1 = \frac{r_C - r_A}{r_C + r_A} = 0.33333 \tag{d}$$

and the true anomaly of B is 90°, it follows from Equation 3.13b that the eccentric anomaly of B is

$$E_B = 2\tan^{-1}\left(\sqrt{\frac{1 - e_1}{1 + e_1}} \tan\frac{\theta_B}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1 - 0.33333}{1 + 0.33333}} \tan\frac{90°}{2}\right) = 1.2310 \text{ rad.} \tag{e}$$

Substituting (b), (d) and (e) into (c) yields

$$t_{AB} = \frac{10{,}252}{2\pi}(1.231 - 0.33333 \cdot \sin 1.231) = 1495.7 \text{ s}$$

It follows from (a) that

$$T_2 = 10{,}252 - 1495.7 = 8756.3 \text{ s}$$

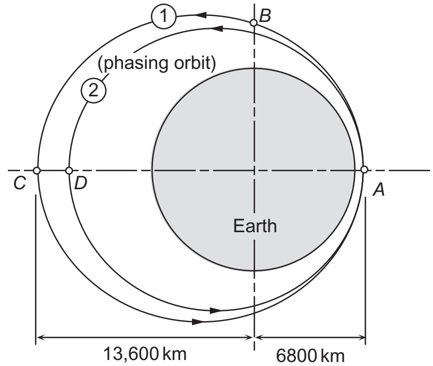


FIGURE 6.11

Phasing maneuver.

This, together with the period formula, Equation 2.83, yields the semimajor axis of orbit 2,

$$a_2 = \left(\frac{\sqrt{\mu}T_2}{2\pi}\right)^{2/3} = \left(\frac{\sqrt{398{,}600}\cdot 8756.2}{2\pi}\right)^{2/3} = 9182.1 \text{ km}$$

Since $2a_2 = r_A + r_D$, we find that the apogee of orbit 2 is

$$r_D = 2a_2 - r_A = 2\cdot 9182.1 - 6800 = 11{,}564 \text{ km}$$

Finally, Equation 6.2 yields the angular momentum of orbit 2,

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_D}{r_A + r_D}} = \sqrt{2\cdot 398{,}600}\sqrt{\frac{6800\cdot 11{,}564}{6800 + 11{,}564}} = 58{,}426 \text{ km/s}^2$$

Velocities at A

Since *A* is the perigee of orbit 1, there is no radial velocity component there. The speed, directed entirely in the transverse direction, is found from the angular momentum formula,

$$v_A)_1 = \frac{h_1}{r_A} = \frac{60{,}116}{6800} = 8.8406 \text{ km/s}$$

Likewise, the speed at the perigee of orbit 2 is

$$v_A)_2 = \frac{h_2}{r_A} = \frac{58{,}426}{6800} = 8.5921 \text{ km/s}$$

At the beginning of the phasing maneuver, the velocity change required to drop into the phasing orbit 2 is

$$\Delta v_A = v_A)_2 - v_A)_1 = 8.5921 - 8.8406 = -0.24851 \text{ km/s}$$

At the end of the phasing maneuver, the velocity change required to return to orbit 1 is

$$\Delta v_A = v_A)_1 - v_A)_2 = 8.8406 - 8.5921 = 0.24851 \text{ km/s}$$

The total delta-v required for the chaser to catch up with the target is

$$\Delta v_{\text{total}} = |-0.24851| + |0.24851| = \boxed{0.4970 \text{ km/s}}$$

The delta-v requirement for a phasing maneuver can be lowered by reducing the difference between the period of the main orbit and that of the phasing orbit. In the previous example, we could make Δv_{total} smaller by requiring the chaser to catch the target after n revolutions of the phasing orbit instead of just one. In that case, we would replace Equation (a) of Example 6.4 by $T_2 = T_1 - t_{AB}/n$.

Example 6.5

It is desired to shift the longitude of a GEO satellite 12° westward in three revolutions of its phasing orbit. Calculate the delta-v requirement.

Solution

This problem is illustrated in Figure 6.12. It may be recalled from Equations 2.67, 2.68 and 2.69 that the angular velocity of the earth, the radius to GEO and the speed in GEO are, respectively

$$\begin{aligned} \omega_E &= \omega_{\text{GEO}} = 72.922 \times 10^{-6} \text{ rad/s} \\ r_{\text{GEO}} &= 42{,}164 \text{ km} \\ v_{\text{GEO}} &= 3.0747 \text{ km/s} \end{aligned} \tag{a}$$

Let $\Delta\Lambda$ be the change in longitude in radians. Then the period T_2 of the phasing orbit can be obtained from the following formula,

$$\omega_E(3T_2) = 3 \cdot 2\pi + \Delta\Lambda \tag{b}$$

which states that after three circuits of the phasing orbit, the original position of the satellite will be $\Delta\Lambda$ radians east of P. In other words, the satellite will end up $\Delta\Lambda$ radians west of its original position in GEO, as desired. From (b) we obtain,

$$T_2 = \frac{1}{3}\frac{\Delta\Lambda + 6\pi}{\omega_E} = \frac{1}{3}\frac{12° \cdot \dfrac{\pi}{180°} + 6\pi}{72.922 \times 10^{-6}} = 87{,}121 \text{ s}$$

Note that the period of GEO is

$$T_{\text{GEO}} = \frac{2\pi}{\omega_{\text{GEO}}} = 86{,}163 \text{ s}$$

The satellite in its slower phasing orbit appears to drift westward at the rate

$$\dot{\Lambda} = \frac{\Delta\Lambda}{3T_2} = 8.0133 \times 10^{-7} \text{ rad/s} = 3.9669 \text{ degrees/day}$$

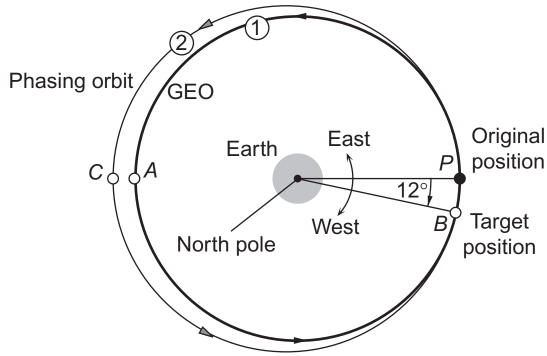


FIGURE 6.12

GEO repositioning.

Having the period, we can use Equation 6.5 to obtain the semimajor axis of orbit 2,

$$a_2 = \left(\frac{T_2\sqrt{\mu}}{2\pi}\right)^{2/3} = \left(\frac{87{,}121\sqrt{398{,}600}}{2\pi}\right)^{2/3} = 42{,}476 \text{ km}$$

From this we find the radius to the apogee C of the phasing orbit,

$$2a_2 = r_P + r_C \Rightarrow r_C = 2\cdot 42{,}476 - 42{,}164 = 42{,}788 \text{ km}$$

The angular momentum of the orbit is given by Equation 6.2

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = \sqrt{2\cdot 398{,}600}\sqrt{\frac{42{,}164\cdot 42{,}788}{42{,}164 + 42{,}788}} = 130{,}120 \text{ km}^2/\text{s}$$

At P the speed in orbit 2 is

$$v_P)_2 = \frac{130{,}120}{42{,}164} = 3.0859 \text{ km/s}$$

Therefore, at the beginning of the phasing orbit,

$$\Delta v = v_P)_2 - v_{\text{GEO}} = 3.0859 - 3.0747 = 0.01126 \text{ km/s}$$

At the end of the phasing maneuver,

$$\Delta v = v_{\text{GEO}} - v_P)_2 = 3.0747 - 3.08597 = -0.01126 \text{ km/s}$$

Therefore,

$$\Delta v_{\text{total}} = |0.01126| + |-0.01126| = \boxed{0.02252 \text{ km/s}}$$

6.6 NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

Figure 6.13 illustrates a transfer between two coaxial, coplanar elliptical orbits in which the transfer trajectory shares the apse line but is not necessarily tangent to either the initial or target orbit. The problem is to determine whether there exists such a trajectory joining points *A* and *B*, and, if so, to find the total delta-v requirement.

r_A and r_B are given, as are the true anomalies θ_A and θ_B. Because of the common apse line assumption, θ_A and θ_B are the true anomalies of points *A* and *B* on the transfer orbit as well. Applying the orbit equation to *A* and *B* on the transfer orbit yields

$$r_A = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_A}$$
$$r_B = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_B}$$

Solving these two equations for *e* and *h*, we get

$$e = -\frac{r_A - r_B}{r_A\cos\theta_A - r_B\cos\theta_B} \tag{6.6a}$$

$$h = \sqrt{\mu r_A r_B}\sqrt{\frac{\cos\theta_A - \cos\theta_B}{r_A\cos\theta_A - r_B\cos\theta_B}} \tag{6.6b}$$

With these, the transfer orbit is determined and the velocity may be found at any true anomaly. Note that for a Hohmann transfer, in which $\theta_A = 0$ and $\theta_B = \pi$, Equations 6.6 become

$$e = \frac{r_B - r_A}{r_B + r_A} \qquad h = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} \qquad \text{(Hohmann transfer)} \tag{6.7}$$

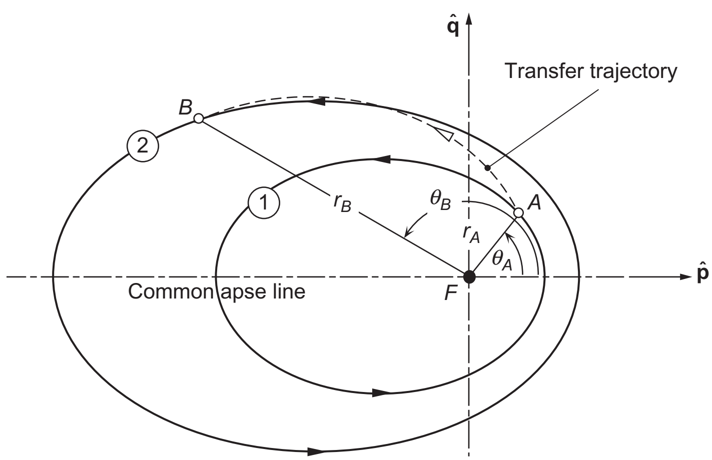


FIGURE 6.13

Non-Hohmann transfer between two coaxial elliptical orbits.

When a delta-v calculation is done for an impulsive maneuver at a point which is not on the apse line, care must be taken to include the change in direction as well as the magnitude of the velocity vector. Figure 6.14 shows a point where an impulsive maneuver changes the velocity vector from $\mathbf{v}_1$ on orbit 1 to $\mathbf{v}_2$ on coplanar orbit 2. The difference in length of the two vectors shows the change in the speed, and the difference in the flight path angles γ_2 and γ_1 indicates the change in the direction. It is important to observe that the Δv we seek is the magnitude of the change in the velocity vector, not the change in its magnitude (speed). That is, from Equation 1.11,

$$\Delta v = \|\Delta\mathbf{v}\| = \sqrt{(\mathbf{v}_2 - \mathbf{v}_1)\cdot(\mathbf{v}_2 - \mathbf{v}_1)}$$

Expanding under the radical we get

$$\Delta v = \sqrt{\mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_2 - 2\mathbf{v}_1\cdot\mathbf{v}_2}$$

Again, according to Equation 1.11, $\mathbf{v}_1\cdot\mathbf{v}_1 = v_1^2$ and $\mathbf{v}_2\cdot\mathbf{v}_2 = v_2^2$. Furthermore, since $\gamma_2 - \gamma_1$ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, Equation 1.7 implies that

$$\mathbf{v}_1\cdot\mathbf{v}_2 = v_1 v_2 \cos\Delta\gamma$$

where $\Delta\gamma = \gamma_2 - \gamma_1$. Therefore, the formula for Δv ***without plane change is***

$$\boxed{\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\Delta\gamma}} \quad \text{(impulsive maneuver, coplanar orbits)} \qquad (6.8)$$

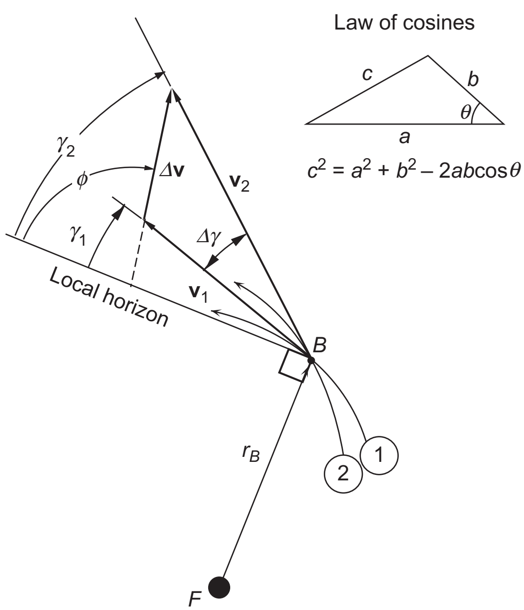


FIGURE 6.14

Vector diagram of the change in velocity and flight path angle at the intersection of two orbits (plus a reminder of the law of cosines).

This is just the familiar law of cosines from trigonometry. Only if $\Delta\gamma = 0$, which means that $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel (as in a Hohmann transfer), is it true that $\Delta v = |v_2 - v_1|$. If $v_2 = v_1 = v$, then Equation 6.8 yields

$$\Delta v = v\sqrt{2(1 - \cos\Delta\gamma)} \text{ (Pure rotation of the velocity vector in the orbital plane)} \tag{6.9}$$

Therefore, fuel expenditure is required to change the direction of the velocity even if its magnitude remains the same.

The direction of $\Delta\mathbf{v}$ shows the required alignment of the thruster that produces the impulse. The orientation of $\Delta\mathbf{v}$ relative to the local horizon is found by replacing v_r and $v_\perp$ in Equation 2.51 by Δv_r and $\Delta v_\perp$, so that

$$\tan\phi = \frac{\Delta v_r}{\Delta v_\perp} \tag{6.10}$$

where ϕ is the angle from the local horizon to the $\Delta\mathbf{v}$ vector.

Finally, recall the formula for specific mechanical energy of an orbit, Equation 2.57,

$$\varepsilon = \frac{\mathbf{v}\cdot\mathbf{v}}{2} - \frac{\mu}{r} \qquad (v^2 = \mathbf{v}\cdot\mathbf{v})$$

An impulsive maneuver results in a change of orbit and, therefore, a change in the specific energy ε. If the expenditure of propellant Δm is negligible compared to the initial mass m_1 of the vehicle, then $\Delta\varepsilon = \varepsilon_2 - \varepsilon_1$. For the situation illustrated in Figure 6.14,

$$\varepsilon_1 = \frac{v_1^{\,2}}{2} - \frac{\mu}{r_P}$$

and

$$\varepsilon_2 = \frac{(\mathbf{v}_1 + \Delta\mathbf{v})\cdot(\mathbf{v}_1 + \Delta\mathbf{v})}{2} - \frac{\mu}{r_P} = \frac{v_1^{\,2} + 2\mathbf{v}_1\cdot\Delta\mathbf{v} + \Delta v^2}{2} - \frac{\mu}{r_P}$$

Hence

$$\Delta\varepsilon = \mathbf{v}_1\cdot\Delta\mathbf{v} + \frac{\Delta v^2}{2}$$

From Figure 6.14 and Equation 1.7 it is apparent that $\mathbf{v}_1\cdot\Delta\mathbf{v} = v_1\Delta v\cos\Delta\gamma$. so that

$$\Delta\varepsilon = v_1\Delta v\cos\Delta\gamma + \frac{\Delta v^2}{2} = v_1\Delta v\left(\cos\Delta\gamma + \frac{1}{2}\frac{\Delta v}{v_1}\right)$$

For consistency with our assumption that $\Delta m \ll m_1$, it must be true (recall Figure 6.1) that $\Delta v \ll v_1$. It follows that

$$\Delta\varepsilon \approx v_1\Delta v\cos\Delta\gamma \tag{6.11}$$

This shows that, for a given Δv, the change in specific energy is larger the faster the spacecraft is moving (unless, of course, the change in flight path angle is 90°). The larger the $\Delta\varepsilon$ associated with a given Δv, the more efficient the maneuver. As we know, a spacecraft has its greatest speed at periapsis.

Example 6.6

A geocentric satellite in orbit 1 of Figure 6.15 executes a delta-v maneuver at A which places it on orbit 2, for re-entry at D. Calculate Δv at A and its direction relative to the local horizon.

Solution

As usual, the strategy is to first obtain the eccentricity and angular momentum for each of the orbits involved. From the figure we see that

$$r_B = 20{,}000 \text{ km} \qquad r_C = 10{,}000 \text{ km} \qquad r_D = 6378 \text{ km}$$

Orbit 1:

The eccentricity is

$$e_1 = \frac{r_B - r_C}{r_B + r_C} = 0.33333$$

The angular momentum is obtained from Equation 6.2, noting that point C is perigee:

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{20{,}000 \cdot 10{,}000}{20{,}000 + 10{,}000}} = 72{,}902 \text{ km}^2/\text{s}$$

With the angular momentum and the eccentricity, we can use the orbit equation to find the radial coordinate of point A,

$$r_A = \frac{72{,}902^2}{398{,}600}\frac{1}{1 + 0.33333 \cdot \cos 150^\circ} = 18{,}744 \text{ km}$$

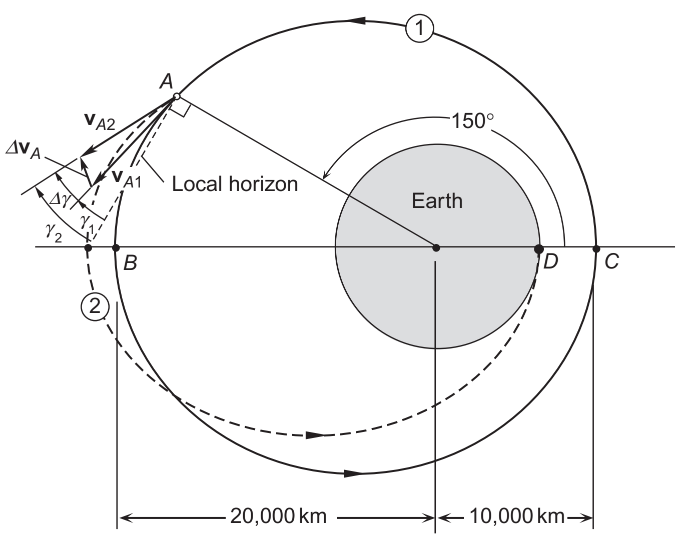


FIGURE 6.15

Non-Hohmann transfer with a common apse line.

Equations 2.31 and 2.49 yield the transverse and radial components of velocity at A on orbit 1,

$$v_{\perp_A})_1 = \frac{h_1}{r_A} = 3.8893 \text{ km/s}$$

$$v_{r_A})_1 = \frac{\mu}{h_1} e_1 \sin 150° = 0.91127 \text{ km/s}$$

From these we find the speed at A

$$v_A)_1 = \sqrt{v_{\perp_A})_1{}^2 + v_{r_A})_1{}^2} = 3.9946 \text{ km/s}$$

and the flight path angle,

$$\gamma_1 = \tan^{-1}\frac{v_{r_A})_1}{v_{\perp_A})_1} = \tan^{-1}\frac{0.91127}{3.8893} = 13.187°$$

Orbit 2:
The radius and true anomaly of points A and D on orbit 2 are known. From Equations 6.6 we find

$$e_2 = -\frac{r_D - r_A}{r_D \cos\theta_D - r_A \cos\theta_A} = -\frac{6378 - 18{,}744}{6378 \cos 0 - 18{,}744 \ \cos 150°} = 0.5469$$

$$h_2 = \sqrt{\mu r_A r_D}\sqrt{\frac{\cos\theta_D - \cos\theta_A}{r_D \cos\theta_D - r_A \cos\theta_A}} = \sqrt{398{,}600 \cdot 18{,}744 \cdot 6378}\sqrt{\frac{\cos 0 - \cos 150°}{6378 \cos 0 - 18{,}744 \cos 150°}}$$
$$= 62{,}711 \text{ km}^2/\text{s}$$

Now we can calculate the radial and perpendicular components of velocity on orbit 2 at point A.

$$v_{\perp_A})_2 = \frac{h_2}{r_A} = 3.3456 \text{ km/s}$$

$$v_{r_A})_2 = \frac{\mu}{h_2} e_2 \sin 150° = 1.7381 \text{ km/s}$$

Hence, the speed and flight path angle at A on orbit 2 are

$$v_A)_2 = \sqrt{v_{\perp_A})_2{}^2 + v_{r_A})_2{}^2} = 3.7702 \text{ km/s}$$

$$\gamma_2 = \tan^{-1}\frac{v_{r_A})_2}{v_{\perp_A})_2} = \tan^{-1}\frac{1.7381}{3.3456} = 27.453°$$

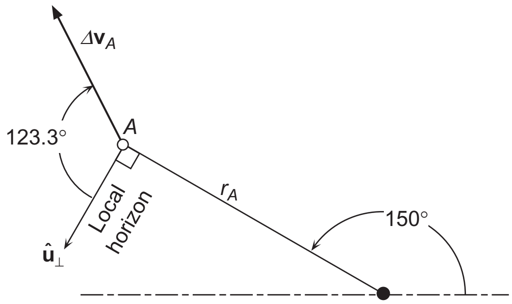


FIGURE 6.16

Orientation of $\Delta\mathbf{v}_A$ to the local horizon.

The change in the flight path angle as a result of the impulsive maneuver is

$$\Delta\gamma = \gamma_2 - \gamma_1 = 27.453° - 13.187° = 14.266°$$

With this we can use Equation 6.8 to finally obtain Δv_A,

$$\Delta v_A = \sqrt{v_A)_1^{\,2} + v_A)_2^{\,2} - 2v_A)_1\, v_A)_2 \cos\Delta\gamma} = \sqrt{3.9946^2 + 3.7702^2 - 2\cdot 3.9946\cdot 3.7702\cdot \cos 14.266°}$$

$$\boxed{\Delta v_A = 0.9896 \text{ km/s}}$$

Note that Δv_A is the magnitude of the change in velocity vector $\Delta\mathbf{v}_A$ at A. That is not the same as the change in the magnitude of the velocity (i.e., the change in speed), which is

$$v_A)_2 - v_A)_1 = 3.9946 - 3.7702 = 0.2244 \text{ km/s}$$

To find the orientation of $\Delta\mathbf{v}_A$, we use Equation 6.10,

$$\tan\phi = \frac{\Delta v_r)_A}{\Delta v_\perp)_A} = \frac{v_{r_A})_2 - v_{r_A})_1}{v_{\perp_A})_2 - v_{\perp_A})_1} = \frac{1.7381 - 0.9113}{3.3456 - 3.8893} = -1.5207$$

so that

$$\boxed{\phi = 123.3°}$$

This angle is illustrated in Figure 6.16. Prior to firing, the spacecraft would have to be rotated so that the centerline of the rocket motor coincides with the line of action of $\Delta\mathbf{v}_A$, with the nozzle aimed in the opposite direction.

6.7 APSE LINE ROTATION

Figure 6.17 shows two intersecting orbits which have a common focus, but their apse lines are not collinear. A Hohmann transfer between them is clearly impossible. The opportunity for transfer from one orbit to the other by a single impulsive maneuver occurs where they intersect, at points I and J in this case. As can be

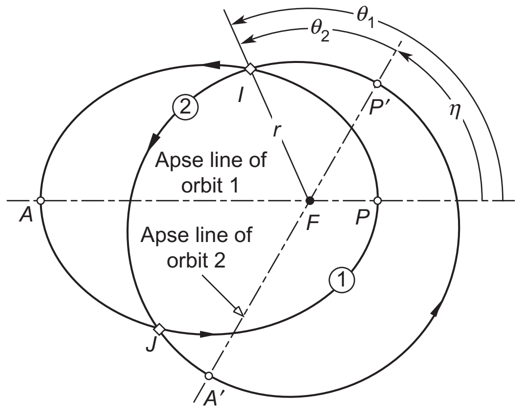


FIGURE 6.17

Two intersecting orbits whose apse lines do not coincide.

seen from the figure, the rotation η of the apse line is the difference between the true anomalies of the point of intersection, measured from periapsis of each orbit. That is,

$$\eta = \theta_1 - \theta_2 \tag{6.12}$$

We will consider two cases of apse line rotation.

The first case is that in which the apse line rotation η is given as well as the orbital parameters e and h of both orbits. The problem is then to find the true anomalies of I and J relative to both orbits. The radius of the point of intersection I is given by either of the following

$$r_I)_1 = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos\theta_1} \qquad r_I)_2 = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos\theta_2}$$

Since $r_I)_1 = r_I)_2$, we can equate these two expressions and rearrange terms to get

$$e_1 h_2^2\cos\theta_1 - e_2 h_1^2\cos\theta_2 = h_1^2 - h_2^2$$

Setting $\theta_2 = \theta_1 - \eta$ and using the trig identity $\cos(\theta_1 - \eta) = \cos\theta_1\cos\eta + \sin\theta_1\sin\eta$ leads to an equation for θ_1,

$$a\cos\theta_1 + b\sin\theta_1 = c \tag{6.13a}$$

where

$$a = e_1 h_2^2 - e_2 h_1^2\cos\eta \qquad b = -e_2 h_1^2\sin\eta \qquad c = h_1^2 - h_2^2 \tag{6.13b}$$

Equation (6.13a) has two roots (see Problem 3.12), corresponding to the two points of intersection I and J of the two orbits:

$$\theta_1 = \phi \pm \cos^{-1}\left(\frac{c}{a}\cos\phi\right) \tag{6.14a}$$

where

$$\phi = \tan^{-1}\frac{b}{a} \tag{6.14b}$$

Having found θ_1 we obtain θ_2 from Equation 6.12. Δv for the impulsive maneuver may then be computed as illustrated in the following example.

Example 6.7

An earth satellite is in a 8000 km by 16,000 km radius orbit (orbit 1 of Figure 6.18). Calculate the delta-v and the true anomaly θ_1 required to obtain a 7000 km by 21,000 km radius orbit (orbit 2) whose apse line is rotated 25° counterclockwise. Indicate the orientation ϕ of $\Delta\mathbf{v}$ to the local horizon.

Solution

First obtain the eccentricity and angular momentum of each orbit. The eccentricities of the two orbits are

$$\begin{aligned} e_1 &= \frac{r_{A_1} - r_{P_1}}{r_{A_1} + r_{P_1}} = \frac{16{,}000 - 8000}{16{,}000 + 8000} = 0.33333 \\ e_2 &= \frac{r_{A_2} - r_{P_2}}{r_{A_2} + r_{P_2}} = \frac{21{,}000 - 7000}{21{,}000 + 7000} = 0.5 \end{aligned} \tag{a}$$

The orbit equation yields the angular momenta

$$\begin{aligned} r_{P_1} &= \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow 8000 = \frac{h_1^2}{398{,}600}\frac{1}{1 + 0.33333} \Rightarrow h_1 = 65{,}205\ \text{km}^2/\text{s} \\ r_{P_2} &= \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos(0)} \Rightarrow 7000 = \frac{h_2^2}{398{,}600}\frac{1}{1 + 0.5} \Rightarrow h_2 = 64{,}694\ \text{km}^2/\text{s} \end{aligned} \tag{b}$$

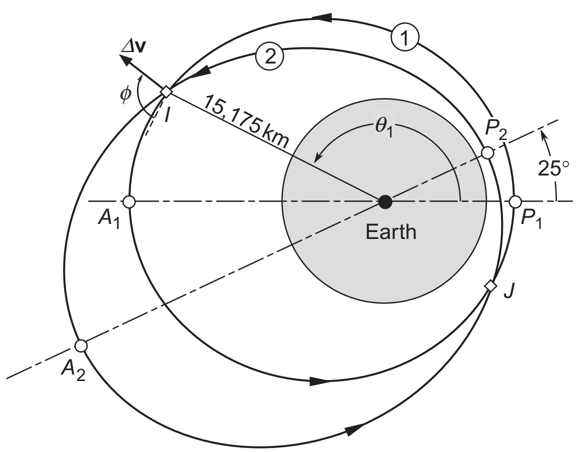


FIGURE 6.18

$\Delta\mathbf{v}$ produces a rotation of the apse line.

Using these orbital parameters and the fact that $\eta = 25°$, we calculate the terms in Equations 6.13b,

$$a = e_1 h_2^{\,2} - e_2 h_1^{\,2}\cos\eta = 0.3333\cdot 64{,}694^2 - 0.5\cdot 65{,}205^2\cdot\cos 25° = -5.3159\times 10^8\ \text{km}^4/\text{s}^2$$

$$b = -e_2 h_1^{\,2}\sin\eta = -0.5\cdot 65{,}205^2\sin 25° = -8.9843\times 10^8\ \text{km}^4/\text{s}^2$$

$$c = h_1^{\,2} - h_2^{\,2} = 65{,}205^2 - 64{,}694^2 = 6.6433\times 10^7\ \text{km}^4/\text{s}^2$$

Then Equations 6.14 yield

$$\phi = \tan^{-1}\frac{-8.9843\times 10^8}{-5.3159\times 10^8} = 59.39°$$

$$\theta_1 = 59.39° \pm \cos^{-1}\left(\frac{6.6433\times 10^7}{-5.3159\times 10^8}\cos 59.39°\right) = 59.39° \pm 93.65°$$

Thus, the true anomaly of point I, the point of interest, is

$$\theta_1 = 153.04° \tag{c}$$

(For point J, $\theta_1 = 325.74°$.)
With the true anomaly available, we can evaluate the radial coordinate of the maneuver point,

$$r = \frac{h_1^{\,2}}{\mu}\frac{1}{1 + e_1\cos 153.04°} = 15{,}175\ \text{km}$$

The velocity components and flight path angle for orbit 1 at point I are

$$v_{\perp_1} = \frac{h_1}{r} = \frac{65{,}205}{15{,}175} = 4.2968\ \text{km/s}$$

$$v_{r_1} = \frac{\mu}{h_1}e_1\sin 153.04° = \frac{398{,}600}{65{,}205}\cdot 0.33333\cdot\sin 153.04° = 0.92393\ \text{km/s}$$

$$\gamma_1 = \tan^{-1}\frac{v_{r_1}}{v_{\perp_1}} = 12.135°$$

The speed of the satellite in orbit 1 is, therefore,

$$v_1 = \sqrt{v_{r_1}^{\,2} + v_{\perp_1}^{\,2}} = 4.3950\ \text{km/s}$$

Likewise, for orbit 2,

$$v_{\perp_2} = \frac{h_2}{r} = \frac{64{,}694}{15{,}175} = 4.2631 \text{ km/s}$$

$$v_{r_2} = \frac{\mu}{h_2} e_2 \sin(153.04° - 25°) = \frac{398{,}600}{64{,}694} \cdot 0.5 \cdot \sin 128.04° = 2.4264 \text{ km/s}$$

$$\gamma_2 = \tan^{-1} \frac{v_{r_2}}{v_{\perp_2}} = 29.647°$$

$$v_2 = \sqrt{v_{r_2}^2 + v_{\perp_2}^2} = 4.9053 \text{ km/s}$$

Equation 6.8 is used to find Δv,

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos(\gamma_2 - \gamma_1)}$$

$$= \sqrt{4.3950^2 + 4.9053^2 - 2 \cdot 4.3950 \cdot 4.9053 \cos\left(29.647° - 12.135°\right)}$$

$$\boxed{\Delta v = 1.503 \text{ km/s}}$$

The angle ϕ which the vector $\Delta\mathbf{v}$ makes with the local horizon is given by Equation 6.10,

$$\phi = \tan^{-1} \frac{\Delta v_r}{\Delta v_\perp} = \tan^{-1} \frac{v_{r_2} - v_{r_1}}{v_{\perp_2} - v_{\perp_1}} = \tan^{-1} \frac{2.4264 - 0.92393}{4.2631 - 4.2968} = \boxed{91.28°}$$

The second case of apse line rotation is that in which the impulsive maneuver takes place at a given true anomaly θ_1 on orbit 1. The problem is to determine the angle of rotation η and the eccentricity e_2 of the new orbit.

The impulsive maneuver creates a change in the radial and transverse velocity components at point *I* of orbit 1. From the angular momentum formula, $h = rv_\perp$, we obtain the angular momentum of orbit 2,

$$h_2 = r(v_\perp + \Delta v_\perp) = h_1 + r\Delta v_\perp \tag{6.15}$$

The formula for radial velocity, $v_r = (\mu/h)e\sin\theta$, applied to orbit 2 at point *I*, where $v_{r_2} = v_{r_1} + \Delta v_r$ and $\theta_2 = \theta_1 - \eta$, yields

$$v_{r_1} + \Delta v_r = \frac{\mu}{h_2} e_2 \sin\theta_2$$

Substituting Equation 6.15 into this expression and solving for $\sin\theta_2$ leads to

$$\sin\theta_2 = \frac{1}{e_2} \frac{(h_1 + r\Delta v_\perp)(\mu e_1 \sin\theta_1 + h_1 \Delta v_r)}{\mu h_1} \tag{6.16}$$

From the orbit equation, we have at point I

$$r = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos\theta_1} \qquad \text{(orbit 1)}$$

$$r = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos\theta_2} \qquad \text{(orbit 2)}$$

Equating these two expressions for r, substituting Equation 6.15, and solving for $\cos\theta_2$, yields

$$\cos\theta_2 = \frac{1}{e_2}\frac{(h_1 + r\Delta v_\perp)^2 e_1\cos\theta_1 + (2h_1 + r\Delta v_\perp)r\Delta v_\perp}{h_1^2} \tag{6.17}$$

Finally, by substituting Equations 6.16 and 6.17 into the trigonometric identity $\tan\theta_2 = \sin\theta_2/\cos\theta_2$ we obtain a formula for θ_2 which does not involve the eccentricity e_2,

$$\tan\theta_2 = \frac{h_1}{\mu}\frac{(h_1 + r\Delta v_\perp)(\mu e_1\sin\theta_1 + h_1\Delta v_r)}{(h_1 + r\Delta v_\perp)^2 e_1\cos\theta_1 + (2h_1 + r\Delta v_\perp)r\Delta v_\perp} \tag{6.18a}$$

Equation 6.18a can be simplified a bit by replacing $\mu e_1\sin\theta_1$ with $h_1 v_{r_1}$ and h_1 with $rv_{\perp_1}$, so that

$$\tan\theta_2 = \frac{(v_{\perp_1} + \Delta v_\perp)(v_{r_1} + \Delta v_r)}{(v_{\perp_1} + \Delta v_\perp)^2 e_1\cos\theta_1 + (2v_{\perp_1} + \Delta v_\perp)\Delta v_\perp}\frac{v_{\perp_1}^2}{(\mu/r)} \tag{6.18b}$$

Equations 6.18 show how the apse line rotation, $\eta = \theta_1 - \theta_2$, is completely determined by the components of $\Delta\mathbf{v}$ imparted at the true anomaly θ_1. Notice that if $\Delta v_r = -v_{r_1}$, then $\theta_2 = 0$, which means that the maneuver point is on the apse line of the new orbit.

After solving Equation 6.18 (a or b), we substitute θ_2 into either Equation 6.16 or 6.17 to calculate the eccentricity e_2 of orbit 2. Therefore, with h_2 from Equation 6.15, the rotated orbit 2 is completely specified.

If the impulsive maneuver takes place at the periapsis of orbit 1, so that $\theta_1 = v_r = 0$, and if it is also true that $\Delta v_\perp = 0$, then Equation 6.18b yields

$$\tan\eta = -\frac{rv_{\perp_1}}{\mu e_1}\Delta v_r \qquad \text{(with radial impulse at periapsis)}$$

Thus, if the velocity vector is given an outward radial component at periapsis, then $\eta < 0$, which means the apse line of the resulting orbit is rotated clockwise relative to the original one. That makes sense, since having acquired $v_r > 0$ means the spacecraft is now flying away from its new periapsis. Likewise, applying an inward radial velocity component at periapsis rotates the apse line counterclockwise.

Example 6.8

An earth satellite in orbit 1 of Figure 6.19 undergoes the indicated delta-v maneuver at its perigee. Determine the rotation η of its apse line as well as the new perigee and apogee.

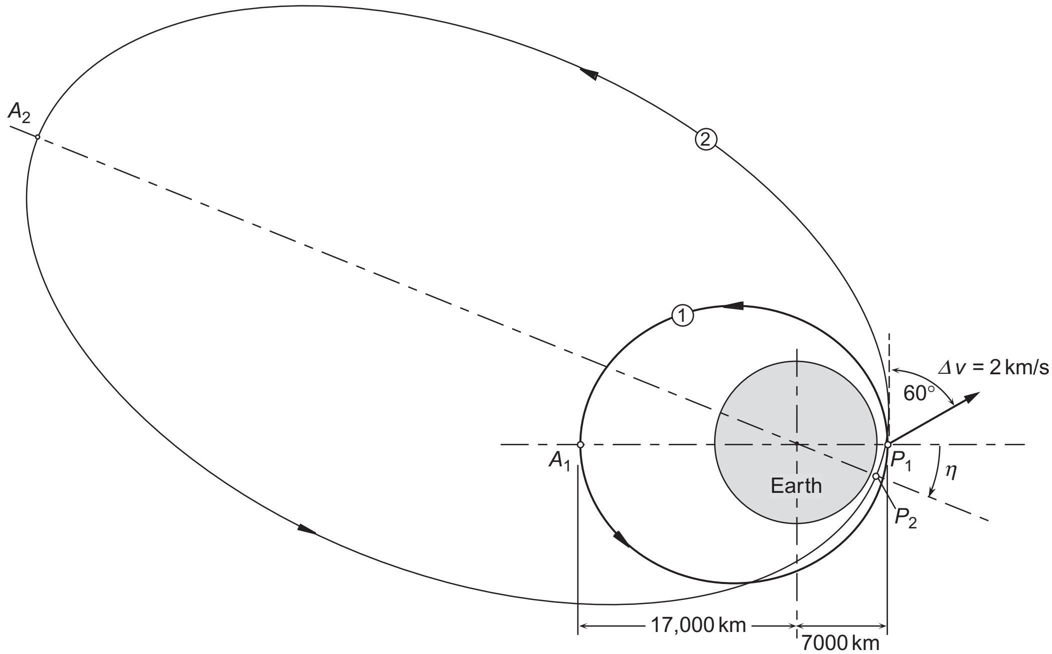


FIGURE 6.19

Apse line rotation maneuver.

Solution

First obtain the eccentricity and angular momentum of orbit 1. From the figure the apogee and perigee of orbit 1 are

$$r_{A_1} = 17{,}000 \text{ km} \qquad r_{P_1} = 7000 \text{ km}$$

Therefore, the eccentricity of orbit 1 is

$$e_1 = \frac{r_{A_1} - r_{P_1}}{r_{A_1} + r_{P_1}} = 0.41667 \tag{a}$$

As usual, we use the orbit equation to find the angular momentum,

$$r_{P_1} = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow 7000 = \frac{h_1^2}{398{,}600}\frac{1}{1 + 0.41667} \Rightarrow h_1 = 62{,}871 \text{ km}^2/\text{s}$$

At the maneuver point P_1, the angular momentum formula and the fact that P_1 is perigee of orbit 1 ($\theta = 0$) imply that

$$\begin{aligned} v_{\perp_1} &= \frac{h_1}{r_{P_1}} = \frac{62{,}871}{7000} = 8.9816 \text{ km/s} \\ v_{r_1} &= 0 \end{aligned} \tag{b}$$

From Figure 6.19 it is clear that

$$\begin{aligned} \Delta v_\perp &= \Delta v \cos 60^\circ = 1 \text{ km/s} \\ \Delta v_r &= \Delta v \sin 60^\circ = 1.7321 \text{ km/s} \end{aligned} \tag{c}$$

The angular momentum of orbit 2 is given by Equation 6.15.

$$h_2 = h_1 + r\Delta v_\perp = 62{,}871 + 7000 \cdot 1 = 69{,}871 \text{ km}^2/\text{s}$$

To compute the true anomaly θ_2, we use Equation 6.18b together with (a), (b) and (c):

$$\begin{aligned} \tan\theta_2 &= \frac{\left(v_{\perp_1} + \Delta v_\perp\right)\left(v_{r_1} + \Delta v_r\right)}{\left(v_{\perp_1} + \Delta v_\perp\right)^2 e_1 \cos\theta_1 + \left(2v_{\perp_1} + \Delta v_\perp\right)\Delta v_\perp} \frac{{v_{\perp_1}}^2}{\left(\mu / r_{P_1}\right)} \\ &= \frac{(8.9816 + 1)(0 + 1.7321)}{(8.9816 + 1)^2 \cdot 0.41667 \cdot \cos(0) + (2 \cdot 8.9816 + 1) \cdot 1} \cdot \frac{8.9816^2}{(398{,}600/7000)} \\ &= 0.4050 \end{aligned}$$

It follows that $\theta_2 = 22.047^\circ$, so that Equation 6.12 yields the apse line rotation angle:

$$\boxed{\eta = -22.05^\circ}$$

This means that the rotation of the apse line is clockwise, as indicated in Figure 6.19.
From Equation 6.17 we obtain the eccentricity of orbit 2,

$$\begin{aligned} e_2 &= \frac{\left(h_1 + r_{P_1}\Delta v_\perp\right)^2 e_1 \cos\theta_1 + \left(2h_1 + r_{P_1}\Delta v_\perp\right) r_{P_1}\Delta v_\perp}{{h_1}^2 \cos\theta_2} \\ &= \frac{(62{,}871 + 7000 \cdot 1)^2 \cdot 0.41667 \cdot \cos(0) + (2 \cdot 62{,}871 + 7000 \cdot 1) \cdot 7000 \cdot 1}{62{,}871^2 \cdot \cos 22.047^\circ} \\ &= 0.808830 \end{aligned}$$

With this and the angular momentum h_2 we find using the orbit equation that the perigee and apogee radii of orbit 2 are

$$\begin{aligned} r_{P_2} &= \frac{{h_2}^2}{\mu}\frac{1}{1 + e_2} = \frac{69{,}871^2}{398{,}600}\frac{1}{1 + 0.808830} = \boxed{6771.1 \text{ km}} \\ r_{A_2} &= \frac{69{,}871^2}{398.600}\frac{1}{1 - 0.808830} = \boxed{64{,}069 \text{ km}} \end{aligned}$$

6.8 CHASE MANEUVERS

Whereas Hohmann transfers and phasing maneuvers are leisurely, energy-efficient procedures that require some preconditions (e.g., coaxial elliptical, orbits) in order to work, a chase or intercept trajectory is one which answers the question, "How do I get from point A to point B in space in a given amount of time?" The nature of the orbit lies in the answer to the question rather than being prescribed at the outset. Intercept trajectories near a planet are likely to require delta-v's beyond the capabilities of today's technology, so they are largely of theoretical rather than practical interest. We might refer to them as "star wars maneuvers." Chase trajectories can be found as solutions to Lambert's problem (Section 5.3).

Example 6.9

Spacecraft B and C are both in the geocentric elliptical orbit (1) shown in Figure 6.20, from which it can be seen that the true anomalies are $\theta_B = 45°$ and $\theta_C = 150°$. At the instant shown, spacecraft B executes a delta-v maneuver, embarking upon a trajectory (2) which will intercept and rendezvous with vehicle C in precisely one hour. Find the orbital parameters (e and h) of the intercept trajectory and the total delta-v required for the chase maneuver.

Solution

First, we must determine the parameters e and h of orbit 1 in the usual way. The eccentricity is found using the orbit's perigee and apogee, shown in Figure 6.20,

$$e_1 = \frac{18{,}900 - 8100}{18{,}900 + 8100} = 0.4000$$

From Equation 6.2,

$$h_1 = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{8100 \cdot 18{,}900}{8100 + 18{,}900}} = 67{,}232\ \text{km}^2/\text{s}$$

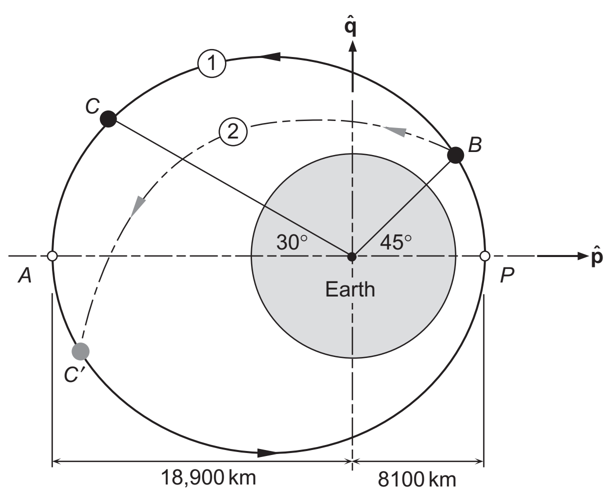


FIGURE 6.20

Intercept trajectory (2) required for B to catch C in one hour.

Using Equation 2.82 yields the period,

$$T_1 = \frac{2\pi}{\mu^2}\left(\frac{h_1}{\sqrt{1-e_1{}^2}}\right)^3 = \frac{2\pi}{398{,}600^2}\left(\frac{67{,}232}{\sqrt{1-0.4^2}}\right)^3 = 15{,}610 \text{ s}$$

In perifocal coordinates (Equation 2.119) the position vector of B is

$$\mathbf{r}_B = \frac{h_1{}^2}{\mu}\frac{1}{1+e_1\cos\theta_B}(\cos\theta_B\hat{\mathbf{p}} + \sin\theta_B\hat{\mathbf{q}}) = \frac{67{,}232^2}{398{,}600}\frac{1}{1+0.4\cos 45^\circ}(\cos 45^\circ\hat{\mathbf{p}} + \sin 45^\circ\hat{\mathbf{q}})$$

or

$$\mathbf{r}_B = 6250.6\hat{\mathbf{p}} + 6250.6\hat{\mathbf{q}} \text{ (km)} \tag{a}$$

Likewise, according to Equation 2.125, the velocity at B on orbit 1 is

$$\mathbf{v}_B)_1 = \frac{\mu}{h}[-\sin\theta_B\hat{\mathbf{p}} + (e+\cos\theta_B)\hat{\mathbf{q}}] = \frac{398{,}600}{67{,}232}[-\sin 45^\circ\hat{\mathbf{p}} + (0.4+\cos 45^\circ)\hat{\mathbf{q}}]$$

so that

$$\mathbf{v}_B)_1 = -4.1922\hat{\mathbf{p}} + 6.5637\hat{\mathbf{q}} \text{ (km/s)} \tag{b}$$

Now we need to move spacecraft C along orbit 1 to the position C' that it will occupy one hour later, when it will presumably be met by spacecraft B. To do that, we must first calculate the time since perigee passage at C. Since we know the true anomaly, the eccentric anomaly follows from Equation 3.13b,

$$E_C = 2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_C}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.4}{1+0.4}}\tan\frac{150^\circ}{2}\right) = 2.3646 \text{ rad}$$

Substituting this value into Kepler's equation (Equations 3.8 and 3.14) yields the time since perigee passage,

$$t_C = \frac{T_1}{2\pi}(E_C - e_1\sin E_C) = \frac{15{,}610}{2\pi}(2.3646 - 0.4\cdot\sin 2.3646) = 5178 \text{ s}$$

One hour later ($\Delta t = 3600$ s), the spacecraft will be in intercept position at C',

$$t_{C'} = t_C + \Delta t = 5178 + 3600 = 8778 \text{ s}$$

The corresponding mean anomaly is

$$M_e)_{C'} = 2\pi\frac{t_{C'}}{T_1} = 2\pi\frac{8778}{15{,}610} = 3.5331 \text{ rad}$$

With this value of the mean anomaly, Kepler's equation becomes

$$E_{C'} - e_1 \sin E_{C'} = 3.5331$$

Applying Algorithm 3.1 to the solution of this equation we get

$$E_{C'} = 3.4223 \text{ rad}$$

Substituting this result into Equation 3.13a yields the true anomaly at C',

$$\tan\frac{\theta_{C'}}{2} = \sqrt{\frac{1+0.4}{1-0.4}}\tan\frac{3.4223}{2} = -10.811 \Rightarrow \theta_{C'} = 190.57°$$

We are now able to calculate the perifocal position and velocity vectors at C' on orbit 1.

$$\begin{aligned}\mathbf{r}_{C'} &= \frac{67{,}232^2}{398{,}600}\frac{1}{1+0.4\cos 190.57°}(\cos 190.57°\,\hat{\mathbf{p}} + \sin 4190.57°\,\hat{\mathbf{q}})\\ &= -18372\hat{\mathbf{p}} - 3428.1\hat{\mathbf{q}} \text{ (km)}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_{C'})_1 &= \frac{398{,}600}{67{,}232}[-\sin 190.57°\,\hat{\mathbf{p}} + (0.4 + \cos 190.57°)\hat{\mathbf{q}}]\\ &= 1.0875\hat{\mathbf{p}} - 3.4566\hat{\mathbf{q}} \text{ (km/s)}\end{aligned} \quad \text{(c)}$$

The intercept trajectory connecting points B and C' is found by solving Lambert's problem. Substituting $\mathbf{r}_B$ and $\mathbf{r}_{C'}$ along with $\Delta t = 3600\,\text{s}$ into Algorithm 5.2 yields

$$\mathbf{v}_B)_2 = -8.1349\hat{\mathbf{p}} + 4.0506\hat{\mathbf{q}} \text{ (km/s)} \quad \text{(d)}$$

$$\mathbf{v}_{C'})_2 = -3.4745\hat{\mathbf{p}} + -4.7943\hat{\mathbf{q}} \text{ (km/s)} \quad \text{(e)}$$

These velocities are most easily obtained by running the following MATLAB® script, which executes Algorithm 5.2 by means of the function M-file *lambert.m* (Appendix D.25).

```
clear
global mu
deg      = pi/180;
mu       = 398600;
e        = 0.4;
h        = 67232;
theta1   = 45*deg;
theta2   = 190.57*deg;
delta_t  = 3600;
rB       = h^2/mu/(1 + e*cos(theta1)). . .
                     *[cos(theta1),sin(theta1),0],
rc_prime = h^2/mu/(1 + e*cos(theta2)). . .
                     *[cos(theta2),sin(theta2),0];
string   =  'pro';
[vB2 vC_prime_2] = lambert(rB, rC_prime, delta_t, string)
```

From (b) and (d) we find

$$\Delta\mathbf{v}_B = \mathbf{v}_B)_2 - \mathbf{v}_B)_1 = -3.9326\hat{\mathbf{p}} - 2.5132\hat{\mathbf{q}} \text{ (km/s)}$$

whereas (c) and (e) yield

$$\Delta\mathbf{v}_{C'} = \mathbf{v}_{C'})_1 - \mathbf{v}_{C'})_2 = -4.5620\hat{\mathbf{p}} + 1.3376\hat{\mathbf{q}} \text{ (km/s)}$$

The anticipated, extremely large delta-v requirement for this chase maneuver is the sum of the magnitudes of these two vectors,

$$\Delta v = \|\Delta\mathbf{v}_B\| + \|\Delta\mathbf{v}_{C'}\| = 4.6755 + 4.7540 = \boxed{9.430 \text{ km/s}}$$

We know that orbit 2 is an ellipse, because the magnitude of $\mathbf{v}_B)_2$ (9.088 km/s) is less than the escape speed $\left(\sqrt{2\mu/r_B} = 9.496 \text{ km/s}\right)$ at B. To pin it down a bit more, we can use $\mathbf{r}_B$ and $\mathbf{v}_B)_2$ to obtain the orbital elements from Algorithm 4.2, which yields

$$\boxed{h_2 = 76{,}167 \text{ km}^2/\text{s}}$$
$$\boxed{e_2 = 0.8500}$$
$$a_2 = 52{,}449 \text{ km}$$
$$\theta_{B2} = 319.52°$$

These may be found quickly by running the following MATLAB script, in which the M-function *coe_from_sv.m* implements Algorithm 4.2 (see Appendix D.18):

```
clear
global mu
mu  = 398600;
rB  = [6250.6    6250.6   0];
vB2 = [-8.1349   4.0506   0];
orbital_elements = coe_from_sv(rB, vB2);
```

The details of the intercept trajectory and the delta-v maneuvers are shown in Figure 6.21. A far less dramatic though more leisurely (and realistic) way for B to catch up with C would be to use a phasing maneuver.

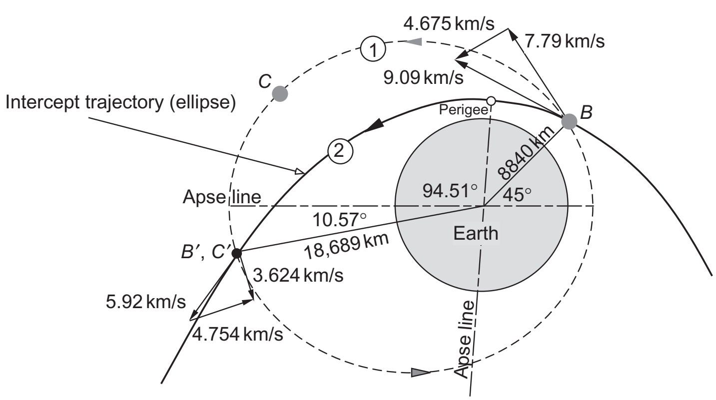


FIGURE 6.21

Details of the large elliptical orbit, a portion of which serves as the intercept trajectory.

6.9 PLANE CHANGE MANEUVERS

Orbits having a common focus F need not, and generally do not, lie in a common plane. Figure 6.22 shows two such orbits and their line of intersection BD. A and P denote the apoapses and periapses. Since the common focus lies in every orbital plane, it must lie on the line of intersection of any two orbits. For a spacecraft in orbit 1 to change its plane to that of orbit 2 by means of a single delta-v maneuver (cranking maneuver), it must do so when it is on the line of intersection of the orbital planes. Those two opportunities occur only at points B and D in Figure 6.22(a).

A view down the line of intersection, from B towards D, is shown in Figure 6.22(b). Here we can see in true view the dihedral angle δ between the two planes. The transverse component of velocity $\mathbf{v}_\perp$ at B is evident in this perspective, whereas the radial component $\mathbf{v}_r$, lying as it does on the line of intersection, is normal to the view plane (thus appearing as a dot). It is apparent that changing the plane of orbit 1 requires simply rotating $\mathbf{v}_\perp$ around the intersection line, through the dihedral angle. If $v_\perp$ and v_r remain unchanged in the process, then we have a rigid body rotation of the orbit. That is, except for its new orientation in space, the orbit remains unchanged. If the magnitudes of $\mathbf{v}_r$ and $\mathbf{v}_\perp$ change in the process, then the rotated orbit acquires a new size and shape.

To find the delta-v associated with a plane change, let $\mathbf{v}_1$ be the velocity before and $\mathbf{v}_2$ the velocity after the impulsive maneuver. Then

$$\begin{aligned}\mathbf{v}_1 &= v_{r_1}\hat{\mathbf{u}}_r + v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}\\ \mathbf{v}_2 &= v_{r_2}\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2}\end{aligned}$$

where $\hat{\mathbf{u}}_r$ is the radial unit vector directed along the line of intersection of the two orbital planes. $\hat{\mathbf{u}}_r$ does not change during the maneuver. As we know, the transverse unit vector $\hat{\mathbf{u}}_\perp$ is perpendicular to $\hat{\mathbf{u}}_r$ and lies in the orbital plane. Therefore, it rotates through the dihedral angle δ from its initial orientation $\hat{\mathbf{u}}_{\perp_1}$ to its final orientation $\hat{\mathbf{u}}_{\perp_2}$.

The change $\Delta\mathbf{v}$ in the velocity vector is

$$\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = (v_{r_2} - v_{r_1})\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2} - v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}$$

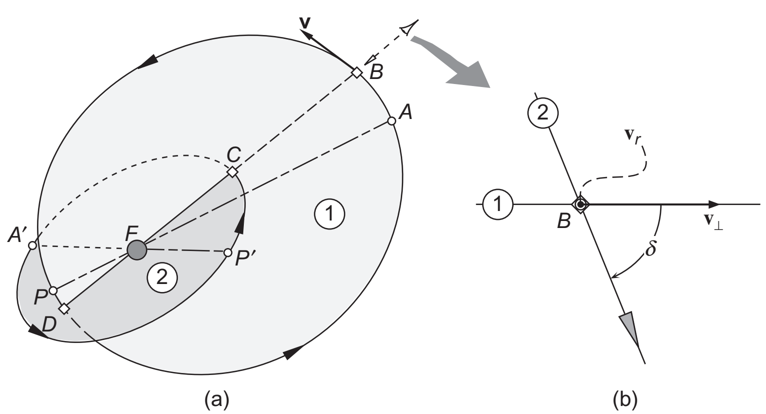


FIGURE 6.22

(a) Two noncoplanar orbits about F. (b) A view down the line of intersection of the two orbital planes.

The magnitude Δv is found by taking the dot product of $\Delta \mathbf{v}$ with itself,

$$\Delta v^2 = \Delta \mathbf{v} \cdot \Delta \mathbf{v} = \left[\left(v_{r_2} - v_{r_1}\right)\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2} - v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}\right] \cdot \left[\left(v_{r_2} - v_{r_1}\right)\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2} - v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}\right]$$

Carrying out the dot products while noting that $\hat{\mathbf{u}}_r \cdot \hat{\mathbf{u}}_r = \hat{\mathbf{u}}_{\perp_1} \cdot \hat{\mathbf{u}}_{\perp_1} = \hat{\mathbf{u}}_{\perp_2} \cdot \hat{\mathbf{u}}_{\perp_2} = 1$ and $\hat{\mathbf{u}}_r \cdot \hat{\mathbf{u}}_{\perp_1} = \hat{\mathbf{u}}_r \cdot \hat{\mathbf{u}}_{\perp_2} = 0$, yields

$$\Delta v^2 = \left(v_{r_2} - v_{r_1}\right)^2 + {v_{\perp_1}}^2 + {v_{\perp_2}}^2 - 2v_{\perp_1}v_{\perp_2}(\hat{\mathbf{u}}_{\perp_1} \cdot \hat{\mathbf{u}}_{\perp_2})$$

But $\hat{\mathbf{u}}_{\perp_1} \cdot \hat{\mathbf{u}}_{\perp_2} = \cos\delta$, so that we finally obtain a general formula for Δv ***with plane change***,

$$\boxed{\Delta v = \sqrt{\left(v_{r_2} - v_{r_1}\right)^2 + {v_{\perp_1}}^2 + {v_{\perp_2}}^2 - 2v_{\perp_1}v_{\perp_2}\cos\delta}} \tag{6.19}$$

From the definition of the flight path angle (cf. Figure 2.12),

$$v_{r_1} = v_1 \sin\gamma_1 \qquad v_{\perp_1} = v_1 \cos\gamma_1$$
$$v_{r_2} = v_2 \sin\gamma_2 \qquad v_{\perp_2} = v_2 \cos\gamma_2$$

Substituting these relations into Equation 6.19, expanding and collecting terms, and using the trigonometric identities

$$\sin^2\gamma_1 + \cos^2\gamma_1 = \sin^2\gamma_2 + \cos^2\gamma_2 = 1$$
$$\cos(\gamma_2 - \gamma_1) = \cos\gamma_2\cos\gamma_1 + \sin\gamma_2\sin\gamma_1$$

leads to another version of the same equation,

$$\Delta v = \sqrt{{v_1}^2 + {v_2}^2 - 2v_1v_2[\cos\Delta\gamma - \cos\gamma_2\cos\gamma_1(1 - \cos\delta)]} \tag{6.20}$$

where $\Delta\gamma = \gamma_2 - \gamma_1$. If there is no plane change ($\delta = 0$), then $\cos\delta = 1$ and Equation 6.20 reduces to

$$\Delta v = \sqrt{{v_1}^2 + {v_2}^2 - 2v_1v_2\cos\Delta\gamma} \qquad \text{No plane change}$$

which is the cosine law we have been using to compute Δv in coplanar maneuvers. Therefore, not surprisingly, Equation 6.19 contains Equation 6.8 as a special case.

To keep Δv at a minimum, it is clear from Equation 6.19 that the radial velocity should remain unchanged during a plane change maneuver. For the same reason, it is apparent that the maneuver should occur where $v_\perp$ is smallest, which is at apoapsis. Figure 6.23 illustrates a plane change maneuver at the apoapsis of both orbits. In this case $v_{r_1} = v_{r_2} = 0$, so that $v_{\perp_1} = v_1$ and $v_{\perp_2} = v_2$, thereby reducing Equation 6.19 to

$$\Delta v = \sqrt{{v_1}^2 + {v_2}^2 - 2v_1v_2\cos\delta} \qquad \text{Rotation about the common apse line} \tag{6.21}$$

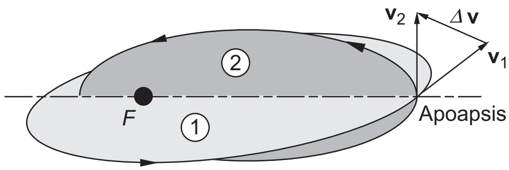


FIGURE 6.23

Impulsive plane change maneuver at apoapsis.

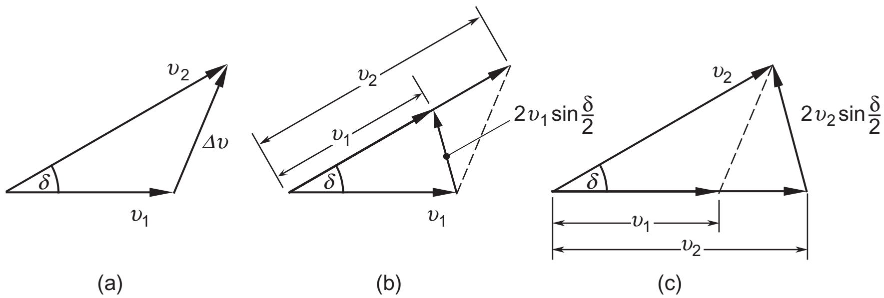


FIGURE 6.24

Orbital plane rotates about the common apse line. (a) Speed change accompanied by plane change. (b) Plane change followed by speed change. (c) Speed change followed by plane change.

Equation 6.21 is for a speed change accompanied by a plane change, as illustrated in Figure 6.24a. Using the trigonometric identity

$$\cos\delta = 1 - 2\sin^2\frac{\delta}{2}$$

we can rewrite Equation 6.21 as follows,

$$\Delta v_{\mathrm{I}} = \sqrt{(v_2 - v_1)^2 + 4v_1 v_2 \sin^2\frac{\delta}{2}} \qquad \text{Rotation about the common apse line} \tag{6.22}$$

If there is no change in the speed, so that $v_2 = v_1$, then Equation 6.22 yields

$$\Delta v_\delta = 2v\sin\frac{\delta}{2} \qquad \text{Pure rotation of the velocity vector} \tag{6.23}$$

The subscript δ reminds us that this is the delta-v for a pure rotation of the velocity vector through the angle δ.

Another plane-change strategy, illustrated in Figure 6.24b, is to rotate the velocity vector and then change its magnitude. In that case, the delta-v is

$$\Delta v_{\mathrm{II}} = 2v_1 \sin\frac{\delta}{2} + |v_2 - v_1|$$

Yet another possibility is to change the speed first, and then rotate the velocity vector (Figure 6.24c). Then

$$\Delta v_{\rm III} = |v_2 - v_1| + 2v_2 \sin\frac{\delta}{2}$$

Since the sum of the lengths of any two sides of a triangle must be greater than the length of the third side, it is evident from Figure 6.24 that both $\Delta v_{\rm II}$ and $\Delta v_{\rm III}$ are greater than $\Delta v_{\rm I}$. It follows that plane change accompanied by speed change is the most efficient of the above three maneuvers.

Equation 6.23, the delta-v formula for pure rotation of the velocity vector, is plotted in Figure 6.25, which shows why significant plane changes are so costly in terms of propellant expenditure. For example, a plane change of just 24° requires a delta-v equal to that needed for an escape trajectory (41.4% velocity boost). A 60° plane change requires a delta-v equal to the speed of the spacecraft itself, which in earth orbit operations is about 7.5 km/s. For such a maneuver in LEO, the most efficient chemical propulsion system would require that well over 80% of the spacecraft mass consist of propellant. The space shuttle is capable of a plane change in orbit of only about 3°, a maneuver which would exhaust its entire fuel capacity. Orbit plane adjustments are therefore made during the powered ascent phase when the energy is available to do so.

For some missions, however, plane changes must occur in orbit. A common example is the maneuvering of GEO satellites into position. These must orbit the earth in the equatorial plane, but it is impossible to throw a satellite directly into an equatorial orbit from a launch site which is not on the equator. That is not difficult to understand when we realize that the plane of the orbit must contain the center of the earth (the focus) as well as the point at which the satellite is inserted into orbit, as illustrated in Figure 6.26. So if the insertion point is anywhere but on the equator, the plane of the orbit will be tilted away from the earth's equator. As we know from Chapter 4, the angle between the equatorial plane and the plane of the orbiting satellite is called the inclination i.

Launching a satellite due east takes full advantage of the earth's rotational velocity, which is 0.46 km/s (about 1000 miles per hour) at the equator and diminishes towards the poles according to the formula

$$v_{\rm rotational} = v_{\rm equatorial} \cos\phi$$

where ϕ is the latitude. Figure 6.26, shows a spacecraft launched due east into low earth orbit at a latitude of 28.6° north, which is the latitude of Kennedy Space Flight Center (KSC). As can be seen from the figure, the

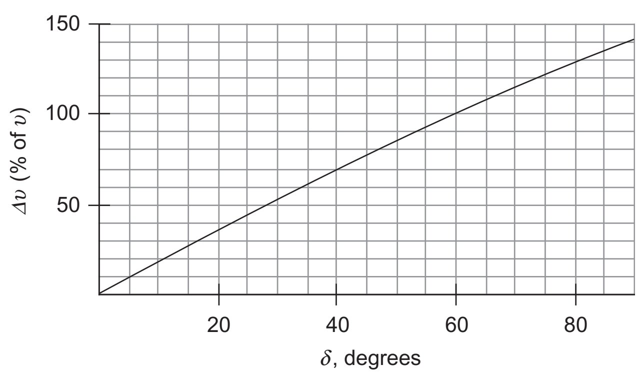


FIGURE 6.25

Δv required to rotate the velocity vector through an angle δ.

inclination of the orbit will be 28.6°. One-fourth of the way around the earth the satellite will cross the equator. Halfway around the earth it reaches its southernmost latitude, $\phi = 28.6°$ south. It then heads north, crossing over the equator at the three-quarters point and returning after one complete revolution to $\phi = 28.6°$ north.

Launch azimuth A is the flight direction at insertion, measured clockwise from north on the local meridian. Thus, $A = 90°$ is due east. If the launch direction is not directly eastward, then the orbit will have an inclination greater than the launch latitude, as illustrated in Figure 6.27 for $\phi = 28.6°$N. Northeasterly ($0 < A < 90°$) or southeasterly ($90° < A < 180°$) launches take only partial advantage of the earth's rotational speed and both produce an inclination i greater than the launch latitude but less than 90°. Since these orbits have an eastward velocity component, they are called prograde orbits. Launches to the west produce retrograde orbits with inclinations between 90° and 180°. Launches directly north or directly south result in polar orbits.

Spherical trigonometry is required to obtain the relationship between orbital ***inclination*** i, launch platform ***latitude*** ϕ, and launch ***azimuth*** A. It turns out that

$$\cos i = \cos\phi \sin A \tag{6.24}$$

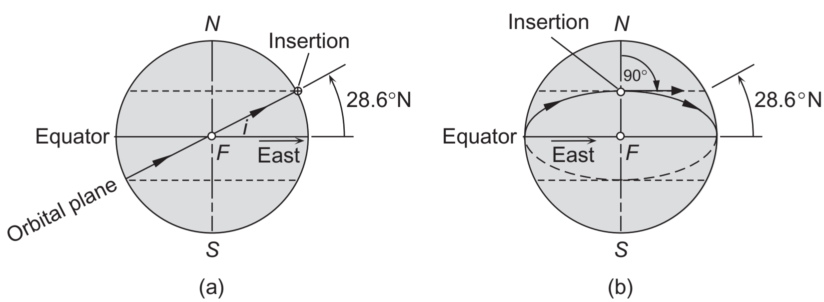


FIGURE 6.26

Two views of the orbit of a satellite launched directly east at 28.6° north latitude. (a) Edge-on view of the orbital plane. (b) View towards insertion point meridian.

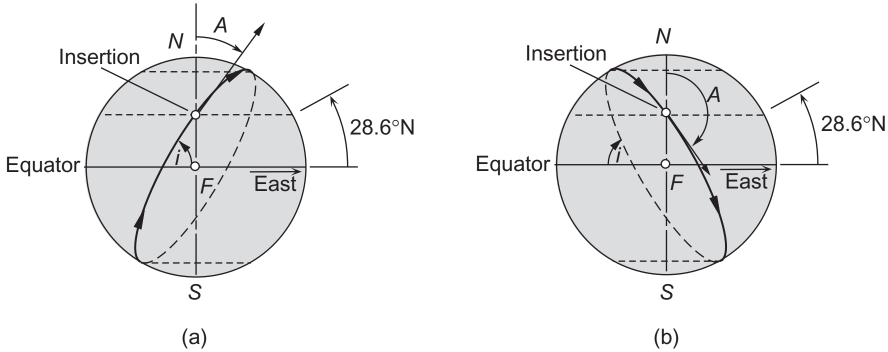


FIGURE 6.27

(a) Northeasterly launch ($0 < A < 90°$) from a latitude of 28.6°N. (b) Southeasterly launch ($90° < A < 270$).

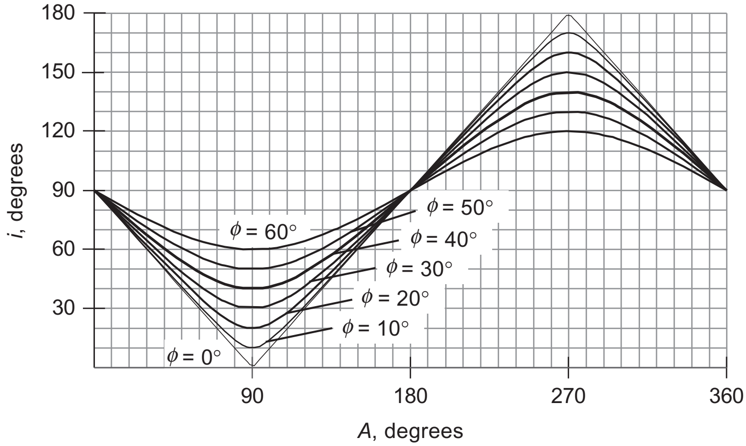


FIGURE 6.28

Orbit inclination i versus launch azimuth A for several latitudes ϕ.

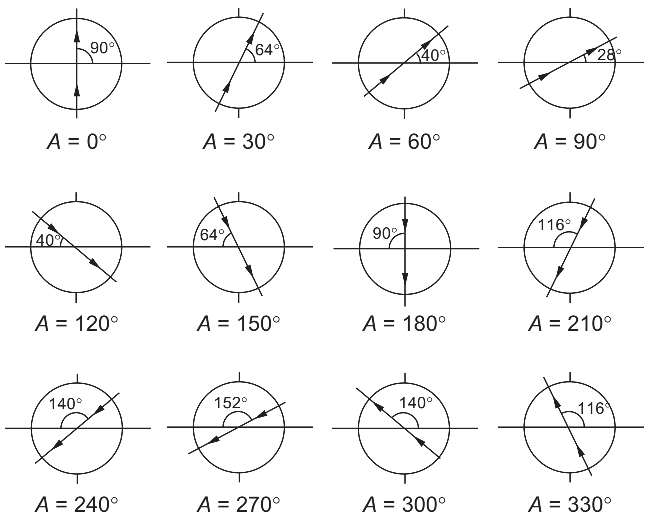


FIGURE 6.29

Variation of orbit inclinations with launch azimuth at $\phi = 28°$. Note the retrograde orbits for $A > 180°$.

From this we verify, for example, that $i = \phi$ when $A = 90°$, as pointed out above. A plot of this relation is presented in Figure 6.28, while Figure 6.29 illustrates the orientation of orbits for a range of launch azimuths at $\phi = 28°$.

Example 6.10

Determine the required launch azimuth for the sun-synchronous satellite of Example 4.9 if it is launched from Vandenburgh AFB on the California coast (latitude = 34.5°N).

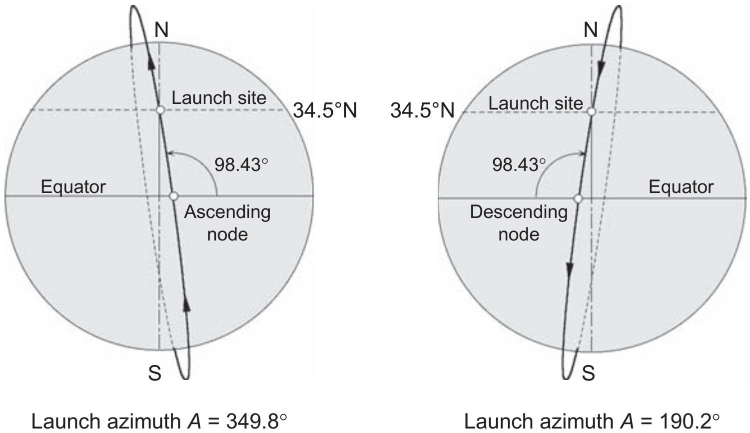


FIGURE 6.30

Effect of launch azimuth on the position of the orbit.

Solution

In Example 4.9 the inclination of the sun-synchronous orbit was determined to be 98.43°. Equation 6.24 is used to calculate the launch azimuth,

$$\sin A = \frac{\cos i}{\cos l} = \frac{\cos 98.43°}{\cos 34.5°} = -0.1779$$

From this, $A = 190.2°$, a launch to the south or $A = 349.8°$, a launch to the north.

Figure 6.30 shows the effect that the choice of launch azimuth has on the orbit. It does not change the fact that the orbit is retrograde; it simply determines whether the ascending node will be in the same hemisphere as the launch site or on the opposite side of the earth. Actually, a launch to the north from Vandenburgh is not an option because of the safety hazard to the populated land lying below the ascent trajectory. Launches to the south, over open water, are not a hazard. Working this problem for Kennedy Space Center (latitude 28.6°N) yields nearly the same values of A. Since safety considerations on the Florida east coast limit launch azimuths to between 35° and 120°, polar and sun-synchronous satellites cannot be launched from the eastern test range.

Example 6.11

Find the delta-v required to transfer a satellite from a circular, 300 km altitude low earth orbit of 28° inclination to a geostationary equatorial orbit. Circularize and change the inclination at altitude. Compare that delta-v requirement with the one in which the plane change is done in the low earth orbit.

Solution

Figure 6.31 shows the 28° inclined low-earth parking orbit (1), the coplanar Hohmann transfer ellipse (2), and the coplanar GEO orbit (3). From the figure we see that

$$r_B = 6678 \text{ km} \qquad r_C = 42{,}164 \text{ km}$$

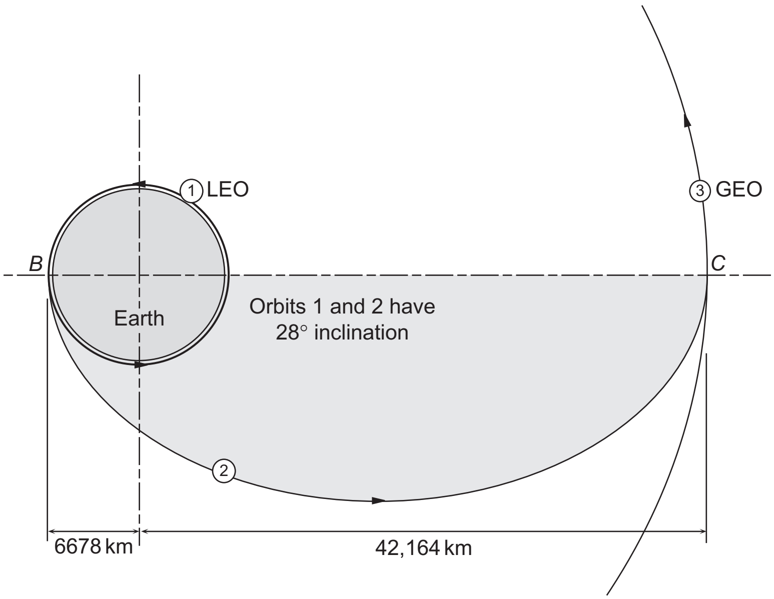


FIGURE 6.31

Transfer from LEO to GEO in an orbit of 28° inclination.

Orbit 1:

For this circular orbit the speed at B is

$$v_B)_1 = \sqrt{\frac{\mu}{r_B}} = \sqrt{\frac{398{,}600}{6678}} = 7.7258 \text{ km/s}$$

Orbit 2:

We first obtain the angular momentum by means of Equation 6.2,

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = 67{,}792 \text{ km/s}$$

The velocities at perigee and apogee of orbit 2 are, from the angular momentum formula,

$$v_B)_2 = \frac{h_2}{r_B} = 10.152 \text{ km/s} \quad v_C)_2 = \frac{h_2}{r_C} = 1.6078 \text{ km/s}$$

At this point we can calculate Δv_B,

$$\Delta v_B = v_B)_2 - v_B)_1 = 10.152 - 7.7258 = 2.4258 \text{ km/s}$$

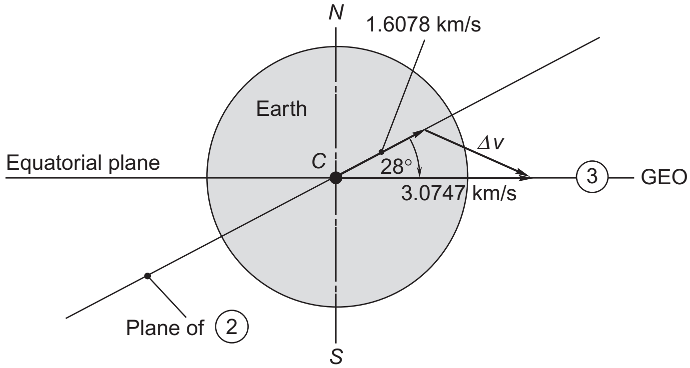


FIGURE 6.32

Plane change maneuver required after the Hohmann transfer.

Orbit 3:

For this GEO orbit, which is circular, the speed at *C* is

$$v_C)_3 = \sqrt{\frac{\mu}{r_C}} = 3.0747 \text{ km/s}$$

The spacecraft in orbit 2 arrives at *C* with a velocity of 1.6078 km/s inclined at 28° to orbit 3. Therefore, both its orbital speed and inclination must be changed at *C*. The most efficient strategy is to combine the plane change with the speed change (Equation 6.21), so that

$$\begin{aligned}\Delta v_C &= \sqrt{v_C)_2{}^2 + v_C)_3{}^2 - 2v_C)_2\, v_C)_3 \cos \Delta i} \\ &= \sqrt{1.6078^2 + 3.0747^2 - 2 \cdot 1.6078 \cdot 3.0747 \cdot \cos 28°} = 1.8191 \text{ km/s}\end{aligned}$$

Therefore, the total delta-v requirement is

$$\Delta v_{\text{total}} = \Delta v_B + \Delta v_C = 2.4258 + 1.8191 = \boxed{4.2449 \text{ km/s}} \quad \text{(plane change at } C\text{)}$$

Suppose we make the plane change at LEO instead of at GEO. In that case, Equation 6.21 provides the initial delta-v,

$$\begin{aligned}\Delta v_B &= \sqrt{v_B)_1{}^2 + v_B)_2{}^2 - 2v_B)_1\, v_B)_2 \cos \Delta i} \\ &= \sqrt{7.7258^2 + 10.152^2 - 2 \cdot 7.7258 \cdot 10.152 \cdot \cos 28°} = 4.9242 \text{ km/s}\end{aligned}$$

The spacecraft travels to *C* in the equatorial plane, so that when it arrives, the delta-v requirement at *C* is simply

$$\Delta v_C = v_C)_3 - v_C)_2 = 3.0747 - 1.6078 = 1.4668 \text{ km/s}$$

Therefore, the total delta-v is

$$\Delta v_{\text{total}} = \Delta v_B + \Delta v_C = 4.9242 + 1.4668 = \boxed{6.3910\ \text{km/s}} \qquad \text{(plane change at } B\text{)}$$

This is a 50% increase over the total delta-v with plane change at GEO. Clearly, it is best to do plane change maneuvers at the largest possible distance (apoapsis) from the primary attractor, where the velocities are smallest.

Example 6.12

Suppose in the previous example that part of the plane change, Δi, takes place at B, the perigee of the Hohmann transfer ellipse, and the remainder, $28° - \Delta i$ occurs at the apogee C. What is the value of Δi which results in the minimum Δv_{total}?

Solution

We found in Example 6.11 that if $\Delta i = 0$, then $\Delta v_{\text{total}} = 4.2449\,\text{km/s}$, whereas $\Delta i = 28°$ made $\Delta v_{\text{total}} = 6.3910\,\text{km/s}$. Here we are to determine if there is a value of Δi between 0 and 28° that yields a Δv_{total} which is smaller than either of those two.

In this case a plane change occurs at both B and C. Recall that the most efficient strategy is to combine the plane change with the speed change, so that the delta-v's at those points are (Equation 6.21)

$$\begin{aligned}
\Delta v_B &= \sqrt{\left.v_B\right)_1^2 + \left.v_B\right)_2^2 - 2\left.v_B\right)_1 \left.v_B\right)_2 \cos\Delta i} \\
&= \sqrt{7.7258^2 + 10.152^2 - 2\cdot 7.7258\cdot 10.152\cdot\cos\Delta i} \\
&= \sqrt{162.74 - 156.86\cos\Delta i}
\end{aligned}$$

and

$$\begin{aligned}
\Delta v_C &= \sqrt{\left.v_C\right)_2^2 + \left.v_C\right)_3^2 - 2\left.v_C\right)_2 \left.v_C\right)_3 \cos(28° - \Delta i)} \\
&= \sqrt{1.6078^2 + 3.0747^2 - 2\cdot 1.6078\cdot 3.0747\cdot\cos(28° - \Delta i)} \\
&= \sqrt{12.039 - 9.8871\cos(28° - \Delta i)}
\end{aligned}$$

Thus,

$$\Delta v_{\text{total}} = \Delta v_B + \Delta v_C = \sqrt{162.74 - 156.86\cos\Delta i} + \sqrt{12.039 - 9.8871\cos(28° - \Delta i)} \qquad \text{(a)}$$

To determine if there is a Δi which minimizes Δv_{total}, we take its derivative with respect to Δi and set it equal to zero:

$$\frac{d\Delta v_{\text{total}}}{d\Delta i} = \frac{78.43\sin\Delta i}{\sqrt{162.74 - 156.86\cos\Delta i}} - \frac{4.9435\sin(28° - \Delta i)}{\sqrt{12.039 - 9.8871\cos(28° - \Delta i)}} = 0$$

This is a transcendental equation which must be solved iteratively. The solution, as the reader may verify, is

$$\boxed{\Delta i = 2.1751°} \qquad \text{(b)}$$

That is, an inclination change of 2.1751° should occur in low earth orbit, while the rest of the plane change, 25.825°, is done at GEO. Substituting (b) into (a) yields

$$\boxed{\Delta v_{\text{total}} = 4.2207 \text{ km/s}}$$

This is only very slightly smaller (less than 1%) than the lowest Δv_{total} computed in Example 6.11.

Example 6.13

A spacecraft is in a 500 km by 10,000 km altitude geocentric orbit which intersects the equatorial plane at a true anomaly of 120° (see Figure 6.33). If the inclination to the equatorial plane is 15°, what is the minimum velocity increment required to make this an equatorial orbit?

Solution

The orbital parameters are

$$e = \frac{r_A - r_P}{r_A + r_P} = \frac{(6378 + 10{,}000) - (6378 + 500)}{(6378 + 10{,}000) + (6378 + 500)} = 0.4085$$

$$h = \sqrt{2\mu}\sqrt{\frac{r_A r_P}{r_A + r_P}} = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{16{,}378 \cdot 6878}{16{,}378 + 6878}} = 62{,}141 \text{ km}^2/\text{s}$$

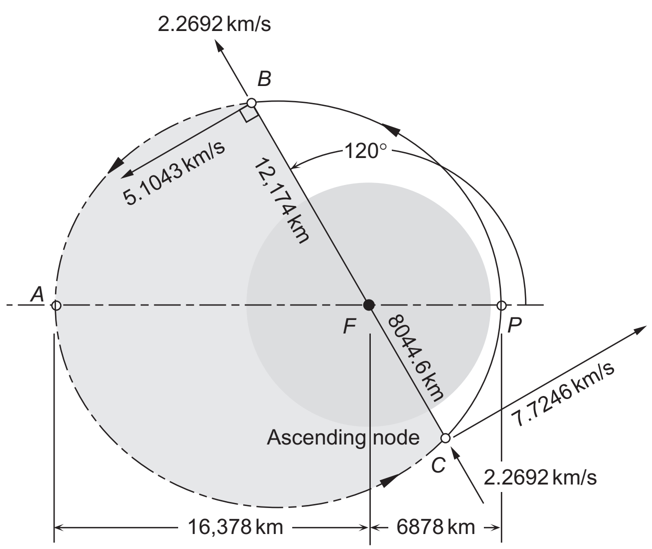


FIGURE 6.33

An orbit which intersects the equatorial plane along line *BC*. The equatorial plane makes an angle of 15° with the plane of the page.

The radial coordinate and velocity components at points B and C, on the line of intersection with the equatorial plane, are

$$r_B = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_B} = \frac{62{,}141^2}{398{,}600}\frac{1}{1+0.4085\cdot\cos 120^\circ} = 12{,}174 \text{ km}$$

$$v_{\perp B} = \frac{h}{r_B} = \frac{62{,}141}{12{,}174} = 5.1043 \text{ km/s}$$

$$v_{rB} = \frac{\mu}{h}e\sin\theta_B = \frac{398{,}600}{62{,}141}\cdot 0.4085\cdot\sin 120^\circ = 2.2692 \text{ km/s}$$

and

$$r_C = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_C} = \frac{62{,}141^2}{398{,}600}\frac{1}{1+0.4085\cdot\cos 300^\circ} = 8044.6 \text{ km}$$

$$v_{\perp C} = \frac{h}{r_C} = \frac{62{,}141}{8044.6} = 7.7246 \text{ km/s}$$

$$v_{rC} = \frac{\mu}{h}e\sin\theta_C = \frac{398{,}600}{62{,}141}\cdot 0.4085\cdot\sin 300^\circ = -2.2692 \text{ km/s}$$

These are all shown in Figure 6.33.

All we wish to do here is rotate the plane of the orbit rigidly around the node line BC. The impulsive maneuver must occur at either B or C. Equation 6.19 applies, and since the radial and transverse velocity components remain fixed, it reduces to

$$\Delta v = v_\perp\sqrt{2(1-\cos\delta)} = 2v_\perp\sin\frac{\delta}{2}$$

where $\delta = 15^\circ$. For the minimum Δv, the maneuver must be done where $v_\perp$ is smallest, which is at B, the point farthest from the center of attraction F. Thus,

$$\Delta v = 2\cdot 5.1043\cdot\sin\frac{15^\circ}{2} = \boxed{1.3325 \text{ km/s}}$$

Example 6.14

Orbit 1 has angular momentum h and eccentricity e. The direction of motion is shown. Calculate the Δv required to rotate the orbit 90° about its latus rectum BC without changing h and e. The required direction of motion in orbit 2 is shown in Figure 6.34.

By symmetry, the required maneuver may occur at either B or C, and it involves a rigid body rotation of the ellipse, so that v_r and $v_\perp$ remain unaltered. Because of the directions of motion shown, the true anomalies of B on the two orbits are

$$\theta_B)_1 = -90^\circ \qquad \theta_B)_2 = +90^\circ$$

The radial coordinate of B is

$$r_B = \frac{h^2}{\mu}\frac{1}{1 + e\cos(\pm 90)} = \frac{h^2}{\mu}$$

For the velocity components at B, we have

$$v_{\perp_B}\Big)_1 = v_{\perp_B}\Big)_2 = \frac{h}{r_B} = \frac{\mu}{h}$$

$$v_{r_B}\Big)_1 = \frac{\mu}{h}e\sin\theta_B)_1 = -\frac{\mu e}{h} \qquad v_{r_B}\Big)_2 = \frac{\mu}{h}e\sin\theta_B)_2 = \frac{\mu e}{h}$$

Substituting these into Equation 6.19 yields

$$\begin{aligned}\Delta v_B &= \sqrt{\left[v_{r_B}\Big)_2 - v_{r_B}\Big)_1\right]^2 + v_{\perp_B}\Big)_1^2 + v_{\perp_B}\Big)_2^2 - 2v_{\perp_B}\Big)_1 v_{\perp_B}\Big)_2 \cos 90^\circ} \\ &= \sqrt{\left[\frac{\mu e}{h} - \left(-\frac{\mu e}{h}\right)\right]^2 + \left(\frac{\mu}{h}\right)^2 + \left(\frac{\mu}{h}\right)^2 - 2\left(\frac{\mu}{h}\right)\left(\frac{\mu}{h}\right)\cdot 0} \\ &= \sqrt{4\frac{\mu^2}{h^2}e^2 + 2\frac{\mu^2}{h^2}}\end{aligned}$$

so that

$$\boxed{\Delta v_B = \frac{\sqrt{2}\mu}{h}\sqrt{1 + 2e^2}} \qquad \text{(a)}$$

If the motion on ellipse 2 were opposite to that shown in Figure 6.34, then the radial velocity components at B (and C) would be in the same rather than in the opposite direction on both ellipses, so that instead of (a) we would find a smaller velocity increment,

$$\Delta v_B = \frac{\sqrt{2}\mu}{h}$$

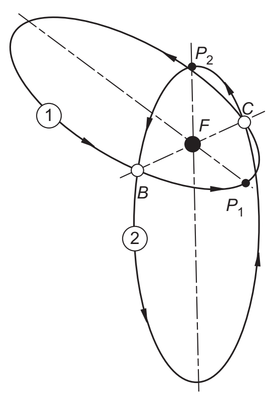


FIGURE 6.34

Identical ellipses intersecting at 90° along their common latus rectum, BC.

6.10 NONIMPULSIVE ORBITAL MANEUVERS

Up to this point we have assumed that delta-v maneuvers take place in zero time, altering the velocity vector but leaving the position vector unchanged. In nonimpulsive maneuvers the thrust acts over a significant time interval and must be included in the equations of motion. According to Problem 2.3, adding an external force **F** to the spacecraft yields the following equation of relative motion

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} + \frac{\mathbf{F}}{m} \tag{6.25}$$

where m is the mass of the spacecraft. This of course reduces to Equation 2.22 when $\mathbf{F} = \mathbf{0}$. If the external force is a thrust T in the direction of the velocity vector **v**, then $\mathbf{F} = T(\mathbf{v}/v)$ and Equation 6.25 becomes

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} + \frac{T}{m}\frac{\mathbf{v}}{v} \qquad (\mathbf{v} = \dot{\mathbf{r}}) \tag{6.26}$$

(Drag forces act opposite to the velocity vector, and so does thrust during a retrofire maneuver.) The Cartesian component form of Equation 6.26 is

$$\ddot{x} = -\mu\frac{x}{r^3} + \frac{T}{m}\frac{\dot{x}}{v} \qquad \ddot{y} = -\mu\frac{y}{r^3} + \frac{T}{m}\frac{\dot{y}}{v} \qquad \ddot{z} = -\mu\frac{z}{r^3} + \frac{T}{m}\frac{\dot{z}}{v} \tag{6.27a}$$

where

$$r = \sqrt{x^2 + y^2 + z^2} \qquad v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{6.27b}$$

While the rocket motor is firing, the spacecraft mass decreases, because propellant combustion products are being discharged into space through the nozzle. According to elementary rocket dynamics (cf. Section 11.3), the mass decreases at a rate given by the formula

$$\frac{dm}{dt} = -\frac{T}{I_{sp}g_o} \tag{6.28}$$

where T and I_{sp} are the thrust and the specific impulse of the propulsion system. g_o is the sea-level acceleration of gravity.

If the thrust is not zero, then Equations 6.27 may not have a straightforward analytical solution. In any case, they can be solved numerically using methods such as those discussed in Section 1.8. For that purpose, Equations 6.27 and 6.28 must be rewritten as a system of linear differential equations in the form

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{6.29}$$

For the case at hand, the vector **y** consists of the six components of the state vector (position and velocity vectors) plus the mass. Therefore, with the aid of Equations 6.27 and 6.28, we have

$$\mathbf{y} = \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ m \end{Bmatrix} \qquad \dot{\mathbf{y}} = \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{m} \end{Bmatrix} \qquad \mathbf{f}(t,\mathbf{y}) = \begin{Bmatrix} y_4 \\ y_5 \\ y_6 \\ -\mu\dfrac{y_1}{r^3} + \dfrac{T}{m}\dfrac{y_4}{v} \\ -\mu\dfrac{y_2}{r^3} + \dfrac{T}{m}\dfrac{y_5}{v} \\ -\mu\dfrac{y_3}{r^3} + \dfrac{T}{m}\dfrac{y_6}{v} \\ -\dfrac{T}{I_{sp}g_o} \end{Bmatrix} \tag{6.30}$$

The numerical solution of Equations 6.30 is illustrated in the following examples.

Example 6.15

Suppose the spacecraft in Example 6.1 (see Figure 6.3) has a restartable onboard propulsion system with a thrust of 10 kN and specific impulse of 300 s. Assuming that the thrust vector remains aligned with the velocity vector, solve Example 6.1 without using impulsive (zero time) delta-v burns. Compare the propellant expenditures for the two solutions.

Solution

Refer to Figure 6.3 as an aid to visualizing the solution procedure described below. Let us assume that the plane of Figure 6.3 is the *xy* plane of an earth-centered inertial frame with the *z*-axis directed out of the page. The apse line of orbit 1 is the *x*-axis, which is directed to the right, and *y* points upwards towards the top of the page.

Transfer from perigee of orbit 1 to apogee of orbit 2

According to Example 6.1, the state vector just before the first delta-v maneuver is

$$\mathbf{y}_0 = \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ m \end{Bmatrix}_{t=0} = \begin{Bmatrix} 6858\text{ km} \\ 0 \\ 0 \\ 0 \\ 7.7102\text{ km/s} \\ 0 \\ 2000\text{ kg} \end{Bmatrix} \tag{a}$$

Using this together with an assumed burn time t_{burn}, we numerically integrate Equations 6.29 from $t = 0$ to $t = t_{\text{burn}}$. This yields **r**, **v** and the mass m at the start of the coasting trajectory (orbit 2). We can find the true anomaly θ at the start of orbit 2 by substituting these values of **r** and **v** into Algorithm 4.2. The spacecraft must coast through a true anomaly of $\Delta\theta = 180° - \theta$ in order to reach apogee. Substituting **r**, **v** and $\Delta\theta$ into Algorithm 2.3 yields the state vector ($\mathbf{r}_a$ and $\mathbf{v}_a$) at apogee.

The apogee radius r_a is the magnitude of $\mathbf{r}_a$. If r_a does not equal the target value of 22,378 km, then we assume a new burn time and repeat the above steps to calculate a new r_a. This trial and error process is repeated until r_a is acceptably close to 22,378 km.

The calculations are done in the MATLAB M-function *integrate_thrust.m*, which is listed in Appendix D.30. *rkf45.m* (see Appendix D.4) was chosen as the numerical integrator. The initial conditions $\mathbf{y}_0$ in Equation (a) above are passed to *rkf45*, which solves the system of Equations 6.29 at discrete times between 0 and t_{burn}. *rkf45.m* employs the subfunction *rates,* embedded in *integrate_thrust.m*, to calculate the vector of derivatives **f** in Equation 6.30. Output is to the command window, and a revised burn time was entered into the code in the MATLAB editor after each calculation of r_a.

The following output of *integrate_thrust.m* shows that a burn time of 261.1127 seconds (4.352 minutes), with a propellant expenditure of 887.5 kg, is required to produce a coasting trajectory with an apogee of 22,378 km. Due to the finite burn time, the apse line in this case is rotated 8.336° counterclockwise from that in Example 6.1 (line *BCA* in Figure 6.3). Notice that the speed boost Δv imparted by the burn is 9.38984 − 7.71020 = 1.6796 km/s, compared to the impulsive Δv_A = 1.7725 km/s in Example 6.1.

```
----------------------------------------------------------

Before ignition:
  Mass = 2000 kg
  State vector:
    r = [    6858,          0,             0](km)
      Radius = 6858
    v = [    0,             7.7102,        0](km/s)
      Speed = 7.7102

Thrust          =           10  kN
Burn time       =    261.112700 s
Mass after burn =  1.112495E+03 kg

End-of-burn-state vector:
    r = [    6551.56,      2185.85,      0] (km)
      Radius = 6906.58
    v = [    −2.42229,     9.07202,      0] (km/s)
      Speed = 9.38984

Post-burn trajectory:
  Eccentricity    = 0.530257
  Semimajor axis  = 14623.7 km
  Apogee state vector:
    r = [−2.2141572950E+04,   −3.2445306214E+03,    0.0000000000E+00] (km)
      Radius = 22378
    v = [ 4.1938999506E-01,   −2.8620331423E+00,   −0.0000000000E+00] (km/s)
      Speed = 2.8926

----------------------------------------------------------
```

Transfer from apogee of orbit 2 to the circular target orbit 3.

The spacecraft mass and state vector at apogee, given by the above output (under "post-burn trajectory"), are entered as new initial conditions in *integrate_thrust.m*, and the manual trial and error process described above is carried out. It is not possible to transfer from the 22,378 km apogee of orbit 2 to a circular orbit of radius 22,378 km using a single finite-time burn. Therefore, the objective in this case is to make the semimajor axis of the final orbit equal to 22,378 km. This was achieved with a burn time of 118.88 s and a propellant expenditure of 404.05 kg, and it yields a nearly circular orbit having an eccentricity of 0.00867 and an apse line rotated 80.85° clockwise from the x-axis.

The computed spacecraft mass at the end of the second delta-v maneuver is 708.44 kg. Therefore, the total propellant expenditure is 2000 − 708.44 = 1291.6 kg. This is essentially the same as the propellant requirement (1291.3 kg) calculated in Example 6.1, in which the two delta-v maneuvers were impulsive.

Let us take the dot product of both sides of Equation 6.26 with the velocity **v**, to obtain

$$\ddot{\mathbf{r}}\cdot\mathbf{v} = -\frac{\mu}{r^3}\mathbf{r}\cdot\mathbf{v} + \frac{T}{m}\frac{\mathbf{v}\cdot\mathbf{v}}{v} \tag{6.31}$$

In Section 2.5, we showed that

$$\ddot{\mathbf{r}}\cdot\mathbf{v} = \frac{1}{2}\frac{dv^2}{dt} \qquad \text{and} \qquad \frac{\mu}{r^3}\mathbf{r}\cdot\mathbf{v} = -\frac{d}{dt}\left(\frac{\mu}{r}\right)$$

Substituting these together with $\mathbf{v}\cdot\mathbf{v} = v^2$ into Equation 6.31 yields the energy equation,

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = \frac{T}{m}v \tag{6.32}$$

This equation may be applied to the approximate solution of a constant tangential thrust orbit transfer problem. If the spacecraft is in a circular orbit, then applying a very low constant thrust T in the forward direction will cause its total energy $\varepsilon = v^2/2 - \mu/r$ to slowly increase over time according to Equation 6.32. This will raise the height after each revolution, resulting in a slow outward spiral (or inward spiral if the thrust is directed aft). If we assume that the speed at any radius of the closely-spaced spiral trajectory is essentially that of a circular orbit of that radius (Wiesel, 1997), then we can replace v by $\sqrt{\mu/r}$ to obtain an approximate version of Equation 6.32,

$$\frac{d}{dt}\left(\frac{1}{2}\frac{\mu}{r} - \frac{\mu}{r}\right) = \frac{T}{m}\sqrt{\frac{\mu}{r}}$$

Simplifying and separating variables leads to

$$\frac{d(\mu/r)}{\sqrt{\mu/r}} = -2\frac{T}{m}dt \tag{6.33}$$

The spacecraft mass is a function of time

$$m = m_0 - \dot{m}_e t \tag{6.34}$$

where m_0 is the mass at the start of the orbit transfer ($t = 0$) and $\dot{m}_e$ is the constant rate at which propellant is expended. Thus,

$$\frac{d(\mu/r)}{\sqrt{\mu/r}} = -2\frac{T}{m_0 - \dot{m}_e t}dt \tag{6.35}$$

Integrating both sides of this equation and setting $r = r_0$ when $t = 0$ results in

$$\sqrt{\frac{\mu}{r}} - \sqrt{\frac{\mu}{r_0}} = \frac{T}{\dot{m}_e}\ln\left(1 - \frac{\dot{m}_e}{m_0}t\right) \tag{6.36}$$

Finally, since $\dot{m}_e = -dm/dt$, Equation 6.28 implies that we can replace $\dot{m}_e$ with $T/(I_{sp}g_0)$, so that

$$\sqrt{\frac{\mu}{r}} - \sqrt{\frac{\mu}{r_0}} = I_{sp}g_0\ln\left(1 - \frac{T}{m_0 g_0 I_{sp}}t\right) \tag{6.37}$$

We may solve this equation for either r or t to get

$$r = \frac{\mu}{\left[\sqrt{\frac{\mu}{r_0}} + I_{sp}g_0\ln\left(1 - \frac{T}{m_0 g_0 I_{sp}}t\right)\right]^2} \tag{6.38}$$

$$t = \frac{m_0 g_0 I_{sp}}{T}\left[1 - e^{\frac{1}{I_{sp}g_0}\left(\sqrt{\frac{\mu}{r}} - \sqrt{\frac{\mu}{r_0}}\right)}\right] \tag{6.39}$$

Although this scalar analysis yields the radius in terms of the elapsed time, it does not provide us the state vector components **r** and **v**.

Example 6.16

A 1000 kg spacecraft is in a 6678 km (300 km altitude) circular equatorial earth orbit. Its ion propulsion system, which has a specific impulse of 10,000 s, exerts a constant tangential thrust of 2500×10^{-6} kN.

(a) How long will it take the spacecraft to reach GEO (42,164 km)?
(b) How much fuel will be expended?

Solution

(a) Using Equation 6.39, and remembering to express the acceleration of gravity in km/s^2, the flight time is

$$t = \frac{1000 \cdot 0.009807 \cdot 10{,}000}{2500 \times 10^{-6}}\left[1 - e^{\frac{1}{10{,}000 \cdot 0.009807}\left(\sqrt{\frac{398{,}600}{6678}} - \sqrt{\frac{398{,}600}{42{,}164}}\right)}\right]$$

$$\boxed{t = 1{,}817{,}000 \text{ s} = 21.03 \text{ days}}$$

(b) The propellant mass m_p used is

$$m_p = \dot{m}_e t = \frac{T}{I_{sp} g_o} t = \frac{2500 \times 10^{-6}}{10{,}000 \cdot 0.009807} \cdot 1{,}817{,}000$$

$$\boxed{m_p = 46.32 \text{ kg}}$$

Previously, in Example 6.10, we found that the total delta-v for a Hohmann transfer from 6678 km to GEO radius, with no plane change, is 3.893 km/s. Assuming a typical chemical rocket specific impulse of 300 seconds, Equation 6.1 reveals that the propellant requirement would be 734 kg if the initial mass is 1000 kg. This is almost 16 times that required for the hypothetical ion-propelled spacecraft of Example 6.16. Because of their efficiency (high specific impulse), ion engines—typically using xenon as the propellant—will play an increasing role in deep space missions and satellite station keeping. However, these extremely low thrust devices cannot replace chemical rockets in high acceleration applications, such as launch vehicles.

Example 6.17

What will be the orbit after the ion engine in Example 6.16 shuts down upon reaching GEO radius?

Solution

This requires a numerical solution using the MATLAB M-function *integrate_thrust.m*, listed in Appendix D.30. According to the data of Example 6.16, the initial state vector in geocentric equatorial coordinates can be written

$$\mathbf{r}_0 = 6678\hat{\mathbf{I}} \text{ (km)} \qquad \mathbf{v}_0 = \sqrt{\frac{\mu}{r_0}}\hat{\mathbf{J}} = 7.72584\hat{\mathbf{J}} \text{ (km/s)}$$

Using these as the initial conditions, we start by assuming that the elapsed time is 21.03 days, as calculated in Example 6.16. *integrate_thrust.m* computes the final radius for that burn time and outputs the results to the command window. Depending on whether the radius is smaller or greater than 42,164 km, we re-enter a slightly larger or slightly smaller time in the MATLAB editor and run the program again. Several of these manual trial and error steps yields the following MATLAB output.

```
--------------------------------------------------------
Before ignition:
  Mass = 1000 kg
  State vector:
    r = [         6678,               0,            0] (km)
      Radius = 6678
    v = [            0,         7.72584,            0] (km/s)
      Speed = 7.72584

Thrust          =        0.0025  kN
Burn time       =     21.037600  days
Mass after burn = 9.536645E+02  kg
```

```
End-of-burn-state vector:
    r = [    -19028,     -37625.9,          0] (km)
      Radius = 42163.6
    v = [   2.71001,     -1.45129,          0] (km/s)
      Speed = 3.07415

Post-burn trajectory:
  Eccentricity = 0.0234559
  Semimajor axis = 42149 km
  Apogee state vector:
    r = [ 3.7727275971E+04,   -2.0917194986E+04,    0.0000000000E+00] (km)
      Radius = 43137.9
    v = [ 1.4565649916E+00,   -2.6271318618E+00,    0.0000000000E+00] (km/s)
      Speed = 3.0039
--------------------------------------------------------
```

From the printout it is evident that to reach GEO radius requires the following time and propellant expenditure:

(a) $\boxed{t = 21.0376 \text{ days}}$

(b) $\boxed{m_p = 46.34\,\text{kg}}$

These are very nearly the same as the values found in the previous example. However, this numerical solution in addition furnishes the end-of-burn state vector, which shows that the post-burn orbit is slightly elliptical, having an eccentricity of 0.02346 and a semimajor axis that is only 15 km less than GEO radius.

PROBLEMS

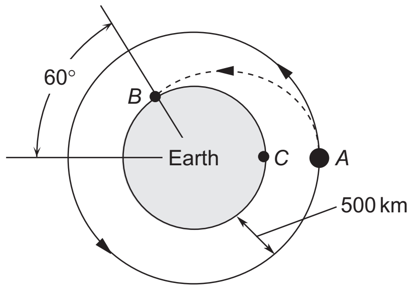
Section 6.2

6.1 The Shuttle orbiter has a mass of 125,000 kg. The two orbital maneuvering engines produce a combined (nonthrottleable) thrust of 53.4 kN. The orbiter is in a 300 km circular orbit. A delta-v maneuver transfers the spacecraft to a coplanar 250 km by 300 km elliptical orbit. Neglecting propellant loss and using elementary physics (linear impulse equals change in linear momentum, distance equals speed times time), estimate

(a) the time required for the Δv burn, and
(b) the distance traveled by the orbiter during the burn.
(c) Calculate the ratio of your answer for (b) to the circumference of the initial circular orbit.
{Ans.: (a) $\Delta t = 34$ sec; (b) 263 km; (c) 0.0063}

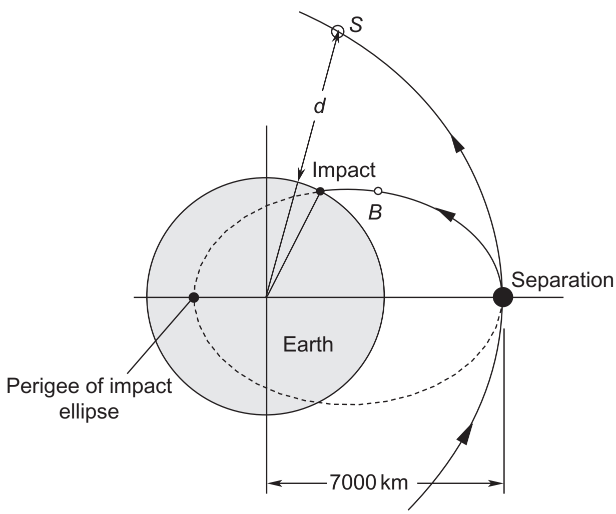
6.2 A satellite traveling 8.2 km/s at a perigee altitude of 480 km fires a retrorocket. What delta-v is necessary to reach a minimum altitude of 160 km during the next orbit?
{Ans.: −668 m/s}

6.3 A spacecraft is in a 500 km altitude circular earth orbit. Neglecting the atmosphere, find the delta-v required at *A* in order to impact the earth at (a) point *B*; (b) point *C*.
{Ans.: (a) 192 m/s ; (b) 7.61 km/s}



6.4 A satellite is in a circular orbit at an altitude of 320 km above the earth's surface. If an onboard rocket provides a delta-v of 500 m/s in the direction of the satellite's motion, calculate the altitude of the new orbit's apogee.
{Ans.: 2390 km}

6.5 A spacecraft S is in a geocentric hyperbolic trajectory with a perigee radius of 7000 km and a perigee speed of $1.3v_{esc}$. At perigee, the spacecraft releases a projectile B with a speed of 7.1 km/s parallel to the spacecraft's velocity. How far d from the earth's surface is S at the instant B impacts the earth? Neglect the atmosphere.
{Ans.: $d = 8978$ km}



6.6 A spacecraft is in a 200 km circular earth orbit. At $t = 0$, it fires a projectile in the direction opposite to the spacecraft's motion. Thirty minutes after leaving the spacecraft, the projectile impacts the earth. What delta-v was imparted to the projectile? Neglect the atmosphere.
{Ans.: $\Delta v = 77.2$ m/s}

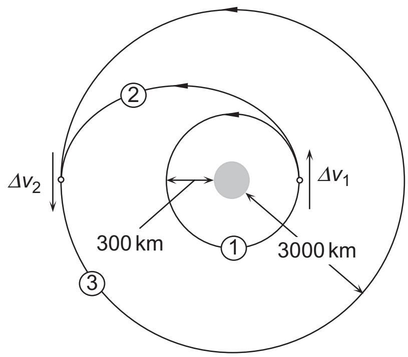
6.7 A spacecraft is in a circular orbit of radius r and speed v around an unspecified planet. A rocket on the spacecraft is fired, instantaneously increasing the speed in the direction of motion by the amount $\Delta v = \alpha v$, where $\alpha > 0$. Calculate the eccentricity of the new orbit.
{Ans.: $e = \alpha(\alpha + 2)$}

Section 6.3

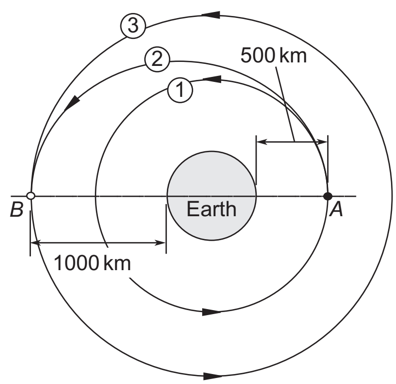
6.8 A spacecraft is in a 300 km circular earth orbit. Calculate

(a) the total delta-v required for a Hohmann transfer to a 3000 km coplanar circular earth orbit and
(b) the transfer orbit time.

{Ans.: (a) 1.198 km/s; (b) 59 m 39 s}



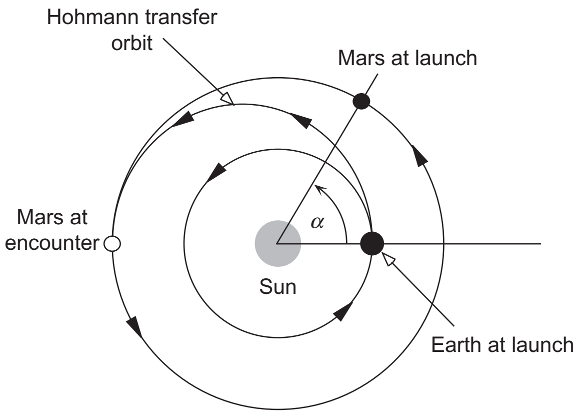
6.9 A space vehicle in a circular orbit at an altitude of 500 km above the earth executes a Hohmann transfer to a 1000 km circular orbit. Calculate the total delta-v requirement.
{Ans.: 0.2624 km/s}



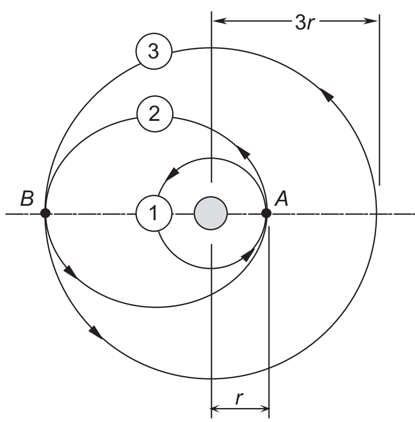
6.10 Assuming the orbits of earth and Mars are circular and coplanar, calculate

(a) the time required for a Hohmann transfer from earth orbit to Mars orbit and
(b) the initial position of Mars (α) in its orbit relative to earth for interception to occur.

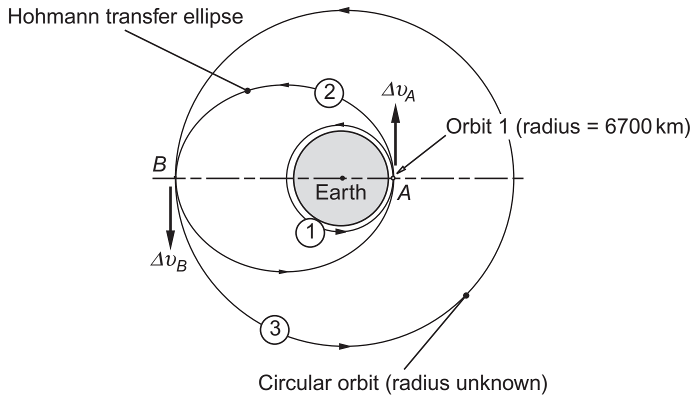
Radius of earth orbit = 1.496×10^8 km. Radius of Mars orbit = 2.279×10^8 km.
$\mu_{Sun} = 1.327 \times 10^{11}\ km^3/s^2$.
{Ans.: (a) 259 days; (b) $\alpha = 44.3°$}



6.11 Calculate the total delta-v required for a Hohmann transfer from the smaller circular orbit to the larger one.
{Ans.: $0.394v_1$, where v_1 is the speed in orbit 1.}



6.12 With a Δv_A of 1.500 km/s, a spacecraft in the circular 6700 km geocentric orbit 1 initiates a Hohmann transfer to the larger circular orbit 3. Calculate Δv_B at apogee of the Hohmann transfer ellipse 2.
{Ans.: $\Delta v_B = 1.877$ km/s}

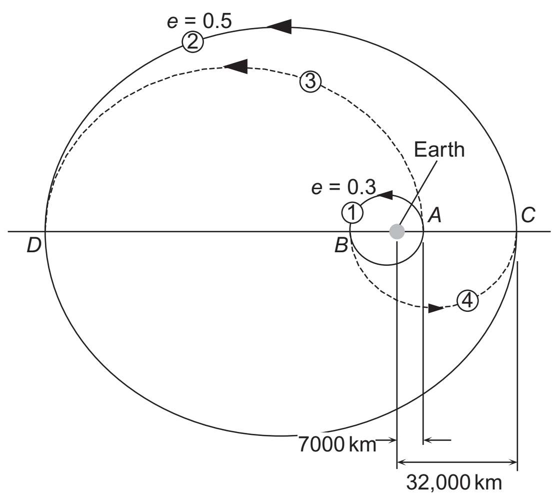


6.13 Two geocentric elliptical orbits have common apse lines and their perigees are on the same side of the earth. The first orbit has a perigee radius of $r_p = 7000$ km and $e = 0.3$, whereas for the second orbit $r_p = 32{,}000$ km and $e = 0.5$.

(a) Find the minimum total delta-v and the time of flight for a transfer from the perigee of the inner orbit to the apogee of the outer orbit.

(b) Do part (a) for a transfer from the apogee of the inner orbit to the perigee of the outer orbit.

{Ans.: (a) $\Delta v_{\text{total}} = 2.388$ km/s, TOF = 16.2 hours; (b) $\Delta v_{\text{total}} = 3.611$ km/s, TOF = 4.66 hours}

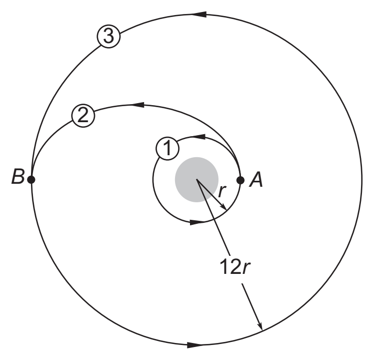


6.14 The Space Shuttle was launched on a fifteen-day mission. There were four orbits after injection, all of them at 39° inclination.

Orbit 1: 302 by 296 km.
Orbit 2 (day 11): 291 by 259 km.

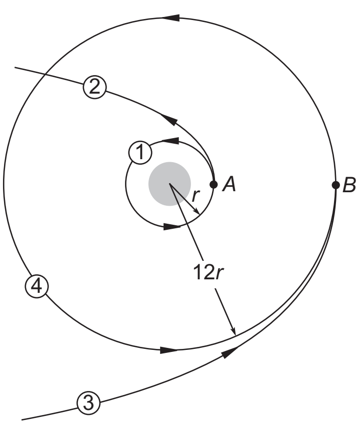
Orbit 3 (day 12): 259 km circular.
Orbit 4 (day 13): 255 by 194 km.
Calculate the total delta-v, which should be as small as possible, assuming Hohmann transfers.
{Ans.: $\Delta v_{\text{total}} = 43.5$ m/s}

6.15 Calculate the total delta-v required for a Hohmann transfer from a circular orbit of radius r to a circular orbit of radius 12r.
{Ans.: $0.5342\sqrt{\mu/r}$ }



Section 6.4

6.16 A spacecraft in circular orbit 1 of radius r leaves for infinity on parabolic trajectory 2 and returns from infinity on a parabolic trajectory 3 to a circular orbit 4 of radius $12r$. Find the total delta-v required for this non-Hohmann orbit change maneuver.
{Ans.: $0.5338\sqrt{\mu/r}$ }

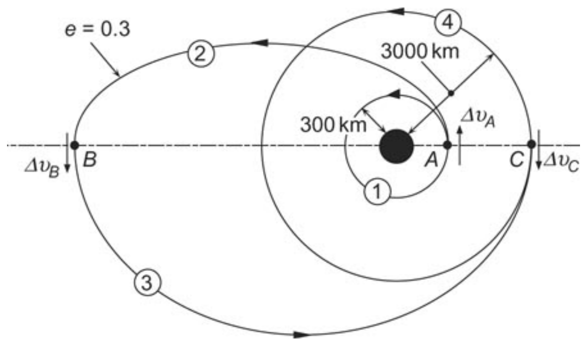


6.17 A spacecraft is in a 300 km circular earth orbit. Calculate

(a) the total delta-v required for the bi-elliptic transfer to a 3000 km altitude coplanar circular orbit shown, and

(b) the total transfer time.

{Ans.: (a) 2.039 km/s; (b) 2.86 hr}



6.18 Verify Equations 6.4.

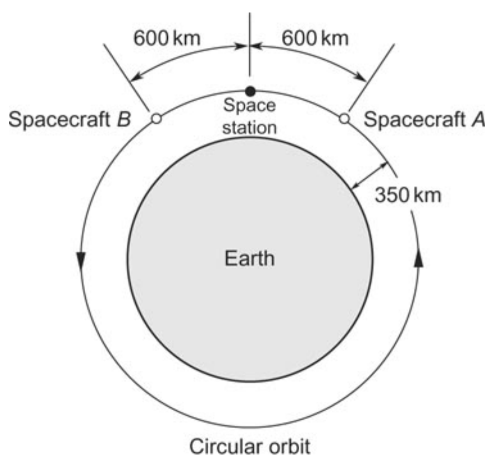
Section 6.5

6.19 The space station and spacecraft *A* and *B* are all in the same circular earth orbit of 350 km altitude. Spacecraft *A* is 600 km behind the space station and Spacecraft *B* is 600 km ahead of the space station. At the same instant, both spacecraft apply a $\Delta v_\perp$ so as to arrive at the space station in one revolution of their phasing orbits.

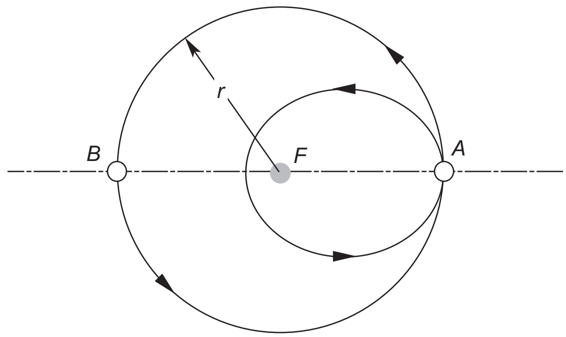
(a) Calculate the times required for each spacecraft to reach the space station.

(b) Calculate the total delta-v requirement for each spacecraft.

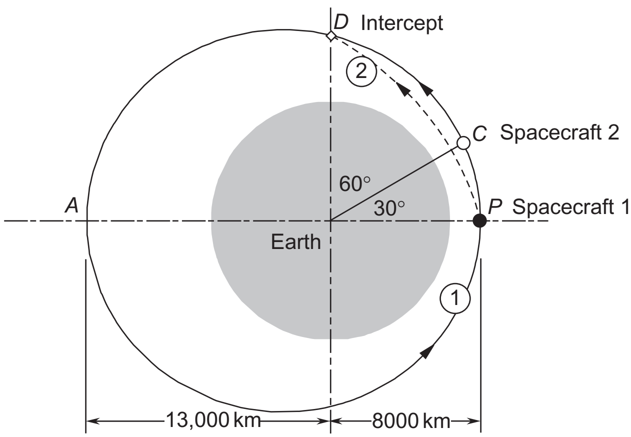
{Ans.: (a) Spacecraft *A*: 90.2 min; Spacecraft *B*: 92.8 min; (b) $\Delta v_A = 73.9$ m/s; $\Delta v_B = 71.5$ m/s}



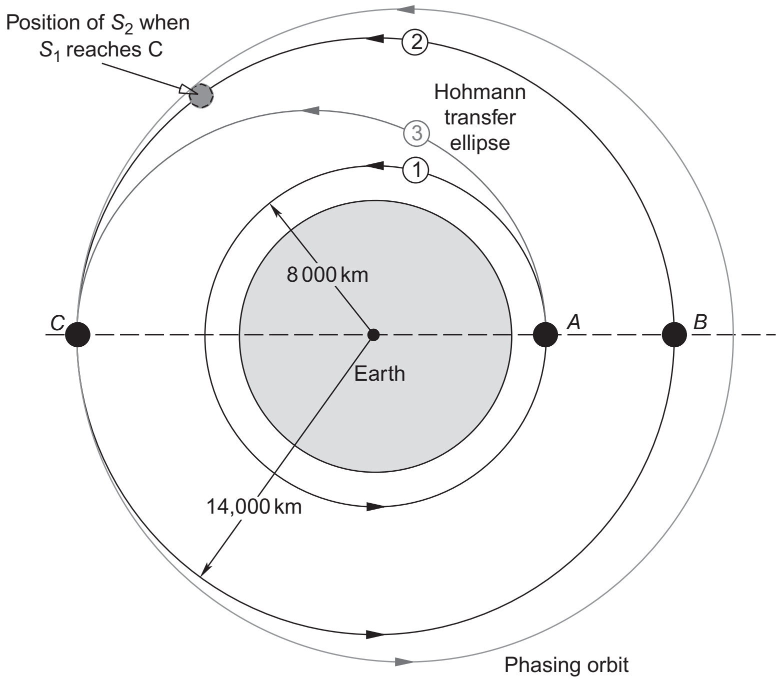
6.20 Satellites A and B are in the same circular orbit of radius r. B is 180° ahead of A. Calculate the semimajor axis of a phasing orbit in which A will rendezvous with B after just one revolution in the phasing orbit.
{Ans.: $a = 0.63r$}



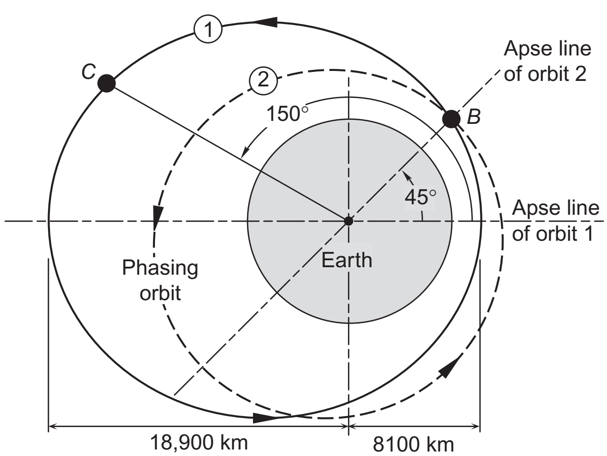
6.21 Two spacecraft are in the same elliptical earth orbit with perigee radius 8000 km and apogee radius 13,000 km. Spacecraft 1 is at perigee and spacecraft 2 is 30° ahead. Calculate the total delta-v required for spacecraft 1 to intercept and rendezvous with spacecraft 2 when spacecraft 2 has traveled 60°.
{Ans.: $\Delta v_{total} = 6.24$ km/s}



6.22 At the instant shown, spacecraft S_1 is at point A of circular orbit 1 and spacecraft S_2 is at point B of circular orbit 2. At that instant, S_1 executes a Hohmann transfer so as to arrive at point C of orbit 2. Upon arriving at C, S_1 immediately executes a phasing maneuver in order to rendezvous with S_2 after one revolution of its phasing orbit. What is the total delta-v requirement?
{Ans.: 2.159 km/s}



6.23 Spacecraft *B* and *C,* which are in the same elliptical earth orbit 1, are located at the true anomalies shown. At this instant, spacecraft *B* executes a phasing maneuver so as to rendezvous with spacecraft *C* after one revolution of its phasing orbit 2. Calculate the total delta-v required. Note that the apse line of orbit 2 is at 45° to that of orbit 1.
{Ans.: 3.405 km/s}

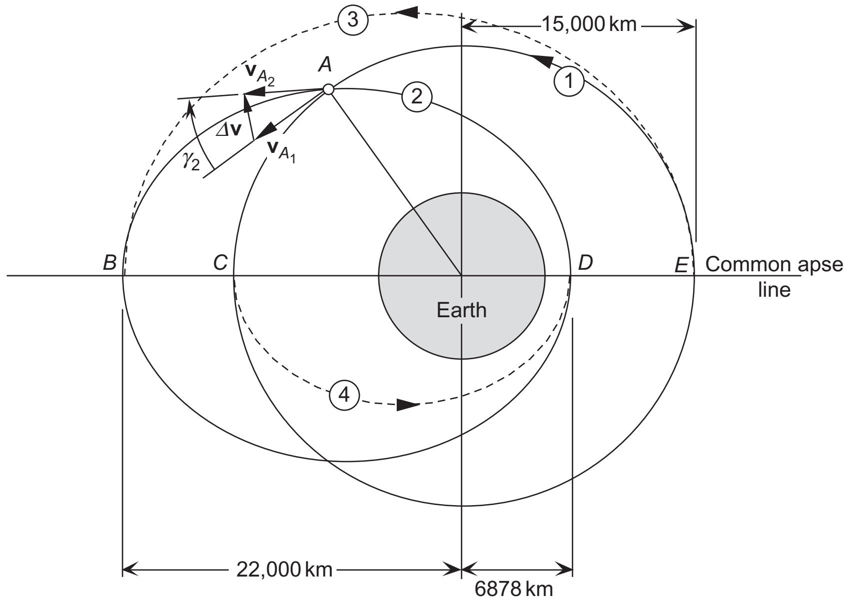


Section 6.6

6.24 (a) With a single delta-v maneuver, the earth orbit of a satellite is to be changed from a circle of radius 15,000 km to a collinear ellipse with perigee altitude of 500 km and apogee radius of 22,000 km. Calculate the magnitude of the required delta-v and the change in the flight path angle $\Delta\gamma$.

(b) What is the minimum total delta-v if the orbit change is accomplished instead by a Hohmann transfer?

{Ans.: (a) $\|\Delta\mathbf{v}\| = 2.77$ km/s , $\Delta\gamma = 31.51°$; (b) $\Delta v_{\text{Hohmann}} = 1.362$ km/s}



6.25 An earth satellite has a perigee altitude of 1270 km and a perigee speed of 9 km/s. It is required to change its orbital eccentricity to 0.4, without rotating the apse line, by a delta-v maneuver at $\theta = 100°$. Calculate the magnitude of the required $\Delta\mathbf{v}$ and the change in flight path angle $\Delta\gamma$.

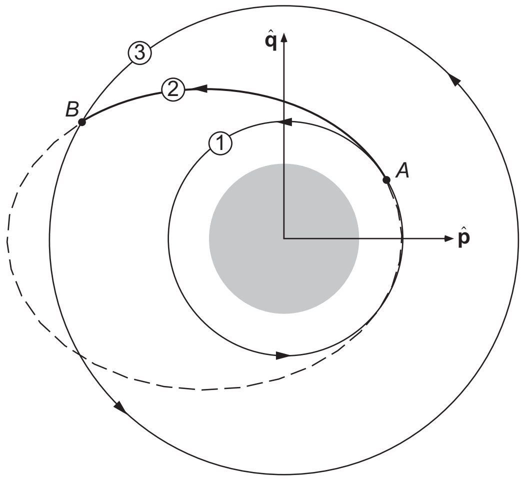
{Ans.: $\|\Delta\mathbf{v}\| = 0.915$ km/s ; $\Delta\gamma = -8.18°$}

6.26 The velocities at points A and B on orbits 1, 2 and 3, respectively, are (relative to the perifocal frame)

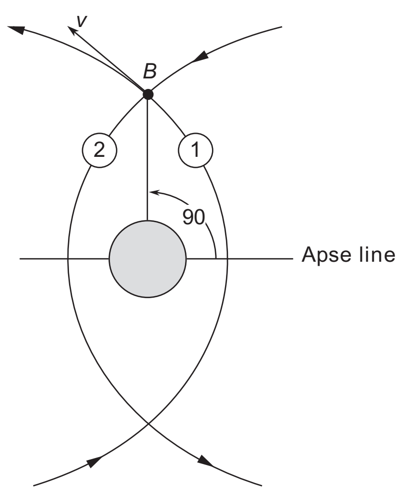
$$\begin{aligned}
\mathbf{v}_A)_1 &= -3.7730\hat{\mathbf{p}} + 6.5351\hat{\mathbf{q}} \text{ (km/s)}\\
\mathbf{v}_A)_2 &= -3.2675\hat{\mathbf{p}} + 8.1749\hat{\mathbf{q}} \text{ (km/s)}\\
\mathbf{v}_B)_2 &= -3.2675\hat{\mathbf{p}} - 3.1442\hat{\mathbf{q}} \text{ (km/s)}\\
\mathbf{v}_B)_3 &= -2.6679\hat{\mathbf{p}} - 4.6210\hat{\mathbf{q}} \text{ (km/s)}
\end{aligned}$$

Calculate the total Δv for a transfer from orbit 1 to orbit 3 by means of orbit 2.

{Ans.: 3.310 km/s}



6.27 Trajectories 1 and 2 are ellipses with eccentricity 0.4 and the same angular momentum h. Their speed at B is v. Calculate, in terms of v, the Δv required at B to transfer from orbit 1 to orbit 2.
{Ans.: $\Delta v = 0.7428v$}



Section 6.7

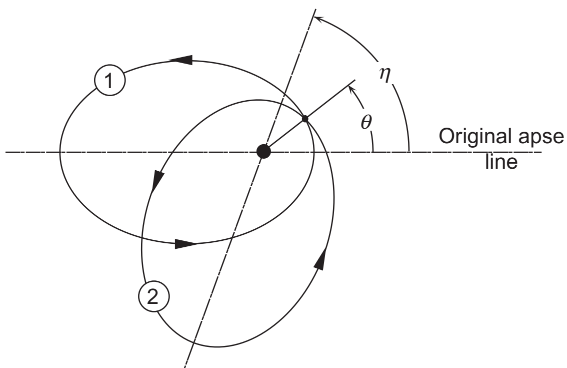
6.28 A satellite is in a circular earth orbit of altitude 400 km. Determine the new perigee and apogee altitudes if the satellite's onboard rocket

(a) Provides a delta-v in the tangential direction of 240 m/s.
(b) Provides a delta-v in the radial (outward) direction of 240 m/s.
{Ans.: (a) $z_A = 1320$ km, $z_P = 400$ km; (b) $z_A = 619$ km, $z_P = 194$ km}

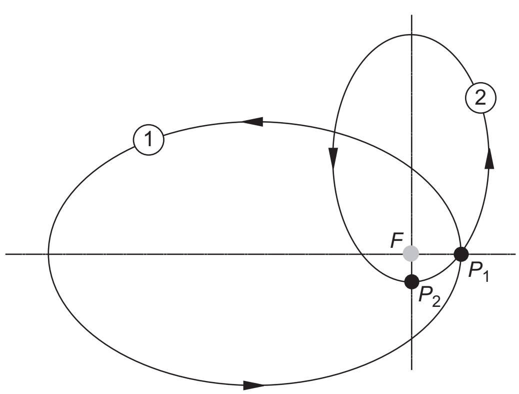
6.29 At point A on its earth orbit, the radius, speed and flight path angle of a satellite are $r_A = 12{,}756\,\text{km}$, $v_A = 6.5992\,\text{km/s}$ and $\gamma_A = 20°$. At point B, at which the true anomaly is 150°, an impulsive maneuver causes $\Delta v_\perp = +0.75820\,\text{km/s}$ and $\Delta v_r = 0$.

(a) What is the time of flight from A to B?
(b) What is the rotation of the apse line as a result of this maneuver?
{Ans.: (a) 2.045 hr; (b) 43.39° counterclockwise}

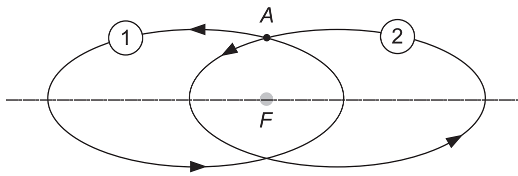
6.30 A satellite is in elliptical orbit 1. Calculate the true anomaly θ (relative to the apse line of orbit 1) of an impulsive maneuver that rotates the apse line an angle η counterclockwise but leaves the eccentricity and the angular momentum unchanged.
{Ans.: $\theta = \eta/2$}



6.31 A satellite in orbit 1 undergoes a delta-v maneuver at perigee P_1 such that the new orbit 2 has the same eccentricity e, but its apse line is rotated 90° clockwise from the original one. Calculate the specific angular momentum of orbit 2 in terms of that of orbit 1 and the eccentricity e.
{Ans.: $h_2 = h_1/\sqrt{1+e}$}

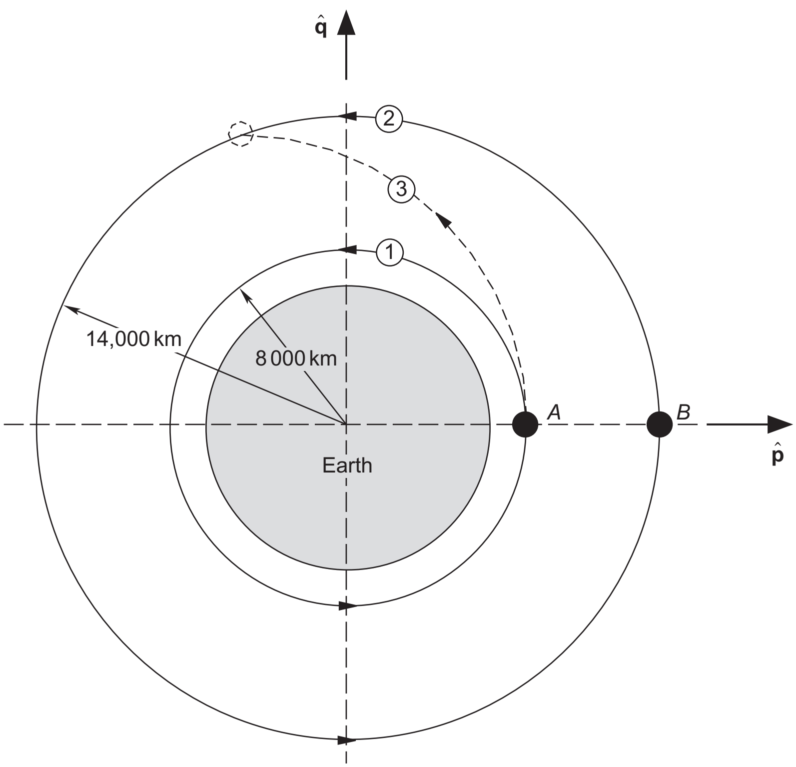


6.32 Calculate the delta-v required at A in orbit 1 for a single impulsive maneuver to rotate the apse line 180° counterclockwise (to become orbit 2), but keep the eccentricity e and the angular momentum $\mathbf{h}$ the same.
{Ans.: $\Delta v = 2\mu e/h$}

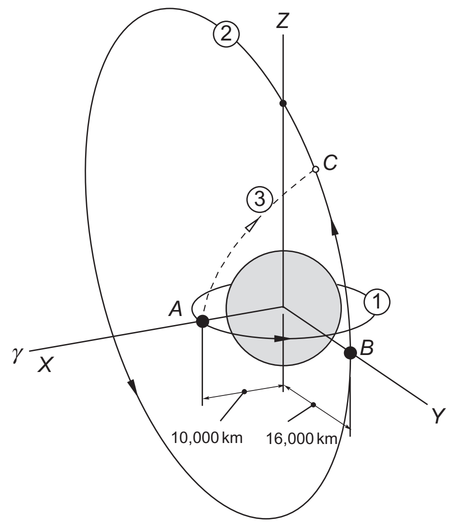


Section 6.8

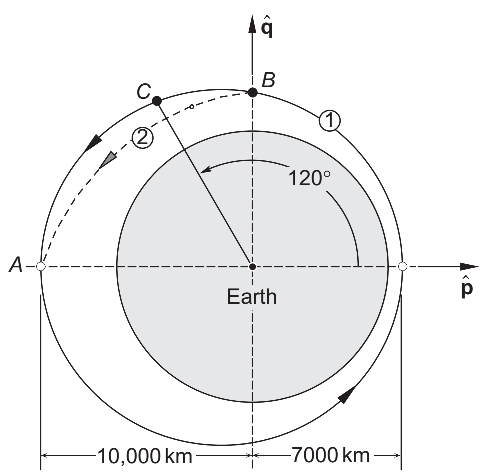
6.33 Spacecraft A and B are in concentric, coplanar circular orbits 1 and 2, respectively. At the instant shown, spacecraft A executes an impulsive delta-v maneuver to embark on orbit 3 in order to intercept and rendezvous with spacecraft B in a time equal to the period of orbit 1. Calculate the total delta-v required.
{Ans.: 3.795 km/s}



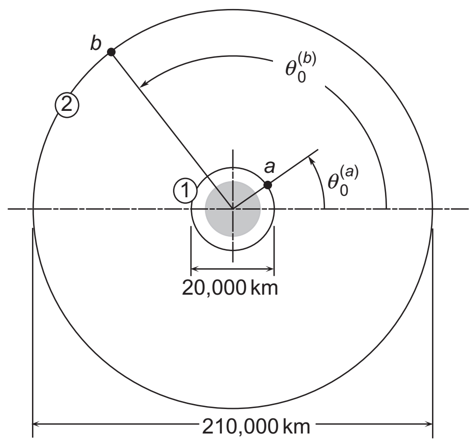
6.34 Spacecraft *A* is in orbit 1, a 10,000 km radius equatorial earth orbit. Spacecraft *B* is in elliptical polar orbit 2, having eccentricity 0.5 and perigee radius 16,000 km. At the instant shown, both spacecraft are in the equatorial plane and *B* is at its perigee. At that instant, spacecraft *A* executes an impulsive delta-v maneuver to intercept spacecraft *B* one hour later at point *C*. Calculate the delta-v required for *A* to switch to the intercept trajectory 3.
{Ans.: 8.117 km/s}



6.35 Spacecraft *B* and *C* are in the same elliptical orbit 1, characterized by a perigee radius of 7000 km and an apogee radius of 10,000 km. The spacecraft are in the positions shown when *B* executes an impulsive transfer to orbit 2 in order to catch and rendezvous with *C* when *C* arrives at apogee *A*. Find the total delta-v requirement.
{Ans.: 5.066 km/s}



6.36 At time $t = 0$, manned spacecraft a and unmanned spacecraft b are at the positions shown in circular earth orbits 1 and 2, respectively. For assigned values of $\theta_0^{(a)}$ and $\theta_0^{(b)}$, design a series of impulsive maneuvers by means of which spacecraft a transfers from orbit 1 to orbit 2 so as to rendezvous with spacecraft b (i.e., occupy the same position in space). The total time and total delta-v required for the transfer should be as small as possible. Consider earth's gravity only.



Section 6.9

6.37 What must the launch azimuth be if the satellite in Example 4.10 is launched from (a) Kennedy Space Center (latitude = 28.5°N); (b) Vandenburgh AFB (latitude = 34.5°N) (c) Kourou, French Guiana (latitude 5.5°N).
{Ans.: (a) 329.4° or 210.6°; (b) 327.1° or 212.9°; (c) 333.3° or 206.7°}

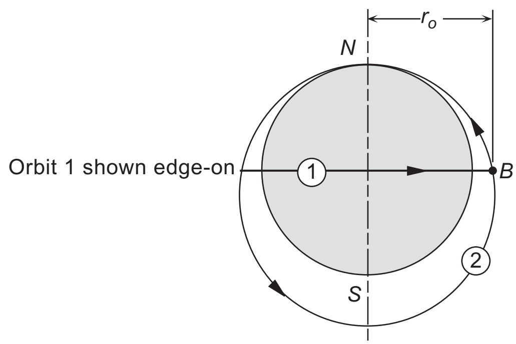
6.38 The state vector of a spacecraft in the geocentric equatorial frame is $\mathbf{r} = r\hat{\mathbf{I}}$ and $\mathbf{v} = v\hat{\mathbf{J}}$. At that instant an impulsive maneuver produces the velocity change $\Delta\mathbf{v} = 0.5v\hat{\mathbf{I}} + 0.5v\hat{\mathbf{K}}$. What is the inclination of the new orbit?
{Ans.: 26.57°}

6.39 An earth satellite has the following orbital elements: $a = 15{,}000\,\text{km}$, $e = 0.5$, $\Omega = 45°$, $\omega = 30°$, $i = 10°$. What minimum delta-v is required to reduce the inclination to zero?
{Ans.: 0.588 km/s}

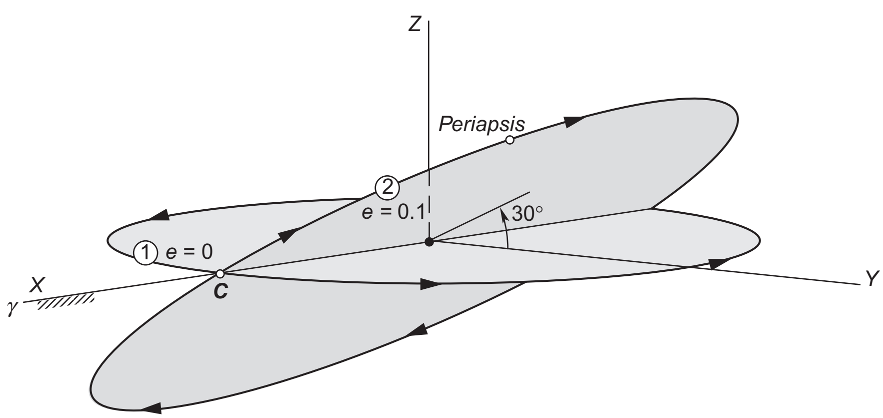
6.40 With a single impulsive maneuver, an earth satellite changes from a 400 km circular orbit inclined at 60° to an elliptical orbit of eccentricity $e = 0.5$ with an inclination of 40°. Calculate the minimum required delta-v.
{Ans.: 3.41 km/s}

6.41 An earth satellite is in an elliptical orbit of eccentricity 0.3 and angular momentum 60,000 km^2/s. Find the delta-v required for a 90° change in inclination at apogee (no change in speed).
{Ans.: 6.58 km/s}

6.42 A spacecraft is in a circular, equatorial orbit 1 of radius r_o about a planet. At point B it impulsively transfers to polar orbit 2, whose eccentricity is 0.25 and whose perigee is directly over the North Pole. Calculate the minimum delta-v required at B for this maneuver.
{Ans.: $1.436\sqrt{\mu/r_o}$}



6.43 A spacecraft is in a circular, equatorial orbit 1 of radius r_0 and speed v_0 about an unknown planet ($\mu \neq 398{,}600\,\text{km}^3/\text{s}^2$). At point C it impulsively transfers to orbit 2, for which the ascending node is point C, the eccentricity is 0.1, the inclination is 30° and the argument of periapsis is 60°. Calculate, in terms of v_0, the single delta-v required at C for this maneuver.
{Ans.: $\Delta v = 0.5313 v_0$}



6.44 A spacecraft is in a 300 km circular parking orbit. It is desired to increase the altitude to 600 km and change the inclination by 20°. Find the total delta-v required if

(a) The plane change is made after insertion into the 600 km orbit (so that there are a total of three delta-v burns).

(b) If the plane change and insertion into the 600 km orbit are accomplished simultaneously (so that the total number of delta-v burns is two).
(c) The plane change is made upon departing the lower orbit (so that the total number of delta-v burns is two).
{Ans.: (a) 2.793 km/s; (b) 2.696 km/s; (c) 2.783 km/s}

Section 6.10

6.45 Calculate the total propellant expenditure for Problem 6.3 using finite-time delta-v maneuvers. The initial spacecraft mass is 4000 kg. The propulsion system has a thrust of 30 kN and a specific impulse of 280 s.

6.46 Calculate the total propellant expenditure for problem Problem 6.14 using finite-time delta-v maneuvers. The initial spacecraft mass is 4000 kg. The propulsion system has a thrust of 30 kN and a specific impulse of 280 s.

6.47 At a given instant t_0, a 1000 kg earth-orbiting satellite has the inertial position and velocity vectors $\mathbf{r}_0 = 436\hat{\mathbf{i}} + 6083\hat{\mathbf{j}} + 2529\hat{\mathbf{k}}$ (km) and $\mathbf{v}_0 = -7.340\hat{\mathbf{i}} - 0.5125\hat{\mathbf{j}} + 2.497\hat{\mathbf{k}}$ (km/s). 89 minutes later a rocket motor with $I_{sp} = 300$ s and 10 kN thrust aligned with the velocity vector ignites and burns for 120 seconds. Use numerical integration to find the maximum altitude reached by the satellite and the time it occurs.

List of Key Terms

Δv with plane change
Δv without plane change
chemical rockets
cranking maneuver
ion population
launch azimuth
launch latitude
orbit inclination
phasing orbit
pumping maneuver
specific impulse
spiral trajectory

CHAPTER

Relative motion and rendezvous

7

Chapter outline

7.1	Introduction	391
7.2	Relative motion in orbit	392
7.3	Linearization of the equations of relative motion in orbit	400
7.4	Clohessy-Wiltshire equations	407
7.5	Two-impulse rendezvous maneuvers	411
7.6	Relative motion in close-proximity circular orbits	419

7.1 INTRODUCTION

Up to now we have mostly referenced the motion of orbiting objects to a nonrotating coordinate system fixed to the center of attraction (e.g., the center of the earth). This platform served as an inertial frame of reference, in which Newton's second law can be written

$$\mathbf{F}_{\text{net}} = m\mathbf{a}_{\text{absolute}}$$

An exception to this rule was the discussion of the restricted three-body problem at the end of Chapter 2, in which we made use of the relative motion equations developed in Chapter 1. In a rendezvous maneuver, two orbiting vehicles observe one another from each of their own free-falling, rotating, clearly noninertial frames of reference. To base impulsive maneuvers on observations made from a moving platform requires transforming relative velocity and acceleration measurements into an inertial frame. Otherwise, the true thrusting forces cannot be sorted out from the fictitious "inertia forces" that appear in Newton's law when it is written incorrectly as

$$\mathbf{F}_{\text{net}} = m\mathbf{a}_{\text{rel}}$$

The purpose of this chapter is to use relative motion analysis to gain some familiarity with the problem of maneuvering one spacecraft relative to another, especially when they are in close proximity.

7.2 RELATIVE MOTION IN ORBIT

A rendezvous maneuver usually involves a target vehicle *A*, which is passive and nonmaneuvering, and a chase vehicle *B* which is active and performs the maneuvers required to bring itself alongside the target. An obvious example is the space shuttle, the chaser, rendezvousing with the international space station, the target. The position vector of the target *A* in the geocentric equatorial frame is $\mathbf{r}_A$. This radial is sometimes called the "*r*-bar". The moving frame of reference has its origin at the target, as illustrated in Figure 7.1. The *x* axis is directed along the outward radial $\mathbf{r}_A$ to the target. Therefore, the unit vector $\hat{\mathbf{i}}$ along the moving *x* axis is

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_A}{r_A} \tag{7.1}$$

The *z* axis is normal to the orbital plane of the target spacecraft and therefore lies in the direction of *A*'s angular momentum vector. It follows that the unit vector along the *z* axis of the moving frame is given by

$$\hat{\mathbf{k}} = \frac{\mathbf{h}_A}{h_A} \tag{7.2}$$

The *y* axis is perpendicular to both $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ and points in the direction of the target satellite's local horizon. Therefore, both the *x* and *y* axes lie in the target's orbital plane, with the *y* unit vector completing a right-triad; that is,

$$\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}} \tag{7.3}$$

We may refer to the co-moving *xyz* frame defined here as a local vertical/local horizontal (LVLH) frame.

The position, velocity and acceleration of *B* relative to *A*, measured in the co-moving frame, are given by

$$\mathbf{r}_{\text{rel}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{7.4a}$$

$$\mathbf{v}_{\text{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{7.4b}$$

$$\mathbf{a}_{\text{rel}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{7.4c}$$

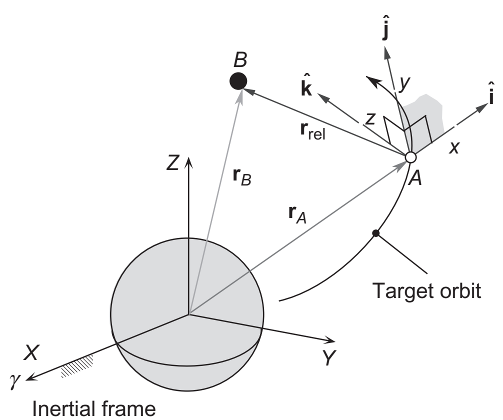


FIGURE 7.1

Co-moving LVLH reference frame attached to *A*, from which the body *B* is observed.

The angular velocity $\boldsymbol{\Omega}$ of the *xyz* axes attached to the target is just the angular velocity of the target's position vector. It is obtained with the aid of Equations 2.31 and 2.46 from the fact that

$$\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A = (r_A v_{A_\perp})\hat{\mathbf{k}} = \left(r_A^2 \Omega\right)\hat{\mathbf{k}} = r_A^2 \boldsymbol{\Omega}$$

from which we obtain the ***angular velocity of the co-moving frame***

$$\boldsymbol{\Omega} = \frac{\mathbf{h}_A}{r_A^2} = \frac{\mathbf{r}_A \times \mathbf{v}_A}{r_A^2} \tag{7.5}$$

To find the angular acceleration $\dot{\boldsymbol{\Omega}}$ of the *xyz* frame, we take the time derivative of $\boldsymbol{\Omega}$ in Equation 7.5 and use the fact that the angular momentum $\mathbf{h}_A$ of the passive target is constant.

$$\dot{\boldsymbol{\Omega}} = \mathbf{h}_A \frac{d}{dt}\left(\frac{1}{r_A^2}\right) = -2\frac{\mathbf{h}_A}{r_A^3}\dot{r}_A$$

Recall from Equation 2.35a that $\dot{r}_A = \mathbf{v}_A \cdot \mathbf{r}_A / r_A$, so the ***angular acceleration of the co-moving frame*** may be written

$$\dot{\boldsymbol{\Omega}} = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A^4}\mathbf{h}_A = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A^2}\boldsymbol{\Omega} \tag{7.6}$$

After first calculating

$$\mathbf{r}_{\text{rel}} = \mathbf{r}_B - \mathbf{r}_A \tag{7.7}$$

we use Equations 7.5 and 7.6 to determine the angular velocity and angular acceleration of the co-moving frame, both of which are required in the relative velocity and acceleration formulas (Equations 1.69 and 1.70),

$$\mathbf{v}_{\text{rel}} = \mathbf{v}_B - \mathbf{v}_A - \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}} \tag{7.8}$$

$$\mathbf{a}_{\text{rel}} = \mathbf{a}_B - \mathbf{a}_A - \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{\text{rel}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}) - 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} \tag{7.9}$$

The vectors in Equations 7.7 through 7.9 are all referred to the inertial *XYZ* frame in Figure 7.1. In order to find their components in the accelerating *xyz* frame at any instant, we must first form the orthogonal transformation matrix $[\mathbf{Q}]_{Xx}$, as discussed in Section 4.5. The rows of this matrix comprise the direction cosines of each of the *xyz* axes with respect to the XYZ axes. That is, from Equations 7.1, 7.2 and 7.3 we find

$$\begin{aligned}\hat{\mathbf{i}} &= l_x\hat{\mathbf{I}} + m_x\hat{\mathbf{J}} + n_x\hat{\mathbf{K}}\\ \hat{\mathbf{j}} &= l_y\hat{\mathbf{I}} + m_y\hat{\mathbf{J}} + n_y\hat{\mathbf{K}}\\ \hat{\mathbf{k}} &= l_z\hat{\mathbf{I}} + m_z\hat{\mathbf{J}} + n_z\hat{\mathbf{K}}\end{aligned} \tag{7.10}$$

where the *l*s, *m*s and *n*s are the direction cosines. Then

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix} \begin{matrix} \leftarrow \text{components of } \hat{\mathbf{i}} \\ \leftarrow \text{components of } \hat{\mathbf{j}} \\ \leftarrow \text{components of } \hat{\mathbf{k}} \end{matrix} \tag{7.11}$$

The components (Equations 7.4) of the relative position, velocity and acceleration in the LVLH frame arecomputed from their components in the inertial XYZ frame as follows:

$$\{\mathbf{r}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{r}_{\text{rel}}\}_X \qquad \{\mathbf{v}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{v}_{\text{rel}}\}_X \qquad \{\mathbf{a}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{a}_{\text{rel}}\}_X \tag{7.12}$$

where, according to Equations 7.7 through 7.9,

$$\{\mathbf{r}_{\text{rel}}\}_X = \begin{Bmatrix} X_B - X_A \\ Y_B - Y_A \\ Z_B - Z_A \end{Bmatrix} \tag{7.13a}$$

$$\{\mathbf{v}_{\text{rel}}\}_X = \begin{Bmatrix} \dot{X}_B - \dot{X}_A + \Omega_Z(Y_B - Y_A) - \Omega_Y(Z_B - Z_A) \\ \dot{Y}_B - \dot{Y}_A + \Omega_Z(X_B - X_A) - \Omega_X(Z_B - Z_A) \\ \dot{Z}_B - \dot{Z}_A + \Omega_Y(X_B - X_A) - \Omega_X(Y_B - Y_A) \end{Bmatrix} \tag{7.13b}$$

$$\{\mathbf{a}_{\text{rel}}\}_X = \begin{Bmatrix} \ddot{X}_B - \ddot{X}_A + 2\Omega_Z(\dot{Y}_B - \dot{Y}_A) - 2\Omega_Y(\dot{Z}_B - \dot{Z}_A) \\ -\left(\Omega_Y^2 + \Omega_Z^2\right)(X_B - X_A) + (\Omega_X\Omega_Y + a\Omega_Z)(Y_B - Y_A) + (\Omega_X\Omega_Z - a\Omega_Y)(Z_B - Z_A) \\ \\ \ddot{Y}_B - \ddot{Y}_A + 2\Omega_Z(\dot{X}_B - \dot{X}_A) - 2\Omega_X(\dot{Z}_B - \dot{Z}_A) \\ +\left(\Omega_X\Omega_Y - a\Omega_Z\right)(X_B - X_A) + (\Omega_X^2 + \Omega_Z^2)(Y_B - Y_A) + (\Omega_Y\Omega_Z + a\Omega_X)(Z_B - Z_A) \\ \\ \ddot{Z}_B - \ddot{Z}_A + 2\Omega_Y(\dot{X}_B - \dot{X}_A) - 2\Omega_Y(\dot{Y}_B - \dot{Y}_A) \\ +\left(\Omega_X\Omega_Z + a\Omega_Y\right)(X_B - X_A) + (\Omega_Y\Omega_Z - a\Omega_X)(Y_B - Y_A) + (\Omega_X^2 - \Omega_Y^2)(Z_B - Z_A) \end{Bmatrix} \tag{7.13c}$$

The components of $\boldsymbol{\Omega}$ are obtained from Equation 7.5, and $\dot{\boldsymbol{\Omega}} = \boldsymbol{a\Omega}$, where, according to Equation 7.6, $a = -2\mathbf{v}_A \cdot \mathrm{r}_A / \mathbf{r}_A^{\,2}$.

Algorithm 7.1 Given the state vectors ($\mathbf{r}_A$, $\mathbf{v}_A$) of target spacecraft A and ($\mathbf{r}_B$, $\mathbf{v}_B$) of chaser spacecraft B, find the position $\{\mathbf{r}_{\text{rel}}\}_x$, velocity $\{\mathbf{v}_{\text{rel}}\}_x$ and acceleration $\{\mathbf{a}_{\text{rel}}\}_x$ of B relative to A along the LVLH axes attached to A. See Appendix D.31 for an implementation of this procedure in MATLAB®.

1. Calculate the angular momentum of A: $\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A$.
2. Calculate the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ of the co-moving frame.

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_A}{r_A} \qquad \hat{\mathbf{k}} = \frac{\mathbf{h}_A}{h_A} \qquad \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$$

3. Calculate the orthogonal transformation matrix $[\mathbf{Q}]_{Xx}$ using Equation 7.11.
4. Calculate $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}}$ from Equations 7.5 and 7.6.
5. Calculate the absolute accelerations of A and B using Equation 2.22.

$$\mathbf{a}_A = -\frac{\mu}{r_A^3}\mathbf{r}_A \qquad \mathbf{a}_B = -\frac{\mu}{r_B^3}\mathbf{r}_B$$

6. Calculate $\mathbf{r}_{\text{rel}}$ using Equation 7.7.
7. Calculate $\mathbf{v}_{\text{rel}}$ using Equations 7.8.
8. Calculate $\mathbf{a}_{\text{rel}}$ using Equation 7.9.
9. Calculate $\{\mathbf{r}_{\text{rel}}\}$, $\{\mathbf{v}_{\text{rel}}\}_x$ and $\{\mathbf{a}_{\text{rel}}\}_x$ using Equations 7.12.

Example 7.1

Spacecraft *A* is in an elliptical earth orbit having the following parameters:

$$h = 52{,}059 \text{ km}^2/\text{s} \;\; e = 0.025724 \;\; i = 60° \;\; \boldsymbol{\Omega} = 40° \;\; \omega = 30° \; \theta = 40° \tag{a}$$

Spacecraft *B* is likewise in an orbit with these parameters:

$$h = 52{,}362 \text{ km}^2/\text{s} \; e = 0.0072696 \; i = 50° \; \boldsymbol{\Omega} = 40° \; \omega = 120° \; \theta = 40° \tag{b}$$

Calculate the position $\mathbf{r}_{\text{rel}})_x$, velocity $\mathbf{v}_{\text{rel}})_x$ and acceleration $\mathbf{a}_{\text{rel}})_x$ of spacecraft *B* relative to spacecraft *A*, measured along the *xyz* axes of the co-moving coordinate system of spacecraft *A*, as defined in Figure 7.1.

Solution

From the orbital elements in (a) and (b) we can use Algorithm 4.5 to find the position and velocity of both spacecraft relative to the geocentric equatorial reference frame. Omitting those familiar calculations here, the reader can verify that, for spacecraft *A*,

$$\mathbf{r}_A = -266.77\hat{\mathbf{I}} + 3865.8\hat{\mathbf{J}} + 5426.2\hat{\mathbf{K}} \text{ (km)} \qquad (r_A = 6667.8 \text{ km}) \tag{c}$$

$$\mathbf{v}_A = -6.4836\hat{\mathbf{I}} - 3.6198\hat{\mathbf{J}} + 2.4156\hat{\mathbf{K}} \text{ (km/s)} \tag{d}$$

and for spacecraft *B*,

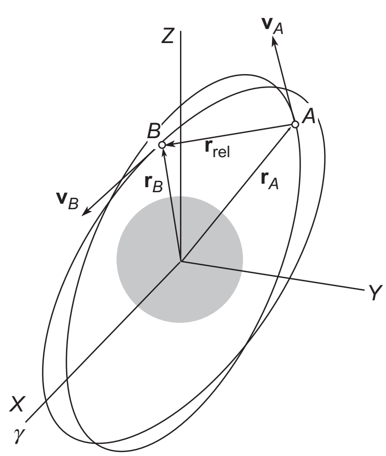


FIGURE 7.2

Spacecraft *A* and *B* in slightly different orbits.

$$\mathbf{r}_B = -5890.7\hat{\mathbf{I}} - 2979.8\hat{\mathbf{J}} + 1792.2\hat{\mathbf{K}} \text{ (km)} \qquad (r_B = 6840.4 \text{ km}) \tag{e}$$

$$\mathbf{v}_B = 0.93583\hat{\mathbf{I}} - 5.2403\hat{\mathbf{J}} - 5.5009\hat{\mathbf{K}} \text{ km/s} \tag{f}$$

Having found the state vectors, which are illustrated in Figure 7.2, we can proceed with Algorithm 7.1.

Step 1:

$$\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -266.77 & 3865.8 & 5426.2 \\ -6.4836 & -3.6198 & 2.4156 \end{vmatrix} = -28{,}980\hat{\mathbf{I}} - 34{,}537\hat{\mathbf{J}} + 26{,}029\hat{\mathbf{K}} \text{ km}^2/\text{s}$$

$(h_A = 52{,}059 \text{ km}^2/\text{s})$

Step 2:

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_A}{r_A} = -0.040009\hat{\mathbf{I}} + 0.57977\hat{\mathbf{J}} + 0.81380\hat{\mathbf{K}}$$

$$\hat{\mathbf{k}} = \frac{\mathbf{h}_A}{h_A} = 0.55667\hat{\mathbf{I}} - 0.66341\hat{\mathbf{J}} - 0.5000\hat{\mathbf{K}}$$

$$\hat{\mathbf{j}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.55667 & -0.66341 & 0.5000 \\ -0.040008 & 0.57977 & 0.81380 \end{vmatrix} = -0.82977\hat{\mathbf{I}} - 0.47302\hat{\mathbf{J}} + 0.29620\hat{\mathbf{K}}$$

Step 3:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -0.040009 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix}$$

Step 4:

$$\boldsymbol{\Omega} = \frac{\mathbf{h}_A}{r_A^2} = 0.00065183\hat{\mathbf{I}} - 0.00077682\hat{\mathbf{J}} + 0.00058547\hat{\mathbf{K}} \text{ (rad/s)}$$

$$\dot{\boldsymbol{\Omega}} = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A^2}\boldsymbol{\Omega} = -2.47533(10^{-8})\hat{\mathbf{I}} + 2.9500(10^{-8})\hat{\mathbf{J}} - 2.2233(10^{-8})\hat{\mathbf{K}} \text{ (rad/s}^2\text{)}$$

Step 5:

$$\mathbf{a}_A = -\mu\frac{\mathbf{r}_A}{r_A^3} = 0.00035870\hat{\mathbf{I}} - 0.0051980\hat{\mathbf{J}} - 0.0072962\hat{\mathbf{K}} \text{ (km/s}^2\text{)}$$

$$\mathbf{a}_B = -\mu\frac{\mathbf{r}_B}{r_B^3} = 0.0073359\hat{\mathbf{I}} + 0.0037108\hat{\mathbf{J}} - 0.0022319\hat{\mathbf{K}} \text{ (km/s}^2\text{)}$$

Step 6:

$$\mathbf{r}_{\text{rel}} = \mathbf{r}_B - \mathbf{r}_A = -5623.9\hat{\mathbf{I}} - 6845.5\hat{\mathbf{J}} - 3634.0\hat{\mathbf{K}} \text{ (km)}$$

Step 7:

$$\mathbf{v}_{rel} = \mathbf{v}_B - \mathbf{v}_A - \boldsymbol{\Omega}\times\mathbf{r}_{rel}$$
$$= (0.93583\mathbf{I} - 5.2403\mathbf{J} - 5.5009\mathbf{K}) - (-6.4836\mathbf{I} - 3.6198\hat{\mathbf{J}} + 2.4156\hat{\mathbf{K}})$$
$$-\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix}$$

$$\mathbf{v}_{rel} = 0.58855\hat{\mathbf{I}} - 0.69663\hat{\mathbf{J}} + 0.91436\hat{\mathbf{K}}\ (\text{km/s})$$

Step 8:

$$\mathbf{a}_{rel} = \mathbf{a}_B - \mathbf{a}_A - \dot{\boldsymbol{\Omega}}\times\mathbf{r}_{rel} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}_{rel}) - 2\boldsymbol{\Omega}\times\mathbf{v}_{rel}$$
$$= (0.0073359\hat{\mathbf{I}} + 0.0037108\hat{\mathbf{J}} - 0.0022319\hat{\mathbf{K}}) - (0.00035870\hat{\mathbf{I}} - 0.0051980\hat{\mathbf{J}} - 0.0072962\mathbf{K})$$
$$-\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -2.4753(10^{-8}) & 2.9500(10^{-8}) & -2.2233(10^{-8}) \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix}$$
$$-(0.00065183\hat{\mathbf{I}} - 0.00077682\hat{\mathbf{J}} + 0.00058547\hat{\mathbf{K}})\times\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix}$$
$$-2\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ 0.58855 & -0.69663 & 0.91436 \end{vmatrix}$$

$$\mathbf{a}_{rel} = 0.00044050\hat{\mathbf{I}} - 0.00037900\hat{\mathbf{J}} + 0.000018581\hat{\mathbf{K}}\ (\text{km/s}^2)$$

Step 9:

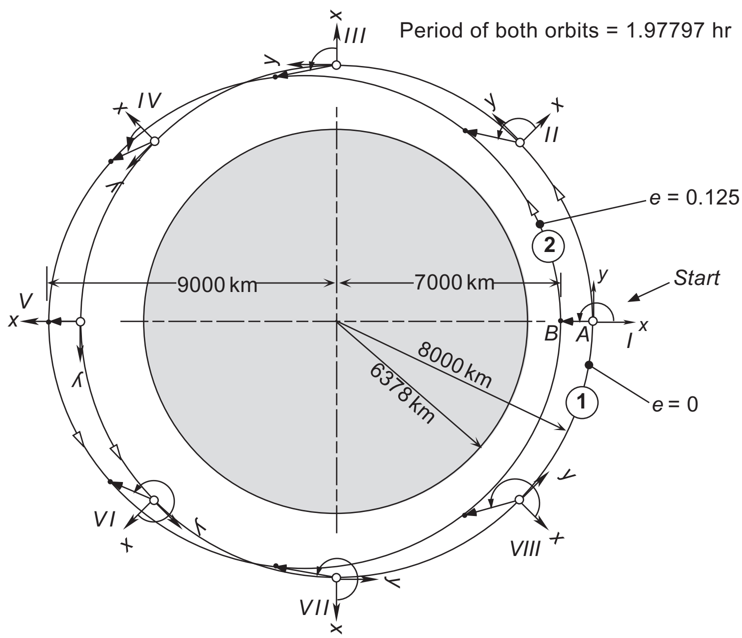
$$\mathbf{r}_{rel})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix}\begin{Bmatrix} -5623.9 \\ -6845.5 \\ -3634.0 \end{Bmatrix} \Rightarrow \mathbf{r}_{rel})_x = \begin{Bmatrix} -6701.2 \\ 6828.3 \\ -406.26 \end{Bmatrix}(\text{km})$$

$$\mathbf{v}_{rel})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix}\begin{Bmatrix} 0.58855 \\ -0.69663 \\ 0.91436 \end{Bmatrix} \Rightarrow \mathbf{v}_{rel})_x = \begin{Bmatrix} 0.31667 \\ 0.11199 \\ 1.2470 \end{Bmatrix}(\text{km/s})$$

$$\mathbf{a}_{rel})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix}\begin{Bmatrix} 0.00044050 \\ -0.00037900 \\ 0.000018581 \end{Bmatrix} \Rightarrow \mathbf{a}_{rel})_x = \begin{Bmatrix} -0.00022222 \\ -0.00018074 \\ 0.00050593 \end{Bmatrix}(\text{km/s}^2)$$

See Appendix D.31 for the MATLAB solution to this problem.

The motion of one spacecraft relative to another in orbit may be hard to visualize at first. Figure 7.3 is offered as an assist. Orbit 1 is circular and orbit 2 is an ellipse with eccentricity 0.125. Both orbits were chosen to have the same semimajor axis length, so that they both have the same period. A co-moving frame is shown attached to the observers *A* in circular orbit 1. At epoch I the spacecraft *B* in elliptical orbit 2 is directly below the observers. In other words, *A* must draw an arrow in the negative local *x* direction to determine the position vector of *B* in the lower orbit. The figure shows eight different epochs (I, II, III,...,VIII), equally spaced around the circular orbit, at which observers *A* construct the position vector pointing from them to *B* in the elliptical orbit. Of course, *A*'s frame is rotating, because its *x*-axis must always be directed away from the earth. Observers *A* cannot sense this rotation and record the set of observations in their (to them) fixed *xy* coordinate system, as shown at the bottom of the figure. Coasting at a uniform speed along his circular orbit, observers *A* see the other vehicle orbiting them clockwise in a sort of bean-shaped path. The distance between the two spacecraft in this case never becomes so great that the earth intervenes.



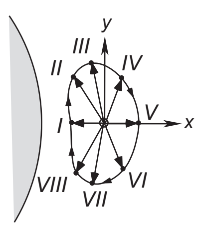


FIGURE 7.3

The spacecraft *B* in elliptical orbit 2 appears to orbit the observers *A* who are in circular orbit 1.

If observers A declared theirs to be an inertial frame of reference, they would be faced with the task of explaining the physical origin of the force holding B in its bean-shaped orbit. Of course, there is no such force. The apparent path is due to the actual, combined motion of both spacecraft in their free fall around the earth. When B is below A (having a negative x-coordinate), conservation of angular momentum demands that B move faster than A, thereby speeding up in A's positive y-direction until the orbits cross ($x = 0$). When B's x coordinate becomes positive, i.e., B is above A, the laws of momentum dictate that B slow down, which it does, progressing in A's negative y-direction until the next crossing of the orbits. B then falls below and begins to pick up speed. The process repeats over and over. From inertial space, the process is the motion of two satellites on intersecting orbits, appearing not at all like the orbiting motion seen by the moving observers A.

Example 7.2

Plot the motion of spacecraft B relative to spacecraft A in Example 7.1.

Solution

In Example 7.1 we found $\mathbf{r}_{\text{rel}})_x$ at a single time. To plot the path of B relative to A we must find $\mathbf{r}_{\text{rel}})_x$ at a large number of times, so that when we "connect the dots" in three-dimensional space a smooth curve results. Let us outline an algorithm and implement it in MATLAB.

1. Given the orbital elements of spacecraft A and B, calculate their state vectors $(\mathbf{r}_{A_0}, \mathbf{v}_{A_0})$ and $(\mathbf{r}_{B_0}, \mathbf{v}_{B_0})$ at the initial time t_0 using Algorithm 4.5 (as we did in Example 7.1).
2. Calculate the period T_A of A's orbit from Equation 2.82. (For the data of Example 7.1, $T_A = 5585\,\text{s}$).
3. Let the final time t_f for the plot be $t_0 + mT_A$, where m is an arbitrary integer.
4. Let n be the number of points to be plotted, so that the time step is $\Delta t = (t_f - t_0)/n$.
5. At time $t \geq t_0$:
   a. Calculate the state vectors $(\mathbf{r}_A, \mathbf{v}_A)$ and $(\mathbf{r}_B, \mathbf{v}_B)$ using Algorithm 3.4.
   b. Calculate $\mathbf{r}_{\text{rel}})_x$ using Algorithm 7.1.
   c. Plot the point $(x_{\text{rel}}, y_{\text{rel}}, z_{\text{rel}})$.
6. Let $t \leftarrow t + \Delta t$ and repeat Step 5 until $t = t_f$.

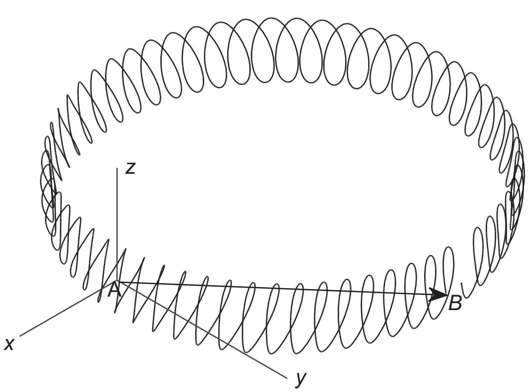


FIGURE 7.4

Trajectory of spacecraft B relative to spacecraft A for the data in Example 7.1. The total time is 60 periods of A's orbit (93.1 hours).

This algorithm is implemented in the MATLAB script *Example_7_02.m* listed in Appendix D.32. The resulting plot of the relative motion for a time interval of 60 periods of spacecraft A is shown in Figure 7.4. The arrow drawn from A to B is the initial position vector $\mathbf{r}_{\text{rel}})_x$ found in Example 7.1. As can be seen, the trajectory of B is a looping, counterclockwise motion around a circular path about 14,000 km in diameter. The closest approach of B to A is 105.5 km at an elapsed time of 25.75 hours.

7.3 LINEARIZATION OF THE EQUATIONS OF RELATIVE MOTION IN ORBIT

Figure 7.5, like Figure 7.1, shows two spacecraft in earth orbit. Let the inertial position vector of the target vehicle A be denoted $\mathbf{R}$, and that of the chase vehicle B be denoted $\mathbf{r}$. The position vector of the chase vehicle relative to the target is $\delta\mathbf{r}$, so that

$$\mathbf{r} = \mathbf{R} + \delta\mathbf{r} \tag{7.14}$$

The symbol δ is used to represent the fact that the relative position vector has a magnitude which is very small compared to the magnitude of $\mathbf{R}$ (and $\mathbf{r}$); that is,

$$\frac{\delta r}{R} \ll 1 \tag{7.15}$$

where $\delta r = \|\delta\mathbf{r}\|$ and $R = \|\mathbf{R}\|$. This is true if the two vehicles are in close proximity to each other, as is the case in a rendezvous maneuver or close formation flight. Our purpose in this section is to seek the equations of motion of the chase vehicle relative to the target when they are close together. Since the relative motion is seen from the target vehicle, its orbit is also called the reference orbit.

The equation of motion of the chase vehicle B relative to the inertial geocentric equatorial frame is

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} \tag{7.16}$$

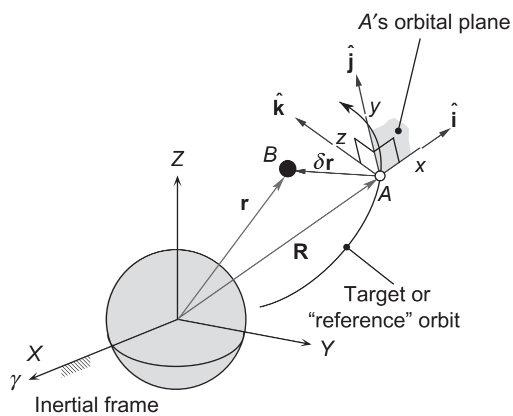


FIGURE 7.5

Position of chaser B relative to the target A.

where $r = \|\mathbf{r}\|$. Substituting Equation 7.14 into Equation 7.16 and writing $\delta\ddot{\mathbf{r}} = (d^2/dt^2)\delta\mathbf{r}$ yields the equation of motion of the chaser relative to the target,

$$\delta\ddot{\mathbf{r}} = -\ddot{\mathbf{R}} - \mu\frac{\mathbf{R} + \delta\mathbf{r}}{r^3} \qquad (\text{where } r = \|\mathbf{R} + \delta\mathbf{r}\|) \tag{7.17}$$

We will simplify this equation by making use of the fact that $\|\delta\mathbf{r}\|$ is very small, as expressed in Equation 7.15.

First, note that

$$r^2 = \mathbf{r}\cdot\mathbf{r} = (\mathbf{R} + \delta\mathbf{r})\cdot(\mathbf{R} + \delta\mathbf{r}) = \mathbf{R}\cdot\mathbf{R} + 2\mathbf{R}\cdot\delta\mathbf{r} + \delta\mathbf{r}\cdot\delta\mathbf{r}$$

Since $\mathbf{R}\cdot\mathbf{R} = R^2$ and $\delta\mathbf{r}\cdot\delta\mathbf{r} = \delta r^2$, we can factor out R^2 on the right to obtain

$$r^2 = R^2\left[1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2} + \left(\frac{\delta r}{R}\right)^2\right]$$

By virtue of Equation 7.15, we can neglect the last term in the brackets, so that

$$r^2 = R^2\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right) \tag{7.18}$$

In fact, we will neglect all powers of $\delta r/R$ greater than unity, wherever they appear. Since $r^{-3} = (r^2)^{-3/2}$, it follows from Equation 7.18 that

$$r^{-3} = R^{-3}\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-\frac{3}{2}} \tag{7.19}$$

Using the binomial theorem (Equation 5.44) and neglecting terms of higher order than 1 in $\delta r/R$, we obtain

$$\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-\frac{3}{2}} = 1 + \left(-\frac{3}{2}\right)\left(\frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)$$

Therefore, to our level of approximation, Equation 7.19 becomes

$$r^{-3} = R^{-3}\left(1 - \frac{3}{R^2}\mathbf{R}\cdot\delta\mathbf{r}\right)$$

which can be written

$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5}\mathbf{R}\cdot\delta\mathbf{r} \tag{7.20}$$

Substituting Equation 7.20 into Equation 7.17 (the equation of motion), we get

$$\begin{aligned}\delta\ddot{\mathbf{r}} &= -\ddot{\mathbf{R}} - \mu\left(\frac{1}{R^3} - \frac{3}{R^5}\mathbf{R}\cdot\delta\mathbf{r}\right)(\mathbf{R}+\delta\mathbf{r}) \\ &= -\ddot{\mathbf{R}} - \mu\left[\frac{\mathbf{R}+\delta\mathbf{r}}{R^3} - \frac{3}{R^5}(\mathbf{R}\cdot\delta\mathbf{r})(\mathbf{R}+\delta\mathbf{r})\right] \\ &= -\ddot{\mathbf{R}} - \mu\left[\frac{\mathbf{R}}{R^3} + \frac{\delta\mathbf{r}}{R^3} - \frac{3}{R^5}(\mathbf{R}\cdot\delta\mathbf{r})\mathbf{R} + \overbrace{\text{terms of higher order than 1 in } \delta\mathbf{r}}^{\text{neglect}}\right]\end{aligned}$$

That is, to our degree of approximation,

$$\delta\ddot{\mathbf{r}} = -\ddot{\mathbf{R}} - \mu\frac{\mathbf{R}}{R^3} - \frac{\mu}{R^3}\left[\delta\mathbf{r} - \frac{3}{R^2}(\mathbf{R}\cdot\delta\mathbf{r})\mathbf{R}\right] \tag{7.21}$$

But the equation of motion of the reference orbit is

$$\ddot{\mathbf{R}} = -\mu\frac{\mathbf{R}}{R^3} \tag{7.22}$$

Substituting this into Equation 7.21 finally yields

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}\left[\delta\mathbf{r} - \frac{3}{R^2}(\mathbf{R}\cdot\delta\mathbf{r})\mathbf{R}\right] \tag{7.23}$$

This is the linearized version of Equation 7.17, the equation which governs the motion of the chaser with respect to the target. The expression is linear because the unknown $\delta\mathbf{r}$ appears only in the numerator and only to the first power throughout. We achieved this by dropping a lot of terms that are insignificant when Equation 7.15 is valid. Equation 7.23 is nonlinear in $\mathbf{R}$, which is not an unknown because it is determined independently by solving Equation 7.22.

In the co-moving frame of Figure 7.5, the x axis lies along the radial $\mathbf{R}$, so that

$$\mathbf{R} = R\hat{\mathbf{i}} \tag{7.24}$$

In terms of its components in the co-moving frame, the relative position vector $\delta\mathbf{r}$ in Figure 7.5 is (cf. Equation 7.4a)

$$\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}} \tag{7.25}$$

Substituting Equations 7.24 and 7.25 into Equation 7.23 yields

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}\left[(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}) - \frac{3}{R^2}\left[(R\hat{\mathbf{i}})\cdot(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}})\right](R\hat{\mathbf{i}})\right]$$

After expanding the dot product on the right and collecting terms, we find that the linearized equation of relative motion takes a rather simple form when the components of $\mathbf{R}$ and $\delta\mathbf{r}$ are given in the co-moving frame,

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}(-2\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}) \tag{7.26}$$

Recall that $\delta\ddot{\mathbf{r}}$ is the acceleration of the chaser B relative to the target A as measured in the inertial frame. That is,

$$\delta\ddot{\mathbf{r}} = \frac{d^2}{dt^2}\delta\mathbf{r} = \frac{d^2}{dt^2}(\mathbf{r}_B - \mathbf{r}_A) = \ddot{\mathbf{r}}_B - \ddot{\mathbf{r}}_A = \mathbf{a}_B - \mathbf{a}_A$$

$\delta\ddot{\mathbf{r}}$ is not to be confused with $\delta\mathbf{a}_{\text{rel}}$, the relative acceleration measured in the co-moving frame. These two quantities are related by Equation 7.9

$$\delta\mathbf{a}_{\text{rel}} = \delta\ddot{\mathbf{r}} - \dot{\boldsymbol{\Omega}}\times\delta\mathbf{r} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\delta\mathbf{r}) - 2\boldsymbol{\Omega}\times\delta\mathbf{v}_{\text{rel}} \tag{7.27}$$

Since we arrived at an expression for $\delta\ddot{\mathbf{r}}$ in Equation 7.26, let us proceed to evaluate each of the three terms on the right of Equation 7.27 that involve $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}}$. First, recall that the angular momentum of A ($\mathbf{h} = \mathbf{R}\times\dot{\mathbf{R}}$) is normal to A's orbital plane, and so is the z axis of the co-moving frame. Therefore, $\mathbf{h} = h\hat{\mathbf{k}}$. It follows that Equations 7.5 and 7.6 may be written

$$\boldsymbol{\Omega} = \frac{h}{R^2}\hat{\mathbf{k}} \tag{7.28}$$

and

$$\dot{\boldsymbol{\Omega}} = -\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\hat{\mathbf{k}} \tag{7.29}$$

where $\mathbf{V} = \dot{\mathbf{R}}$.

From Equations 7.25, 7.28 and 7.29 we find

$$\dot{\boldsymbol{\Omega}}\times\delta\mathbf{r} = \left[-\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\hat{\mathbf{k}}\right]\times(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}) = \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}(\delta y\hat{\mathbf{i}} - \delta x\hat{\mathbf{j}}) \tag{7.30}$$

and

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\delta\mathbf{r}) = \frac{h}{R^2}\hat{\mathbf{k}}\times\left[\frac{h}{R^2}\hat{\mathbf{k}}\times(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}})\right] = -\frac{h^2}{R^4}(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}) \tag{7.31}$$

According to Equation 7.4b, $\delta\mathbf{v}_{\text{rel}} = \delta\dot{x}\hat{\mathbf{i}} + \delta\dot{y}\hat{\mathbf{j}} + \delta\dot{z}\hat{\mathbf{k}}$, where $\delta\dot{x} = (d/dt)\delta x$, etc. It follows that

$$2\boldsymbol{\Omega}\times\delta\mathbf{v}_{\text{rel}} = 2\frac{h}{R^2}\hat{\mathbf{k}}\times(\delta\dot{x}\hat{\mathbf{i}} + \delta\dot{y}\hat{\mathbf{j}} + \delta\dot{z}\hat{\mathbf{k}}) = 2\frac{h}{R^2}(\delta\dot{x}\hat{\mathbf{j}} - \delta\dot{y}\hat{\mathbf{k}}) \tag{7.32}$$

Substituting Equation 7.26 along with Equations 7.30 through 7.32 into Equation 7.27 yields

$$\delta\mathbf{a}_{\text{rel}} = \overbrace{-\frac{\mu}{R^3}(-2\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}})}^{\delta\ddot{\mathbf{r}}} - \overbrace{\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}(\delta y\hat{\mathbf{i}} - \delta x\hat{\mathbf{j}})}^{\dot{\boldsymbol{\Omega}}\times\delta\mathbf{r}} - \overbrace{\left[-\frac{h^2}{R^4}(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}})\right]}^{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\delta\mathbf{r})} - \overbrace{2\frac{h}{R^2}(\delta\dot{x}\hat{\mathbf{j}} - \delta\dot{y}\hat{\mathbf{i}})}^{2\boldsymbol{\Omega}\times\delta\mathbf{v}_{\text{rel}}}$$

Referring to Equation 7.4c, we set $\delta\mathbf{a}_{\text{rel}} = \delta\ddot{x}\hat{\mathbf{i}} + \delta\ddot{y}\hat{\mathbf{j}} + \delta\ddot{z}\hat{\mathbf{k}}$ [where $\delta\ddot{x} = (d^2/dt^2)\delta x$, etc.] and collect the terms on the right to obtain

$$\begin{aligned}\delta\ddot{x}\hat{\mathbf{i}} + \delta\ddot{y}\hat{\mathbf{j}} + \delta\ddot{z}\hat{\mathbf{k}} &= \left[\left(\frac{2\mu}{R^3} + \frac{h^2}{R^4}\right)\delta x - \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta y + 2\frac{h}{R^2}\delta\dot{y}\right]\hat{\mathbf{i}} \\ &+ \left[\left(\frac{h^2}{R^4} - \frac{\mu}{R^3}\right)\delta y + \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta x - 2\frac{h}{R^2}\delta\dot{x}\right]\hat{\mathbf{j}} \\ &- \frac{\mu}{R^3}\delta z\hat{\mathbf{k}}\end{aligned} \quad (7.33)$$

Finally, by equating the coefficients of the three unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, this vector equation yields the three scalar equations,

$$\delta\ddot{x} - \left(\frac{2\mu}{R^3} + \frac{h^2}{R^4}\right)\delta x + \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta y - 2\frac{h}{R^2}\delta\dot{y} = 0 \quad (7.34a)$$

$$\delta\ddot{y} + \left(\frac{\mu}{R^3} - \frac{h^2}{R^4}\right)\delta y - \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta x + 2\frac{h}{R^2}\delta\dot{x} = 0 \quad (7.34b)$$

$$\delta\ddot{z} + \frac{\mu}{R^3}\delta z = 0 \quad (7.34c)$$

This set of linear second order differential equations must be solved in order to obtain the relative position coordinates δx, δy and δz as a function of time. Equations 7.34a and 7.34b are coupled since δx and δy appear in each one. δz appears by itself in Equation 7.34c and nowhere else, which means the relative motion in the z direction is independent of that in the other two directions. If the reference orbit is an ellipse, then **R** and **V** vary with time (although the angular momentum h is constant). In that case, the coefficients in Equations 7.34 are time-dependent, so there is not an easy analytical solution. However, we can solve Equations 7.34 numerically using the methods of Section 1.8.

To that end, we recast Equations 7.34 as a set of first order differential equations in the standard form

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \quad (7.35)$$

where

$$\mathbf{y} = \begin{Bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta \dot{x} \\ \delta \dot{y} \\ \delta \dot{z} \end{Bmatrix} \qquad \dot{\mathbf{y}} = \begin{Bmatrix} \delta \dot{x} \\ \delta \dot{y} \\ \delta \dot{z} \\ \delta \ddot{x} \\ \delta \ddot{y} \\ \delta \ddot{z} \end{Bmatrix} \qquad \mathbf{f}(t,\mathbf{y}) = \begin{Bmatrix} y_4 \\ y_5 \\ y_6 \\ \left(\dfrac{2\mu}{R^3} + \dfrac{h^2}{R^4}\right) y_1 - \dfrac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4} y_2 + 2\dfrac{h}{R^2} y_5 \\ \left(\dfrac{h^2}{R^4} - \dfrac{\mu}{R^3}\right) y_2 + \dfrac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4} y_1 - 2\dfrac{h}{R^2} y_4 \\ -\dfrac{\mu}{R^3} y_3 \end{Bmatrix} \tag{7.36}$$

These can be solved by Algorithm 1.1 (Runge-Kutta), Algorithm 1.2 (Heun) or Algorithm 1.3 (Runge-Kutta-Fehlberg). In any case, the state vector of the target orbit must be updated at each time step to provide the current values of **R** and **V**. This is done with the aid of Algorithm 3.4. (Alternatively, Equation 7.22, the equations of motion of the target, can be integrated along with Equations 7.36 in order to provide **R** and **V** as a function of time.)

Example 7.3

At time $t = 0$, the orbital parameters of target vehicle A in an equatorial earth orbit are

$$r_p = 6678\,\text{km} \qquad e = 0.1 \qquad i = \Omega = \omega = \theta = 0^\circ \tag{a}$$

where r_p is the perigee radius. At that same instant, the state vector of the chaser vehicle B relative to A is

$$\delta\mathbf{r}_0 = -1\hat{\mathbf{i}}\ (\text{km}) \qquad \delta\mathbf{v}_{\text{rel}})_0 = 2n\hat{\mathbf{j}}\ (\text{km/s}) \tag{b}$$

where n is the mean motion of A. Plot the path of B relative to A in the co-moving frame for five periods of the reference orbit.

Solution

1. Use Algorithm 4.5 to obtain the initial state vector ($\mathbf{R}_0$,$\mathbf{V}_0$) of the target vehicle from the orbital parameters given in (a).
2. Starting with the initial conditions given in (b), use Algorithm 1.3 to integrate Equations 7.36 over the specified time interval. Use Algorithm 3.4 to obtain the reference orbit state vector (**R**,**V**) at each time step in order to evaluate the coefficients in Equation 7.36.
3. Graph the trajectory $\delta y(t)$ versus $\delta x(t)$.

This procedure is implemented in the MATLAB function *Example_7_03.m* listed in Appendix D.33. The output of the program is shown in Figure 7.6. Observe that since $\delta z_0 = \delta \dot{z}_0 = 0$, no movement develops in the z-direction. The motion of the chaser therefore lies in the plane of the target vehicle's orbit. The figure

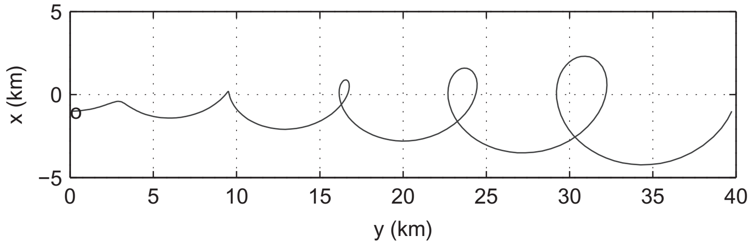


FIGURE 7.6

Trajectory of *B* relative to *A* in the co-moving frame during five revolutions of the reference orbit. Eccentricity of the reference orbit = 0.1. The small "o" marks the beginning of the simulation.

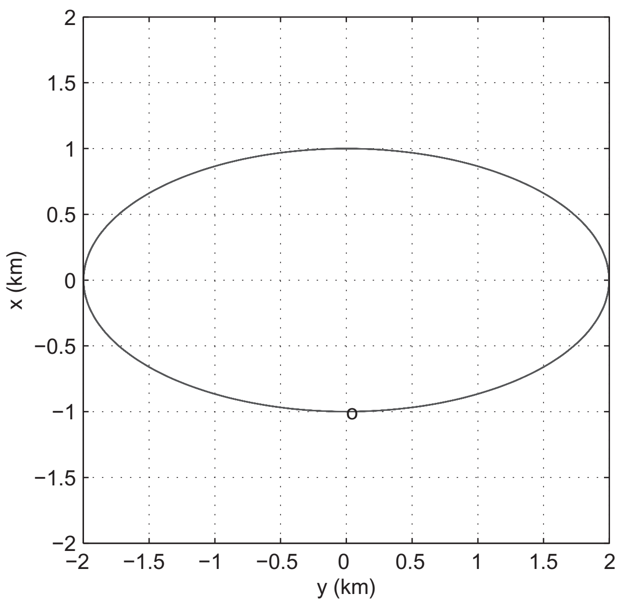


FIGURE 7.7

Trajectory of *B* relative to *A* in the co-moving frame during five revolutions of the reference orbit. Eccentricity of reference orbit = 0. The small "o" marks the beginning of the simulation.

shows that *B* rapidly moves away from *A* along the *y* direction and that the amplitude of its looping motion about the *x* axis continuously increases. The accuracy of this solution degrades over time because, eventually, the criterion in Equation 7.15 is no longer satisfied.

It is interesting to note that if we change the eccentricity of *A* to zero, so that the reference orbit is a circle, then Figure 7.7 results. That is, for the same initial conditions, *B* orbits the target vehicle instead of drifting away from it.

7.4 CLOHESSY-WILTSHIRE EQUATIONS

If the orbit of the target vehicle A in Figure 7.5 is a circle, then our LVLH frame is called a Clohessy-Wiltshire (CW) frame. In such a frame Equations 7.34 simplify considerably. For a circular target orbit, $\mathbf{V}\cdot\mathbf{R} = 0$ and $h = \sqrt{\mu R}$. Substituting these into Equations 7.34 yields

$$\begin{aligned} \delta\ddot{x} - 3\frac{\mu}{R^3}\delta x - 2\sqrt{\frac{\mu}{R^3}}\delta\dot{y} &= 0 \\ \delta\ddot{y} + 2\sqrt{\frac{\mu}{R^3}}\delta\dot{x} &= 0 \\ \delta\ddot{z} + \frac{\mu}{R^3}\delta z &= 0 \end{aligned} \tag{7.37}$$

It is furthermore true for circular orbits that the angular velocity (mean motion) is

$$n = \frac{V}{R} = \frac{\sqrt{\mu/R}}{R} = \sqrt{\frac{\mu}{R^3}}$$

Therefore, Equations 7.37 may be written

$$\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = 0 \tag{7.38a}$$

$$\delta\ddot{y} + 2n\delta\dot{x} = 0 \tag{7.38b}$$

$$\delta\ddot{z} + n^2\delta z = 0 \tag{7.38c}$$

These are known as the ***Clohessy-Wiltshire equations*** (CW equations). Unlike Equations 7.34, where the target orbit is an ellipse, the coefficients in Equations 7.38 are constant. Therefore, a straightforward analytical solution exists.

We start with the first two equations, which are coupled and define the motion of the chaser in the xy plane of the reference orbit. First, observe that Equation 7.38b can be written $(d/dt)(\delta\dot{y} + 2n\delta x) = 0$, which means that $\delta\dot{y} + 2n\delta x = C_1$, where C_1 is a constant. Therefore,

$$\delta\dot{y} = C_1 - 2n\delta x \tag{7.39}$$

Substituting this expression into Equation 7.38a yields

$$\delta\ddot{x} + n^2\delta x = 2nC_1 \tag{7.40}$$

This familiar differential equation has the following solution, which can be easily verified by substitution:

$$\delta x = \frac{2}{n}C_1 + C_2\sin nt + C_3\cos nt \tag{7.41}$$

Differentiating this expression gives the x-component of the relative velocity,

$$\delta\dot{x} = C_2 n \cos nt - C_3 n \sin nt \tag{7.42}$$

Substituting Equation 7.41 into Equation 7.39 yields the y-component of the relative velocity.

$$\delta\dot{y} = -3C_1 - 2C_2 n \sin nt - 2C_3 n \cos nt \tag{7.43}$$

Integrating this equation with respect to time yields

$$\delta y = -3C_1 t + 2C_2 \cos nt - 2C_3 \sin nt + C_4 \tag{7.44}$$

The constants C_1 through C_4 are found by applying the initial conditions, namely,

$$\text{At } t = 0 \qquad \delta x = \delta x_0 \qquad \delta y = \delta y_0 \qquad \delta\dot{x} = \delta\dot{x}_0 \qquad \delta\dot{y} = \delta\dot{y}_0$$

Evaluating Equations 7.41 through 7.44, respectively, at $t = 0$, we get

$$\begin{aligned}
\frac{2}{n}C_1 + C_3 &= \delta x_0 \\
C_2 n &= \delta\dot{x}_0 \\
-3C_1 - 2C_3 n &= \delta\dot{y}_0 \\
2C_2 + C_4 &= \delta y_0
\end{aligned}$$

Solving for C_1 through C_4 yields

$$C_1 = 2n\delta x_0 + \delta\dot{y}_0 \quad C_2 = \frac{1}{n}\delta\dot{x}_0 \quad C_3 = -3\delta x_0 - \frac{2}{n}\delta\dot{y}_0 \quad C_4 = -\frac{2}{n}\delta\dot{x}_0 + \delta y_0 \tag{7.45}$$

Finally we turn our attention to Equation 7.38c, which governs the relative motion normal to the plane of the circular reference orbit. Equation 7.38c has the same form as Equation 7.40 with $C_1 = 0$. Therefore, its solution is

$$\delta z = C_5 \sin nt + C_6 \cos nt \tag{7.46}$$

It follows that the velocity normal to the reference orbit is

$$\delta\dot{z} = C_5 n \cos nt - C_6 n \sin nt \tag{7.47}$$

The initial conditions are $\delta z = \delta z_0$ and $\delta\dot{z} = \delta\dot{z}_0$ at $t = 0$, which means

$$C_5 = \frac{\delta\dot{z}_0}{n} \qquad C_6 = \delta z_0 \tag{7.48}$$

Substituting Equations 7.45 and 7.48 into Equations 7.41, 7.44 and 7.46 yields the trajectory of the chaser in the CW frame,

$$\delta x = 4\delta x_0 + \frac{2}{n}\delta \dot{y}_0 + \frac{\delta \dot{x}_0}{n}\sin nt - \left(3\delta x_0 + \frac{2}{n}\delta \dot{y}_0\right)\cos nt \tag{7.49a}$$

$$\delta y = \delta y_0 - \frac{2}{n}\delta \dot{x}_0 - 3(2n\delta x_0 + \delta \dot{y}_0)t + 2\left(3\delta x_0 + \frac{2}{n}\delta \dot{y}_0\right)\sin nt + \frac{2}{n}\delta \dot{x}_0 \cos nt \tag{7.49b}$$

$$\delta z = \frac{1}{n}\delta \dot{z}_0 \sin nt + \delta z_0 \cos nt \tag{7.49c}$$

Observe that all three components of $\delta\mathbf{r}$ oscillate with a frequency equal to the frequency of revolution (mean motion n) of the CW frame. Only δy has a secular term, which grows linearly with time. Therefore, unless $2n\delta x_0 + \delta \dot{y}_0 = 0$, the chaser will drift away from the target and the distance δr will increase without bound. The accuracy of Equations 7.49 will consequently degrade as the criterion (Equation 7.15) on which this solution is based eventually ceases to be valid. Figure 7.8 shows the motion of a particle relative to a Clohessy-Wiltshire frame of orbital radius 6678 km. The particle started at the origin with a velocity of 0.01 km/s in the negative y direction. This delta-v dropped the particle into a lower energy, slightly elliptical orbit. The subsequent actual relative motion of the particle in the Clohessy-Wiltshire frame is graphed in Figure 7.8 as is the motion given by Equations 7.49, the linearized Clohessy-Wiltshire solution. Clearly the two solutions diverge markedly after one orbit of the reference frame, when the distance of the particle from the origin exceeds 150 km.

Now that we have finished solving the Clohessy-Wiltshire equations, let us simplify our notation a bit and denote the x, y and z components of relative velocity in the moving frame as δu, δv and δw, respectively. That is, let

$$\delta u = \delta \dot{x} \qquad \delta v = \delta \dot{y} \qquad \delta w = \delta \dot{z} \tag{7.50a}$$

The initial conditions on the relative velocity components are then written

$$\delta u_0 = \delta \dot{x}_0 \qquad \delta v_0 = \delta \dot{y}_0 \qquad \delta w_0 = \delta \dot{z}_0 \tag{7.50b}$$

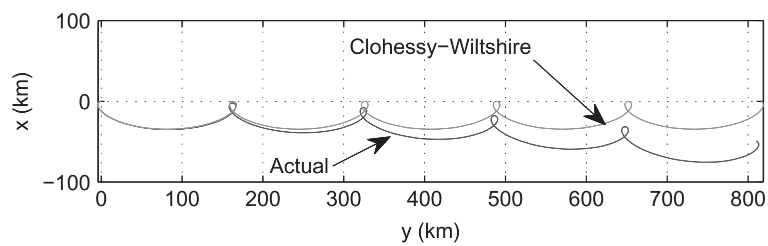


FIGURE 7.8

Relative motion of a particle and its Clohessy-Wiltshire approximation.

Using this notation in Equations 7.49 and rearranging terms we get

$$\begin{aligned}
\delta x &= (4 - 3\cos nt)\delta x_0 + \frac{\sin nt}{n}\delta u_0 + \frac{2}{n}(1 - \cos nt)\delta v_0 \\
\delta y &= 6(\sin nt - nt)\delta x_0 + \delta y_0 + \frac{2}{n}(\cos nt - 1)\delta u_0 + \frac{1}{n}(4\sin nt - 3nt)\delta v_0 \\
\delta z &= \cos nt\delta z_0 + \frac{1}{n}\sin nt\delta w_0
\end{aligned} \tag{7.51a}$$

Differentiating each of these with respect to time and using Equation 7.50a yields

$$\begin{aligned}
\delta u &= 3n\sin nt\delta x_0 + \cos nt\delta u_0 + 2\sin nt\delta v_0 \\
\delta v &= 6n(\cos nt - 1)\delta x_0 - 2\sin nt\delta u_0 + (4\cos nt - 3)\delta v_0 \\
\delta w &= -n\sin nt\delta z_0 + \cos nt\delta w_0
\end{aligned} \tag{7.51b}$$

Let us introduce matrix notation to define the relative position and velocity vectors

$$\{\delta \mathbf{r}(t)\} = \begin{Bmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{Bmatrix} \qquad \{\delta \mathbf{v}(t)\} = \begin{Bmatrix} \delta u(t) \\ \delta v(t) \\ \delta w(t) \end{Bmatrix}$$

and their initial values (at $t = 0$)

$$\{\delta \mathbf{r}_0\} = \begin{Bmatrix} \delta x_0 \\ \delta y_0 \\ \delta z_0 \end{Bmatrix} \qquad \{\delta \mathbf{v}_0\} = \begin{Bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{Bmatrix}$$

Observe that we have dropped the subscript *rel* introduced in Equations 7.17 because it is superfluous in rendezvous analysis, where all kinematic quantities are relative to the Clohessy-Wiltshire frame. In matrix notation Equations 7.51 appear more compactly as

$$\{\delta \mathbf{r}(t)\} = [\mathbf{\Phi}_{rr}(t)]\{\delta \mathbf{r}_0\} + [\mathbf{\Phi}_{rv}(t)]\{\delta \mathbf{v}_0\} \tag{7.52a}$$

$$\{\delta \mathbf{v}(t)\} = [\mathbf{\Phi}_{vr}(t)]\{\delta \mathbf{r}_0\} + [\mathbf{\Phi}_{vv}(t)]\{\delta \mathbf{v}_0\} \tag{7.52b}$$

where, from Equations 7.51, the ***Clohessy-Wiltshire matrices*** are

$$[\mathbf{\Phi}_{rr}(t)] = \left[\begin{array}{cc|c} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ \hline 0 & 0 & \cos nt \end{array}\right] \tag{7.53a}$$

$$[\mathbf{\Phi}_{\mathrm{rv}}(t)] = \left[\begin{array}{cc|c} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ \hline 0 & 0 & \frac{1}{n}\sin nt \end{array}\right] \tag{7.53b}$$

$$[\mathbf{\Phi}_{\mathrm{vr}}(t)] = \left[\begin{array}{cc|c} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ \hline 0 & 0 & -n\sin nt \end{array}\right] \tag{7.53c}$$

$$[\mathbf{\Phi}_{\mathrm{vv}}(t)] = \left[\begin{array}{cc|c} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ \hline 0 & 0 & \cos nt \end{array}\right] \tag{7.53c}$$

The subscripts on $\mathbf{\Phi}$ remind us which of the vectors $\delta\mathbf{r}$ and $\delta\mathbf{v}$ is related by that matrix to which of initial conditions $\delta\mathbf{r}_0$ and $\delta\mathbf{v}_0$. For example, $[\mathbf{\Phi}_{\mathrm{rv}}]$ relates $\delta\mathbf{r}$ to $\delta\mathbf{v}_0$. The partition lines remind us that motion in the xy plane is independent of that in the z direction normal to the target's orbit. In problems where there is no motion in the z direction ($\delta z_0 = \delta w_0 = 0$), we need only use the upper 2 by 2 corners of the Clohessy-Wiltshire matrices. Finally, note also that

$$[\mathbf{\Phi}_{\mathrm{vr}}(t)] = \frac{d}{dt}[\mathbf{\Phi}_{\mathrm{rr}}(t)] \quad \text{and} \quad [\mathbf{\Phi}_{\mathrm{vv}}(t)] = \frac{d}{dt}[\mathbf{\Phi}_{\mathrm{rv}}(t)]$$

7.5 TWO-IMPULSE RENDEZVOUS MANEUVERS

Figure 7.9 illustrates the ***two-impulse rendezvous*** problem. At time $t = 0^-$ (the instant preceding $t = 0$), the position $\delta\mathbf{r}_0$ and velocity $\delta\mathbf{v}_0^-$ of the chase vehicle B relative to the target A are known. At $t = 0$ an impulsive maneuver instantaneously changes the relative velocity to $\delta\mathbf{v}_0^+$ at $t = 0^+$ (the instant after $t = 0$). The components of $\delta\mathbf{v}_0^+$ are shown in Figure 7.6. We must determine the values of δu_0^+, δv_0^+ and δw_0^+, at the beginning of the rendezvous trajectory, so that B will arrive at the target in a specified time t_f. The delta-v required to place B on the rendezvous trajectory is

$$\Delta\mathbf{v}_0 = \delta\mathbf{v}_0^+ - \delta\mathbf{v}_0^- = (\delta u_0^+ - \delta u_0^-)\hat{\mathbf{i}} + (\delta v_0^+ - \delta v_0^-)\hat{\mathbf{j}} + (\delta w_0^+ - \delta w_0^-)\hat{\mathbf{k}} \tag{7.54}$$

At time t_f, B arrives at A, at the origin of the CW frame, which means $\delta\mathbf{r}_f = \delta\mathbf{r}(t_f) = \mathbf{0}$. Evaluating Equation 7.52a at t_f, we find

$$\{\mathbf{0}\} = [\mathbf{\Phi}_{\mathrm{rr}}(t_f)]\{\delta\mathbf{r}_0\} + [\Phi_{\mathrm{rv}}(t_f)]\{\delta\mathbf{v}_0^+\} \tag{7.55}$$

Solving this for $\{\delta\mathbf{v}_0^+\}$ yields

$$\{\delta\mathbf{v}_0^+\} = -[\mathbf{\Phi}_{\mathrm{rv}}(t_f)]^{-1}[\mathbf{\Phi}_{\mathrm{rr}}(t_f)]\{\delta\mathbf{r}_0\} \qquad (\delta\mathbf{v}_0^+ = \delta u_0^+\hat{\mathbf{i}} + \delta v_0^+\hat{\mathbf{j}} + \delta w_0^+\hat{\mathbf{k}}) \tag{7.56}$$

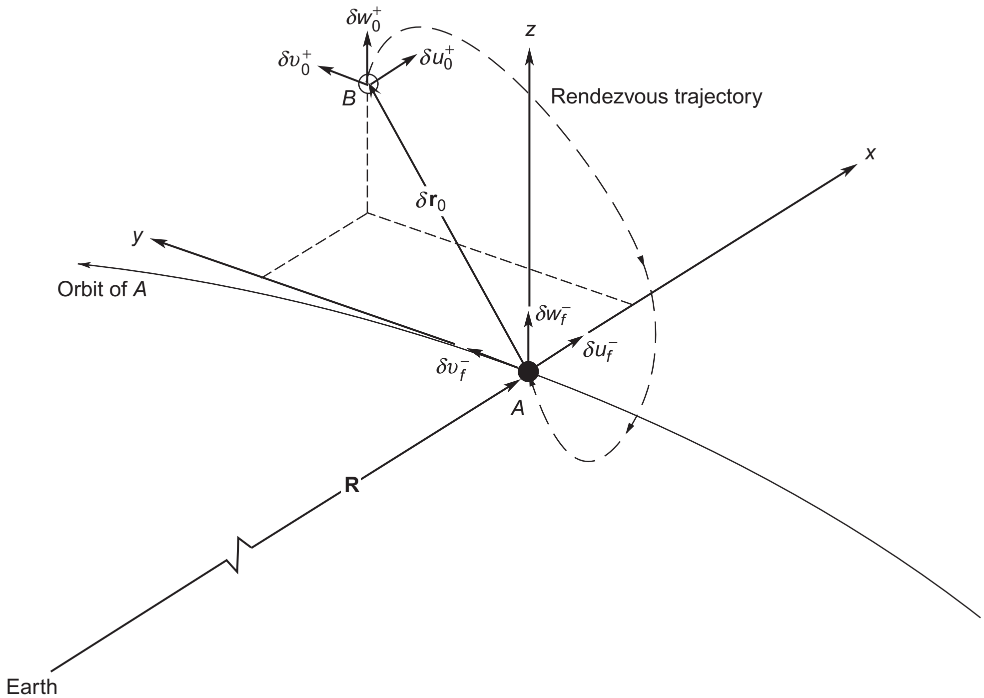


FIGURE 7.9

Rendezvous with a target *A* in the neighborhood of the chase vehicle *B*.

where $[\boldsymbol{\Phi}_{\text{rv}}(t_f)]^{-1}$ is the matrix inverse of $[\boldsymbol{\Phi}_{\text{rv}}(t_f)]$. Thus, we now have the velocity $\delta\mathbf{v}_0^+$ at the beginning of the rendezvous path. We substitute Equation 7.56 into Equation 7.52b to obtain the velocity $\delta\mathbf{v}_f^-$ at $t = t_f^{\;-}$, when *B* arrives at the target *A*:

$$\begin{aligned}\{\delta\mathbf{v}_f^-\} &= [\boldsymbol{\Phi}_{\text{vr}}(t_f)]\{\delta\mathbf{r}_0\} + [\boldsymbol{\Phi}_{\text{vv}}(t_f)]\{\delta\mathbf{r}_0\}\\ &= [\boldsymbol{\Phi}_{\text{vr}}(t_f)]\{\delta\mathbf{r}_0\} + [\boldsymbol{\Phi}_{\text{vv}}(t_f)](-[\boldsymbol{\Phi}_{\text{rv}}(t_f)]^{-1}[\boldsymbol{\Phi}_{\text{rr}}(t_f)]\{\delta\mathbf{r}_0\})\end{aligned}$$

Collecting terms, we get

$$\{\delta\mathbf{v}_f^-\} = ([\boldsymbol{\Phi}_{\text{vr}}(t_f)] - [\boldsymbol{\Phi}_{\text{vv}}(t_f)][\boldsymbol{\Phi}_{\text{rv}}(t_f)]^{-1}[\boldsymbol{\Phi}_{\text{rr}}(t_f)])\{\delta\mathbf{r}_0\} \qquad (\delta\mathbf{v}_f^- = \delta u_f^-\hat{\mathbf{i}} + \delta v_f^-\hat{\mathbf{j}} + \delta w_f^-\hat{\mathbf{k}}) \tag{7.57}$$

Obviously, an impulsive delta-v maneuver is required at $t = t_f$ to bring vehicle *B* to rest relative to *A* $(\delta\mathbf{v}_f^+ = \mathbf{0})$:

$$\Delta\mathbf{v}_f = \delta\mathbf{v}_f^+ - \delta\mathbf{v}_f^- = \mathbf{0} - \delta\mathbf{v}_f^- = -\delta\mathbf{v}_f^- \tag{7.58}$$

Note that in Equations 7.54 and 7.58 we are using the difference between relative velocities to calculate delta-v, which is the difference in absolute velocities. To show that this is valid, use Equation 1.66, to write

$$\begin{aligned}\mathbf{v}^- &= \mathbf{V}^- + \boldsymbol{\Omega}^- \times \mathbf{r}_{\text{rel}}^- + \mathbf{v}_{\text{rel}}^-\\ \mathbf{v}^+ &= \mathbf{V}^+ + \boldsymbol{\Omega}^+ \times \mathbf{r}_{\text{rel}}^+ + \mathbf{v}_{\text{rel}}^+\end{aligned} \tag{7.59}$$

Since the target is passive, the impulsive maneuver has no effect on its state of motion, which means $\mathbf{V}^+ = \mathbf{V}^-$ and $\boldsymbol{\Omega}^+ = \boldsymbol{\Omega}^-$. Furthermore, by definition of an impulsive maneuver, there is no change in the position, that is, $\mathbf{r}_{\text{rel}}^+ = \mathbf{r}_{\text{rel}}^-$. It follows from Equation 7.59 that

$$\mathbf{v}^+ - \mathbf{v}^- = \mathbf{v}_{\text{rel}}^+ - \mathbf{v}_{\text{rel}}^- \qquad \text{or} \qquad \Delta\mathbf{v} = \Delta\mathbf{v}_{\text{rel}}$$

Example 7.4

A space station and spacecraft are in orbits with the following parameters:

	Space station	Spacecraft
Perigee × apogee (altitude)	300 km	320.06 km × 513.86 km
Period (computed using above data)	1.5086 hr	1.5484 hr
True anomaly, θ	60°	349.65°
Inclination, i	40°	40.130°
RAAN, Ω	20°	19.819°
Argument of perigee, ω	0° (arbitrary)	70.662°

Compute the total delta-v required for an 8-hour, two-impulse rendezvous trajectory.

Solution

We use the given data in Algorithm 4.5 to obtain the state vectors of the two spacecraft in the geocentric equatorial frame.

Space station:

$$\mathbf{R} = 1622.39\hat{\mathbf{I}} + 5305.10\hat{\mathbf{J}} + 3717.55\hat{\mathbf{K}} \text{ (km)}$$
$$\mathbf{V} = -7.29936\hat{\mathbf{I}} + 0.492329\hat{\mathbf{J}} + 2.48304\hat{\mathbf{K}} \text{ (km/s)}$$

Spacecraft:

$$\mathbf{r} = 1612.75\hat{\mathbf{I}} + 5310.19\hat{\mathbf{J}} + 3750.33\hat{\mathbf{K}} \text{ (km)}$$
$$\mathbf{v} = -7.35170\hat{\mathbf{I}} + 0.463828\hat{\mathbf{J}} + 2.46906\hat{\mathbf{K}} \text{ (km/s)}$$

The space station reference frame unit vectors (at this instant) are, by definition:

$$\hat{\mathbf{i}} = \frac{\mathbf{R}}{\|\mathbf{R}\|} = 0.242945\hat{\mathbf{I}} + 0.794415\hat{\mathbf{J}} + 0.556670\hat{\mathbf{K}}$$
$$\hat{\mathbf{j}} = \frac{\mathbf{V}}{\|\mathbf{V}\|} = -0.944799\hat{\mathbf{I}} + 0.063725\hat{\mathbf{J}} + 0.321394\hat{\mathbf{K}}$$
$$\hat{\mathbf{k}} = \hat{\mathbf{i}} \times \hat{\mathbf{j}} = 0.219846\hat{\mathbf{I}} - 0.604023\hat{\mathbf{J}} + 0.766044\hat{\mathbf{K}}$$

Therefore, the transformation matrix from the geocentric equatorial frame into space station frame is (at this instant)

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{matrix} \leftarrow & \text{direction cosines of } \mathbf{i} \\ \leftarrow & \text{direction cosines of } \hat{\mathbf{j}} \\ \leftarrow & \text{direction cosines of } \hat{\mathbf{k}} \end{matrix}$$

The position vector of the spacecraft relative to space station (in the geocentric equatorial frame) is

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{R} = -9.64015\hat{\mathbf{I}} + 5.08235\hat{\mathbf{J}} + 32.8822\hat{\mathbf{K}} \text{ (km)}$$

The relative velocity is given by the formula (Equation 1.66)

$$\delta\mathbf{v} = \mathbf{v} - \mathbf{V} - \boldsymbol{\Omega}_{\text{space station}} \times \delta\mathbf{r}$$

where $\boldsymbol{\Omega}_{\text{space station}} = n\hat{\mathbf{k}}$ and n, the mean motion of the space station, is

$$n = \frac{V}{R} = \frac{7.72627}{6678} = 0.00115691 \text{ rad/s} \quad \text{(a)}$$

Thus,

$$\delta\mathbf{v} = \overbrace{-7.35170\hat{\mathbf{I}} + 0.463828\hat{\mathbf{J}} + 2.46906\hat{\mathbf{K}}}^{\mathbf{v}} - \overbrace{(-7.29936\hat{\mathbf{I}} + 0.492329\hat{\mathbf{J}} + 2.48304\hat{\mathbf{K}})}^{\mathbf{V}_0}$$
$$\overbrace{-(0.00115691)\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.219846 & -0.604023 & 0.766044 \\ -9.64015 & 5.08235 & 32.8822 \end{vmatrix}}^{\boldsymbol{\Omega}_{\text{space station}} \times \delta\mathbf{r}}$$

so that

$$\delta\mathbf{v} = -0.024854\hat{\mathbf{I}} - 0.01159370\hat{\mathbf{J}} - 0.00853575\text{v}\hat{\mathbf{K}} \text{ (km/s)}$$

In space station coordinates, the relative position vector $\delta\mathbf{r}_0$ at the beginning of the rendezvous maneuver is

$$\{\delta\mathbf{r}_0\} = [\mathbf{Q}]_{Xx}\{\delta\mathbf{r}\} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{Bmatrix} -9.64015 \\ 5.08235 \\ 32.8822 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix} \text{(km)} \quad \text{(b)}$$

Likewise, the relative velocity $\delta\mathbf{v}_0^-$ just before launch into the rendezvous trajectory is

$$\{\delta\mathbf{v}_0^-\} = [\mathbf{Q}]_{Xx}\{\delta\mathbf{v}\} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{Bmatrix} -0.024854 \\ -0.0115937 \\ -0.00853575 \end{Bmatrix} = \begin{Bmatrix} -0.02000 \\ 0.02000 \\ -0.005000 \end{Bmatrix} \text{(km/s)}$$

The Clohessy-Wiltshire matrices, for $t = t_f = 8$ hr $= 28{,}800$ s and $n = 0.00115691$ rad/s [from (a)], are

$$[\boldsymbol{\Phi}_{rr}] = \begin{bmatrix} 4-3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 4.97849 & 0 & 0 \\ -194.242 & 1.000 & 0 \\ 0 & 0 & -0.326163 \end{bmatrix}$$

$$[\boldsymbol{\Phi}_{rv}] = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \frac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 817.102 & 2292.60 & 0 \\ -2292.60 & -83131.6 & 0 \\ 0 & 0 & 817.103 \end{bmatrix}$$

$$[\boldsymbol{\Phi}_{vr}] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} 0.00328092 & 0 & 0 \\ -0.00920550 & 0 & 0 \\ 0 & 0 & -0.00109364 \end{bmatrix}$$

$$[\boldsymbol{\Phi}_{vv}] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} -0.326164 & 1.89063 & 0 \\ -1.89063 & -4.30466 & 0 \\ 0 & 0 & -0.326164 \end{bmatrix}$$

From Equation 7.56 and (b) we find $\delta\mathbf{v}_0^+$:

$$\begin{Bmatrix} \delta u_o^+ \\ \delta v_o^+ \\ \delta w_o^+ \end{Bmatrix} = -\begin{bmatrix} 817.102 & 2292.60 & 0 \\ -2292.60 & -83131.6 & 0 \\ 0 & 0 & 817.103 \end{bmatrix}^{-1} \begin{bmatrix} 4.97849 & 0 & 0 \\ -194.242 & 1.000 & 0 \\ 0 & 0 & -0.326163 \end{bmatrix} \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix}$$

$$= -\begin{bmatrix} 817.102 & 2292.60 & 0 \\ -2292.60 & -83131.6 & 0 \\ 0 & 0 & 817.103 \end{bmatrix}^{-1} \begin{Bmatrix} 99.5698 \\ -3864.84 \\ -6.52386 \end{Bmatrix} = \begin{Bmatrix} 0.00930458 \\ -0.0467472 \\ 0.00798343 \end{Bmatrix} \text{ (km/s)} \tag{c}$$

From Equation 7.52b, evaluated at $t = t_f$, we have

$$\{\delta\mathbf{v}_f^-\} = [\boldsymbol{\Phi}_{vr}(t_f)]\{\delta\mathbf{r}_0\} + [\boldsymbol{\Phi}_{vv}(t_f)]\{\delta\mathbf{v}_0^+\}$$

Substituting (b) and (c),

$$\begin{Bmatrix} \delta u_f^- \\ \delta v_f^- \\ \delta w_f^- \end{Bmatrix} = \begin{bmatrix} 0.00328092 & 0 & 0 \\ -0.00920550 & 0 & 0 \\ 0 & 0 & -0.00109364 \end{bmatrix} \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix} + \begin{bmatrix} -0.326164 & 1.89063 & 0 \\ -1.89063 & -4.30466 & 0 \\ 0 & 0 & -0.326164 \end{bmatrix} \begin{Bmatrix} 0.00930458 \\ -0.0467472 \\ 0.00798343 \end{Bmatrix}$$

$$\begin{Bmatrix} \delta u_f^- \\ \delta v_f^- \\ \delta w_f^- \end{Bmatrix} = \begin{Bmatrix} -0.0257978 \\ -0.000470870 \\ -0.0244767 \end{Bmatrix} \text{ (km/s)} \tag{d}$$

The delta-v at the beginning of the rendezvous maneuver is found as

$$\{\Delta\mathbf{v}_0\} = \{\delta\mathbf{v}_0^+\} - \{\delta\mathbf{v}_0^-\} = \begin{Bmatrix} 0.00930458 \\ -0.0467472 \\ 0.00798343 \end{Bmatrix} - \begin{Bmatrix} -.02 \\ 0.02 \\ -0.005 \end{Bmatrix} = \begin{Bmatrix} .0293046 \\ -.0667472 \\ 0.0129834 \end{Bmatrix} \text{ (km/s)}$$

The delta-v at the conclusion of the maneuver is

$$\{\Delta\mathbf{v}_f\} = \{\delta\mathbf{v}_f^+\} - \{\delta\mathbf{v}_f^-\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -0.0257978 \\ -0.000470870 \\ -0.0244767 \end{Bmatrix} = \begin{Bmatrix} 0.0257978 \\ 0.000470870 \\ 0.0244767 \end{Bmatrix} \text{ (km/s)}$$

The total delta-v requirement is

$$\Delta v_{\text{total}} = \left\|\Delta\mathbf{v}_0\right\| + \left\|\Delta\mathbf{v}_f\right\| = 0.0740440 + 0.0355649 = 0.109609 \text{ km/s} = 109.6 \text{ m/s}$$

From Equation 7.52a, we have, for $0 < t < t_f$,

$$\begin{Bmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(\text{t}) \end{Bmatrix} = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{n}\sin nt & \dfrac{2}{n}(1 - \cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \dfrac{1}{n}\sin nt \end{bmatrix} \begin{Bmatrix} 0.00930458 \\ -0.0467472 \\ 0.00798343 \end{Bmatrix}$$

Substituting n from (a), we obtain the relative position vector as a function of time. It is plotted in Figure 7.10.

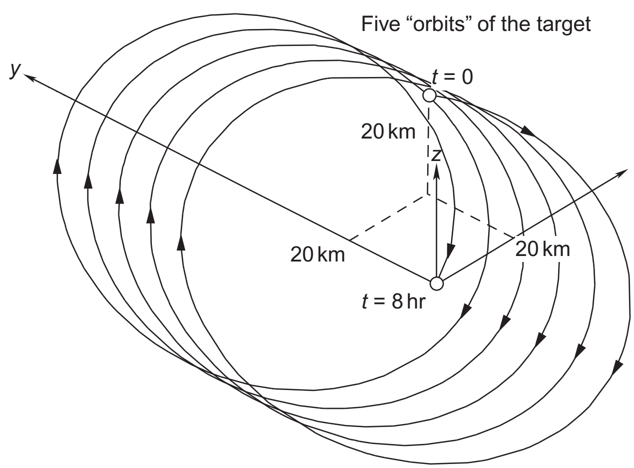


FIGURE 7.10

Rendezvous trajectory of the chase vehicle relative to the target.

Example 7.5

A target and a chase vehicle are in the same 300 km circular earth orbit. The chaser is 2 km behind the target when the chaser initiates a two-impulse rendezvous maneuver so as to rendezvous with the target in 1.49 hours. Find the total delta-v requirement.

Solution

For the circular reference orbit

$$V = \sqrt{\frac{\mu}{R}} = \sqrt{\frac{398{,}600}{6378 + 300}} = 7.7258 \text{ km/s} \tag{a}$$

so that the mean motion is

$$n = \frac{V}{R} = \frac{7.7258}{6678} = 0.0011569 \text{ rad/s} \tag{b}$$

For this mean motion and the rendezvous trajectory time $t = 1.49$ hr $= 5364$ s, the Clohessy-Wiltshire matrices are

$$[\boldsymbol{\Phi}_{rr}] = \left[\begin{array}{cc|c} 1.0090 & 0 & 0 \\ -37.699 & 1 & 0 \\ \hline 0 & 0 & 0.99700 \end{array}\right] \qquad [\boldsymbol{\Phi}_{rv}] = \left[\begin{array}{cc|c} -66.946 & 5.1928 & 0 \\ -5.1928 & -16{,}360 & 0 \\ \hline 0 & 0 & -66.946 \end{array}\right]$$

$$[\boldsymbol{\Phi}_{vr}] = \left[\begin{array}{cc|c} -2.6881\times10^{-4} & 0 & 0 \\ -2.0851\times10^{-5} & 0 & 0 \\ \hline 0 & 0 & 8.9603\times10^{-5} \end{array}\right] \qquad [\boldsymbol{\Phi}_{vv}] = \left[\begin{array}{cc|c} 0.99700 & -0.15490 & 0 \\ 0.15490 & 0.98798 & 0 \\ \hline 0 & 0 & 0.99700 \end{array}\right] \tag{c}$$

The initial and final positions of the chaser in the CW frame are

$$[\delta\mathbf{r}_0] = \begin{Bmatrix} 0 \\ -2 \\ 0 \end{Bmatrix} \text{(km)} \qquad \left\{\delta\mathbf{r}_f\right\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{(km)} \tag{d}$$

Since $\delta z_0 = \delta w_0 = 0$, there is no motion in the z direction $[\delta z(t) = 0]$, so we need employ only the upper left 2 by 2 corners of the Clohessy-Wiltshire matrices and treat this as a two dimensional problem in the plane of the reference orbit. Thus, solving the first CW equation, $\{\delta\mathbf{r}_f\} = [\boldsymbol{\Phi}_{rr}]\{\delta\mathbf{r}_0\} + [\boldsymbol{\Phi}_{rv}]\{\delta\mathbf{v}_0^+\}$ for $\{\delta\mathbf{v}_0^+\}$, we get

$$\{\delta\mathbf{v}_0^+\} = -[\boldsymbol{\Phi}_{rv}]^{-1}[\boldsymbol{\Phi}_{rr}]\{\delta\mathbf{r}_0\} = -\begin{bmatrix} -0.014937 & -4.7412\times10^{-6} \\ 4.7412\times10^{-6} & -6.1124\times10^{-5} \end{bmatrix}\begin{bmatrix} 1.0090 & 0 \\ -37.699 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -2 \end{Bmatrix} = \begin{Bmatrix} -9.4824\times10^{-6} \\ -1.2225\times10^{-4} \end{Bmatrix}$$

or

$$\delta\mathbf{v}_0^+ = -9.4824\times10^{-6}\,\hat{\mathbf{i}} - 1.2225\times10^{-4}\,\hat{\mathbf{j}} \text{ (km/s)} \tag{e}$$

Therefore, the second CW equation, $\{\delta\mathbf{v}_f^-\} = [\mathbf{\Phi}_{vr}]\{\delta\mathbf{r}_0\} + [\mathbf{\Phi}_{vv}]\{\delta\mathbf{v}_0^+\}$, yields

$$\{\delta\mathbf{v}_f^-\} = \begin{bmatrix} -2.6881\times10^{-4} & 0 \\ -2.0851\times10^{-5} & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ -2 \end{Bmatrix} + \begin{bmatrix} 0.99700 & -0.15490 \\ 0.15490 & 0.98798 \end{bmatrix}\begin{Bmatrix} -9.4824\times10^{-6} \\ -1.2225\times10^{-4} \end{Bmatrix} = \begin{Bmatrix} 9.4824\times10^{-6} \\ -1.2225\times10^{-4} \end{Bmatrix}$$

or

$$\delta\mathbf{v}_f^- = 9.4824\times10^{-6}\hat{\mathbf{i}} - 1.2225\times10^{-4}\hat{\mathbf{j}} \text{ (km/s)} \tag{f}$$

Since the chaser is in the same circular orbit as the target, its relative velocity is initially zero, that is $\delta\mathbf{v}_0^- = \mathbf{0}$. (See also Equation 7.68 at the end of the next section.) Thus,

$$\begin{aligned}\Delta\mathbf{v}_0 &= \delta\mathbf{v}_0^+ - \delta\mathbf{v}_0^- = (-9.4824\times10^{-6}\hat{\mathbf{i}} - 1.2225\times10^{-4}\hat{\mathbf{j}}) - \mathbf{0} \\ &= -9.4824\times10^{-6}\hat{\mathbf{i}} - 1.2225\times10^{-4}\hat{\mathbf{j}} \text{ (km/s)}\end{aligned}$$

which implies

$$\|\Delta\mathbf{v}_0\| = 0.1226 \text{ m/s} \tag{g}$$

At the end of the rendezvous maneuver, $\delta\mathbf{v}_f^+ = \mathbf{0}$, so that

$$\Delta\mathbf{v}_f = \delta\mathbf{v}_f^+ - \delta\mathbf{v}_f^- = \mathbf{0} - (9.4824\times10^{-6}\hat{\mathbf{i}} - 1.2225\times10^{-4}\hat{\mathbf{j}}) = -9.4824\times10^{-6}\hat{\mathbf{i}} + 1.2225\times10^{-4}\hat{\mathbf{j}} \text{ (km/s)}$$

Therefore,

$$\|\Delta\mathbf{v}_f\| = 0.1226 \text{ m/s} \tag{h}$$

The total delta-v required is

$$\Delta v_{total} = \|\Delta\mathbf{v}_0\| + \|\Delta\mathbf{v}_f\| = \boxed{0.2452 \text{ m/s}} \tag{i}$$

The coplanar rendezvous trajectory relative to the CW frame is sketched in Figure 7.11.

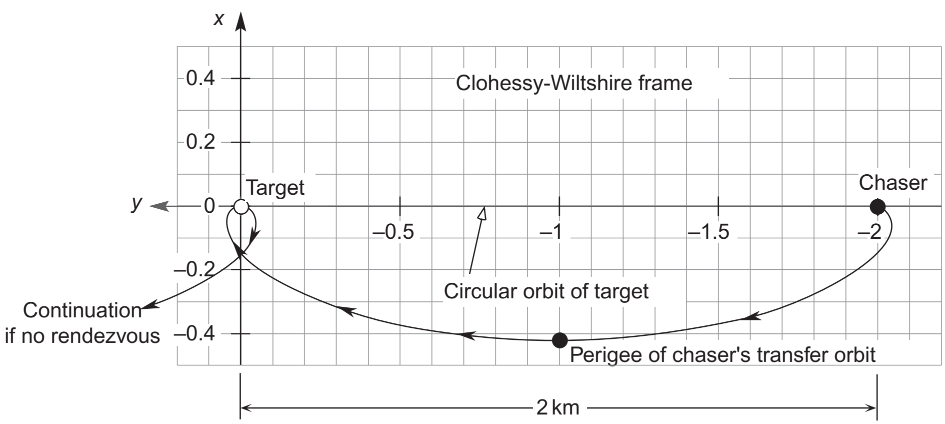


FIGURE 7.11

Motion of the chaser relative to the target.

7.6 RELATIVE MOTION IN CLOSE-PROXIMITY CIRCULAR ORBITS

Figure 7.12 shows two spacecraft in coplanar circular orbits. Let us calculate the velocity $\delta\mathbf{v}$ of the chase vehicle B relative to the target A when they are in close proximity. "Close proximity" means that

$$\frac{\delta r}{R} \ll 1$$

To solve this problem, we must use the relative velocity equation,

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \delta\mathbf{r} + \delta\mathbf{v} \tag{7.60}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the CW frame attached to A,

$$\boldsymbol{\Omega} = n\hat{\mathbf{k}}$$

n is the mean motion of the target vehicle,

$$n = \frac{v_A}{R} \tag{7.61}$$

where, by virtue of the circular orbit,

$$v_A = \sqrt{\frac{\mu}{R}} \tag{7.62}$$

Solving Equation 7.60 for the relative velocity δv yields

$$\delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A - (n\hat{\mathbf{k}}) \times \delta\mathbf{r} \tag{7.63}$$

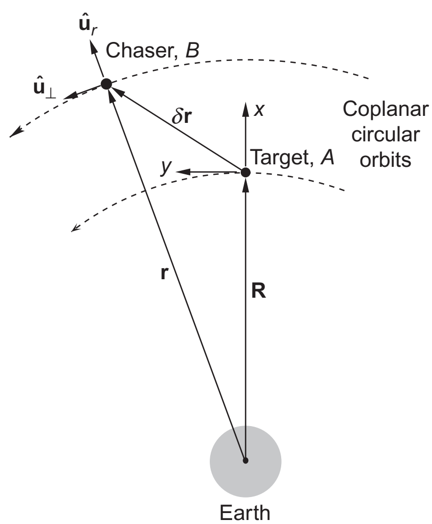


FIGURE 7.12

Two spacecraft in close proximity.

The chase orbit is circular. Therefore, for the first term on the right-hand side of Equation 7.63 we have

$$\mathbf{v}_B = \sqrt{\frac{\mu}{r}}\hat{\mathbf{u}}_\perp = \sqrt{\frac{\mu}{r}}\,(\hat{\mathbf{k}} \times \hat{\mathbf{u}}_r) = \sqrt{\mu}\hat{\mathbf{k}} \times \left(\frac{1}{\sqrt{r}}\frac{\mathbf{r}}{r}\right) \tag{7.64}$$

Since, as is apparent from Figure 7.9, $\mathbf{r} = \mathbf{R} + \delta\mathbf{r}$, we can write this expression for $\mathbf{v}_B$ as follows:

$$\mathbf{v}_B = \sqrt{\mu}\hat{\mathbf{k}} \times r^{-3/2}(\mathbf{R} + \delta\mathbf{r}) \tag{7.65}$$

Now

$$r^{-3/2} = (r^2)^{-3/4} = \left[\overbrace{R^2\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)}^{\text{See Equation 7.18}}\right]^{-3/4} = R^{-3/2}\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-3/4} \tag{7.66}$$

Using the binomial theorem (Equation 5.44), and retaining terms at most linear in $\delta\mathbf{r}$, we get

$$\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-3/4} = 1 - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^2}$$

Substituting this into Equation 7.66, leads to

$$r^{-3/2} = R^{-3/2} - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^{7/2}}$$

Upon substituting this result into Equation 7.65, we get

$$\mathbf{v}_B = \sqrt{\mu}\hat{\mathbf{k}} \times (\mathbf{R} + \delta\mathbf{r})\left(R^{-3/2} - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^{7/2}}\right)$$

Retaining terms at most linear in $\delta\mathbf{r}$, we can write this as

$$\mathbf{v}_B = \hat{\mathbf{k}} \times \left\{\sqrt{\frac{\mu}{R}}\frac{\mathbf{R}}{R} + \frac{\sqrt{\mu/R}}{R}\delta\mathbf{r} - \frac{3}{2}\frac{\sqrt{\mu/R}}{R}\left[\left(\frac{\mathbf{R}}{R}\right)\cdot\delta\mathbf{r}\right]\frac{\mathbf{R}}{R}\right\}$$

Using Equations 7.61 and 7.62, together with the facts that $\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}$ and $\mathbf{R}/R = \hat{\mathbf{i}}$, this reduces to

$$\begin{aligned}\mathbf{v}_B &= \hat{\mathbf{k}} \times \left\{v_A\hat{\mathbf{i}} + \frac{v_A}{R}(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}) - \frac{3}{2}\frac{v_A}{R}[\hat{\mathbf{i}}\cdot(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}})]\hat{\mathbf{i}}\right\} \\ &= v_A\hat{\mathbf{j}} + (-n\delta y\hat{\mathbf{i}} + n\delta x\hat{\mathbf{j}}) - \tfrac{3}{2}n\delta x\hat{\mathbf{j}}\end{aligned}$$

so that

$$\mathbf{v}_B = -n\delta y\hat{\mathbf{i}} + \left(v_A - \tfrac{1}{2}n\delta x\right)\hat{\mathbf{j}} \tag{7.67}$$

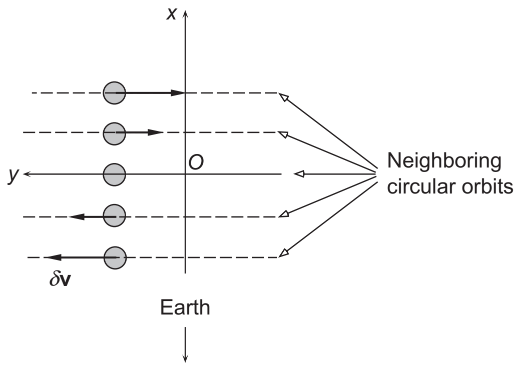


FIGURE 7.13

Circular orbits, with relative velocity directions, in the vicinity of the Clohessy-Wiltshire frame.

This is the absolute velocity of the chaser resolved into components in the target's Clohessy-Wiltshire frame.

Substituting Equation 7.67 into 7.63 and using the fact that $\mathbf{v}_A = v_A\hat{\mathbf{j}}$ yields

$$\begin{aligned}\delta\mathbf{v} &= \left[-n\delta y\hat{\mathbf{i}} + \left(v_A - \tfrac{1}{2}n\delta x\right)\hat{\mathbf{j}}\right] - (v_A\hat{\mathbf{j}}) - (n\hat{\mathbf{k}}) \times (\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}) \\ &= -n\delta y\hat{\mathbf{i}} + v_A\hat{\mathbf{j}} - \tfrac{1}{2}n\delta x\hat{\mathbf{j}} - v_A\hat{\mathbf{j}} - n\delta x\hat{\mathbf{j}} + n\delta y\hat{\mathbf{i}}\end{aligned}$$

so that

$$\delta\mathbf{v} = -\tfrac{3}{2}n\delta x\hat{\mathbf{j}} \tag{7.68}$$

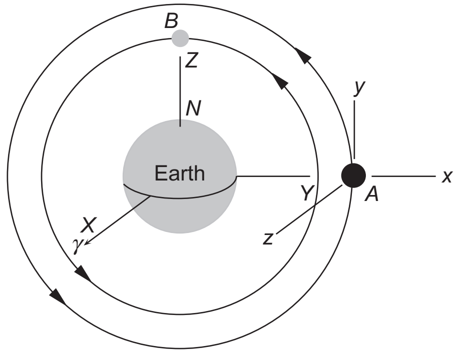
This is the velocity of the chaser as measured in the moving reference frame of the neighboring target. Keep in mind that circular orbits were assumed at the outset.

In the Clohessy-Wiltshire frame, neighboring coplanar circular orbits appear to be straight lines parallel to the *y* axis, which is the orbit of the origin. Figure 7.13 illustrates this point, showing also the linear velocity variation according to Equation 7.68.

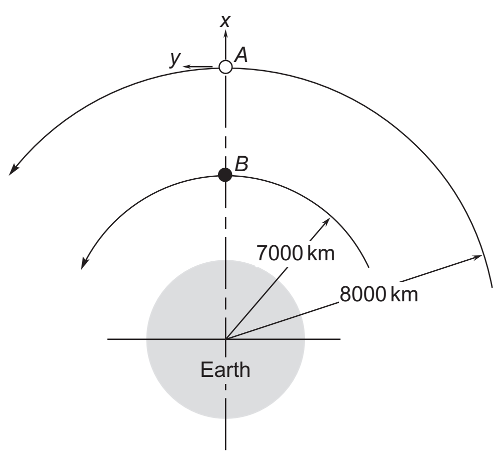
PROBLEMS

Section 7.2

7.1 Two manned spacecraft, *A* and *B* (see figure), are in circular polar ($i = 90°$) orbits around the earth. *A*'s orbital altitude is 300 km; *B*'s is 250 km. At the instant shown (*A* over the equator, *B* over the north pole), calculate (a) the position (b) velocity and (c) the acceleration of *B* relative to *A*. *A*'s *y*-axis points always in the flight direction, and its *x* axis is directed radially outward at all times.
{Ans.: (a) $\mathbf{r}_{\text{rel}})_{xyz} = -6678\hat{\mathbf{i}} + 6628\hat{\mathbf{j}}$ km; (b) $\mathbf{v}_{\text{rel}})_{xyz} = -0.08693\mathbf{i}$ km/s;
(c) $\mathbf{a}_{\text{rel}})_{xyz} = -1.140 \times 10^{-6}\,\mathbf{j}$ km/s^2}

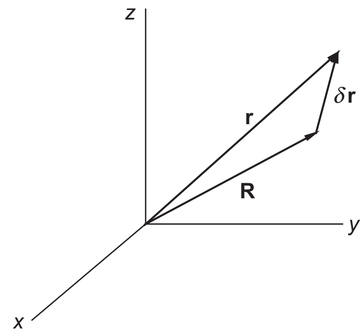


7.2 Spacecraft *A* and *B* are in coplanar, circular geocentric orbits. The orbital radii are shown in the figure. When *B* is directly below *A*, as shown, calculate *B*'s speed $v_B)_{rel}$ relative to *A*.
{Ans.: $v_B)_{rel}$ =1.370 km/s}



Section 7.3

7.3 Use the order of magnitude analysis in this chapter as a guide to answer the following questions.
(a) If $\mathbf{r} = \mathbf{R} + \delta\mathbf{r}$, express $\sqrt{r}$ (where $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$) to the first order in $\delta\mathbf{r}$ (i.e., to the first order in the components of $\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}$). In other words, find $O(\delta\mathbf{r})$, such that $\sqrt{r} = \sqrt{R} + O(\delta\mathbf{r})$, where $O(\delta\mathbf{r})$ is linear in $\delta\mathbf{r}$.
(b) For the special case $\mathbf{R} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\delta\mathbf{r} = 0.01\mathbf{i} - 0.01\mathbf{j} + 0.03\mathbf{k}$, calculate $\sqrt{r} - \sqrt{R}$ and compare that result with $O(\delta\mathbf{r})$.
(c) Repeat part (b) using $\delta\mathbf{r} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and compare the results.
{Ans.: (a) $O(\delta\mathbf{r}) = \mathbf{R}\cdot\delta\mathbf{r}/(2R^{3/2})$; (b) $O(\delta\mathbf{r})/\left(\sqrt{r} - \sqrt{R}\right) = 0.998$; (c) $O(\delta\mathbf{r})/\left(\sqrt{r} - \sqrt{R}\right) = 0.903$}



7.4 Write the expression $r = a(1 - e^2)/1 + e\cos\theta$ as a linear function of e, valid for small values of e ($e \ll 1$).

Section 7.4

7.5 Given $\ddot{x} + 9x = 10$, with the initial conditions $x = 5$ and $\dot{x} = -3$ at $t = 0$, find x and $\dot{x}$ at $t = 1.2$.
{Ans.: $x(1.2) = -1.934$, $\dot{x}(1.2) = 7.853$}

7.6 Given that

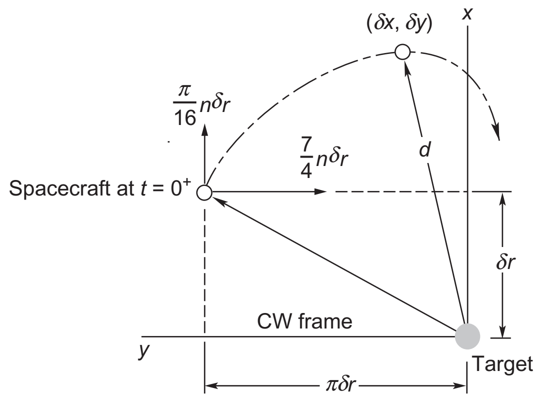
$$\ddot{x} + 10x + 2\dot{y} = 0$$
$$\ddot{y} + 3\dot{x} = 0$$

with initial conditions $x(0) = 1$, $y(0) = 2$, $\dot{x}(0) = -3$ and $\dot{y}(0) = 4$, find x and y at $t = 5$.
{Ans.: $x(5) = -6.460$, $y(5) = 97.31$}

7.7 A space station is in a 90-minute period earth orbit. At $t = 0$, a satellite has the following position and velocity components relative to a Clohessy-Wiltshire frame attached to the space station: $\delta\mathbf{r} = \hat{\mathbf{i}}$ (km), $\delta\mathbf{v} = 10\hat{\mathbf{j}}$ (m/s). How far is the satellite from the space station 15 minutes later?
{Ans.: 11.2 km}

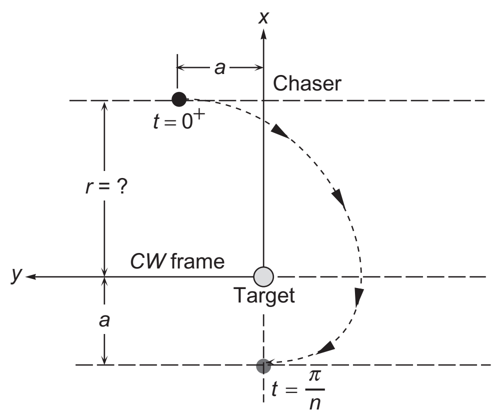
7.8 Spacecraft A and B are in the same circular earth orbit with a period of 2 hours. B is 6 km ahead of A. At $t = 0$, B applies an in-track delta-v (retrofire) of 3 m/s. Using a Clohessy-Wiltshire frame attached to A, determine the distance between A and B at $t = 30$ minutes and the velocity of B relative to A.
{Ans.: $\delta r = 10.9$ km, $\delta v = 10.8$ m/s}

7.9 The Clohessy-Wiltshire coordinates and velocities of a spacecraft upon entering a rendezvous trajectory with the target vehicle are shown. The spacecraft orbits are co-planar. Calculate the distance d of the spacecraft from the target when $t = \pi/2n$, where n is the mean motion of the target's circular orbit.
{Ans.: $0.900\delta r$}



7.10 At time $t = 0$ a particle is at the origin of a CW frame with a relative velocity $\delta\mathbf{v}_0 = v\hat{\mathbf{j}}$. What will be the relative speed of the particle after one-half orbital period of the C-W frame?
{Ans.: $7v$}

7.11 The chaser and the target are in close-proximity, coplanar circular orbits. At $t = 0$, the position of the chaser relative to the target is $\delta\mathbf{r}_0 = r\hat{\mathbf{i}} + a\hat{\mathbf{j}}$, where a is given and r is unknown. The relative velocity at $t = 0^+$ is $\delta\mathbf{v}_0^+ = v_0\hat{\mathbf{j}}$ (v_0 is unknown), and the chaser ends up at $\delta\mathbf{r}_f = -a\hat{\mathbf{i}}$ when $t = \pi/n$, where n is the mean motion of the target. Use the Clohessy-Wiltshire equations to find the required value of the orbital spacing r.
{Ans.: $1.424a$}



Section 7.5

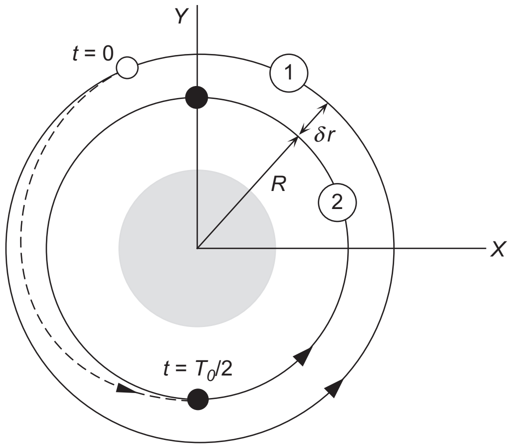
7.12 A space station is in a circular earth orbit of radius 6600 km. An approaching spacecraft executes a delta-v burn when its position vector relative to the space station is $\delta\mathbf{r}_0 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ (km). Just before the burn the relative velocity of the spacecraft was $\delta\mathbf{v}_0^- = 5\hat{\mathbf{k}}$ (m/s). Calculate the total delta-v required for the space shuttle to rendezvous with the station in one third period of the space station orbit.
{Ans.: 6.21 m/s}

7.13 A space station is in circular orbit 2 of radius R. A spacecraft is in coplanar circular orbit 1 of radius $R + \delta r$. At $t = 0$ the spacecraft executes an impulsive maneuver to rendezvous with the space station at time t_f = one-half the period T_0 of the space station. If $\delta u_0^+ = 0$, find

(a) the initial position of the spacecraft relative to the space station, and

(b) the relative velocity of the spacecraft when it arrives at the target. Sketch the rendezvous trajectory relative to the target.

{Ans.: (a) $\delta \mathbf{r}_0 = \delta r\hat{\mathbf{i}} + \left(3\pi\delta r/4\right)\hat{\mathbf{j}}$, (b) $\delta \mathbf{v}_f^- = \left(\pi\delta r/2T_0\right)\mathbf{j}$}

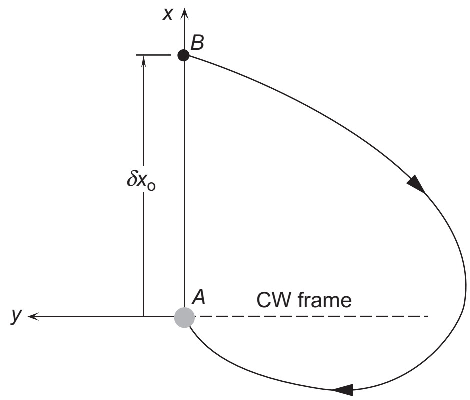


7.14 If $\delta u_0^+ = 0$, calculate the total delta-v required for rendezvous if $\delta \mathbf{r}_0 = \delta y_0\hat{\mathbf{j}}$, $\delta \mathbf{v}_0^- = \mathbf{0}$ and t_f = the period of the circular target orbit. Sketch the rendezvous trajectory relative to the target.
{Ans.: $\Delta v_{\text{tot}} = 2\delta y_o/(3T)$}

7.15 A GEO satellite strikes some orbiting debris and is found 2 hours afterwards to have drifted to the position $\delta \mathbf{r} = -10\hat{\mathbf{i}} + 10\hat{\mathbf{j}}$ (km) relative to its original location. At that time the only slightly damaged satellite initiates a two-impulse maneuver to return to its original location in 6 hours. Find the total delta-v for this maneuver.
{Ans.: 3.5 m/s}

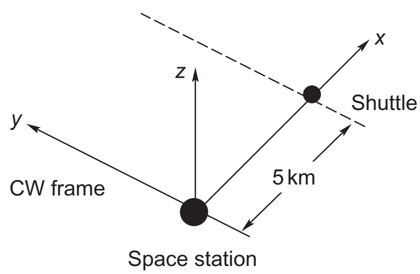
7.16 A space station is in a 245 km circular earth orbit inclined at 30°.The right ascension of its node line is 40°. Meanwhile, a space shuttle has been launched into a 280 km by 250 km orbit inclined at 30.1°, with a nodal right ascension of 40° and argument of perigee equal to 60°.When the shuttle's true anomaly is 40°, the space station is 99° beyond its node line. At that instant, the space shuttle executes a delta-v burn to rendezvous with the space station in (precisely) t_f hours, where t_f is selected by you or assigned by the instructor. Calculate the total delta-v required and sketch the projection of the rendezvous trajectory on the xy plane of the space station coordinates.

7.17 The target A is in a circular earth orbit with mean motion n. The chaser B is directly above A in a slightly larger circular orbit having the same plane as A's. What relative initial velocity $\delta \mathbf{v}_0^+$ is required so that B arrives at the target A at time t_f = one-half the target's period?
{Ans.: $\delta \mathbf{v}_0^+ = -0.589n\delta x_0\mathbf{i} - 1.75n\delta x_0\mathbf{j}$}

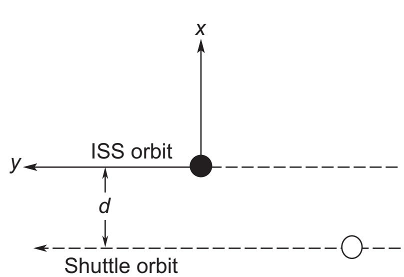


Section 7.6

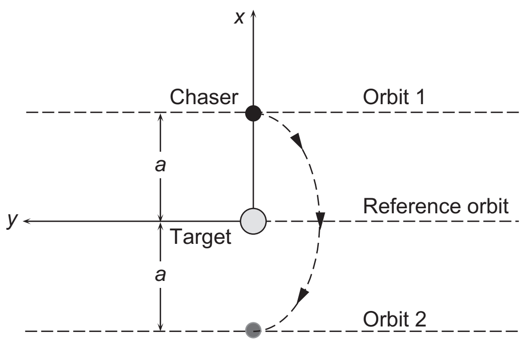
7.18 The space station is in a circular earth orbit of radius 6600 km. The space shuttle is also in a circular orbit in the same plane as the space station. At the instant that the position of the shuttle relative to the space station, in Clohessy-Wiltshire coordinates, is $\delta\mathbf{r} = 5\hat{\mathbf{i}}\,(\text{km})$. What is the relative velocity $\delta\mathbf{v}$ of the space shuttle?
{Ans.: 8.83 m/s}



7.19 The Space Shuttle and the International Space Station are in coplanar circular orbits. The space station has an orbital radius R and a mean motion n. The shuttle's radius is $R - d$ ($d < < R$). If a two-impulse rendezvous maneuver with $t_f = \pi/(4n)$ is initiated with zero relative velocity in the x-direction $(\delta u_0^+ = 0)$, calculate the total delta-v.
{Ans.: 4.406nd}



7.20 The chaser and the target are in close-proximity, coplanar circular orbits. At $t = 0$, the position of the chaser relative to the target is $\delta \mathbf{r}_0 = a\hat{\mathbf{i}}$. Use the Clohessy-Wiltshire equations to find the total delta-v required for the chaser to end up in circular orbit 2 at $\delta \mathbf{r}_f = -a\hat{\mathbf{i}}$ when $t = \pi/n$, where n is the mean motion of the target.
{Ans.: na}



List of Key Terms

angular acceleration of co-moving frame
angular velocity of co-moving frame
Clohessy-Wiltshire equations
Clohessy-Wiltshire matrices
LVLH frame
r-bar
relative acceleration in the inertial frame
relative acceleration in the co-moving frame
two-impulse rendezvous

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CHAPTER

Interplanetary trajectories

8

Chapter outline

8.1	Introduction	429
8.2	Interplanetary Hohmann transfers	430
8.3	Rendezvous opportunities	432
8.4	Sphere of influence	437
8.5	Method of patched conics	441
8.6	Planetary departure	442
8.7	Sensitivity analysis	448
8.8	Planetary rendezvous	451
8.9	Planetary flyby	458
8.10	Planetary ephemeris	470
8.11	Non-Hohmann interplanetary trajectories	475

8.1 INTRODUCTION

In this chapter we consider some basic aspects of planning interplanetary missions. We begin by considering Hohmann transfers, which are the easiest to analyze and the most energy-efficient. The orbits of the planets involved must lie in the same plane and the planets must be positioned just right for a Hohmann transfer to be used. The time between such opportunities is derived. The method of patched conics is employed to divide the mission up into three parts: the hyperbolic departure trajectory relative to the home planet; the cruise ellipse relative to the sun; and the hyperbolic arrival trajectory, relative to the target plane.

The use of patched conics is justified by calculating the radius of a planet's sphere of influence and showing how small it is on the scale of the solar system. Matching the velocity of the spacecraft at the home planet's sphere of influence to that required to initiate the outbound cruise phase and then specifying the periapsis radius of the departure hyperbola determines the delta-v requirement at departure. The sensitivity of the target radius to the burnout conditions is discussed. Matching the velocities at the target planet's sphere of influence and specifying the periapsis of the arrival hyperbola yields the delta-v at the target for a planetary rendezvous or the direction of the outbound hyperbola for a planetary flyby. Flyby maneuvers are discussed, including the effect of leading and trailing side flybys, and some noteworthy examples of the use of gravity assist maneuvers are presented.

The chapter concludes with an analysis of the situation in which the planets' orbits are not coplanar and the transfer ellipse is tangent to neither orbit. This is akin to the chase maneuver in Chapter 6 and requires the solution of Lambert's problem using Algorithm 5.2.

8.2 INTERPLANETARY HOHMANN TRANSFERS

As can be seen from Table A.1, the orbits of most of the planets in the solar system lie very close to the earth's orbital plane (the ecliptic plane). The innermost planet, Mercury, and the outermost dwarf planet, Pluto, differ most in inclination (7° and 17°, respectively). The orbital planes of the other planets lie within 3.5° of the ecliptic. It is also evident from Table A.1 that most of the planetary orbits have small eccentricities, the exceptions once again being Mercury and Pluto. To simplify the beginning of our study of interplanetary trajectories, we will assume that all of the planets' orbits are circular and coplanar. Later on, in Section 8.10, we will relax this assumption.

The most energy efficient way for a spacecraft to transfer from one planet's orbit to another is to use a Hohmann transfer ellipse (Section 6.2). Consider Figure 8.1, which shows a Hohmann transfer from an inner planet 1 to an outer planet 2. The departure point D is at periapsis (perihelion) of the transfer ellipse and the arrival point is at apoapsis (aphelion). The circular orbital speed of planet 1 relative to the sun is given by Equation 2.63,

$$V_1 = \sqrt{\frac{\mu_{\text{sun}}}{R_1}} \tag{8.1}$$

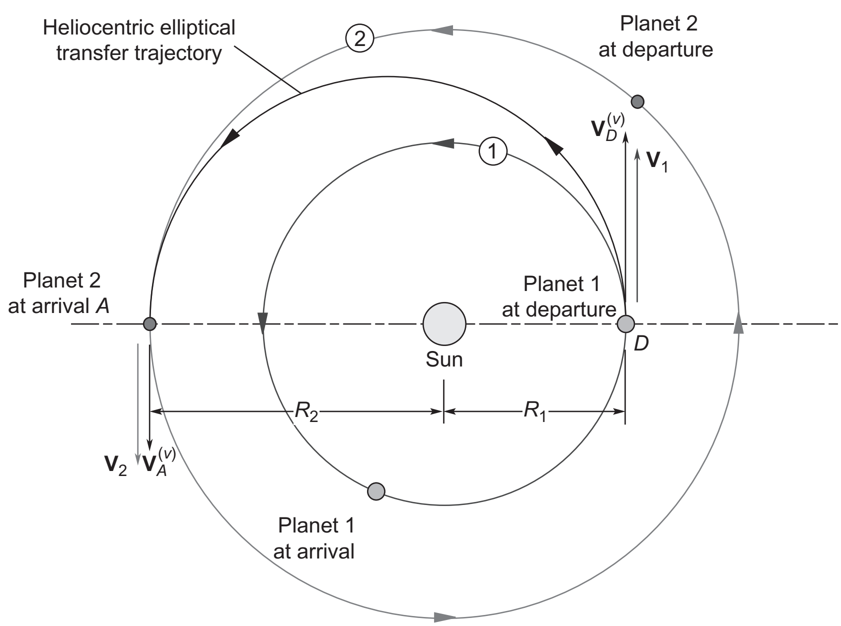

FIGURE 8.1

Hohmann transfer from inner planet 1 to outer planet 2.

The specific angular momentum h of the transfer ellipse relative to the sun is found from Equation 6.2, so that the speed of the space vehicle on the transfer ellipse at the departure point D is

$$V_D^{(v)} = \frac{h}{R_1} = \sqrt{2\mu_{\text{sun}}}\sqrt{\frac{R_2}{R_1(R_1 + R_2)}} \tag{8.2}$$

This is greater than the speed of the planet. Therefore, the required delta-v at D is

$$\Delta V_D = V_D^{(v)} - V_1 = \sqrt{\frac{\mu_{\text{sun}}}{R_1}}\left(\sqrt{\frac{2R_2}{R_1 + R_2}} - 1\right) \tag{8.3}$$

Likewise, the delta-v at the arrival point A is

$$\Delta V_A = V_2 - V_A^{(v)} = \sqrt{\frac{\mu_{\text{sun}}}{R_2}}\left(1 - \sqrt{\frac{2R_1}{R_1 + R_2}}\right) \tag{8.4}$$

This velocity increment, like that at point D, is positive since planet 2 is traveling faster than the spacecraft at point A.

For a mission from an outer planet to an inner planet, as illustrated in Figure 8.2, the delta-v's computed using Equations 8.3 and 8.4 will both be negative instead of positive. That is because the departure point and arrival point are now at aphelion and perihelion, respectively, of the transfer ellipse. The speed of the spacecraft must be reduced for it to drop into the lower-energy transfer ellipse at the departure point D, and it must be reduced again at point A in order to arrive in the lower-energy circular orbit of planet 2.

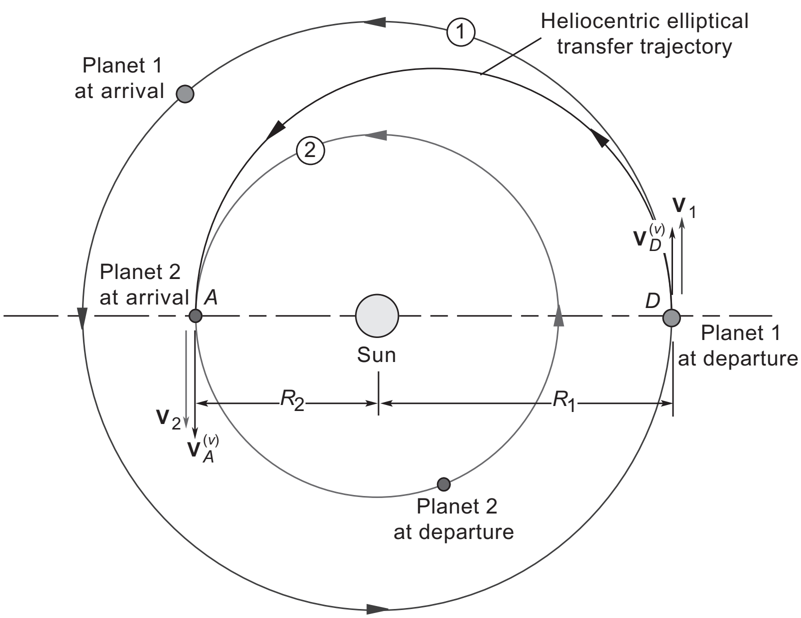

FIGURE 8.2

Hohmann transfer from outer planet 1 to inner planet 2.

8.3 RENDEZVOUS OPPORTUNITIES

The purpose of an interplanetary mission is for the spacecraft not only to intercept a planet's orbit but also to rendezvous with the planet when it gets there. For rendezvous to occur at the end of a Hohmann transfer, the location of planet 2 in its orbit at the time of the spacecraft's departure from planet 1 must be such that planet 2 arrives at the apse line of the transfer ellipse at the same time the spacecraft does. Phasing maneuvers (Section 6.5) are clearly not practical, especially for manned missions, due to the large periods of the heliocentric orbits.

Consider planet 1 and planet 2 in circular orbits around the sun, as shown in Figure 8.3. Since the orbits are circular, we can choose a common horizontal apse line from which to measure the true anomaly θ. The true anomalies of planets 1 and 2, respectively, are

$$\theta_1 = \theta_{1_0} + n_1 t \tag{8.5}$$

$$\theta_2 = \theta_{2_0} + n_2 t \tag{8.6}$$

where n_1 and n_2 are the mean motions (angular velocities) of the planets and θ_{1_0} and θ_{2_0} are their true anomalies at time $t = 0$. The phase angle between the position vectors of the two planets is defined as

$$\phi = \theta_2 - \theta_1 \tag{8.7}$$

ϕ is the angular position of planet 2 relative to planet 1. Substituting Equations 8.5 and 8.6 into 8.7 we get

$$\phi = \phi_0 + (n_2 - n_1)t \tag{8.8}$$

ϕ_0 is the phase angle at time zero. $n_2 - n_1$ is the orbital angular velocity of planet 2 relative to planet 1. If the orbit of planet 1 lies inside that of planet 2, as in Figure 8.3(a), then $n_1 > n_2$. Therefore, the relative angular velocity $n_2 - n_1$ is negative, which means planet 2 moves clockwise relative to planet 1. On the other hand, if planet 1 is outside of planet 2 then $n_2 - n_1$ is positive, so that the relative motion is counterclockwise.

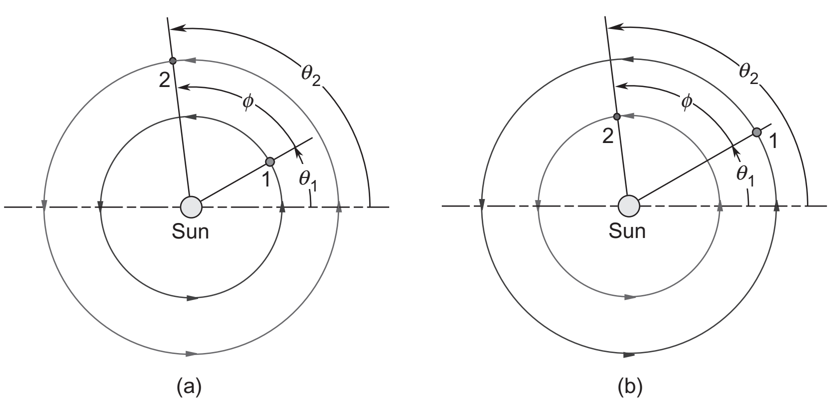

FIGURE 8.3

Planets in circular orbits around the sun. (a) Planet 2 outside the orbit of planet 1. (b) Planet 2 inside the orbit of planet 1.

The phase angle obviously varies linearly with time according to Equation 8.8. If the phase angle is ϕ_0 at $t = 0$, how long will it take to become ϕ_0 again? The answer: when the position vector of planet 2 rotates through 2π radians relative to planet 1. The time required for the phase angle to return to its initial value is called the ***synodic period,*** which is denoted T_{syn}. For the case shown in Figure 8.3(a) in which the relative motion is clockwise, T_{syn} is the time required for ϕ to change from ϕ_0 to $\phi_0 - 2\pi$. From Equation 8.8 we have

$$\phi_0 - 2\pi = \phi_0 + (n_2 - n_1)T_{syn}$$

so that

$$T_{syn} = \frac{2\pi}{n_1 - n_2} \quad (n_1 > n_2)$$

For the situation illustrated in Figure 8.3(b) ($n_2 > n_1$), T_{syn} is the time required for ϕ to go from ϕ_0 to $\phi_0 + 2\pi$, in which case Equation 8.8 yields

$$T_{syn} = \frac{2\pi}{n_2 - n_1} \quad (n_2 > n_1)$$

Both cases are covered by writing

$$T_{syn} = \frac{2\pi}{|n_1 - n_2|} \tag{8.9}$$

Recalling Equation 3.9, we can write $n_1 = 2\pi/T_1$ and $n_2 = 2\pi/T_2$. Thus, in terms of the orbital periods of the two planets,

$$T_{syn} = \frac{T_1 T_2}{|T_1 - T_2|} \tag{8.10}$$

Observe that T_{syn} is the orbital period of planet 2 relative to planet 1.

Example 8.1

Calculate the synodic period of Mars relative to the earth.

Solution

In Table A.1 we find the orbital periods of earth and Mars:

$$T_{earth} = 365.26 \text{ days (1 year)}$$
$$T_{Mars} = 1 \text{ year } 321.73 \text{ days} = 687.99 \text{ days}$$

Hence,

$$T_{syn} = \frac{T_{earth} T_{Mars}}{|T_{earth} - T_{Mars}|} = \frac{365.26 \times 687.99}{|365.26 - 687.99|} = 777.9 \text{ days}$$

These are earth days (1 day = 24 hours). Therefore, it takes 2.13 years for a given configuration of Mars relative to the earth to occur again.

Figure 8.4 depicts a mission from planet 1 to planet 2. Following a heliocentric Hohmann transfer, the spacecraft intercepts and rendezvous with planet 2. Later it returns to planet 1 by means of another Hohmann transfer. The major axis of the heliocentric transfer ellipse is the sum of the radii of the two planets' orbits, $R_1 + R_2$. The time t_{12} required for the transfer is one-half the period of the ellipse. Hence, according to Equation 2.83,

$$t_{12} = \frac{\pi}{\sqrt{\mu_{\text{sun}}}}\left(\frac{R_1 + R_2}{2}\right)^{\frac{3}{2}} \tag{8.11}$$

During the time it takes the spacecraft to fly from orbit 1 to orbit 2, through an angle of π radians, planet 2 must move around its circular orbit and end up at a point directly opposite of planet 1's position when the spacecraft departed. Since planet 2's angular velocity is n_2, the angular distance traveled by the planet during the spacecraft's trip is $n_2 t_{12}$. Hence, as can be seen from Figure 8.4(a), the initial phase angle ϕ_0 between the two planets is

$$\phi_0 = \pi - n_2 t_{12} \tag{8.12}$$

When the spacecraft arrives at planet 2, the phase angle will be ϕ_f, which is found using Equations 8.8 and 8.12.

$$\begin{aligned} \phi_f &= \phi_0 + (n_2 - n_1)t_{12} = (\pi - n_2 t_{12}) + (n_2 - n_1)t_{12} \\ \phi_f &= \pi - n_1 t_{12} \end{aligned} \tag{8.13}$$

For the situation illustrated in Figure 8.4, planet 2 ends up being behind planet 1 by an amount equal to the magnitude of ϕ_f.

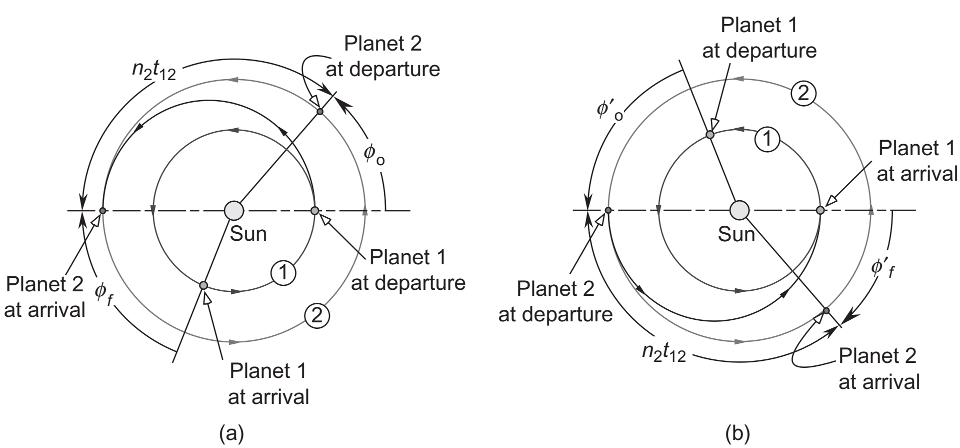

FIGURE 8.4

Round-trip mission, with layover, to planet 2. (a) Departure and rendezvous with planet 2. (b) Return and rendezvous with planet 1.

At the start of the return trip, illustrated in Figure 8.4(b), planet 2 must be ϕ_0' radians ahead of planet 1. Since the spacecraft flies the same Hohmann transfer trajectory back to planet 1, the time of flight is t_{12}, the same as the outbound leg. Therefore, the distance traveled by planet 1 during the return trip is the same as the outbound leg, which means

$$\phi_0' = -\phi_f \tag{8.14}$$

In any case, the phase angle at the beginning of the return trip must be the negative of the phase angle at arrival from planet 1. The time required for the phase angle to reach its proper value is called the ***wait time***, t_{wait}. Setting time equal to zero at the instant we arrive at planet 2, Equation 8.8 becomes

$$\phi = \phi_f + (n_2 - n_1)t$$

ϕ becomes $-\phi_f$ after the time t_{wait}. That is

$$-\phi_f = \phi_f + (n_2 - n_1)t_{wait}$$

or

$$t_{wait} = \frac{-2\phi_f}{n_2 - n_1} \tag{8.15}$$

where ϕ_f is given by Equation 8.13. Equation 8.15 may yield a negative result, which means the desired phase relation occurred in the past. Therefore, we must add or subtract an integral multiple of 2π to the numerator in order to get a positive value for t_{wait}. Specifically, if $N = 0,1,2,\ldots$, then

$$t_{wait} = \frac{-2\phi_f - 2\pi N}{n_2 - n_1} \quad (n_1 > n_2) \tag{8.16}$$

$$t_{wait} = \frac{-2\phi_f + 2\pi N}{n_2 - n_1} \quad (n_1 < n_2) \tag{8.17}$$

where N is chosen to make t_{wait} positive. t_{wait} would probably be the smallest positive number thus obtained.

Example 8.2

Calculate the minimum wait time for initiating a return trip from Mars to earth.

Solution

From Tables A.1 and A.2 we have

$$R_{earth} = 149.6 \times 10^6 \text{ km}$$
$$R_{Mars} = 227.9 \times 10^6 \text{ km}$$
$$\mu_{sun} = 132.71 \times 10^9 \text{ km}^3/\text{s}^2$$

According to Equation 8.11, the time of flight from earth to Mars is

$$t_{12} = \frac{\pi}{\sqrt{\mu_{sun}}}\left(\frac{R_{earth} + R_{Mars}}{2}\right)^{\frac{3}{2}}$$

$$= \frac{\pi}{\sqrt{132.71 \times 10^9}}\left(\frac{149.6 \times 10^6 + 227.9 \times 10^6}{2}\right)^{\frac{3}{2}} = 2.2362 \times 10^7\,\text{s}$$

or

$$t_{12} = 258.82 \text{ days}$$

From Equation 3.9 and the orbital periods of earth and Mars (see Example 8.1 above) we obtain the mean motions of the earth and Mars.

$$n_{earth} = \frac{2\pi}{365.26} = 0.017202 \text{ rad/day}$$

$$n_{Mars} = \frac{2\pi}{687.99} = 0.0091327 \text{ rad/day}$$

The phase angle between earth and Mars when the spacecraft reaches Mars is given by Equation 8.13.

$$\phi_f = \pi - n_{earth}t_{12} = \pi - 0.017202 \cdot 258.82 = -1.3107 \text{ (rad)}$$

Since $n_{earth} > n_{Mars}$, we choose Equation 8.16 to find the wait time.

$$t_{wait} = \frac{-2\phi_f - 2\pi N}{n_{Mars} - n_{earth}} = \frac{-2(-1.3107) - 2\pi N}{0.0091327 - 0.017202} = 778.65N - 324.85 \text{ (days)}$$

$N = 0$ yields a negative value, which we cannot accept. Setting $N = 1$, we get

$$\boxed{t_{wait} = 453.8 \text{ days}}$$

This is the minimum wait time. Obviously, we could set $N = 2,3,\ldots$ to obtain longer wait times.

In order for a spacecraft to depart on a mission to Mars by means of a Hohmann (minimum energy) transfer, the phase angle between earth and Mars must be that given by Equation 8.12. Using the results of Example 8.2, we find it to be

$$\phi_o = \pi - n_{Mars}t_{12} = \pi - 0.0091327 \cdot 258.82 = 0.7778 \text{ rad} = 44.57°$$

This opportunity occurs once every synodic period, which we found to be 2.13 years in Example 8.1. In Example 8.2 we found that the time to fly to Mars is 258.8 days, followed by a wait time of 453.8 days, followed by a return trip time of 258.8 days. Hence, the minimum total time for a manned Mars mission is

$$t_{total} = 258.8 + 453.8 + 258.8 = 971.4 \text{ days} = 2.66 \text{ years}$$

Remember that this result is for Hohmann transfer trajectories for the spacecraft and circular coplanar orbits for earth and Mars.

8.4 SPHERE OF INFLUENCE

The sun, of course, is the dominant celestial body in the solar system. It is over one thousand times more massive than the largest planet, Jupiter, and has a mass of over 300 000 earths. The sun's gravitational pull holds all of the planets in its grasp according to Newton's law of gravity, Equation 1.31. However, near a given planet the influence of its own gravity exceeds that of the sun. For example, at its surface the earth's gravitational force is over 1600 times greater than the sun's. The inverse-square nature of the law of gravity means that the force of gravity F_g drops off rapidly with distance r from the center of attraction. If F_{g_0} is the gravitational force at the surface of a planet with radius r_0, then Figure 8.5 shows how rapidly the force diminishes with distance. At ten body radii, the force is 1% of its value at the surface. Eventually, the force of the sun's gravitational field overwhelms that of the planet.

In order to estimate the radius of a planet's gravitational sphere of influence, consider the three-body system comprising a planet p of mass m_p, the sun s of mass m_s and a space vehicle v of mass m_v illustrated in Figure 8.6. The position vectors of the planet and spacecraft relative to an inertial frame centered at the sun are $\mathbf{R}$ and $\mathbf{R}_v$, respectively. The position vector of the space vehicle relative to the planet is $\mathbf{r}$. (Throughout this chapter we will use upper case letters to represent position, velocity and acceleration measured relative to the sun and lower case letters when they are measured relative to a planet.) The gravitational force exerted on the vehicle by the planet is denoted $\mathbf{F}_p^{(v)}$, and that exerted by the sun is $\mathbf{F}_s^{(v)}$. Likewise, the forces on the planet are $\mathbf{F}_s^{(p)}$ and $\mathbf{F}_v^{(p)}$, whereas on the sun we have $\mathbf{F}_v^{(s)}$ and $\mathbf{F}_p^{(s)}$. According to Newton's law of gravitation (Equation 2.10), these forces are

$$\mathbf{F}_p^{(v)} = -\frac{Gm_v m_p}{r^3}\mathbf{r} \tag{8.18a}$$

$$\mathbf{F}_s^{(v)} = -\frac{Gm_v m_s}{R_V^3}\mathbf{R}_v \tag{8.18b}$$

$$\mathbf{F}_s^{(p)} = -\frac{Gm_p m_s}{R^3}\mathbf{R} \tag{8.18c}$$

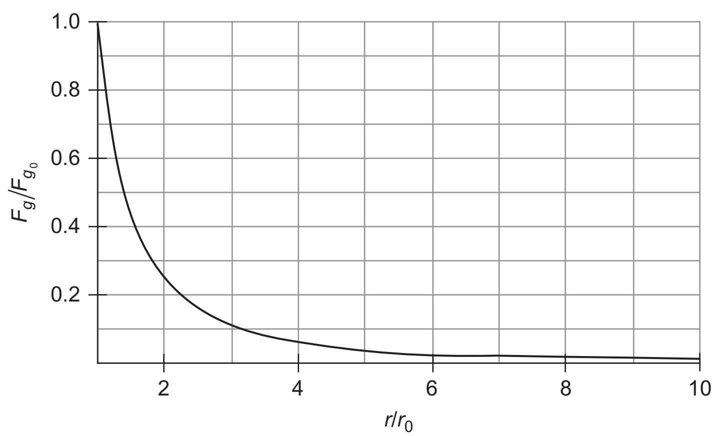

FIGURE 8.5

Decrease of gravitational force with distance from a planet's surface.

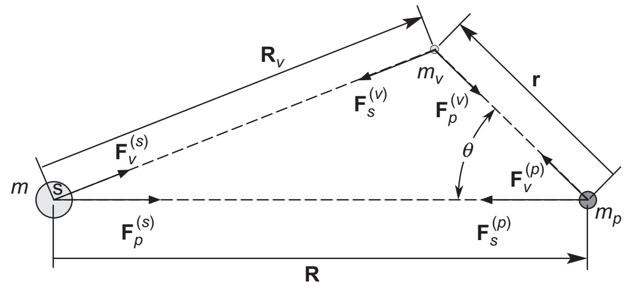

FIGURE 8.6

Relative position and gravitational force vectors among the three bodies.

Observe that

$$\mathbf{R}_v = \mathbf{R} + \mathbf{r} \tag{8.19}$$

From Figure 8.6 and the law of cosines we see that the magnitude of $\mathbf{R}_v$ is

$$R_v = (R^2 + r^2 - 2Rr\cos\theta)^{\frac{1}{2}} = R\left[1 - 2\frac{r}{R}\cos\theta + \left(\frac{r}{R}\right)^2\right]^{\frac{1}{2}} \tag{8.20}$$

We expect that within the planet's sphere of influence, $r/R \ll 1$. In that case, the terms involving r/R in Equation 8.20 can be neglected, so that, approximately

$$R_v = R \tag{8.21}$$

The equation of motion of the spacecraft relative to the sun-centered inertial frame is

$$m_v\ddot{\mathbf{R}}_v = \mathbf{F}_s^{(v)} + \mathbf{F}_p^{(v)}$$

Solving for $\ddot{\mathbf{R}}_v$ and substituting the gravitational forces given by Equations 8.18a and 8.18b, we get

$$\ddot{\mathbf{R}}_v = \frac{1}{m_v}\left(-\frac{Gm_vm_s}{R_v^{\ 3}}\mathbf{R}_v\right) + \frac{1}{m_v}\left(-\frac{Gm_vm_p}{r^3}\mathbf{r}\right) = -\frac{Gm_s}{R_v^{\ 3}}\mathbf{R}_v - \frac{Gm_p}{r^3}\mathbf{r} \tag{8.22}$$

Let us write this as

$$\ddot{\mathbf{R}}_v = \mathbf{A}_s + \mathbf{P}_p \tag{8.23}$$

where

$$\mathbf{A}_s = -\frac{Gm_s}{R_v^{\ 3}}\mathbf{R}_v \quad \mathbf{P}_p = -\frac{Gm_p}{r^3}\mathbf{r} \tag{8.24}$$

$\mathbf{A}_s$ is the primary gravitational acceleration of the vehicle due to the sun, whereas $\mathbf{P}_p$ is the secondary or perturbing acceleration due to the planet. The magnitudes of $\mathbf{A}_s$ and $\mathbf{P}_p$ are

$$A_s = \frac{Gm_s}{R^2} \qquad P_p = \frac{Gm_p}{r^2} \tag{8.25}$$

where we made use of the approximation given by Equation 8.21. The ratio of the perturbing acceleration to the primary acceleration is, therefore,

$$\frac{P_p}{A_s} = \frac{\dfrac{Gm_p}{r^2}}{\dfrac{Gm_s}{R^2}} = \frac{m_p}{m_s}\left(\frac{R}{r}\right)^2 \tag{8.26}$$

The equation of motion of the planet relative to the inertial frame is

$$m_p\ddot{\mathbf{R}} = \mathbf{F}_v^{(p)} + \mathbf{F}_s^{(p)}$$

Solving for $\ddot{\mathbf{R}}$, noting that $\mathbf{F}_v^{(p)} = -\mathbf{F}_p^{(v)}$, and using Equations 8.18b and 8.18c yields

$$\ddot{\mathbf{R}} = \frac{1}{m_p}\left(\frac{Gm_v m_p}{r^3}\mathbf{r}\right) + \frac{1}{m_p}\left(-\frac{Gm_p m_s}{R^3}\mathbf{R}\right) = \frac{Gm_v}{r^3}\mathbf{r} - \frac{Gm_s}{R^3}\mathbf{R} \tag{8.27}$$

Subtracting Equation 8.27 from 8.22 and collecting terms, we find

$$\ddot{\mathbf{R}}_v - \ddot{\mathbf{R}} = -\frac{Gm_p}{r^3}\mathbf{r}\left(1 + \frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v{}^3}\left[\mathbf{R}_v - \left(\frac{R_v}{R}\right)^3\mathbf{R}\right]$$

Recalling Equation 8.19, we can write this as

$$\ddot{\mathbf{r}} = -\frac{Gm_p}{r^3}\mathbf{r}\left(1 + \frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v{}^3}\left\{\mathbf{r} + \left[1 - \left(\frac{R_v}{R}\right)^3\right]\mathbf{R}\right\} \tag{8.28}$$

This is the equation of motion of the vehicle relative to the planet. By using Equation 8.21 and the fact that $m_v \ll m_p$, we can write this in approximate form as

$$\ddot{\mathbf{r}} = \mathbf{a}_p + \mathbf{p}_s \tag{8.29}$$

where

$$\mathbf{a}_p = -\frac{Gm_p}{r^3}\mathbf{r} \qquad \mathbf{p}_s = -\frac{Gm_s}{R^3}\mathbf{r} \tag{8.30}$$

In this case $\mathbf{a}_p$ is the primary gravitational acceleration of the vehicle due to the planet, and $\mathbf{p}_s$ is the perturbation caused by the sun. The magnitudes of these vectors are

$$a_p = \frac{Gm_p}{r^2} \qquad p_s = \frac{Gm_s}{R^3}r \tag{8.31}$$

The ratio of the perturbing acceleration to the primary acceleration is

$$\frac{p_s}{a_p} = \frac{Gm_s \dfrac{r}{R^3}}{\dfrac{Gm_p}{r^2}} = \frac{m_s}{m_p}\left(\frac{r}{R}\right)^3 \tag{8.32}$$

For motion relative to the planet, the ratio p_s/a_p is a measure of the deviation of the vehicle's orbit from the Keplerian orbit arising from the planet acting by itself ($p_s/a_p = 0$). Likewise, P_p/A_s is a measure of the planet's influence on the orbit of the vehicle relative to the sun. If

$$\frac{p_s}{a_p} < \frac{P_p}{A_s} \tag{8.33}$$

then the perturbing effect of the sun on the vehicle's orbit around the planet is less than the perturbing effect of the planet on the vehicle's orbit around the sun. We say that the vehicle is therefore within the planet's sphere of influence. Substituting Equations 8.26 and 8.32 into 8.33 yields

$$\frac{m_s}{m_p}\left(\frac{r}{R}\right)^3 < \frac{m_p}{m_s}\left(\frac{R}{r}\right)^2$$

which means

$$\left(\frac{r}{R}\right)^5 < \left(\frac{m_p}{m_s}\right)^2$$

or

$$\frac{r}{R} < \left(\frac{m_p}{m_s}\right)^{2/5}$$

Let r_{SOI} be the radius of the sphere of influence. Within the planet's sphere of influence, defined by

$$\frac{r_{\mathrm{SOI}}}{R} = \left(\frac{m_p}{m_s}\right)^{2/5} \tag{8.34}$$

the motion of the spacecraft is determined by its equations of motion relative to the planet (Equation 8.28). Outside of the sphere of influence, the path of the spacecraft is computed relative to the sun (Equation 8.22).

The sphere of influence radius presented in Equation 8.34 is not an exact quantity. It is simply a reasonable estimate of the distance beyond which the sun's gravitational attraction dominates that of a planet. The spheres of influence of all of the planets and the earth's moon are listed in Table A.2.

Example 8.3

Calculate the radius of the earth's sphere of influence.

In Table A.1 we find

$$m_{\text{earth}} = 5.974 \times 10^{24} \text{ kg}$$
$$m_{\text{sun}} = 1.989 \times 10^{30} \text{ kg}$$
$$R_{\text{earth}} = 149.6 \times 10^{6} \text{ km}$$

Substituting this data into Equation 8.34 yields

$$r_{\text{SOI}} = 149.6 \times 10^{6} \left(\frac{5.974 \times 10^{24}}{1.989 \times 10^{24}} \right)^{\frac{2}{5}} = 925 \times 10^{6} \text{ km}$$

Since the radius of the earth is 6378 km,

$$r_{\text{SOI}} = 145 \text{ earth radii}$$

Relative to the earth, its sphere of influence is very large. However, relative to the sun it is tiny, as illustrated in Figure 8.7.

FIGURE 8.7

The earth's sphere of influence and the sun, drawn to scale.

8.5 METHOD OF PATCHED CONICS

"Conics" refers to the fact that two-body, or Keplerian orbits, are conic sections with the focus at the attracting body. To study an interplanetary trajectory we assume that when the spacecraft is outside the sphere of influence of a planet it follows an unperturbed Keplerian orbit around the sun. Because interplanetary distances are so vast, for heliocentric orbits we may neglect the size of the spheres of influence and consider them, like the planets they surround, to be just points in space coinciding with the planetary centers. Within each planetary sphere of influence, the spacecraft travels an unperturbed Keplerian path about the planet. While the sphere of influence appears as a mere speck on the scale of the solar system, from the point of view of the planet it is very large indeed and may be considered to lie at infinity.

To analyze a mission from planet 1 to planet 2 using the method of patched conics, we first determine the heliocentric trajectory—such as the Hohmann transfer ellipse discussed in Section 8.2—that will intersect the desired positions of the two planets in their orbits. This trajectory takes the spacecraft from the sphere of influence of planet 1 to that of planet 2. At the spheres of influence, the heliocentric velocities of the transfer orbit are computed relative to the planet to establish the velocities "at infinity" which are then used to determine planetocentric departure trajectory at planet 1 and arrival trajectory at planet 2.

In this way we "patch" together the three conics, one centered at the sun and the other two centered at the planets in question.

Whereas the method of patched conics is remarkably accurate for interplanetary trajectories, such is not the case for lunar rendezvous and return trajectories. The orbit of the moon is determined primarily by the earth, whose sphere of influence extends well beyond the moon's 384,400 km orbital radius. To apply patched conics to lunar trajectories we ignore the sun and consider the motion of a spacecraft as influenced by just the earth and moon, as in the restricted three-body problem discussed in Section 2.12. The size of the moon's sphere of influence is found using Equation 8.34, with the earth playing the role of the sun:

$$r_{\mathrm{SOI}} = R\left(\frac{m_{\mathrm{moon}}}{m_{\mathrm{earth}}}\right)^{\frac{2}{5}}$$

where R is the radius of the moon's orbit. Thus, using Table A.1,

$$r_{\mathrm{SOI}} = 384{,}400\left(\frac{73.48\times10^{21}}{5974\times10^{21}}\right)^{\frac{2}{5}} = 66{,}200\ \mathrm{km}$$

as recorded in Table A.2. The moon's sphere of influence extends out to over one-sixth of the distance to the earth. We can hardly consider it to be a mere speck relative to the earth. Another complication is the fact that the earth and the moon are somewhat comparable in mass, so that their center of mass lies almost three quarters of an earth radius from the center of the earth. The motion of the moon cannot be accurately described as rotating around the center of the earth.

Complications such as these place the analysis of cislunar trajectories beyond the scope of this chapter. (In Example 2.18 we did a lunar trajectory calculation not by using patched conics but by integrating the equations of motion of a spacecraft within the context of the restricted three-body problem.) Extensions of the patched conic technique to lunar trajectories may be found in references such as Bate, Mueller and White (1971), Kaplan (1976) and Battin (1999).

8.6 PLANETARY DEPARTURE

In order to escape the gravitational pull of a planet, the spacecraft must travel a hyperbolic trajectory relative to the planet, arriving at its sphere of influence with a relative velocity $\mathbf{v}_\infty$ (hyperbolic excess velocity) greater than zero. On a parabolic trajectory, according to Equation 2.91, the spacecraft will arrive at the sphere of influence ($r = \infty$) with a relative speed of zero. In that case the spacecraft remains in the same orbit as the planet and does not embark upon a heliocentric elliptical path.

Figure 8.8 shows a spacecraft departing on a Hohmann trajectory from a planet 1 towards a target planet 2 which is farther away from the sun (as in Figure 8.1). At the sphere of influence crossing, the heliocentric velocity $\mathbf{V}_D^{(v)}$ of the spacecraft is parallel to the asymptote of the departure hyperbola as well as to the planet's heliocentric velocity vector $\mathbf{V}_1$. $\mathbf{V}_D^{(v)}$ and $\mathbf{V}_1$ must be parallel and in the same direction for a Hohmann transfer such that ΔV_D in Equation 8.3 is positive. Clearly, ΔV_D is the hyperbolic excess speed of the departure hyperbola,

$$v_\infty = \sqrt{\frac{\mu_{\mathrm{sun}}}{R_1}}\left(\sqrt{\frac{2R_2}{R_1+R_2}}-1\right) \tag{8.35}$$

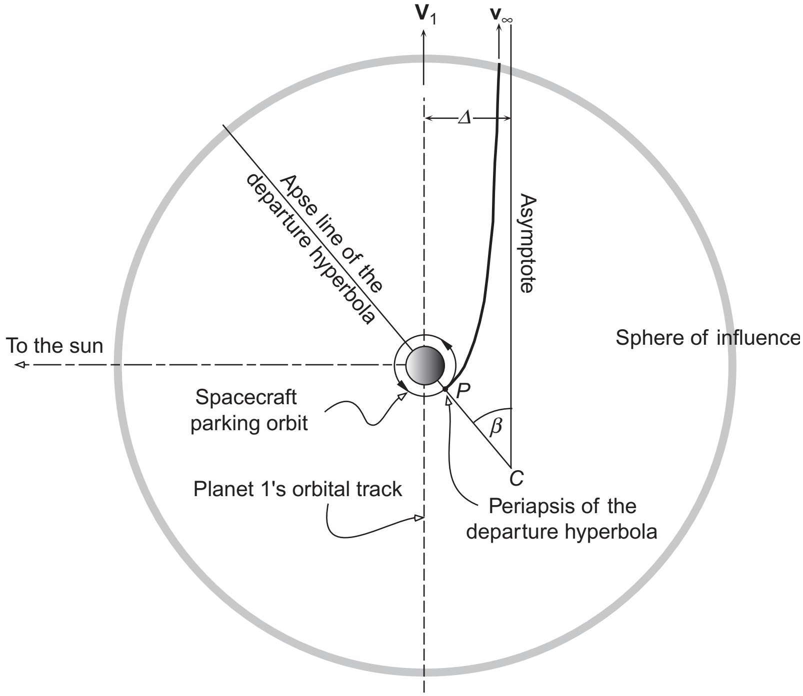

FIGURE 8.8

Departure of a spacecraft on a mission from an inner planet to an outer planet.

It would be well at this point for the reader to review Section 2.9 on hyperbolic trajectories and compare Figures 8.8 and 2.25. Recall that point *C* is the center of the hyperbola.

A space vehicle is ordinarily launched into an interplanetary trajectory from a circular parking orbit. The radius of this parking orbit equals the periapsis radius r_p of the departure hyperbola. According to Equation 2.50, the periapsis radius is given by

$$r_p = \frac{h^2}{\mu_1}\frac{1}{1+e} \tag{8.36}$$

where h is the angular momentum of the departure hyperbola (relative to the planet), e is the eccentricity of the hyperbola and μ_1 is the planet's gravitational parameter. The hyperbolic excess speed is found in Equation 2.115, from which we obtain

$$h = \frac{\mu_1\sqrt{e^2 - 1}}{v_\infty} \tag{8.37}$$

Substituting this expression for the angular momentum into Equation 8.36 and solving for the eccentricity yields

$$e = 1 + \frac{r_p {v_\infty}^2}{\mu_1} \tag{8.38}$$

We place this result back into Equation 8.37 to obtain the following expression for the angular momentum.

$$h = r_p \sqrt{v_\infty{}^2 + \frac{2\mu_1}{r_p}} \tag{8.39}$$

The hyperbolic excess speed v_∞ is specified by the mission requirements (Equation 8.35). Equations 8.38 and 8.39 show that choosing a departure periapsis r_p yields the orbital parameters e and h of the departure hyperbola. From the angular momentum we get the periapsis speed,

$$v_p = \frac{h}{r_p} = \sqrt{v_\infty{}^2 + \frac{2\mu_1}{r_p}} \tag{8.40}$$

which can also be found from an energy approach using Equation 2.113. With Equation 8.40 and the speed of the circular parking orbit (Equation 2.63),

$$v_c = \sqrt{\frac{\mu_1}{r_p}} \tag{8.41}$$

we can calculate the delta-v required to put the vehicle onto the hyperbolic departure trajectory,

$$\Delta v = v_p - v_c = v_c\left(\sqrt{2 + \left(\frac{v_\infty}{v_c}\right)^2} - 1\right) \tag{8.42}$$

The location of periapsis, where the delta-v maneuver must occur, is found using Equations 2.99 and 8.38,

$$\beta = \cos^{-1}\left(\frac{1}{e}\right) = \cos^{-1}\left(\frac{1}{1 + \dfrac{r_p v_\infty{}^2}{\mu_1}}\right) \tag{8.43}$$

β gives the orientation of the apse line of the hyperbola to the planet's heliocentric velocity vector.

It should be pointed out that the only requirement on the orientation of the plane of the departure hyperbola is that it contain the center of mass of the planet as well as the relative velocity vector $\mathbf{v}_\infty$. Therefore, as shown in Figure 8.9, the hyperbola can be rotated about a line *A-A* which passes through the planet's center of mass and is parallel to $\mathbf{v}_\infty$ (or $\mathbf{V}_1$, which of course is parallel to $\mathbf{v}_\infty$ for Hohmann transfers). Rotating the hyperbola in this way sweeps out a surface of revolution on which all possible departure hyperbolas lie. The periapsis of the hyperbola traces out a circle which, for the specified periapsis radius r_p, is the locus of all possible points of injection into a departure trajectory towards the target planet. This circle is the base of a cone with vertex at the center of the planet. From Figure 2.25 we can determine that its radius is $r_p \sin\beta$, where β is given just above in Equation 8.43.

The plane of the parking orbit, or direct ascent trajectory, must contain the line *A-A* and the launch site at the time of launch. The possible inclinations of a prograde orbit range from a minimum of $i_{\min}$, where $i_{\min}$ is the latitude of the launch site, to $i_{\max}$, which cannot exceed 90°. Launch site safety considerations may place additional limits on that range. For example, orbits originating from the Kennedy Space Center

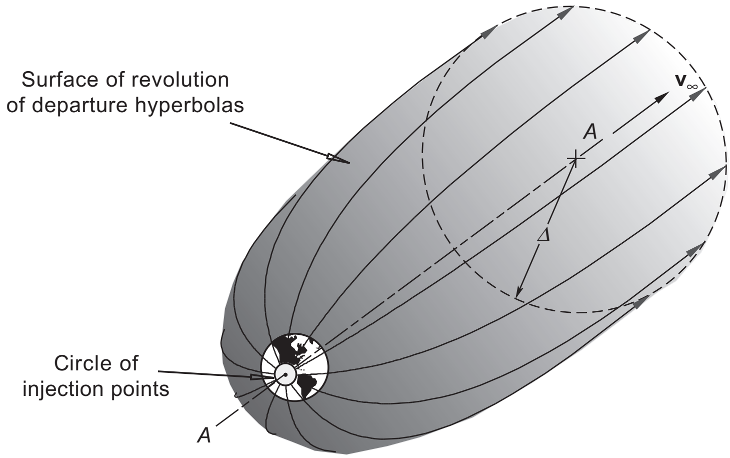

FIGURE 8.9

Locus of possible departure trajectories for a given v_∞ and r_p.

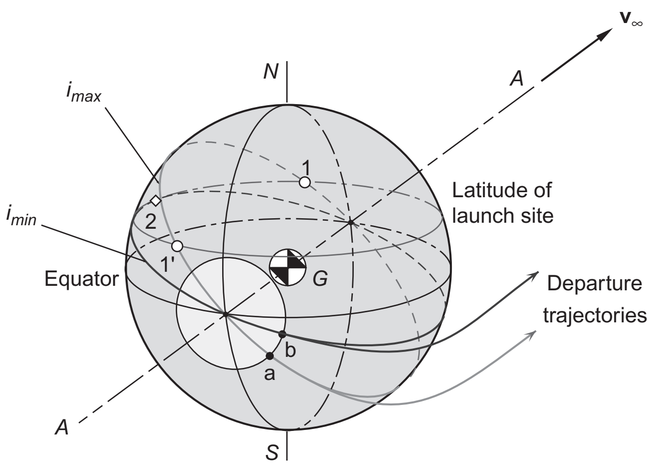

FIGURE 8.10

Parking orbits and departure trajectories for a launch site at a given latitude.

in Florida, USA, (latitude 28.5°) are limited to inclinations between 28.5° and 52.5°. For the scenario illustrated in Figure 8.10 the location of the launch site limits access to just the departure trajectories having periapses lying between *a* and *b*. The figure shows that there are two times per day—when the planet rotates the launch site through positions 1 and 1′—that a spacecraft can be launched into a parking orbit. These times are closer together (the launch window is smaller) the lower the inclination of the parking orbit.

Once a spacecraft is established in its parking orbit, then an opportunity for launch into the departure trajectory occurs each orbital circuit.

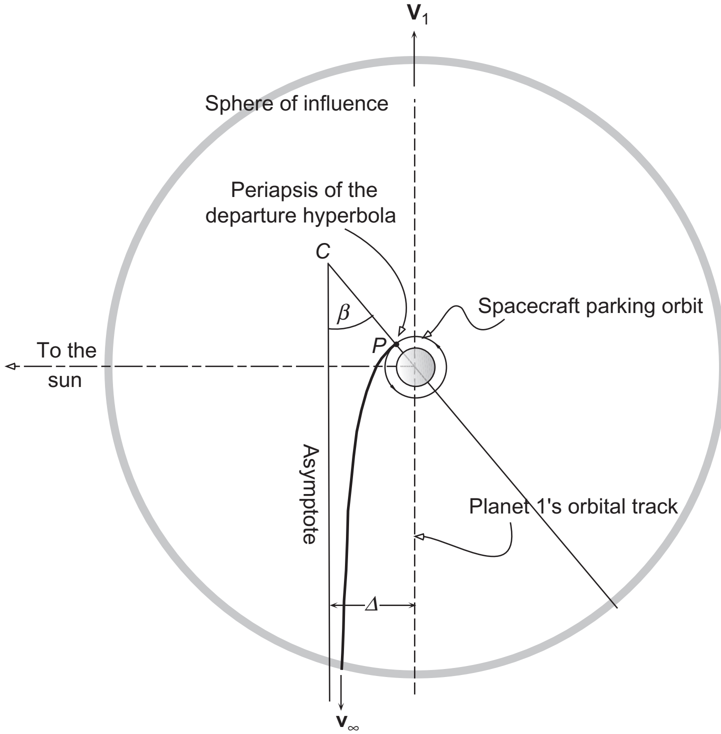

FIGURE 8.11

Departure of a spacecraft on a trajectory from an outer planet to an inner planet.

If the mission is to send a spacecraft from an outer planet to an inner planet, as in Figure 8.2, then the spacecraft's heliocentric speed $V_D^{(v)}$ at departure must be less than that of the planet. That means the spacecraft must emerge from the back side of the sphere of influence with its relative velocity vector $\mathbf{v}_\infty$ directed opposite to $\mathbf{V}_1$, as shown in Figure 8.11. Figures 8.9 and 8.10 apply to this situation as well.

Example 8.4

A spacecraft is launched on a mission to Mars starting from a 300 km circular parking orbit. Calculate (a) the delta-v required; (b) the location of perigee of the departure hyperbola; (c) the amount of propellant required as a percentage of the spacecraft mass before the delta-v burn, assuming a specific impulse of 300 seconds.

Solution

From Tables A.1 and A.2 we obtain the gravitational parameters for the sun and the earth,

$$\begin{aligned} \mu_{\text{sun}} &= 1.327 \times 10^{11}\ \text{km}^3/\text{s}^2 \\ \mu_{\text{earth}} &= 398{,}600\ \text{km}^3/\text{s}^2 \end{aligned}$$

and the orbital radii of the earth and Mars,

$$R_{\text{earth}} = 149.6 \times 10^6 \text{ km}$$
$$R_{\text{Mars}} = 227.9 \times 10^6 \text{ km}$$

(a) According to Equation 8.35, the hyperbolic excess speed is

$$v_\infty = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{earth}}}}\left(\sqrt{\frac{2R_{\text{Mars}}}{R_{\text{earth}} + R_{\text{Mars}}}} - 1\right) = \sqrt{\frac{1.327 \times 10^{11}}{149.6 \times 10^6}}\left(\sqrt{\frac{2(227.9 \times 10^6)}{149.6 \times 10^6 + 227.9 \times 10^6}} - 1\right)$$

from which

$$v_\infty = 2.943 \text{ km/s}$$

The speed of the spacecraft in its 300 km circular parking orbit is given by Equation 8.41,

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r_{\text{earth}} + 300}} = \sqrt{\frac{398{,}600}{6678}} = 7.726 \text{ km/s}$$

Finally, we use Equation 8.42 to calculate the delta-v required to step up to the departure hyperbola.

$$\Delta v = v_p - v_c = v_c\left(\sqrt{2 + \left(\frac{v_\infty}{v_c}\right)^2} - 1\right) = 7.726\left(\sqrt{2 + \left(\frac{2.943}{7.726}\right)^2} - 1\right)$$

$$\boxed{\Delta v = 3.590 \text{ km/s}}$$

(b) Perigee of the departure hyperbola, relative to the earth's orbital velocity vector, is found using Equation 8.43,

$$\beta = \cos^{-1}\left(\frac{1}{1 + \dfrac{r_p v_\infty{}^2}{\mu_{\text{earth}}}}\right) = \cos^{-1}\left(\frac{1}{1 + \dfrac{6678 \cdot 2.943^2}{368{,}600}}\right)$$

$$\boxed{\beta = 29.16^\circ}$$

Figure 8.12 shows that the perigee can be located on either the sunlit or dark side of the earth. It is likely that the parking orbit would be a prograde orbit (west to east), which would place the burnout point on the dark side.

(c) From Equation 6.1 we have

$$\frac{\Delta m}{m} = 1 - e^{-\frac{\Delta v}{I_{sp} g_o}}$$

Substituting $\Delta v = 3.590$ km/s, $I_{sp} = 300$ s and $g_o = 9.81 \times 10^{-3}$ km/s^2, this yields

$$\frac{\Delta m}{m} = 0.705$$

That is, prior to the delta-v maneuver, $\boxed{\text{over 70\% of the spacecraft mass must be propellant.}}$

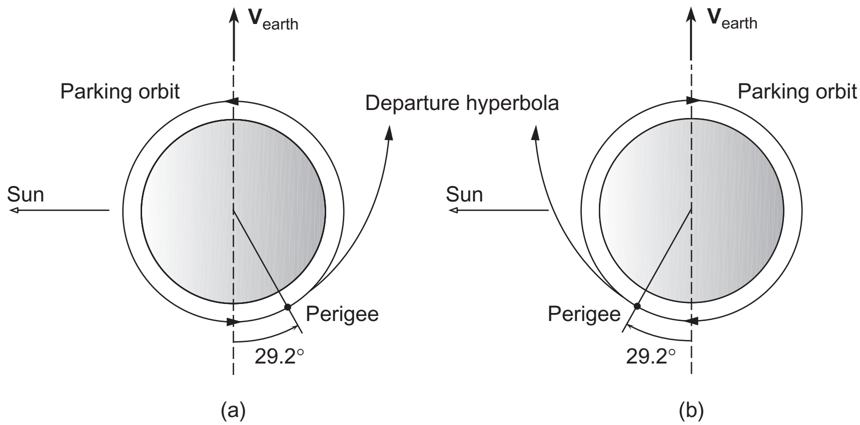

FIGURE 8.12

Departure trajectory to Mars initiated from (a) the dark side (b) the sunlit side of the earth.

8.7 SENSITIVITY ANALYSIS

The initial maneuvers required to place a spacecraft on an interplanetary trajectory occur well within the sphere of influence of the departure planet. Since the sphere of influence is just a point on the scale of the solar system, one may ask what effects small errors in position and velocity at the maneuver point have on the trajectory. Assuming the mission is from an inner to an outer planet, let us consider the effect which small changes in the burnout velocity v_p and radius r_p have on the target radius R_2 of the heliocentric Hohmann transfer ellipse (see Figures 8.1 and 8.8).

R_2 is the radius of aphelion, so we use Equation 2.70 to obtain

$$R_2 = \frac{h^2}{\mu_{\text{sun}}}\frac{1}{1-e}$$

Substituting $h = R_1 V_D^{(v)}$ and $e = (R_2 - R_1)/(R_2 + R_1)$, and solving for R_2 yields

$$R_2 = \frac{R_1{}^2\left[V_D^{(v)}\right]^2}{2\mu_{\text{sun}} - R_1\left[V_D^{(v)}\right]^2} \tag{8.44}$$

(This expression holds as well for a mission from an outer to inner planet.) The change δR_2 in R_2 due to a small variation $\delta V_D^{(v)}$ of $V_D^{(v)}$ is

$$\delta R_2 = \frac{dR_2}{dV_D^{(v)}}\delta V_D^{(v)} = \frac{4R_1{}^2\mu_{\text{sun}}}{\left\{2\mu_{\text{sun}} - R_1\left[V_D^{(v)}\right]^2\right\}^2} V_D^{(v)}\delta V_D^{(v)}$$

Dividing this equation by Equation 8.44 leads to

$$\frac{\delta R_2}{R_2} = \frac{2}{1 - \frac{R_1\left[V_D^{(v)}\right]^2}{2\mu_{\text{sun}}}}\frac{\delta V_D^{(v)}}{V_D^{(v)}} \tag{8.45}$$

The departure speed $V_D^{(v)}$ of the space vehicle is the sum of the planet's speed V_1 and excess speed v_∞.

$$V_D^{(v)} = V_1 + v_\infty$$

We can solve Equation 8.40 for v_∞,

$$v_\infty = \sqrt{{v_p}^2 - \frac{2\mu_1}{r_p}}$$

Hence,

$$V_D^{(v)} = V_1 + \sqrt{{v_p}^2 - \frac{2\mu_1}{r_p}} \tag{8.46}$$

The change in $V_D^{(v)}$ due to variations δr_p and δv_p of the burnout position (periapsis) r_p and speed v_p is given by

$$\delta V_D^{(v)} = \frac{\partial V_D^{(v)}}{\partial r_p}\delta r_p + \frac{\partial V_D^{(v)}}{\partial v_p}\delta v_p \tag{8.47}$$

From Equation 8.46 we obtain

$$\frac{\partial V_D^{(v)}}{\partial r_p} = \frac{\mu_1}{v_\infty {r_p}^2} \quad \frac{\partial V_D^{(v)}}{\partial v_p} = \frac{v_p}{v_\infty}$$

Therefore,

$$\delta V_D^{(v)} = \frac{\mu_1}{v_\infty {r_p}^2}\delta r_p + \frac{v_p}{v_\infty}\delta v_p$$

Once again making use of Equation 8.40, this can be written as follows

$$\frac{\delta V_D^{(v)}}{V_D^{(v)}} = \frac{\mu_1}{V_D^{(v)} v_\infty r_p}\frac{\delta r_p}{r_p} + \frac{v_\infty + \frac{2\mu_1}{r_p v_\infty}}{V_D^{(v)}}\frac{\delta v_p}{v_p} \tag{8.48}$$

Substituting this into Equation 8.45 finally yields the desired result, an expression for the variation of R_2 due to variations in r_p and v_p.

$$\frac{\delta R_2}{R_2} = \frac{2}{1 - \dfrac{R_1[V_D^{(v)}]^2}{2\mu_{\text{sun}}}}\left(\frac{\mu_1}{V_D^{(v)} v_\infty r_p}\frac{\delta r_p}{r_p} + \frac{v_\infty + \dfrac{2\mu_1}{r_p v_\infty}}{V_D^{(v)}}\frac{\delta v_p}{v_p}\right) \tag{8.49}$$

Consider a mission from earth to Mars, starting from a 300 km parking orbit. We have

$$\begin{aligned}
\mu_{\text{sun}} &= 1.327 \times 10^{11} \text{ km}^3/\text{s}^2 \\
\mu_1 &= \mu_{\text{earth}} = 398{,}600 \text{ km}^3/\text{s}^2 \\
R_1 &= 149.6 \times 10^6 \text{ km} \\
R_2 &= 227.9 \times 10^6 \text{ km} \\
r_p &= 6678 \text{ km}
\end{aligned}$$

In addition, from Equations 8.1 and 8.2,

$$V_1 = V_{\text{earth}} = \sqrt{\frac{\mu_{\text{sun}}}{R_1}} = \sqrt{\frac{1.327 \times 10^{11}}{149.6 \times 10^6}} = 29.78 \text{ km/s}$$

$$V_D^{(v)} = \sqrt{2\mu_{\text{sun}}}\sqrt{\frac{R_2}{R_1(R_1 + R_2)}} = \sqrt{2 \cdot 1.327 \times 10^{11}}\sqrt{\frac{227.9 \times 10^6}{149.6 \times 10^6(149.6 \times 10^6 + 227.9 \times 10^6)}} = 32.73 \text{ km/s}$$

Therefore,

$$v_\infty = V_D^{(v)} - V_{\text{earth}} = 2.943 \text{ km/s}$$

and, from Equation 8.40

$$v_p = \sqrt{{v_\infty}^2 + \frac{2\mu_{\text{earth}}}{r_p}} = \sqrt{2.943^2 + \frac{2 \cdot 398{,}600}{6678}} = 11.32 \text{ km/s}$$

Substituting these values into Equation 8.49 yields

$$\frac{\delta R_2}{R_2} = 3.127\frac{\delta r_p}{r_p} + 6.708\frac{\delta v_p}{v_p}$$

This expression shows that a 0.01% variation (1.1 m/s) in the burnout speed ν_p changes the target radius R_2 by 0.067% or 153,000 km! Likewise, an error of 0.01% (0.67 km) in burnout radius r_p produces an error of over 70,000 km. Thus small errors which are likely to occur in the launch phase of the mission must be corrected by midcourse maneuvers during the coasting flight along the elliptical transfer trajectory.

8.8 PLANETARY RENDEZVOUS

A spacecraft arrives at the sphere of influence of the target planet with a hyperbolic excess velocity $\mathbf{v}_\infty$ relative to the planet. In the case illustrated in Figure 8.1, a mission from an inner planet 1 to an outer planet 2 (e.g., earth to Mars), the spacecraft's heliocentric approach velocity $\mathbf{V}_A^{(v)}$ is smaller in magnitude than that of the planet, $\mathbf{V}_2$. Therefore, it crosses the forward portion of the sphere of influence, as shown in Figure 8.13. For a Hohmann transfer, $\mathbf{V}_A^{(v)}$ and $\mathbf{V}_2$ are parallel, so the magnitude of the hyperbolic excess velocity is, simply

$$v_\infty = V_2 - V_A^{(v)} \tag{8.50}$$

If the mission is as illustrated in Figure 8.2, from an outer planet to an inner one (e.g., earth to Venus), then $V_A^{(v)}$ is greater than V_2, and the spacecraft must cross the rear portion of the sphere of influence, as shown in Figure 8.14. In that case

$$v_\infty = V_A^{(v)} - V_2 \tag{8.51}$$

What happens after crossing the sphere of influence depends on the nature of the mission. If the goal is to impact the planet (or its atmosphere), the aiming radius Δ of the approach hyperbola must be such

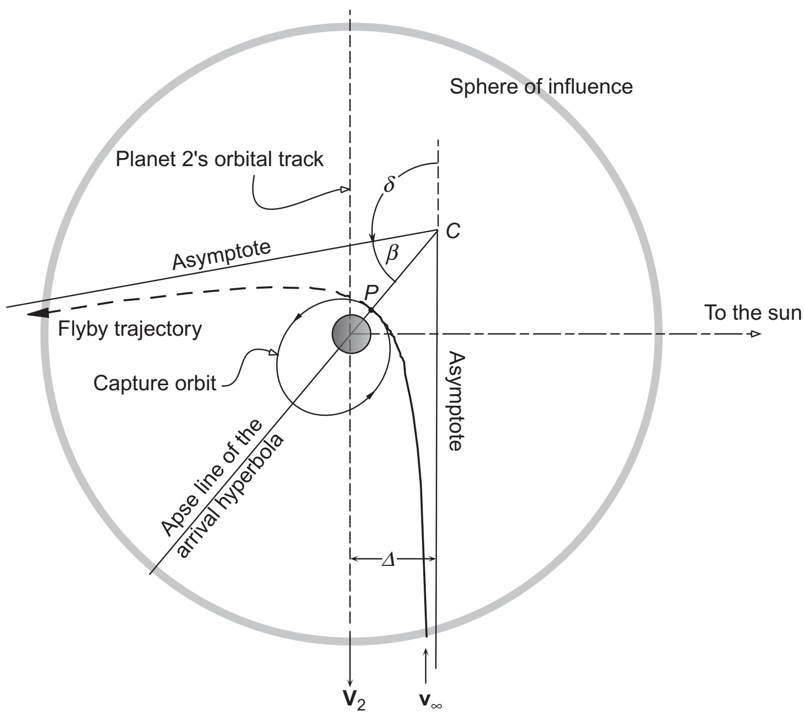

FIGURE 8.13

Spacecraft approach trajectory for a Hohmann transfer to an outer planet from an inner one. *P* is the periapsis of the approach hyperbola.

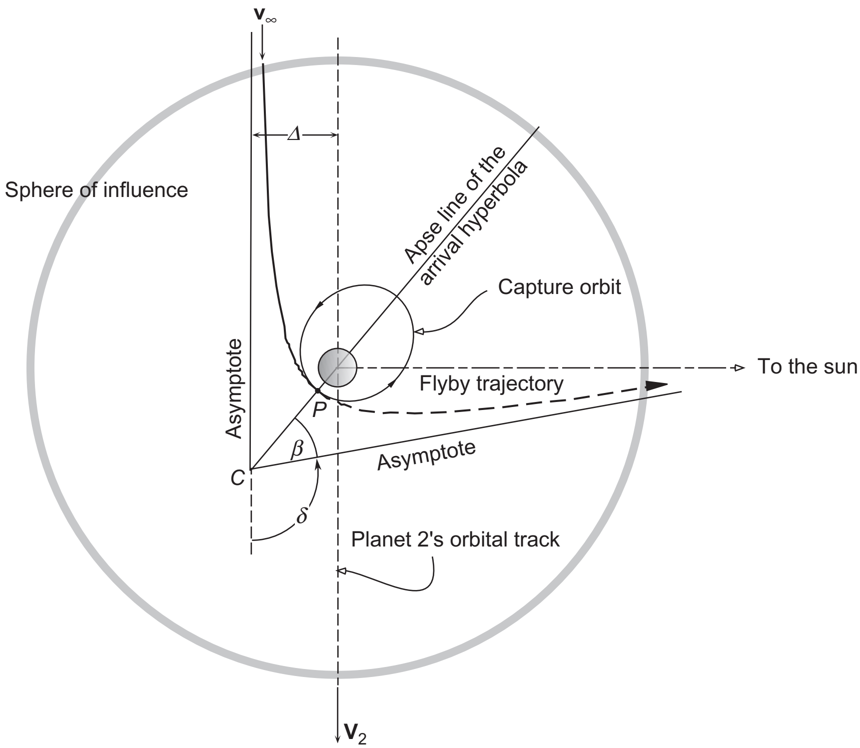

FIGURE 8.14

Spacecraft approach trajectory for a Hohmann transfer to an inner planet from an outer one. *P* is the periapsis of the approach hyperbola.

that hyperbola's periapsis radius r_p equals essentially the radius of the planet. If the intent is to go into orbit around the planet, then Δ must be chosen so that the delta-v burn at periapsis will occur at the correct altitude above the planet. If there is no impact with the planet and no drop into a capture orbit around the planet, then the spacecraft will simply continue past periapsis on a flyby trajectory, exiting the sphere of influence with the same relative speed v_∞ it entered, but with the velocity vector rotated through the turn angle δ, given by Equation 2.100,

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) \tag{8.52}$$

With the hyperbolic excess speed v_∞ and the periapsis radius r_p specified, the eccentricity of the approach hyperbola is found from Equation 8.38,

$$e = 1 + \frac{r_p {v_\infty}^2}{\mu_2} \tag{8.53}$$

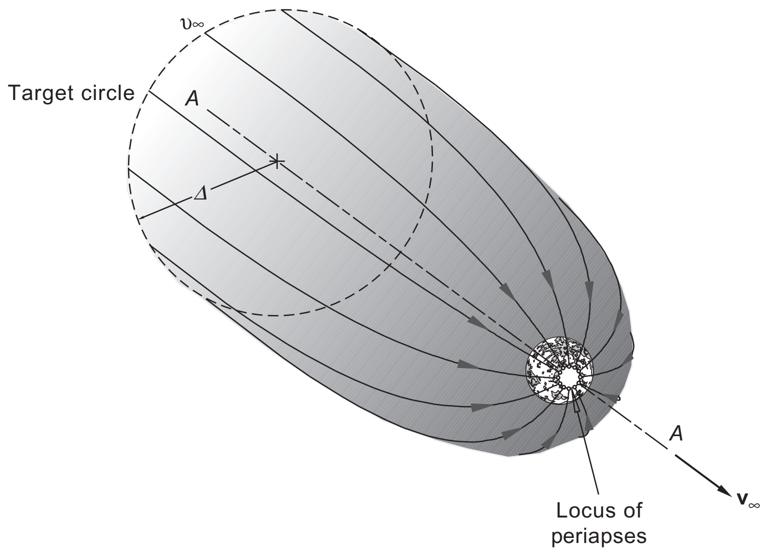

FIGURE 8.15

Locus of approach hyperbolas to the target planet.

where μ_2 is the gravitational parameter of planet 2. Hence, the turn angle is

$$\delta = 2\sin^{-1}\left(\frac{1}{1 + \dfrac{r_p {v_\infty}^2}{\mu_2}}\right) \tag{8.54}$$

We can combine Equations 2.103 and 2.107 to obtain the following expression for the aiming radius,

$$\Delta = \frac{h^2}{\mu_2}\frac{1}{\sqrt{e^2 - 1}} \tag{8.55}$$

The angular momentum of the approach hyperbola relative to the planet is found using Equation 8.39,

$$h = r_p\sqrt{{v_\infty}^2 + \frac{2\mu_2}{r_p}} \tag{8.56}$$

Substituting Equations 8.53 and 8.56 into 8.55 yields the aiming radius in terms of the periapsis radius and the hyperbolic excess speed,

$$\Delta = r_p\sqrt{1 + \frac{2\mu_2}{r_p {v_\infty}^2}} \tag{8.57}$$

Just as we observed when discussing departure trajectories, the approach hyperbola does not lie in a unique plane. We can rotate the hyperbolas illustrated in Figures 8.11 and 8.12 about a line *A*-*A* parallel to $\mathbf{v}_\infty$ and passing through the target planet's center of mass, as shown in Figure 8.15. The approach hyperbolas in

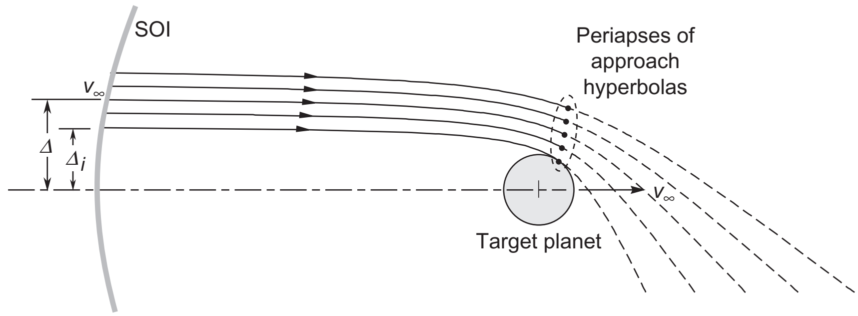

FIGURE 8.16

Family of approach hyperbolas having the same v_∞ but different Δ.

that figure terminate at the circle of periapses. Figure 8.16 is a plane through the solid of revolution revealing the shape of hyperbolas having a common $\mathbf{v}_\infty$ but varying Δ.

Let us suppose that the purpose of the mission is to enter an elliptical orbit of eccentricity e around the planet. This will require a Δv maneuver at periapsis P (Figures 8.13 and 8.14), which is also periapsis of the ellipse. The speed in the hyperbolic trajectory at periapsis is given by Equation 8.40

$$v_p\big)_{\text{hyp}} = \sqrt{{v_\infty}^2 + \frac{2\mu_2}{r_p}} \tag{8.58}$$

The velocity at periapsis of the capture orbit is found by setting $h = r_p v_p$ in Equation 2.50 and solving for v_p.

$$v_p\big)_{\text{capture}} = \sqrt{\frac{\mu_2(1+e)}{r_p}} \tag{8.59}$$

Hence, the required delta-v is

$$\Delta v = v_p\big)_{\text{hyp}} - v_p\big)_{\text{capture}} = \sqrt{{v_\infty}^2 + \frac{2\mu_2}{r_p}} - \sqrt{\frac{\mu_2(1+e)}{r_p}} \tag{8.60}$$

For a given v_∞, Δv clearly depends upon the choice of periapsis radius r_p and capture orbit eccentricity e. Requiring the maneuver point to remain the periapsis of the capture orbit means that Δv is maximum for a circular capture orbit and decreases with increasing eccentricity until $\Delta v = 0$, which, of course, means no capture (flyby).

In order to determine optimal capture radius, let us write Equation 8.60 in nondimensional form as

$$\frac{\Delta v}{v_\infty} = \sqrt{1 + \frac{2}{\xi}} - \sqrt{\frac{1+e}{\xi}} \tag{8.61}$$

where

$$\xi = \frac{r_p {v_\infty}^2}{\mu_2} \tag{8.62}$$

The first and second derivatives of $\Delta v/v_\infty$ with respect to ξ are

$$\frac{d}{d\xi}\frac{\Delta v}{v_\infty} = \left(-\frac{1}{\sqrt{\xi+2}} + \frac{\sqrt{1+e}}{2}\right)\frac{1}{\xi^{\frac{3}{2}}} \tag{8.63}$$

$$\frac{d^2}{d\xi^2}\frac{\Delta v}{v_\infty} = \left[\frac{2\xi+3}{(\xi+2)^{\frac{3}{2}}} - \frac{3}{4}\sqrt{1+e}\right]\frac{1}{\xi^{\frac{5}{2}}} \tag{8.64}$$

Setting the first derivative equal to zero and solving for ξ yields

$$\xi = 2\frac{1-e}{1+e} \tag{8.65}$$

Substituting this value of ξ into Equation 8.64, we get

$$\frac{d^2}{d\xi^2}\frac{\Delta v}{v_\infty} = \frac{\sqrt{2}}{64}\frac{(1+e)^3}{(1-e)^{\frac{3}{2}}} \tag{8.66}$$

This expression is positive for elliptical orbits ($0 \le e < 1$), which means that when ξ is given by Equation 8.65 Δv is a minimum. Therefore, from Equation 8.62, the optimal periapsis radius as far as fuel expenditure is concerned is

$$r_p = \frac{2\mu_2}{{v_\infty}^2}\frac{1-e}{1+e} \tag{8.67}$$

We can combine Equations 2.50 and 2.70 to get

$$\frac{1-e}{1+e} = \frac{r_p}{r_a} \tag{8.68}$$

where r_a is the apoapsis radius. Thus, Equation 8.67 implies

$$r_a = \frac{2\mu_2}{{v_\infty}^2} \tag{8.69}$$

That is, the apoapsis of this capture ellipse is independent of the eccentricity and equals the radius of the optimal circular orbit.

Substituting Equation 8.65 back into Equation 8.61 yields the minimum Δv.

$$\Delta v = v_\infty\sqrt{\frac{1-e}{2}} \tag{8.70}$$

Finally, placing the optimal r_p into Equation 8.57 leads to an expression for the aiming radius required for minimum Δv,

$$\Delta = 2\sqrt{2}\frac{\sqrt{1-e}}{1+e}\frac{\mu_2}{{v_\infty}^2} = \sqrt{\frac{2}{1-e}}r_p \tag{8.71}$$

Clearly, the optimal Δv (and periapsis height) are reduced for highly eccentric elliptical capture orbits ($e \to 1$). However, it should be pointed out that the use of optimal Δv may have to be sacrificed in favor of a variety of other mission requirements.

If a planet has an atmosphere and the periapsis lies within it, then a spacecraft might be designed to employ aerobraking, where atmospheric drag is used to reduce the speed instead of dependence solely on rocket engines. The reduced propellent requirement would allow for increased payload or a smaller vehicle. See, for example, Hale (1994), Tewari (2007), and Wiesel (1997) for introductory discussions of this subject.

Example 8.5

After a Hohmann transfer from earth to Mars, calculate

(a) the minimum delta-v required to place a spacecraft in an orbit with a period of seven hours.
(b) the periapsis ("periareion") radius.
(c) the aiming radius.
(d) the angle between periapsis and Mars' velocity vector.

Solution

The following data are required from Tables A.1 and A.2:

$$\begin{aligned}
\mu_{\text{sun}} &= 1.327\times 10^{11}\ \text{km}^3/\text{s}^2\\
\mu_{\text{Mars}} &= 42.830\ \text{km}^3/\text{s}^2\\
R_{\text{earth}} &= 149.6\times 10^{6}\ \text{km}\\
R_{\text{Mars}} &= 227.9\times 10^{6}\ \text{km}\\
r_{\text{Mars}} &= 3396\ \text{km}
\end{aligned}$$

(a) The hyperbolic excess speed is found using Equation 8.4,

$$v_\infty = \Delta V_A = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{Mars}}}}\left(1-\sqrt{\frac{2R_{\text{earth}}}{R_{\text{earth}}+R_{\text{Mars}}}}\right) = \sqrt{\frac{1.327\times 10^{11}}{227.9\times 10^{6}}}\left(1-\sqrt{\frac{2\cdot 149.6\times 10^{6}}{149.6\times 10^{6}+227.9\times 10^{6}}}\right)$$

$$v_\infty = 2.648\ \text{km/s}$$

We can use Equation 2.83 to express the semimajor axis a of the capture orbit in terms of its period T,

$$a = \left(\frac{T\sqrt{\mu_{\text{Mars}}}}{2\pi}\right)^{\frac{2}{3}}$$

Substituting $T = 7{\cdot}3600\,\text{s}$ yields

$$a = \left(\frac{25{,}200\sqrt{42{,}830}}{2\pi}\right)^{\frac{2}{3}} = 8832\ \text{km}$$

From Equation 2.73 we obtain

$$a = \frac{r_p}{1-e}$$

Upon substituting the optimal periapsis radius, Equation 8.67, this becomes

$$a = \frac{2\mu_{\text{Mars}}}{{v_\infty}^2}\frac{1}{1+e}$$

from which

$$e = \frac{2\mu_{\text{Mars}}}{a\,{v_\infty}^2} - 1 = \frac{2\cdot 42{,}830}{8832\cdot 2.648^2} - 1 = 0.3833$$

Thus, using Equation 8.70, we find

$$\Delta v = v_\infty\sqrt{\frac{1-e}{2}} = 2.648\sqrt{\frac{1-0.3833}{2}} = \boxed{1.470\ \text{km/s}}$$

(b) From Equation 8.67 we obtain the periapsis radius

$$r_p = \frac{2\mu_{\text{Mars}}}{{v_\infty}^2}\frac{1-e}{1+e} = \frac{2\cdot 42{,}830}{2.648^2}\frac{1-0.3833}{1+0.3833} = \boxed{5447\ \text{km}}$$

(c) The aiming radius is given by Equation 8.71

$$\Delta = r_p\sqrt{\frac{2}{1-e}} = 5447\sqrt{\frac{2}{1-0.3833}} = \boxed{9809\ \text{km}}$$

(d) Using Equation 8.43, we get the angle to periapsis

$$\beta = \cos^{-1}\left(\frac{1}{1+\dfrac{r_p {v_\infty}^2}{\mu_{\text{Mars}}}}\right) = \cos^{-1}\left(\frac{1}{1+\dfrac{5447\cdot 2.648^2}{42830}}\right) = \boxed{58.09°}$$

Mars, the approach hyperbola, and the capture orbit are shown to scale in Figure 8.17. The approach could also be made from the dark side of the planet instead of the sunlit side. The approach hyperbola and capture ellipse would be the mirror image of that shown, as is the case in Figure 8.12.

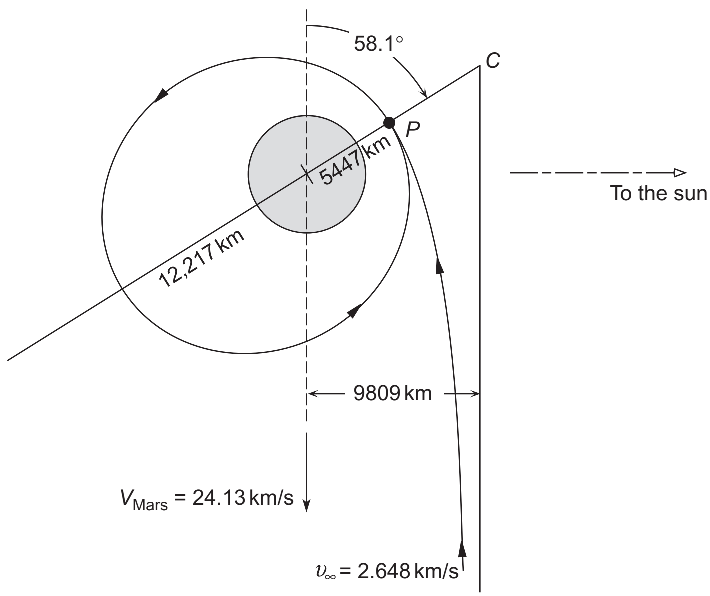

FIGURE 8.17

An optimal approach to a Mars capture orbit with a seven hour period. $r_{Mars} = 3396\,\text{km}$.

8.9 PLANETARY FLYBY

A spacecraft which enters a planet's sphere of influence and does not impact the planet or go into orbit around it will continue in its hyperbolic trajectory through periapsis P and exit the sphere of influence. Figure 8.18 shows a hyperbolic flyby trajectory along with the asymptotes and apse line of the hyperbola. It is a leading-side flyby because the periapsis is on the side of the planet facing into the direction of motion. Likewise, Figure 8.19 illustrates a trailing-side flyby. At the inbound crossing point, the heliocentric velocity $\mathbf{V}_1^{(v)}$ of the spacecraft equals the planet's heliocentric velocity $\mathbf{V}$ plus the hyperbolic excess velocity $\mathbf{v}_{\infty_1}$ of the spacecraft (relative to the planet),

$$\mathbf{V}_1^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_1} \tag{8.72}$$

Similarly, at the outbound crossing we have

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} \tag{8.73}$$

The change $\Delta\mathbf{V}^{(v)}$ in the spacecraft's heliocentric velocity is

$$\Delta\mathbf{V}^{(v)} = \mathbf{V}_2^{(v)} - \mathbf{V}_1^{(v)} = (\mathbf{V} + \mathbf{v}_{\infty_2}) - (\mathbf{V} + \mathbf{v}_{\infty_1})$$

which means

$$\Delta\mathbf{V}^{(v)} = \mathbf{v}_{\infty_2} - \mathbf{v}_{\infty_1} = \Delta\mathbf{v}_\infty \tag{8.74}$$

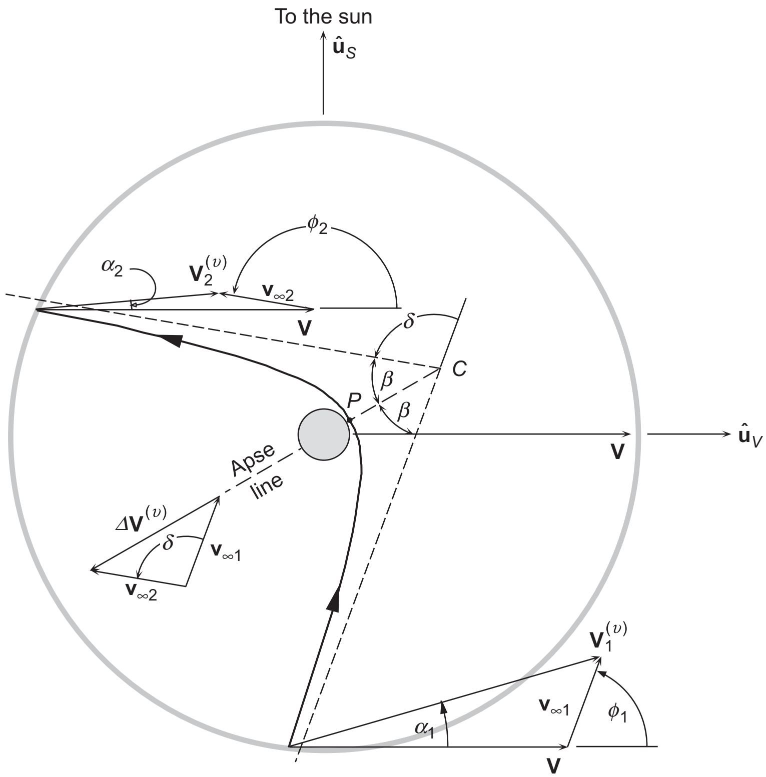

FIGURE 8.18

Leading-side planetary flyby.

The excess velocities $\mathbf{v}_{\infty_1}$ and $\mathbf{v}_{\infty_2}$ lie along the asymptotes of the hyperbola and are therefore inclined at the same angle β to the apse line (see Figure 2.25), with $\mathbf{v}_{\infty_1}$ pointing towards and $\mathbf{v}_{\infty_2}$ pointing away from the center C. They both have the same magnitude v_∞, with $\mathbf{v}_{\infty_2}$ having rotated relative to $\mathbf{v}_{\infty_1}$ by the turn angle δ. Hence, $\Delta\mathbf{v}_\infty$—and therefore $\Delta\mathbf{V}^{(v)}$—is a vector which lies along the apse line and always points away from periapsis, as illustrated in Figures 8.18 and 8.19. From those figures it can be seen that, in a leading-side flyby, the component of $\Delta\mathbf{V}^{(v)}$ in the direction of the planet's velocity is negative, whereas for the trailing-side flyby it is positive. This means that a leading-side flyby results in a decrease in the spacecraft's heliocentric speed. On the other hand, a trailing-side flyby increases that speed.

In order to analyze a flyby problem, we proceed as follows. First, let $\hat{\mathbf{u}}_V$ be the unit vector in the direction of the planet's heliocentric velocity $\mathbf{V}$ and let $\hat{\mathbf{u}}_S$ be the unit vector pointing from the planet to the sun. At the inbound crossing of the sphere of influence, the heliocentric velocity $\mathbf{V}_1^{(v)}$ of the spacecraft is

$$\mathbf{V}_1^{(v)} = \left[V_1^{(v)}\right]_V \hat{\mathbf{u}}_V + \left[V_1^{(v)}\right]_S \hat{\mathbf{u}}_S \tag{8.75}$$

where the scalar components of $\mathbf{V}_1^{(v)}$ are

$$\left[V_1^{(v)}\right]_V = V_1^{(v)} \cos\alpha_1 \qquad \left[V_1^{(v)}\right]_S = V_1^{(v)} \sin\alpha_1 \tag{8.76}$$

α_1 is the angle between $\mathbf{V}_1^{(v)}$ and $\mathbf{V}$. All angles are measured positive counterclockwise. Referring to Figure 2.12, we see that the magnitude of α_1 is the flight path angle γ of the spacecraft's heliocentric trajectory when it encounters the planet's sphere of influence (a mere speck) at the planet's distance R from the sun. Furthermore,

$$\left[V_1^{(v)}\right]_V = V_{\perp_1} \qquad \left[V_1^{(v)}\right]_S = -V_{r_1} \tag{8.77}$$

$V_{\perp_1}$ and V_{r_1} are furnished by Equations 2.48 and 2.49

$$V_{\perp_1} = \frac{\mu_{\text{sun}}}{h_1}(1 + e_1 \cos\theta_1) \qquad V_{r_1} = \frac{\mu_{\text{sun}}}{h_1} e_1 \sin\theta_1 \tag{8.78}$$

in which e_1, h_1 and θ_1 are the eccentricity, angular momentum and true anomaly of the heliocentric approach trajectory.

The velocity of the planet relative to the sun is

$$\mathbf{V} = V\hat{\mathbf{u}}_V \tag{8.79}$$

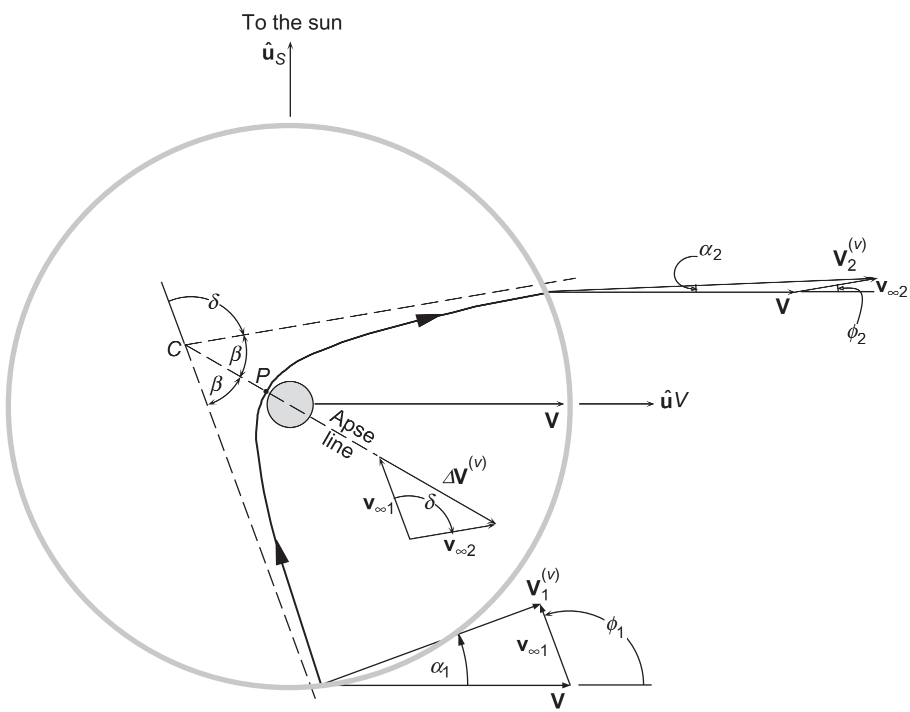

FIGURE 8.19

Trailing-side planetary flyby.

where $V = \sqrt{\mu_{sun}/R}$. At the inbound crossing of the planet's sphere of influence, the hyperbolic excess velocity of the spacecraft is obtained from Equation 8.72

$$\mathbf{v}_{\infty_1} = \mathbf{V}_1^{(v)} - \mathbf{V}$$

Using this we find

$$\mathbf{v}_{\infty_1} = (v_{\infty_1})_V \hat{\mathbf{u}}_V + (v_{\infty_1})_S \hat{\mathbf{u}}_S \tag{8.80}$$

where the scalar components of $\mathbf{v}_{\infty_1}$ are

$$(v_{\infty_1})_V = V_1^{(v)} \cos\alpha_1 - V \qquad (v_{\infty_1})_S = V_1^{(v)} \sin\alpha_1 \tag{8.81}$$

v_∞ is the magnitude of $\mathbf{v}_{\infty_1}$,

$$v_\infty = \sqrt{\mathbf{v}_{\infty_1} \cdot \mathbf{v}_{\infty_1}} = \sqrt{\left[V_1^{(v)}\right]^2 + V^2 - 2V_1^{(v)} V \cos\alpha_1} \tag{8.82}$$

At this point v_∞ is known, so that upon specifying the periapsis radius r_p we can compute the angular momentum and eccentricity of the flyby hyperbola (relative to the planet), using Equations 8.38 and 8.39.

$$h = r_p \sqrt{v_\infty{}^2 + \frac{2\mu}{r_p}} \qquad e = 1 + \frac{r_p v_\infty{}^2}{\mu} \tag{8.83}$$

where μ is the gravitational parameter of the planet.

The angle between $\mathbf{v}_{\infty_1}$ and the planet's heliocentric velocity is ϕ_1. It is found using the components of $\mathbf{v}_{\infty_1}$ in Equation 8.81,

$$\phi_1 = \tan^{-1} \frac{(v_{\infty_1})_S}{(v_{\infty_1})_V} = \tan^{-1} \frac{V_1^{(v)} \sin\alpha_1}{V_1^{(v)} \cos\alpha_1 - V} \tag{8.84}$$

At the outbound crossing the angle between $\mathbf{v}_{\infty_2}$ and $\mathbf{V}$ is ϕ_2, where

$$\phi_2 = \phi_1 + \delta \tag{8.85}$$

For the leading-side flyby in Figure 8.18, the turn angle is δ positive (counterclockwise) whereas in Figure 8.19 it is negative. Since the magnitude of $\mathbf{v}_{\infty_2}$ is v_∞, we can express $\mathbf{v}_{\infty_2}$ in components as

$$\mathbf{v}_{\infty_2} = v_\infty \cos\phi_2 \hat{\mathbf{u}}_V + v_\infty \sin\phi_2 \hat{\mathbf{u}}_S \tag{8.86}$$

Therefore, the heliocentric velocity of the spacecraft at the outbound crossing is

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} = \left[V_2^{(v)}\right]_V \hat{\mathbf{u}}_V + \left[V_2^{(v)}\right]_S \hat{\mathbf{u}}_S \tag{8.87}$$

where the components of $\mathbf{V}_2^{(v)}$ are

$$\left[V_2^{(v)}\right]_V = V + v_\infty \cos\phi_2 \qquad \left[V_2^{(v)}\right]_S = v_\infty \sin\phi_2 \tag{8.88}$$

From this we obtain the radial and transverse heliocentric velocity components,

$$V_{\perp_2} = \left[V_2^{(v)}\right]_V \qquad V_{r_2} = -\left[V_2^{(v)}\right]_S \tag{8.89}$$

Finally, we obtain the three elements e_2, h_2 and θ_2 of the new heliocentric departure trajectory by means of Equation 2.31,

$$h_2 = RV_{\perp_2} \tag{8.90}$$

Equation 2.45,

$$R = \frac{h_2^{\,2}}{\mu_{\text{sun}}} \frac{1}{1 + e_2 \cos\theta_2} \tag{8.91}$$

and Equation 2.49,

$$V_{r_2} = \frac{\mu_{\text{sun}}}{h_2} e_2 \sin\theta_2 \tag{8.92}$$

Notice that the flyby is considered to be an impulsive maneuver during which the heliocentric radius of the spacecraft, which is confined within the planet's sphere of influence, remains fixed at R. The heliocentric velocity analysis is similar to that described in Section 6.7.

Example 8.6

A spacecraft departs earth with a velocity perpendicular to the sun line on a flyby mission to Venus. Encounter occurs at a true anomaly in the approach trajectory of –30°. Periapsis ("pericytherion") altitude is to be 300 km.

(a) For an approach from the dark side of the planet, show that the post-flyby orbit is as illustrated in Figure 8.20.

(b) For an approach from the sunlit side of the planet, show that the post-flyby orbit is as illustrated in Figure 8.21.

Solution

The following data is found in Tables A.1 and A.2.

$$\begin{aligned}
\mu_{\text{sun}} &= 1.3271 \times 10^{11}\ \text{km}^3/\text{s}^2 \\
\mu_{\text{Venus}} &= 324{,}900\ \text{km}^3/\text{s}^2 \\
R_{\text{earth}} &= 149.6 \times 10^6\ \text{km} \\
R_{\text{Venus}} &= 108.2 \times 10^6\ \text{km} \\
r_{\text{Venus}} &= 6052\ \text{km}
\end{aligned}$$

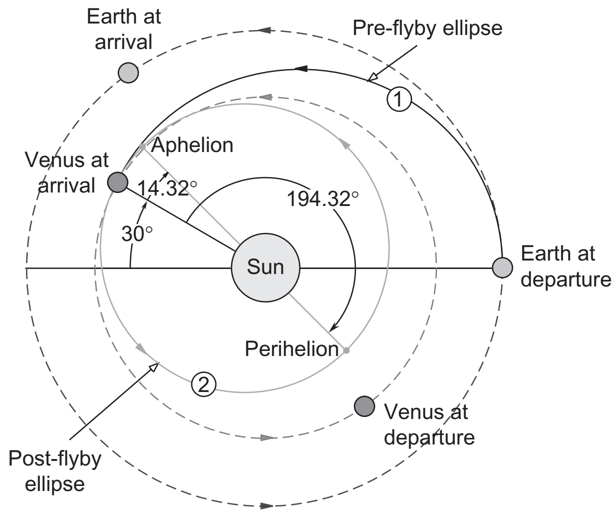

FIGURE 8.20

Spacecraft orbits before and after a flyby of Venus, approaching from the dark side.

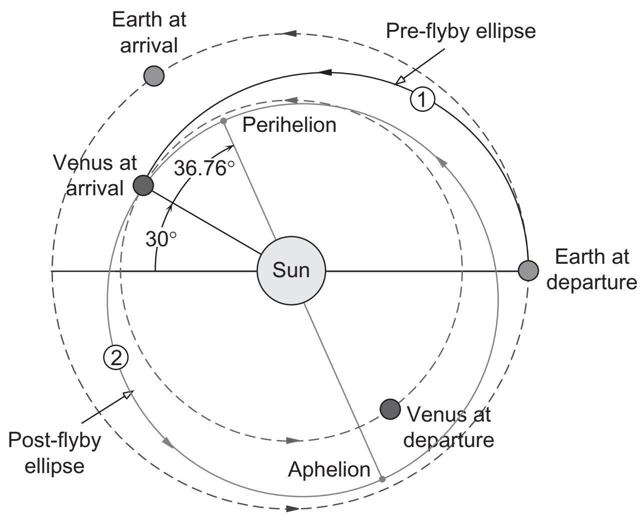

FIGURE 8.21

Spacecraft orbits before and after a flyby of Venus, approaching from the sunlit side.

Pre-flyby ellipse (orbit 1)

Evaluating the orbit formula, Equation 2.45, at aphelion of orbit 1 yields

$$R_{\text{earth}} = \frac{h_1^{\,2}}{\mu_{\text{sun}}} \frac{1}{1 - e_1}$$

Thus,

$$h_1^2 = \mu_{sun} R_{earth} (1 - e_1) \tag{a}$$

At intercept

$$R_{Venus} = \frac{h_1^2}{\mu_{sun}} \frac{1}{1 + e_1 \cos(\theta_1)}$$

Substituting Equation (a) and $\theta_1 = -30°$ and solving the resulting expression for e_1 leads to

$$e_1 = \frac{R_{earth} - R_{Venus}}{R_{earth} + R_{Venus} \cos(\theta_1)} = \frac{149.6 \times 10^6 - 108.2 \times 10^6}{149.6 \times 10^6 + 108.2 \times 10^6 \cos(-30°)} = 0.1702$$

With this result, Equation (a) yields

$$h_1 = \sqrt{1.327 \times 10^{11} \cdot 149.6 \times 10^6 (1 - 0.1702)} = 4.059 \times 10^9 \ \text{km}^2/\text{s}$$

Now we can use Equations 2.31 and 2.49 to calculate the radial and transverse components of the spacecraft's heliocentric velocity at the inbound crossing of Venus's sphere of influence.

$$V_{\perp_1} = \frac{h_1}{R_{Venus}} = \frac{4.059 \times 10^9}{108.2 \times 10^6} = 37.51 \text{ km/s}$$

$$V_{r_1} = \frac{\mu_{sun}}{h_1} e_1 \sin(\theta_1) = \frac{1.327 \times 10^{11}}{4.059 \times 10^9} \cdot 0.1702 \cdot \sin(-30°) = -2.782 \text{ km/s}$$

The flight path angle, from Equation 2.51, is

$$\gamma_1 = \tan^{-1} \frac{V_{r_1}}{V_{\perp_1}} = \tan^{-1}\left(\frac{-2.782}{37.51}\right) = -4.241°$$

The negative sign is consistent with the fact that the spacecraft is flying towards perihelion of the pre-flyby elliptical trajectory (orbit 1).

The speed of the space vehicle at the inbound crossing is

$$V_1^{(v)} = \sqrt{V_{r_1}^{\ 2} + V_{\perp_1}^{\ 2}} = \sqrt{(-2.782)^2 + 37.51^2} = 37.62 \text{ km/s} \tag{b}$$

Flyby hyperbola

From Equations 8.75 and 8.77 we obtain

$$\mathbf{V}_1^{(v)} = 37.51\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S \text{ (km/s)}$$

The velocity of Venus in its presumed circular orbit around the sun is

$$\mathbf{V} = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{Venus}}}}\hat{\mathbf{u}}_V = \sqrt{\frac{1.327 \times 10^{11}}{108.2 \times 10^6}}\hat{\mathbf{u}}_V = 35.02\hat{\mathbf{u}}_V \text{ (km/s)} \tag{c}$$

Hence,

$$\mathbf{v}_{\infty_1} = \mathbf{V}_1^{(v)} - \mathbf{V} = (37.51\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S) - 35.02\hat{\mathbf{u}}_V = 2.490\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S \text{ (km/s)} \tag{d}$$

It follows that

$$v_\infty = \sqrt{\mathbf{v}_{\infty_1} \cdot \mathbf{v}_{\infty_1}} = 3.733 \text{ km/s}$$

The periapsis radius is

$$r_p = r_{\text{Venus}} + 300 = 6352 \text{ km}$$

Equations 8.38 and 8.39 are used to compute the angular momentum and eccentricity of the planetocentric hyperbola.

$$h = 6352\sqrt{v_\infty{}^2 + \frac{2\mu_{\text{Venus}}}{6352}} = 6352\sqrt{3.733^2 + \frac{2 \cdot 324{,}900}{6352}} = 68{,}480 \text{ km}^2\text{/s}$$

$$e = 1 + \frac{r_p v_\infty{}^2}{\mu_{\text{Venus}}} = 1 + \frac{6352 \cdot 3.733^2}{324{,}900} = 1.272$$

The turn angle and true anomaly of the asymptote are

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\sin^{-1}\left(\frac{1}{1.272}\right) = 103.6°$$

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right) = \cos^{-1}\left(-\frac{1}{1.272}\right) = 141.8°$$

From Equations 2.50, 2.103 and 2.107, the aiming radius is

$$\Delta = r_p\sqrt{\frac{e+1}{e-1}} = 6352\sqrt{\frac{1.272+1}{1.272-1}} = 18{,}340 \text{ km} \tag{e}$$

Finally, from Equation (d) we obtain the angle between $\mathbf{v}_{\infty_1}$ and $\mathbf{V}$,

$$\phi_1 = \tan^{-1}\frac{2.782}{2.490} = 48.17° \tag{f}$$

There are two flyby approaches, as shown in Figure 8.22. In the dark side approach, the turn angle is counterclockwise (+103.6°) whereas for the sunlit side approach it is clockwise (–103.6°).

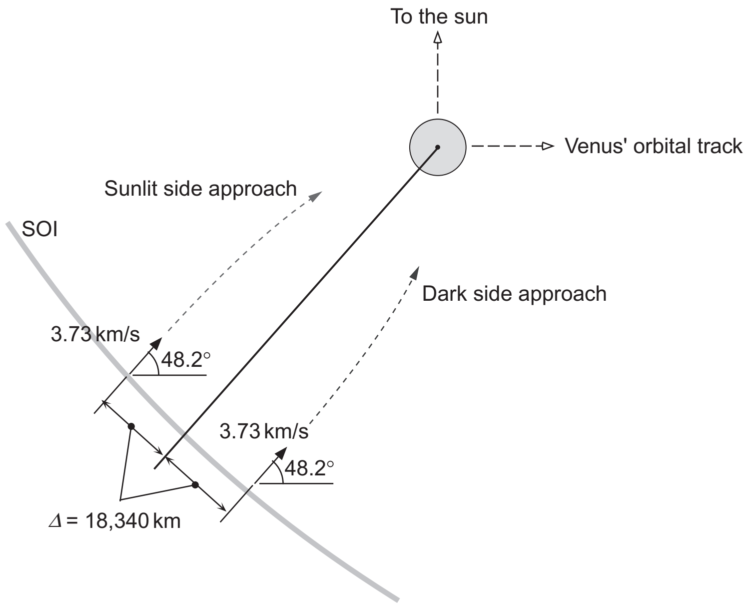

FIGURE 8.22

Initiation of a sunlit side approach and dark side approach at the inbound crossing.

Dark side approach

According to Equation 8.85, the angle between $\mathbf{v}_\infty$ and $\mathbf{V}_{\text{Venus}}$ at the outbound crossing is

$$\phi_2 = \phi_1 + \delta = 48.17° + 103.6° = 151.8°$$

Hence, by Equation 8.86,

$$\mathbf{v}_{\infty_2} = 3.733(\cos 151.8°\hat{\mathbf{u}}_V + \sin 151.8°\hat{\mathbf{u}}_S) = -3.289\hat{\mathbf{u}}_V + 1.766\hat{\mathbf{u}}_S \text{ (km/s)}$$

Using this and Equation (c) above, we compute the spacecraft's heliocentric velocity at the outbound crossing.

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} = 31.73\hat{\mathbf{u}}_V + 1.766\hat{\mathbf{u}}_S \text{ (km/s)}$$

It follows from Equation 8.89 that

$$V_{\perp_2} = 31.73 \text{ km/s} \qquad V_{r_2} = -1.766 \text{ km/s} \tag{g}$$

The speed of the spacecraft at the outbound crossing is

$$V_2^{(v)} = \sqrt{{V_{r_2}}^2 + {V_{\perp_2}}^2} = \sqrt{(-1.766)^2 + 31.73^2} = 31.78 \text{ km/s}$$

This is 5.83 km/s less than the inbound speed.

Post-flyby ellipse (orbit 2) for the dark side approach

For the heliocentric post flyby trajectory, labeled orbit 2 in Figure 8.20, the angular momentum is found using Equation 8.90

$$h_2 = R_{\text{Venus}} V_{\perp_2} = (108.2 \times 10^6) \cdot 31.73 = 3.434 \times 10^9 \ (\text{km}^2/\text{s}) \tag{h}$$

From Equation 8.91,

$$e\cos\theta_2 = \frac{h_2{}^2}{\mu_{\text{sun}} R_{\text{Venus}}} - 1 = \frac{(3.434 \times 10^6)^2}{1.327 \times 10^{11} \cdot 108.2 \times 10^6} - 1 = -0.1790 \tag{i}$$

and from Equation 8.92

$$e\sin\theta_2 = \frac{V_{r_2} h_2}{\mu_{\text{sun}}} = \frac{-1.766 \cdot 3.434 \times 10^9}{1.327 \times 10^{11}} = -0.04569 \tag{j}$$

Thus,

$$\tan\theta_2 = \frac{e\sin\theta_2}{e\cos\theta_2} = \frac{-0.045{,}69}{-0.1790} = 0.2553 \tag{k}$$

which means

$$\theta_2 = 14.32° \quad or \quad 194.32° \tag{l}$$

But θ_2 must lie in the third quadrant since, according to Equations (i) and (j), both the sine and cosine are negative. Hence,

$$\theta_2 = 194.32° \tag{m}$$

With this value of θ_2, we can use either Equation (i) or (j) to calculate the eccentricity,

$$e_2 = 0.1847 \tag{n}$$

Perihelion of the departure orbit lies 194.32° clockwise from the encounter point (so that aphelion is 14.32° therefrom), as illustrated in Figure 8.20. The perihelion radius is given by Equation 2.50,

$$R_{\text{perihelion}} = \frac{h_2{}^2}{\mu_{\text{sun}}} \frac{1}{1 + e_2} = \frac{(3.434 \times 10^9)^2}{1.327 \times 10^{11}} \frac{1}{1 + 0.1847} = 74.98 \times 10^6 \ \text{km}$$

which is well within the orbit of Venus.

Sunlit side approach

In this case the angle between $\mathbf{v}_\infty$ and $\mathbf{V}_{\text{Venus}}$ at the outbound crossing is

$$\phi_2 = \phi_1 - \delta = 48.17° - 103.6° = -55.44°$$

Therefore,

$$\mathbf{v}_{\infty_2} = 3.733\left[\cos(-55.44°)\hat{\mathbf{u}}_V + \sin(-55.44°)\hat{\mathbf{u}}_S\right] = 2.118\hat{\mathbf{u}}_V - 3.074\hat{\mathbf{u}}_S \text{ (km/s)}$$

The spacecraft's heliocentric velocity at the outbound crossing is

$$\mathbf{V}_2^{(v)} = \mathbf{V}_{\text{Venus}} + \mathbf{v}_{\infty_2} = 37.14\hat{\mathbf{u}}_V - 3.074\hat{\mathbf{u}}_S \text{ (km/s)}$$

which means

$$V_{\perp_2} = 37.14 \text{ km/s} \qquad V_{r_2} = 3.074 \text{ km/s}$$

The speed of the spacecraft at the outbound crossing is

$$V_2^{(v)} = \sqrt{{V_{r_2}}^2 + {V_{\perp_2}}^2} = \sqrt{3.074^2 + 37.14^2} = 37.27 \text{ km/s}$$

This speed is just 0.348 km/s less than inbound crossing speed. The relatively small speed change is due to the fact that the apse line of this hyperbola is nearly perpendicular to Venus' orbital track, as shown in Figure 8.23. Nevertheless, the periapses of both hyperbolas are on the leading side of the planet.

Post-flyby ellipse (orbit 2) for the sunlit side approach

To determine the heliocentric post-flyby trajectory, labeled orbit 2 in Figure 8.21, we repeat steps (h) through (n) above.

$$h_2 = R_{\text{Venus}}V_{\perp_2} = (108.2 \times 10^6)\cdot 37.14 = 4.019 \times 10^9 \text{ (km}^2\text{/s)}$$

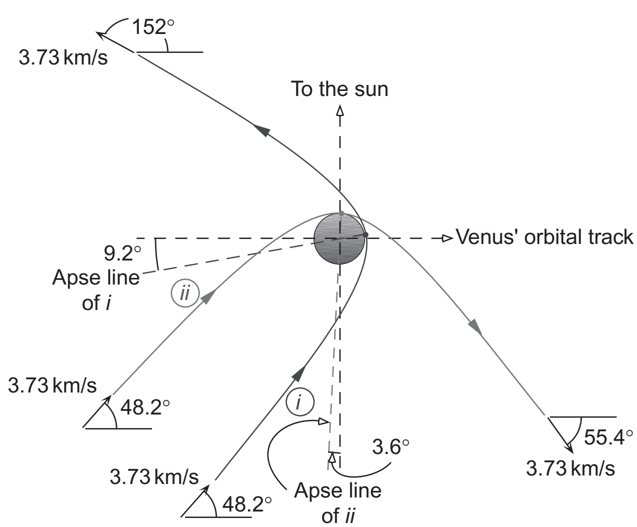

FIGURE 8.23

Hyperbolic flyby trajectories for (i) the dark side approach and (ii) the sunlit side approach.

$$e\cos\theta_2 = \frac{{h_2}^2}{\mu_{\text{sun}} R_{\text{Venus}}} - 1 = \frac{(4.019 \times 10^9)^2}{1.327 \times 10^{11} \cdot 108.2 \times 10^6} - 1 = 0.1246 \quad \text{(o)}$$

$$e\sin\theta_2 = \frac{V_{r_2} h_2}{\mu_{\text{sun}}} = \frac{3.074 \cdot 4.019 \times 10^9}{1.327 \times 10^{11}} = 0.09309 \quad \text{(p)}$$

$$\tan\theta_2 = \frac{e\sin\theta_2}{e\cos\theta_2} = \frac{0.09309}{0.1246} = 0.7469$$

$$\theta_2 = 36.08° \quad or \quad 216.08°$$

θ_2 must lie in the first quadrant since both the sine and cosine are positive. Hence,

$$\theta_2 = 36.76° \quad \text{(q)}$$

With this value of θ_2, we can use either Equation (o) or (p) to calculate the eccentricity,

$$e_2 = 0.1556$$

Perihelion of the departure orbit lies 36.76° clockwise from the encounter point as illustrated in Figure 8.21. The perihelion radius is

$$R_{\text{perihelion}} = \frac{{h_2}^2}{\mu_{sun}} \frac{1}{1+e_2} = \frac{(4.019 \times 10^9)^2}{1.327 \times 10^{11}} \frac{1}{1+0.1556} = 105.3 \times 10^6 \text{ km}$$

which is just within the orbit of Venus. Aphelion lies between the orbits of earth and Venus.

Gravity assist maneuvers are used to add momentum to a spacecraft over and above that available from a spacecraft's on-board propulsion system. A sequence of flybys of planets can impart the delta-v needed to reach regions of the solar system that would be inaccessible using only existing propulsion technology. The technique can also reduce the flight time. Interplanetary missions using gravity assist flybys must be carefully designed in order to take advantage of the relative positions of planets.

The 260 kg spacecraft Pioneer 11, launched in April 1973, used a December 1974 flyby of Jupiter to gain the momentum required to carry it to the first ever flyby encounter with Saturn on 1 September 1979.

Following its September 1977 launch, Voyager 1 likewise used a flyby of Jupiter (March 1979) to reach Saturn in November 1980. In August 1977 Voyager 2 was launched on its "grand tour" of the outer planets and beyond. This involved gravity assist flybys of Jupiter (July 1979), Saturn (August 1981), Uranus (January 1986) and Neptune (August 1989), after which the spacecraft departed the solar system at an angle of 30° to the ecliptic.

With a mass nine times that of Pioneer 11, the dual-spin Galileo spacecraft departed on 18 October 1989 for an extensive international exploration of Jupiter and its satellites lasting until September 2003. Galileo used gravity assist flybys of Venus (February 1990), earth (December 1990) and earth again (December 1992) before arriving at Jupiter in December 1995.

The international Cassini mission to Saturn also made extensive use of gravity assist flyby maneuvers. The Cassini spacecraft was launched on 15 October 1997 from Cape Canaveral, Florida, and arrived at

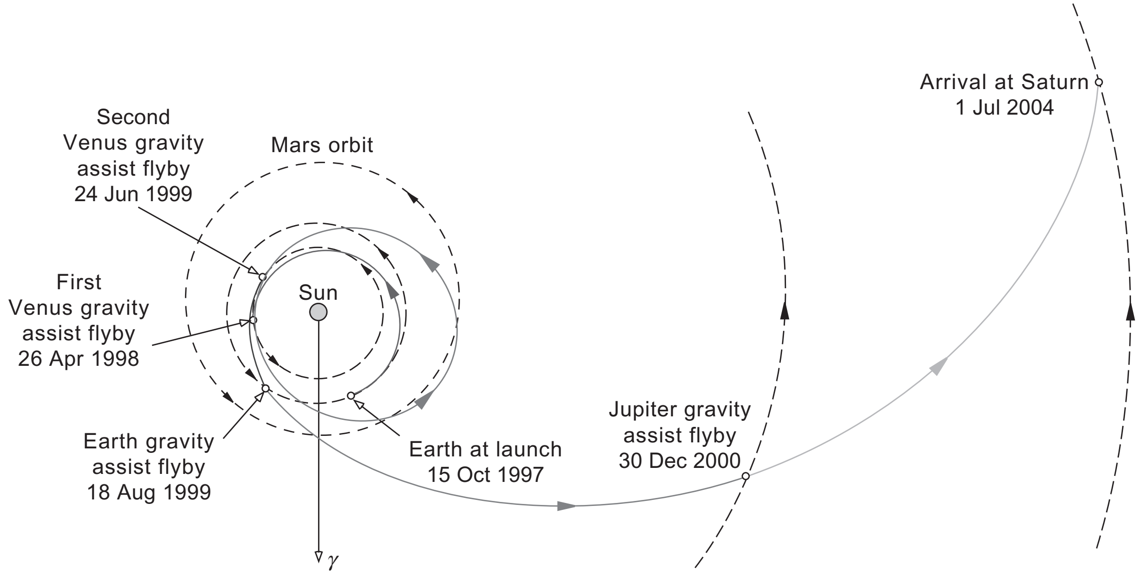

FIGURE 8.24

Cassini seven-year mission to Saturn.

Saturn nearly seven years later, on 1 July 2004. The mission involved four flybys, as illustrated in Figure 8.24. A little over eight months after launch, on 26 April 1998, Cassini flew by Venus at a periapsis altitude of 284 km and received a speed boost of about 7 km/s. This placed the spacecraft in an orbit which sent it just outside the orbit of Mars (but well away from the planet) and returned it to Venus on 24 June 1999 for a second flyby, this time at an altitude of 600 km. The result was a trajectory that vectored Cassini toward the earth for an 18 August 1999 flyby at an altitude of 1171 km. The 5.5 km/s speed boost at earth sent the spacecraft toward Jupiter for its next flyby maneuver. This occurred on 30 December 2000 at a distance of 9.7 million km from Jupiter, boosting Cassini's speed by about 2 km/s and adjusting its trajectory so as to rendezvous with Saturn about three and a half years later.

8.10 PLANETARY EPHEMERIS

The state vector **R**, **V** of a planet is defined relative to the heliocentric ecliptic frame of reference illustrated in Figure 8.25. This is very similar to the geocentric equatorial frame of Figure 4.7. The sun replaces the earth as the center of attraction, and the plane of the ecliptic replaces the earth's equatorial plane. The vernal equinox continues to define the inertial X axis.

In order to design realistic interplanetary missions we must be able to determine the state vector of a planet at any given time. Table 8.1 provides the orbital elements of the planets and their rates of change per century with respect to the J2000 epoch (1 January 2000, 12 h UT). The table, covering the years 1800 to 2050, is sufficiently accurate for our needs. From the orbital elements we can infer the state vector using Algorithm 4.5.

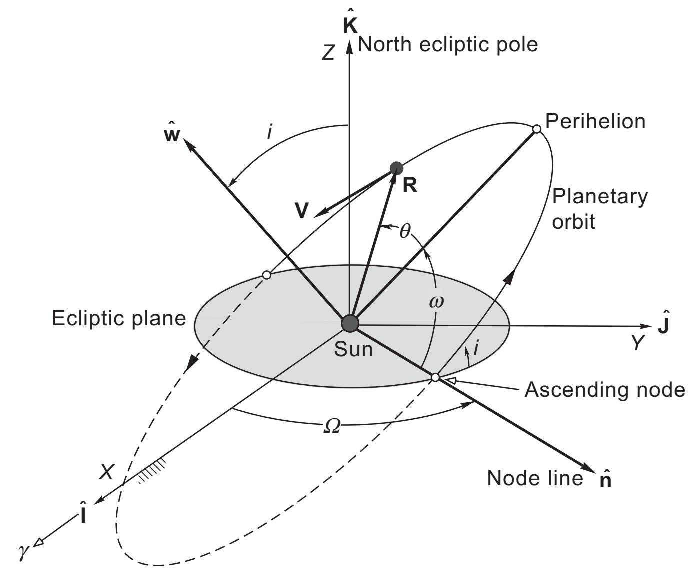

FIGURE 8.25

Planetary orbit in the heliocentric ecliptic frame.

In order to interpret the Table 8.1, observe the following:
1 astronomical unit (1 AU) is 1.49597871×10^8 km, the average distance between the earth and the sun.
1 arcsecond (1″) is 1/3600 of a degree.
a is the semimajor axis.
e is the eccentricity.
i is the inclination to the ecliptic plane.
Ω is the right ascension of the ascending node (relative to the J2000 vernal equinox).
ϖ, the longitude of perihelion, is defined as $\varpi = \omega + \Omega$, where ω is the argument of perihelion.
L, the mean longitude, is defined as $L = \varpi + M$, where M is the mean anomaly.
$\dot{a}$, $\dot{e}$, $\dot{\Omega}$, etc., are the rates of change of the above orbital elements per Julian century. 1 century (Cy) equals 36,525 days.

Algorithm 8.1 Determine the state vector of a planet at a given date and time. All angular calculations must be adjusted so that they lie in the range 0 to 360°. Recall that the gravitational parameter of the sun is $\mu = 1.327 \times 10^{11}$ km^3/s^2. This procedure is implemented in MATLAB® as the function *planet_elements_and_sv.m* in Appendix D.35.

1. Use Equations 5.47 and 5.48 to calculate the Julian day number JD.
2. Calculate T_0, the number of Julian centuries between J2000 and the date in question

$$T_0 = \frac{JD - 2{,}451{,}545}{36{,}525} \tag{8.93a}$$

3. If Q is any one of the six planetary orbital elements listed in Table 8.1, then calculate its value at JD by means of the formula

$$Q = Q_0 + \dot{Q}T_0 \tag{8.93b}$$

Table 8.1 Planetary Orbital Elements and their Centennial Rates
From Standish et al. (1992). Used with permission.

	a, AU $\dot{a}$, AU/Cy	e $\dot{e}$, 1/Cy	i, deg $\dot{i}$, ″/Cy	Ω, deg $\dot{\Omega}$, ″/Cy	ϖ, deg $\dot{\varpi}$, ″/Cy	L, deg $\dot{L}$, ″/Cy
Mercury	0.38709893	0.20563069	7.00487	48.33167	77.4545	252.25084
	0.00000066	0.00002527	–23.51	–446.30	573.57	538101628.29
Venus	0.72333199	0.00677323	3.39471	76.68069	131.53298	181.97973
	0.00000092	–0.00004938	–2.86	–996.89	–108.80	210 664,136.06
Earth	1.00000011	0.01671022	0.00005	–11.26064	102.94719	100.46435
	–0.00000005	–0.00003804	–46.94	–18 228.25	1198.28	129,597,740.63
Mars	1.52366231	0.09341233	1.85061	49.57854	336.04084	355.45332
	–0.00007221	0.00011902	–25.47	–1020.19	1560.78	68,905,103.78
Jupiter	5.20336301	0.04839266	1.30530	100.55615	14.75385	34.40438
	0.00060737	–0.00012880	–4.15	1217.17	839.93	10,925,078.35
Saturn	9.53707032	0.05415060	2.48446	113.71504	92.43194	49.94432
	–0.00301530	–0.00036762	6.11	–1591.05	–1948.89	4,401,052.95
Uranus	19.19126393	0.04716771	0.76986	74.22988	170.96424	313.23218
	0.00152025	–0.00019150	–2.09	–1681.4	1312.56	1,542,547.79
Neptune	30.06896348	0.00858587	1.76917	131.72169	44.97135	304.88003
	–0.00125196	0.00002514	–3.64	–151.25	–844.43	786 449.21
(Pluto)	39.48168677	0.24880766	17.14175	110.30347	224.06676	238.92881
	–0.00076912	0.00006465	11.07	–37.33	–132.25	522,747.90

where Q_0 is the value listed for J2000 and $\dot{Q}$ is the tabulated rate. All angular quantities must be adjusted to lie in the range 0 to 360°.

4. Use the semimajor axis a and the eccentricity e to calculate the angular momentum h at JD from Equation 2.71

$$h = \sqrt{\mu a(1 - e^2)}$$

5. Obtain the argument of perihelion ω and mean anomaly M at JD from the results of step 3 by means of the definitions

$$\omega = \varpi - \Omega$$
$$M = L - \varpi$$

6. Substitute the eccentricity e and the mean anomaly M at JD into Kepler's equation (Equation 3.14) and calculate the eccentric anomaly E.
7. Calculate the true anomaly θ using Equation 3.13.
8. Use h, e, Ω, i, ω and θ to obtain the heliocentric position vector **R** and velocity **V** by means of Algorithm 4.5, with the heliocentric ecliptic frame replacing the geocentric equatorial frame.

Example 8.7

Find the distance between the earth and Mars at 12h UT on 27 August 2003. Use Algorithm 8.1.

Step 1:

According to Equation 5.48, the Julian day number J_0 for midnight (0h UT) of this date is

$$\begin{aligned} J_0 &= 367 \cdot 2003 - \mathrm{INT}\left\{\frac{7\left[2003 + \mathrm{INT}\left(\dfrac{8+9}{12}\right)\right]}{4}\right\} + \mathrm{INT}\left(\frac{275 \cdot 8}{9}\right) + 27 + 1{,}721{,}013.5 \\ &= 735{,}101 - 3507 + 244 + 27 + 1{,}721{,}013.5 \\ &= 2{,}452{,}878.5 \end{aligned}$$

At $UT = 12$, the Julian day number is

$$JD = 2{,}452{,}878.5 + \frac{12}{24} = 2{,}452{,}879.0$$

Step 2:

The number of Julian centuries between J2000 and this date is

$$T_0 = \frac{JD - 2{,}451{,}545}{36{,}525} = \frac{2{,}452{,}879 - 2{,}451{,}545}{36{,}525} = 0.036523 \text{ Cy}$$

Step 3:

Table 8.1 and Equation 8.93b yield the orbital elements of earth and Mars at 12h UT on 27 August 2003.

	***a*, km**	***e***	***i*, deg**	***Ω*, deg**	***ϖ*, deg**	***L*, deg**
Earth	1.4960×10^8	0.016709	0.00042622	348.55	102.96	335.27
Mars	2.2794×10^8	0.093417	1.8504	49.568	336.06	334.51

Step 4:

$$\begin{aligned} h_{\text{earth}} &= 4.4451 \times 10^9 \text{ km}^2/\text{s} \\ h_{\text{Mars}} &= 5.4760 \times 10^9 \text{ km}^2/\text{s} \end{aligned}$$

Step 5:

$$\begin{aligned} \omega_{\text{earth}} &= (\varpi - \Omega)_{\text{earth}} = 102.96 - 348.55 = -245.59^\circ\ (114.1^\circ) \\ \omega_{\text{Mars}} &= (\varpi - \Omega)_{\text{Mars}} = 336.06 - 49.568 = 286.49^\circ \end{aligned}$$

$$\begin{aligned} M_{\text{earth}} &= (L - \varpi)_{\text{earth}} = 335.27 - 102.96 = 232.31^\circ \\ M_{\text{Mars}} &= (L - \varpi)_{\text{Mars}} = 334.51 - 336.06 = -1.55^\circ\ (358.45^\circ) \end{aligned}$$

Step 6:

$$E_{\text{earth}} - 0.016709 \sin E_{\text{earth}} = 232.31°(\pi/180) \Rightarrow E_{\text{earth}} = 231.56°$$
$$E_{\text{Mars}} - 0.093417 \sin E_{\text{Mars}} = 358.45°(\pi/180) \Rightarrow E_{\text{Mars}} = 358.30°$$

Step 7:

$$\begin{aligned}\theta_{\text{earth}} &= 2\tan^{-1}\left(\sqrt{\frac{1-0.016709}{1+0.016709}}\tan\frac{231.56°}{2}\right) = -129.19° \Rightarrow \theta_{\text{earth}} = 230.81° \\ \theta_{\text{Mars}} &= 2\tan^{-1}\left(\sqrt{\frac{1-0.093417}{1+0.093417}}\tan\frac{358.30°}{2}\right) = -1.8669° \Rightarrow \theta_{\text{Mars}} = 358.13°\end{aligned} \qquad \text{(d)}$$

Step 8:

From Algorithm 4.5,

$$\mathbf{R}_{\text{earth}} = (135.59\hat{\mathbf{I}} - 66.803\hat{\mathbf{J}} - 0.00028691\,\hat{\mathbf{K}}) \times 10^6 \text{ (km)}$$
$$\mathbf{V}_{\text{earth}} = 12.680\hat{\mathbf{I}} + 26.61\hat{\mathbf{J}} - 0.00021273\,\hat{\mathbf{K}} \text{ (km/s)}$$

$$\mathbf{R}_{\text{Mars}} = (185.95\hat{\mathbf{I}} - 89.916\hat{\mathbf{J}} - 6.4566\hat{\mathbf{K}}) \times 10^6 \text{ (km)}$$
$$\mathbf{V}_{\text{Mars}} = 11.474\hat{\mathbf{I}} + 23.884\hat{\mathbf{J}} + 0.21826\hat{\mathbf{K}} \text{ (km/s)}$$

The distance d between the two planets is therefore,

$$\begin{aligned} d &= \left\|\mathbf{R}_{\text{Mars}} - \mathbf{R}_{\text{earth}}\right\| \\ &= \sqrt{(185.95 - 135.59)^2 + [-89.916 - (-66.803)]^2 + (-6.4566 - 0.00028691)^2} \times 10^6 \end{aligned}$$

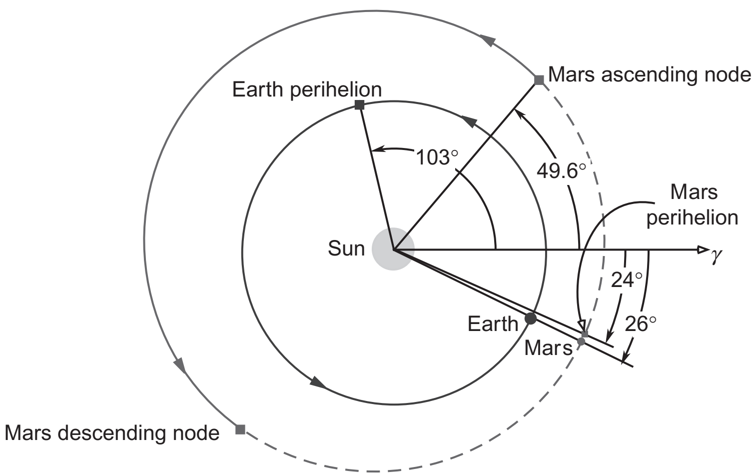

FIGURE 8.26

Earth and Mars on 27 August 2003. Angles shown are heliocentric latitude, measured in the plane of the ecliptic counterclockwise from the vernal equinox of J2000.

or

$$\boxed{d = 55.79 \times 10^6 \text{ km}}$$

The positions of earth and Mars are illustrated in Figure 8.26. It is a rare event for Mars to be in opposition (lined up with earth on the same side of the sun) when Mars is at or near perihelion. The two planets had not been this close in recorded history.

8.11 NON-HOHMANN INTERPLANETARY TRAJECTORIES

To implement a systematic patched conic procedure for three-dimensional trajectories, we will use vector notation and the procedures described in Sections 4.4 and 4.6 (Algorithms 4.2 and 4.5), together with the solution of Lambert's problem presented in Section 5.3 (Algorithm 5.2). The mission is to send a spacecraft from planet 1 to planet 2 in a specified time t_{12}. As previously in this chapter, we break the mission down into three parts: the departure phase, the cruise phase and the arrival phase. We start with the cruise phase.

The frame of reference that we use is the heliocentric ecliptic frame shown in Figure 8.27. The first step is to obtain the state vector of planet 1 at departure (time t) and the state vector of planet 2 at arrival (time $t + t_{12}$). That is accomplished by means of Algorithm 8.1.

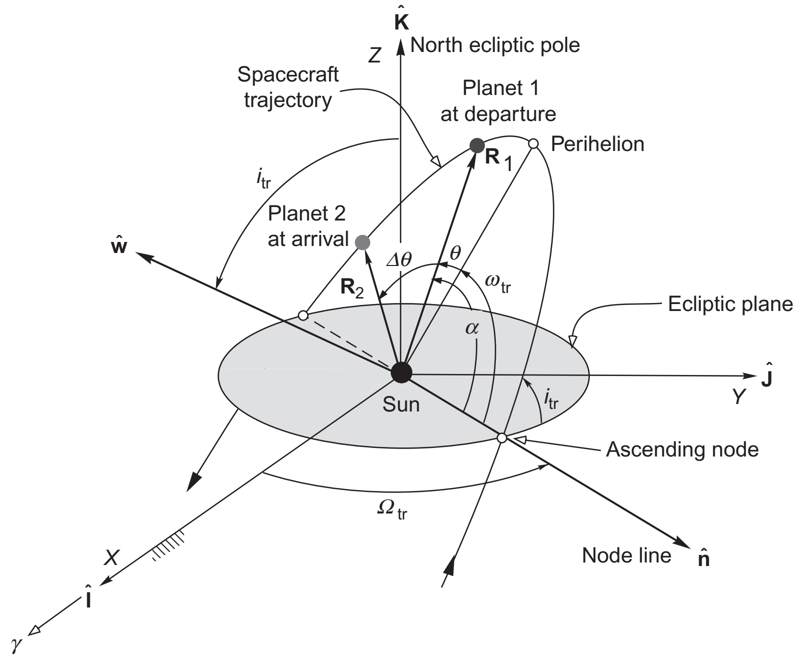

FIGURE 8.27

Heliocentric orbital elements of a three-dimensional transfer trajectory from planet 1 to planet 2.

The next step is to determine the spacecraft's transfer trajectory from planet 1 to planet 2. We first observe that, according to the patched conic procedure, the heliocentric position vector of the spacecraft at time t is that of planet 1 ($\mathbf{R}_1$) and at time $t + t_{12}$ its position vector is that of planet 2 ($\mathbf{R}_2$). With $\mathbf{R}_1$, $\mathbf{R}_2$ and the time of flight t_{12} we can use Algorithm 5.2 (Lambert's problem) to obtain the spacecraft's departure and arrival velocities $\mathbf{V}_D^{(v)}$ and $\mathbf{V}_A^{(v)}$ relative to the sun. Either of the state vectors $\mathbf{R}_1, \mathbf{V}_D^{(v)}$ or $\mathbf{R}_2, \mathbf{V}_A^{(v)}$ can be used to obtain the transfer trajectory's six orbital elements by means of Algorithm 4.2.

The spacecraft's hyperbolic excess velocity upon exiting the sphere of influence of planet 1 is

$$\mathbf{v}_\infty\big)_{\text{Departure}} = \mathbf{V}_D^{(v)} - \mathbf{V}_1 \tag{8.94a}$$

and its excess speed is

$$v_\infty\big)_{\text{Departure}} = \left\|\mathbf{V}_D^{(v)} - \mathbf{V}_1\right\| \tag{8.94b}$$

Likewise, at the sphere of influence crossing at planet 2,

$$\mathbf{v}_\infty\big)_{\text{Arrival}} = \mathbf{V}_A^{(v)} - \mathbf{V}_2 \tag{8.95a}$$

$$v_\infty\big)_{\text{Arrival}} = \left\|\mathbf{V}_A^{(v)} - \mathbf{V}_2\right\| \tag{8.95b}$$

Algorithm 8.2 Given the departure and arrival dates (and, therefore, the time of flight), determine the trajectory for a mission from planet 1 to planet 2. This procedure is implemented as the MATLAB function *interplanetary.m* in Appendix D.36.

1. Use Algorithm 8.1 to determine the state vector $\mathbf{R}_1, \mathbf{V}_1$ of planet 1 at departure and the state vector $\mathbf{R}_2, \mathbf{V}_2$ of planet 2 at arrival.
2. Use $\mathbf{R}_1$, $\mathbf{R}_2$ and the time of flight in Algorithm 5.2 to find the spacecraft velocity $\mathbf{V}_D{}^{(v)}$ at departure from planet 1's sphere of influence and its velocity $\mathbf{V}_A{}^{(v)}$ upon arrival at planet 2's sphere of influence.
3. Calculate the hyperbolic excess velocities at departure and arrival using Equations 8.94 and 8.95.

Example 8.8

A spacecraft departed earth's sphere of influence on 7 November 1996 (0 hr UT) on a prograde coasting flight to Mars, arriving at Mars' sphere of influence on 12 September 1997 (0 hr UT). Use Algorithm 8.2 to determine the trajectory and then compute the hyperbolic excess velocities at departure and arrival.

Solution

Step 1:

Algorithm 8.1 yields the state vectors for earth and Mars.

$$\begin{aligned} \mathbf{R}_{\text{earth}} &= 1.0500 \times 10^8 \hat{\mathbf{I}} + 1.0466 \times 10^8 \hat{\mathbf{J}} + 988.33 \hat{\mathbf{K}} \text{ (km)} & (R_{\text{earth}} &= 1.482 \times 10^8 \text{ km}) \\ \mathbf{V}_{\text{earth}} &= -21.516 \hat{\mathbf{I}} + 20.987 \hat{\mathbf{J}} + 0.00013228 \hat{\mathbf{K}} \text{ (km/s)} & (V_{\text{earth}} &= 30.06 \text{ km/s}) \end{aligned}$$

$$\begin{aligned} \mathbf{R}_{\text{Mars}} &= -2.0833 \times 10^7 \hat{\mathbf{I}} - 2.1840 \times 10^8 \hat{\mathbf{J}} - 4.0629 \times 10^6 \hat{\mathbf{K}} \text{ (km)} & (R_{\text{Mars}} &= 2.194 \times 10^8 \text{ km}) \\ \mathbf{V}_{\text{Mars}} &= 25.047 \hat{\mathbf{I}} - 0.22029 \hat{\mathbf{J}} - 0.62062 \hat{\mathbf{K}} \text{ (km/s)} & (V_{\text{Mars}} &= 25.05 \text{ km/s}) \end{aligned}$$

Step 2:
The position vector $\mathbf{R}_1$ of the spacecraft at crossing the earth's sphere of influence is just that of the earth,

$$\mathbf{R}_1 = \mathbf{R}_{\text{earth}} = 1.0500 \times 10^8 \hat{\mathbf{I}} + 1.0466 \times 10^8 \hat{\mathbf{J}} + 988.33\hat{\mathbf{K}} \text{ (km)}$$

Upon arrival at Mars' sphere of influence the spacecraft's position vector is

$$\mathbf{R}_2 = \mathbf{R}_{\text{Mars}} = -2.0833 \times 10^7 \hat{\mathbf{I}} - 2.1840 \times 10^8 \hat{\mathbf{J}} - 4.0629 \times 10^6 \hat{\mathbf{K}} \text{ (km)}$$

According to Equations 5.47 and 5.48

$$\begin{aligned} JD_{\text{Departure}} &= 2{,}450{,}394.5 \\ JD_{\text{Arrival}} &= 2{,}450{,}703.5 \end{aligned}$$

Hence, the time of flight is

$$t_{12} = 2{,}450{,}703.5 - 2{,}450{,}394.5 = 309 \text{ days}$$

Entering $\mathbf{R}_1$, $\mathbf{R}_2$ and t_{12} into Algorithm 5.2 yields

$$\begin{aligned} \mathbf{V}_D^{(v)} &= -24.427\hat{\mathbf{I}} + 21.781\hat{\mathbf{J}} + 0.94803\hat{\mathbf{K}} \text{ (km/s)} \quad \left[V_D^{(v)} = 32.741 \text{ km/s}\right] \\ \mathbf{V}_A^{(v)} &= 22.158\hat{\mathbf{I}} - 0.19668\hat{\mathbf{J}} - 0.45785\hat{\mathbf{K}} \text{ (km/s)} \quad \left[V_A^{(v)} = 22.164 \text{ km/s}\right] \end{aligned}$$

Using the state vector $\mathbf{R}_1, \mathbf{V}_{\text{D}}^{(v)}$ we employ Algorithm 4.2 to find the orbital elements of the transfer trajectory.

$$\begin{aligned} h &= 4.8456 \times 10^6 \text{ km}^2/\text{s} \\ e &= 0.20579 \\ \Omega &= 44.895^\circ \\ i &= 1.6621^\circ \\ \omega &= 19.969^\circ \\ \theta_1 &= 340.04^\circ \\ a &= 1.8474 \times 10^8 \text{ km} \end{aligned}$$

Step 3:
At departure the hyperbolic excess velocity is

$$\mathbf{v}_\infty\big)_{\text{Departure}} = \mathbf{V}_D^{(v)} - \mathbf{V}_{\text{earth}} = -2.913\hat{\mathbf{I}} + 0.7958\hat{\mathbf{J}} + 0.9480\hat{\mathbf{K}} \text{ (km/s)}$$

Therefore, the hyperbolic excess speed is

$$v_\infty\Big)_{\text{Departure}} = \left\| \mathbf{v}_\infty\Big)_{\text{Departure}} \right\| = \boxed{3.1651 \text{ km/s}} \tag{a}$$

Likewise, at arrival

$$\mathbf{v}_\infty\Big)_{\text{Arrival}} = \mathbf{V}_A^{(v)} - \mathbf{V}_{\text{Mars}} = -2.8804\hat{\mathbf{I}} + 0.0239\,76\hat{\mathbf{J}} + 0.16277\hat{\mathbf{K}} \text{ (km/s)}$$

so that

$$v_\infty\Big)_{\text{Arrival}} = \left\| \mathbf{v}_\infty\Big) \text{Arrival} \right\| = \boxed{2.8851 \text{ km/s}} \tag{b}$$

For the previous example, Figure 8.28 shows the orbits of earth, Mars and the spacecraft from directly above the ecliptic plane. Dotted lines indicate the portions of an orbit which are below the plane. λ is the heliocentric longitude measured counterclockwise from the vernal equinox of J2000. Also shown are the position of Mars at departure and the position of earth at arrival.

The transfer orbit resembles that of the Mars Global Surveyor, which departed earth on 7 November 1996 and arrived at Mars 309 days later, on 12 September 1997.

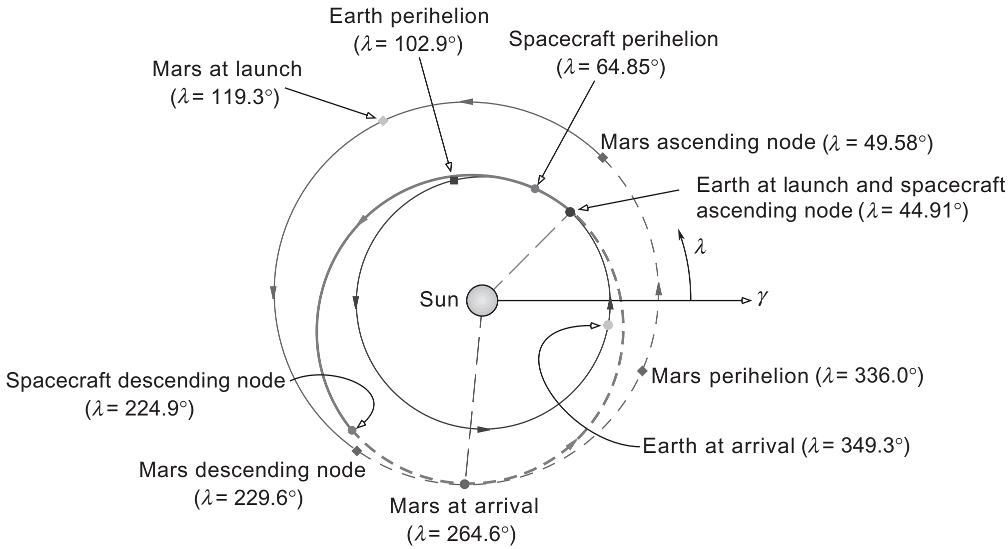

FIGURE 8.28

The transfer trajectory, together with the orbits of earth and Mars, as viewed from directly above the plane of the ecliptic.

Example 8.9

In Example 8.8, calculate the delta-v required to launch the spacecraft onto its cruise trajectory from a 180 km circular parking orbit. Sketch the departure trajectory.

Solution

Recall that

$$r_{\text{earth}} = 6378 \text{ km}$$
$$\mu_{\text{earth}} = 398{,}600 \text{ km}^3/\text{s}^2$$

The radius to periapsis of the departure hyperbola is the radius of the earth plus the altitude of the parking orbit,

$$r_p = 6378 + 180 = 6558 \text{ km}$$

Substituting this and Equation (a) from Example 8.8 into Equation 8.40 we get the speed of the spacecraft at periapsis of the departure hyperbola,

$$\begin{aligned} v_p &= \sqrt{\left[\left(v_\infty\right)_{\text{Departure}}\right]^2 + \frac{2\mu_{\text{earth}}}{r_p}} \\ &= \sqrt{3.1651^2 + \frac{2 \cdot 398{,}600}{6558}} = 11.47 \text{ km/s} \end{aligned}$$

The speed of the spacecraft in its circular parking orbit is

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r_p}} = \sqrt{\frac{398{,}600}{6558}} = 7.796 \text{ km/s}$$

Hence, the delta-v requirement is

$$\Delta v = v_p - v_c = 3.674 \text{ km/s}$$

The eccentricity of the hyperbola is given by Equation 8.38,

$$\begin{aligned} e &= 1 + \frac{r_p {v_\infty}^2}{\mu_{\text{earth}}} \\ &= 1 + \frac{6558 \cdot 3.1651^2}{398{,}600} = 1.165 \end{aligned}$$

If we assume that the spacecraft is launched from a parking orbit of 28° inclination, then the departure appears as shown in the three-dimensional sketch in Figure 8.29.

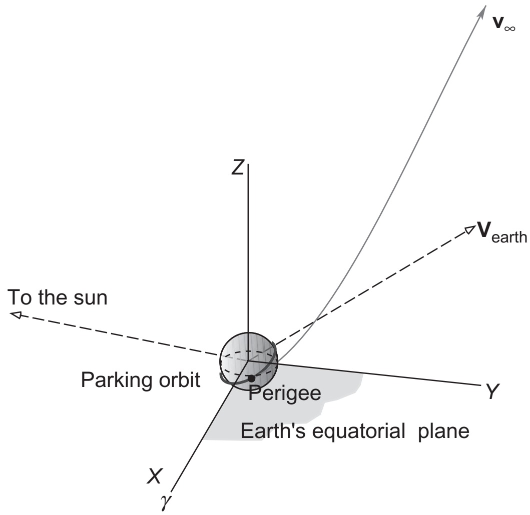

FIGURE 8.29

The departure hyperbola, assumed to be at 28° inclination to earth's equator.

Example 8.10

In Example 8.8, calculate the delta-v required to place the spacecraft in an elliptical capture orbit around Mars with a periapsis altitude of 300 km and a period of 48 hours. Sketch the approach hyperbola.

Solution

From Tables A.1 and A.2 we know that

$$r_{\text{Mars}} = 3380\,\text{km}$$
$$\mu_{\text{Mars}} = 42{,}830\,\text{km}^3/\text{s}^2$$

The radius to periapsis of the arrival hyperbola is the radius of Mars plus the periapsis of the elliptical capture orbit,

$$r_p = 3380 + 300 = 3680\,\text{km}$$

According to Equation 8.40 and Equation (b) of Example 8.8, the speed of the spacecraft at periapsis of the arrival hyperbola is

$$v_p)_{\text{hyp}} = \sqrt{[v_\infty)_{\text{Arrival}}]^2 + \frac{2\mu_{\text{Mars}}}{r_p}} = \sqrt{2.8851^2 + \frac{2\cdot 42{,}830}{3680}} = 5.621\,\text{km/s}$$

To find the speed $v_p)_{\text{el}}$ at periapsis of the capture ellipse, we use the required period (48 hours) to determine the ellipse's semimajor axis by means of Equation 2.83,

$$a_{\text{ell}} = \left(\frac{T\sqrt{\mu_{\text{Mars}}}}{2\pi}\right)^{\frac{2}{3}} = \left(\frac{48\cdot 3600\cdot\sqrt{42{,}830}}{2\pi}\right)^{\frac{2}{3}} = 31{,}880\,\text{km}$$

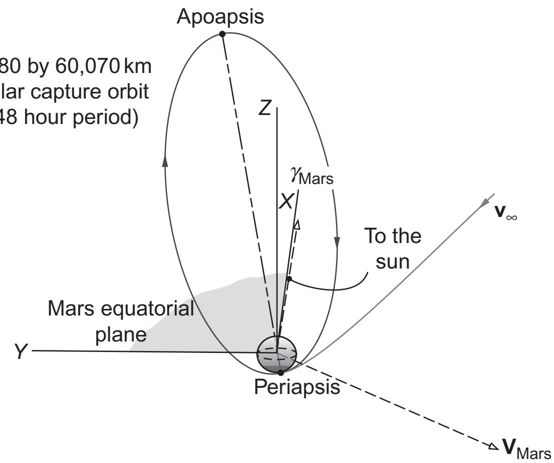

FIGURE 8.30

The approach hyperbola and capture ellipse.

From Equation 2.63 we obtain the eccentricity of the capture ellipse

$$e_{\text{ell}} = 1 - \frac{r_p}{a_{\text{ell}}} = 1 - \frac{3680}{31{,}880} = 0.8846$$

Then Equation 8.59 yields

$$v_p)_{\text{ell}} = \sqrt{\frac{\mu_{\text{Mars}}}{r_p}(1 + e_{\text{ell}})} = \sqrt{\frac{42{,}830}{3680}(1 + 0.8846)} = 4.683\ \text{km/s}$$

Hence, the delta-v requirement is

$$\Delta v = v_p)_{\text{hyp}} - v_p)_{\text{ell}} = 0.9382\ \text{km/s}$$

The eccentricity of the approach hyperbola is given by Equation 8.38,

$$e = 1 + \frac{r_p v_\infty{}^2}{\mu_{\text{Mars}}} = 1 + \frac{3680 \cdot 2.8851^2}{42{,}830} = 1.715$$

Assuming that the capture ellipse is a polar orbit of Mars, then the approach hyperbola is as illustrated in Figure 8.30. Note that Mars' equatorial plane is inclined 25° to the plane of its orbit around the sun. Furthermore, the vernal equinox of Mars lies at an angle of 85° from that of the earth.

PROBLEMS

Section 8.2

8.1 Find the total delta-v required for a Hohmann transfer from earth orbit to Saturn's orbit.
{Ans.: 15.74 km/s}

8.2 Find the total delta-v required for a Hohmann transfer from Mars' orbit to Jupiter's orbit.
{Ans.: 10.15 km/s}

Section 8.3

8.3 Calculate the synodic period of Venus relative to the earth.
{Ans.: 1.599 y}

8.4 Calculate the synodic period of Jupiter relative to Mars.
{Ans.: 2.236 y}

Section 8.4

8.5 Calculate the radius of the spheres of influence of Mercury, Venus, Mars and Jupiter.
{Ans.: See Table A.2}

8.6 Calculate the radius of the spheres of influence of Saturn, Uranus and Neptune.
{Ans.: See Table A.2}

Section 8.6

8.7 On a date when the earth was 147.4×10^6 km from the sun, a spacecraft parked in a 200 km altitude circular earth orbit was launched directly into an elliptical orbit around the sun with perihelion of 120×10^6 km and aphelion equal to the earth's distance from the sun on the launch date. Calculate the delta-v required and v_∞ of the departure hyperbola.
{Ans.: $v_\infty = 1.814$ km/s, $\Delta v = 3.373$ km/s.}

8.8 Calculate the propellant mass required to launch a 2000 kg spacecraft from a 180 km circular earth orbit on a Hohmann transfer trajectory to the orbit of Saturn. Calculate the time required for the mission and compare it to that of Cassini. Assume the propulsion system has a specific impulse of 300 s.
{Ans.: 6.03 y; 21,810 kg}

Section 8.7

8.9 An earth orbit has a perigee radius of 7000 km and a perigee velocity of 9 km/s. Calculate the change in apogee radius due to a change of

(a) 1 km in the perigee radius.
(b) 1 m/s in the perigee speed.
{Ans.: (a) 7.33 km (b) 7.57 km}

8.10 An earth orbit has a perigee radius of 7000 km and a perigee velocity of 9 km/s. Calculate the change in apogee speed due to a change of

(a) 1 km in the perigee radius.
(b) 1 m/s in the perigee speed.
{Ans.: (a) −1.81 m/s (b) −0.406 m/s}

Section 8.8

8.11 Estimate the total delta-v requirement for a Hohmann transfer from earth to Mercury, assuming a 150 km circular parking orbit at earth and a 150 km circular capture orbit at Mercury. Furthermore, assume that the planets have coplanar circular orbits with radii equal to the semimajor axes listed in Table A.1.
{Ans.: 13.08 km/s}

Section 8.9

8.12 Suppose a spacecraft approaches Jupiter on a Hohmann transfer ellipse from earth. If the spacecraft flies by Jupiter at an altitude of 200,000 km on the sunlit side of the planet, determine the orbital elements of the post-flyby trajectory and the delta-v imparted to the spacecraft by Jupiter's gravity. Assume that all of the orbits lie in the same (ecliptic) plane.
{Ans.: $\Delta V = 10.6$ km/s, $a = 4.79 \times 10^6$ km, $e = 0.8453$}

Section 8.10

8.13 Use Table 8.1 to verify that the orbital elements for earth and Mars presented in Example 8.7.
8.14 Use Table 8.1 to determine the day of the year 2005 when the earth was farthest from the sun.
{Ans.: 4 July.}

Section 8.11

8.15 On 1 December 2005 a spacecraft left a 180 km altitude circular orbit around the earth on a mission to Venus. It arrived at Venus 121 days later on 1 April 2006, entering a 300 km by 9000 km capture ellipse around the planet. Calculate the total delta-v requirement for this mission.
{Ans.: 6.75 km/s}

8.16 On 15 August 2005 a spacecraft in a 190 km, 52° inclination circular parking orbit around the earth departed on a mission to Mars, arriving at the red planet on 15 March 2006, whereupon retro rockets place it into a highly elliptic orbit with a periapsis of 300 km and a period of 35 hours. Determine the total delta-v required for this mission.
{Ans.: 4.86 km/s}

List of Key Terms

aerobraking
gravity assist maneuver

hyperbolic excess velocity
Keplerian orbits
leading side flyby
longitude of perihelion
mean longitude
perturbing acceleration
synodic period
trailing side flyby
wait time

CHAPTER

Rigid-body dynamics

9

Chapter outline

9.1 Introduction 485
9.2 Kinematics 486
9.3 Equations of translational motion 495
9.4 Equations of rotational motion 497
9.5 Moments of inertia 501
9.6 Euler's equations 524
9.7 Kinetic energy 530
9.8 The spinning top 533
9.9 Euler angles 538
9.10 Yaw, pitch and roll angles 549
9.11 Quaternions 552

9.1 INTRODUCTION

Just as Chapter 1 provides a foundation for the development of the equations of orbital mechanics, this chapter serves as a basis for developing the equations of satellite attitude dynamics. Chapter 1 deals with particles, whereas here we are concerned with rigid bodies. Those familiar with rigid body dynamics can move on to the next chapter, perhaps returning from time to time to review concepts.

The kinematics of rigid bodies is presented first. The subject depends on a theorem of the French mathematician Michel Chasles (1793–1880). Chasles' theorem states that the motion of a rigid body can be described by the displacement of any point of the body (the base point) plus a rotation about a unique axis through that point. The magnitude of the rotation does not depend on the base point. Thus, at any instant a rigid body in a general state of motion has an angular velocity vector whose direction is that of the instantaneous axis of rotation. Describing the rotational component of the motion of a rigid body in three dimensions requires taking advantage of the vector nature of angular velocity and knowing how to take the time derivative of moving vectors, which is explained in Chapter 1. Several examples illustrate how this is done.

We then move on to study the interaction between the motion of a rigid body and the forces acting on it. Describing the translational component of the motion requires simply concentrating all of the mass at a point, the center of mass, and applying the methods of particle mechanics to determine its motion. Indeed, our study of the two-body problem up to this point has focused on the motion of their centers of mass without regard to the rotational aspect. Analyzing the rotational dynamics requires computing the body's angular momentum, and that in turn requires accounting for how the mass is distributed throughout the body. The mass distribution is described by the six components of the moment of inertia tensor.

Writing the equations of rotational motion relative to coordinate axes embedded in the rigid body and aligned with the principle axes of inertia yields the nonlinear Euler equations of motion, which are applied to a study of the dynamics of a spinning top (or one-axis gyro).

The expression for the kinetic energy of a rigid body is derived because it will be needed in the following chapter.

The chapter next describes two sets of three angles commonly employed to specify the orientation of a body in three-dimensional space. One of these comprises the Euler angles, which are the same as the right ascension of the node (Ω), argument of periapsis (ω) and inclination (i) introduced in Chapter 4 to orient orbits in space. The other set comprises the yaw, pitch and roll angles, which are suitable for describing the orientation of an airplane. Both the Euler angles and yaw-pitch-roll angles will be employed in Chapter 10.

The chapter concludes with a brief discussion of quaternions and an example of how they are used to describe the evolution of the attitude of a rigid body.

9.2 KINEMATICS

Figure 9.1 shows a moving rigid body and its instantaneous axis of rotation, which defines the direction of the absolute angular velocity vector $\boldsymbol{\omega}$. The XYZ axes are a fixed, inertial frame of reference. The position vectors $\mathbf{R}_A$ and $\mathbf{R}_B$ of two points on the rigid body are measured in the inertial frame. The vector $\mathbf{R}_{B/A}$ drawn from point A to point B is the position vector of B relative to A. Since the body is rigid, $\mathbf{R}_{B/A}$ has a constant magnitude even though its direction is continuously changing. Clearly,

$$\mathbf{R}_B = \mathbf{R}_A + \mathbf{R}_{B/A}$$

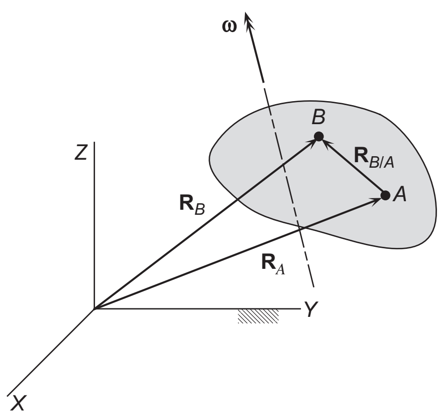

FIGURE 9.1

Rigid body and its instantaneous axis of rotation.

Differentiating this equation through with respect to time, we get

$$\dot{\mathbf{R}}_B = \dot{\mathbf{R}}_A + \frac{d\mathbf{R}_{B/A}}{dt} \tag{9.1}$$

$\dot{\mathbf{R}}_A$ and $\dot{\mathbf{R}}_B$ are the absolute velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of points A and B. Because the magnitude of $\mathbf{R}_{B/A}$ does not change, its time derivative is given by Equation 1.52,

$$\frac{d\mathbf{R}_{B/A}}{dt} = \boldsymbol{\omega} \times \mathbf{R}_{B/A}$$

Thus, from Equation 9.1 we obtain the ***relation between velocities of points on a rigid body***

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{R}_{B/A} \tag{9.2}$$

Taking the time derivative of Equation 9.1 yields

$$\ddot{\mathbf{R}}_B = \ddot{\mathbf{R}}_A + \frac{d^2\mathbf{R}_{B/A}}{dt^2} \tag{9.3}$$

$\ddot{\mathbf{R}}_A$ and $\ddot{\mathbf{R}}_B$ are the absolute accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$ of the two points of the rigid body, while from Equation 1.53 we have

$$\frac{d^2\mathbf{R}_{B/A}}{dt^2} = \boldsymbol{\alpha} \times \mathbf{R}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}_{B/A})$$

in which $\boldsymbol{\alpha}$ is the angular acceleration, $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$. Therefore, Equation 9.3 yields the ***relation between accelerations of points on a rigid body***

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{R}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}_{B/A}) \tag{9.4}$$

Equations 9.2 and 9.4 are the relative velocity and acceleration formulas. Note that all quantities in these expressions are measured in the same inertial frame of reference.

When the rigid body under consideration is connected to and moving relative to another rigid body, computation of its inertial angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ must be done with care. The key is to remember that angular velocity is a vector. It may be found as the vector sum of a sequence of angular velocities, each measured relative to another, starting with one measured relative to an absolute frame, as illustrated in Figure 9.2. In that case, the absolute angular velocity $\boldsymbol{\omega}$ of body 4 is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_{2/1} + \boldsymbol{\omega}_{3/2} + \boldsymbol{\omega}_{4/3} \tag{9.5}$$

Each of these angular velocities is resolved into components along the axes of the moving frame of reference xyz shown in Figure 9.2, so that the vector sum may be written

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} \tag{9.6}$$

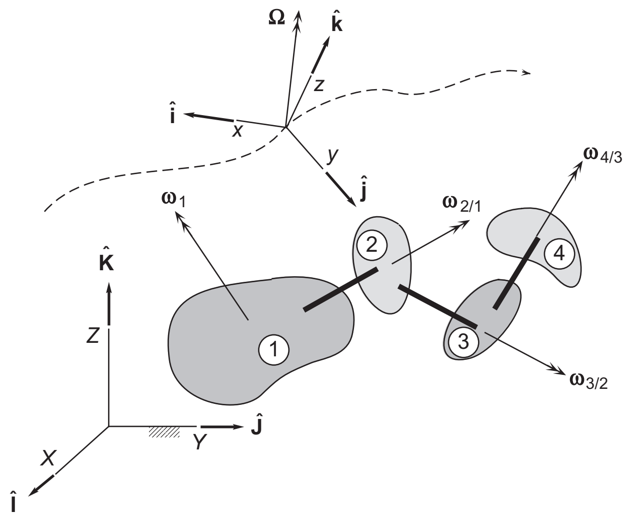

FIGURE 9.2

Angular velocity is the vector sum of the relative angular velocities starting with $\boldsymbol{\omega}_1$, measured relative to the inertial frame.

The moving frame is chosen for convenience of the analysis, and its own inertial angular velocity is denoted $\boldsymbol{\Omega}$, as discussed in Section 1.6. According to Equation 1.56, the absolute angular acceleration $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ is obtained from Equation 9.6 by means of the following calculation,

$$\boldsymbol{\alpha} = \left.\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rel}} + \boldsymbol{\Omega} \times \boldsymbol{\omega} \tag{9.7}$$

where

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rel}} = \dot{\omega}_x \hat{\mathbf{i}} + \dot{\omega}_y \hat{\mathbf{j}} + \dot{\omega}_z \hat{\mathbf{k}} \tag{9.8}$$

and $\dot{\omega}_x = d\omega_x/dt$, etc.

Being able to express the absolute angular velocity vector in an appropriately chosen moving reference frame, as in Equation 9.6, is crucial to the analysis of rigid body motion. Once we have the components of $\boldsymbol{\omega}$, we simply differentiate them with respect to time to arrive at Equation 9.8. Observe that the absolute angular acceleration $\boldsymbol{\alpha}$ and $d\boldsymbol{\omega}/dt)_{\text{rel}}$, the angular acceleration relative to the moving frame, are the same if and only if $\boldsymbol{\Omega} = \boldsymbol{\omega}$. That occurs if the moving reference is a body-fixed frame, that is, a set of *xyz* axes imbedded in the rigid body itself.

Example 9.1

An airplane flies at constant speed v while simultaneously undergoing a constant yaw rate ω_{yaw} about a vertical axis and describing a circular loop in the vertical plane with a radius ρ. The constant propeller spin rate is ω_{spin} relative to the airframe. Find the velocity and acceleration of the tip P of the propeller relative to the hub H, when P is directly above H. The propeller radius is l.

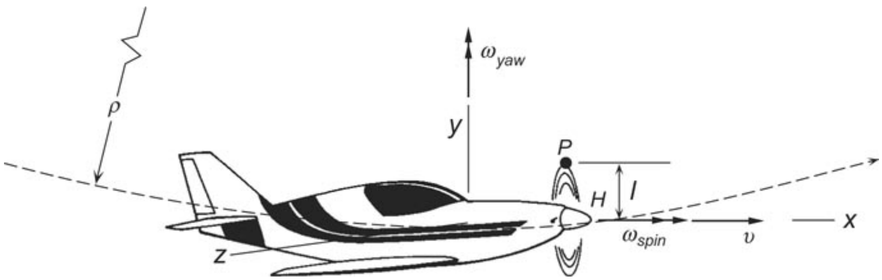

FIGURE 9.3

Airplane with attached *xyz* body frame.

Solution

The *xyz* axes are rigidly attached to the airplane. The *x* axis is aligned with the propeller's spin axis. The *y* axis is vertical, and the *z* axis is in the spanwise direction, so that *xyz* forms a right-handed triad. Although the *xyz* frame is not inertial, we can imagine it to instantaneously coincide with an inertial frame.

The absolute angular velocity of the airplane has two components, the yaw rate and the counterclockwise pitch angular velocity v/ρ of its rotation in the circular loop,

$$\boldsymbol{\omega}_{\text{airplane}} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \omega_{\text{pitch}}\hat{\mathbf{k}} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}$$

The angular velocity of the body-fixed moving frame is that of the airplane, $\boldsymbol{\Omega} = \boldsymbol{\omega}_{\text{airplane}}$, so that

$$\boldsymbol{\Omega} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}} \tag{a}$$

The absolute angular velocity of the propeller is that of the airplane plus the angular velocity of the propeller relative to the airplane,

$$\boldsymbol{\omega}_{\text{prop}} = \boldsymbol{\omega}_{\text{airplane}} + \omega_{\text{spin}}\hat{\mathbf{i}} = \boldsymbol{\Omega} + \omega_{\text{spin}}\hat{\mathbf{i}}$$

which means

$$\boldsymbol{\omega}_{\text{prop}} = \omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}} \tag{b}$$

From Equation 9.2, the velocity of point *P* on the propeller relative to *H* on the hub, $\mathbf{v}_{P/H}$, is given by

$$\mathbf{v}_{P/H} = \mathbf{v}_P - \mathbf{v}_H = \boldsymbol{\omega}_{\text{prop}} \times \mathbf{r}_{P/H}$$

where $\mathbf{r}_{P/H}$ is the position vector of *P* relative to *H* at this instant,

$$\mathbf{r}_{P/H} = l\hat{\mathbf{j}} \tag{c}$$

Thus, using (b) and (c),

$$\mathbf{v}_{P/H} = \left(\omega_{spin}\hat{\mathbf{i}} + \omega_{yaw}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times (l\hat{\mathbf{j}})$$

from which

$$\boxed{\mathbf{v}_{P/H} = -\frac{v}{\rho}l\hat{\mathbf{i}} + \omega_{spin}l\hat{\mathbf{k}}}$$

The absolute angular acceleration of the propeller is found by substituting (a) and (b) into Equation 9.7,

$$\begin{aligned}\boldsymbol{\alpha}_{\text{prop}} &= \left.\frac{d\boldsymbol{\omega}_{\text{prop}}}{dt}\right)_{\text{rel}} + \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{prop}} \\ &= \left(\frac{d\omega_{\text{spin}}}{dt}\hat{\mathbf{i}} + \frac{d\omega_{\text{yaw}}}{dt}\hat{\mathbf{j}} + \frac{d(v/\rho)}{dt}\hat{\mathbf{k}}\right) + \left(\omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right)\end{aligned}$$

Since ω_{spin}, ω_{yaw}, v and ρ are all constant, this reduces to

$$\boldsymbol{\alpha}_{\text{prop}} = \left(\omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right)$$

Carrying out the cross product yields

$$\boldsymbol{\alpha}_{\text{prop}} = \frac{v}{\rho}\omega_{\text{spin}}\hat{\mathbf{j}} - \omega_{\text{yaw}}\omega_{\text{spin}}\hat{\mathbf{k}} \tag{d}$$

From Equation 9.4, the acceleration of P relative to H, $\mathbf{a}_{P/H}$, is given by

$$\mathbf{a}_{P/H} = \mathbf{a}_P - \mathbf{a}_H = \boldsymbol{\alpha}_{\text{prop}} \times \mathbf{r}_{P/H} + \boldsymbol{\omega}_{\text{prop}} \times (\boldsymbol{\omega}_{\text{prop}} \times \mathbf{r}_{P/H})$$

Substituting (b), (c) and (d) into this expression yields

$$\mathbf{a}_{P/H} = \left(\frac{v}{\rho}\omega_{\text{spin}}\hat{\mathbf{j}} - \omega_{\text{yaw}}\omega_{\text{spin}}\hat{\mathbf{k}}\right) \times (l\hat{\mathbf{j}}) + \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left[\left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times (l\hat{\mathbf{j}})\right]$$

From this we find

$$\begin{aligned}\mathbf{a}_{P/H} &= (\omega_{\text{yaw}}\omega_{\text{spin}}l\hat{\mathbf{i}}) + \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left[-\frac{v}{\rho}l\hat{\mathbf{i}} + \omega_{\text{spin}}l\hat{\mathbf{k}}\right] \\ &= (\omega_{\text{yaw}}\omega_{\text{spin}}l\hat{\mathbf{i}}) + \left[\omega_{\text{yaw}}\omega_{\text{spin}}l\hat{\mathbf{i}} - \left(\frac{v^2}{\rho^2} + \omega_{\text{spin}}^2\right)l\hat{\mathbf{j}} + \omega_{\text{yaw}}\frac{v}{\rho}l\hat{\mathbf{k}}\right]\end{aligned}$$

so that finally,

$$\boxed{\mathbf{a}_{P/H} = 2\omega_{\text{yaw}}\omega_{\text{spin}}l\hat{\mathbf{i}} - \left(\frac{v^2}{\rho^2} + \omega_{\text{spin}}^2\right)l\hat{\mathbf{j}} + \omega_{\text{yaw}}\frac{v}{\rho}l\hat{\mathbf{k}}}$$

Example 9.2

The satellite is rotating about the z-axis at a constant rate N. The xyz axes are attached to the spacecraft, and the z axis has a fixed orientation in inertial space. The solar panels rotate at a constant rate $\dot{\theta}$ clockwise around the positive y-axis, as shown in Figure 9.4. Relative to point O, which lies at the center of the spacecraft and on the centerline of the panels, calculate for point A on the panel

(a) Its absolute velocity and

(b) Its absolute acceleration.

Solution

(a) Since the moving xyz frame is attached to the body of the spacecraft, its angular velocity is

$$\boldsymbol{\Omega} = N\hat{\mathbf{k}} \tag{a}$$

The absolute angular velocity of the panel is the absolute angular velocity of the spacecraft plus the angular velocity of the panel relative to the spacecraft,

$$\boldsymbol{\omega}_{\text{panel}} = -\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}} \tag{b}$$

The position vector of A relative to O is

$$\mathbf{r}_{A/O} = -\frac{w}{2}\sin\theta\hat{\mathbf{i}} + d\hat{\mathbf{j}} + \frac{w}{2}\cos\theta\hat{\mathbf{k}} \tag{c}$$

According to Equation 9.2, the velocity of A relative to O is

$$\mathbf{v}_{A/O} = \mathbf{v}_A - \mathbf{v}_O = \boldsymbol{\omega}_{\text{panel}} \times \mathbf{r}_{A/O} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\dot{\theta} & N \\ -\frac{w}{2}\sin\theta & d & \frac{w}{2}\cos\theta \end{vmatrix}$$

from which

$$\boxed{\mathbf{v}_{A/O} = -\left(\frac{w}{2}\dot{\theta}\cos\theta + Nd\right)\hat{\mathbf{i}} - \frac{w}{2}N\sin\theta\hat{\mathbf{j}} - \frac{w}{2}\dot{\theta}\sin\theta\hat{\mathbf{k}}}$$

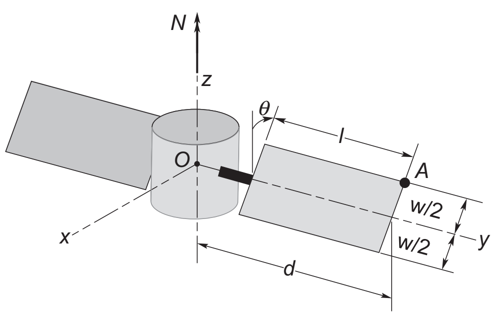

FIGURE 9.4

Rotating solar panel on a rotating satellite.

(b) The absolute angular acceleration of the panel is found by substituting (a) and (b) into Equation 9.7,

$$\begin{aligned}\boldsymbol{\alpha}_{\text{panel}} &= \left.\frac{d\boldsymbol{\omega}_{\text{panel}}}{dt}\right)_{\text{rel}} + \boldsymbol{\Omega}\times\boldsymbol{\omega}_{\text{panel}}\\ &= \left[\frac{d(-\dot{\theta})}{dt}\hat{\mathbf{j}} + \frac{dN}{dt}\hat{\mathbf{k}}\right] + (N\hat{\mathbf{k}})\times(-\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}})\end{aligned}$$

Since N and $\dot{\theta}$ are constants, this reduces to

$$\boldsymbol{\alpha}_{panel} = \dot{\theta}N\hat{\mathbf{i}} \tag{d}$$

To find the acceleration of A relative to O, we substitute (b), (c) and (d) into Equation 9.4,

$$\begin{aligned}\mathbf{a}_{A/O} &= \mathbf{a}_A - \mathbf{a}_O = \boldsymbol{\alpha}_{\text{panel}}\times\mathbf{r}_{A/O} + \boldsymbol{\omega}_{\text{panel}}\times(\boldsymbol{\omega}_{\text{panel}}\times\mathbf{r}_{A/O})\\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \dot{\theta}N & 0 & 0\\ -\frac{w}{2}\sin\theta & d & \frac{w}{2}\cos\theta\end{vmatrix} + (-\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}})\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & -\dot{\theta} & N\\ -\frac{w}{2}\sin\theta & d & \frac{w}{2}\cos\theta\end{vmatrix}\\ &= \left(-\frac{w}{2}N\dot{\theta}\cos\theta\hat{\mathbf{j}} + N\dot{\theta}d\hat{\mathbf{k}}\right) + \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & -\dot{\theta} & N\\ -\frac{w}{2}\dot{\theta}\cos\theta - Nd & -N\frac{w}{2}\sin\theta & -\frac{w}{2}\dot{\theta}\sin\theta\end{vmatrix}\end{aligned}$$

which leads to

$$\boxed{\mathbf{a}_{A/O} = \frac{w}{2}(N^2 + \dot{\theta}^2)\sin\theta\hat{\mathbf{i}} - N(Nd + w\dot{\theta}\cos\theta)\hat{\mathbf{j}} - \frac{w}{2}\dot{\theta}^2\cos\theta\hat{\mathbf{k}}}$$

Example 9.3

The gyro rotor shown has a constant spin rate ω_{spin} around axis $b-a$ in the direction shown. The XYZ axes are fixed. The xyz axes are attached to the gimbal ring, whose angle θ with the vertical is increasing at the constant rate $\dot{\theta}$ in the direction shown. The assembly is forced to precess at the constant rate N around the vertical, as shown. For the rotor in the position shown, calculate

(a) The absolute angular velocity and
(b) The absolute angular acceleration.

Express the results in both the fixed XYZ frame and the moving xyz frame.

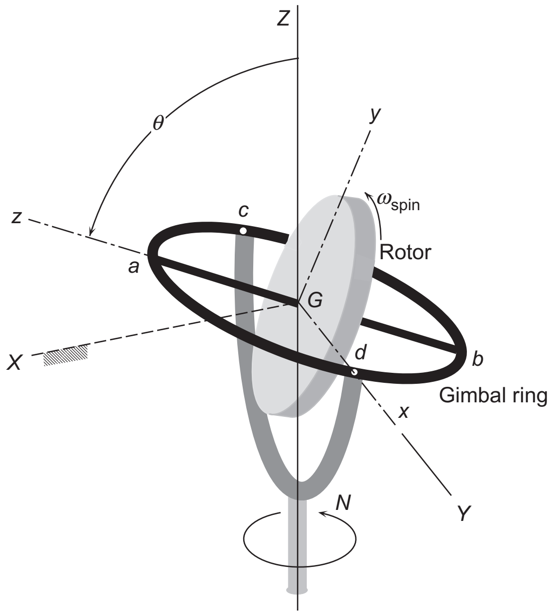

FIGURE 9.5

Rotating, precessing, nutating gyro.

Solution

(a) We will need the instantaneous relationship between the unit vectors of the inertial *XYZ* axes and the co-moving *xyz* frame, which by inspecting Figure 9.6 can be seen to be

$$\begin{aligned}\hat{\mathbf{I}} &= -\cos\theta\hat{\mathbf{j}} + \sin\theta\hat{\mathbf{k}} \\ \hat{\mathbf{J}} &= \hat{\mathbf{i}} \\ \hat{\mathbf{K}} &= \sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\end{aligned} \qquad \text{(a)}$$

so that the matrix of the transformation from *xyz* to *XYZ* is (see Section 4.5)

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} 0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad \text{(b)}$$

The absolute angular velocity of the gimbal ring is that of the base plus the angular velocity of the gimbal relative to the base,

$$\boldsymbol{\omega}_{\text{gimbal}} = N\hat{\mathbf{K}} + \dot{\theta}\hat{\mathbf{i}} = N(\sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}) + \dot{\theta}\hat{\mathbf{i}} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + N\cos\theta\hat{\mathbf{k}} \qquad \text{(c)}$$

where we made use of $(\text{a})_3$. Since the moving *xyz* frame is attached to the gimbal, $\boldsymbol{\Omega} = \boldsymbol{\omega}_{\text{gimbal}}$, so that,

$$\boldsymbol{\Omega} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + N\cos\theta\hat{\mathbf{k}} \qquad \text{(d)}$$

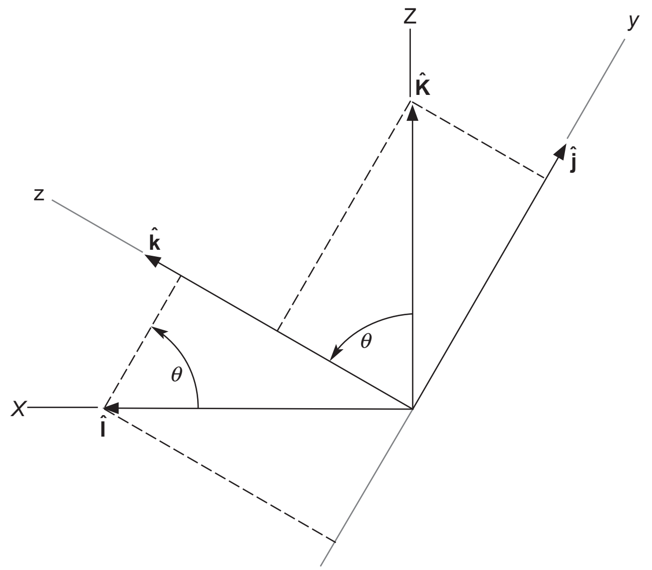

FIGURE 9.6

Orientation of the fixed *XZ* axes relative to the rotating *xz* axes.

The absolute angular velocity of the rotor is its spin relative to the gimbal, plus the angular velocity of the gimbal,

$$\boldsymbol{\omega}_{\text{rotor}} = \boldsymbol{\omega}_{\text{gimbal}} + \omega_{\text{spin}}\hat{\mathbf{k}} \tag{e}$$

From (c) it follows that,

$$\boxed{\boldsymbol{\omega}_{\text{rotor}} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + (N\cos\theta + \omega_{\text{spin}})\hat{\mathbf{k}}} \tag{f}$$

Because $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ move with the gimbal, this expression is valid for any time, not just the instant shown in Figure 9.5. Alternatively, applying the vector transformation

$$\{\boldsymbol{\omega}_{\text{rotor}}\}_{XYZ} = [\mathbf{Q}]_{xX}\{\boldsymbol{\omega}_{\text{rotor}}\}_{xyz} \tag{g}$$

we obtain the angular velocity of the rotor in the inertial frame, but only at the instant shown in the figure, i.e., when the *x* axis aligns with the *Y* axis

$$\begin{Bmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{Bmatrix} = \begin{bmatrix}0 & -\cos\theta & \sin\theta\\ 1 & 0 & 0\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}\dot{\theta}\\ N\sin\theta\\ N\cos\theta + \omega_{\text{spin}}\end{Bmatrix} = \begin{Bmatrix}-N\sin\theta\cos\theta + N\sin\theta\cos\theta + \omega_{\text{spin}}\sin\theta\\ \dot{\theta}\\ N\sin^2\theta + N\cos^2\theta + \omega_{\text{spin}}\cos\theta\end{Bmatrix}$$

or

$$\boxed{\boldsymbol{\omega}_{\text{rotor}} = \omega_{\text{spin}}\sin\theta\hat{\mathbf{I}} + \dot{\theta}\hat{\mathbf{J}} + (N + \omega_{\text{spin}}\cos\theta)\hat{\mathbf{K}}} \tag{h}$$

(b) The angular acceleration of the rotor is obtained by substituting (d) and (f) into Equation 9.7, recalling that N, $\dot{\theta}$, and ω_{spin} are independent of time:

$$\begin{aligned}
\boldsymbol{\alpha}_{\text{rotor}} &= \left.\frac{d\boldsymbol{\omega}_{\text{rotor}}}{dt}\right)_{rel} + \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rotor}} \\
&= \left[\frac{d(\dot{\theta})}{dt}\hat{\mathbf{i}} + \frac{d(N\sin\theta)}{dt}\hat{\mathbf{j}} + \frac{d(N\cos\theta + \omega_{\text{spin}})}{dt}\hat{\mathbf{k}}\right] + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{\theta} & N\sin\theta & N\cos\theta \\ \dot{\theta} & N\sin\theta & N\cos\theta + \omega_{\text{spin}} \end{vmatrix} \\
&= (N\dot{\theta}\cos\theta\hat{\mathbf{j}} - N\dot{\theta}\sin\theta\hat{\mathbf{k}}) + [\hat{\mathbf{i}}(N\omega_{\text{spin}}\sin\theta) - \hat{\mathbf{j}}(\omega_{\text{spin}}\dot{\theta}) + \hat{\mathbf{k}}(0)]
\end{aligned}$$

Upon collecting terms, we get

$$\boxed{\boldsymbol{\alpha}_{\text{rotor}} = N\omega_{\text{spin}}\sin\theta\hat{\mathbf{i}} + \dot{\theta}(N\cos\theta - \omega_{\text{spin}})\hat{\mathbf{j}} - N\dot{\theta}\sin\theta\hat{\mathbf{k}}} \tag{i}$$

This expression, like (f), is valid at any time.

The components of $\boldsymbol{\alpha}_{\text{rotor}}$ along the XYZ axes are found in the same way as for $\boldsymbol{\omega}_{\text{rotor}}$,

$$\{\boldsymbol{\alpha}_{\text{rotor}}\}_{XYZ} = [\mathbf{Q}]_{xX}\{\boldsymbol{\alpha}_{\text{rotor}}\}_{xyz}$$

which means

$$\begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix} = \begin{bmatrix} 0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} N\omega_{\text{spin}}\sin\theta \\ \dot{\theta}(N\cos\theta - \omega_{\text{spin}}) \\ -N\dot{\theta}\sin\theta \end{Bmatrix} = \begin{Bmatrix} -N\dot{\theta}\cos^2\theta + \dot{\theta}\omega_{\text{spin}}\cos\theta - N\dot{\theta}\sin^2\theta \\ N\omega_{\text{spin}}\sin\theta \\ N\dot{\theta}\sin\theta\cos\theta - \dot{\theta}\omega_{\text{spin}}\sin\theta - N\dot{\theta}\sin\theta\cos\theta \end{Bmatrix}$$

or

$$\boxed{\boldsymbol{\alpha}_{\text{rotor}} = \dot{\theta}(\omega_{\text{spin}}\cos\theta - N)\hat{\mathbf{I}} + N\omega_{\text{spin}}\sin\theta\hat{\mathbf{J}} - \dot{\theta}\omega_{\text{spin}}\sin\theta\hat{\mathbf{K}}} \tag{j}$$

Note carefully that (j) is not simply the time derivative of (h). Equations (h) and (j) are valid only at the instant that the *xyz* and *XYZ* axes have the alignments shown in Figure 9.5.

9.3 EQUATIONS OF TRANSLATIONAL MOTION

Figure 9.7 again shows an arbitrary, continuous, three-dimensional body of mass *m*. "Continuous" means that as we zoom in on a point it remains surrounded by a continuous distribution of matter having the infinitesimal mass *dm* in the limit. The point never ends up in a void. In particular, we ignore the actual atomic and molecular microstructure in favor of this continuum hypothesis, as it is called. Molecular microstructure does not bear upon the overall dynamics of a finite body. We will use *G* to denote the center of mass.

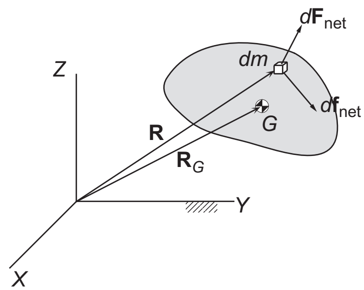

FIGURE 9.7

Forces on the mass element *dm* of a continuous medium.

Position vectors of points relative to the origin of the inertial frame will be designated by capital letters. Thus, the position of the center of mass is $\mathbf{R}_G$, defined as

$$m\mathbf{R}_G = \int_m \mathbf{R}\,dm \tag{9.9}$$

$\mathbf{R}$ is the position of a mass element dm within the continuum. Each element of mass is acted upon by a net external force $d\mathbf{F}_{net}$ and a net internal force $d\mathbf{f}_{net}$. The external force comes from direct contact with other objects and from action at a distance, such as gravitational attraction. The internal forces are those exerted from within the body by neighboring particles. These are the forces that hold the body together. For each mass element, Newton's second law, Equation 1.38, is written

$$d\mathbf{F}_{net} + d\mathbf{f}_{net} = dm\ddot{\mathbf{R}} \tag{9.10}$$

Writing this equation for the infinite number of mass elements of which the body is composed, and then summing them all together leads to the integral,

$$\int d\mathbf{F}_{net} + \int d\mathbf{f}_{net} = \int_m \ddot{\mathbf{R}}\,dm$$

Because the internal forces occur in action-reaction pairs, $\int d\mathbf{f}_{net} = \mathbf{0}$. (External forces on the body are those without an internal reactant; the reactant lies outside the body and, hence, outside our purview.) Thus,

$$\mathbf{F}_{net} = \int_m \ddot{\mathbf{R}}\,dm \tag{9.11}$$

where $\mathbf{F}_{net}$ is the resultant external force on the body, $\mathbf{F}_{net} = \int d\mathbf{F}_{net}$. From Equation 9.9

$$\int_m \ddot{\mathbf{R}}\,dm = m\ddot{\mathbf{R}}_G$$

where $\ddot{\mathbf{R}}_G = \mathbf{a}_G$, the absolute acceleration of the center of mass. Therefore, Equation 9.11, the ***equation of translational motion of a rigid body***, can be written

$$\mathbf{F}_{net} = m\ddot{\mathbf{R}}_G \tag{9.12}$$

We are therefore reminded that the motion of the center of mass of a body is determined solely by the resultant of the external forces acting on it. So far our study of orbiting bodies has focused exclusively on the motion of their centers of mass. In this chapter we will turn our attention to rotational motion around the center of mass. To simplify things, we will ultimately assume that the body is not only continuous, but that it is also rigid. That means all points of the body remain a fixed distance from each other and there is no flexing, bending or twisting deformation.

9.4 EQUATIONS OF ROTATIONAL MOTION

Our development of the rotational dynamics equations does not require at the outset that the body under consideration be rigid. It may be a solid, fluid or gas.

Point P in the Figure 9.8 is arbitrary; it need not be fixed in space nor attached to a point on the body. Then the moment about P of the forces on mass element dm (cf. Figure 9.7) is

$$d\mathbf{M}_P = \mathbf{r} \times d\mathbf{F}_{\text{net}} + \mathbf{r} \times d\mathbf{f}_{\text{net}}$$

where $\mathbf{r}$ is the position vector of the mass element dm relative to the point P. Writing the right hand side as $\mathbf{r} \times (d\mathbf{F}_{\text{net}} + d\mathbf{f}_{\text{net}})$, substituting Equation 9.10, and integrating over all of the mass elements of the body yields

$$\mathbf{M}_{P_{\text{net}}} = \int_m \mathbf{r} \times \ddot{\mathbf{R}} dm \tag{9.13}$$

where $\ddot{\mathbf{R}}$ is the absolute acceleration of dm relative to the inertial frame and

$$\mathbf{M}_{P_{\text{net}}} = \int \mathbf{r} \times d\mathbf{F}_{\text{net}} + \int \mathbf{r} \times d\mathbf{f}_{\text{net}}$$

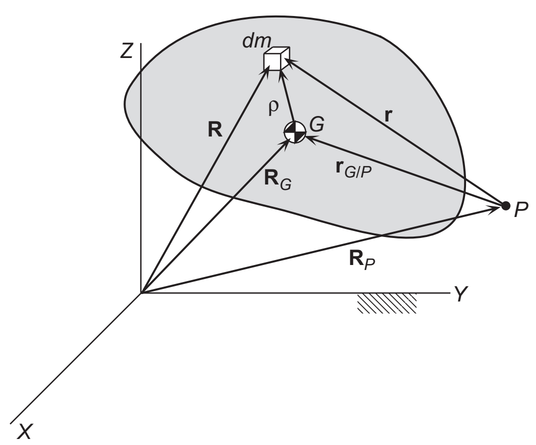

FIGURE 9.8

Position vectors of a mass element in a continuum from several key reference points.

But $\int \mathbf{r} \times d\mathbf{f}_{\text{net}} = \mathbf{0}$ because the internal forces occur in action-reaction pairs. Thus,

$$\mathbf{M}_{P_{\text{net}}} = \int \mathbf{r} \times d\mathbf{F}_{\text{net}}$$

which means the net moment includes only the moment of all of the external forces on the body.

From the product rule of calculus we know that $d(\mathbf{r} \times \dot{\mathbf{R}})/dt = \mathbf{r} \times \ddot{\mathbf{R}} + \dot{\mathbf{r}} \times \dot{\mathbf{R}}$, so that the integrand in Equation 9.13 may be written

$$\mathbf{r} \times \ddot{\mathbf{R}} = \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{R}}) - \dot{\mathbf{r}} \times \dot{\mathbf{R}} \tag{9.14}$$

Furthermore, Figure 9.8 shows that $\mathbf{r} = \mathbf{R} - \mathbf{R}_P$, where $\mathbf{R}_P$ is the absolute position vector of P. It follows that

$$\dot{\mathbf{r}} \times \dot{\mathbf{R}} = (\dot{\mathbf{R}} - \dot{\mathbf{R}}_P) \times \dot{\mathbf{R}} = -\dot{\mathbf{R}}_P \times \dot{\mathbf{R}} \tag{9.15}$$

Substituting Equation 9.15 into Equation 9.14, then moving that result into Equation 9.13 yields

$$\mathbf{M}_{P_{\text{net}}} = \frac{d}{dt}\int_m \mathbf{r} \times \dot{\mathbf{R}}\, dm + \dot{\mathbf{R}}_P \times \int_m \dot{\mathbf{R}}\, dm \tag{9.16}$$

Now, $\mathbf{r} \times \dot{\mathbf{R}} dm$ is the moment of the absolute linear momentum of mass element dm about P. The moment of momentum, or angular momentum, of the entire body is the integral of this cross product over all of its mass elements. That is, the absolute angular momentum of the body relative to point P is

$$\mathbf{H}_P = \int_m \mathbf{r} \times \dot{\mathbf{R}}\, dm \tag{9.17}$$

Observing from Figure 9.8 that $\mathbf{r} = \mathbf{r}_{G/P} + \boldsymbol{\rho}$, we can write Equation 9.17 as

$$\mathbf{H}_P = \int_m (\mathbf{r}_{G/P} + \boldsymbol{\rho}) \times \dot{\mathbf{R}}\, dm = \mathbf{r}_{G/P} \times \int_m \dot{\mathbf{R}}\, dm + \int_m \boldsymbol{\rho} \times \dot{\mathbf{R}}\, dm \tag{9.18}$$

The last term is the absolute angular momentum relative to the center of mass G,

$$\mathbf{H}_G = \int \boldsymbol{\rho} \times \dot{\mathbf{R}} dm \tag{9.19}$$

Furthermore, by the definition of center of mass, Equation 9.9,

$$\int_m \dot{\mathbf{R}} dm = m\dot{\mathbf{R}}_G \tag{9.20}$$

Equations 9.19 and 9.20 allow us to write Equation 9.18 as

$$\mathbf{H}_P = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_G \tag{9.21}$$

This useful relationship shows how to obtain the absolute angular momentum about any point P once $\mathbf{H}_G$ is known.

For calculating the angular momentum about the center of mass, Equation 9.19 can be cast in a much more useful form by making the substitution (cf. Figure 9.8) $\mathbf{R} = \mathbf{R}_G + \boldsymbol{\rho}$, so that

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times (\dot{\mathbf{R}}_G + \dot{\boldsymbol{\rho}})dm = \int_m \boldsymbol{\rho} \times \dot{\mathbf{R}}_G dm + \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\,dm$$

In the two integrals on the right, the variable is $\boldsymbol{\rho}$. $\dot{\mathbf{R}}_G$ is fixed and can therefore be factored out of the first integral to obtain

$$\mathbf{H}_G = \left(\int_m \boldsymbol{\rho}\,dm\right) \times \dot{\mathbf{R}}_G + \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\,dm$$

By definition of the center of mass, $\int_m \boldsymbol{\rho}\,dm = \mathbf{0}$ (the position vector of the center of mass relative to itself is zero), which means

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\,dm \tag{9.22}$$

Since $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ are the position and velocity relative to the center of mass G, $\int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\,dm$ is the total moment about the center of mass of the linear momentum relative to the center of mass, $\mathbf{H}_{G_{\text{rel}}}$. In other words,

$$\mathbf{H}_G = \mathbf{H}_{G_{\text{rel}}} \tag{9.23}$$

This is a rather surprising fact, hidden in Equation 9.19 and true in general for no other point of the body.

Another useful angular momentum formula, similar to Equation 9.21, may be found by substituting $\mathbf{R} = \mathbf{R}_P + \mathbf{r}$ into Equation 9.17,

$$\mathbf{H}_P = \int_m \mathbf{r} \times (\dot{\mathbf{R}}_P + \dot{\mathbf{r}})\,dm = \left(\int_m \mathbf{r}dm\right) \times \dot{\mathbf{R}}_P + \int_m \mathbf{r} \times \dot{\mathbf{r}}dm \tag{9.24}$$

The term on the far right is the net moment of relative linear momentum about P,

$$\mathbf{H}_{P_{\text{rel}}} = \int_m \mathbf{r} \times \dot{\mathbf{r}}dm \tag{9.25}$$

Also, $\int_m \mathbf{r}\,dm = m\mathbf{r}_{G/P}$, where $\mathbf{r}_{G/P}$ is the position of the center of mass relative to P. Thus, Equation 9.24 can be written

$$\mathbf{H}_P = \mathbf{H}_{P_{\text{rel}}} + \mathbf{r}_{G/P} \times m\mathbf{v}_P \tag{9.26}$$

Finally, substituting this into Equation 9.21, solving for $\mathbf{H}_{P_{\text{rel}}}$, and noting that $\mathbf{v}_G - \mathbf{v}_P = \mathbf{v}_{G/P}$ yields

$$\mathbf{H}_{P_{\text{rel}}} = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_{G/P} \tag{9.27}$$

This expression is useful when the absolute velocity $\mathbf{v}_G$ of the center of mass, which is required in Equation 9.21, is not available.

So far we have written down some formulas for calculating the angular momentum about an arbitrary point in space and about the center of mass of the body itself. Let us now return to the problem of relating angular momentum to the applied torque. Substituting Equations 9.17 and 9.20 into 9.16, we obtain

$$\mathbf{M}_{P_{\text{net}}} = \dot{\mathbf{H}}_P + \dot{\mathbf{R}}_P \times m\dot{\mathbf{R}}_G$$

Thus, for an arbitrary point P,

$$\mathbf{M}_{P_{\text{net}}} = \dot{\mathbf{H}}_P + \mathbf{v}_P \times m\mathbf{v}_G \tag{9.28}$$

where $\mathbf{v}_P$ and $\mathbf{v}_G$ are the absolute velocities of points P and G, respectively. This expression is applicable to two important special cases.

If the point P is at rest in inertial space ($\mathbf{v}_P = \mathbf{0}$), then Equation 9.28 reduces to

$$\mathbf{M}_{P_{\text{net}}} = \dot{\mathbf{H}}_P \tag{9.29}$$

This equation holds as well if $\mathbf{v}_P$ and $\mathbf{v}_G$ are parallel, e.g., if P is the point of contact of a wheel rolling while slipping in the plane. Note that the validity of Equation 9.29 depends neither on the body's being rigid nor on its being in pure rotation about P.

If point P is chosen to be the center of mass, then, $\mathbf{v}_P = \mathbf{v}_G$, and Equation 9.28 becomes the equation of rotational motion of a continuous medium

$$\boxed{\mathbf{M}_{G_{\text{net}}} = \dot{\mathbf{H}}_G} \tag{9.30}$$

This equation is valid for any state of motion.

If Equation 9.30 is integrated over a time interval, then we obtain the angular impulse-momentum principle,

$$\int_{t_1}^{t_2} \mathbf{M}_{G_{\text{net}}} dt = \mathbf{H}_{G_2} - \mathbf{H}_{G_1} \tag{9.31}$$

A similar expression follows from Equation 9.29. $\int \mathbf{M}dt$ is the ***angular impulse***. If the net angular impulse is zero, then $\Delta\mathbf{H} = \mathbf{0}$, which is a statement of the conservation of angular momentum. Keep in mind that the angular impulse–momentum principle is not valid for just any reference point.

Additional versions of Equations 9.29 and 9.30 can be obtained which may prove useful in special circumstances. For example, substituting the expression for $\mathbf{H}_P$ (Equation 9.21) into Equation 9.28 yields

$$\begin{aligned}
\mathbf{M}_{P_{\text{net}}} &= \left[\dot{\mathbf{H}}_G + \frac{d}{dt}(\mathbf{r}_{G/P} \times m\mathbf{v}_G)\right] + \mathbf{v}_P \times m\mathbf{v}_G \\
&= \dot{\mathbf{H}}_G + \frac{d}{dt}\left[(\mathbf{r}_G - \mathbf{r}_P) \times m\mathbf{v}_G\right] + \mathbf{v}_P \times m\mathbf{v}_G \\
&= \dot{\mathbf{H}}_G + (\mathbf{v}_G - \mathbf{v}_P) \times m\mathbf{v}_G + \mathbf{r}_{G/P} \times m\mathbf{a}_G + \mathbf{v}_P \times m\mathbf{v}_G
\end{aligned}$$

or, finally,

$$\mathbf{M}_{P_{\text{net}}} = \dot{\mathbf{H}}_G + \mathbf{r}_{G/P} \times m\mathbf{a}_G \tag{9.32}$$

This expression is useful when it is convenient to compute the net moment about a point other than the center of mass. Alternatively, by simply differentiating Equation 9.27 we get

$$\dot{\mathbf{H}}_{P_{\text{rel}}} = \dot{\mathbf{H}}_G + \overbrace{\mathbf{v}_{G/P} \times m\mathbf{v}_{G/P}}^{=\mathbf{0}} + \mathbf{r}_{G/P} \times m\mathbf{a}_{G/P}$$

Solving for $\dot{\mathbf{H}}_G$, invoking Equation 9.30, and using the fact that $\mathbf{a}_{P/G} = -\mathbf{a}_{G/P}$ leads to

$$\mathbf{M}_{G_{\text{net}}} = \dot{\mathbf{H}}_{P_{\text{rel}}} + \mathbf{r}_{G/P} \times m\mathbf{a}_{P/G} \tag{9.33}$$

Finally, if the body is rigid, the magnitude of the position vector $\boldsymbol{\rho}$ of any point relative to the center of mass does not change with time. Therefore, Equation 1.52 requires that $\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho}$, leading us to conclude from Equation 9.22 that the ***angular momentum of a rigid body*** is

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\, dm \qquad \text{(Rigid body)} \tag{9.34}$$

Again, the absolute angular momentum about the center of mass depends only on the absolute angular velocity and not on the absolute translational velocity of any point of the body.

No such simplification of Equation 9.17 exists for an arbitrary reference point P. However, if the point P is fixed in inertial space and the rigid body is rotating about P, then the position vector $\mathbf{r}$ from P to any point of the body is constant. It follows from Equation 1.52 that $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$. According to Figure 9.8,

$$\mathbf{R} = \mathbf{R}_p + \mathbf{r}$$

Differentiating with respect to time gives

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_p + \dot{\mathbf{r}} = \mathbf{0} + \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}$$

Substituting this into Equation 9.17 yields the formula for angular momentum in this special case,

$$\mathbf{H}_P = \int_m \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm \qquad \text{(Rigid body rotating about fixed point } P\text{)} \tag{9.35}$$

Although Equations 9.34 and 9.35 are mathematically identical, one must keep in mind the notation of Figure 9.8. Equation 9.35 applies only if the rigid body is in pure rotation about a stationary point in inertial space, whereas Equation 9.34 applies unconditionally to any situation.

9.5 MOMENTS OF INERTIA

To use Equation 9.29 or 9.30 to solve problems, the vectors within them have to be resolved into components. To find the components of angular momentum, we must appeal to its definition. We will focus on the

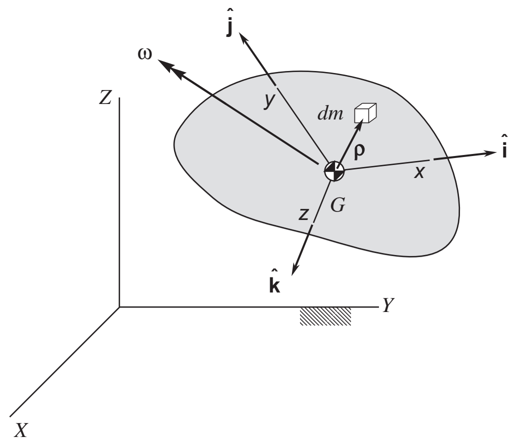

FIGURE 9.9

Co-moving *xyz* frame used to compute the moments of inertia.

formula for angular momentum of a rigid body about its center of mass, Equation 9.34, because the expression for fixed-point rotation (Equation 9.35) is mathematically the same. The integrand of Equation 9.34 can be rewritten using the *bac-cab* vector identity presented in Equation 1.20,

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \boldsymbol{\omega}\rho^2 - \boldsymbol{\rho}(\boldsymbol{\omega} \cdot \boldsymbol{\rho}) \tag{9.36}$$

Let the origin of a co-moving *xyz* coordinate system be attached to the center of mass G, as shown in Figure 9.9. The unit vectors of this frame are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. The vectors $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ can be resolved into components in the *xyz* directions to get $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$. Substituting these vector expressions into the right side of Equation 9.36 yields

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = (\omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}})(x^2 + y^2 + z^2) - (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})(\omega_x x + \omega_y y + \omega_z z)$$

Expanding the right side and collecting terms having the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ in common, we get

$$\begin{aligned}\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = {} & [(y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z]\hat{\mathbf{i}} \\ & + [-yx\omega_x + (x^2 + z^2)\omega_y - yz\omega_z]\hat{\mathbf{j}} \\ & + [-zx\omega_x - zy\omega_y + (x^2 + y^2)\omega_z]\hat{\mathbf{k}}\end{aligned} \tag{9.37}$$

We put this result into the integrand of Equation 9.34 to obtain

$$\mathbf{H}_G = H_x\hat{\mathbf{i}} + H_y\hat{\mathbf{j}} + H_z\hat{\mathbf{k}} \tag{9.38}$$

where

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \tag{9.39a}$$

or, in matrix notation,

$$\{\mathbf{H}\} = [\mathbf{I}]\{\boldsymbol{\omega}\} \tag{9.39b}$$

The nine components of the matrix [**I**] of mass moments of inertia about the center of mass are

$$\begin{aligned} I_x &= \int (y^2 + z^2)\,dm & I_{xy} &= -\int xy\,dm & I_{xz} &= -\int xz\,dm \\ I_{yx} &= -\int yx\,dm & I_y &= \int (x^2 + z^2)\,dm & I_{yz} &= -\int yz\,dm \\ I_{zx} &= -\int zx\,dm & I_{zy} &= -\int zy\,dm & I_z &= \int (x^2 + y^2)\,dm \end{aligned} \tag{9.40}$$

Since $I_{yx} = I_{xy}$, $I_{zx} = I_{xz}$ and $I_{zy} = I_{yz}$, it follows that [**I**] is a symmetric matrix ($[\mathbf{I}]^T = [\mathbf{I}]$). Therefore, [**I**] has six independent components instead of nine. Observe that, whereas the products of inertia I_{xy}, I_{xz} and I_{xz} can be positive, negative or zero, the moments of inertia I_x, I_y and I_z are always positive (never zero or negative) for bodies of finite dimensions. For this reason, [**I**] is a symmetric positive-definite matrix. Keep in mind that Equations 9.38 and 9.39 are valid as well for axes attached to a fixed point *P* about which the body is rotating.

The moments of inertia reflect how the mass of a rigid body is distributed. They manifest a body's rotational inertia, that is, its resistance to being set into rotary motion or stopped once rotation is underway. It is not an object's mass alone but how that mass is distributed which determines how the body will respond to applied torques.

If the *xy* plane is a plane of symmetry, then for any *x* and *y* within the body there are identical mass elements located at $+z$ and $-z$. That means the products of inertia with *z* in the integrand vanish. Similar statements are true if *xz* or *yz* are symmetry planes. In summary, we conclude:

If the *xy* plane is a plane of symmetry of the body, then $I_{xz} = I_{yz} = 0$.

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It follows that if the body has two planes of symmetry relative to the *xyz* frame of reference, then all three products of inertia vanish, and [**I**] becomes a diagonal matrix,

$$[\mathbf{I}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{9.41}$$

A, *B* and *C* are the principal moments of inertia (all positive), and the *xyz* axes are the body's principal axes of inertia. In this case, relative to either the center of mass or a fixed point of rotation, we have

$$H_x = A\omega_x \quad H_y = B\omega_y \quad H_z = C\omega_z \tag{9.42}$$

In general, the angular velocity $\boldsymbol{\omega}$ and the angular momentum **H** are not parallel. However, if (for example) $\boldsymbol{\omega} = \omega\hat{\mathbf{i}}$, then according to Equations 9.42, $\mathbf{H} = A\boldsymbol{\omega}$. In other words, if the angular velocity is aligned with a principal direction, so is the angular momentum. In that case the two vectors $\boldsymbol{\omega}$ and **H** are indeed parallel.

Each of the three principal moments of inertia can be expressed as follows:

$$A = mk_x^2 \qquad B = mk_y^2 \qquad C = mk_z^2 \tag{9.43}$$

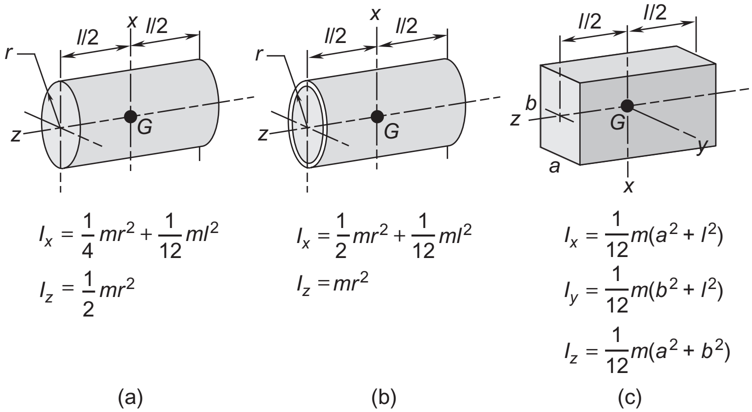


FIGURE 9.10

Moments of inertia for three common homogeneous solids of mass *m*. (a) Solid circular cylinder; (b) Circular cylindrical shell; (c) Rectangular parallelepiped.

where m is the mass of the body and k_x, k_y and k_z are the three ***radii of gyration***. One may imagine the mass of a body to be concentrated around a principal axis at a distance equal to the radius of gyration.

The moments of inertia for several common shapes are listed in Figure 9.10. By symmetry, their products of inertia vanish for the coordinate axes used. Formulas for other solid geometries can be found in engineering handbooks and in dynamics textbooks.

For a mass concentrated at a point, the moments of inertia in Equation 9.40 are just the mass times the integrand evaluated at the point. That is, the matrix $[\mathbf{I}^{(m)}]$ containing the components of the ***moments of inertia of a point mass*** m is given by

$$\left[\mathbf{I}^{(m)}\right] = \begin{bmatrix} m(y^2+z^2) & -mxy & -mxz \\ -mxy & m(x^2+z^2) & -myz \\ -mxz & -myz & m(x^2+y^2) \end{bmatrix} \tag{9.44}$$

Example 9.4

The following table lists mass and coordinates of seven point masses. Find the center of mass of the system and the moments of inertia about the origin.

Point, i	Mass m_i (kg)	x_i (m)	y_i (m)	z_i (m)
1	3	–0.5	0.2	0.3
2	7	0.2	0.75	–0.4
3	5	1	–0.8	0.9
4	6	1.2	–1.3	1.25
5	2	–1.3	1.4	–0.8
6	4	–0.3	1.35	0.75
7	1	1.5	–1.7	0.85

Solution

The total mass of this system is

$$m = \sum_{i=1}^{7} m_i = 28 \text{ kg}$$

For concentrated masses the integral in Equation 9.9 is replaced by the mass times its position vector. Therefore, in this case the three components of the position vector of the center of mass are $x_G = (1/m)\sum_{i=1}^{7} m_i x_i$, $y_G = (1/m)\sum_{i=1}^{7} m_i y_i$ and $z_G = (1/m)\sum_{i=1}^{7} m_i z_i$, so that

$$\boxed{x_G = 0.35 \text{ m} \qquad y_G = 0.01964 \text{ m} \qquad z_G = 0.4411 \text{ m}}$$

The total moment of inertia is the sum over all of the particles of Equation 9.44 evaluated at each point. Thus,

$$[\mathbf{I}] = \overbrace{\begin{bmatrix} 0.39 & 0.3 & 0.45 \\ 0.3 & 1.02 & -0.18 \\ 0.45 & -0.18 & 0.87 \end{bmatrix}}^{(1)} + \overbrace{\begin{bmatrix} 5.0575 & -1.05 & 0.56 \\ -1.05 & 1.4 & 2.1 \\ 0.56 & 2.1 & 4.2175 \end{bmatrix}}^{(2)} + \overbrace{\begin{bmatrix} 7.25 & 4 & -4.5 \\ 4 & 9.05 & 3.6 \\ -4.5 & 3.6 & 8.2 \end{bmatrix}}^{(3)} + \overbrace{\begin{bmatrix} 19.515 & 9.36 & -9 \\ 9.36 & 18.015 & 9.75 \\ -9 & 9.75 & 18.78 \end{bmatrix}}^{(4)}$$
$$+ \overbrace{\begin{bmatrix} 5.2 & 3.64 & -2.08 \\ 3.64 & 4.66 & 2.24 \\ -2.08 & 2.24 & 7.3 \end{bmatrix}}^{(5)} + \overbrace{\begin{bmatrix} 9.54 & 1.62 & 0.9 \\ 1.62 & 2.61 & -4.05 \\ 0.9 & -4.05 & 7.65 \end{bmatrix}}^{(6)} + \overbrace{\begin{bmatrix} 3.6125 & 2.55 & -1.275 \\ 2.55 & 2.9725 & 1.445 \\ -1.275 & 1.445 & 5.14 \end{bmatrix}}^{(7)}$$

or

$$\boxed{[\mathbf{I}] = \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} (\text{kg} \cdot \text{m}^2)}$$

Example 9.5

Calculate the moments of inertia of a slender, homogeneous straight rod of length l and mass m. One end of the rod is at the origin and the other has coordinates (a,b,c).

Solution

A slender rod is one whose cross sectional dimensions are negligible compared with its length. The mass is concentrated along its centerline. Since the rod is homogeneous, the mass per unit length ρ is uniform and given by

$$\rho = \frac{m}{l} \tag{a}$$

The length of the rod is

$$l = \sqrt{a^2 + b^2 + c^2}$$

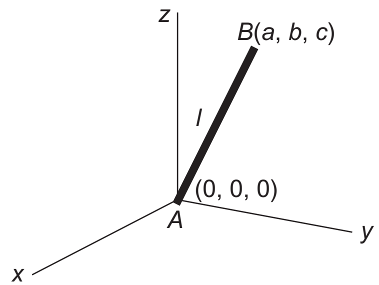


FIGURE 9.11

Uniform slender bar of mass *m* and length *l*.

Starting with I_x, we have from Equations 9.40,

$$I_x = \int_0^l (y^2 + z^2)\rho ds$$

in which we replaced the element of mass *m* by ρds, where *ds* is the element of length along the rod. The distance *s* is measured from end *A* of the rod, so that the *x*, *y* and *z* coordinates of any point along it are found in terms of *s* by the following relations,

$$x = \frac{s}{l}a \qquad y = \frac{s}{l}b \qquad z = \frac{s}{l}c$$

Thus,

$$I_x = \int_0^l \left(\frac{s^2}{l^2}b^2 + \frac{s^2}{l^2}c^2\right)\rho\, ds = \rho\frac{b^2 + c^2}{l^2}\int_0^l s^2 ds = \frac{1}{3}\rho(b^2 + c^2)l$$

Substituting (a) yields

$$\boxed{I_x = \frac{1}{3}m(b^2 + c^2)}$$

In precisely the same way we find

$$\boxed{I_y = \frac{1}{3}m(a^2 + c^2)} \qquad \boxed{I_z = \frac{1}{3}m(a^2 + b^2)}$$

For I_{xy} we have

$$I_{xy} = -\int_0^l xy\rho ds = -\int_0^l \frac{s}{l}a \cdot \frac{s}{l}b\rho ds = -\rho\frac{ab}{l^2}\int_0^l s^2 ds = -\frac{1}{3}\rho abl$$

Once again using (a),

$$\boxed{I_{xy} = -\frac{1}{3}mab}$$

Likewise,

$$\boxed{I_{xz} = -\frac{1}{3}mac} \qquad \boxed{I_{yz} = -\frac{1}{3}mbc}$$

Example 9.6

The gyro rotor in Example 9.3 has a mass m of 5 kg, radius r of 0.08 m, and thickness t of 0.025 m. If $N = 2.1$ rad/s, $\dot{\theta} = 4$ rad/s, $\omega = 10.5$ rad/s, and $\theta = 60°$, calculate

(a) The angular momentum of the rotor about its center of mass G in the body-fixed xyz frame.
(b) The angle between the rotor's angular velocity vector and its angular momentum vector.

Solution

Example 9.3 (f) gives the components of the absolute angular velocity of the rotor in the moving xyz frame.

$$\begin{aligned}
\omega_x &= \dot{\theta} = 4 \text{ rad/s} \\
\omega_y &= N\sin\theta = 2.1 \cdot \sin 60° = 1.819 \text{ rad/s} \\
\omega_z &= \omega_{\text{spin}} + N\cos\theta = 10.5 + 2.1 \cdot \cos 60° = 11.55 \text{ rad/s}
\end{aligned} \tag{a}$$

Therefore,

$$\boldsymbol{\omega} = 4\hat{\mathbf{i}} + 1.819\hat{\mathbf{j}} + 11.55\hat{\mathbf{k}} \text{ (rad/s)} \tag{b}$$

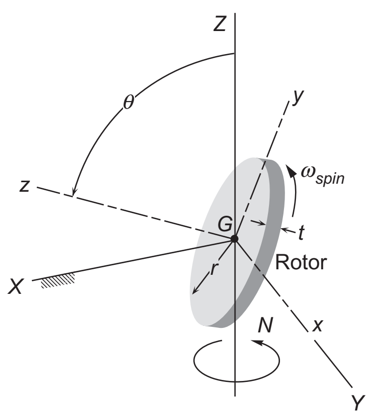


FIGURE 9.12

Rotor of the gyroscope in Figure 9.5.

All three coordinate planes of the body-fixed *xyz* frame contain the center of mass *G* and all are planes of symmetry of the circular cylindrical rotor. Therefore, $I_{xy} = I_{zx} = I_{yz} = 0$.

From Figure 9.10a we see that the nonzero diagonal entries in the moment of inertia tensor are

$$A = B = \frac{1}{12}mt^2 + \frac{1}{4}mr^2 = \frac{1}{12}\cdot 5\cdot 0.025^2 + \frac{1}{4}\cdot 5\cdot 0.08^2 = 0.008260 \text{ kg}\cdot\text{m}^2 \tag{c}$$

$$C = \frac{1}{2}mr^2 = \frac{1}{2}\cdot 5\cdot 0.08^2 = 0.0160 \text{ kg}\cdot\text{m}^2$$

We can use Equation 9.42 to calculate the angular momentum, because the origin of the *xyz* frame is the rotor's center of mass (which in this case also happens to be a fixed point of rotation, which is another reason we can use Equation 9.42). Substituting (a) and (c) into Equation 9.42 yields

$$\begin{aligned} H_x &= A\omega_x = 0.008260\cdot 4 = 0.03304 \text{ kg}\cdot\text{m}^2/\text{s} \\ H_y &= B\omega_y = 0.008260\cdot 1.819 = 0.0150 \text{ kg}\cdot\text{m}^2/\text{s} \\ H_z &= C\omega_z = 0.0160\cdot 11.55 = 0.1848 \text{ kg}\cdot\text{m}^2/\text{s} \end{aligned} \tag{d}$$

so that

$$\boxed{\mathbf{H} = 0.03304\hat{\mathbf{i}} + 0.0150\hat{\mathbf{j}} + 0.1848\hat{\mathbf{k}}\,(\text{kg}\cdot\text{m}^2/\text{s})} \tag{e}$$

The angle ϕ between $\mathbf{H}$ and $\boldsymbol{\omega}$ is found by taking the dot product of the two vectors,

$$\phi = \cos^{-1}\left(\frac{\mathbf{H}\cdot\boldsymbol{\omega}}{H\omega}\right) = \cos^{-1}\left(\frac{2.294}{0.1883\cdot 12.36}\right) = \boxed{9.717^\circ} \tag{f}$$

As this problem illustrates, the angular momentum and the angular velocity are in general not collinear.

Consider a coordinate system $x'y'z'$ with the same origin as *xyz*, but different orientation. Let $[\mathbf{Q}]$ be the orthogonal matrix ($[\mathbf{Q}]^{-1} = [\mathbf{Q}]^T$) that transforms the components of a vector from the *xyz* system to the $x'y'z'$ frame. Recall from Section 4.5 that the rows of $[\mathbf{Q}]$ are the direction cosines of the $x'y'z'$ axes relative to *xyz*. If $\{\mathbf{H}'\}$ comprises the components of the angular momentum vector along the $x'y'z'$ axes, then $\{\mathbf{H}'\}$ is obtained from its components $\{\mathbf{H}\}$ in the *xyz* frame by the relation

$$\{\mathbf{H}'\} = [\mathbf{Q}]\{\mathbf{H}\}$$

From Equation 9.39 we can write this as

$$\{\mathbf{H}'\} = [\mathbf{Q}][\mathbf{I}]\{\boldsymbol{\omega}\} \tag{9.45}$$

where $[\mathbf{I}]$ is the moment of inertia matrix, Equation 9.39, in *xyz* coordinates. Like the angular momentum vector, the components $\{\boldsymbol{\omega}\}$ of the angular velocity vector in the *xyz* system are related to those in the primed system ($\{\boldsymbol{\omega}'\}$) by the expression

$$\{\boldsymbol{\omega}'\} = [\boldsymbol{Q}]\{\boldsymbol{\omega}\}$$

The inverse relation is simply

$$\{\omega\} = [\mathbf{Q}]^{-1}\{\omega'\} = [\mathbf{Q}]^{T}\{\omega'\} \tag{9.46}$$

Substituting this into Equation 9.45, we get

$$\{\mathbf{H}'\} = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^{T}\{\omega'\} \tag{9.47}$$

But the components of angular momentum and angular velocity in the $x'y'z'$ frame are related by an equation of the same form as Equation 9.39, so that

$$\{\mathbf{H}'\} = [\mathbf{I}']\{\omega'\} \tag{9.48}$$

where $[\mathbf{I}']$ comprises the components of the inertia matrix in the primed system. Comparing the right-hand sides of Equations 9.47 and 9.48, we conclude that

$$[\mathbf{I}'] = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^{T} \tag{9.49a}$$

That is,

$$\begin{bmatrix} I_{x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} I_{x} & I_{xy} & I_{xz} \\ I_{yx} & I_{y} & I_{yz} \\ I_{zx} & I_{zy} & I_{z} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{32} & Q_{33} \end{bmatrix} \tag{9.49b}$$

This shows how to transform the components of the inertia matrix from the xyz coordinate system to any other orthogonal system with a common origin. Thus, for example,

$$\begin{aligned} I_{x'} &= \overbrace{\lfloor Q_{11} \quad Q_{12} \quad Q_{13} \rfloor}^{\lfloor \text{Row 1} \rfloor} \begin{bmatrix} I_{x} & I_{xy} & I_{xz} \\ I_{yx} & I_{y} & I_{yz} \\ I_{zx} & I_{zy} & I_{z} \end{bmatrix} \overbrace{\begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{13} \end{Bmatrix}}^{\lfloor \text{Row 1} \rfloor^{T}} \\ I_{y'z'} &= \overbrace{\lfloor Q_{21} \quad Q_{22} \quad Q_{23} \rfloor}^{\lfloor \text{Row 2} \rfloor} \begin{bmatrix} I_{x} & I_{xy} & I_{xz} \\ I_{yx} & I_{y} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \overbrace{\begin{Bmatrix} Q_{31} \\ Q_{32} \\ Q_{33} \end{Bmatrix}}^{\lfloor \text{Row 3} \rfloor^{T}} \end{aligned} \tag{9.50}$$

etc.

Any object represented by a square matrix whose components transform according to Equation 9.49 is called a second order tensor. We may therefore refer to $[\mathbf{I}]$ as the ***inertia tensor***.

Example 9.7

Find the mass moment of inertia of the system of point masses in Example 9.4 about an axis from the origin through the point with coordinates (2 m, –3 m, 4 m).

Solution

From Example 9.4 the moment of inertia tensor for the system of point masses is

$$[\mathbf{I}] = \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} (\text{kg}\cdot\text{m}^2)$$

The vector **V** connecting the origin with (2 m, –3 m, 4 m) is

$$\mathbf{V} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

The unit vector in the direction of **V** is

$$\hat{\mathbf{u}}_V = \frac{\mathbf{V}}{\|\mathbf{V}\|} = 0.3714\hat{\mathbf{i}} - 0.5571\hat{\mathbf{j}} + 0.7428\hat{\mathbf{k}}$$

We may consider $\hat{\mathbf{u}}_V$ as the unit vector along the x' axis of a rotated Cartesian coordinate system. Then, from Equation 9.50,

$$\begin{aligned} I_{V'} &= \lfloor 0.3714 \quad -0.5571 \quad 0.7428 \rfloor \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} \begin{Bmatrix} 0.3714 \\ -0.5571 \\ 0.7428 \end{Bmatrix} \\ &= \lfloor 0.3714 \quad -0.5571 \quad 0.7428 \rfloor \begin{Bmatrix} -3.695 \\ -3.482 \\ 24.90 \end{Bmatrix} = \boxed{19.06\ \text{kg}\cdot\text{m}^2} \end{aligned}$$

Example 9.8

For the satellite of Example 9.2, which is reproduced in Figure 9.13, the data are as follows: $N = 0.1$ rad/s and $\dot{\theta} = 0.01$ rad/s, in the directions shown. $\theta = 40°$. $d_0 = 1.5$ m. The length, width and thickness of the panel are $l = 6$ m, $\omega = 2$ m and $t = 0.025$ m. The uniformly distributed mass of the panel is 50 kg. Find the angular momentum of the panel relative to the center of mass O of the satellite.

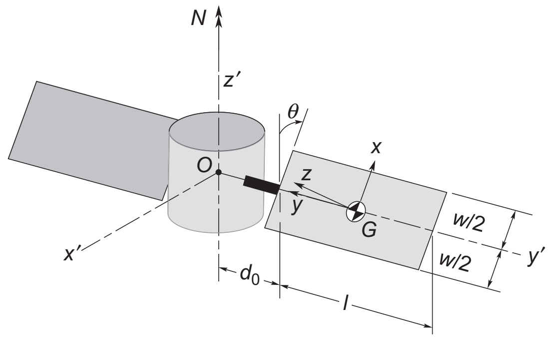


FIGURE 9.13

Satellite and solar panel.

Solution

We can treat the panel as a thin parallelepiped. The panel's *xyz* axes have their origin at the center of mass *G* of the panel and are parallel to its three edge directions. According to Figure 9.10(c), the moments of inertia relative to the *xyz* coordinate system are

$$\begin{aligned}
I_{G_x} &= \frac{1}{12}m(l^2+t^2) = \frac{1}{12}\cdot 50\cdot(6^2+0.025^2) = 150.0\ \text{kg}\cdot\text{m}^2\\
I_{G_y} &= \frac{1}{12}m(w^2+t^2) = \frac{1}{12}\cdot 50\cdot(2^2+0.025^2) = 16.67\ \text{kg}\cdot\text{m}^2\\
I_{G_z} &= \frac{1}{12}m(w^2+l^2) = \frac{1}{12}\cdot 50\cdot(2^2+6^2) = 166.7\ \text{kg}\cdot\text{m}^2\\
I_{G_{xy}} &= I_{G_{xz}} = I_{G_{yz}} = 0
\end{aligned} \tag{a}$$

In matrix notation,

$$[\mathbf{I}_G] = \begin{bmatrix} 150.0 & 0 & 0 \\ 0 & 16.67 & 0 \\ 0 & 0 & 166.7 \end{bmatrix} (\text{kg}\cdot\text{m}^2) \tag{b}$$

The unit vectors of the satellite's $x'y'z'$ system are related to those of panel's *xyz* frame by inspection,

$$\begin{aligned}
\hat{\mathbf{i}}' &= -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{k}} = -0.6428\hat{\mathbf{i}} + 0.7660\hat{\mathbf{k}}\\
\hat{\mathbf{j}}' &= -\hat{\mathbf{j}}\\
\hat{\mathbf{k}}' &= \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{k}} = 0.7660\hat{\mathbf{i}} + 0.6428\hat{\mathbf{k}}
\end{aligned} \tag{c}$$

The matrix [**Q**] of the transformation from *xyz* to $x'y'z'$ comprises the direction cosines of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$:

$$[\mathbf{Q}] = \begin{bmatrix} -0.6428 & 0 & 0.7660 \\ 0 & -1 & 0 \\ 0.7660 & 0 & 0.6428 \end{bmatrix} \tag{d}$$

In Example 9.2 we found that the absolute angular velocity of the panel, in the satellite's $x'y'z'$ frame of reference, is

$$\boldsymbol{\omega} = -\dot{\theta}\hat{\mathbf{j}}' + N\hat{\mathbf{k}}' = -0.01\hat{\mathbf{j}}' + 0.1\hat{\mathbf{k}}' \text{ (rad/s)}$$

That is,

$$\{\boldsymbol{\omega}'\} = \begin{Bmatrix} 0 \\ -0.01 \\ 0.1 \end{Bmatrix} \text{ (rad/s)} \tag{e}$$

To find the absolute angular momentum $\{\mathbf{H}'_G\}$ in the satellite system requires using Equation 9.39,

$$\{\mathbf{H}'_G\} = [\mathbf{I}'_G]\{\boldsymbol{\omega}'\} \tag{f}$$

Before doing so, we must transform the components of the moments of inertia tensor in (b) from the unprimed system to the primed system, by means of Equation 9.49,

$$[\mathbf{I}'_G] = [\mathbf{Q}][\mathbf{I}_G][\mathbf{Q}]^T = \begin{bmatrix} -0.6428 & 0 & 0.7660 \\ 0 & -1 & 0 \\ 0.7660 & 0 & 0.6428 \end{bmatrix}\begin{bmatrix} 150.0 & 0 & 0 \\ 0 & 16.67 & 0 \\ 0 & 0 & 166.7 \end{bmatrix}\begin{bmatrix} -0.6428 & 0 & 0.7660 \\ 0 & -1 & 0 \\ 0.7660 & 0 & 0.6428 \end{bmatrix}$$

so that

$$[\mathbf{I}'_G] = \begin{bmatrix} 159.8 & 0 & 8.205 \\ 0 & 16.67 & 0 \\ 8.205 & 0 & 156.9 \end{bmatrix} (\text{kg}\cdot\text{m}^2) \tag{g}$$

Then (f) yields

$$\{\mathbf{H}'_G\} = \begin{bmatrix} 159.8 & 0 & 8.205 \\ 0 & 16.67 & 0 \\ 8.205 & 0 & 156.9 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.01 \\ 0.1 \end{Bmatrix} = \begin{Bmatrix} 0.8205 \\ -0.1667 \\ 15.69 \end{Bmatrix} (\text{kg}\cdot\text{m}^2/\text{s})$$

or, in vector notation,

$$\mathbf{H}_G = 0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 15.69\hat{\mathbf{k}}' \,(\text{kg}\cdot\text{m}^2/\text{s}) \tag{h}$$

This is the absolute angular momentum of the panel about its own center of mass G, and it is used in Equation 9.27 to calculate the angular momentum $\mathbf{H}_{O_{\text{rel}}}$ relative to the satellite's center of mass O,

$$\mathbf{H}_{O_{\text{rel}}} = \mathbf{H}_G + \mathbf{r}_{G/O} \times m\mathbf{v}_{G/O} \tag{i}$$

$\mathbf{r}_{G/O}$ is the position vector from O to G,

$$\mathbf{r}_{G/O} = \left(d_O + \frac{l}{2}\right)\hat{\mathbf{j}}' = \left(1.5 + \frac{6}{2}\right)\hat{\mathbf{j}}' = 4.5\hat{\mathbf{j}}' \text{ (m)} \tag{j}$$

The velocity of G relative to O, $\mathbf{v}_{G/O}$, is found from Equation 9.2,

$$\mathbf{v}_{G/O} = \boldsymbol{\omega}_{\text{satellite}} \times \mathbf{r}_{G/O} = N\hat{\mathbf{k}}' \times \mathbf{r}_{G/O} = 0.1\hat{\mathbf{k}}' \times 4.5\hat{\mathbf{j}}' = -0.45\hat{\mathbf{i}}' \text{ (m/s)} \quad \text{(k)}$$

Substituting (h), (j) and (k) into (i) finally yields

$$\begin{aligned}\mathbf{H}_{O_{\text{rel}}} &= (0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 15.69\hat{\mathbf{k}}') + 4.5\hat{\mathbf{j}}' \times \left[50(-0.45\hat{\mathbf{i}}')\right] \\ &= \boxed{0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 116.9\hat{\mathbf{k}} \text{ (kg}\cdot\text{m}^2\text{/s)}}\end{aligned} \quad \text{(l)}$$

Note that we were unable to use Equation 9.21 to find the absolute angular momentum $\mathbf{H}_O$ because that requires knowing the absolute velocity $\mathbf{v}_G$, which in turn depends on the absolute velocity of O, which was not provided.

How can we find that direction cosine matrix [**Q**] such that Equation 9.49 will yield a moment of inertia matrix [**I**′] that is diagonal, i.e., of the form given by Equation 9.41? In other words, how do we find the principal directions (***eigenvectors***) and the corresponding principal values (***eigenvalues***) of the moment of inertia tensor?

Let the angular velocity vector $\boldsymbol{\omega}$ be parallel to the principal direction defined by the vector $\mathbf{e}$, so that $\boldsymbol{\omega} = \beta\mathbf{e}$, where β is a scalar. Since $\boldsymbol{\omega}$ points in a principal direction of the inertia tensor, so must $\mathbf{H}$, which means $\mathbf{H}$ is also parallel to $\mathbf{e}$. Therefore, $\mathbf{H} = \alpha\mathbf{e}$, where α is a scalar. From Equation 9.39 it follows that

$$\alpha\{\mathbf{e}\} = [\mathbf{I}](\beta\{\mathbf{e}\}$$

or

$$[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$$

where $\lambda = \alpha/\beta$ (a scalar). That is,

$$\begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \lambda \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix}$$

This can be written

$$\begin{bmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (9.51)$$

The trivial solution of Equation 9.51 is $\mathbf{e} = \mathbf{0}$, which is of no interest. The only way that Equation 9.51 will not yield the trivial solution is if the coefficient matrix on the left is singular. That will occur if its determinant vanishes, that is, if

$$\begin{vmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{vmatrix} = 0 \quad (9.52)$$

Expanding the determinant, we find

$$\begin{vmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{vmatrix} = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 \tag{9.53}$$

where

$$\begin{aligned} I_1 &= I_x + I_y + I_z \\ I_2 &= \begin{vmatrix} I_x & I_{xy} \\ I_{xy} & I_y \end{vmatrix} + \begin{vmatrix} I_x & I_{xz} \\ I_{xz} & I_z \end{vmatrix} + \begin{vmatrix} I_y & I_{yz} \\ I_{yz} & I_z \end{vmatrix} \\ I_3 &= \begin{vmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{vmatrix} \end{aligned} \tag{9.54}$$

I_1, I_2 and I_3 are invariants; that is, they have the same value in every Cartesian coordinate system.

Equations 9.52 and 9.53 yields the characteristic equation of the tensor [**I**]

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{9.55}$$

The three roots λ_p ($p = 1,2,3$) of this cubic equation are real, since [**I**] is symmetric; furthermore they are all positive, since [**I**] is a positive-definite matrix. We substitute each root, or eigenvalue, λ_p back into Equation 9.51 to obtain

$$\begin{bmatrix} I_x - \lambda_p & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda_p & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda_p \end{bmatrix} \begin{Bmatrix} e_x^{(p)} \\ e_y^{(p)} \\ e_z^{(p)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad p = 1,2,3 \tag{9.56}$$

Solving this system yields the three eigenvectors $\mathbf{e}^{(p)}$ corresponding to each of the three eigenvalues λ_p. The three eigenvectors are orthogonal, also due to the symmetry of the matrix [**I**]. Each eigenvalue is a principal moment of inertia, and its corresponding eigenvector is a principal direction.

Example 9.9

Find the principal moments of inertia and the principal axes of inertia of the inertia tensor

$$[\mathbf{I}] = \begin{bmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{bmatrix} \text{kg}\cdot\text{m}^2$$

Solution

We seek the nontrivial solutions of the system $[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$, that is,

$$\begin{bmatrix} 100-\lambda & -20 & -100 \\ -20 & 300-\lambda & -50 \\ -100 & -50 & 500-\lambda \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{a}$$

From Equation 9.54,

$$\begin{aligned} I_1 &= 100 + 300 + 500 = 900 \\ I_2 &= \begin{vmatrix} 100 & -20 \\ -20 & 300 \end{vmatrix} + \begin{vmatrix} 100 & -100 \\ -100 & 500 \end{vmatrix} + \begin{vmatrix} 300 & -50 \\ -50 & 500 \end{vmatrix} = 217{,}100 \\ I_3 &= \begin{vmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{vmatrix} = 11{,}350{,}000 \end{aligned} \tag{b}$$

Thus, the characteristic equation is

$$\lambda^3 - 900\lambda^2 + 217{,}100\lambda - 11{,}350{,}000 = 0 \tag{c}$$

The three roots are the principal moments of inertia, which are found to be

$$\boxed{\lambda_1 = 532.052 \qquad \lambda_2 = 295.840 \qquad \lambda_3 = 72.1083 \ \left(\text{kg}\cdot\text{m}^2\right)} \tag{d}$$

Each of these is substituted, in turn, back into (a) to find its corresponding principal direction.

Substituting $\lambda_1 = 532.052\,\text{kg}\cdot\text{m}^2$ into (a) we obtain

$$\begin{bmatrix} -432.052 & -20.0000 & -100.0000 \\ -20.0000 & -232.052 & -50.0000 \\ -100.0000 & -50.0000 & -32.0519 \end{bmatrix} \begin{Bmatrix} e_x^{(1)} \\ e_y^{(1)} \\ e_z^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{e}$$

Since the determinant of the coefficient matrix is zero, at most two of the three equations in (e) are independent. Thus, at most two of the three components of the vector $\mathbf{e}^{(1)}$ can be found in terms of the third. We can therefore arbitrarily set $e_x^{(1)} = 1$ and solve for $e_y^{(1)}$ and $e_z^{(1)}$ using any two of the independent equations in (e). With $e_x^{(1)} = 1$, the first two of Equations (e) become

$$\begin{aligned} -20.0000e_y^{(1)} - 100.000e_z^{(1)} &= 432.052 \\ -232.052e_y^{(1)} - 50.000e_z^{(1)} &= 20.0000 \end{aligned} \tag{f}$$

Solving these two equations for $e_y^{(1)}$ and $e_z^{(1)}$ yields, together with the assumption that $e_x^{(1)} = 1$,

$$e_x^{(1)} = 1.00000 \quad e_y^{(1)} = 0.882793 \quad e_z^{(1)} = -4.49708 \tag{g}$$

The unit vector in the direction of $\mathbf{e}^{(1)}$

$$\hat{\mathbf{i}}_1 = \frac{\mathbf{e}^{(1)}}{\left\|\mathbf{e}^{(1)}\right\|} = \frac{1.00000\hat{\mathbf{i}} + 0.882793\hat{\mathbf{j}} - 4.49708\hat{\mathbf{k}}}{\sqrt{1.00000^2 + 0.882793^2 + (-4.49708)^2}}$$

or

$$\boxed{\hat{\mathbf{i}}_1 = 0.213186\hat{\mathbf{i}} + 0.188199\hat{\mathbf{j}} - 0.958714\hat{\mathbf{k}} \qquad (\lambda_1 = 532.052\ \text{kg}\cdot\text{m}^2)} \tag{h}$$

Substituting $\lambda_2 = 295.840\,\text{kg}\cdot\text{m}^2$ into (a) and proceeding as above we find

$$\boxed{\hat{\mathbf{i}}_2 = 0.176732\hat{\mathbf{i}} - 0.972512\hat{\mathbf{j}} - 0.151609\hat{\mathbf{k}} \qquad (\lambda_2 = 295.840\,\text{kg}\cdot\text{m}^2)} \tag{i}$$

The two unit vectors $\hat{\mathbf{i}}_1$ and $\hat{\mathbf{i}}_2$ define two of the three principal directions of the inertia tensor. Observe that $\hat{\mathbf{i}}_1 \cdot \hat{\mathbf{i}}_2 = 0$, as must be the case for symmetric matrices.

To obtain the third principal direction $\hat{\mathbf{i}}_3$, we can substitute $\lambda_3 = 72.1083\,\text{kg}\cdot\text{m}^2$ into (a) and proceed as above. However, since the inertia tensor is symmetric, we know that the three principal directions are mutually orthogonal. That means $\hat{\mathbf{i}}_3 = \hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_2$. Substituting Equations (h) and (i) into the cross product, we find that

$$\boxed{\hat{\mathbf{i}}_3 = -0.960894\hat{\mathbf{i}} - 0.137114\hat{\mathbf{j}} - 0.240587\hat{\mathbf{k}} \qquad (\lambda_3 = 72.1083\,\text{kg}\cdot\text{m}^2)} \tag{j}$$

We can check our work by substituting λ_3 and $\hat{\mathbf{i}}_3$ into (a) and verify that it is indeed satisfied:

$$\begin{bmatrix} 100 - 72.1083 & -20 & -100 \\ -20 & 300 - 72.1083 & -50 \\ -100 & -50 & 500 - 72.1083 \end{bmatrix} \begin{Bmatrix} -0.960894 \\ -0.137114 \\ -0.240587 \end{Bmatrix} \overset{\checkmark}{=} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{k}$$

The components of the vectors $\hat{\mathbf{i}}_1$, $\hat{\mathbf{i}}_2$ and $\hat{\mathbf{i}}_3$ define the three rows of the orthogonal transformation $[\mathbf{Q}]$ from the xyz system into the $x'y'z'$ system aligned along the three principal directions:

$$[\mathbf{Q}] = \begin{bmatrix} 0.213186 & 0.188199 & -0.958714 \\ 0.176732 & -0.972512 & -0.151609 \\ -0.960894 & -0.137114 & -0.240587 \end{bmatrix} \tag{l}$$

Indeed, if we apply the transformation in Equation 9.49, $[\mathbf{I}'] = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^T$, we find

$$[\mathbf{I}'] = \begin{bmatrix} 0.213186 & 0.188199 & -0.958714 \\ 0.176732 & -0.972512 & -0.151609 \\ -0.960894 & -0.137114 & -0.240587 \end{bmatrix} \begin{bmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{bmatrix} \begin{bmatrix} 0.213186 & 0.176732 & -0.960894 \\ 0.188199 & -0.972512 & -0.137114 \\ -0.958714 & -0.151609 & -0.240587 \end{bmatrix}$$

$$= \begin{bmatrix} 532.052 & 0 & 0 \\ 0 & 295.840 & 0 \\ 0 & 0 & 72.1083 \end{bmatrix} \ (\mathrm{kg \cdot m^2})$$

An alternative to the above hand calculations in Example 9.9 is to type the following lines into the MATLAB® Command Window:

```
I = [100  -20  -100
     -20  300   -50
    -100  -50   500];

[eigenVectors, eigenValues] = eig(I)
```

Hitting the Enter (or Return) key yields the following output to the Command Window:

```
eigenVectors =

    0.9609    0.1767   -0.2132
    0.1371   -0.9725   -0.1882
    0.2406   -0.1516    0.9587

eigenValues =

   72.1083         0         0
         0  295.8398         0
         0         0  532.0519
```

Two of the eigenvectors delivered by MATLAB are opposite in direction to those calculated in Example 9.9. This illustrates the fact that we can determine an eigenvector only to within an arbitrary scalar factor. That is, suppose $\mathbf{e}$ is an eigenvector of the tensor $[\mathbf{I}]$ so that $[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$. Multiplying this equation through by an arbitrary scalar a yields $[\mathbf{I}]\{\mathbf{e}\}a = \lambda\{\mathbf{e}\}a$, or $[\mathbf{I}]\{a\mathbf{e}\} = \lambda\{a\mathbf{e}\}$, which means that $\{a\mathbf{e}\}$ is an eigenvector corresponding to the same eigenvalue λ.

9.5.1 Parallel axis theorem

Suppose the rigid body in Figure 9.14 is in pure rotation about point P. Then, according to Equation 9.39,

$$\{\mathbf{H}_{P_{\mathrm{rel}}}\} = [\mathbf{I}_P]\{\boldsymbol{\omega}\} \tag{9.57}$$

where $[\mathbf{I}_p]$ is the moment of inertia about P, given by Equations 9.40 with

$$x = x_{G/P} + \xi \qquad y = y_{G/P} + \eta \qquad z = z_{G/P} + \zeta$$

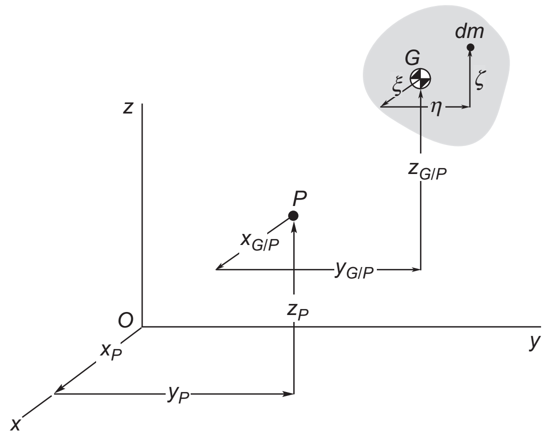


FIGURE 9.14

The moments of inertia are to be computed at *P*, given their values at *G*.

On the other hand, we have from Equation 9.27 that

$$\mathbf{H}_{P_{\text{rel}}} = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_{G/P} \tag{9.58}$$

The vector $\mathbf{r}_{G/P} \times m\mathbf{v}_{G/P}$ is the angular momentum about P of the concentrated mass m located at G. Using matrix notation, it is computed as follows,

$$\{\mathbf{r}_{G/P} \times m\mathbf{v}_{G/P}\} \equiv \left\{\mathbf{H}_{P_{\text{rel}}}^{(m)}\right\} = \left[\mathbf{I}_P^{(m)}\right]\{\omega\} \tag{9.59}$$

where $[\mathbf{I}_P^{(m)}]$, the moment of inertia of m about P, is obtained from Equation 9.44, with $x = x_{G/P}$, $y = y_{G/P}$ and $z = z_{G/P}$. That is,

$$[\mathbf{I}_P^{(m)}] = \begin{bmatrix} m({y_{G/P}}^2 + {z_{G/P}}^2) & -mx_{G/P}y_{G/P} & -mx_{G/P}z_{G/P} \\ -mx_{G/P}y_{G/P} & m({x_{G/P}}^2 + {z_{G/P}}^2) & -my_{G/P}z_{G/P} \\ -mx_{G/P}z_{G/P} & -my_{G/P}z_{G/P} & m({x_{G/P}}^2 + {y_{G/P}}^2) \end{bmatrix} \tag{9.60}$$

Of course, Equation 9.39 requires

$$\{\mathbf{H}_G\} = [\mathbf{I}_G]\{\omega\}$$

Substituting this together with Equations 9.57 and 9.59 into Equation 9.58 yields

$$[\mathbf{I}_P]\{\omega\} = [\mathbf{I}_G]\{\omega\} + [\mathbf{I}_P^{(m)}]\{\omega\} = ([\mathbf{I}_G] + [\mathbf{I}_P^{(m)}])\{\omega\}$$

From this we may infer the ***parallel axis theorem***,

$$[\mathbf{I}_P] = [\mathbf{I}_G] + [\mathbf{I}_P^{(m)}] \tag{9.61}$$

The moment of inertia about P is the moment of inertia about parallel axes through the center of mass plus the moment of inertia of the center of mass about P. That is,

$$\begin{aligned} I_{P_x} &= I_{G_x} + m({y_{G/P}}^2 + {z_{G/P}}^2) & I_{P_y} &= I_{G_y} + m({x_{G/P}}^2 + {z_{G/P}}^2) & I_{P_z} &= I_{G_z} + m({x_{G/P}}^2 + {y_{G/P}}^2) \\ I_{P_{xy}} &= I_{G_{xy}} - m x_{G/P} y_{G/P} & I_{P_{xz}} &= I_{G_{xz}} - m x_{G/P} z_{G/P} & I_{P_{yz}} &= I_{G_{yz}} - m y_{G/P} z_{G/P} \end{aligned} \tag{9.62}$$

Example 9.10

Find the moments of inertia of the rod in Example 9.5 (Figure 9.15) about its center of mass G.

Solution

From Example 9.5,

$$[\mathbf{I}_A] = \begin{bmatrix} \frac{1}{3}m(b^2 + c^2) & -\frac{1}{3}mab & -\frac{1}{3}mac \\ -\frac{1}{3}mab & \frac{1}{3}m(a^2 + c^2) & -\frac{1}{3}mbc \\ -\frac{1}{3}mac & -\frac{1}{3}mbc & \frac{1}{3}m(a^2 + b^2) \end{bmatrix}$$

Using Equation 9.62_1, and noting the coordinates of the center of mass in Figure 9.15,

$$I_{G_x} = I_{A_x} - m\left[(y_G - 0)^2 + (z_G - 0)^2\right] = \frac{1}{3}m(b^2 + c^2) - m\left[\left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2\right] = \frac{1}{12}m(b^2 + c^2)$$

Equation 9.62_4 yields

$$I_{G_{xy}} = I_{A_{xy}} + m(x_G - 0)(y_G - 0) = -\frac{1}{3}mab + m \cdot \frac{a}{2} \cdot \frac{b}{2} = -\frac{1}{12}mab$$

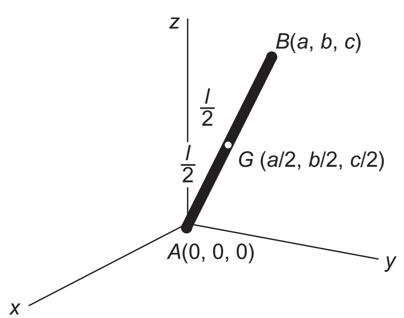


FIGURE 9.15

Uniform slender rod.

The remaining four moments of inertia are found in a similar fashion, so that

$$[\mathbf{I}_G] = \begin{bmatrix} \frac{1}{12}m(b^2+c^2) & -\frac{1}{12}mab & -\frac{1}{12}mac \\ -\frac{1}{12}mab & \frac{1}{12}m(a^2+c^2) & -\frac{1}{12}mbc \\ -\frac{1}{12}mac & -\frac{1}{12}mbc & \frac{1}{12}m(a^2+b^2) \end{bmatrix} \tag{9.63}$$

Example 9.11

Calculate the principal moments of inertia about the center of mass and the corresponding principal directions for the bent rod in Figure 9.16. Its mass is uniformly distributed at 2 kg/m.

Solution

The mass of each of the rod segments is

$$m_1 = 2\cdot 0.4 = 0.8\text{ kg} \quad m_2 = 2\cdot 0.5 = 1\text{ kg} \quad m_3 = 2\cdot 0.3 = 0.6\text{ kg} \quad m_4 = 2\cdot 0.2 = 0.4\text{ kg} \tag{a}$$

The total mass of the system is

$$m = \sum_{i=1}^{4} m_i = 2.8\text{ kg} \tag{b}$$

The coordinates of each segment's center of mass are

$$\begin{array}{lll} x_{G_1} = 0 & y_{G_1} = 0 & z_{G_1} = 0.2\text{ m} \\ x_{G_2} = 0 & y_{G_2} = 0.25\text{ m} & z_{G_2} = 0.2\text{ m} \\ x_{G_3} = 0.15\text{ m} & y_{G_3} = 0.5\text{ m} & z_{G_3} = 0 \\ x_{G_4} = 0.3\text{ m} & y_{G_4} = 0.4\text{ m} & z_{G_4} = 0 \end{array} \tag{c}$$

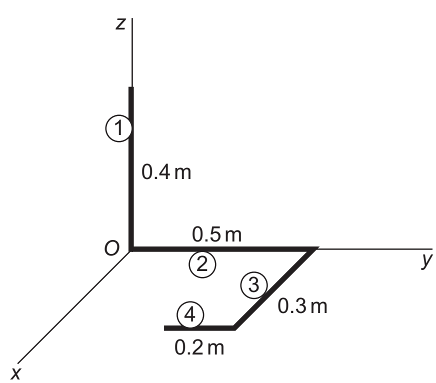


FIGURE 9.16

Bent rod for which the principal moments of inertia are to be determined.

If the slender rod of Figure 9.15 is aligned with, say, the x axis, then $a = l$ and $b = c = 0$, so that according to Equation 9.63,

$$[\boldsymbol{I}_G] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix}$$

That is, the moment of inertia of a slender rod about axes normal to the rod at its center of mass is $\frac{1}{12}ml^2$, where m and l are the mass and length of the rod, respectively. Since the mass of a slender bar is assumed to be concentrated along the axis of the bar (its cross sectional dimensions are infinitesimal), the moment of inertia about the centerline is zero. By symmetry, the products of inertia about axes through the center of mass are all zero. Using this information and the parallel axis theorem, we find the moments and products of inertia of each rod segment about the origin O of the xyz system as follows.

Rod 1:

$$\begin{aligned}
I_x^{(1)} &= \left.I_{G_1}^{(1)}\right)_x + m_1(y_{G_1}^2 + z_{G_1}^2) = \left(\frac{1}{12}\cdot 0.8\cdot 0.4^2\right) + \left[0.8(0 + 0.2^2)\right] = 0.04267\ \text{kg-m}^2 \\
I_y^{(1)} &= \left.I_{G_1}^{(1)}\right)_y + m_1(x_{G_1}^2 + z_{G_1}^2) = \left(\frac{1}{12}\cdot 0.8\cdot 0.4^2\right) + \left[0.8(0 + 0.2^2)\right] = 0.04267\ \text{kg-m}^2 \\
I_z^{(1)} &= \left.I_{G_1}^{(1)}\right)_z + m_1(x_{G_1}^2 + y_{G_1}^2) = 0 + [0.8(0 + 0)] = 0 \\
I_{xy}^{(1)} &= \left.I_{G_1}^{(1)}\right)_{xy} - m_1 x_{G_1} y_{G_1} = 0 - [0.8(0)(0)] = 0 \\
I_{xz}^{(1)} &= \left.I_{G_1}^{(1)}\right)_{xz} - m_1 x_{G_1} z_{G_1} = 0 - \left[0.8(0)(0.2)\right] = 0 \\
I_{yz}^{(1)} &= \left.I_{G_1}^{(1)}\right)_{yz} - m_1 y_{G_1} z_{G_1} = 0 - \left[0.8(0)(0)\right] = 0
\end{aligned}$$

Rod 2:

$$\begin{aligned}
I_x^{(2)} &= \left.I_{G_2}^{(2)}\right)_x + m_2\left(y_{G_2}^2 + z_{G_2}^2\right) = \left(\frac{1}{12}\cdot 1.0\cdot 0.5^2\right) + [1.0(0 + 0.25^2)] = 0.08333\ \text{kg-m}^2 \\
I_y^{(2)} &= \left.I_{G_2}^{(2)}\right)_y + m_2\left(x_{G_2}^2 + z_{G_2}^2\right) = 0 + [1.0(0 + 0)] = 0 \\
I_z^{(2)} &= \left.I_{G_2}^{(2)}\right)_z + m_2\left(x_{G_2}^2 + y_{G_2}^2\right) = \left(\frac{1}{12}\cdot 1.0\cdot 0.5^2\right) + [1.0(0 + 0.5^2)] = 0.08333\ \text{kg-m}^2 \\
I_{xy}^{(2)} &= \left.I_{G_2}^{(2)}\right)_{xy} - m_2 x_{G_2} y_{G_2} = 0 - [1.0(0)(0.5)] = 0 \\
I_{xz}^{(2)} &= \left.I_{G_2}^{(2)}\right)_{xz} - m_2 x_{G_2} z_{G_2} = 0 - \left[1.0(0)(0)\right] = 0 \\
I_{yz}^{(2)} &= \left.I_{G_2}^{(2)}\right)_{yz} - m_2 y_{G_2} z_{G_2} = 0 - \left[1.0(0.5)(0)\right] = 0
\end{aligned}$$

Rod 3:

$$\begin{aligned}
I_x^{(3)} &= I_{G_3}^{(3)}\Big)_x + m_3(y_{G_3}^2 + z_{G_3}^2) = 0 + [0.6(0.5^2 + 0)] = 0.15\ \text{kg-m}^2\\
I_y^{(3)} &= I_{G_3}^{(3)}\Big)_y + m_3(x_{G_3}^2 + z_{G_3}^2) = \left(\frac{1}{12}\cdot 0.6\cdot 0.3^2\right) + [0.6(0.15^2 + 0)] = 0.018\ \text{kg-m}^2\\
I_z^{(3)} &= I_{G_3}^{(3)}\Big)_z + m_3(x_{G_3}^2 + y_{G_3}^2) = \left(\frac{1}{12}\cdot 0.6\cdot 0.3^2\right) + [0.6(0.15^2 + 0.5^2)] = 0.1680\ \text{kg-m}^2\\
I_{xy}^{(3)} &= I_{G_3}^{(3)}\Big)_{xy} - m_3 x_{G_3} y_{G_3} = 0 - [0.6(0.15)(0.5)] = -0.045\ \text{kg-m}^2\\
I_{xz}^{(3)} &= I_{G_3}^{(3)}\Big)_{xz} - m_3 x_{G_3} z_{G_3} = 0 - [0.6(0.15)(0)] = 0\\
I_{yz}^{(3)} &= I_{G_3}^{(3)}\Big)_{yz} - m_3 y_{G_3} z_{G_3} = 0 - [0.6(0.5)(0)] = 0
\end{aligned}$$

Rod 4:

$$\begin{aligned}
I_x^{(4)} &= I_{G_4}^{(4)}\Big)_x + m_4(y_{G_4}^2 + z_{G_4}^2) = \left(\frac{1}{12}\cdot 0.4\cdot 0.2^2\right) + [0.4(0.4^2 + 0)] = 0.06533\ \text{kg-m}^2\\
I_y^{(4)} &= I_{G_4}^{(4)}\Big)_y + m_4(x_{G_4}^2 + z_{G_4}^2) = 0 + [0.4(0.3^2 + 0)] = 0.0360\ \text{kg-m}^2\\
I_z^{(4)} &= I_{G_4}^{(4)}\Big)_z + m_4(x_{G_4}^2 + y_{G_4}^2) = \left(\frac{1}{12}\cdot 0.4\cdot 0.2^2\right) + [0.4(0.3^2 + 0.4^2)] = 0.1013\ \text{kg-m}^2\\
I_{xy}^{(4)} &= I_{G_4}^{(4)}\Big)_{xy} - m_4 x_{G_4} y_{G_4} = 0 - [0.4(0.3)(0.4)] = -0.0480\ \text{kg-m}^2\\
I_{xz}^{(4)} &= I_{G_4}^{(4)}\Big)_{xz} - m_4 x_{G_4} z_{G_4} = 0 - [0.4(0.3)(0)] = 0\\
I_{yz}^{(4)} &= I_{G_4}^{(4)}\Big)_{yz} - m_4 y_{G_4} z_{G_4} = 0 - [0.4(0.4)(0)] = 0
\end{aligned}$$

The total moments of inertia for all four of the rods about O are

$$\begin{aligned}
I_x &= \sum_{i=1}^{4} I_x^{(i)} = 0.3413\ \text{kg-m}^2 & I_y &= \sum_{i=1}^{4} I_y^{(i)} = 0.09667\ \text{kg-m}^2 & I_z &= \sum_{i=1}^{4} I_z^{(i)} = 0.3527\ \text{kg-m}^2\\
I_{xy} &= \sum_{i=1}^{4} I_{xy}^{(i)} = -0.0930\ \text{kg-m}^2 & I_{xz} &= \sum_{i=1}^{4} I_{xz}^{(i)} = 0 & I_{yz} &= \sum_{i=1}^{4} I_{yz}^{(i)} = 0
\end{aligned} \tag{d}$$

The coordinates of the center of mass of the system of four rods are, from (a), (b) and (c),

$$\begin{aligned}
x_G &= \frac{1}{m}\sum_{i=1}^{4} m_i x_{G_i} = \frac{1}{2.8}\cdot 0.21 = 0.075\ \text{m}\\
y_G &= \frac{1}{m}\sum_{i=1}^{4} m_i y_{G_i} = \frac{1}{2.8}\cdot 0.71 = 0.2536\ \text{m}\\
z_G &= \frac{1}{m}\sum_{i=1}^{4} m_i z_{G_i} = \frac{1}{2.8}\cdot 0.16 = 0.05714\ \text{m}
\end{aligned} \tag{e}$$

We use the parallel axis theorems to shift the moments of inertia in (d) to the center of mass G of the system.

$$\begin{aligned}
I_{G_x} &= I_x - m({y_G}^2 + {z_G}^2) = 0.3413 - 0.1892 = 0.1522 \text{ kg-m}^2 \\
I_{G_y} &= I_y - m({x_G}^2 + {z_G}^2) = 0.09667 - 0.02489 = 0.07177 \text{ kg-m}^2 \\
I_{G_z} &= I_z - m({x_G}^2 + {y_G}^2) = 0.3527 - 0.1958 = 0.1569 \text{ kg-m}^2 \\
I_{G_{xy}} &= I_{xy} + m x_G y_G = -0.093 + 0.05325 = -0.03975 \text{ kg-m}^2 \\
I_{G_{xz}} &= I_{xz} + m x_G z_G = 0 + 0.012 = 0.012 \text{ kg-m}^2 \\
I_{G_{yz}} &= I_{yz} + m y_G z_G = 0 + 0.04057 = 0.04057 \text{ kg-m}^2
\end{aligned}$$

Therefore the inertia tensor, relative to the center of mass, is

$$[\mathbf{I}] = \begin{bmatrix} I_{G_x} & I_{G_{xy}} & I_{G_{xz}} \\ I_{G_{xy}} & I_{G_y} & I_{G_{yz}} \\ I_{G_{xz}} & I_{G_{yz}} & I_{G_z} \end{bmatrix} = \begin{bmatrix} 0.1522 & -0.03975 & 0.012 \\ -0.03975 & 0.07177 & 0.04057 \\ 0.012 & 0.04057 & 0.1569 \end{bmatrix} (\text{kg} \cdot \text{m}^2) \qquad \text{(f)}$$

To find the three principal moments of inertia, we may proceed as in Example 9.9, or simply enter the following lines in the MATLAB command window

```
I = [ 0.1522  -0.03975   0.012
     -0.03975  0.07177   0.04057
      0.012    0.04057   0.1569 ];

[eigenVectors, eigenValues] = eig(IG)
```

to obtain

```
eigenVectors =

   0.3469  -0.8482  -0.4003
   0.8742   0.1378   0.4656
  -0.3397  -0.5115   0.7893

eigenValues =

   0.0402        0        0
        0   0.1658        0
        0        0   0.1747
```

Hence, the three principal moments of inertia and their principal directions are

$$\boxed{\begin{aligned}
\lambda_1 &= 0.04023 \text{ kg-m}^2 & \mathbf{e}^{(1)} &= 0.3469\hat{\mathbf{i}} + 0.8742\hat{\mathbf{j}} - 0.3397\hat{\mathbf{k}} \\
\lambda_2 &= 0.1658 \text{ kg-m}^2 & \mathbf{e}^{(2)} &= -0.8482\hat{\mathbf{i}} + 0.1378\hat{\mathbf{j}} - 0.5115\hat{\mathbf{k}} \\
\lambda_3 &= 0.1747 \text{ kg-m}^2 & \mathbf{e}^{(3)} &= -0.4003\hat{\mathbf{i}} + 0.4656\hat{\mathbf{j}} + 0.7893\hat{\mathbf{k}}
\end{aligned}}$$

9.6 EULER'S EQUATIONS

For either the center of mass G or a fixed point P about which the body is in pure rotation, we know from Equations 9.29 and 9.30 that

$$\mathbf{M}_{\text{net}} = \dot{\mathbf{H}} \tag{9.64}$$

Using a co-moving coordinate system, with angular velocity $\boldsymbol{\Omega}$ and its origin located at the point (G or P), the angular momentum has the analytical expression

$$\mathbf{H} = H_x\hat{\mathbf{i}} + H_y\hat{\mathbf{j}} + H_z\hat{\mathbf{k}} \tag{9.65}$$

We shall henceforth assume, for simplicity, that

(a) The moving xyz axes are the principal axes of inertia, and (9.66a)

(b) The moments of inertia relative to xyz are constant in time. (9.66b)

Equations 9.42 and 9.66a imply that

$$\mathbf{H} = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{9.67}$$

where A, B and C are the principal moments of inertia.

According to Equation 1.56, the time derivative of $\mathbf{H}$ is $\dot{\mathbf{H}} = \dot{\mathbf{H}})_{\text{rel}} + \boldsymbol{\Omega}\times\mathbf{H}$, so that Equation 9.64 can be written

$$\mathbf{M}_{\text{net}} = \dot{\mathbf{H}})_{\text{rel}} + \boldsymbol{\Omega}\times\mathbf{H} \tag{9.68}$$

Keep in mind that, whereas $\boldsymbol{\Omega}$ (the angular velocity of the moving xyz coordinate system) and $\boldsymbol{\omega}$ (the angular velocity of the rigid body itself) are both absolute kinematic quantities, Equation 9.68 contains their components as projected onto the axes of the noninertial xyz frame,

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$$
$$\boldsymbol{\Omega} = \Omega_x\hat{\mathbf{i}} + \Omega_y\hat{\mathbf{j}} + \Omega_z\hat{\mathbf{k}}$$

The absolute angular acceleration $\boldsymbol{\alpha}$ is obtained using Equation 1.56

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \overbrace{\frac{d\omega_x}{dt}\hat{\mathbf{i}} + \frac{d\omega_y}{dt}\hat{\mathbf{j}} + \frac{d\omega_z}{dt}\hat{\mathbf{k}}}^{\boldsymbol{\alpha}_{\text{rel}}} + \boldsymbol{\Omega}\times\boldsymbol{\omega}$$

that is,

$$\boldsymbol{\alpha} = (\dot{\omega}_x + \Omega_y\omega_z - \Omega_z\omega_y)\hat{\mathbf{i}} + (\dot{\omega}_y + \Omega_z\omega_x - \Omega_x\omega_z)\hat{\mathbf{j}} + (\dot{\omega}_z + \Omega_x\omega_y - \Omega_y\omega_x)\hat{\mathbf{k}} \tag{9.69}$$

Clearly, it is generally true that

$$\alpha_x \neq \dot{\omega}_x \quad \alpha_y \neq \dot{\omega}_y \quad \alpha_z \neq \dot{\omega}_z$$

From Equations 1.66 and 9.67

$$\dot{\mathbf{H}})_{\text{rel}} = \frac{d(A\omega_x)}{dt}\hat{\mathbf{i}} + \frac{d(B\omega_y)}{dt}\hat{\mathbf{j}} + \frac{d(C\omega_z)}{dt}\hat{\mathbf{k}}$$

Since A, B and C are constant, this becomes

$$\dot{\mathbf{H}})_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{9.70}$$

Substituting Equations 9.67 and 9.70 into Equation 9.68 yields

$$\mathbf{M}_{\text{net}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Omega_x & \Omega_y & \Omega_z \\ A\omega_x & B\omega_y & C\omega_z \end{vmatrix}$$

Expanding the cross product and collecting terms leads to

$$\begin{aligned} M_{x_{\text{net}}} &= A\dot{\omega}_x + C\Omega_y\omega_z - B\Omega_z\omega_y \\ M_{y_{\text{net}}} &= B\dot{\omega}_y + A\Omega_z\omega_x - C\Omega_x\omega_z \\ M_{z_{\text{net}}} &= C\dot{\omega}_z + B\Omega_x\omega_y - A\Omega_y\omega_x \end{aligned} \tag{9.71}$$

If the co-moving frame is a rigidly attached body frame, then its angular velocity is the same as that of the body, i.e., $\mathbf{\Omega} = \boldsymbol{\omega}$. In that case, Equations 9.68 reduce to the classical ***Euler equations*** of motion,

$$\mathbf{M}_{\text{net}} = \dot{\mathbf{H}})_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H} \tag{9.72a}$$

the three components of which are obtained from Equation 9.71,

$$\begin{aligned} M_{x_{\text{net}}} &= A\dot{\omega}_x + (C - B)\omega_y\omega_z \\ M_{y_{\text{net}}} &= B\dot{\omega}_y + (A - C)\omega_z\omega_x \\ M_{z_{\text{net}}} &= C\dot{\omega}_z + (B - A)\omega_x\omega_y \end{aligned} \tag{9.72b}$$

Equations 9.71 are sometimes referred to as the ***modified Euler equations***.

For a body-fixed frame, $\mathbf{\Omega} = \boldsymbol{\omega}$. It follows from Equation 9.69 that

$$\dot{\omega}_x = \alpha_x \quad \dot{\omega}_y = \alpha_y \quad \dot{\omega}_z = \alpha_z \tag{9.73}$$

That is, in a body-fixed frame, the relative angular acceleration equals the absolute angular acceleration. Rather than calculating the time derivatives $\dot{\omega}_x$, $\dot{\omega}_y$and $\dot{\omega}_z$ for use in Equation 9.72, we may in this case first compute $\boldsymbol{\alpha}$ in the absolute XYZ frame

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = \frac{d\omega_X}{dt}\hat{\mathbf{I}} + \frac{d\omega_Y}{dt}\hat{\mathbf{J}} + \frac{d\omega_Z}{dt}\hat{\mathbf{K}}$$

and then project these components onto the xyz body frame, so that

$$\begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} = [\mathbf{Q}]_{Xx} \begin{Bmatrix} d\omega_X/dt \\ d\omega_Y/dt \\ d\omega_Z/dt \end{Bmatrix} \tag{9.74}$$

where $[\mathbf{Q}]_{Xx}$ is the time-dependent orthogonal transformation from the inertial XYZ frame to the noninertial xyz frame.

Example 9.12

Calculate the net moment on the solar panel of Examples 9.2 and 9.8.

Solution

Since the co-moving frame is rigidly attached to the panel, Euler's equation (Equation 9.72a), applies to this problem,

$$\mathbf{M}_{G_{\text{net}}} = \dot{\mathbf{H}}_G)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_G \tag{a}$$

where

$$\mathbf{H}_G = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{b}$$

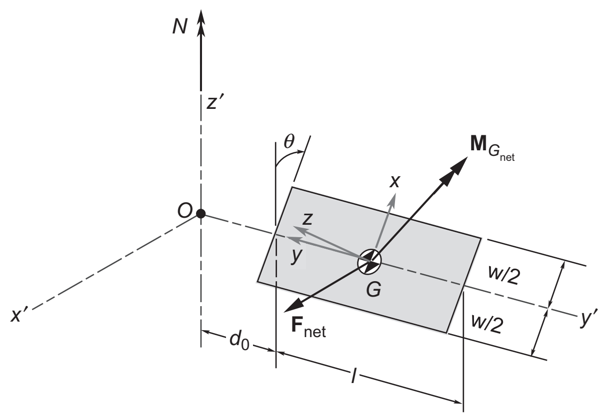


FIGURE 9.17

Free-body diagram of the solar panel in Examples 9.2 and 9.8.

and

$$\dot{\mathbf{H}}_G)_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{c}$$

In Example 9.2, the angular velocity of the panel in the satellite's $x'y'z'$ frame was found to be

$$\boldsymbol{\omega} = -\dot{\theta}\hat{\mathbf{j}}' + N\hat{\mathbf{k}}' \tag{d}$$

In Example 9.8 we showed that the transformation from the panel's xyz frame to that of the satellite is represented by the matrix

$$[\mathbf{Q}] = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix} \tag{e}$$

We use the transpose of $[\mathbf{Q}]$ to transform the components of $\boldsymbol{\omega}$ into the panel frame of reference,

$$\{\boldsymbol{\omega}\}_{xyz} = [\mathbf{Q}]^T\{\boldsymbol{\omega}\}_{x'y'z'} = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix}\begin{Bmatrix} 0 \\ -\dot{\theta} \\ N \end{Bmatrix} = \begin{Bmatrix} N\cos\theta \\ \dot{\theta} \\ N\sin\theta \end{Bmatrix}$$

or

$$\omega_x = N\cos\theta \qquad \omega_y = \dot{\theta} \qquad \omega_z = N\sin\theta \tag{f}$$

In Example 9.2, N and $\dot{\theta}$ were said to be constant. Therefore, the time derivatives of (f) are

$$\begin{aligned} \dot{\omega}_x &= \frac{d(N\cos\theta)}{dt} = -N\dot{\theta}\sin\theta \\ \dot{\omega}_y &= \frac{d\dot{\theta}}{dt} = 0 \\ \dot{\omega}_z &= \frac{d(N\sin\theta)}{dt} = N\dot{\theta}\cos\theta \end{aligned} \tag{g}$$

In Example 9.8 the moments of inertia in the panel frame of reference were listed as

$$\begin{gathered} A = \frac{1}{12}m(l^2 + t^2) \qquad B = \frac{1}{12}m(w^2 + t^2) \qquad C = \frac{1}{12}m(w^2 + l^2) \\ (I_{Gxy} = I_{Gxz} = I_{Gyz} = 0) \end{gathered} \tag{h}$$

Substituting (b), (c), (f), (g) and (h) into (a) yields,

$$\mathbf{M}_{G_{\text{net}}} = \tfrac{1}{12}m(l^2 + t^2)(-N\dot{\theta}\sin\theta)\hat{\mathbf{i}} + \tfrac{1}{12}m(w^2 + t^2)\cdot 0\cdot\hat{\mathbf{j}} + \tfrac{1}{12}m(w^2 + l^2)(N\dot{\theta}\cos\theta)\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ N\cos\theta & \dot{\theta} & N\sin\theta \\ \tfrac{1}{12}m(l^2 + t^2)(N\cos\theta) & \tfrac{1}{12}m(w^2 + t^2)\dot{\theta} & \tfrac{1}{12}m(w^2 + l^2)(N\sin\theta) \end{vmatrix}$$

Upon expanding the cross product and collecting terms, this reduces to

$$\mathbf{M}_{G\text{net}} = -\tfrac{1}{6}mt^2N\dot{\theta}\sin\theta\,\hat{\mathbf{i}} + \tfrac{1}{24}m(t^2 - w^2)N^2\sin 2\theta\,\hat{\mathbf{j}} + \tfrac{1}{6}mw^2N\dot{\theta}\cos\theta\,\hat{\mathbf{k}}$$

Using the numerical data of Example 9.8 ($m = 50\,\text{kg}$, $N = 0.1$ rad/s, $\theta = 40°$, $\dot{\theta} = 0.01$ rad/s, $l = 6\,\text{m}$, $w = 2\,\text{m}$ and $t = 0.025\,\text{m}$), we find

$$\boxed{\mathbf{M}_{G\text{net}} = -3.348\times 10^{-6}\,\hat{\mathbf{i}} - 0.08205\,\hat{\mathbf{j}} + 0.02554\,\hat{\mathbf{k}}\ (\text{N}\cdot\text{m})}$$

Example 9.13

Calculate the net moment on the gyro rotor of Examples 9.3 and 9.6.

Solution

Figure 9.18 is a free-body diagram of the rotor. Since in this case the co-moving frame is not rigidly attached to the rotor, we must use Equation 9.68 to find the net moment about G,

$$\mathbf{M}_{G_{\text{net}}} = \dot{\mathbf{H}}_G)_{\text{rel}} + \boldsymbol{\Omega}\times\mathbf{H}_G \tag{a}$$

where

$$\mathbf{H}_G = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{b}$$

and

$$\dot{\mathbf{H}}_G)_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{c}$$

From Equation (f) of Example 9.3 we know that the components of the angular velocity of the rotor in the moving reference frame are

$$\begin{aligned} \omega_x &= \dot{\theta} \\ \omega_y &= N\sin\theta \\ \omega_z &= \omega_{\text{spin}} + N\cos\theta \end{aligned} \tag{d}$$

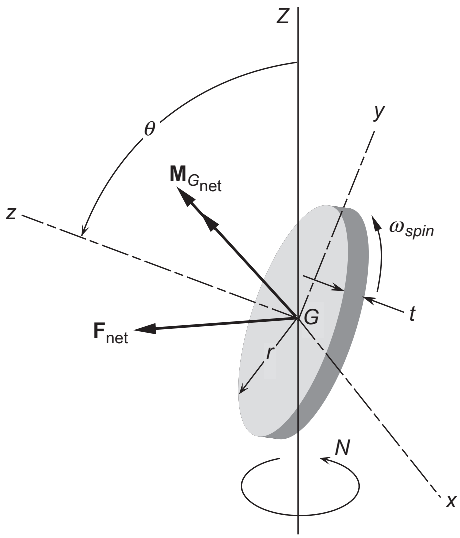


FIGURE 9.18

Free-body diagram of the gyro rotor of Examples 9.6 and 9.3.

Since, as specified in Example 9.3, $\dot{\theta}$, N and ω_{spin} are all constant, it follows that

$$\begin{aligned}
\dot{\omega}_x &= \frac{d\dot{\theta}}{dt} = 0 \\
\dot{\omega}_y &= \frac{d(N\sin\theta)}{dt} = N\dot{\theta}\cos\theta \\
\dot{\omega}_z &= \frac{d(\omega_{\text{spin}} + N\cos\theta)}{dt} = -N\dot{\theta}\sin\theta
\end{aligned} \tag{e}$$

The angular velocity $\mathbf{\Omega}$ of the co-moving xyz frame is that of the gimbal ring, which equals the angular velocity of the rotor minus its spin. Therefore,

$$\begin{aligned}
\Omega_x &= \dot{\theta} \\
\Omega_y &= N\sin\theta \\
\Omega_z &= N\cos\theta
\end{aligned} \tag{f}$$

In Example 9.6 we found that

$$\begin{aligned}
A = B &= \tfrac{1}{12}mt^2 + \tfrac{1}{4}mr^2 \\
C &= \tfrac{1}{2}mr^2
\end{aligned} \tag{g}$$

Substituting (b) through (g) into (a), we get

$$\mathbf{M}_{G_{\text{net}}} = \left(\tfrac{1}{12}mt^2 + \tfrac{1}{4}mr^2\right)\cdot 0\hat{\mathbf{i}} + \left(\tfrac{1}{12}mt^2 + \tfrac{1}{4}mr^2\right)(N\dot{\theta}\cos\theta)\hat{\mathbf{j}} + \tfrac{1}{2}mr^2(-N\dot{\theta}\sin\theta)\hat{\mathbf{k}}$$
$$+ \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{\theta} & N\sin\theta & N\cos\theta \\ \left(\tfrac{1}{12}mt^2 + \tfrac{1}{4}mr^2\right)\dot{\theta} & \left(\tfrac{1}{12}mt^2 + \tfrac{1}{4}mr^2\right)N\sin\theta & \tfrac{1}{2}mr^2(\omega_{spin} + N\cos\theta) \end{vmatrix}$$

Expanding the cross product, collecting terms, and simplifying leads to

$$\begin{aligned} \mathbf{M}_{G_{\text{net}}} &= \left[\tfrac{1}{2}\omega_{spin} + \tfrac{1}{12}\left(3 - \frac{t^2}{r^2}\right)N\cos\theta\right]mr^2N\sin\theta\hat{\mathbf{i}} \\ &+ \left(\tfrac{1}{6}\frac{t^2}{r^2}N\cos\theta - \tfrac{1}{2}\omega_{spin}\right)mr^2\dot{\theta}\hat{\mathbf{j}} - \tfrac{1}{2}N\dot{\theta}\sin\theta mr^2\hat{\mathbf{k}} \end{aligned} \tag{h}$$

In Example 9.3 the following numerical data were provided: $m = 5\,\text{kg}$, $r = 0.08\,\text{m}$, $t = 0.025\,\text{m}$, $N = 2.1\,\text{rad/s}$, $\theta = 60°$, $\dot{\theta} = 4$ rad/s and $\omega_{spin} = 105\,\text{rad/s}$. For this set of numbers, (h) becomes

$$\boxed{\mathbf{M}_{G_{\text{net}}} = 0.3203\hat{\mathbf{i}} - 0.6698\hat{\mathbf{j}} - 0.1164\hat{\mathbf{k}}\ (\text{N}\cdot\text{m})}$$

9.7 KINETIC ENERGY

The kinetic energy T of a rigid body is the integral of the kinetic energy $\frac{1}{2}v^2dm$ of its individual mass elements,

$$T = \int_m \frac{1}{2}v^2dm = \int_m \frac{1}{2}\mathbf{v}\cdot\mathbf{v}dm \tag{9.75}$$

where $\mathbf{v}$ is the absolute velocity $\dot{\mathbf{R}}$ of the element of mass dm. From Figure 9.8 we infer that $\dot{\mathbf{R}} = \dot{\mathbf{R}}_G + \dot{\rho}$. Furthermore, Equation 1.52 requires that $\dot{\boldsymbol{\rho}} = \boldsymbol{\omega}\times\boldsymbol{\rho}$. Thus, $\mathbf{v} = \mathbf{v}_G + \boldsymbol{\omega}\times\boldsymbol{\rho}$, which means

$$\mathbf{v}\cdot\mathbf{v} = [\mathbf{v}_G + \boldsymbol{\omega}\times\boldsymbol{\rho}]\cdot[\mathbf{v}_G + \boldsymbol{\omega}\times\boldsymbol{\rho}] = v_G{}^2 + 2\mathbf{v}_G\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}) + (\boldsymbol{\omega}\times\boldsymbol{\rho})\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho})$$

We can apply the vector identity introduced in Equation 1.21,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) \tag{9.76}$$

to the last term to get

$$\mathbf{v}\cdot\mathbf{v} = v_G^{\;2} + 2\mathbf{v}_G\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}) + \boldsymbol{\omega}\cdot[\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})]$$

Therefore, Equation 9.75 becomes

$$T = \int_m \frac{1}{2}v_G^{\;2}dm + \mathbf{v}_G\cdot\left(\boldsymbol{\omega}\times\int_m \boldsymbol{\rho}dm\right) + \frac{1}{2}\boldsymbol{\omega}\cdot\int_m \boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})dm$$

Since $\boldsymbol{\rho}$ is measured from the center of mass, $\int_m \boldsymbol{\rho}dm = \mathbf{0}$. Recall that, according to Equation 9.34,

$$\int_m \boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})\,dm = \mathbf{H}_G$$

It follows that the kinetic energy may be written

$$T = \frac{1}{2}mv_G^{\;2} + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G \tag{9.77}$$

The second term is the ***rotational kinetic energy*** T_R,

$$T_R = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G \tag{9.78}$$

If the body is rotating about a point P which is at rest in inertial space, we have from Equation 9.2 and Figure 9.8 that

$$\mathbf{v}_G = \mathbf{v}_P + \boldsymbol{\omega}\times\mathbf{r}_{G/P} = \mathbf{0} + \boldsymbol{\omega}\times\mathbf{r}_{G/P} = \boldsymbol{\omega}\times\mathbf{r}_{G/P}$$

It follows that

$$v_G^{\;2} = \mathbf{v}_G\cdot\mathbf{v}_G = (\boldsymbol{\omega}\times\mathbf{r}_{G/P})\cdot(\boldsymbol{\omega}\times\mathbf{r}_{G/P})$$

Making use once again of the vector identity in Equation 9.76, we find

$$v_G^{\;2} = \boldsymbol{\omega}\cdot[\mathbf{r}_{G/P}\times(\boldsymbol{\omega}\times\mathbf{r}_{G/P})] = \boldsymbol{\omega}\cdot(\mathbf{r}_{G/P}\times\mathbf{v}_G)$$

Substituting this into Equation 9.77 yields

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot[\mathbf{H}_G + \mathbf{r}_{G/P}\times m\mathbf{v}_G]$$

Equation 9.21 shows that this can be written

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_P \tag{9.79}$$

In this case, of course, all of the kinetic energy is rotational.

In terms of the components of $\boldsymbol{\omega}$ and $\mathbf{H}$, whether it is $\mathbf{H}_P$ or $\mathbf{H}_G$, the rotational kinetic energy expression becomes, with the aid of Equation 9.39,

$$T_R = \frac{1}{2}(\omega_x H_x + \omega_y H_y + \omega_y H_z) = \frac{1}{2}\lfloor \omega_x \quad \omega_y \quad \omega_z \rfloor \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

Expanding, we obtain

$$T_R = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z \tag{9.80}$$

Obviously, if the *xyz* axes are principal axes of inertia, then Equation 9.80 simplifies considerably,

$$T_R = \frac{1}{2}A\omega_x^2 + \frac{1}{2}B\omega_y^2 + \frac{1}{2}C\omega_z^2 \tag{9.81}$$

Example 9.14

A satellite in circular geocentric orbit of 300 km altitude has a mass of 1500 kg, and the moments of inertia relative to a body frame with origin at the center of mass G are

$$[\mathbf{I}] = \begin{bmatrix} 2000 & -1000 & 2500 \\ -1500 & 3000 & -1500 \\ 2500 & -1500 & 4000 \end{bmatrix} (\text{kg} \cdot \text{m}^2)$$

If at a given instant the components of angular velocity in this frame of reference are

$$\boldsymbol{\omega} = 1\hat{\mathbf{i}} - 0.9\hat{\mathbf{j}} + 1.5\hat{\mathbf{k}} \text{ (rad/s)}$$

calculate the total kinetic energy of the satellite.

Solution

The speed of the satellite in its circular orbit is

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398{,}600}{6378 + 300}} = 7.7258 \text{ km/s}$$

The angular momentum of the satellite is

$$\{\mathbf{H}_G\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\} = \begin{bmatrix} 2000 & -1000 & 2500 \\ -1500 & 3000 & -1500 \\ 2500 & -1500 & 4000 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.9 \\ 1.5 \end{Bmatrix} = \begin{Bmatrix} 6650 \\ -5950 \\ 9850 \end{Bmatrix} (\text{kg} \cdot \text{m}^2/\text{s})$$

Therefore, the total kinetic energy is

$$T = \frac{1}{2}mv_G{}^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G = \frac{1}{2}\cdot 1500\cdot 7725.8^2 + \frac{1}{2}\begin{bmatrix}1 & -0.9 & 1.5\end{bmatrix}\begin{Bmatrix}6650\\ -5950\\ 9850\end{Bmatrix} = 44.766\times 10^6 + 13\,390$$

$$\boxed{T = 44.766\ \text{MJ}}$$

Obviously, the kinetic energy is dominated by that due to the orbital motion.

9.8 THE SPINNING TOP

Let us analyze the motion of the simple axisymmetric top in Figure 9.19. It is constrained to rotate about point O.

The moving coordinate system is chosen to have its origin at O. The z axis is aligned with the spin axis of the top (the axis of rotational symmetry). The x axis is the node line, which passes through O and is perpendicular to the plane defined by the inertial Z axis and the spin axis of the top. The y axis is then perpendicular to x and z, such that $\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{i}}$. By symmetry, the moment of inertia matrix of the top relative to the xyz frame is diagonal, with $I_x = I_y = A$ and $I_Z = C$. From Equations 9.68 and 9.70, we have

$$\mathbf{M}_{0_{\text{net}}} = A\dot{\omega}_x\hat{\mathbf{i}} + A\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} + \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \Omega_x & \Omega_y & \Omega_z\\ A\omega_x & A\omega_y & C\omega_z\end{vmatrix} \tag{9.82}$$

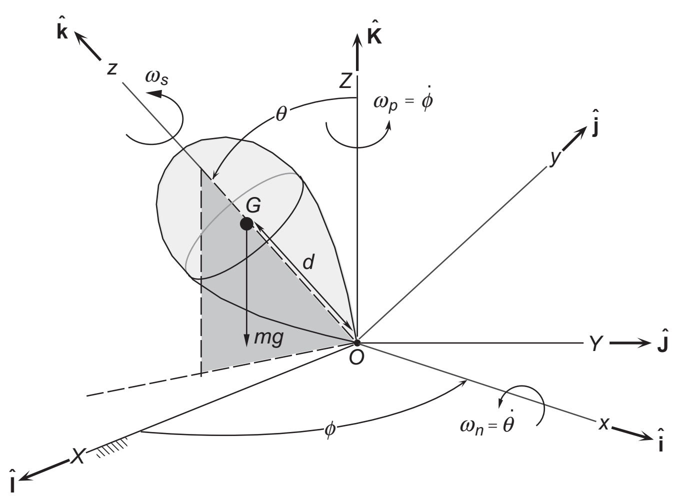


FIGURE 9.19

Simple top rotating about the fixed point O.

The angular velocity $\boldsymbol{\omega}$ of the top is the vector sum of the spin rate ω_s and the rates of precession ω_p and nutation ω_n, where

$$\omega_p = \dot{\phi} \quad \omega_n = \dot{\theta} \tag{9.83}$$

Thus,

$$\boldsymbol{\omega} = \omega_n \hat{\mathbf{i}} + \omega_p \hat{\mathbf{K}} + \omega_s \hat{\mathbf{k}}$$

From the geometry we see that

$$\hat{\mathbf{K}} = \sin\theta \hat{\mathbf{j}} + \cos\theta \hat{\mathbf{k}} \tag{9.84}$$

Therefore, relative to the co-moving system,

$$\boldsymbol{\omega} = \omega_n \hat{\mathbf{i}} + \omega_p \sin\theta \hat{\mathbf{j}} + (\omega_s + \omega_p \cos\theta)\hat{\mathbf{k}} \tag{9.85}$$

It follows from this equation that

$$\omega_x = \omega_n \qquad \omega_y = \omega_p \sin\theta \qquad \omega_z = \omega_s + \omega_p \tag{9.86}$$

Computing the time rates of these three expressions yields the components of angular acceleration relative to the xyz frame,

$$\dot{\omega}_x = \dot{\omega}_n \qquad \dot{\omega}_y = \dot{\omega}_p \sin\theta + \omega_p \omega_n \cos\theta \qquad \dot{\omega}_z = \dot{\omega}_s + \dot{\omega}_p \cos\theta - \omega_p \omega_n \sin\theta \tag{9.87}$$

The angular velocity $\boldsymbol{\Omega}$ of the xyz system is $\boldsymbol{\Omega} = \omega_p \hat{\mathbf{K}} + \omega_n \hat{\mathbf{i}}$, so that, using Equation 9.84,

$$\boldsymbol{\Omega} = \omega_n \hat{\mathbf{i}} + \omega_p \sin\theta \hat{\mathbf{j}} + \omega_p \cos\theta \hat{\mathbf{k}} \tag{9.88}$$

From this we obtain

$$\Omega_x = \omega_n \qquad \Omega_y = \omega_p \sin\theta \qquad \Omega_z = \omega_p \cos\theta \tag{9.89}$$

The moment about O in Figure 9.19 is that of the weight vector acting through the center of mass G:

$$\mathbf{M}_{0_{\text{net}}} = (d\hat{\mathbf{k}}) \times (-mg\hat{\mathbf{K}}) = -mgd\hat{\mathbf{k}} \times (\sin\theta \hat{\mathbf{j}} + \cos\theta \hat{\mathbf{k}})$$

or

$$\mathbf{M}_{0_{\text{net}}} = mgd \sin\theta \hat{\mathbf{i}} \tag{9.90}$$

Substituting Equations 9.86, 9.87, 9.89 and 9.90 into Equation 9.82, we get

$$mgd\sin\theta\hat{\mathbf{i}} = A\dot{\omega}_n\hat{\mathbf{i}} + A(\dot{\omega}_p\sin\theta + \omega_p\omega_n\cos\theta)\hat{\mathbf{j}} + C(\dot{\omega}_s + \dot{\omega}_p\cos\theta - \omega_p\omega_n\sin\theta)\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_n & \omega_p\sin\theta & \omega_p\cos\theta \\ A\omega_n & A\omega_p\sin\theta & C(\omega_s + \omega_p\cos\theta) \end{vmatrix} \tag{9.91}$$

Let us consider the special case in which θ is constant, i.e., there is no nutation, so that $\omega_n = \dot{\omega}_n = 0$. Then Equation 9.91 reduces to

$$mgd\sin\theta\hat{\mathbf{i}} = A\dot{\omega}_p\sin\theta\hat{\mathbf{j}} + C(\dot{\omega}_s + \dot{\omega}_p\cos\theta)\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & \omega_p\sin\theta & \omega_p\cos\theta \\ 0 & A\omega_p\sin\theta & C(\omega_s + \omega_p\cos\theta) \end{vmatrix} \tag{9.92}$$

Expanding the determinant yields

$$mgd\sin\theta\hat{\mathbf{i}} = A\dot{\omega}_p\sin\theta\hat{\mathbf{j}} + C(\dot{\omega}_s + \dot{\omega}_p\cos\theta)\hat{\mathbf{k}} + [C\omega_p\omega_s\sin\theta + (C - A)\omega_p{}^2\cos\theta\sin\theta]\hat{\mathbf{i}}$$

Equating the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ on each side of the equation and assuming that $0 < \theta < 180°$ leads to

$$mgd = C\omega_p\omega_s + (C - A)\omega_p{}^2\cos\theta \tag{9.93a}$$

$$A\dot{\omega}_p = 0 \tag{9.93b}$$

$$C(\dot{\omega}_s + \dot{\omega}_p\cos\theta) = 0 \tag{9.93c}$$

Equation 9.93b implies $\dot{\omega}_p = 0$, and from Equation 9.93c it follows that $\dot{\omega}_s = 0$. Therefore, the rates of spin and precession are both constant. From Equation 9.93a we find

$$(A - C)\cos\theta\omega_p{}^2 - C\omega_s\omega_p + mgd = 0 \tag{9.94}$$

If the spin rate is zero, Equation 9.94 yields

$$\omega_p)_{\omega_s=0} = \pm\sqrt{\frac{mgd}{(C - A)\cos\theta}} \qquad \text{if} \quad (C - A)\cos\theta > 0 \tag{9.95}$$

In this case, the top rotates about O at this rate, without spinning. If $A > C$ (prolate), its symmetry axis must make an angle between 90° and 180° to the vertical; otherwise ω_p is imaginary. On the other hand, if $A < C$ (oblate), the angle lies between 0° and 90°. Thus, in steady rotation without spin, the top's axis sweeps out a cone that lies either below the horizontal plane ($A > C$) or above the plane ($A < C$).

In the special case $(A - C)\cos\theta = 0$, Equation 9.94 yields a steady precession rate that is inversely proportional to the spin rate,

$$\omega_p = \frac{mgd}{C\omega_s} \qquad \text{if } (A - C)\cos\theta = 0 \tag{9.96}$$

If $A = C$, this precession apparently occurs irrespective of tilt angle θ. If $A \neq C$, this rate of precession occurs at $\theta = 90°$, i.e., the spin axis is perpendicular to the precession axis.

In general, Equation 9.94 is a quadratic equation in ω_p, so we can use the quadratic formula to find

$$\omega_p = \frac{C}{2(A - C)\cos\theta}\left(\omega_s \pm \sqrt{\omega_s^2 - \frac{4mgd(A - C)\cos\theta}{C^2}}\right) \tag{9.97}$$

Thus, for a given spin rate and tilt angle θ ($\theta \neq 90°$), there are two rates of precession $\dot{\phi}$.

Observe that if $(A - C)\cos\theta > 0$, then ω_p is imaginary when $\omega_p^2 < 4mgd(A - C)\cos\theta/C^2$. Therefore, the minimum spin rate required for steady precession at a constant inclination θ is

$$\omega_s)_{\text{min}} = \frac{2}{C}\sqrt{mgd(A - C)\cos\theta} \qquad \text{if } (A - C)\cos\theta > 0 \tag{9.98}$$

If $(A - C)\cos\theta < 0$, the radical in Equation 9.97 is real for all ω_s. In this case, as $\omega_s \rightarrow 0$, ω_p approaches the value given above in Equation 9.95.

Example 9.15

Calculate the precession rate ω_p for the top of Figure 9.19 if $m = 0.5\,\text{kg}$, $A\ (=I_x = I_y) = 12 \times 10^{-4}\,\text{kg}\cdot\text{m}^2$, $C\ (=I_z) = 4.5 \times 10^{-4}\,\text{kg m}^2$ and $d = 0.05\,\text{m}$.

Solution

For an inclination of, say, 60°, $(A - C)\cos\theta > 0$, so that Equation 9.98 requires $\omega_s)_{\text{min}} = 407.01$ rpm. Let us choose the spin rate to be $\omega_s = 1000$ rpm $= 104.7$ rad/sec. Then, from Equation 9.97, the precession rate as a function of the inclination θ is given by either one of the following formulas

$$\omega_p = 31.42\frac{1 + \sqrt{1 - 0.3312\cos\theta}}{\cos\theta} \qquad \text{and} \qquad \omega_p = 31.42\frac{1 - \sqrt{1 - 0.3312\cos\theta}}{\cos\theta} \tag{a}$$

These are plotted in Figures 9.20.

Figure 9.21 shows an axisymmetric rotor mounted so that its spin axis (z) remains perpendicular to the precession axis (y). In that case Equation 9.85 with $\theta = 90°$ yields

$$\boldsymbol{\omega} = \omega_p\hat{\mathbf{j}} + \omega_s\hat{\mathbf{i}} \tag{9.99}$$

Likewise, from Equation 9.88, the angular velocity of the co-moving xyz system is $\boldsymbol{\Omega} = \omega_p\hat{\mathbf{j}}$. If we assume that the spin rate and precession rate are constant ($d\omega_p/dt = d\omega_s/dt = 0$), then Equation 9.68, written for the center of mass G, becomes

$$\mathbf{M}_{G_{\text{net}}} = \boldsymbol{\Omega} \times \mathbf{H} = (\omega_p\hat{\mathbf{j}}) \times (A\omega_p\hat{\mathbf{j}} + C\omega_s\hat{\mathbf{k}}) \tag{9.100}$$

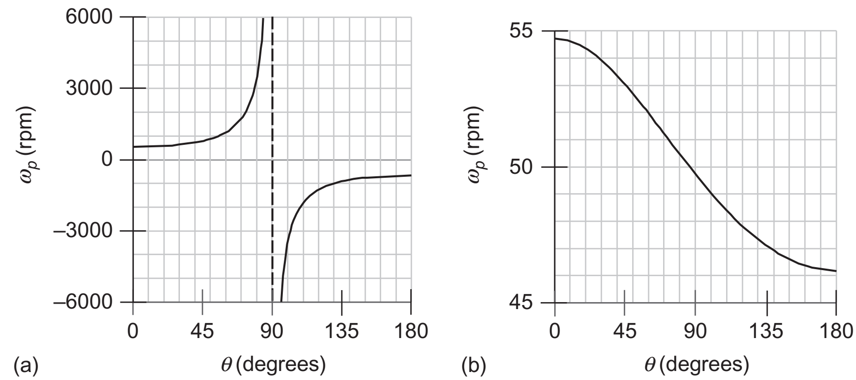


FIGURE 9.20

(a) High-energy precession rate (unlikely to be observed); (b) Low energy precession rate (the one most always seen).

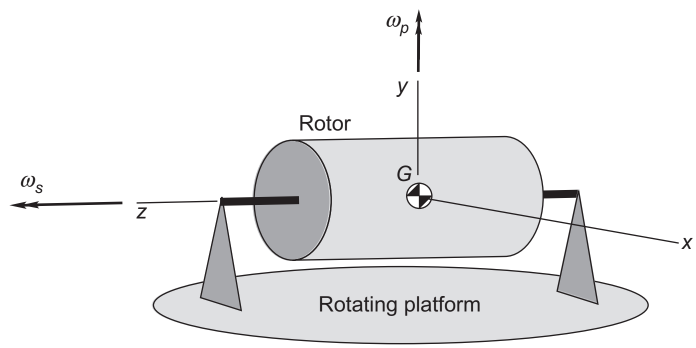


FIGURE 9.21

A spinning rotor on a rotating platform.

where A and C are the moments of inertia of the rotor about the x and z axes, respectively. Setting $C\omega_s\hat{\mathbf{k}} = \mathbf{H}_s$, the spin angular momentum, and $\omega_p\hat{\mathbf{j}} = \omega_p$, we obtain the ***gyroscopic moment***

$$\mathbf{M}_{G_{\text{net}}} = \boldsymbol{\omega}_p \times \mathbf{H}_s \qquad (\mathbf{H}_s = C\omega_s\hat{\mathbf{k}}) \tag{9.101}$$

Since the center of mass G is the reference point, there is no restriction on the motion for which Equation 9.101 is valid. Observe that the net gyroscopic moment $\mathbf{M}_{G_{\text{net}}}$ exerted on the rotor by its supports is perpendicular to the plane of the spin and precession vectors. If a spinning rotor is forced to precess, the gyroscopic moment $\mathbf{M}_{G_{\text{net}}}$ develops. Or, if a moment is applied normal to the spin axis of a rotor, it will precess so as to cause the spin axis to turn towards the moment axis.

Example 9.16

A uniform cylinder of radius r, length L and mass m spins at a constant angular velocity ω_s. It rests on simple supports (which cannot exert couples), mounted on a platform that rotates at an angular velocity of ω_p. Find the reactions at A and B. Neglect the weight (i.e., calculate the reactions due just to gyroscopic effects).

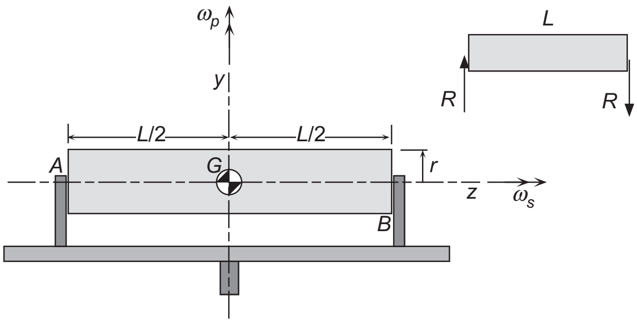


FIGURE 9.22

Illustration of the gyroscopic effect.

Solution

The net vertical force on the cylinder is zero, so the reactions at each end must be equal and opposite in direction, as shown on the free-body diagram insert in Figure 9.22. Noting that the moment of inertia of a uniform cylinder about its axis of rotational symmetry is $\frac{1}{2}mr^2$, Equation 9.101 yields

$$RL\hat{\mathbf{i}} = (\omega_p\hat{\mathbf{j}}) \times \left(\frac{1}{2}mr^2\omega_s\hat{\mathbf{k}}\right) = \frac{1}{2}mr^2\omega_p\omega_s\hat{\mathbf{i}}$$

so that

$$\boxed{R = \frac{mr^2\omega_p\omega_s}{2L}}$$

9.9 EULER ANGLES

Three angles are required to specify the orientation of a rigid body relative to an inertial frame. The choice is not unique, but there are two sets in common use: the Euler angles and the yaw, pitch and roll angles. We will discuss each of them in turn. The reader is urged to review Section 4.5 on orthogonal coordinate transformations and, in particular, the discussion of Euler angle sequences.

The three Euler angles ϕ, θ and ψ shown in Figure 9.23 give the orientation of a body-fixed *xyz* frame of reference relative to the *XYZ* inertial frame of reference. The *xyz* frame is obtained from the *XYZ* frame by a sequence of rotations through each of the Euler angles in turn. The first rotation is around the Z ($=z_1$) axis through the precession angle ϕ. This takes X into x_1 and Y into y_1. The second rotation is around the x_2 ($=x_1$) axis through the nutation angle θ. This carries y_1 and z_1 into y_2 and z_2, respectively. The third and final rotation is around the z ($=z_2$) axis through the spin angle ψ, which takes x_2 into x and y_2 into y.

The direction cosine matrix $[\mathbf{Q}]_{Xx}$ of the transformation from the inertial frame to the body-fixed frame is given by the classical Euler angle sequence, Equation 4.37:

$$[\mathbf{Q}]_{Xx} = [\mathbf{R}_3(\psi)][\mathbf{R}_1(\theta)][\mathbf{R}_3(\phi)] \tag{9.102}$$

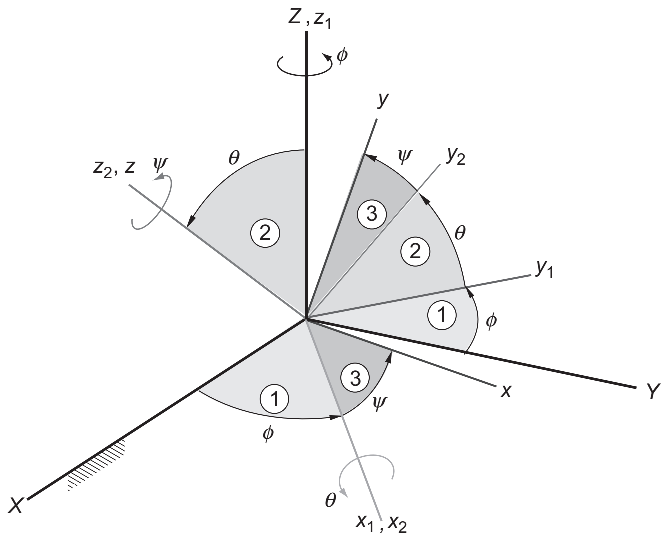


FIGURE 9.23

Classical Euler angle sequence. See also Figure 4.14.

From Equations (4.32) and (4.34) we have

$$[\mathbf{R}_3(\psi)] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\mathbf{R}_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad [\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.103}$$

According to Equation 4.38, the direction cosine matrix is

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \sin\theta\sin\psi \\ -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{bmatrix} \tag{9.104}$$

Since this is an orthogonal matrix, the inverse transformation from *xyz* to *XYZ* is

$$[\mathbf{Q}]_{xX} = ([\mathbf{Q}]_{Xx})^T,$$

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \sin\phi\sin\theta \\ \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \tag{9.105}$$

Algorithm 4.3 is used to find the three Euler angles θ,ϕ,ψ from a given direction cosine matrix $[\mathbf{Q}]_{Xx}$.

Example 9.17

The direction cosine matrix of an orthogonal transformation from *XYZ* to *xyz* is

$$[\mathbf{Q}] = \begin{bmatrix} -0.32175 & 0.89930 & -0.29620 \\ 0.57791 & -0.061275 & -0.81380 \\ -0.75000 & -0.43301 & -0.5000 \end{bmatrix}$$

Use Algorithm 4.3 to find the Euler angles ϕ, θ and ψ for this transformation.

Solution

Step 1 (precession angle):

$$\phi = \tan^{-1}\left(\frac{Q_{31}}{-Q_{32}}\right) = \tan^{-1}\left(\frac{-0.75000}{0.43301}\right) \qquad (0 \le \phi < 360°)$$

Since the numerator is negative and the denominator is positive, the angle ϕ lies in the fourth quadrant.

$$\boxed{\phi = \tan^{-1}(-1.7320) = 300°}$$

Step 2 (nutation angle):

$$\theta = \cos^{-1} Q_{33} = \cos^{-1}(-0.5000) \quad (0 \le \theta \le 180°)$$

$$\boxed{\theta = 120°}$$

Step 3 (spin angle):

$$\psi = \tan^{-1}\frac{Q_{13}}{Q_{23}} = \tan^{-1}\left(\frac{-0.29620}{-0.81380}\right) \qquad (0 \le \psi < 360°)$$

Since both the numerator and denominator are negative, the angle ψ lies in the third quadrant.

$$\boxed{\psi = \tan^{-1}(0.36397) = 200°}$$

The time rates of change of the Euler angles ϕ, θ and ψ are, respectively, the precession rate ω_p, the nutation rate ω_n and the spin ω_s. That is,

$$\omega_p = \dot{\phi} \qquad \omega_n = \dot{\theta} \qquad \omega_s = \dot{\psi} \tag{9.106}$$

The absolute angular velocity $\boldsymbol{\omega}$ of a rigid body can be resolved into components ω_x, ω_y and ω_z along the body-fixed *xyz* axes, so that

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} \tag{9.107}$$

Figure 9.23 shows that precession is measured around the inertial Z axis (unit vector $\hat{\mathbf{K}}$); nutation is measured around the intermediate x_1 axis (node line) with unit vector $\hat{\mathbf{i}}_1$; and spin is measured around the body-fixed z axis (unit vector $\hat{\mathbf{k}}$). Therefore, the absolute angular velocity can alternatively be written in terms of the non-orthogonal Euler angle rates as

$$\boldsymbol{\omega} = \omega_p \hat{\mathbf{K}} + \omega_n \hat{\mathbf{i}}_1 + \omega_s \mathbf{k} \tag{9.108}$$

In order to find the relationship between the body rates ω_x, ω_y, ω_z and the Euler angle rates ω_p, ω_n, ω_s, we must express $\hat{\mathbf{K}}$ and $\hat{\mathbf{i}}_1$ in terms of the unit vectors $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ of the body-fixed frame. To accomplish that, we proceed as follows.

The first rotation $[\mathrm{R}_3(\phi)]$ in Equation (9.102) rotates the unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ of the inertial frame into the unit vectors $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ of the intermediate $x_1y_1z_1$ axes in Figure 9.23. Hence $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ are rotated into $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ by the inverse transformation

$$\begin{Bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{Bmatrix} = [\mathbf{R}_3(\phi)]^T \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} \tag{9.109}$$

The second rotation $[\mathbf{R}_1(\theta)]$ rotates $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ into the unit vectors $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ of the second intermediate frame $x_2y_2z_2$ in Figure 9.23. The inverse transformation rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ back into $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$:

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = [\mathbf{R}_1(\theta)]^T \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} \tag{9.110}$$

Finally, the third rotation $[\mathbf{R}_3(\psi)]$ rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$, the target unit vectors of the body-fixed xyz frame. $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ are obtained from $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ by the reverse rotation,

$$\begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = [\mathbf{R}_3(\psi)]^T \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} \tag{9.111}$$

From Equations 9.109, 9.110 and 9.111 we observe that

$$\hat{\mathbf{K}} \overset{\overbrace{9.109}}{=} \hat{\mathbf{k}}_1 \overset{\overbrace{9.110}}{=} \sin\theta\hat{\mathbf{j}}_2 + \cos\theta\hat{\mathbf{k}}_2 \overset{\overbrace{9.111}}{=} \sin\theta(\sin\psi\hat{\mathbf{i}} + \cos\psi\hat{\mathbf{j}}) + \cos\theta\hat{\mathbf{k}}$$

or

$$\hat{\mathbf{K}} = \sin\theta\sin\psi\hat{\mathbf{i}} + \sin\theta\cos\psi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{9.112}$$

Similarly, Equations 9.110 and 9.111 imply that

$$\hat{\mathbf{i}}_1 = \hat{\mathbf{i}}_2 = \cos\psi\hat{\mathbf{i}} - \sin\psi\hat{\mathbf{j}} \tag{9.113}$$

Substituting Equations 9.112 and 9.113 into 9.108 yields

$$\boldsymbol{\omega} = \omega_p(\sin\theta\sin\psi\hat{\mathbf{i}} + \sin\theta\cos\psi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}) + \omega_n(\cos\psi\hat{\mathbf{i}} - \sin\psi\hat{\mathbf{j}}) + \omega_s\mathbf{k}$$

or

$$\boldsymbol{\omega} = (\omega_p\sin\theta\sin\psi + \omega_n\cos\psi)\hat{\mathbf{i}} + (\omega_p\sin\theta\cos\psi - \omega_n\sin\psi)\hat{\mathbf{j}} + (\omega_s + \omega_p\cos\theta)\hat{\mathbf{k}} \tag{9.114}$$

Comparing Equations 9.107 and 9.114, we obtain the ***angular velocities in terms of the Euler angle rates***,

$$\begin{aligned}\omega_x &= \omega_p\sin\theta\sin\psi + \omega_n\cos\psi\\ \omega_y &= \omega_p\sin\theta\cos\psi - \omega_n\sin\psi\\ \omega_z &= \omega_s + \omega_p\cos\theta\end{aligned} \tag{9.115}$$

(Notice that the angle ϕ does not appear.) We can solve these three equations to obtain the ***Euler angle rates in terms of the angular velocities*** ω_x, ω_y and ω_z:

$$\begin{aligned}\omega_p &= \dot{\phi} = \frac{1}{\sin\theta}(\omega_x\sin\psi + \omega_y\cos\psi)\\ \omega_n &= \dot{\theta} = \omega_x\cos\psi - \omega_y\sin\psi\\ \omega_s &= \dot{\psi} = -\frac{1}{\tan\theta}(\omega_x\sin\psi + \omega_y\cos\psi) + \omega_z\end{aligned} \tag{9.116}$$

Observe that if ω_x, ω_y and ω_z are given functions of time, found by solving Euler's equations of motion (Equations 9.72), then Equations 9.116 are three coupled differential equations which may be solved to obtain the three time-dependent Euler angles

$$\phi = \phi(\mathrm{t}) \qquad \theta = \theta(t) \qquad \psi = \psi(t)$$

With this solution, the orientation of the xyz frame, and hence the body to which it is attached, is known for any given time t. Note, however, that Equations 9.116 "blow up" when $\theta = 0$, i.e., when the xy plane is parallel to the XY plane.

Example 9.18

At a given instant, the unit vectors of a body frame are

$$\begin{aligned}\hat{\mathbf{i}} &= 0.40825\hat{\mathbf{I}} - 0.40825\hat{\mathbf{J}} + 0.81649\hat{\mathbf{K}}\\ \hat{\mathbf{j}} &= -0.10102\hat{\mathbf{I}} - 0.90914\hat{\mathbf{J}} - 0.40405\hat{\mathbf{K}}\\ \hat{\mathbf{k}} &= 0.90726\hat{\mathbf{I}} + 0.082479\hat{\mathbf{J}} - 0.41240\hat{\mathbf{K}}\end{aligned} \tag{a}$$

and the angular velocity is

$$\boldsymbol{\omega} = -3.1\hat{\mathbf{I}} + 2.5\hat{\mathbf{J}} + 1.7\hat{\mathbf{K}} \text{ (rad/s)} \tag{b}$$

Calculate ω_p, ω_n and ω_s (the precession, nutation and spin rates) at this instant.

Solution

We will ultimately use Equation 9.116 to find ω_p, ω_n and ω_s. To do so we must first obtain the Euler angles ϕ, θ and ψ as well as the components of the angular velocity in the body frame.

The three rows of the direction cosine matrix $[\mathbf{Q}]_{Xx}$ comprise the components of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0.40825 & -0.40825 & 0.81649 \\ -0.10102 & -0.90914 & -0.40405 \\ 0.90726 & 0.082479 & -0.41240 \end{bmatrix} \tag{c}$$

Therefore, the components of the angular velocity in the body frame are

$$\{\boldsymbol{\omega}\}_x = [\mathbf{Q}]_{Xx}\{\omega\}_X = \begin{bmatrix} 0.40825 & -0.40825 & 0.81649 \\ -0.10102 & -0.90914 & -0.40405 \\ 0.90726 & 0.082479 & -0.41240 \end{bmatrix} \begin{Bmatrix} -3.1 \\ 2.5 \\ 1.7 \end{Bmatrix} = \begin{Bmatrix} -0.89817 \\ -2.6466 \\ -3.3074 \end{Bmatrix}$$

or

$$\omega_x = -0.89817 \text{ rad/s} \qquad \omega_y = -2.6466 \text{ rad/s} \qquad \omega_z = -3.3074 \text{ rad/s} \tag{d}$$

To obtain the Euler angles ϕ, θ and ψ from the direction cosine matrix in (c), we use Algorithm 4.3, as illustrated in Example 9.17. That algorithm is implemented as the MATLAB function *dcm_to_Euler.m* in Appendix D.20. Typing the following lines in the MATLAB Command Window

```
>> Q = [ .40825  -.40825    .81649
       -.10102  -.90914   -.40405
        .90726   .082479  -.41240];
>> [phi theta psi] = dcm_to_euler(Q)
```

produces the following output:

```
phi =
   95.1945
theta =
   114.3557
psi =
   116.3291
```

Substituting $\theta = 114.36°$ and $\psi = 116.33°$ together with the angular velocities (d) into Equations 9.116 yields

$$\omega_p = \frac{1}{\sin 114.36°}[-0.89817 \cdot \sin 116.33° + (-2.6466) \cdot \cos 116.33°] = \boxed{0.40492 \text{ rad/s}}$$

$$\omega_n = -0.89817 \cdot \cos 116.33° - (-2.6466) \cdot \sin 116.33° = \boxed{2.7704 \text{ rad/s}}$$

$$\omega_s = -\frac{1}{\tan 114.36°}[-0.89817 \cdot \sin 116.33° + (-2.6466) \cdot \cos 116.33°] + (-3.3074) = \boxed{-3.1404 \text{ rad/s}}$$

Example 9.19

The mass moments of inertia of a body about the principal body frame axes with origin at the center of mass G are

$$A = 1000\,\text{kg}\cdot\text{m}^2 \qquad B = 2000\,\text{kg}\cdot\text{m}^2 \qquad C = 3000\,\text{kg}\cdot\text{m}^2 \tag{a}$$

The Euler angles in radians are given as functions of time in seconds as follows:

$$\begin{aligned}\phi &= 2te^{-0.05t}\\ \theta &= 0.02 + 0.3\sin 0.25t\\ \psi &= 0.6t\end{aligned} \tag{b}$$

At $t = 10$ s, find

(a) The net moment about G

(b) The components α_X, α_Y and α_Z of the absolute angular acceleration in the inertial frame.

Solution

(a) We must use Euler's equations (Equations 9.72) to calculate the net moment, which means we must first obtain ω_x, ω_y, ω_z, $\dot{\omega}_x$, $\dot{\omega}_y$ and $\dot{\omega}_z$. Since we are given the Euler angles as functions of time, we can compute their time derivatives and then use Equation 9.115 to find the body frame angular velocity components and their derivatives.

Starting with $(b)_1$, we get

$$\begin{aligned}\omega_p &= \frac{d\phi}{dt} = \frac{d}{dt}(2te^{-0.05t}) = 2e^{-0.05t} - 0.1te^{-0.05t}\\ \dot{\omega}_p &= \frac{d\omega_p}{dt} = \frac{d}{dt}(2e^{-0.05t} - 0.1e^{-0.05t}) = -0.2e^{-0.05t} + 0.005te^{-0.05t}\end{aligned}$$

Proceeding to the remaining two Euler angles leads to

$$\begin{aligned}\omega_n &= \frac{d\theta}{dt} = \frac{d}{dt}(0.02 + 0.3\sin 0.25t) = 0.075\cos 0.25t\\ \dot{\omega}_n &= \frac{d\omega_n}{dt} = \frac{d}{dt}(0.075\cos 0.25t) = -0.01875\sin 0.25t\\ \omega_s &= \frac{d\psi}{dt} = \frac{d}{dt}(0.6t) = 0.6\\ \dot{\omega}_s &= \frac{d\omega_s}{dt} = 0\end{aligned}$$

Evaluating all of these quantities, including those in (b), at $t = 10$ s yields

$$\begin{array}{lll}\phi = 335.03^\circ & \omega_p = 0.60653 \text{ rad/s} & \dot{\omega}_p = -0.09098 \text{ rad/s}^2\\ \theta = 11.433^\circ & \omega_n = -0.060086 \text{ rad/s} & \dot{\omega}_n = -0.011221 \text{ rad/s}^2\\ \psi = 343.77 & \omega_s = 0.6 \text{ rad/s} & \dot{\omega}_s = 0\end{array} \tag{c}$$

Equation 9.115 relates the Euler angle rates to the angular velocity components,

$$\begin{aligned}\omega_x &= \omega_p \sin\theta \sin\psi + \omega_n \cos\psi \\ \omega_y &= \omega_p \sin\theta \cos\psi - \omega_n \sin\psi \\ \omega_z &= \omega_s + \omega_p \cos\theta\end{aligned} \tag{d}$$

Taking the time derivative of each of these equations in turn leads to the following three equations,

$$\begin{aligned}\dot{\omega}_x &= \omega_p\omega_n \cos\theta \sin\psi + \omega_p\omega_s \sin\theta\cos\psi - \omega_n\omega_s \sin\psi + \dot{\omega}_p \sin\theta \sin\psi + \dot{\omega}_n \cos\psi \\ \dot{\omega}_y &= \omega_p\omega_n \cos\theta \cos\psi - \omega_p\omega_s \sin\theta\sin\psi - \omega_n\omega_s \cos\psi + \dot{\omega}_p \sin\theta \cos\psi - \dot{\omega}_n \sin\psi \\ \dot{\omega}_z &= -\omega_p\omega_n \sin\theta + \dot{\omega}_p \cos\theta + \dot{\omega}_s\end{aligned} \tag{e}$$

Substituting the data in (c) into (d) and (e) yields

$$\begin{aligned}&\omega_x = -0.091286 \text{ rad/s} \quad \omega_y = 0.098649 \text{ rad/s} \quad \omega_z = 1.1945 \text{ rad/s} \\ &\dot{\omega}_x = 0.063435 \text{ rad/s}^2 \quad \dot{\omega}_y = 2.2346 \times 10^{-5} \text{ rad/s}^2 \quad \dot{\omega}_z = -0.08195 \text{ rad/s}^2\end{aligned} \tag{f}$$

With (a) and (f) we have everything we need for Euler's equations,

$$\begin{aligned}M_{x_{\text{net}}} &= A\dot{\omega}_x + (C - B)\omega_y\omega_z \\ M_{y_{\text{net}}} &= B\dot{\omega}_y + (A - C)\omega_z\omega_x \\ M_{z_{\text{net}}} &= C\dot{\omega}_z + (B - A)\omega_x\omega_y\end{aligned}$$

from which we find

$$\boxed{\begin{aligned}M_{x_{\text{net}}} &= 181.27 \text{ N}\cdot\text{m} \\ M_{y_{\text{net}}} &= 218.12 \text{ N}\cdot\text{m} \\ M_{z_{\text{net}}} &= -254.86 \text{ N}\cdot\text{m}\end{aligned}}$$

(b) Since the co-moving *xyz* frame is a body frame, rigidly attached to the solid, we know from Equation 9.73 that

$$\begin{Bmatrix}\alpha_X \\ \alpha_Y \\ \alpha_Z\end{Bmatrix} = [\mathbf{Q}]_{xX} \begin{Bmatrix}\dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z\end{Bmatrix} \tag{g}$$

In other words, the absolute angular acceleration and the relative angular acceleration of the body are the same. All we have to do is project the components of relative acceleration in (f) onto the axes of the inertial frame. The required orthogonal transformation matrix is given in Equation 9.105,

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \sin\phi\sin\theta \\ \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

Upon substituting the numerical values of the Euler angles from (c), this becomes

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} 0.75484 & 0.65055 & -0.083668 \\ -0.65356 & 0.73523 & -0.17970 \\ -0.055386 & 0.19033 & 0.98016 \end{bmatrix}$$

Substituting this and the relative angular velocity rates from (f) into (g) yields

$$\begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix} = \begin{bmatrix} 0.75484 & 0.65055 & -0.083668 \\ -0.65356 & 0.73523 & -0.17970 \\ -0.055386 & 0.19033 & 0.98016 \end{bmatrix} \begin{Bmatrix} 0.063435 \\ 2.2345 \times 10^{-5} \\ -0.08195 \end{Bmatrix} = \boxed{\begin{Bmatrix} 0.054755 \\ -0.026716 \\ -0.083833 \end{Bmatrix} (\text{rad/s}^2)}$$

Example 9.20

Figure 9.24 shows a rotating platform on which is mounted a rectangular parallelepiped shaft (with dimensions b, h and l) spinning about the inclined axis DE. If the mass of the shaft is m, and the angular velocities ω_p and ω_s are constant, calculate the bearing forces at D and E as a function of ϕ and ψ. Neglect gravity, since we are interested only in the gyroscopic forces. (The small extensions shown at each end of the parallelepiped are just for clarity; the distance between the bearings at D and E is l.)

Solution

The inertial XYZ frame is centered at O on the platform, and it is right-handed ($\hat{\mathbf{I}} \times \hat{\mathbf{J}} = \hat{\mathbf{K}}$). The origin of the right-handed co-moving body frame xyz is at the shaft's center of mass G, and it is aligned with the symmetry axes of the parallelepiped. The three Euler angles ϕ, θ and ψ are shown in Figure 9.24. Since θ is constant, the nutation rate is zero ($\omega_n = 0$). Thus, Equations 9.115 reduce to

$$\omega_x = \omega_p \sin\theta \sin\psi \qquad \omega_y = \omega_p \sin\theta \cos\psi \qquad \omega_z = \omega_p \cos\theta + \omega_s \tag{a}$$

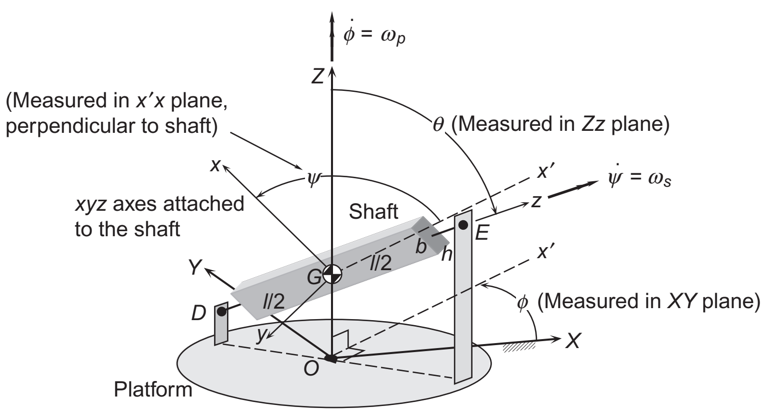


FIGURE 9.24

Spinning block mounted on rotating platform.

Since ω_p, ω_s and θ are constant, it follows (recalling Equations 9.106) that

$$\dot{\omega}_x = \omega_p\omega_s \sin\theta\cos\psi \qquad \dot{\omega}_y = -\omega_p\omega_s \sin\theta\sin\psi \qquad \dot{\omega}_z = 0 \tag{b}$$

The principal moments of inertia of the parallelepiped are [see Figure 9.10(c)]

$$\begin{aligned} A = I_x &= \frac{1}{12}m(h^2 + l^2) \\ B = I_y &= \frac{1}{12}m(b^2 + l^2) \\ C = I_z &= \frac{1}{12}m(b^2 + h^2) \end{aligned} \tag{c}$$

Figure 9.25 is a free-body diagram of the shaft. Let us assume that the bearings at D and E are such as to exert just the six body frame components of force shown. Thus, D is a thrust bearing to which the axial torque T_D is applied from, say, a motor of some kind. At E there is a simple journal bearing.

From Newton's laws of motion we have $\mathbf{F}_{\text{net}} = m\mathbf{a}_G$. But G is fixed in inertial space, so $\mathbf{a}_G = \mathbf{0}$. Thus,

$$(D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}} + D_z\hat{\mathbf{k}}) + (E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}}) = \mathbf{0}$$

It follows that

$$E_x = -D_x \qquad E_y = -D_y \qquad D_z = 0 \tag{d}$$

Summing moments about G we get

$$\begin{aligned} \mathbf{M}_{G_{\text{net}}} &= \frac{l}{2}\hat{\mathbf{k}} \times (E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}}) + \left(-\frac{l}{2}\hat{\mathbf{k}}\right) \times (D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}}) + T_D\hat{\mathbf{k}} \\ &= \left(D_y\frac{l}{2} - E_y\frac{l}{2}\right)\hat{\mathbf{i}} + \left(-D_x\frac{l}{2} + E_x\frac{l}{2}\right)\hat{\mathbf{j}} + T_D\hat{\mathbf{k}} \\ &= D_y l\hat{\mathbf{i}} - D_x l\hat{\mathbf{j}} + T_D\hat{\mathbf{k}} \end{aligned}$$

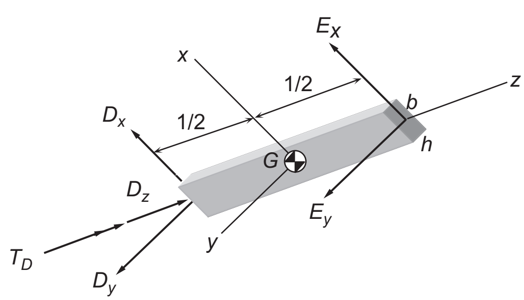


FIGURE 9.25

Free-body diagram of the block in Figure 9.24.

where we made use of Equation $(d)_2$. Thus,

$$M_{x_{net}} = D_y l \qquad M_{y_{net}} = -D_x l \qquad M_{z_{net}} = T_D \tag{e}$$

We substitute (a), (b), (c), and (e) into Euler's equations (Equations 9.72):

$$\begin{aligned} M_{x_{net}} &= A\dot{\omega}_x + (C - B)\omega_y\omega_z \\ M_{y_{net}} &= B\dot{\omega}_y + (A - C)\omega_x\omega_z \\ M_{z_{net}} &= C\dot{\omega}_z + (B - A)\omega_x\omega_y \end{aligned} \tag{f}$$

After making the substitutions and simplifying, the first Euler equation, Equation $(f)_1$, becomes

$$D_x = \left\{\frac{1}{12}\frac{m}{l}[(l^2 - h^2)\omega_p \cos\theta - 2h^2\omega_s]\omega_p \sin\theta\right\}\cos\psi \tag{g}$$

Likewise, from Equation $(f)_2$ we obtain

$$D_y = \left\{\frac{1}{12}\frac{m}{l}[(l^2 - b^2)\omega_p \cos\theta - 2b^2\omega_s]\omega_p \sin\theta\right\}\sin\psi \tag{h}$$

Finally, Equation $(f)_3$ yields

$$T_D = \left[\frac{1}{24}m(b^2 - h^2)\omega_p{}^2 \sin^2\theta\right]\sin 2\psi \tag{i}$$

This completes the solution, since $E_y = -D_y$ and $E_z = -D_z$. Note that the resultant transverse bearing load V at D (and E) is

$$V = \sqrt{D_x{}^2 + D_y{}^2} \tag{j}$$

As a numerical example, let

$$l = 1\,\text{m} \quad h = 0.1\,\text{m} \quad b = 0.025\,\text{m} \quad \theta = 30° \quad m = 10\,\text{kg}$$

and

$$\omega_p = 100\,\text{rpm} = 10.47\,\text{rads/s} \quad \omega_s = 2000\,\text{rpm} = 209.4\,\text{rad/s}$$

For these numbers, the variation of V and T_D with ψ is as shown in Figure 9.26.

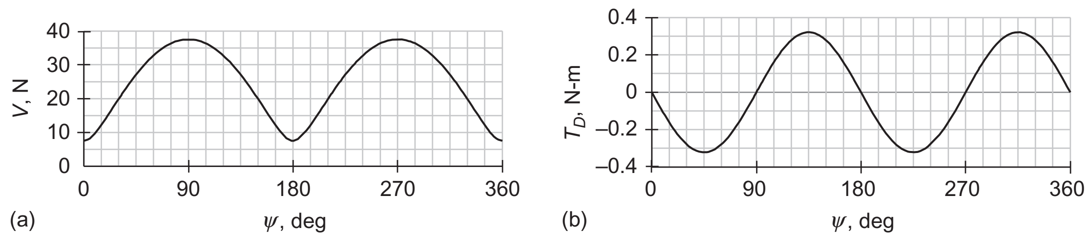


FIGURE 9.26

(a) Transverse bearing load. (b) Axial torque at *D*.

9.10 YAW, PITCH AND ROLL ANGLES

The problem of the Euler angle relations, Equations 9.116, becoming singular when the nutation angle θ is zero can be alleviated by using the yaw, pitch and roll angles discussed in Section 4.5. As in the classical Euler sequence, yaw-pitch-roll sequence rotates the inertial *XYZ* axes into the body fixed *xyz* axes triad by means of a series of three elementary rotations illustrated in Figure 9.27. Like the classical Euler sequence, the first rotation is around the Z ($=z_1$) axis through the yaw angle ϕ. This takes X into x_1 and Y into y_1. The second rotation is around the y_2 ($=y_1$) axis through the pitch angle θ. This carries x_1 and z_1 into x_2 and z_2, respectively. The third and final rotation is around the x ($=x_2$) axis through the pitch angle ψ, which takes y_2 into y and z_2 into z.

Equation 4.40 gives the matrix $[\mathbf{Q}]_{Xx}$ of the transformation from the inertial frame into the body-fixed frame,

$$[\mathbf{Q}]_{Xx} = [\mathbf{R}_1(\psi)][\mathbf{R}_2(\theta)][\mathbf{R}_3(\phi)] \tag{9.117}$$

From Equations 4.32, 4.33 and 4.34, the elementary rotation matrices are

$$[\mathbf{R}_1(\psi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \quad [\mathbf{R}_2(\theta)] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad [\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.118}$$

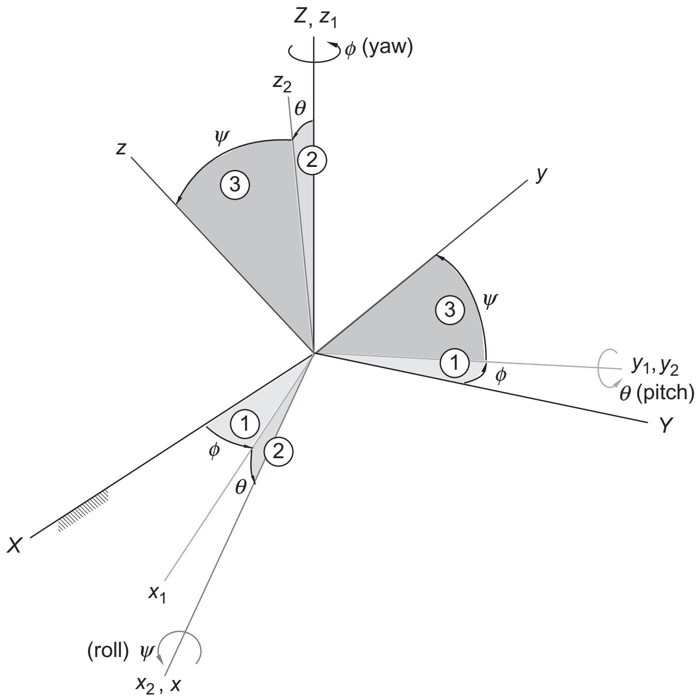


FIGURE 9.27

Yaw, pitch and roll sequence. See also Figure 4.15.

According to Equation 4.41, the multiplication on the right of Equation 9.117 yields the following direction cosine matrix for the yaw-pitch-roll sequence,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\theta\sin\psi \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\theta\cos\psi \end{bmatrix} \tag{9.119}$$

The inverse matrix $[\mathbf{Q}]_{xX}$, which transforms xyz into XYZ, is just the transpose

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{bmatrix} \tag{9.120}$$

Algorithm 4.4 (*dcm_to_ypr.m* in Appendix D.21) is used to determine the yaw, pitch and roll angles for a given direction cosine matrix. The following brief MATLAB session reveals that the yaw, pitch and roll angles for the direction cosine matrix in Example 9.17 are $\phi = 109.69°$, $\theta = 17.230°$ and $\psi = 238.43°$.

```
 >> Q = [ -0.32175    0.89930   -0.29620
           0.57791  -0.061275 -0.81380
          -0.75000  -0.43301   -0.5000];
 >> [yaw pitch roll] = dcm_to_ypr(Q)
yaw =
   109.6861
pitch =
   17.2295
roll =
   238.4334
```

Figure 9.27 shows that yaw ϕ is measured around the inertial Z axis (unit vector $\hat{\mathbf{K}}$), pitch θ is measured around the intermediate y_1 axis (unit vector $\hat{\mathbf{j}}_1$) and roll ψ is measured around the body-fixed x axis (unit vector $\hat{\mathbf{i}}$). The angular velocity $\boldsymbol{\omega}$, expressed in terms of the rates of yaw, pitch and roll, is

$$\boldsymbol{\omega} = \omega_{\text{yaw}}\hat{\mathbf{K}} + \omega_{\text{pitch}}\hat{\mathbf{j}}_2 + \omega_{\text{roll}}\hat{\mathbf{i}} \tag{9.121}$$

in which

$$\omega_{\text{yaw}} = \dot{\phi} \qquad \omega_{\text{pitch}} = \dot{\theta} \qquad \omega_{\text{roll}} = \dot{\psi} \tag{9.122}$$

The first rotation $[\mathbf{R}_3(\phi)]$ in Equation 9.117 rotates the unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ of the inertial frame into the unit vectors $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ of the intermediate $x_1y_1z_1$ axes in Figure 9.27. Thus, $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ are rotated into $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ by the inverse transformation

$$\begin{Bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} \tag{9.123}$$

The second rotation $[\mathbf{R}_2(\theta)]$ rotates $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ into the unit vectors $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ of the second intermediate frame $x_2y_2z_2$ in Figure 9.27. The inverse transformation rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ back into $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$:

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} \tag{9.124}$$

Lastly, the third rotation $[\mathbf{R}_1(\psi)]$ rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$, the unit vectors of the body fixed xyz frame. $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ are obtained from $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ by the reverse transformation,

$$\begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} \tag{9.125}$$

From Equations 9.123 through 9.125 we see that

$$\hat{\mathbf{K}} \overset{9.123}{=} \hat{\mathbf{k}}_1 \overset{9.124}{=} -\sin\theta\hat{\mathbf{i}}_2 + \cos\theta\hat{\mathbf{k}}_2 \overset{9.125}{=} -\sin\theta\hat{\mathbf{i}} + \cos\theta(\sin\psi\hat{\mathbf{j}} + \cos\psi\hat{\mathbf{k}})$$

or

$$\hat{\mathbf{K}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\sin\psi\hat{\mathbf{j}} + \cos\theta\cos\psi\hat{\mathbf{k}} \tag{9.126}$$

From Equation 9.125,

$$\hat{\mathbf{j}}_2 = \cos\psi\hat{\mathbf{j}} - \sin\psi\hat{\mathbf{k}} \tag{9.127}$$

Substituting Equations 9.126 and 9.127 into 9.121 yields

$$\boldsymbol{\omega} = \omega_{\text{yaw}}(-\sin\theta\hat{\mathbf{i}} + \cos\theta\sin\psi\hat{\mathbf{j}} + \cos\theta\cos\psi\hat{\mathbf{k}}) + \omega_{\text{pitch}}(\cos\psi\hat{\mathbf{j}} - \sin\psi\hat{\mathbf{k}}) + \omega_{\text{roll}}\hat{\mathbf{i}}$$

or

$$\boldsymbol{\omega} = (-\omega_{\text{yaw}}\sin\theta + \omega_{\text{roll}})\hat{\mathbf{i}} + (\omega_{\text{yaw}}\cos\theta\sin\psi + \omega_{\text{pitch}}\cos\psi)\hat{\mathbf{j}} + (\omega_{\text{yaw}}\cos\theta\cos\psi - \omega_{\text{pitch}}\sin\psi)\hat{\mathbf{k}} \tag{9.128}$$

Comparing Equations 9.107 and 9.128 we see that the body ***angular velocities are related to the yaw, pitch and roll rates*** as follows:

$$\begin{aligned} \omega_x &= \omega_{\text{roll}} - \omega_{\text{yaw}}\sin\theta_{\text{pitch}} \\ \omega_y &= \omega_{\text{yaw}}\cos\theta_{\text{pitch}}\sin\psi_{\text{roll}} + \omega_{\text{pitch}}\cos\psi_{\text{roll}} \\ \omega_z &= \omega_{\text{yaw}}\cos\theta_{\text{pitch}}\cos\psi_{\text{roll}} - \omega_{\text{pitch}}\sin\psi_{\text{roll}} \end{aligned} \tag{9.129}$$

wherein the subscript on each symbol helps us remember the rotation it describes. The inverse of these equations ***yields the yaw, pitch and roll rates in terms of angular velocities***

$$\begin{aligned}\omega_{\text{yaw}} &= \frac{1}{\cos\theta_{\text{pitch}}}(\omega_y \sin\psi_{\text{roll}} + \omega_z \cos\psi_{\text{roll}})\\ \omega_{\text{pitch}} &= \omega_y \cos\psi_{\text{roll}} - \omega_z \sin\psi_{\text{roll}}\\ \omega_{\text{roll}} &= \omega_x + \omega_y \tan\theta_{\text{pitch}} \sin\psi_{\text{roll}} + \omega_z \tan\theta_{\text{pitch}} \cos\psi_{\text{roll}}\end{aligned} \tag{9.130}$$

Notice that this system becomes singular ($\cos\theta_{\text{pitch}} = 0$) when the pitch angle is $\pm 90°$.

9.11 QUATERNIONS

In Chapter 4 we showed that the transformation from any Cartesian coordinate frame to another having the same origin can be accomplished by a sequence of three transformations, each being an elementary rotation about one of the three coordinate axes. We have focused on the commonly used classical Euler angle sequence ($[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$) and the yaw-pitch-roll sequence ($[\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)]$).

Another of Euler's theorems, which we used in Section 1.6, states that any two Cartesian coordinate frames are related by a unique rotation about a single line through their common origin. This line is called the ***Euler axis*** and the angle is referred to as the ***principal angle***.

Let $\hat{\mathbf{u}}$ be the unit vector along the Euler axis. A vector $\mathbf{v}$ can be resolved into orthogonal components $\mathbf{v}_\perp$ normal to $\hat{\mathbf{u}}$ and $\mathbf{v}_\parallel$ parallel to $\hat{\mathbf{u}}$, so that we may write

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \tag{9.131}$$

The component of $\mathbf{v}$ along $\hat{\mathbf{u}}$ is given by $\mathbf{v}\cdot\hat{\mathbf{u}}$. That is,

$$\mathbf{v}_\parallel = (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{9.132}$$

From Equations 9.131 and 9.132 we have

$$\mathbf{v}_\perp = \mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{9.133}$$

Let $\mathbf{v}'$ be the vector obtained by rotating $\mathbf{v}$ through an angle θ around $\hat{\mathbf{u}}$, as illustrated in Figure 9.28. This rotation leaves the magnitude of $\mathbf{v}_\perp$ and its component along $\hat{\mathbf{u}}$ unchanged. That is

$$\left\|\mathbf{v}'_\perp\right\| = \left\|\mathbf{v}_\perp\right\| \tag{9.134}$$

$$\mathbf{v}'_\parallel = (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{9.135}$$

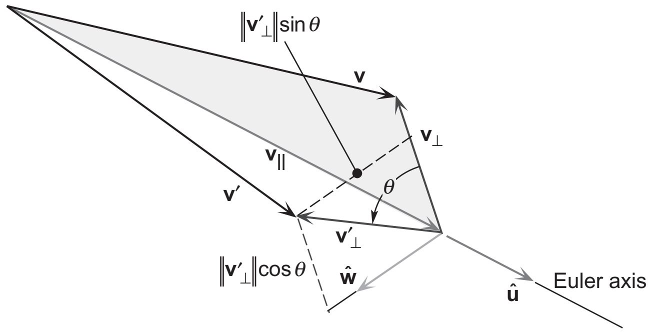


FIGURE 9.28

Rotation of a vector $\mathbf{v}$ through an angle θ about an axis with unit vector $\hat{\mathbf{u}}$.

$\mathbf{v}'_\perp$, having been rotated about $\hat{\mathbf{u}}$, has the component $\|\mathbf{v}'_\perp\|\cos\theta$ along $\mathbf{v}_\perp$ and the component $\|\mathbf{v}'_\perp\|\sin\theta$ along the vector normal to the plane of $\hat{\mathbf{u}}$ and $\mathbf{v}$. Let $\hat{\mathbf{w}}$ be the unit vector normal to that plane. Then

$$\hat{\mathbf{w}} = \hat{\mathbf{u}} \times \frac{\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|} \tag{9.136}$$

Thus, as can be seen from Figure 9.28,

$$\mathbf{v}'_\perp = \|\mathbf{v}'_\perp\|\cos\theta\frac{\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|} + \|\mathbf{v}'_\perp\|\sin\theta\frac{\mathbf{u}\times\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|}$$

According to Equation 9.134, this reduces to

$$\mathbf{v}'_\perp = \cos\theta\,\mathbf{v}_\perp + \sin\theta\,\mathbf{u}\times\mathbf{v}_\perp \tag{9.137}$$

Observe that

$$\hat{\mathbf{u}}\times\mathbf{v}_\perp = \hat{\mathbf{u}}\times\left[\mathbf{v}-\mathbf{v}_{||}\right] = \hat{\mathbf{u}}\times\mathbf{v}$$

since $\mathbf{v}_{||}$ is parallel to $\hat{\mathbf{u}}$. This, together with Equation 9.133, means we can write Equation 9.137 as

$$\mathbf{v}'_\perp = \cos\theta[\mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}] + \sin\theta(\hat{\mathbf{u}}\times\mathbf{v}) \tag{9.138}$$

Since $\mathbf{v}' = \mathbf{v}'_\perp + \mathbf{v}'_{||}$, we find, upon substituting Equation 9.135 and 9.138 and collecting terms, that

$$\mathbf{v}' = \cos\theta\,\mathbf{v} + (1-\cos\theta)(\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}} + \sin\theta(\hat{\mathbf{u}}\times\mathbf{v}) \tag{9.139}$$

This is a useful formula for determining the result of rotating a vector about a line.

We can obtain the body fixed *xyz* Cartesian frame from the inertial *XYZ* frame by a single rotation through the principal angle θ about the Euler axis $\hat{\mathbf{u}}$. The unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ are thereby rotated into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$. The two sets of unit vectors are related by Equation 9.139. Thus,

$$\begin{aligned}
\hat{\mathbf{i}} &= \cos\theta\hat{\mathbf{I}} + (1-\cos\theta)(\hat{\mathbf{u}}\cdot\hat{\mathbf{I}})\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{I}} \\
\hat{\mathbf{j}} &= \cos\theta\hat{\mathbf{J}} + (1-\cos\theta)(\hat{\mathbf{u}}\cdot\hat{\mathbf{J}})\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{J}} \\
\hat{\mathbf{k}} &= \cos\theta\hat{\mathbf{K}} + (1-\cos\theta)(\hat{\mathbf{u}}\cdot\hat{\mathbf{K}})\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{K}}
\end{aligned} \tag{9.140}$$

Let us express the unit vector $\hat{\mathbf{u}}$ in terms of its direction cosines *l*, *m* and *n* along the original *XYZ* axes,

$$\hat{\mathbf{u}} = l\hat{\mathbf{I}} + m\hat{\mathbf{J}} + n\hat{\mathbf{K}} \tag{9.141}$$

Substituting this into Equation 9.140, carrying out the vector operations and collecting terms yields

$$\begin{aligned}
\hat{\mathbf{i}} &= [l^2(1-\cos\theta)+\cos\theta]\hat{\mathbf{I}} + [lm(1-\cos\theta)+n\sin\theta]\hat{\mathbf{J}} + [ln(1-\cos\theta)-m\sin\theta]\hat{\mathbf{K}} \\
\hat{\mathbf{j}} &= [lm(1-\cos\theta)-n\sin\theta]\hat{\mathbf{I}} + [m^2(1-\cos\theta)+\cos\theta]\hat{\mathbf{J}} + [mn(1-\cos\theta)+l\sin\theta]\hat{\mathbf{K}} \\
\hat{\mathbf{k}} &= [ln(1-\cos\theta)+m\sin\theta]\hat{\mathbf{I}} + [mn(1-\cos\theta)-l\sin\theta]\hat{\mathbf{J}} + [n^2(1-\cos\theta)+\cos\theta]\hat{\mathbf{K}}
\end{aligned} \tag{9.142}$$

Recall that the rows of the matrix $[\mathbf{Q}]_{Xx}$ of the transformation from *XYZ* to *xyz* comprise the direction cosines of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively. That is,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} l^2(1-\cos\theta)+\cos\theta & lm(1-\cos\theta)+n\sin\theta & ln(1-\cos\theta)-m\sin\theta \\ lm(1-\cos\theta)-n\sin\theta & m^2(1-\cos\theta)+\cos\theta & mn(1-\cos\theta)+l\sin\theta \\ ln(1-\cos\theta)+m\sin\theta & mn(1-\cos\theta)-l\sin\theta & n^2(1-\cos\theta)+\cos\theta \end{bmatrix} \tag{9.143}$$

The direction cosine matrix is thus expressed in terms of the Euler axis direction cosines and the principal angle.

Quaternions (also known as Euler symmetric parameters) were introduced in 1843 by the Irish mathematician Sir William R. Hamilton (1805–1865). They provide an alternative to the use of direction cosine matrices for describing the orientation of a body frame in three-dimensional space. Quaternions can be used to avoid encountering the singularities we observed for the classical Euler angle sequence when the nutation angle θ becomes zero (Equations 9.116) or for the yaw-pitch-roll sequence when the pitch angle θ approaches 90° (Equations 9.126).

As the name implies, a quaternion $\{\hat{\mathbf{q}}\}$ comprises four numbers

$$\{\hat{\mathbf{q}}\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \mathbf{q} \\ q_4 \end{Bmatrix} \tag{9.144}$$

$\mathbf{q}$ is called the vector part ($\mathbf{q} = q_1\hat{\mathbf{i}} + q_2\hat{\mathbf{j}} + q_3\hat{\mathbf{k}}$ and q_4 is the scalar part. (It is equally common to see the scalar part listed first, so that $\{\hat{\mathbf{q}}\} = \lfloor q_4, \mathbf{q} \rfloor^T$ or $\{\hat{\mathbf{q}}\} = \lfloor q_0, \mathbf{q} \rfloor^T$.)

The norm $\|\widehat{\mathbf{q}}\|$ of the quaternion $\{\widehat{\mathbf{q}}\}$ is defined as

$$\|\widehat{\mathbf{q}}\| = \sqrt{\mathbf{q}\cdot\mathbf{q} + q_4^{\,2}} = \sqrt{q_1^{\,2} + q_2^{\,2} + q_3^{\,2} + q_4^{\,2}} \tag{9.145}$$

We will restrict our attention to ***unit quaternions***, which are such that $\|\widehat{\mathbf{q}}\| = 1$. In that case

$$\mathbf{q} = \sin\frac{\theta}{2}\,\hat{\mathbf{u}} \qquad q_4 = \cos\frac{\theta}{2} \tag{9.146}$$

$\hat{\mathbf{u}}$ is the unit vector along the Euler axis around which the inertial reference frame is rotated into the body-fixed frame. θ is the Euler principal rotation angle. Recalling Equation 9.141, we observe that

$$q_1 = l\sin\frac{\theta}{2} \qquad q_2 = m\sin\frac{\theta}{2} \qquad q_3 = n\sin\frac{\theta}{2} \qquad q_4 = \cos\frac{\theta}{2} \tag{9.147}$$

Employing these and the trigonometric identities

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

one can show that the direction cosine matrix $[\mathbf{Q}]_{Xx}$ of the body frame in Equation 9.143 is obtained from the quaternion $\{\widehat{\mathbf{q}}\}$ by means of the following algorithm (Kuipers, 1999).

Algorithm 9.1 Obtain $[\mathbf{Q}]_{Xx}$ from the unit quaternion $\{\widehat{\mathbf{q}}\}$. This procedure is implemented in the MATLAB function *dcm_from_q.m* in Appendix D.37.

1. Write the quaternion as

$$\{\widehat{\mathbf{q}}\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

   where $\lfloor q_1 \quad q_2 \quad q_3 \rfloor^T$ is the vector part and q_4 is the scalar part and $\|\widehat{\mathbf{q}}\| = 1$.
2. Compute the direction cosine matrix of the transformation from *XYZ* to *xyz* as follows:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} q_1^{\,2} - q_2^{\,2} - q_3^{\,2} + q_4^{\,2} & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^{\,2} + q_2^{\,2} - q_3^{\,2} + q_4^{\,2} & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^{\,2} - q_2^{\,2} + q_3^{\,2} + q_4^{\,2} \end{bmatrix} \tag{9.148}$$

One can verify by carrying out the matrix multiplication and using Equation 9.145 that $[\mathbf{Q}]_{Xx}$ in Equation 9.148 has the required orthogonality property,

$$[\mathbf{Q}]_{Xx}[\mathbf{Q}]_{Xx}{}^{T} = [\mathbf{Q}]_{Xx}{}^{T}[\mathbf{Q}]_{Xx} = [\mathbf{1}]$$

To find the unit quaternion ($q_1{}^2 + q_2{}^2 + q_3{}^2 + q_4{}^2 = 1$) for a given direction cosine matrix, we observe from Equation 9.148 that

$$\begin{aligned} q_4 &= \frac{1}{2}\sqrt{1 + Q_{11} + Q_{22} + Q_{33}} \\ q_1 &= \frac{Q_{23} - Q_{32}}{4q_4} \qquad q_2 = \frac{Q_{31} - Q_{13}}{4q_4} \qquad q_3 = \frac{Q_{12} - Q_{21}}{4q_4} \end{aligned} \tag{9.149}$$

This procedure obviously fails if $q_4 = 0$. The following algorithm (Bar-Itzhack, 2000) avoids having to deal with this situation.

Algorithm 9.2 Obtain the (unit) quaternion from the direction cosine matrix $[\mathbf{Q}]_{Xx}$. This procedure is implemented as the MATLAB function *q_from_dcm.m* in Appendix D.38.

1. Form the symmetric matrix

$$[\mathbf{K}] = \frac{1}{3}\begin{bmatrix} Q_{11} - Q_{22} - Q_{33} & Q_{21} + Q_{12} & Q_{31} + Q_{13} & Q_{23} - Q_{32} \\ Q_{21} + Q_{12} & -Q_{11} + Q_{22} - Q_{33} & Q_{32} + Q_{23} & Q_{31} - Q_{13} \\ Q_{31} + Q_{13} & Q_{32} + Q_{23} & -Q_{11} - Q_{22} + Q_{33} & Q_{12} - Q_{21} \\ Q_{23} - Q_{32} & Q_{31} - Q_{13} & Q_{12} - Q_{21} & Q_{11} + Q_{22} + Q_{33} \end{bmatrix} \tag{9.150}$$

2. Solve the eigenvalue problem $[\mathbf{K}]\{\mathbf{x}\} = \lambda\{\mathbf{x}\}$ for the largest eigenvalue λ_{max}. The corresponding eigenvector is the quaternion, $\{\hat{\mathbf{q}}\} = \{\mathbf{x}\}$.

The ***time derivative of a quaternion*** is given by (Sidi, 1997)

$$\frac{d}{dt}\{\hat{\mathbf{q}}\} = \frac{1}{2}[\boldsymbol{\Omega}]\{\hat{\mathbf{q}}\} \tag{9.151a}$$

where

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \tag{9.151b}$$

ω_x, ω_y and ω_z are the body frame components of the angular velocity.

Example 9.21

(a) Write down the quaternion for a rotation about the x axis through an angle θ. (b) Obtain the corresponding direction cosine matrix.

Solution

(a) According to Equation 9.146,

$$\mathbf{q} = \sin(\theta/2)\,\hat{\mathbf{i}} \qquad q_4 = \cos(\theta/2) \tag{a}$$

so that

$$\boxed{\{\hat{\mathbf{q}}\} = \begin{Bmatrix} \sin(\theta/2) \\ 0 \\ 0 \\ \cos(\theta/2) \end{Bmatrix}} \tag{b}$$

(b) Substituting $\hat{q}_1 = \sin(\theta/2)$, $\hat{q}_2 = \hat{q}_3 = 0$ and $\hat{q}_4 = \cos(\theta/2)$ into Equation 9.148 yields

$$[\mathbf{Q}] = \begin{bmatrix} \sin^2(\theta/2) + \cos^2(\theta/2) & 0 & 0 \\ 0 & -\sin^2(\theta/2) + \cos^2(\theta/2) & 2\sin(\theta/2)\cos(\theta/2) \\ 0 & -2\,\mathrm{in}(\theta/2)\cos(\theta/2) & -\sin^2(\theta/2) + \cos^2(\theta/2) \end{bmatrix} \tag{c}$$

From trigonometry we recall that

$$\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1 \quad 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta \quad \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$$

Therefore, (c) becomes

$$\boxed{[\mathbf{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}} \tag{d}$$

We recognize this as the direction cosine matrix $[\mathbf{R}_1(\theta)]$ for a rotation θ around the x axis.

Example 9.22

For the yaw-pitch-roll sequence $\phi_{yaw} = 50°$, $\theta_{pitch} = 90°$ and $\psi_{roll} = 120°$, calculate

(a) The quaternion.
(b) The rotation angle and axis of rotation.

Solution

(a) Substituting the given angles into Equation 9.119 yields the direction cosine matrix

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0 & 0 & -1 \\ 0.93969 & 0.34202 & 0 \\ 0.34202 & -0.93969 & 0 \end{bmatrix} \quad \text{(a)}$$

Substituting the components of $[\mathbf{Q}]_{Xx}$ into Equation 9.150, we get

$$[\mathbf{K}] = \begin{bmatrix} -0.11401 & 0.31323 & -0.21933 & 0.31323 \\ 0.31323 & 0.11401 & -0.31323 & 0.44734 \\ -0.21933 & -0.31323 & 0.11401 & -0.31323 \\ 0.31323 & 0.44734 & -0.31323 & 0.11401 \end{bmatrix} \quad \text{(b)}$$

The eigenvalues of this matrix are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = -1/3$. Hence, λ_1 is the largest eigenvalue. The quaternion $\{\hat{\mathbf{q}}\}$ is the vector $\{\mathbf{x}\}$ such that $[\mathbf{K}]\{\mathbf{x}\} = 1 \cdot \{\mathbf{x}\}$. Therefore, as one can verify,

$$\boxed{\{\hat{\mathbf{q}}\} = \begin{Bmatrix} 0.40558 \\ 0.57923 \\ -0.40558 \\ 0.57923 \end{Bmatrix}}$$

(b) From Equation 9.146 we find that the principal angle is

$$\theta = 2\cos^{-1}(q_4) = 2\cos^{-1}(0.57923) = \boxed{54.604^\circ}$$

and the Euler axis is

$$\hat{\mathbf{u}} = \frac{0.40558\hat{\mathbf{I}} + 0.57923\hat{\mathbf{J}} - 0.40558\hat{\mathbf{K}}}{\sin(54.604^\circ/2)} = \boxed{0.49754\hat{\mathbf{I}} + 0.71056\hat{\mathbf{J}} - 0.49754\hat{\mathbf{K}}}$$

Example 9.23

Use quaternions to numerically solve the problem of the spinning top shown in Figure 9.19 and having the properties listed in Example 9.15. The top is released with an angular velocity of 1000 revolutions per minute (rpm) around its axis of symmetry (the body z axis), which makes an angle of 60° with the vertical (the inertial Z axis).

Solution

We will use Equations 9.72b (Euler's equations) to compute the angular velocity derivatives:

$$\begin{aligned} \frac{d\omega_x}{dt} &= \frac{M_{x_{\text{net}}}}{A} - \frac{C-B}{A}\omega_y\omega_z \\ \frac{d\omega_y}{dt} &= \frac{M_{y_{\text{net}}}}{B} - \frac{A-C}{B}\omega_z\omega_x \\ \frac{d\omega_z}{dt} &= \frac{M_{z_{\text{net}}}}{C} - \frac{B-A}{C}\omega_x\omega_y \end{aligned} \quad \text{(a)}$$

These require that the moment to be expressed in components along the body-fixed axes instead of the co-moving frame shown in Figure 9.19. From Figure 9.19, the moment of the weight vector about O is

$$\mathbf{M}_{0_{\text{net}}} = d\hat{\mathbf{k}} \times (-mg\hat{\mathbf{K}}) = -mgd(\hat{\mathbf{k}} \times \hat{\mathbf{K}}) \tag{b}$$

where

$$\hat{\mathbf{k}} = Q_{31}\hat{\mathbf{I}} + Q_{32}\hat{\mathbf{J}} + Q_{33}\hat{\mathbf{K}} \tag{c}$$

The Qs are components of the direction cosine matrix $[\mathbf{Q}]_{Xx}$ in Equation 9.148. Carrying out the cross product in (b) yields, in matrix form, the components of the moment in the inertial frame,

$$\{\mathbf{M}_0\}_X = \begin{Bmatrix} -mgdQ_{32} \\ mgdQ_{31} \\ 0 \end{Bmatrix} \tag{d}$$

To arrive at the components of $\{\mathbf{M}_0\}_x$ in the body frame, we carry out the transformation

$$\{\mathbf{M}_0\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{M}_0\}_X \tag{e}$$

The MATLAB implementation of the following procedure is listed in Appendix D.39.

Step 1:

Specify the initial orientation of the xyz axes of the body frame, thereby defining the initial value of the direction cosine matrix $[\mathbf{Q}]_{Xx}$.

$$\begin{aligned}\hat{\mathbf{i}} &= \hat{\mathbf{J}} \\ \hat{\mathbf{j}} &= -\cos 60°\hat{\mathbf{I}} + \sin 60°\hat{\mathbf{K}} \\ \hat{\mathbf{k}} &= \sin 60°\hat{\mathbf{I}} + \cos 60°\hat{\mathbf{K}}\end{aligned}$$

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$$

Step 2:

Compute the initial unit quaternion $\{\hat{\mathbf{q}}_0\}$ using Algorithm 9.2.

$$\{\hat{\mathbf{q}}_0\} = \begin{Bmatrix} 0.35355 \\ 0.35355 \\ 0.61237 \\ 0.61237 \end{Bmatrix}$$

Step 3:

Specify the initial value of the angular velocity $\{\boldsymbol{\omega}_0\} = \lfloor \omega_{x0} \quad \omega_{y0} \quad \omega_{z0} \rfloor^T$ in body frame components (1000 rpm = 104.72 rad/s):

$$\{\boldsymbol{\omega}_0\} = \begin{Bmatrix} 0 \\ 0 \\ 104.72 \end{Bmatrix} \text{(rad/s)} \tag{f}$$

Step 4:

Supply $\{\boldsymbol{\omega}_0\}$ and $\{\hat{\mathbf{q}}_0\}$ as initial conditions to the Runge-Kutta-Fehlberg 4(5) numerical integration procedure (Algorithm 1.3) to find the angular velocity $\{\boldsymbol{\omega}\}$ and quaternion $\{\mathbf{q}\}$ as functions of time. At each step of the numerical integration process:

(i) Use the current value of $\{\hat{\mathbf{q}}\}$ to compute $[\mathbf{Q}]_{Xx}$ from Equation 9.148.
(ii) Use the current values of $[\mathbf{Q}]_{Xx}$ and $\{\boldsymbol{\omega}\}$ to compute $d\{\boldsymbol{\omega}\}/dt$ from (a), (d) and (e).
(iii) Use the current values of $\{\hat{\mathbf{q}}\}$ and $\{\boldsymbol{\omega}\}$ to compute $d\{\hat{\mathbf{q}}\}/dt$ from Equations 9.151.

Step 5:

At each solution time:

(i) Use Equation 9.148 (Algorithm 9.1) to compute the DCM $[\mathbf{Q}]_{Xx}$.
(ii) Use Algorithm 4.3 to compute the Euler angles ϕ (precession), θ (nutation) and ψ (spin) from the DCM $[\mathbf{Q}]_{Xx}$.

Step 6:

Plot the results.

Figure 9.29 shows the precession (ϕ), nutation (θ) and spin (ψ) angles as a function of time. Although ϕ and ψ steadily increase with time, Algorithm 4.3 limits their range to between 0° and 360°, which explains their sawtooth-like appearance. In this simulation, the top is spinning about its stationary symmetry axis (z axis) when it is released. Precession begins as the spin axis falls through 15° before reversing and returning to its original orientation. This nutation repeats with a frequency of slightly less than 6 cycles per second.

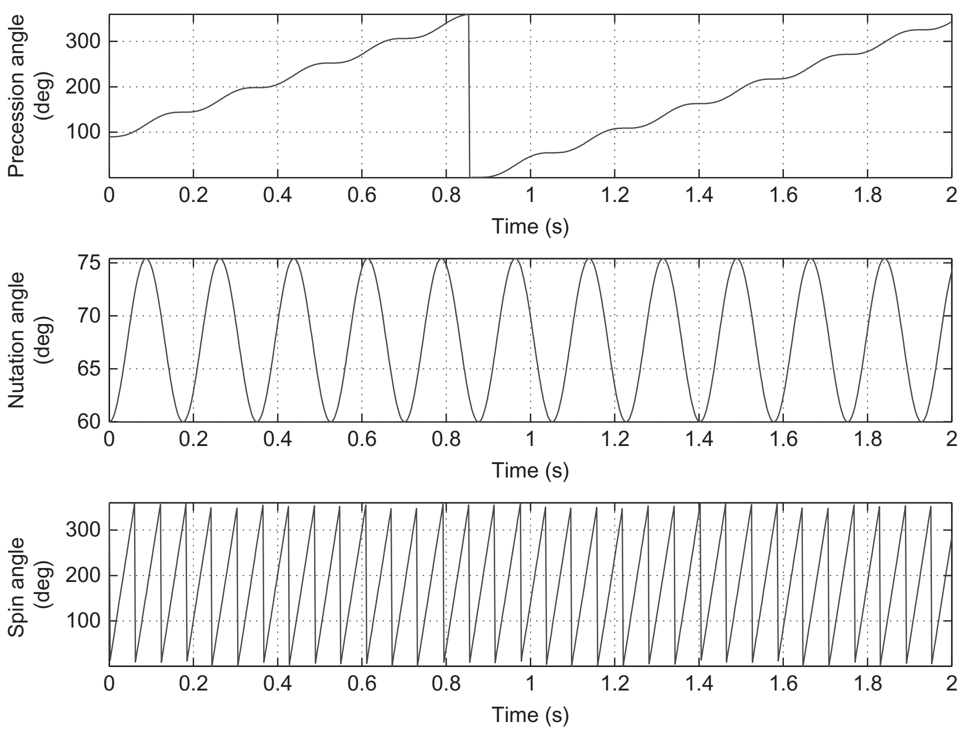


FIGURE 9.29

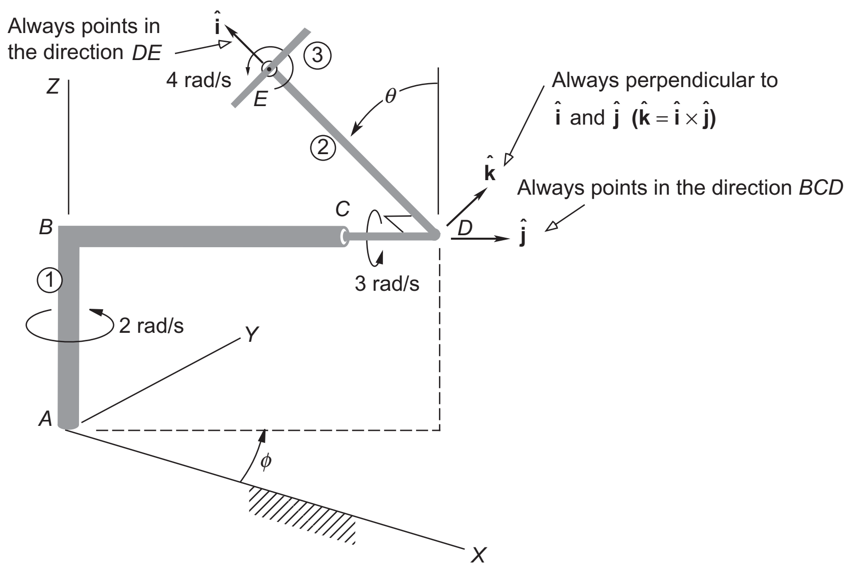
Precession, nutation and spin angles of a prolate top ($A = B = 0.0012\,\text{kg}\cdot\text{m}^2$, $C = 0.00045\,\text{kg}\cdot\text{m}^2$).

In Example 9.15 we assumed that the nutation was zero so that the inclination angle θ remained fixed at its initial value of 60°. Figure 9.20b shows this requires the top to precess at a rate of 51.9 rpm. Running the above numerical simulation with this precession component included in the initial angular velocity vector [Equation (f)] virtually eliminates the nutation evident in Figure 9.29.

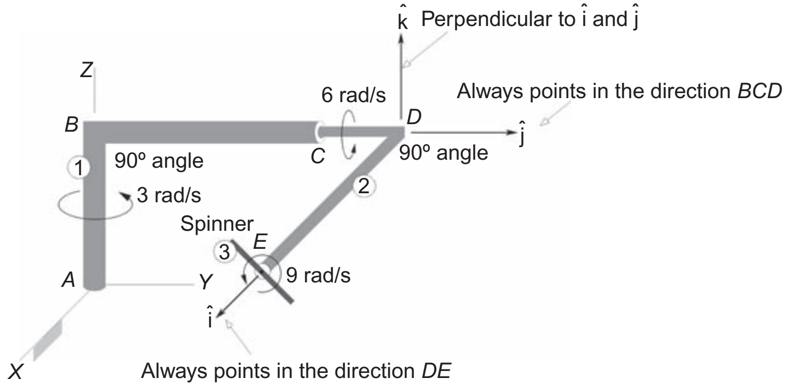
PROBLEMS

Section 9.2

9.1 Rigid, bent shaft 1 (*ABC*) rotates at a constant angular velocity of $2\hat{\mathbf{K}}$ rad/s around the positive *Z* axis of the inertial frame. Bent shaft 2 (*CDE*) rotates around *BC* with a constant angular velocity of $3\hat{\mathbf{j}}$ rad/s, relative to *BC*. Spinner 3 at *E* rotates around *DE* with a constant angular velocity of $4\hat{\mathbf{i}}$ rad/s relative to *DE*. Calculate the magnitude of the absolute angular acceleration $\boldsymbol{\alpha}_3$ of the spinner at the instant shown.
{Ans.: $\|\boldsymbol{\alpha}_3\| = \sqrt{180 + 64\sin^2\theta - 144\cos\theta}$ (rad/s^2)}



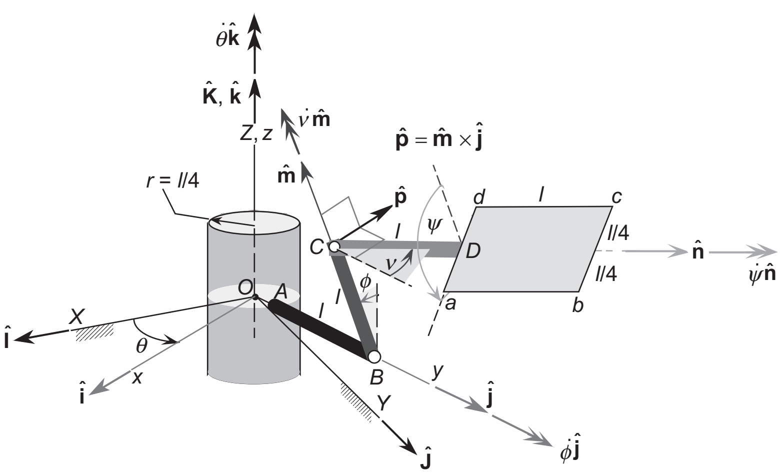
9.2 All of the spin rates shown are constant. Calculate the magnitude of the absolute angular acceleration $\boldsymbol{\alpha}_3$ of the spinner at the instant shown (i.e., at the instant when the unit vector $\hat{\mathbf{i}}$ is parallel to the *X* axis and the unit vector $\hat{\mathbf{j}}$ is parallel to the *Y* axis).
{Ans.: $\|\boldsymbol{\alpha}_3\| = 63$ rad/s^2}



9.3 The body-fixed *xyz* frame is attached to the cylinder as shown. The cylinder rotates around the inertial *Z* axis, which is collinear with the *z* axis, with a constant absolute angular velocity $\dot{\theta}\hat{\mathbf{k}}$. Rod *AB* is attached to the cylinder and aligned with the y-axis. Rod *BC* is perpendicular to *AB* and rotates around *AB* with the constant angular velocity $\dot{\phi}\hat{\mathbf{j}}$ relative to the cylinder. Rod *CD* is perpendicular to *BC* and rotates around *BC* with the constant angular velocity $\dot{\nu}\hat{\mathbf{m}}$ relative to *BC*, where $\hat{\mathbf{m}}$ is the unit vector in the direction of *BC*. The plate *abcd* rotates around *CD* with a constant angular velocity $\dot{\psi}\hat{\mathbf{n}}$ relative to *CD*, where the unit vector $\hat{\mathbf{n}}$ points in the direction of *CD*. Thus, the absolute angular velocity of the plate is $\boldsymbol{\omega}_{\text{plate}} = \dot{\theta}\hat{\mathbf{k}} + \dot{\phi}\hat{\mathbf{j}} + \dot{\nu}\hat{\mathbf{m}} + \dot{\psi}\hat{\mathbf{n}}$. Show that

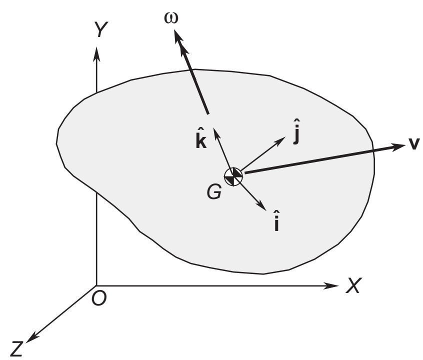
(a) $\boldsymbol{\omega}_{\text{plate}} = (\dot{\nu}\sin\phi - \dot{\psi}\cos\phi\sin\nu)\hat{\mathbf{i}} + (\dot{\phi} + \dot{\psi}\cos\nu)\hat{\mathbf{j}} + (\dot{\theta} + \dot{\nu}\cos\phi + \dot{\psi}\sin\phi\sin\nu)\hat{\mathbf{k}}$

(b)
$$\begin{aligned}\boldsymbol{\alpha}_{\text{plate}} = \frac{d\boldsymbol{\omega}_{\text{plate}}}{dt} &= [\dot{\nu}(\dot{\phi}\cos\phi - \dot{\psi}\cos\phi\cos\nu) + \dot{\psi}\dot{\phi}\sin\phi\sin\nu - \dot{\psi}\dot{\theta}\cos\nu - \dot{\phi}\dot{\theta}]\hat{\mathbf{i}} \\ &+ [\dot{\nu}(\dot{\theta}\sin\phi - \dot{\psi}\sin\nu) - \dot{\psi}\dot{\theta}\cos\phi\sin\nu]\hat{\mathbf{j}} \\ &+ [\dot{\psi}\dot{\nu}\cos\nu\sin\phi + \dot{\psi}\dot{\phi}\cos\phi\sin\nu - \dot{\phi}\dot{\nu}\sin\phi]\hat{\mathbf{k}}\end{aligned}$$



(c) $\mathbf{a}_C = -l(\dot{\phi}^2 + \dot{\theta}^2)\sin\phi\hat{\mathbf{i}} + \left(2l\dot{\phi}\dot{\theta}\cos\phi - \frac{5}{4}l\dot{\theta}^2\right)\hat{\mathbf{j}} - l\dot{\phi}^2\cos\phi\hat{\mathbf{k}}$

9.4 The mass center G of a rigid body has a velocity $\mathbf{v} = t^3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ meters per second and an angular velocity $\boldsymbol{\omega} = 2t^2\hat{\mathbf{k}}$ radians per second, where t is time in seconds. The $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ unit vectors are attached to and rotate with the rigid body. Calculate the magnitude of the acceleration $\mathbf{a}_G$ of the center of mass at $t = 2$ seconds.
{Ans.: $\mathbf{a}_G = -20\hat{\mathbf{i}} + 64\hat{\mathbf{j}}\ (\text{m/s}^2)$}



9.5 Relative to a body-fixed xyz frame $[I_G] = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} (\text{kg}\cdot\text{m}^2)$ and $\boldsymbol{\omega} = 2t^2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}\ (\text{rad/s})$, where t is the time in seconds. Calculate the magnitude of the net moment about the center of mass G at $t = 3$ s.
{Ans.: 3374 N·m}

9.6 The inertial angular velocity of a rigid body is $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are the unit vectors of a co-moving frame whose inertial angular velocity is $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}}$. Calculate the components of angular acceleration of the rigid body in the moving frame, assuming that ω_x, ω_y and ω_z are all constant.
{Ans.: $\boldsymbol{\alpha} = \omega_y\omega_z\hat{\mathbf{i}} - \omega_x\omega_z\hat{\mathbf{j}}$}

Section 9.5

9.7 Find the moments of inertia about the center of mass of the system of six point masses listed in the table.

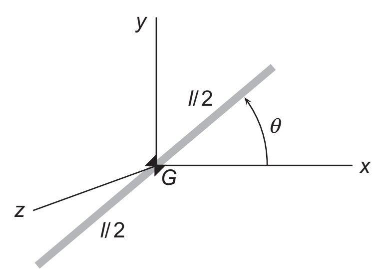
Point, i	Mass m_i (kg)	x_i (m)	y_i (m)	z_i (m)
1	10	1	1	1
2	10	−1	−1	−1
3	8	4	−4	4
4	8	−2	2	−2
5	12	3	−3	−3
6	12	−3	3	3

$$\{\text{Ans.: } [\mathbf{I}_G] = \begin{bmatrix} 783.5 & 351.7 & 40.27 \\ 351.7 & 783.5 & -80.27 \\ 40.27 & -80.27 & 783.5 \end{bmatrix} (\text{kg-m}^2)\}$$

9.8 Find the mass moment of inertia of the configuration of Problem 9.7 about an axis through the origin and the point with coordinates (1 m, 2 m, 2 m).
{Ans.: 898.7 kg-m^2}

9.9 A uniform slender rod of mass m and length l lies in the xy plane inclined to the x axis by the angle θ. Use the results of Example 9.10 to find the mass moments of inertia about the xyz axes passing through the center of mass G.

$$\{\text{Ans.: } [\mathbf{I}_G] = \frac{1}{12}ml^2 \begin{bmatrix} \sin^2\theta & -\frac{1}{2}\sin 2\theta & 0 \\ -\frac{1}{2}\sin 2\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$$



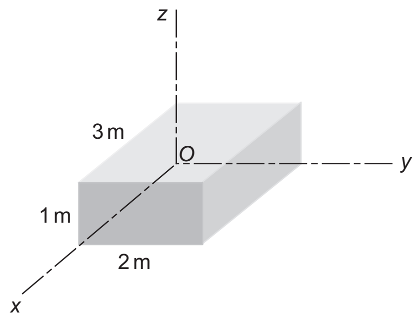
9.10 The uniform rectangular box has a mass of 1000 kg. The dimensions of its edges are shown.

(a) Find the mass moments of inertia about the xyz axes.

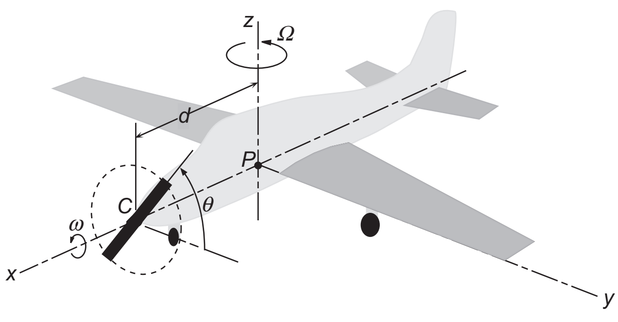
$$\{\text{Ans.: } [\mathbf{I}_O] = \begin{bmatrix} 1666.7 & -1500 & -750 \\ -1500 & 3333.3 & -500 \\ -750 & -500 & 4333.3 \end{bmatrix} (\text{kg-m}^2)\}$$

(b) Find the principal moments of inertia and the principal directions about the *xyz* axes through *O*.

{Partial ans.: $\boldsymbol{I}_1 = 568.9$ kg-m^2, $\hat{\mathbf{v}}_1 = 0.8366\hat{\mathbf{i}} + 0.4960\hat{\mathbf{j}} + 0.2326\hat{\mathbf{k}}$}
(c) Find the moment of inertia about the line through *O* and the point with coordinates (3 m, 2 m, 1 m).
{Ans.: 583.3 kg·m^2}



9.11 A taxing airplane turns about its vertical axis with an angular velocity Ω while its propeller spins at an angular velocity $\omega = \dot{\theta}$. Determine the components of the angular momentum of the propeller about the body-fixed *xyz* axes centered at *P*. Treat the propeller as a uniform slender rod of mass m and length *l*.
{Ans.: $\mathbf{H}_P = \frac{1}{12}m\omega l^2\hat{\mathbf{i}} - \frac{1}{24}m\Omega l^2 \sin 2\theta\hat{\mathbf{j}} + \left(\frac{1}{12}ml^2\cos^2\theta + md^2\right)\Omega\hat{\mathbf{k}}$}



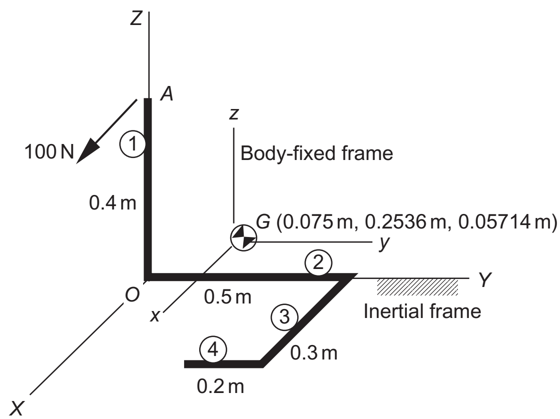
9.12 Relative to an *xyz* frame of reference the components of angular momentum **H** are given by

$$\{\mathbf{H}\} = \begin{bmatrix} 1000 & 0 & -300 \\ 0 & 1000 & 500 \\ -300 & 500 & 1000 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \quad \text{(kg-m}^2\text{/s)}$$

where ω_x, ω_y and ω_z are the components of the angular velocity $\boldsymbol{\omega}$. Find the components of $\boldsymbol{\omega}$ such that $\{\mathbf{H}\} = 1000\,\{\boldsymbol{\omega}\}$, where the magnitude of $\boldsymbol{\omega}$ is 20 radians/second.
{Ans.: $\boldsymbol{\omega} = 17.15\mathbf{i} + 10.29\mathbf{j}$ (rad/s)}

9.13 Relative to a body-fixed *xyz* frame $[\mathbf{I}_G] = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} (\text{kg}\cdot\text{m}^2)$ and $\boldsymbol{\omega} = 2t^2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}$ (rad/s), where t is the time in seconds. Calculate the magnitude of the net moment about the center of mass G at $t = 3$ s.
{Ans.: 3374 N-m}

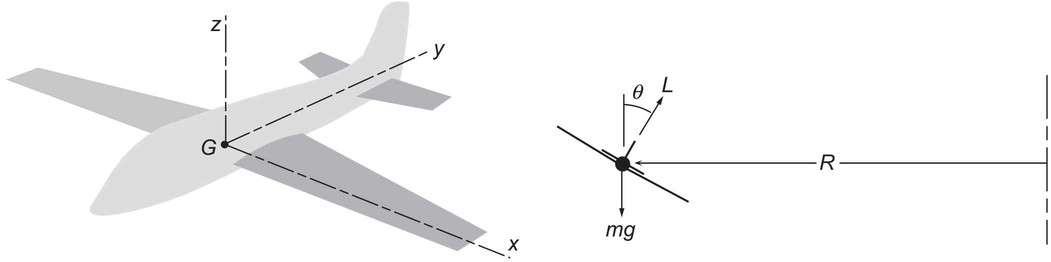
9.14 In Example 9.11, the system is at rest when a 100 N force is applied to point A as shown. Calculate the inertial components of angular acceleration at that instant.
{Ans.: $\alpha_X = 143.9\,\text{rad/s}^2$, $\alpha_Y = 553.1\,\text{rad/s}^2$, $\alpha_Z = 7.61\,\text{rad/s}^2$}



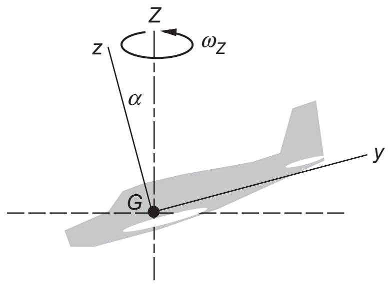
9.15 The body-fixed *xyz* axes pass through the center of mass G of the airplane and are the principal axes of inertia. The moments of inertia about these axes are A, B and C, respectively. The airplane is in a level turn of radius R with a speed v.

(a) Calculate the bank angle θ.
(b) Use Euler's equations to calculate the rolling moment M_y that must be applied by the aerodynamic surfaces.

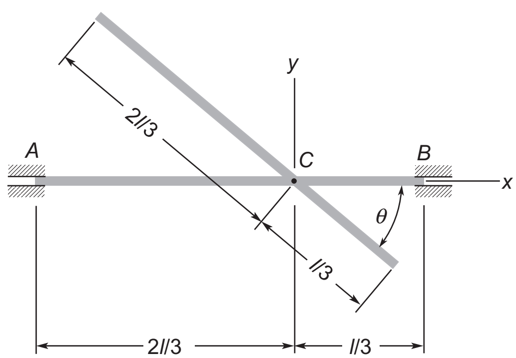
{Ans.: (a) $\theta = \tan^{-1} v^2/Rg$; (b) $M_y = v^2 \sin 2\theta(C - A)/2R^2$}



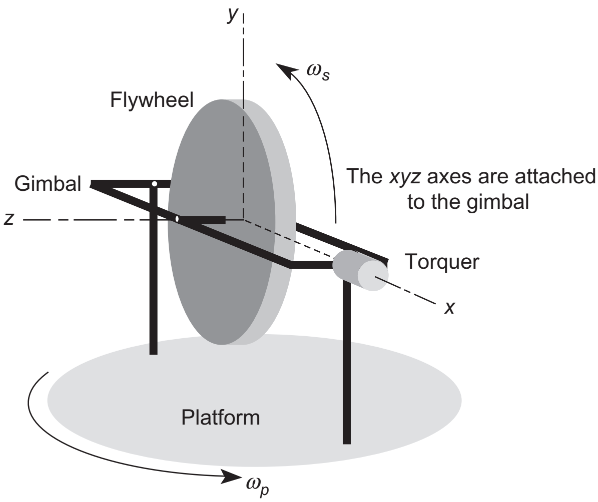
9.16 The airplane in Problem 9.15 is spinning with an angular velocity ω_Z about the vertical Z axis. The nose is pitched down at the angle α. What external moments must accompany this maneuver?
{Ans.: $M_y = M_z = 0, \quad M_x = \omega_Z^2 \sin 2\alpha (C - B)/2$}



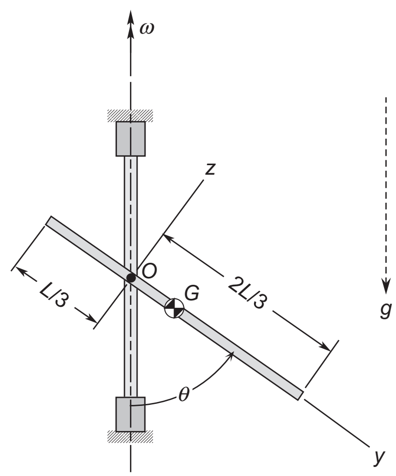
9.17 Two identical slender rods of mass m and length l are rigidly joined together at an angle θ at point C, their 2/3 point. Determine the bearing reactions at A and B if the shaft rotates at a constant angular velocity ω. Neglect gravity and assume that the only bearing forces are normal to rod AB.
{Ans.: $\|\mathbf{F}_A\| = m\omega^2 l \sin\theta(1 + 2\cos\theta)/18, \ \|\mathbf{F}_B\| = m\omega^2 l \sin\theta(1 - \cos\theta)/9$}



9.18 The flywheel ($A = B = 5$ kg-m^2, $C = 10$ kg-m^2) spins at a constant angular velocity of $\boldsymbol{\omega}_s = 100\hat{\mathbf{k}}$ (rad/s). It is supported by a massless gimbal that is mounted on the platform as shown. The gimbal is initially stationary relative to the platform, which rotates with a constant angular velocity of $\boldsymbol{\omega}_p = 0.5\hat{\mathbf{j}}$ (rad/s). What will be the gimbal's angular acceleration when the torquer applies a torque of $600\hat{\mathbf{i}}$ (N-m) to the flywheel?
{Ans.: $70\hat{\mathbf{i}}$ rad/s^2}

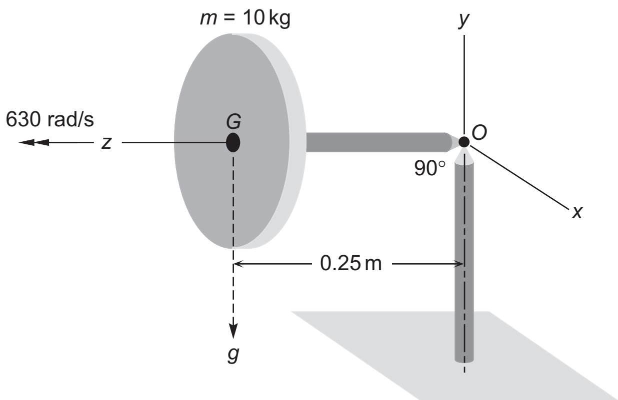


9.19 A uniform slender rod of length L and mass m is attached by a smooth pin at O to a vertical shaft that rotates at constant angular velocity ω. Use Euler's equations and the body frame shown to calculate ω at the instant shown.
{Ans.: $\omega = \sqrt{3g/(2L\cos\theta)}$}



9.20 A uniform, thin circular disk of mass 10 kg spins at a constant angular velocity of 630 rad/s about axis OG, which is normal to the disk, and pivots about the frictionless ball joint at O. Neglecting the mass of the shaft OG, determine the rate of precession if OG remains horizontal as shown. Gravity acts

down, as shown. G is the center of mass, and the y-axis remains fixed in space. The moments of inertia about G are $I_{G_z} = 0.02812 \text{ kg} \cdot \text{m}^2$, and $I_{G_x} = I_{G_y} = 0.01406 \text{ kg} \cdot \text{m}^2$.
{Ans.: 1.38 rad/s}



Section 9.7

9.21 Consider a rigid body experiencing rotational motion associated with angular velocity $\boldsymbol{\omega}$. The inertia tensor (relative to body-fixed axes though the center of mass G) is

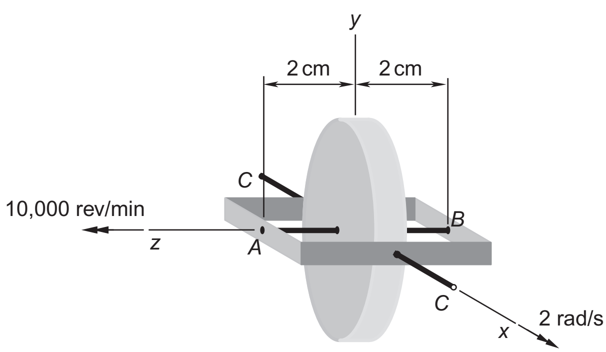
$$\begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix} \quad (\text{kg} - \text{m}^2)$$

and $\boldsymbol{\omega} = 10\hat{\mathbf{i}} + 20\hat{\mathbf{j}} + 30\hat{\mathbf{k}}$ (rad/sec). Calculate
(a) the angular momentum $\mathbf{H}_G$, and
(b) the rotational kinetic energy (about G).
{Partial ans.: (b) $T_R = 23{,}000$ J}

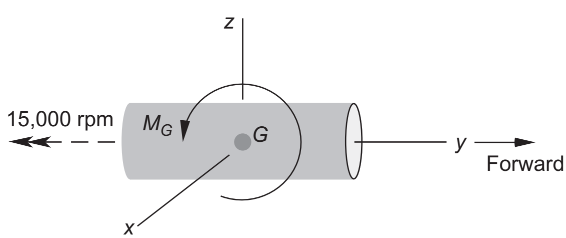
Section 9.8

9.22 At the end of its take-off run, an airplane with retractable landing gear leaves the runway with a speed of 130 km/hr. The gear rotate into the wing with an angular velocity of 0.8 rad/s with the wheels still spinning. Calculate the gyroscopic bending moment in the wheel bearing B. The wheels have a diameter of 0.6 m, a mass of 25 kg and a radius of gyration of 0.2 m.
{Ans.: 96.3 N-m}

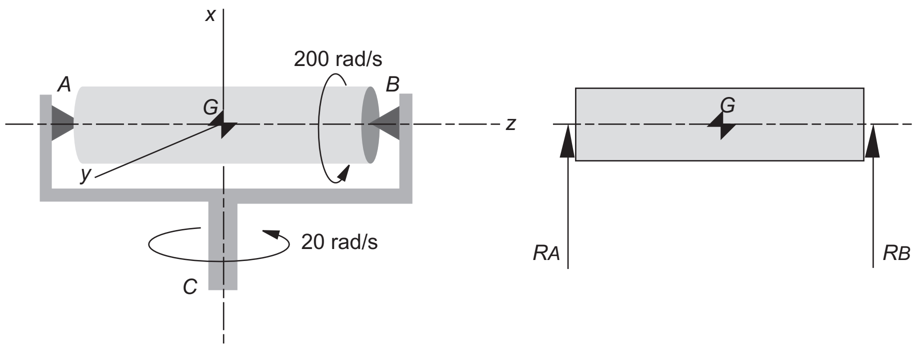
9.23 The gyro rotor, including shaft *AB*, has a mass of 4 kg and a radius of gyration 7 cm around *AB*. The rotor spins at 10 000 revolutions per minute while also being forced to rotate around the gimbal axis *CC* at 2 radians per second. What are the transverse forces exerted on the shaft at *A* and *B*? Neglect gravity.
{Ans.: 1.03 kN}



9.24 A jet aircraft is making a level, 2.5 km radius turn to the left at a speed of 650 km/hr. The rotor of the turbojet engine has a mass of 200 kg, a radius of gyration of 0.25 m and rotates at 15 000 revolutions per minute clockwise as viewed from the front of the airplane. Calculate the gyroscopic moment that the engine exerts on the airframe and specify whether it tends to pitch the nose up or down.
{Ans.: 1.418 kN-m; pitch down}



9.25 A cylindrical rotor of mass 10 kg, radius 0.05 m and length 0.60 m is simply supported at each end in a cradle that rotates at a constant 20 rad/s counterclockwise as viewed from above. Relative to the cradle, the rotor spins at 200 rad/s counterclockwise as viewed from the right (from *B* towards *A*). Assuming there is no gravity, calculate the bearing reactions R_A and R_B. Use the co-moving *xyz* frame shown, which is attached to the cradle but not to the rotor.
{Ans.: $R_A = -R_B = 83.3\,\text{N}$}



Section 9.9

9.26 The Euler angles of a rigid body are $\phi = 50°$, $\theta = 25°$ and $\psi = 70°$. Calculate the angle (a positive number) between the body-fixed *x* axis and the inertial *X* axis.
{Ans.: 115.6°}

List of Key Terms

angular impulse
angular momentum of a rigid body
angular velocities in terms of Euler angle rates
angular velocities in terms of yaw, pitch and roll rates
direction cosine matrix from quaternion
eigenvalue
eigenvector
equation of rotational motion of continuous medium
equation of translational motion of a rigid body
Euler angle rates in terms of angular velocities
Euler axis and principal angle
Eulers equations
gyroscopic moment
inertia tensor
mass moments of inertia
modified Euler equations

moments of inertia of a point mass
quaternion from direction cosine matrix
radii of gyration
relation between accelerations of points on a rigid body
relation between velocities of points on a rigid body
rotational kinetic energy
time derivative of a quaternion
unit quaternion
yaw, pitch and roll rates in terms of angular velocities

CHAPTER

Satellite attitude dynamics

10

Chapter outline

10.1	Introduction	573
10.2	Torque-free motion	574
10.3	Stability of torque-free motion	584
10.4	Dual-spin spacecraft	589
10.5	Nutation damper	593
10.6	Coning maneuver	601
10.7	Attitude control thrusters	605
10.8	Yo-yo despin mechanism	608
10.9	Gyroscopic attitude control	615
10.10	Gravity gradient stabilization	631

10.1 INTRODUCTION

In this chapter we apply the equations of rigid body motion presented in Chapter 9 to the study of the attitude dynamics of satellites. We begin with spin-stabilized spacecraft. Spinning a satellite around its axis is a very simple way to keep the vehicle pointed in a desired direction. We investigate the stability of a spinning satellite to show that only oblate spinners are stable over long times. Overcoming this restriction on the shape of spin-stabilized spacecraft led to the development of dual-spin vehicles, which consist of two interconnected segments rotating at different rates about a common axis. We consider the stability of that type of configuration as well. The nutation damper and its effect on the stability of spin-stabilized spacecraft is covered next.

The rest of the chapter is devoted to some of the common means of changing the attitude or motion of a spacecraft by applying external or internal forces or torques. The coning maneuver changes the attitude of a spinning spacecraft by using thrusters to apply impulsive torque, which alters the angular momentum and hence the orientation of the spacecraft. The much-used yo-yo despin maneuver reduces or eliminates the spin rate by releasing small masses attached to cords initially wrapped around the cylindrical vehicle.

An alternative to spin stabilization is three-axis stabilization by gyroscopic attitude control. In this case, the vehicle does not continuously rotate. Instead, the desired attitude is maintained by the spin of small wheels within the spacecraft. These are called reaction wheels or momentum wheels. If allowed to pivot relative to the vehicle, they are known as control moment gyros. The attitude of the vehicle can be changed by varying the speed or orientation of these internal gyros. Small thrusters may also be used to supplement the gyroscopic attitude control and to hold the spacecraft orientation fixed when it is necessary to despin or reorient gyros that have become saturated (reached their maximum spin rate or deflection) over time.

The chapter concludes with a discussion of how the earth's gravitational field by itself can stabilize the attitude of large satellites such as the space shuttle or space station in low earth orbits.

10.2 TORQUE-FREE MOTION

Gravity is the only force acting on a satellite coasting in orbit (if we neglect secondary drag forces and the gravitational influence of bodies other than the planet being orbited). Unless the satellite is unusually large, the gravitational force is concentrated at the center of mass G. Since the net moment about the center of mass is zero, the satellite is "torque-free," and according to Equation 9.30,

$$\dot{\mathbf{H}}_G = \mathbf{0} \tag{10.1}$$

The angular momentum $\mathbf{H}_G$ about the center of mass does not depend on time. It is a vector fixed in inertial space. We will use $\mathbf{H}_G$ to define the Z axis of an inertial frame, as shown in Figure 10.1. The xyz axes in the figure comprise the principal body frame, centered at G. The angle between the z axis and $\mathbf{H}_G$ is (by definition of the Euler angles) the nutation angle θ. Let us determine the conditions for which θ is constant. From the dot product operation we know that

$$\cos\theta = \frac{\mathbf{H}_G}{\|\mathbf{H}_G\|} \cdot \hat{\mathbf{k}}$$

Differentiating this expression with respect to time, keeping in mind Equation 10.1, we get

$$\frac{d\cos\theta}{dt} = \frac{\mathbf{H}_G}{\|\mathbf{H}_G\|} \cdot \frac{d\hat{\mathbf{k}}}{dt}$$

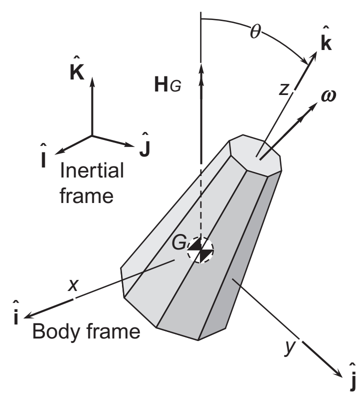


FIGURE 10.1

Rotationally symmetric satellite in torque-free motion.

But $d\hat{\mathbf{k}}/dt = \boldsymbol{\omega} \times \hat{\mathbf{k}}$, according to Equation 1.52, so

$$\frac{d\cos\theta}{dt} = \frac{\mathbf{H}_G \cdot (\boldsymbol{\omega} \times \hat{\mathbf{k}})}{\|\mathbf{H}_G\|} \tag{10.2}$$

Now

$$\boldsymbol{\omega} \times \hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ 0 & 0 & 1 \end{vmatrix} = \omega_y \hat{\mathbf{i}} - \omega_x \hat{\mathbf{j}}$$

Furthermore, we know from Equation 9.67 that the angular momentum is related to the angular velocity in the principal body frame by the expression

$$\mathbf{H}_G = A\omega_x \hat{\mathbf{i}} + B\omega_y \hat{\mathbf{j}} + C\omega_z \hat{\mathbf{k}}$$

Thus,

$$\mathbf{H}_G \cdot (\boldsymbol{\omega} \times \hat{\mathbf{k}}) = (A\omega_x \hat{\mathbf{i}} + B\omega_y \hat{\mathbf{j}} + C\omega_z \hat{\mathbf{k}}) \cdot (\omega_y \hat{\mathbf{i}} - \omega_x \hat{\mathbf{j}}) = (A - B)\omega_x \omega_y$$

so that Equation 10.2 can be written

$$\dot{\theta} = \omega_n = -\frac{(A - B)\omega_x \omega_y}{\|\mathbf{H}_G\| \sin\theta} \tag{10.3}$$

From this we see that the nutation rate vanishes only if $A = B$. If $A \neq B$, the nutation angle θ will not in general be constant.

Relative to the body frame, Equation 10.1 is written (cf. Equation 1.56)

$$\dot{\mathbf{H}}_G)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_G = \mathbf{0}$$

Euler's equations for torque-free motion are the three components of this vector equation, and they are given by Equations 9.72b:

$$\begin{aligned} A\dot{\omega}_x + (C - B)\omega_z \omega_y &= 0 \\ B\dot{\omega}_y + (A - C)\omega_x \omega_z &= 0 \\ C\dot{\omega}_z + (B - A)\omega_y \omega_x &= 0 \end{aligned} \tag{10.4}$$

In the interest of simplicity, let us consider the special case illustrated in Figure 10.1, namely that in which the z axis is an axis of rotational symmetry, so that $A = B$. Then Equations 10.4 yield ***Euler's equations for torque-free motion with rotational symmetry***:

$$\begin{aligned} A\dot{\omega}_x + (C - A)\omega_y \omega_z &= 0 \\ A\dot{\omega}_y + (A - C)\omega_z \omega_x &= 0 \\ C\dot{\omega}_z &= 0 \end{aligned} \tag{10.5}$$

From Equation 10.5_3 we see that the body frame z component of the angular velocity is constant:

$$\omega_z = \omega_o \quad \text{(constant)} \tag{10.6}$$

The assumption of rotational symmetry therefore reduces the three differential equations 10.4 to just two. Substituting Equation 10.6 into Equations 10.5_1 and 10.5_2 and introducing the notation

$$\lambda = \frac{A - C}{A}\omega_o \tag{10.7}$$

they can be written

$$\begin{aligned} \dot{\omega}_x - \lambda\omega_y &= 0 \\ \dot{\omega}_y + \lambda\omega_x &= 0 \end{aligned} \tag{10.8}$$

Notice that the sign of λ depends on the relative values of the principal moments of inertia A and B.

To reduce Equations 10.8 in ω_x and ω_y to just one equation in ω_x, we first differentiate Equation 10.8_1 with respect to time to get

$$\ddot{\omega}_x - \lambda\dot{\omega}_y = 0 \tag{10.9}$$

We then solve Equation 10.8_2 for $\dot{\omega}_y$ and substitute that result into Equation 10.9, which leads to

$$\ddot{\omega}_x + \lambda^2\omega_x = 0 \tag{10.10}$$

The solution of this well-known differential equation is

$$\omega_x = \omega_{xy}\sin\lambda t \tag{10.11}$$

where the constant amplitude ω_{xy} ($\omega_{xy} \neq 0$) has yet to be determined. (Without loss of generality, we have set the phase angle, the other constant of integration, equal to zero.) Substituting Equation 10.11 back into Equation 10.81 yields the solution for ω_y,

$$\omega_y = \frac{1}{\lambda}\frac{d\omega_x}{dt} = \frac{1}{\lambda}\frac{d}{dt}(\omega_{xy}\sin\lambda t)$$

or

$$\omega_y = \omega_{xy}\cos\lambda t \tag{10.12}$$

Equations 10.6, 10.11 and 10.12 give the components of the absolute angular velocity $\boldsymbol{\omega}$ along the three principal body axes,

$$\boldsymbol{\omega} = \omega_{xy}\sin\lambda t\hat{\mathbf{i}} + \omega_{xy}\cos\lambda t\hat{\mathbf{j}} + \omega_o\hat{\mathbf{k}}$$

or

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\perp + \omega_o\hat{\mathbf{k}} \tag{10.13}$$

where

$$\boldsymbol{\omega}_\perp = \omega_{xy}(\sin\lambda t\,\hat{\mathbf{i}} + \cos\lambda t\,\hat{\mathbf{j}}) \tag{10.14}$$

$\boldsymbol{\omega}_\perp$ ("omega-perp") is the component of $\boldsymbol{\omega}$ normal to the z axis. It sweeps out a circle of radius ω_{xy} in the xy plane at an angular velocity λ. Thus, $\boldsymbol{\omega}$ sweeps out a cone, as illustrated in Figure 10.2. If ω_0 is positive, then the body has an inertial counterclockwise rotation around the positive z axis if $A > C$ ($\lambda > 0$). However, an observer fixed in the body would see the world rotating in the opposite direction, clockwise around positive z, as the figure shows. Interestingly, the situation is reversed if $A < C$.

From Equations 9.115, the three Euler orientation angles (and their rates) are related to the body angular velocity components ω_x, ω_y and ω_z by

$$\begin{aligned}
\omega_p &= \dot{\phi} = \frac{1}{\sin\theta}(\omega_x \sin\psi + \omega_y \cos\psi) \\
\omega_n &= \dot{\theta} = \omega_x \cos\psi - \omega_y \sin\psi \\
\omega_s &= \dot{\psi} = -\frac{1}{\tan\theta}(\omega_x \sin\psi + \omega_y \cos\psi) + \omega_z
\end{aligned}$$

Substituting Equations 10.6, 10.11 and 10.12 into these three equations yields

$$\begin{aligned}
\omega_p &= \frac{\omega_{xy}}{\sin\theta}\cos(\lambda t - \psi) \\
\omega_n &= \omega_{xy}\sin(\lambda t - \psi) \\
\omega_s &= \omega_o - \frac{\omega_{xy}}{\tan\theta}\cos(\lambda t - \psi)
\end{aligned} \tag{10.15}$$

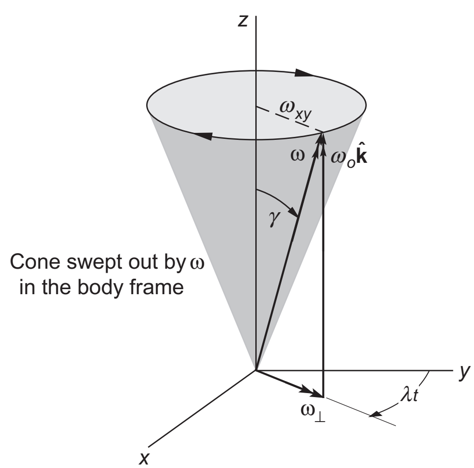


FIGURE 10.2

Components of the angular velocity $\boldsymbol{\omega}$ in the body frame.

Since $A = B$, we know from Equation 10.3 that $\omega_n = 0$. It follows from Equation 10.15_2 that

$$\psi = \lambda t \tag{10.16}$$

(Actually, $\lambda t - \psi = n\pi$, $n = 0,1,2, \ldots$. We can set $n = 0$ without loss of generality.) Substituting Equation 10.16 into Equations 10.15_1 and 10.15_3 yields

$$\omega_p = \frac{\omega_{xy}}{\sin\theta} \tag{10.17}$$

and

$$\omega_s = \omega_o - \frac{\omega_{xy}}{\tan\theta} \tag{10.18}$$

We have thus obtained the Euler angle rates ω_p and ω_s in terms of the components of the angular velocity $\boldsymbol{\omega}$ in the body frame.

Differentiating Equation 10.16 with respect to time shows that

$$\lambda = \dot{\psi} = \omega_s \tag{10.19}$$

That is, the rate λ at which $\boldsymbol{\omega}$ rotates around the body z axis equals the spin rate. Substituting the spin rate for λ in Equation 10.7 shows that ω_s is related to ω_o alone,

$$\omega_s = \frac{A - C}{A}\omega_o \tag{10.20}$$

Observe that ω_s and ω_0 are opposite in sign if $A < C$.

Eliminating ω_s from Equations 10.18 and 10.20 yields the relationship between the magnitudes of the orthogonal components of the angular velocity in Equation 10.13,

$$\omega_{xy} = \frac{C}{A}\omega_o \tan\theta \tag{10.21}$$

A similar relationship exists between ω_p and ω_s, which generally are not orthogonal. Substitute Equation 10.21 into Equation 10.17 to obtain

$$\omega_o = \frac{A}{C}\omega_p \cos\theta \tag{10.22}$$

Placing this result in Equation 10.20 leaves an expression involving only ω_p and ω_s, from which we get a useful formula relating the precession of a torque-free body to its spin,

$$\omega_p = \frac{C}{A - C}\frac{\omega_s}{\cos\theta} \tag{10.23}$$

Observe that if $A > C$ (i.e., the body is ***prolate***, like a soup can or an American football), then ω_p has the same sign as ω_s, which means the precession is prograde. For an ***oblate*** body (like a tuna fish can or a frisbee), $A < C$ and the precession is retrograde.

The components of angular momentum along the body frame axes are obtained from the body frame components of $\boldsymbol{\omega}$,

$$\mathbf{H}_G = A\omega_x\hat{\mathbf{i}} + A\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}}$$

or

$$\mathbf{H}_G = \mathbf{H}_\perp + C\omega_o\hat{\mathbf{k}} \tag{10.24}$$

where

$$\mathbf{H}_\perp = A\omega_{xy}(\sin\omega_s t\,\hat{\mathbf{i}} + \cos\omega_s t\,\hat{\mathbf{j}}) = A\boldsymbol{\omega}_\perp \tag{10.25}$$

Since $\omega_o\hat{\mathbf{k}}$ and $C\omega_o\hat{\mathbf{k}}$ are colinear, as are $\boldsymbol{\omega}_\perp$ and $A\boldsymbol{\omega}_\perp$, it follows that $\hat{\mathbf{k}}$, $\boldsymbol{\omega}$ and $\mathbf{H}_G$ all lie in the same plane. $\mathbf{H}_G$ and $\boldsymbol{\omega}$ both rotate around the z axis at the same rate ω_s. These details are illustrated in Figure 10.3. See how the precession and spin angular velocities, $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_s$, add up vectorially to give $\boldsymbol{\omega}$. Note also that from the point of view of inertial space, where $\mathbf{H}_G$ is fixed, $\boldsymbol{\omega}$ and $\hat{\mathbf{k}}$ rotate around $\mathbf{H}_G$ with angular velocity ω_p.

Let γ be the angle between $\boldsymbol{\omega}$ and the spin axis z, as shown in Figures 10.2 and 10.3. γ is sometimes referred to as the ***wobble angle***. From the figures it is clear that $\tan\gamma = \omega_{xy}/\omega_0$ and $\tan\theta = A\omega_{xy}/C\omega_0$. It follows that

$$\tan\theta = \frac{A}{C}\tan\gamma \tag{10.26}$$

From this we conclude that if $A > C$, then $\gamma < \theta$, whereas $C > A$ means $\gamma > \theta$. That is, the angular velocity vector $\boldsymbol{\omega}$ lies between the z axis and the angular momentum vector $\mathbf{H}_G$ when $A > C$ (prolate body). On the other hand, when $C > A$ (oblate body), $\mathbf{H}_G$ lies between the z axis and $\boldsymbol{\omega}$. These two situations are

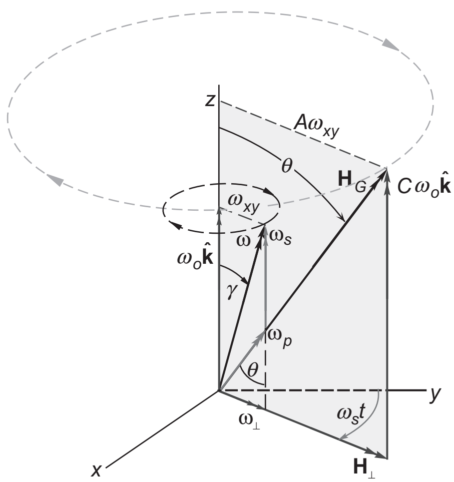


FIGURE 10.3

Angular velocity and angular momentum vectors in the body frame ($A > C$).

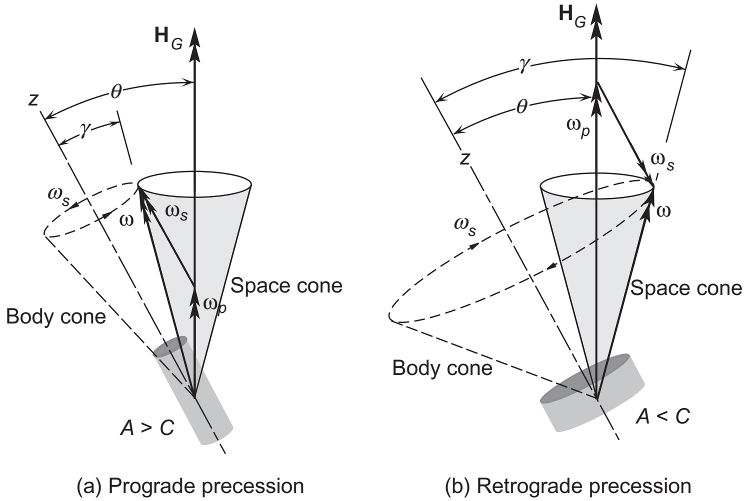


FIGURE 10.4

Space and body cones for a rotationally symmetric body in torque-free motion. (a) Prolate body. (b) Oblate body.

illustrated in Figure 10.4, which also shows the ***body cone*** and ***space cone***. The space cone is swept out in inertial space by the angular velocity vector as it rotates with angular velocity ω_p around $\mathbf{H}_G$, whereas the body cone is the trace of $\boldsymbol{\omega}$ in the body frame as it rotates with angular velocity ω_s about the z axis. From inertial space, the motion may be visualized as the body cone rolling on the space cone, with the line of contact being the angular velocity vector. From the body frame it appears as though the space cone rolls on the body cone. Figure 10.4 graphically confirms our deduction from Equation 10.23, namely, that precession and spin are in the same direction for prolate bodies and opposite in direction for oblate shapes.

Finally, we know from Equations 10.24 and 10.25 that the magnitude $\|\mathbf{H}_G\|$ of the angular momentum is

$$\|\mathbf{H}_G\| = \sqrt{A^2{\omega_{xy}}^2 + C^2{\omega_o}^2}$$

Using Equations 10.17 and 10.22, we can write this as

$$\|\mathbf{H}_G\| = \sqrt{A^2(\omega_p \sin\theta)^2 + C^2\left(\frac{A}{C}\omega_p \cos\theta\right)^2} = \sqrt{A^2{\omega_p}^2(\sin^2\theta + \cos^2\theta)}$$

so that we obtain a surprisingly simple formula for the magnitude of the angular momentum in torque-free motion,

$$\|\mathbf{H}_G\| = A\omega_p \tag{10.27}$$

Example 10.1

A cylindrical shell is rotating in torque-free motion about its longitudinal axis. If the axis is wobbling slightly, determine the ratios of l/r for which the precession will be prograde or retrograde.

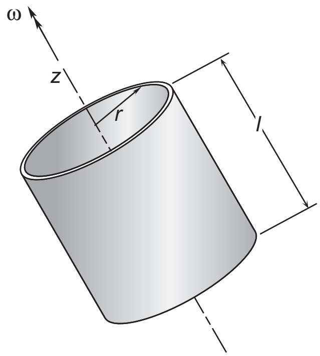


FIGURE 10.5

Cylindrical shell in torque-free motion.

Solution

Figure 9.9(b) shows the moments of inertia of a thin-walled circular cylinder,

$$C = mr^2 \qquad A = \frac{1}{2}mr^2 + \frac{1}{12}ml^2$$

According to Equation 10.23 and Figure 10.4, direct (prograde) precession exists if $A > C$, that is, if

$$\frac{1}{2}mr^2 + \frac{1}{12}ml^2 > mr^2$$

or

$$\frac{1}{12}ml^2 > \frac{1}{2}mr^2$$

Thus,

$$\boxed{\begin{array}{l} l > 2.45r \Rightarrow \text{Direct precession.} \\ l < 2.45r \Rightarrow \text{Retrograde precession.} \end{array}}$$

Example 10.2

In the previous example, let $r = 1\,\text{m}$, $l = 3\,\text{m}$, $m = 100\,\text{kg}$, and let the nutation angle θ be 20°. How long does it take the cylinder to precess through 180° if the spin rate is 2π radians per minute?

Solution

Since $l > 2.45r$, it follows from Example 10.1 that the precession is direct. Furthermore,

$$C = mr^2 = 100 \cdot 1^2 = 100\ \text{kg} \cdot \text{m}^2$$
$$A = \frac{1}{2}mr^2 + \frac{1}{12}ml^2 = \frac{1}{2} \cdot 100 \cdot 1^2 + \frac{1}{12}100 \cdot 3^2 = 125\ \text{kg} \cdot \text{m}^2$$

Thus, Equation 10.23 yields

$$\omega_p = \frac{C}{A - C}\frac{\omega_s}{\cos\theta} = \frac{100}{125 - 100}\frac{2\pi}{\cos 20°} = 26.75 \text{ rad/min}.$$

At this rate, the time for the spin axis to precess through an angle of 180° is

$$t = \frac{\pi}{\omega_p} = \boxed{0.1175 \text{ min}}\ (7.05\,\text{s})$$

Example 10.3

What is the torque-free motion of a satellite for which $A = B = C$?

Solution

If $A = B = C$, the satellite is spherically symmetric. Any orthogonal triad at G is a principal body frame, so $\mathbf{H}_G$ and $\boldsymbol{\omega}$ are colinear,

$$\mathbf{H}_G = C\boldsymbol{\omega}$$

Substituting this and $\mathbf{M}_{G_{\text{net}}} = \mathbf{0}$ into Euler's equations, Equation 10.72a, yields

$$C\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (C\boldsymbol{\omega}) = \mathbf{0}$$

That is, $\boldsymbol{\omega}$ is constant. The angular velocity vector of a spherically symmetric satellite is fixed in magnitude and direction.

Example 10.4

The inertial components of the angular momentum of a torque-free rigid body are

$$\mathbf{H}_G = 320\hat{\mathbf{I}} - 375\hat{\mathbf{J}} + 450\hat{\mathbf{K}}\ (\text{kg}\cdot\text{m}^2/\text{s}) \tag{a}$$

The Euler angles are

$$\phi = 20° \quad \theta = 50° \quad \psi = 75° \tag{b}$$

If the inertia tensor in the body-fixed principal frame is

$$[\mathbf{I}_G] = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} (\text{kg}\cdot\text{m}^2) \tag{c}$$

calculate the inertial components of the (absolute) angular acceleration.

Solution

Substituting the Euler angles from (b) into Equation 9.117, we obtain the matrix of the transformation from the inertial frame to the body-fixed frame,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0.03086 & 0.6720 & 0.7399 \\ -0.9646 & -0.1740 & 0.1983 \\ 0.2620 & -0.7198 & 0.6428 \end{bmatrix} \tag{d}$$

We use this to obtain the components of $\mathbf{H}_G$ in the body frame,

$$\{\mathbf{H}_G\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{H}_G\}_X = \begin{bmatrix} 0.03086 & 0.6720 & 0.7399 \\ -0.9646 & -0.1740 & 0.1983 \\ 0.2620 & -0.7198 & 0.6428 \end{bmatrix} \begin{Bmatrix} 320 \\ -375 \\ 450 \end{Bmatrix} = \begin{Bmatrix} 90.86 \\ -154.2 \\ 643.0 \end{Bmatrix} (\text{kg} \cdot \text{m}^2/\text{s}) \tag{e}$$

In the body frame $\{\mathbf{H}_G\}_x = [\mathbf{I}_G]\{\boldsymbol{\omega}\}_x$, where $\{\boldsymbol{\omega}\}_x$ are the components of angular velocity in the body frame. Thus,

$$\begin{Bmatrix} 90.86 \\ -154.2 \\ 643.0 \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \{\boldsymbol{\omega}\}_x$$

or, solving for $\{\boldsymbol{\omega}\}_x$,

$$\{\boldsymbol{\omega}\}_x = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix}^{-1} \begin{Bmatrix} 90.86 \\ -154.2 \\ 643.0 \end{Bmatrix} = \begin{Bmatrix} 0.09086 \\ -0.07709 \\ 0.2144 \end{Bmatrix} (\text{rad/s}) \tag{f}$$

Euler's equations of motion (Equation 9.72a) may be written for the case at hand as

$$[\mathbf{I}_G]\{\boldsymbol{\alpha}\}_x + \{\boldsymbol{\omega}\}_x \times ([\mathbf{I}_G]\{\boldsymbol{\omega}\}_x) = \{\mathbf{0}\} \tag{g}$$

where $\{\boldsymbol{\alpha}\}_x$ is the absolute acceleration in body frame components. Substituting (c) and (f) into this expression, we get

$$\begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \{\boldsymbol{\alpha}\}_x + \begin{Bmatrix} 0.09086 \\ -0.07709 \\ 0.2144 \end{Bmatrix} \times \left(\begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \begin{Bmatrix} 0.09086 \\ -0.07709 \\ 0.2144 \end{Bmatrix} \right) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \{\boldsymbol{\alpha}\}_x + \begin{Bmatrix} -16.52 \\ -38.95 \\ -7.005 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

so that, finally,

$$\{\alpha\}_x = -\begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix}^{-1} \begin{Bmatrix} -16.52 \\ -38.95 \\ -7.005 \end{Bmatrix} = \begin{Bmatrix} 0.01652 \\ 0.01948 \\ 0.002335 \end{Bmatrix} (\text{rad/s}^2) \tag{h}$$

These are the components of the angular acceleration in the body frame. To transform them into the inertial frame we use

$$\begin{aligned} \{\alpha\}_X &= [\mathbf{Q}]_{xX}\{\alpha\}_x = ([\mathbf{Q}]_{Xx})^T\{\alpha\}_x \\ &= \begin{bmatrix} 0.03086 & -0.9646 & 0.2620 \\ 0.6720 & -0.1740 & -0.7198 \\ 0.7399 & 0.1983 & 0.6428 \end{bmatrix} \begin{Bmatrix} 0.01652 \\ 0.01948 \\ 0.002335 \end{Bmatrix} = \begin{Bmatrix} -0.01766 \\ 0.006033 \\ 0.01759 \end{Bmatrix} (\text{rad/s}^2) \end{aligned}$$

That is,

$$\boxed{\alpha = -0.01766\hat{\mathbf{I}} + 0.006033\hat{\mathbf{J}} + 0.01759\hat{\mathbf{K}}\ (\text{rad/s}^2)}$$

10.3 STABILITY OF TORQUE-FREE MOTION

Let a rigid body be in torque-free motion with its angular velocity vector directed along the principal body z axis, so that $\boldsymbol{\omega} = \omega_o\hat{\mathbf{k}}$, where ω_o is constant. The nutation angle is zero and there is no precession. Let us perturb the motion slightly, as illustrated in Figure 10.6, so that

$$\omega_x = \delta\omega_x \quad \omega_y = \delta\omega_y \quad \omega_z = \omega_o + \delta\omega_z \tag{10.28}$$

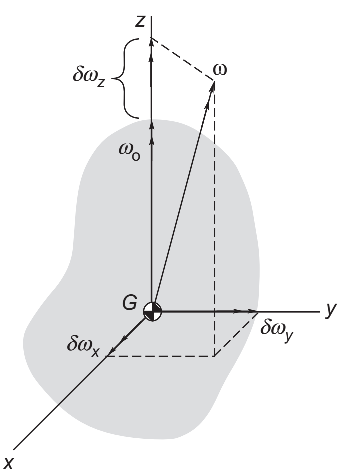


FIGURE 10.6

Principal body axes of a rigid body rotating primarily about the body z axis.

As in Chapter 7, "δ" means a very small quantity. In this case, $\delta\omega_x << \omega_o$ and $\delta\omega_y << \omega_o$. Thus, the angular velocity vector has become slightly inclined to the z axis. For torque-free motion, $M_{G_x} = M_{G_y} = M_{G_z} = 0$, so that Euler's equations (Equations 9.72b) become

$$\begin{aligned} A\dot{\omega}_x + (C - B)\omega_y\omega_z &= 0 \\ B\dot{\omega}_y + (A - C)\omega_x\omega_z &= 0 \\ C\dot{\omega}_z + (B - A)\omega_x\omega_y &= 0 \end{aligned} \tag{10.29}$$

Observe that we have not assumed $A = B$, as we did in the previous section. Substituting Equations 10.28 into Equations 10.29 and keeping in mind our assumption that $\dot{\omega}_o = 0$, we get

$$\begin{aligned} A\delta\dot{\omega}_x + (C - B)\omega_o\delta\omega_y + (C - B)\delta\omega_y\delta\omega_z &= 0 \\ B\delta\dot{\omega}_y + (A - C)\omega_o\delta\omega_x + (C - B)\delta\omega_x\delta\omega_z &= 0 \\ C\delta\dot{\omega}_z + (B - A)\delta\omega_x\delta\omega_y &= 0 \end{aligned} \tag{10.30}$$

Neglecting all products of the $\delta\omega's$ (because they are arbitrarily small), Equations 10.30 become

$$\begin{aligned} A\delta\dot{\omega}_x + (C - B)\omega_o\delta\omega_y &= 0 \\ B\delta\dot{\omega}_y + (A - C)\omega_o\delta\omega_x &= 0 \\ C\delta\dot{\omega}_z &= 0 \end{aligned} \tag{10.31}$$

Equation 10.31_3 implies that $\delta\omega_z$ does not vary with time.

Differentiating Equation 10.31_1 with respect to time, we get

$$A\delta\ddot{\omega}_x + (C - B)\omega_o\delta\dot{\omega}_y = 0 \tag{10.32}$$

Solving Equation 10.31_2 for $\delta\dot{\omega}_y$ yields $\delta\dot{\omega}_y = -[(A - C)/B]\omega_o\delta\omega_x$, and substituting this into Equation 10.32 gives

$$\delta\ddot{\omega}_x - \frac{(A - C)(C - B)}{AB}\omega_o{}^2\delta\omega_x = 0 \tag{10.33}$$

Likewise, by differentiating Equation 10.31_2 and then substituting $\delta\dot{\omega}_x$ from Equation 10.31_1 yields

$$\delta\ddot{\omega}_y - \frac{(A - C)(C - B)}{AB}\omega_o{}^2\delta\omega_y = 0 \tag{10.34}$$

If we define

$$k = \frac{(A - C)(B - C)}{AB}\omega_o{}^2 \tag{10.35}$$

then both Equations 10.33 and 10.34 may be written in the form

$$\delta\ddot{\omega} + k\delta\omega = 0 \tag{10.36}$$

If $k > 0$, then $\delta\omega = c_1 e^{i\sqrt{k}t} + c_2 e^{-i\sqrt{k}t}$, which means $\delta\omega_x$ and $\delta\omega_y$ vary sinusoidally with small amplitude. The motion is therefore bounded and neutrally stable. That means the amplitude does not die out with time, but it does not exceed the small amplitude of the perturbation. Observe from Equation 10.35 that $k > 0$ if either $C > A$ and $B > C$ or $C < A$ and $C < B$. This means that the spin axis (z axis) is either the major axis of inertia or the minor axis of inertia. That is, if the spin axis is either the major or minor axis of inertia, the motion is stable. The stability is neutral for a rigid body, because there is no damping.

On the other hand, if $k < 0$, then $\delta\omega = c_1 e^{\sqrt{k}t} + c_2 e^{-\sqrt{k}t}$, which means that the initially small perturbations $\delta\omega_x$ and $\delta\omega_y$ increase without bound. The motion is unstable. From Equation 10.35 we see that $k < 0$ if either $A > C > B$ or $A < C < B$. This means that the spin axis is the intermediate axis of inertia. If the spin axis is the intermediate axis of inertia, the motion is unstable.

If the angular velocity of a satellite lies in the direction of its major axis of inertia, the satellite is called a ***major axis spinner*** or oblate spinner. A ***minor axis spinner*** or prolate spinner has its minor axis of inertia aligned with the angular velocity. "Intermediate axis spinners" are unstable, causing a continual 180° reorientation of the spin axis, if the satellite is a rigid body. However, the flexibility inherent in any real satellite leads to an additional instability, as we shall now see.

Consider again the rotationally symmetric satellite in torque-free motion discussed in Section 10.2. From Equations 10.24 and 10.25, we know that the angular momentum $\mathbf{H}_G$ is given by

$$\mathbf{H}_G = A\boldsymbol{\omega}_\perp + C\omega_z\hat{\mathbf{k}} \tag{10.37}$$

Hence,

$$H_G^2 = A^2\omega_\perp^2 + C^2\omega_z^2 \qquad (\omega_\perp = \omega_{xy}) \tag{10.38}$$

Differentiating this equation with respect to time yields

$$\frac{dH_G^2}{dt} = A^2\frac{d\omega_\perp^2}{dt} + 2C^2\omega_z\dot{\omega}_z \tag{10.39}$$

But, according to Equation 10.1, $\mathbf{H}_G$ is constant, so that $dH_G^2/dt = 0$ and Equation 10.39 can be written

$$\frac{d\omega_\perp^2}{dt} = -2\frac{C^2}{A^2}\omega_z\dot{\omega}_z \tag{10.40}$$

The rotary kinetic energy of a rotationally symmetric body ($A = B$) is found using Equation 9.81,

$$T_R = \frac{1}{2}A\omega_x^2 + \frac{1}{2}A\omega_y^2 + \frac{1}{2}C\omega_z^2 = \frac{1}{2}A(\omega_x^2 + \omega_y^2) + \frac{1}{2}C\omega_z^2$$

From Equation 10.13 we know that $\omega_x^2 + \omega_y^2 = \omega_\perp^2$, which means

$$T_R = \frac{1}{2}A\omega_\perp^2 + \frac{1}{2}C\omega_z^2 \tag{10.41}$$

The time derivative of T_R is, therefore,

$$\dot{T}_R = \frac{1}{2}A\frac{d\omega_\perp{}^2}{dt} + C\omega_z\dot{\omega}_z$$

Solving this for $\dot{\omega}_z$, we get

$$\dot{\omega}_z = \frac{1}{C\omega_z}\left(\dot{T}_R - \tfrac{1}{2}A\frac{d\omega_\perp{}^2}{dt}\right)$$

Substituting this expression for $\dot{\omega}_z$ into Equation 10.40 and solving for $d\omega_\perp^2/dt$ yields

$$\frac{d\omega_\perp{}^2}{dt} = 2\frac{C}{A}\frac{\dot{T}_R}{C-A} \tag{10.42}$$

Real bodies are not completely rigid, and their flexibility, however slight, gives rise to small dissipative effects which cause the kinetic energy to decrease over time. That is,

$$\dot{T}_R < 0 \qquad \text{For spacecraft with dissipation} \tag{10.43}$$

Substituting this inequality into Equation 10.42 leads us to conclude that

$$\begin{aligned} \frac{d\omega_\perp{}^2}{dt} &< 0 \quad \text{if} \quad C > A \quad \text{(oblate spinner)} \\ \frac{d\omega_\perp{}^2}{dt} &> 0 \quad \text{if} \quad C < A \quad \text{(prolate spinner)} \end{aligned} \tag{10.44}$$

If $d\omega_\perp^2/dt$ is negative, the spin is asymptotically stable. Should a nonzero value of $\omega_\perp$ develop for some reason, it will drift back to zero over time so that once again the angular velocity lies completely in the spin direction. On the other hand, if $d\omega_\perp^2/dt$ is positive, the spin is unstable. $\omega_\perp$ does not damp out, and the angular velocity vector drifts away from the spin axis as $\omega_\perp$ increases without bound. We pointed out above that spin about a minor axis of inertia is stable with respect to small disturbances. Now we see that only major axis spin is stable in the long run if dissipative mechanisms exist.

For some additional insight into this phenomenon, solve Equation 10.38 for $\omega_\perp^2$,

$$\omega_\perp{}^2 = \frac{H_G{}^2 - C^2\omega_z{}^2}{A^2}$$

and substitute this result into the expression for kinetic energy, Equation 10.41, to obtain

$$T_R = \frac{1}{2}\frac{H_G{}^2}{A} + \frac{1}{2}\frac{(A-C)C}{A}\omega_z{}^2 \tag{10.45}$$

According to Equation 10.24,

$$\omega_z = \frac{H_{G_z}}{C} = \frac{H_G \cos\theta}{C}$$

Substituting this into Equation 10.45 yields the kinetic energy as a function of just the inclination angle θ,

$$T_R = \frac{1}{2}\frac{H_G{}^2}{A}\left(1 + \frac{A - C}{C}\cos^2\theta\right) \tag{10.46}$$

The extreme values of T_R occur at $\theta = 0$ or $\theta = \pi$,

$$T_R = \frac{1}{2}\frac{H_G{}^2}{C} \quad \text{(major axis spinner)}$$

and $\theta = \pi/2$,

$$T_R = \frac{1}{2}\frac{H_G{}^2}{A} \quad \text{(minor axis spinner)}$$

Clearly, the kinetic energy of a torque-free satellite is smallest when the spin is around the major axis of inertia. We may think of a satellite with dissipation ($dT_R/dt < 0$) as seeking the state of minimum kinetic energy, which occurs when it spins about its major axis.

Example 10.5

A rigid spacecraft is modeled by the solid cylinder B which has a mass of 300 kg and the slender rod R which passes through the cylinder and has a mass of 30 kg. Which of the principal axes x, y, z can be an axis about which stable torque-free rotation can occur?

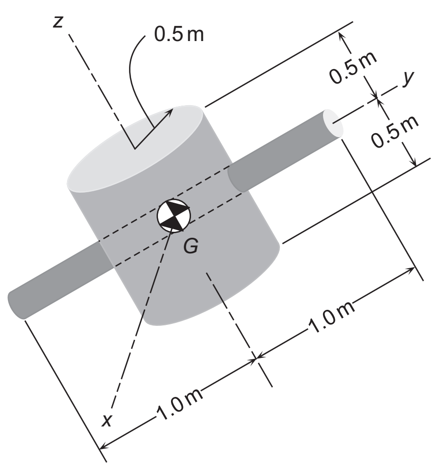


FIGURE 10.7

Built-up satellite structure.

Solution

For the cylindrical shell A, we have

$$r_B = 0.5 \text{ m} \qquad l_B = 1.0 \text{ m} \qquad m_B = 300 \text{ kg}$$

The principal moments of inertia about the center of mass are found in Figure 9.9(b),

$$\begin{aligned} I_{Bx} &= \frac{1}{4}m_B{r_B}^2 + \frac{1}{12}m_B{l_B}^2 = 43.75 \text{ kg}\cdot\text{m}^2 \\ I_{By} &= I_{Bxx} = 43.75 \text{ kg}\cdot\text{m}^2 \\ I_{Bz} &= \frac{1}{2}m_B{r_B}^2 = 37.5 \text{ kg}\cdot\text{m}^2 \end{aligned}$$

The properties of the transverse rod are

$$l_R = 1.0 \text{ m} \qquad m_R = 30 \text{ kg}$$

Figure 9.9(a), with $r = 0$, yields the moments of inertia,

$$\begin{aligned} I_{Ry} &= 0 \\ I_{Rz} &= I_{Rx} = \frac{1}{12}m_A{r_A}^2 = 10.0 \text{ kg}\cdot\text{m}^2 \end{aligned}$$

The moments of inertia of the assembly is the sum of the moments of inertia of the cylinder and the rod,

$$\begin{aligned} I_x &= I_{Bx} + I_{Rx} = 53.75 \text{ kg}\cdot\text{m}^2 \\ I_y &= I_{By} + I_{Ry} = 43.75 \text{ kg}\cdot\text{m}^2 \\ I_z &= I_{Bz} + I_{Rz} = 47.50 \text{ kg}\cdot\text{m}^2 \end{aligned}$$

Since I_z is the intermediate mass moment of inertia, rotation about the z axis is unstable. With energy dissipation, rotation is stable in the long term only about the major axis, which in this case is the x axis.

10.4 DUAL-SPIN SPACECRAFT

If a satellite is to be spin stabilized, it must be an oblate spinner. The diameter of the spacecraft is restricted by the cross section of the launch vehicle's upper stage, and its length is limited by stability requirements. Therefore, oblate spinners cannot take full advantage of the payload volume available in a given launch vehicle, which after all are slender, prolate shapes for aerodynamic reasons. The dual-spin design permits spin stabilization of a prolate shape.

The axisymmetric, dual-spin configuration, or gyrostat, consists of an axisymmetric rotor and a smaller axisymmetric platform joined together along a common longitudinal spin axis at a bearing, as shown in Figure 10.8. The platform and rotor have their own components of angular velocity, ω_p and ω_r respectively, along the spin axis direction $\hat{\mathbf{k}}$. The platform spins at a much slower rate than the rotor. The assembly acts

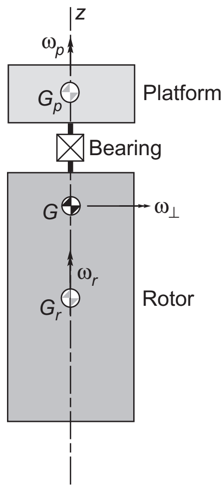


FIGURE 10.8

Axisymmetric, dual-spin satellite.

like a rigid body as far as transverse rotations are concerned; that is, the rotor and the platform have $\boldsymbol{\omega}'_\perp$ in common. An electric motor integrated into the axle bearing connecting the two components acts to overcome frictional torque that would otherwise eventually cause the relative angular velocity between the rotor and platform to go to zero. If that should happen, the satellite would become a single spin unit, probably an unstable prolate spinner, since the rotor of a dual-spin spacecraft is likely to be prolate.

The first dual-spin satellite was OSO-I (Orbiting Solar Observatory), which NASA launched in 1962. It was a major-axis spinner. The first prolate dual-spin spacecraft was the two-story tall TACSAT I (Tactical Communications Satellite). It was launched into geosynchronous orbit by the U. S. Air Force in 1969. Typical of many of today's communications satellites, TACSAT's platform rotated at one revolution per day to keep its antennas pointing towards the earth. The rotor spun at about one revolution per second. Of course, the axis of the spacecraft was normal to the plane of its orbit. The first dual-spin interplanetary spacecraft was Galileo, which we discussed briefly in Section 8.9. Galileo's platform was completely despun to provide a fixed orientation for cameras and other instruments. The rotor spun at three revolutions per minute.

The equations of motion of a dual-spin spacecraft will be developed later on in Section 10.8. Let us determine the stability of the motion by following the same ***energy sink analysis*** procedure employed in the previous section for a single-spin stabilized spacecraft. The angular momentum of the dual-spin configuration about the spacecraft's center of mass G is the sum of the angular momenta of the rotor (r) and the platform (p) about G,

$$\mathbf{H}_G = \mathbf{H}_G^{(p)} + \mathbf{H}_G^{(r)} \tag{10.47}$$

The angular momentum of the platform about the spacecraft center of mass is

$$\mathbf{H}_G^{(p)} = C_p\omega_p\hat{\mathbf{k}} + A_p\boldsymbol{\omega}_\perp \tag{10.48}$$

where C_p is the moment of inertia of the platform about the spacecraft spin axis, and A_p is its transverse moment of inertia about G (not G_p). Likewise, for the rotor,

$$\mathbf{H}_G^{(r)} = C_r\omega_r\hat{\mathbf{k}} + A_r\boldsymbol{\omega}_\perp \tag{10.49}$$

where C_r and A_r are its longitudinal and transverse moments of inertia about axes through G. Substituting Equations 10.48 and 10.49 into 10.47 yields

$$\mathbf{H}_G = (C_r\omega_r + C_p\omega_p)\hat{\mathbf{k}} + A_\perp\boldsymbol{\omega}_\perp \tag{10.50}$$

where $A_\perp$ is the total transverse moment of inertia,

$$A_\perp = A_p + A_r$$

From this it follows that

$$H_G{}^2 = (C_r\omega_r + C_p\omega_p)^2 + A_\perp{}^2\omega_\perp{}^2$$

For torque-free motion, $\dot{\mathbf{H}}_G = \mathbf{0}$, so that $dH_G^2/dt = 0$, or

$$2(C_r\omega_r + C_p\omega_p)(C_r\dot{\omega}_r + C_p\dot{\omega}_p) + A_\perp{}^2\frac{d\omega_\perp{}^2}{dt} = 0 \tag{10.51}$$

Solving this for $d\omega'_\perp{}^2/dt$ yields

$$\frac{d\omega_\perp{}^2}{dt} = -\frac{2}{A_\perp{}^2}(C_r\omega_r + C_p\omega_p)(C_r\dot{\omega}_r + C_p\dot{\omega}_p) \tag{10.52}$$

The total rotational kinetic energy of the dual-spin spacecraft is the sum of that of the rotor and the platform,

$$T = \frac{1}{2}C_r\omega_r{}^2 + \frac{1}{2}C_p\omega_p{}^2 + \frac{1}{2}A_\perp\omega_\perp{}^2$$

Differentiating this expression with respect to time and solving for $d\omega'_\perp{}^2/dt$ yields

$$\frac{d\omega_\perp{}^2}{dt} = \frac{2}{A_\perp}(\dot{T} - C_r\omega_r\dot{\omega}_r - C_p\omega_p\dot{\omega}_p) \tag{10.53}$$

$\dot{T}$ is the sum of the power $P^{(r)}$ dissipated in the rotor and the power $P^{(p)}$ dissipated in the platform,

$$\dot{T} = P^{(r)} + P^{(p)} \tag{10.54}$$

Substituting Equation 10.54 into 10.53 we find

$$\frac{d\omega_\perp{}^2}{dt} = \frac{2}{A_\perp}(P^{(r)} - C_r\omega_r\dot{\omega}_r + P^{(p)} - C_p\omega_p\dot{\omega}_p) \tag{10.55}$$

Equating the two expressions for $d\omega'_\perp{}^2/dt$ in Equations 10.52 and 10.55 yields

$$\frac{2}{A_\perp}(\dot{T} - C_r\omega_r\dot{\omega}_r - C_p\omega_p\dot{\omega}_p) = -\frac{2}{A_\perp{}^2}(C_r\omega_r + C_p\omega_p)(C_r\dot{\omega}_r + C_p\dot{\omega}_p)$$

Solve this for $\dot{T}$ to obtain

$$\dot{T} = \frac{C_r}{A_\perp}[(A_\perp - C_r)\omega_r - C_p\omega_p]\dot{\omega}_r + \frac{C_p}{A_\perp}[(A_\perp - C_p)\omega_p - C_r\omega_r]\dot{\omega}_p \tag{10.56}$$

Following Likens (1967), we identify the terms containing $\dot{\omega}_r$ and $\dot{\omega}_p$ as the power dissipation in the rotor and platform, respectively. That is, comparing Equations 10.54 and 10.56,

$$P^{(r)} = \frac{C_r}{A_\perp}[(A_\perp - C_r)\omega_r - C_p\omega_p]\dot{\omega}_r \tag{10.57a}$$

$$P^{(p)} = \frac{C_p}{A_\perp}[(A_\perp - C_p)\omega_p - C_r\omega_r]\dot{\omega}_p \tag{10.57b}$$

Solving these two expressions for $\dot{\omega}_r$ and $\dot{\omega}_p$, respectively, yields

$$\dot{\omega}_r = \frac{A_\perp}{C_r\omega_r}\frac{P^{(r)}}{(A_\perp - C_r) - C_p(\omega_p/\omega_r)} \tag{10.58a}$$

$$\dot{\omega}_p = \frac{A_\perp}{C_p\omega_r}\frac{P^{(p)}}{(A_\perp - C_p)(\omega_p/\omega_r) - C_r} \tag{10.58b}$$

Substituting these angular velocity rates into Equation 10.55 leads to

$$\frac{d\omega_\perp{}^2}{dt} = \frac{2}{A_\perp}\left[\frac{P^{(r)}}{C_p(\omega_p/\omega_r) - (A_\perp - C_r)} + \frac{P^{(p)}}{C_r - (A_\perp - C_p)(\omega_p/\omega_r)}\right][C_r + C_p(\omega_p/\omega_r)] \tag{10.59}$$

As pointed out above, for geosynchronous dual-spin communication satellites,

$$\frac{\omega_p}{\omega_r} \approx \frac{2\pi \text{ radians/day}}{2\pi \text{ radians/second}} \approx 10^{-5}$$

whereas for interplanetary dual-spin spacecraft, $\omega_p = 0$. Therefore, there is an important class of spin stabilized spacecraft for which $\omega_p/\omega_r \approx 0$. For a despun platform wherein ω_p is zero (or nearly so), Equation 10.59 yields

$$\frac{d\omega_\perp{}^2}{dt} = \frac{2}{A_\perp}\left[P^{(p)} + \frac{C_r}{C_r - A_\perp}P^{(r)}\right] \tag{10.60}$$

If the rotor is oblate ($C_r > A_\perp$), then, since $P^{(r)}$ and $P^{(p)}$ are both negative, it follows from Equation 10.60 that $d\omega_\perp{}^2/dt < 0$. That is, the oblate dual-spin configuration with a despun platform is unconditionally stable. In practice, however, the rotor is likely to be prolate ($C_r < A_\perp$), so that

$$\frac{C_r}{C_r - A_\perp}P^{(r)} > 0$$

In that case, $d\omega_\perp{}^2/dt < 0$ only if the dissipation in the platform is significantly greater than that of the rotor. Specifically, for a prolate design it must be true that

$$\left|P^{(p)}\right| > \left|\frac{C_r}{C_r - A_\perp}P^{(r)}\right|$$

The platform dissipation rate $P^{(p)}$ can be augmented by adding nutation dampers, which are discussed in the next section.

For the despun prolate dual-spin configuration, Equations 10.58 imply

$$\dot{\omega}_r = \frac{P^{(r)}}{(A_\perp - C_r)}\frac{A_\perp}{C_r\omega_r}$$

$$\dot{\omega}_p = -\frac{P^{(p)}}{C_p}\frac{A_\perp}{C_r\omega_r}$$

Clearly, the signs of $\dot{\omega}_r$ and $\dot{\omega}_p$ are opposite. If $\omega_r > 0$, then dissipation causes the spin rate of the rotor to decrease and that of the platform to increase. Were it not for the action of the motor on the shaft connecting the two components of the spacecraft, eventually $\omega_p = \omega_r$. That is, the relative motion between the platform and rotor would cease and the dual-spinner would become an unstable single spin spacecraft. Setting $\omega_p = \omega_r$ in Equation 10.59 yields

$$\frac{d\omega_\perp{}^2}{dt} = 2\frac{C_r + C_p}{A_\perp}\frac{P^{(r)} + P^{(p)}}{(C_r + C_p) - A_\perp}$$

which is the same as Equation 10.42, the energy sink conclusion for a single spinner.

10.5 NUTATION DAMPER

Nutation dampers are passive means of dissipating energy. A common type consists essentially of a tube filled with viscous fluid and containing a mass attached to springs, as illustrated in Figure 10.9. Dampers may contain just fluid, only partially filling the tube so it can slosh around. In either case, the purpose is to dissipate energy through fluid friction. The wobbling of the spacecraft due to nonalignment of the angular velocity with the principal spin axis induces accelerations throughout the satellite, giving rise to the sloshing of fluids, stretching and flexing of nonrigid components, etc., all of which dissipate energy to one degree or another. Nutation dampers are added to deliberately increase energy dissipation, which is desirable for stabilizing oblate single spinners and dual-spin spacecraft.

Let us focus on the motion of the mass within the nutation damper of Figure 10.9 in order to gain some insight into how relative motion and deformation are induced by the satellite's precession. Note that point P is the center of mass of the rigid satellite body itself. The center of mass G of the satellite-damper mass combination lies between P and m, as shown in Figure 10.9. We suppose that the tube is lined up with the z axis of the body-fixed xyz frame, as shown. The mass m in the tube is therefore constrained by the tube walls to move only in the z direction. When the springs are undeformed, the mass lies in the xy plane. In general, the position vector of m in the body frame is

$$\mathbf{r} = R\hat{\mathbf{i}} + z_m\hat{\mathbf{k}} \tag{10.61}$$

where z_m is the z coordinate of m and R is the distance of the damper from the centerline of the spacecraft. The velocity and acceleration of m relative to the satellite are, therefore,

$$\mathbf{v}_{\text{rel}} = \dot{z}_m\hat{\mathbf{k}} \tag{10.62}$$

$$\mathbf{a}_{\text{rel}} = \ddot{z}_m\hat{\mathbf{k}} \tag{10.63}$$

The absolute angular velocity $\boldsymbol{\omega}$ of the satellite (and, therefore, of the body-fixed frame) is

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} \tag{10.64}$$

Recall Equation 9.73, which states that when $\boldsymbol{\omega}$ is given in a body frame, we find the absolute angular acceleration by taking the time derivative of $\boldsymbol{\omega}$, holding the unit vectors fixed. Thus,

$$\dot{\boldsymbol{\omega}} = \dot{\omega}_x\hat{\mathbf{i}} + \dot{\omega}_y\hat{\mathbf{j}} + \dot{\omega}_z\hat{\mathbf{k}} \tag{10.65}$$

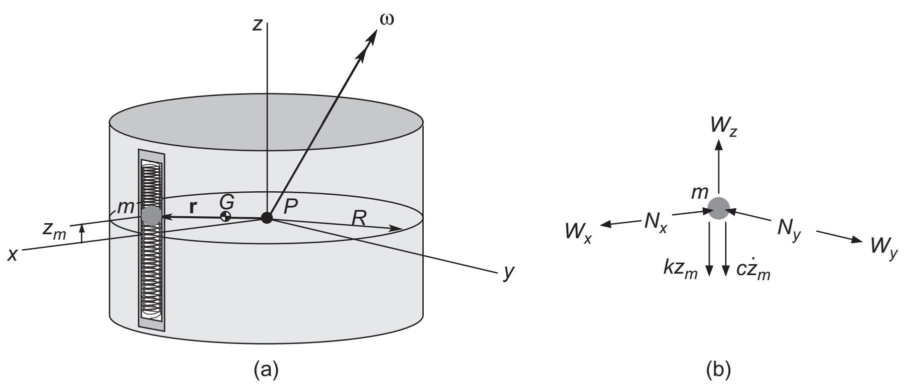


FIGURE 10.9

(a) Precessing oblate spacecraft with a nutation damper aligned with the z axis. (b) Free-body diagram of the moving mass in the nutation damper.

The absolute acceleration of m is found using Equation 1.70, which for the case at hand becomes

$$\mathbf{a} = \mathbf{a}_P + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{10.66}$$

in which $\mathbf{a}_p$ is the absolute acceleration of the reference point P. Substituting Equations 10.61 through 10.65 into Equation 10.66, carrying out the vector operations, combining terms, and simplifying leads to the following expressions for the three components of the inertial acceleration of m,

$$\begin{aligned} a_x &= a_{Px} - R(\omega_y{}^2 + \omega_z{}^2) + z_m\dot{\omega}_y + z_m\omega_x\omega_z + 2\dot{z}_m\omega_y \\ a_y &= a_{Py} + R\dot{\omega}_z + R\omega_x\omega_y - z_m\dot{\omega}_x + z_m\omega_y\omega_z - 2\dot{z}_m\omega_x \\ a_z &= a_{Pz} - z_m(\omega_x{}^2 + \omega_y{}^2) - R\dot{\omega}_y + R\omega_x\omega_z + \ddot{z}_m \end{aligned} \tag{10.67}$$

Figure 10.9(b) shows the free-body diagram of the damper mass m. In the x and y directions the forces on m are the components of the force of gravity (W_x and W_y) and the components N_x and N_y of the force of contact with the smooth walls of the damper tube. The directions assumed for these components are, of course, arbitrary. In the z direction, we have the z component W_z of the weight, plus the force of the springs and the viscous drag of the fluid. The spring force ($-kz_m$) is directly proportional and opposite in direction to the displacement z_m. k is the net spring constant. The viscous drag $(-c\dot{z}_m)$ is directly proportional and opposite in direction to the velocity $\dot{z}_m$ of m relative to the tube. c is the damping constant. Thus, the three components of the net force on the damper mass m are

$$\begin{aligned} F_{\text{net}_x} &= W_x - N_x \\ F_{\text{net}_y} &= W_y - N_y \\ F_{\text{net}_z} &= W_z - kz_m - c\dot{z}_m \end{aligned} \tag{10.68}$$

Substituting Equations 10.67 and 10.68 into Newton's second law, $\mathbf{F}_{\text{net}} = m\mathbf{a}$, yields

$$\begin{aligned} N_x &= mR(\omega_y{}^2 + \omega_z{}^2) - mz_m\dot{\omega}_y - mz_m\omega_x\omega_y - 2m\dot{z}_m\omega_y + \overbrace{(W_x - ma_{Px})}^{=0} \\ N_y &= -mR\dot{\omega}_z - mR\omega_x\omega_y + mz_m\dot{\omega}_x - mz_m\omega_y\omega_z + 2m\dot{z}_m\omega_x + \overbrace{(W_y - ma_{Py})}^{=0} \\ m\ddot{z}_m + c\dot{z}_m + [k - m(\omega_x{}^2 + \omega_y{}^2)]z_m &= mR(\dot{\omega}_y - \omega_x\omega_z) + \overbrace{(W_z - ma_{Pz})}^{=0} \end{aligned} \tag{10.69}$$

The last terms in parentheses in each of these expressions vanish if the acceleration of gravity is the same at m as at the reference point P of the spacecraft. This will be true unless the satellite is of enormous size.

If the damper mass m is vanishingly small compared to the mass M of the rigid spacecraft body, then it will have little effect on the rotary motion. If the rotational state is that of an axisymmetric satellite in torque free motion, then we know from Equations 10.13, 10.14 and 10.19 that

$$\begin{aligned} \omega_x &= \omega_{xy}\sin\omega_s t & \omega_y &= \omega_{xy}\cos\omega_s t & \omega_z &= \omega_o \\ \dot{\omega}_x &= \omega_{xy}\omega_s\cos\omega_s t & \dot{\omega}_y &= -\omega_{xy}\omega_s\sin\omega_s t & \dot{\omega}_z &= 0 \end{aligned}$$

in which case Equations 10.69 become

$$
\begin{aligned}
N_x &= mR(\omega_o{}^2 + \omega_{xy}{}^2 \cos^2 \omega_s t) + m(\omega_s - \omega_o)\omega_{xy} z_m \sin\omega_s t - 2m\omega_{xy}\dot{z}_m \cos\omega_s t \\
N_y &= -mR\omega_{xy}{}^2 \cos\omega_s t \sin\omega_s t + m(\omega_s - \omega_o)\omega_{xy} z_m \cos\omega_s t + 2m\omega_{xy}\dot{z}_m \sin\omega_s t \\
m\ddot{z}_m &+ c\dot{z}_m + (k - m\omega_{xy}{}^2)z_m = -mR(\omega_s + \omega_o)\omega_{xy} \sin\omega_s t
\end{aligned}
\tag{10.70}
$$

Equation 10.70_3 is that of a single degree of freedom, damped oscillator with a sinusoidal forcing function, which was discussed in Section 1.8. The precession produces a force of amplitude $m(\omega_o + \omega_s)\omega_{xy}R$ and frequency ω_s which causes the damper mass m to oscillate back and forth in the tube, such that (see the steady-state part of Equation 1.115a)

$$
z_m = \frac{mR\omega_{xy}(\omega_s + \omega_o)}{[k - m(\omega_s{}^2 + \omega_{xy}{}^2)]^2 + (c\omega_s)^2}\{c\omega_s \cos\omega_s t - [k - m(\omega_s{}^2 + \omega_{xy}{}^2)\sin\omega_s t]\}
$$

Observe that the contact forces N_x and N_y depend exclusively on the amplitude and frequency of the precession. If the angular velocity lines up with the spin axis, so that $\omega_{xy} = 0$ (precession vanishes), then

$$
\left.
\begin{aligned}
N_x &= m\omega_o{}^2 R \\
N_y &= 0 \\
z_m &= 0
\end{aligned}
\right\}\quad \text{No precession}
$$

If precession is eliminated, so there is pure spin around the principal axis, the time-varying motions and forces vanish throughout the spacecraft, which thereafter rotates as a rigid body with no energy dissipation.

Now, the whole purpose of a nutation damper is to interact with the rotational motion of the spacecraft so as to damp out any tendencies to precess. Therefore, its mass should not be ignored in the equations of motion of the spacecraft. We will derive the equations of motion of the rigid spacecraft with nutation damper to show how rigid body mechanics is brought to bear upon the problem and, simply, to discover precisely what we are up against in even this extremely simplified system. We will continue to use P as the origin of our body frame. Since a moving mass has been added to the rigid spacecraft and since we are not using the center of mass of the system as our reference point, we cannot use Euler's equations. Applicable to the case at hand is Equation 9.33, according to which the equation of rotational motion of the system of satellite plus damper is

$$
\dot{\mathbf{H}}_{P_{\text{rel}}} + \mathbf{r}_{G/P} \times (M + m)\mathbf{a}_{P/G} = \mathbf{M}_{G_{\text{net}}} \tag{10.71}
$$

The angular momentum of the satellite body plus that of the damper mass, relative to point P on the spacecraft, is

$$
\mathbf{H}_{P_{\text{rel}}} = \overbrace{A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}}}^{\text{body of the spacecraft}} + \overbrace{\mathbf{r} \times m\dot{\mathbf{r}}}^{\text{damper mass}} \tag{10.72}
$$

where the position vector **r** is given by Equation 10.61. According to Equation 1.59,

$$\dot{\mathbf{r}} = \left.\frac{d\mathbf{r}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega}\times\mathbf{r} = \dot{z}_m\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ R & 0 & z \end{vmatrix} = \omega_y z_m\hat{\mathbf{i}} + (\omega_z R - \omega_x z_m)\hat{\mathbf{j}} + (\dot{z}_m - \omega_y R)\hat{\mathbf{k}}$$

After substituting this into Equation 10.72 and collecting terms we obtain

$$\mathbf{H}_{P_{\text{rel}}} = [(A + mz_m{}^2)\omega_x - mRz_m\omega_z]\hat{\mathbf{i}} + [(B + mR^2 + mz_m{}^2)\omega_y - mR\dot{z}_m]\hat{\mathbf{j}} + [(C + mR^2)\omega_z - mRz_m\omega_x]\hat{\mathbf{k}} \quad (10.73)$$

To calculate $\dot{\mathbf{H}}_{P_{\text{rel}}}$, we again use Equation 1.56,

$$\dot{\mathbf{H}}_{P_{\text{rel}}} = \left.\frac{d\mathbf{H}_{P_{\text{rel}}}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega}\times\mathbf{H}_{P_{\text{rel}}}$$

Substituting Equation 10.73 and carrying out the operations on the right leads eventually to

$$\begin{aligned}\dot{\mathbf{H}}_{P_{\text{rel}}} = &\left[\left(A + mz_m{}^2\right)\dot{\omega}_x - mRz_m\dot{\omega}_z + \left(C - B - mz_m{}^2\right)\omega_y\omega_z - mRz_m\omega_x\omega_y + 2mz_m\dot{z}_m\omega_x\right]\hat{\mathbf{i}} \\ &+ \left\{\left(B + mR^2 + mz_m{}^2\right)\dot{\omega}_y + mRz_m\left(\omega_x{}^2 - \omega_z{}^2\right) + \left[A + mz_m{}^2 - \left(C + mR^2\right)\right]\omega_x\omega_z \right. \\ &\quad \left. + 2mz_m\dot{z}_m\omega_y - mR\ddot{z}_m\right\}\hat{\mathbf{j}} \\ &+ \left[-mRz_m\dot{\omega}_x + \left(C + mR^2\right)\dot{\omega}_z + \left(B + mR^2 - A\right)\omega_x\omega_y + mRz_m\omega_y\omega_z - 2mR\dot{z}_m\omega_x\right]\hat{\mathbf{k}}\end{aligned} \quad (10.74)$$

To calculate the second term on the left of Equation 10.71, we keep in mind that P is the center of mass of the body of the satellite and first determine the position vector of the center of mass G of the vehicle plus damper relative to P,

$$(M + m)\mathbf{r}_{G/P} = M(\mathbf{0}) + m\mathbf{r} \quad (10.75)$$

where **r**, the position of the damper mass m relative to P, is given by Equation 10.61. Thus,

$$\mathbf{r}_{G/P} = \frac{m}{m + M}\mathbf{r} = \mu\mathbf{r} = \mu(R\hat{\mathbf{i}} + z_m\hat{\mathbf{k}}) \quad (10.76)$$

in which

$$\mu = \frac{m}{m + M} \quad (10.77)$$

Thus,

$$\mathbf{r}_{G/P}\times(M + m)\mathbf{a}_{P/G} = \left(\frac{m}{M + m}\right)\mathbf{r}\times(M + m)\mathbf{a}_{P/G} = \mathbf{r}\times m\mathbf{a}_{P/G} \quad (10.78)$$

The acceleration of P relative to G is found with the aid of Equation 1.60,

$$\mathbf{a}_{P/G} = -\ddot{\mathbf{r}}_{G/P} = -\mu\frac{d^2\mathbf{r}}{dt^2} = -\mu\left[\left.\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{rel}} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\left.\frac{d\mathbf{r}}{dt}\right)_{\text{rel}}\right] \tag{10.79}$$

where

$$\left.\frac{d\mathbf{r}}{dt}\right)_{\text{rel}} = \frac{dR}{dt}\hat{\mathbf{i}} + \frac{dz_m}{dt}\hat{\mathbf{k}} = \dot{z}_m\hat{\mathbf{k}} \tag{10.80}$$

and

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{rel}} = \frac{d^2R}{dt^2}\hat{\mathbf{i}} + \frac{d^2z_m}{dt^2}\hat{\mathbf{k}} = \ddot{z}_m\hat{\mathbf{k}} \tag{10.81}$$

Substituting Equations 10.61, 10.64, 10.65, 10.80 and 10.81 into Equation 10.79 yields

$$\begin{aligned}\mathbf{a}_{P/G} &= \mu[-z_m\dot{\omega}_y + R(\omega_y{}^2 + \omega_z{}^2) - z_m\omega_x\omega_z - 2\dot{z}_m\omega_y]\hat{\mathbf{i}} \\ &\quad + \mu(z_m\dot{\omega}_x - R\dot{\omega}_z - R\omega_x\omega_y - z_m\omega_y\omega_z + 2\dot{z}_m\omega_x)\hat{\mathbf{j}} \\ &\quad + \mu[R\dot{\omega}_y + z_m(\omega_x{}^2 + \omega_y{}^2) - R\omega_x\omega_z - \ddot{z}_m]\hat{\mathbf{k}}\end{aligned} \tag{10.82}$$

We move this expression into Equation 10.78 to get

$$\begin{aligned}\mathbf{r}_{G/P}\times(M+m)\mathbf{a}_{P/G} &= \mu m[-z_m{}^2\dot{\omega}_x - 2z_m\dot{z}_m\omega_x + Rz_m(\omega_x\omega_y + \dot{\omega}_z) + z_m{}^2\omega_y\omega_z]\hat{\mathbf{i}} \\ &\quad + \mu m[-(R^2 + z_m{}^2)\dot{\omega}_y - 2z_m\dot{z}_m\omega_y + Rz_m(\omega_z{}^2 - \omega_x{}^2) + (R^2 - z_m{}^2)\omega_x\omega_z + R\ddot{z}_m]\hat{\mathbf{j}} \\ &\quad + \mu m(Rz_m\dot{\omega}_x - R^2\dot{\omega}_z + 2R\dot{z}_m\omega_x - R^2\omega_x\omega_y - Rz_m\omega_y\omega_z)\hat{\mathbf{k}}\end{aligned}$$

Placing this result and Equation 10.74 in Equation 10.71, and using the fact that $\mathbf{M}_{G_{\text{net}}} = \mathbf{0}$, yields a vector equation whose three components are

$$\begin{aligned}&A\dot{\omega}_x + (C-B)\omega_y\omega_z + (1-\mu)m[z_m{}^2(\dot{\omega}_x - \omega_y\omega_z) - Rz_m(\dot{\omega}_z + \omega_x\omega_y) + 2z_m\dot{z}_m\omega_x] = 0 \\ &[(B + mR^2) - \mu mR^2]\dot{\omega}_y + [(A + \mu mR^2) - (C + mR^2)]\omega_x\omega_z + (1-\mu)m[z_m{}^2(\omega_x\omega_z + \dot{\omega}_y) \\ &\qquad + 2z_m\dot{z}_m\omega_y - R\ddot{z}_m + Rz_m(\omega_x{}^2 - \omega_z{}^2)] = 0 \\ &[(C + mR^2) - \mu mR^2]\dot{\omega}_z + [(B + mR^2) - (A + \mu mR^2)]\omega_x\omega_y + (1-\mu)mR[z_m(\omega_y\omega_z - \dot{\omega}_x) \\ &\qquad - 2\dot{z}_m\omega_x] = 0\end{aligned} \tag{10.83}$$

These are three equations in the four unknowns ω_x, ω_y, ω_z and z_m. The fourth equation is that of the motion of the damper mass m in the z direction,

$$W_z - kz_m - c\dot{z}_m = ma_z \tag{10.84}$$

where a_z is given by Equation 10.67_3, in which $a_{Pz} = a_{Pz} - a_{Gz} + a_{Gz} = a_{P/Gz} + a_{Gz}$, so that

$$a_z = a_{P/Gz} + a_{Gz} - z_m({\omega_x}^2 + {\omega_y}^2) - R\dot{\omega}_y + R\omega_x\omega_z + \ddot{z}_m \tag{10.85}$$

Substituting the z component of Equation 10.82 into this expression and that result into Equation 10.84 leads (with $W_z = ma_{G_z}$) to

$$(1-\mu)m\ddot{z}_m + c\dot{z}_m + [k - (1-\mu)m({\omega_x}^2 + {\omega_y}^2)]z_m = (1-\mu)mR[\dot{\omega}_y - \omega_x\omega_z] \tag{10.86}$$

Compare Equation 10.69_3 with this expression, which is the fourth equation of motion we need.

Equations 10.83 and 10.86 are a rather complicated set of nonlinear, second-order differential equations that must be solved (numerically) to obtain a precise description of the motion of the semirigid spacecraft. The procedures of Section 1.8 may be employed. To study the stability of Equations 10.83 and 10.86, we can linearize them in much the same way as we did in Section 10.3. (Note that Equations 10.83 reduce to 10.29 when $m = 0$.) With that as our objective, we assume that the spacecraft is in pure spin with angular velocity ω_0 about the z axis and that the damper mass is at rest ($z_m = 0$). This motion is slightly perturbed, in such a way that

$$\omega_x = \delta\omega_x \qquad \omega_y = \delta\omega_y \qquad \omega_z = \omega_o + \delta\omega_z \qquad z_m = \delta z_m \tag{10.87}$$

It will be convenient for this analysis to introduce operator notation for the time derivative, $D = d/dt$. Thus, given a function of time $f(t)$, for any integer n, $D^n f = d^n f/dt^n$, and $D^0 f(t) = f(t)$. Then the various time derivatives throughout the equations will, in accordance with Equation 10.87, be replaced as follows:

$$\dot{\omega}_x = D\delta\omega_x \qquad \dot{\omega}_y = D\delta\omega_y \qquad \dot{\omega}_z = D\delta\omega_z \qquad \dot{z}_m = D\delta z_m \qquad \ddot{z}_m = D^2\delta z_m \tag{10.88}$$

Substituting Equations 10.87 and 10.88 into Equations 10.83 and 10.86 and retaining only those terms which are at most linear in the small perturbations leads to

$$\begin{aligned} AD\delta\omega_x + (C-B)\omega_o\delta\omega_y &= 0 \\ [A - C - (1-\mu)mR^2]\omega_o\delta\omega_x + [B + (1-\mu)mR^2]D\delta\omega_y - (1-\mu)mR(D^2 + {\omega_o}^2)\delta z_m &= 0 \\ [C + (1-\mu)mR^2]D\delta\omega_z &= 0 \\ (1-\mu)mR\omega_o\delta\omega_x - (1-\mu)mRD\delta\omega_y + [(1-\mu)mD^2 + cD + k]\delta z_m &= 0 \end{aligned} \tag{10.89}$$

$\delta\omega_z$ appears only in the third equation, which states that $\delta\omega_z$ = constant. The first, second and fourth equations may be combined in matrix notation,

$$\begin{bmatrix} AD & (C-B)\omega_o & 0 \\ [A - C - (1-\mu)mR^2]\omega_o & [B + (1-\mu)mR^2]D & -(1-\mu)mR(D^2 + {\omega_o}^2) \\ (1-\mu)mR\omega_o & -(1-\mu)mRD & (1-\mu)mD^2 + cD + k \end{bmatrix} \begin{Bmatrix} \delta\omega_x \\ \delta\omega_y \\ \delta z_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{10.90}$$

This is a set of three linear differential equations in the perturbations $\delta\omega_x$, $\delta\omega_y$ and δz_m. We won't try to solve them, since all we are really interested in is the stability of the satellite-damper system. It can be shown that the determinant Δ of the 3 by 3 matrix in Equation 10.90 is

$$\Delta = a_4 D^4 + a_3 D^3 + a_2 D^2 + a_1 D + a_0 \tag{10.91}$$

in which the coefficients of the characteristic equation $\Delta = 0$ are

$$\begin{aligned}
a_4 &= (1-\mu)mAB \\
a_3 &= cA[B + (1-\mu)mR^2] \\
a_2 &= k[B + (1-\mu)mR^2]A + (1-\mu)m[(A-C)(B-C) - (1-\mu)AmR^2]\omega_o^{\ 2} \\
a_1 &= c\{[A - C - (1-\mu)mR^2](B-C)\}\omega_o^{\ 2} \\
a_0 &= k\{[A - C - (1-\mu)mR^2](B-C)\}\omega_o^{\ 2} + [(B-C)(1-\mu)^2]m^2R^2\omega_o^{\ 4}
\end{aligned} \tag{10.92}$$

According to the ***Routh-Hurwitz stability criteria*** (see any text on control systems, e.g., Palm, 1983), the motion represented by Equations 10.90 is asymptotically stable if and only if the signs of all of the following quantities, defined in terms of the coefficients of the characteristic equation, are the same

$$r_1 = a_4 \qquad r_2 = a_3 \qquad r_3 = a_2 - \frac{a_4 a_1}{a_3} \qquad r_4 = a_1 - \frac{a_3 a_0}{a_3 a_2 - a_4 a_1} \qquad r_5 = a_0 \tag{10.93}$$

Example 10.6

A satellite is spinning about the z of its principal body frame at 2π radians per second. The principal moments of inertia about its center of mass are

$$A = 300\,\text{kg}\cdot\text{m}^2 \qquad B = 400\,\text{kg}\cdot\text{m}^2 \qquad C = 500\,\text{kg}\cdot\text{m}^2 \tag{a}$$

For the nutation damper, the following properties are given

$$R = 1\,\text{m} \qquad \mu = 0.01 \qquad m = 10\,\text{kg} \qquad k = 10{,}000\,\text{N/m} \qquad c = 150\,\text{N-s/m} \tag{b}$$

Use the Routh-Hurwitz stability criteria to assess the stability of the satellite as a major-axis spinner, a minor-axis spinner, and an intermediate-axis spinner.

Solution

The data in (a) are for a major axis spinner. Substituting into Equations 10.92 and 10.93, we find

$$\begin{aligned}
r_1 &= +1.188 \times 10^6\ \text{kg}^3\text{m}^4 \\
r_2 &= +18.44 \times 10^6\ \text{kg}^3\text{m}^4/\text{s} \\
r_3 &= +1.228 \times 10^9\ \text{kg}^3\text{m}^4/\text{s}^2 \\
r_4 &= +92{,}820\,\text{kg}^3\text{m}^4/\text{s}^3 \\
r_5 &= +8.271 \times 10^9\ \text{kg}^3\text{m}^4/\text{s}^4
\end{aligned} \tag{c}$$

Since every r is positive, spin about the major axis is asymptotically stable. As we know from Section 10.3, without the damper the motion is neutrally stable.

For spin about the minor axis,

$$A = 500\,\mathrm{kg\cdot m^2} \qquad B = 400\,\mathrm{kg\cdot m^2} \qquad C = 300\,\mathrm{kg\cdot m^2} \tag{d}$$

For these moment of inertia values, we obtain

$$\begin{aligned} r_1 &= +1.980 \times 10^6\ \mathrm{kg^3m^4} \\ r_2 &= +30.74 \times 10^6\ \mathrm{kg^3m^4/s} \\ r_3 &= +2.048 \times 10^9\ \mathrm{kg^3m^4/s^2} \\ r_4 &= -304{,}490\,\mathrm{kg^3m^4/s^3} \\ r_5 &= +7.520 \times 10^9\ \mathrm{kg^3m^4/s^4} \end{aligned} \tag{e}$$

Since the rs are not all of the same sign, spin about the minor axis is not asymptotically stable. Recall that for the rigid satellite, such a motion was neutrally stable.

Finally, for spin about the intermediate axis,

$$A = 300\,\mathrm{kg\cdot m^2} \qquad B = 500\,\mathrm{kg\cdot m^2} \qquad C = 400\,\mathrm{kg\cdot m^2} \tag{f}$$

We know this motion is unstable, even without the nutation damper, but doing the Routh-Hurwitz stability check anyway, we get

$$\begin{aligned} r_1 &= +1.485 \times 10^6\ \mathrm{kg^3m^4} \\ r_2 &= +22.94 \times 10^6\ \mathrm{kg^3m^4/s} \\ r_3 &= +1.529 \times 10^9\ \mathrm{kg^3m^4/s^2} \\ r_4 &= -192{,}800\ \mathrm{kg^3m^4/s^3} \\ r_5 &= -4.323 \times 10^9\ \mathrm{kg^3m^4/s^4} \end{aligned}$$

The motion, as we expected, is not stable.

10.6 CONING MANEUVER

Like the use of nutation dampers, the coning maneuver is an example of the attitude control of spinning spacecraft. In this case, the angular momentum is changed by the use of on board thrusters (small rockets) to apply pure torques.

Consider a spacecraft in pure spin with angular velocity ω_0 about its body-fixed z axis, which is an axis of rotational symmetry. The angular momentum is $\mathbf{H}_{G_0} = C\omega_0\hat{\mathbf{k}}$. Suppose we wish to maintain the magnitude of the angular momentum but change its direction by rotating the spin axis through an angle θ, as

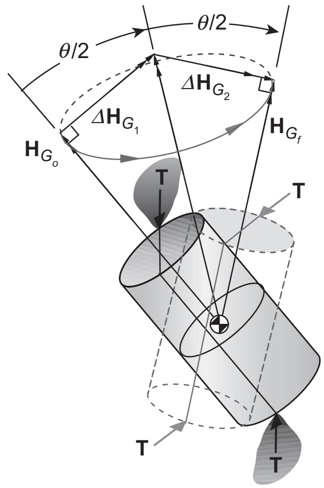


FIGURE 10.10

Impulsive coning maneuver.

illustrated in Figure 10.10. Recall from Section 9.4 that to change the angular momentum of the spacecraft requires applying an external moment,

$$\Delta \mathbf{H}_G = \int_0^{\Delta t} \mathbf{M}_G \, dt$$

Thrusters may be used to provide the external impulsive torque required to produce an angular momentum increment $\Delta \mathbf{H}_{G_1}$ normal to the spin axis. Since the spacecraft is spinning, this induces coning (precession) of the spacecraft about an axis at an angle of $\theta/2$ to the direction of $\mathbf{H}_{G_0}$. Since the external couple is normal to the z axis, the maneuver produces no change in the z component of the angular velocity, which remains ω_0. However, after the impulsive moment, the angular velocity comprises a spin component ω_s and a precession component ω_p. Whereas before the impulsive moment $\omega_s = \omega_0$, afterwards, during coning, the spin component is given by Equation 10.20,

$$\omega_s = \frac{A - C}{A}\omega_0$$

The precession rate is given by Equation 10.22,

$$\omega_p = \frac{C}{A}\frac{\omega_0}{\cos(\theta/2)} \tag{10.94}$$

Notice that before the impulsive maneuver, the magnitude of the angular momentum is $C\omega_0$. Afterwards, it has increased to

$$H_G = A\omega_p = \frac{C\omega_0}{\cos(\theta/2)}$$

Observe that a deflection angle θ of 180° is impossible since it would require an infinite torque impulse.

After precessing 180°, an angular momentum increment $\Delta\mathbf{H}_{G_2}$ normal to the spin axis and in the same direction relative to the spacecraft as the initial torque impulse, with $\|\Delta\mathbf{H}_{G_2}\| = \|\Delta\mathbf{H}_{G_1}\|$, stabilizes the spin vector in the desired direction. Since the spin rate ω_s is not in general the same as the precession rate ω_p, the second angular impulse must be delivered by another pair of thrusters which have rotated into the position to apply the torque impulse in the proper direction. With only one pair of thrusters, both the spin axis and the spacecraft must rotate though 180° in the same time interval, which means $\omega_p = \omega_s$, that is

$$\frac{A-C}{A}\omega_0 = \frac{C}{A}\frac{\omega_0}{\cos(\theta/2)}$$

This requires the deflection angle to be

$$\theta = 2\cos^{-1}\left(\frac{C}{A-C}\right)$$

and limits the values of the moments of inertia A and C to those that do not cause the magnitude of the cosine to exceed unity.

The time required for an angular reorientation θ using a single coning maneuver is found by simply dividing the precession angle, π radians, by the precession rate ω_p,

$$t_1 = \frac{\pi}{\omega_p} = \pi\frac{A}{C\omega_0}\cos\frac{\theta}{2} \tag{10.95}$$

Propellant expenditure is reflected in the magnitude of the individual angular momentum increments, in obvious analogy to delta-v calculations for orbital maneuvers. The total delta-H required for the single coning maneuver is therefore given by

$$\Delta H_{\text{total}} = \|\Delta\mathbf{H}_{G_1}\| + \|\Delta\mathbf{H}_{G_2}\| = 2\left(\|\mathbf{H}_{G_0}\|\tan\frac{\theta}{2}\right) \tag{10.96}$$

Figure 10.11 illustrates the fact that ΔH_{total} can theoretically be reduced by using a sequence of small coning maneuvers (small θs) rather than one big θ. The large number of small ΔHs approximates a circular arc of radius $\|\mathbf{H}_{G_0}\|$, subtended by the angle θ. Therefore, approximately,

$$\Delta H_{\text{total}} = 2\left(\|\mathbf{H}_{G_0}\|\frac{\theta}{2}\right) = \|\mathbf{H}_{G_0}\|\theta \tag{10.97}$$

This expression becomes more precise as the number of intermediate maneuvers increases. Figure 10.12 reveals the extent to which the multiple coning maneuver strategy reduces energy requirements. The difference is quite significant for large reorientation angles.

One of the prices to be paid for the reduced energy of the multiple coning maneuver is time. (The other is the complexity mentioned above, to say nothing of the risk involved in repeating the maneuver over and

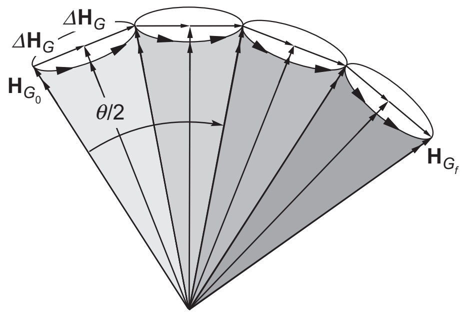


FIGURE 10.11

A sequence of small coning maneuvers.

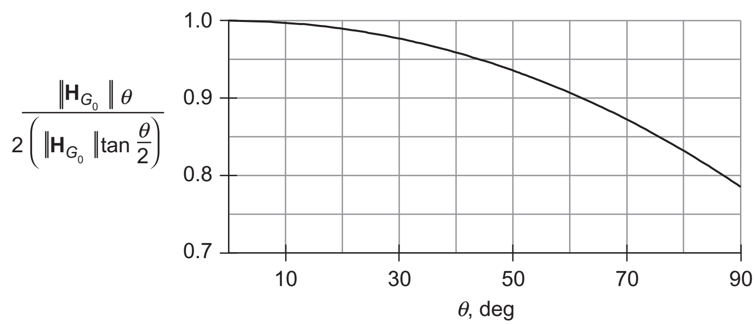


FIGURE 10.12

Ratio of delta-H for a sequence of small coning maneuvers to that for a single coning maneuver, as a function of the angle of swing of the spin axis.

over again.) From Equation 10.95, the time required for n small-angle coning maneuvers through a total angle of θ is

$$t_n = n\pi \frac{A}{C\omega_0} \cos\frac{\theta}{2n} \tag{10.98}$$

The ratio of this to the time t_1 required for a single coning maneuver is

$$\frac{t_n}{t_1} = n \frac{\cos\dfrac{\theta}{2n}}{\cos\dfrac{\theta}{2}} \tag{10.99}$$

The time is directly proportional to the number of intermediate coning maneuvers, as illustrated in Figure 10.13.

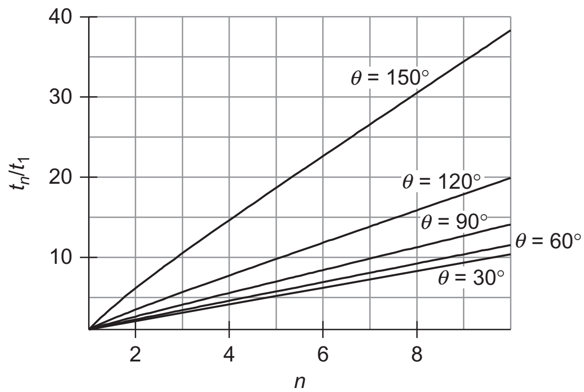


FIGURE 10.13

Time for a coning maneuver versus the number of intermediate steps.

10.7 ATTITUDE CONTROL THRUSTERS

As mentioned above, thrusters are small jets mounted in pairs on a spacecraft to control its rotational motion about the center of mass. These thruster pairs may be mounted in principal planes (planes normal to the principal axes) passing through the center of mass. Figure 10.14 illustrates a pair of thrusters for producing a torque about the positive y axis. These would be accompanied by another pair of reaction motors pointing in the opposite directions to exert torque in the negative y direction. If the position vectors of the thrusters relative to the center of mass are $\mathbf{r}$ and $-\mathbf{r}$, and if $\mathbf{T}$ is their thrust, then the impulsive moment they exert during a brief time interval Δt is

$$\mathbf{M} = \mathbf{r} \times \mathbf{T}\Delta t + (-\mathbf{r}) \times (-\mathbf{T}\Delta t) = 2\mathbf{r} \times \mathbf{T}\Delta t \tag{10.100}$$

If the angular velocity was initially zero, then after the firing, according to Equation 9.31, the angular momentum becomes

$$\mathbf{H} = 2\mathbf{r} \times \mathbf{T}\Delta t \tag{10.101}$$

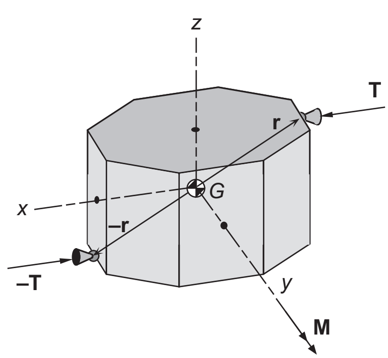


FIGURE 10.14

Pair of attitude control thrusters mounted in the xz plane of the principal body frame.

For **H** in the principal x direction, as in the figure, the corresponding angular velocity acquired by the vehicle is, from Equation 9.67,

$$\omega_y = \frac{\|\mathbf{H}\|}{B} \tag{10.102}$$

Example 10.7

A spacecraft of mass m and with the dimensions shown in Figure 10.15 is spinning without precession at the rate ω_0 about the z axis of the principal body frame. At the instant shown in part (a) of the figure, the spacecraft initiates a coning maneuver to swing its spin axis through 90°, so that at the end of the maneuver the vehicle is oriented as illustrated in Figure 10.15b. Calculate the total delta-H required, and compare it with that required for the same reorientation without coning. Motion is to be controlled exclusively by the pairs of attitude thrusters shown, all of which have identical thrust T.

Solution

According to Figure 9.9c, the moments of inertia about the principal body axes are

$$A = B = \frac{1}{12}m\left[w^2 + \left(\frac{w}{3}\right)^2\right] = \frac{5}{54}mw^2 \qquad C = \frac{1}{12}m(w^2 + w^2) = \frac{1}{6}mw^2$$

The initial angular momentum $\mathbf{H}_{G_1}$ points in the spin direction, along the positive z axis of the body frame,

$$\mathbf{H}_{G_1} = C\omega_z\hat{\mathbf{k}} = \frac{1}{6}mw^2\omega_0\hat{\mathbf{k}}$$

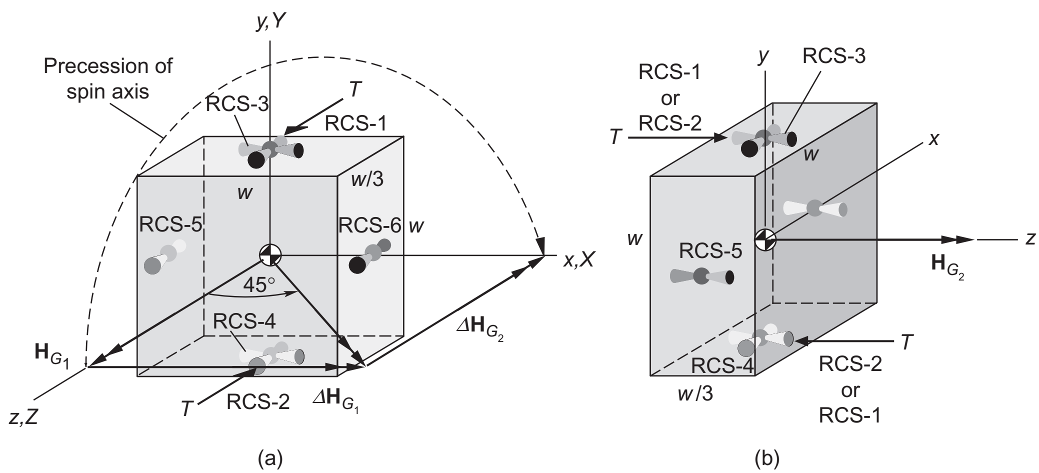


FIGURE 10.15

(a) Initial orientation of spinning spacecraft. (b) Final configuration, with spin axis rotated 90°.

We can presume that in the initial orientation, the body frame happens to coincide instantaneously with inertial frame *XYZ*. The coning motion is initiated by briefly firing the pair of thrusters RCS-1 and RCS-2, aligned with the body *z* axis and lying in the *yz* plane. The impulsive torque will cause a change $\Delta\mathbf{H}_{G_1}$ in angular momentum directed normal to the plane of the thrusters, in the positive body *x* direction. The resultant angular momentum vector must lie at 45° to the *x* and *z* axes, bisecting the angle between the initial and final angular momenta. Thus,

$$\left\|\Delta\mathbf{H}_{G_1}\right\| = \left\|\mathbf{H}_{G_1}\right\|\tan 45^\circ = \frac{1}{6}mw^2\omega_0$$

After the coning is underway, the body axes of course move away from the *XYZ* frame. Since the spacecraft is oblate ($C > A$), the precession of the spin axis will be opposite to the spin direction, as indicated in Figure 10.15. When the spin axis, after 180° of precession, lines up with the *X* axis, the thrusters must fire again for the same duration as before so as to produce the angular momentum change $\Delta\mathbf{H}_{G_2}$, equal in magnitude but perpendicular to $\Delta\mathbf{H}_{G_1}$, so that

$$\mathbf{H}_{G_1} + \Delta\mathbf{H}_{G_1} + \Delta\mathbf{H}_{G_2} = \mathbf{H}_{G_2}$$

where

$$\mathbf{H}_{G_2} = \left\|\mathbf{H}_{G_1}\right\|\hat{\mathbf{I}} = \frac{1}{6}mw^2\omega_0\hat{\mathbf{k}}$$

For this to work, the plane of thrusters RCS-1 and RCS-2—the *yz* plane—must be parallel to the *XY* plane when they fire, as illustrated in Figure 10.15b. Since the thrusters can fire fore or aft, it does not matter which of them ends up on top or bottom. The vehicle must therefore spin through an integral number *n* of half rotations while it precesses to the desired orientation. That is, the total spin angle ψ between the initial and final configurations is

$$\psi = n\pi = \omega_s t \tag{a}$$

where ω_s is the spin rate and *t* is the time for the proper final configuration to be achieved. In the meantime, the precession angle must be ϕ must be π or 3π or 5π, or, in general,

$$\phi = (2m-1)\pi = \omega_p t \tag{b}$$

where *m* is an integer and *t* is, of course, the same as that in (a). Eliminating *t* from both (a) and (b) yields

$$n\pi = (2m-1)\pi\frac{\omega_s}{\omega_p}$$

Substituting Equation 10.23, with $\theta = \pi/2$, gives

$$n = (1-2m)\frac{4}{9}\frac{1}{\sqrt{2}} \tag{c}$$

Obviously, this equation cannot be valid if both *m* and *n* are integers. However, by tabulating *n* as a function of *m* we find that when $m = 18$, n $= -10.999$. The minus sign simply reminds us that spin and precession are in opposite directions. Thus, the eighteenth time that the spin axis lines up with the *X* axis the thrusters may be fired to almost perfectly align the angular momentum vector with the body z axis. The slight misalignment due to the fact that $|n|$ is not precisely 11 would probably occur in reality anyway. Passive or active nutation damping can drive this deviation to zero.

Since $\left\|\mathbf{H}_{G_1}\right\| = \left\|\mathbf{H}_{G_2}\right\|$, we conclude that

$$\Delta H_{\text{total}} = 2\left(\frac{1}{6}mw^2\omega_0\right) = \frac{2}{3}mw^2\omega_0 \tag{d}$$

An obvious alternative to the coning maneuver is to use thrusters RCS-3 and 4 to despin the craft completely, thrusters RCS-5 and 6 to initiate roll around the y axis and stop it after 90°, and then RCS-3 and 4 to respin the spacecraft to ω_0 around the z axis. The combined delta-H for the first and last steps equals that of (d). Additional fuel expenditure is required to start and stop the roll around the y axis. Hence, the coning maneuver is more fuel efficient.

10.8 YO-YO DESPIN MECHANISM

A simple, inexpensive way to despin an axisymmetric spacecraft is to deploy small masses attached to cords wound around the girth of the spacecraft near the transverse plane through the center of mass. As the masses unwrap in the direction of the spacecraft's angular velocity, they exert centrifugal force through the cords on the periphery of the vehicle, creating a moment opposite to the spin direction, thereby slowing down the rotational motion. The cord forces are internal to the system of spaceraft plus weights, so as the strings unwind, the total angular momentum must remain constant. Since the total moment of inertia increases as the yo-yo masses spiral further away, the angular velocity must drop. Not only angular momentum but also rotational kinetic energy is conserved during this process. Yo-yo despin devices were introduced early in unmanned space flight (e.g., 1959 Transit 1-A) and continued to be used thereafter (e.g., 1996 Mars Pathfinder, 1998 Mars Climate Orbiter, 1999 Mars Polar Lander, 2003 Mars Exploration Rover).

The problem is to determine the length of cord required to reduce the spacecraft's angular velocity a specified amount. Because it is easier than solving the equations of motion, we will apply the principles of conservation of energy and angular momentum to the system comprising the spacecraft and yo-yo masses. To maintain the position of the center of mass, two identical yo-yo masses are wound around the spacecraft in a symmetrical fashion, as illustrated in Figure 10.16. Both masses are released simultaneously by explosive bolts and unwrap in the manner shown (for only one of the weights) in the figure. In so doing, the point of tangency T moves around the circumference towards the split hinge device where the cord is attached to the spacecraft. When T and T' reach the hinges H and H', the cords automatically separate from the spacecraft.

Let each yo-yo weight have mass $m/2$. By symmetry, we need to track only one of the masses, to which we can ascribe the total mass m. Let the xyz system be a body frame rigidly attached to the spacecraft, as shown in Figure 10.16. As usual, the z axis lies in the spin direction, pointing out of the page. The x axis is directed from the center of mass of the system through the initial position of the yo-yo mass. The spacecraft and the yo-yo masses, prior to release, are rotating as a single rigid body with angular velocity $\boldsymbol{\omega}_0 = \omega_0\hat{\mathbf{k}}$. The moment of inertia of the satellite, excluding the yo-yo mass, is C, so that the angular momentum of the satellite by itself is $C\omega_0$. The concentrated yo-yo masses are fastened a distance R from the spin axis, so their total moment of inertia is mR^2. Therefore, the initial angular momentum of the satellite plus yo-yo system is

$$H_{G_0} = C\omega_0 + mR^2\omega_0$$

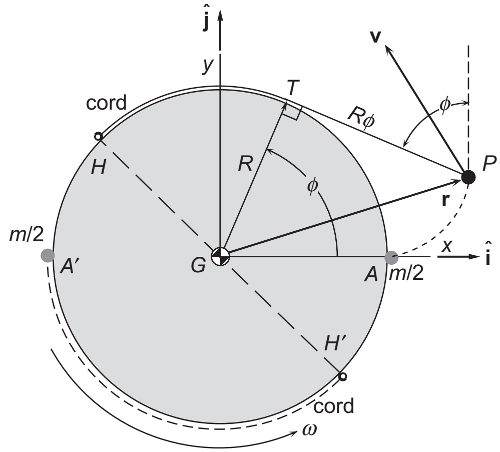


FIGURE 10.16

Two identical string and mass systems wrapped symmetrically around the periphery of an axisymmetric spacecraft. For simplicity, only one is shown being deployed.

It will be convenient to write this as

$$H_{G_0} = KmR^2\omega_0 \tag{10.103}$$

where the nondimensional factor K is defined as

$$K = 1 + \frac{C}{mR^2} \tag{10.104}$$

$\sqrt{K}R$ is the initial radius of gyration of the system.

The initial rotational kinetic energy of the system, before the masses are released, is

$$T_0 = \frac{1}{2}C{\omega_0}^2 + \frac{1}{2}mR^2{\omega_0}^2 = \frac{1}{2}KmR^2{\omega_0}^2 \tag{10.105}$$

At any state between the release of the weights and the release of the cords at the hinges, the velocity of the yo-yo mass must be found in order to compute the new angular momentum and kinetic energy. Observe that when the string has unwrapped an angle ϕ, the free length of string (between the point of tangency T and the yo-yo mass P) is $R\phi$. From the geometry shown in Figure 10.16, the position vector of the mass relative to the body frame is seen to be

$$\begin{aligned}\mathbf{r} &= \overbrace{(R\cos\phi\hat{\mathbf{i}} + R\sin\phi\hat{\mathbf{j}})}^{\mathbf{r}_{T/G}} + \overbrace{(R\phi\sin\phi\hat{\mathbf{i}} - R\phi\cos\phi\hat{\mathbf{j}})}^{\mathbf{r}_{P/T}} \\ &= (R\cos\phi + R\phi\sin\phi)\hat{\mathbf{i}} + (R\sin\phi - R\phi\cos\phi)\hat{\mathbf{j}}\end{aligned} \tag{10.106}$$

Since $\mathbf{r}$ is measured in the moving reference, the absolute velocity $\mathbf{v}$ of the yo-yo mass is found using Equation 1.56

$$\mathbf{v} = \left.\frac{d\mathbf{r}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega}\times\mathbf{r} \tag{10.107}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the xyz axes, which, of course, is the angular velocity $\boldsymbol{\omega}$ of the spacecraft at that instant,

$$\boldsymbol{\Omega} = \boldsymbol{\omega} \tag{10.108}$$

To calculate $d\mathbf{r}/dt)_{\text{rel}}$, we hold $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ constant in Equation 10.106, obtaining

$$\begin{aligned}\left.\frac{d\mathbf{r}}{dt}\right)_{\text{rel}} &= (-R\dot{\phi}\sin\phi + R\dot{\phi}\sin\phi + R\phi\dot{\phi}\cos\phi)\hat{\mathbf{i}} + (R\dot{\phi}\cos\phi - R\dot{\phi}\cos\phi + R\phi\dot{\phi}\sin\phi)\hat{\mathbf{j}} \\ &= R\phi\dot{\phi}\cos\phi\hat{\mathbf{i}} + R\phi\dot{\phi}\sin\phi\hat{\mathbf{j}}\end{aligned}$$

Thus,

$$\mathbf{v} = R\phi\dot{\phi}\cos\phi\hat{\mathbf{i}} + R\phi\dot{\phi}\sin\phi\hat{\mathbf{j}} + \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega \\ R\cos\phi + R\phi\sin\phi & R\sin\phi - R\phi\cos\phi & 0\end{vmatrix}$$

or

$$\mathbf{v} = [R\phi(\omega + \dot{\phi})\cos\phi - R\omega\sin\phi]\hat{\mathbf{i}} + [R\omega\cos\phi + R\phi(\omega + \dot{\phi})\sin\phi]\hat{\mathbf{j}} \tag{10.109}$$

From this we find the speed of the yo-yo weights,

$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = R\sqrt{\omega^2 + (\omega + \dot{\phi})^2\phi^2} \tag{10.110}$$

The angular momentum of the spacecraft plus the weights at an intermediate stage of the despin process is

$$\mathbf{H}_G = C\omega\hat{\mathbf{k}} + \mathbf{r}\times m\mathbf{v} = C\omega\hat{\mathbf{k}} + m\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ R\cos\phi + R\phi\sin\phi & R\sin\phi - R\phi\cos\phi & \omega \\ R\phi(\omega + \dot{\phi})\cos\phi - R\omega\sin\phi & R\omega\cos\phi + R\phi(\omega + \dot{\phi})\sin\phi & 0\end{vmatrix}$$

Carrying out the cross product, combining terms and simplifying, leads to

$$H_G = C\omega + mR^2[\omega + (\omega + \dot{\phi})\phi^2]$$

which, using Equation 10.104, can be written

$$H_G = mR^2[K\omega + (\omega + \dot{\phi})\phi^2] \tag{10.111}$$

The kinetic energy of the spacecraft plus the yo-yo mass is

$$T = \frac{1}{2}C\omega^2 + \frac{1}{2}mv^2$$

Substituting the speed from Equation 10.110 and making use again of Equation 10.104, we find

$$T = \frac{1}{2}mR^2[K\omega^2 + (\omega + \dot{\phi})^2\phi^2] \tag{10.112}$$

By the conservation of angular momentum, $H_G = H_{G_0}$, we obtain from Equations 10.103 and 10.111,

$$mR^2[K\omega + (\omega + \dot{\phi})\phi^2] = KmR^2\omega_0$$

which we can write as

$$K(\omega_0 - \omega) = (\omega + \dot{\phi})\phi^2 \qquad \text{Conservation of angular momentum} \tag{10.113}$$

Equations 10.105 and 10.112 and the conservation of kinetic energy, $T = T_0$, combine to yield

$$\frac{1}{2}mR^2[K\omega^2 + (\omega + \dot{\phi})^2\phi^2] = \frac{1}{2}KmR^2\omega_0{}^2$$

or

$$K(\omega_0{}^2 - \omega^2) = (\omega + \dot{\phi})^2\phi^2 \qquad \text{Conservation of energy} \tag{10.114}$$

Since $\omega_0{}^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega)$, this can be written

$$K(\omega_0 - \omega)(\omega_0 + \omega) = (\omega + \dot{\phi})^2\phi^2$$

Replacing the factor $K(\omega_0 - \omega)$ on the left using Equation 10.113 yields

$$(\omega + \dot{\phi})\phi^2(\omega_0 + \omega) = (\omega + \dot{\phi})^2\phi^2$$

After canceling terms, we find $\omega_0 + \omega = \omega + \dot{\phi}$, or, simply

$$\dot{\phi} = \omega_0 \qquad \text{Conservation of energy } \textit{and} \text{ momentum} \tag{10.115}$$

In other words, the cord unwinds at a constant rate (relative to the spacecraft), equal to the vehicle's initial angular velocity. Thus at any time t after the release of the weights,

$$\phi = \omega_0 t \tag{10.116}$$

By substituting Equation 10.115 into Equation 10.113,

$$K(\omega_0 - \omega) = (\omega + \omega_0)\phi^2$$

we find that

$$\phi = \sqrt{K\frac{\omega_0 - \omega}{\omega_0 + \omega}} \qquad \text{Partial despin} \tag{10.117}$$

Recall that the unwrapped length l of the cord is $R\phi$, which means

$$l = R\sqrt{K\frac{\omega_0 - \omega}{\omega_0 + \omega}} \qquad \text{Partial despin} \tag{10.118}$$

We use Equation 10.118 to find the length of cord required to despin the spacecraft from ω_0 to ω. To remove all of the spin ($\omega = 0$),

$$\phi = \sqrt{K} \Rightarrow l = R\sqrt{K} \qquad \text{Complete despin} \tag{10.119}$$

Surprisingly, the length of cord required to reduce the angular velocity to zero is independent of the initial angular velocity.

We can solve Equation 10.117 for ω in terms of ϕ,

$$\omega = \left(\frac{2K}{K + \phi^2} - 1\right)\omega_0 \tag{10.120}$$

By means of Equation 10.116, this becomes an expression for the angular velocity as a function of time,

$$\omega = \left(\frac{2K}{K + {\omega_0}^2 t^2} - 1\right)\omega_0 \tag{10.121}$$

Alternatively, since $\phi = l/R$, Equation 10.120 yields the angular velocity as a function of cord length,

$$\omega = \left(\frac{2KR^2}{KR^2 + l^2} - 1\right)\omega_0 \tag{10.122}$$

Differentiating ω with respect to time in Equation 10.121 gives us an expression for the angular acceleration of the spacecraft,

$$\alpha = \frac{d\omega}{dt} = -\frac{4K{\omega_0}^3 t}{(K + {\omega_0}^2 t^2)^2} \tag{10.123}$$

whereas integrating ω with respect to time yields the angle rotated by the spacecraft since release of the yo-yo mass,

$$\theta = 2\sqrt{K}\tan^{-1}\frac{\omega_0 t}{\sqrt{K}} - \omega_0 t = 2\sqrt{K}\tan^{-1}\frac{\phi}{\sqrt{K}} - \phi \tag{10.124}$$

For complete despin, this expression, together with Equation 10.119, yields

$$\theta = \sqrt{K}\left(\frac{\pi}{2} - 1\right) \tag{10.125}$$

From the free-body diagram of the spacecraft shown in Figure 10.17, it is clear that the torque exerted by the yo-yo weights is

$$M_{Gz} = -2RN \tag{10.126}$$

where N is the tension in the cord. From Euler's equations of motion, Equations 9.72b,

$$M_{G_z} = C\alpha \tag{10.127}$$

Combining Equations 10.123, 10.126 and 10.127 leads to a formula for the tension in the yo-yo cables,

$$N = \frac{C}{R}\frac{2K\omega_0{}^3 t}{(K + \omega_0{}^2 t^2)^2} = \frac{C\omega_0{}^2}{R}\frac{2K\phi}{(K + \phi^2)^2} \tag{10.128}$$

10.8.1 Radial Release

Finally, we note that instead of releasing the yo-yo masses when the cables are tangent at the split hinges (H and H'), they can be forced to pivot about the hinge and released when the string is directed radially outward, as illustrated in Figure 10.18. The above analysis must be then extended to include the pivoting of the cord around the hinges. It turns out that in this case, the length of the cord as a function of the final angular velocity is

$$l = R\left(\sqrt{\frac{[(\omega_0 - \omega)K + \omega]^2}{(\omega_0{}^2 - \omega^2)K + \omega^2}} - 1\right) \quad \text{Partial despin, radial release} \tag{10.129}$$

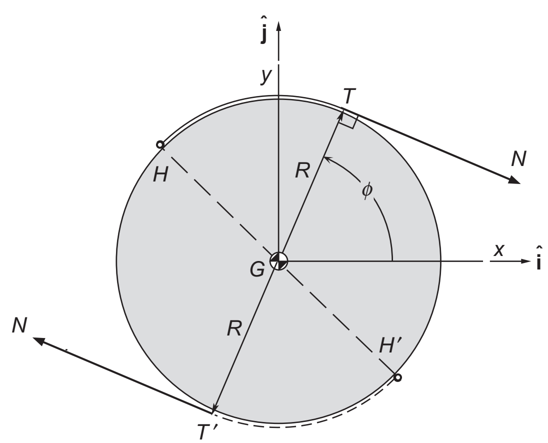


FIGURE 10.17

Free-body diagram of the satellite during the despin process.

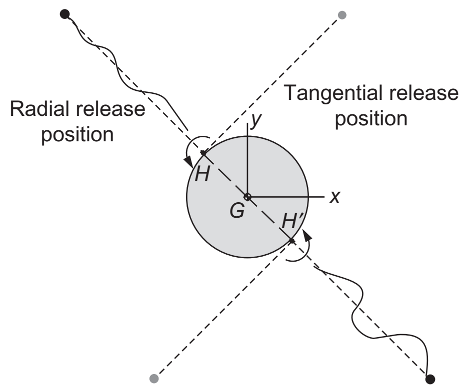


FIGURE 10.18

Radial versus tangential release of yo-yo masses.

so that for $\omega = 0$,

$$l = R(\sqrt{K} - 1) \qquad \text{Complete despin, radial release} \tag{10.130}$$

Example 10.8

A satellite is to be completely despun using a two-mass yo-yo device with tangential release. Assume the spin axis of moment of inertia of the satellite is $C = 200\,\text{kg}\cdot\text{m}^2$ and the initial spin rate is $\omega_0 = 5$ rad/s. The total yo-yo mass is 4 kg, and the radius of the spacecraft is 1 meter. Find

(a) the required cord length l;
(b) the time t to despin;
(c) the maximum tension in the yo-yo cables;
(d) the speed of the masses at release;
(e) the angle rotated by the satellite during the despin;
(f) the cord length required for radial release.

Solution

(a) From Equation 10.104,

$$K = 1 + \frac{C}{mR^2} = 1 + \frac{200}{4\cdot 1^2} = 51 \tag{a}$$

From Equation 10.119 it follows that the cord length required for complete despin is

$$l = R\sqrt{K} = 1\cdot\sqrt{51} = \boxed{7.1414\ \text{m}} \tag{b}$$

(b) The time for complete despin is obtained from Equations 10.116 and 10.119,

$$\omega_0 t = \sqrt{K} \Rightarrow t = \frac{\sqrt{K}}{\omega_0} = \frac{\sqrt{51}}{5} = \boxed{1.4283\ \text{s}}$$

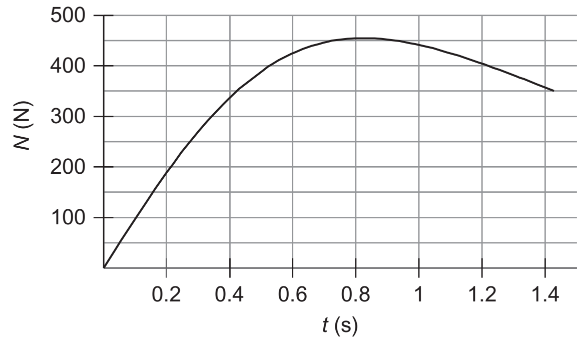


FIGURE 10.19

Variation of cable tension N up to point of release.

(c) A graph of Equation 10.128 is shown in Figure 10.19, from which we see that

$$\boxed{\text{The maximum tension is } 455\text{ N}}$$

which occurs at 0.825 s.

(d) From Equation 10.110, the speed of the yo-yo masses is

$$v = R\sqrt{\omega^2 + (\omega + \dot{\phi})^2\phi^2}$$

According to Equation 10.115, $\dot{\phi} = \omega_0$ and at the time of release ($\omega = 0$) Equation 10.117 states that $\phi = \sqrt{K}$. Thus,

$$v = R\sqrt{\omega^2 + (\omega + \omega_0)^2\sqrt{K}^2} = 1 \cdot \sqrt{0^2 + (0+5)^2\sqrt{51}^2} = \boxed{35.71\text{ m/s}}$$

(e) The angle through which the satellite rotates before coming to rotational rest is given by Equation 10.125,

$$\theta = \sqrt{K}\left(\frac{\pi}{2} - 1\right) = \sqrt{51}\left(\frac{\pi}{2} - 1\right) = \boxed{4.076\text{ rad } (233.5\text{ degrees})}$$

(f) Allowing the cord to detach radially reduces the cord length required for complete despin from 7.141 m to (Equation 10.130)

$$l = R(\sqrt{K} - 1) = 1 \cdot (\sqrt{51} - 1) = \boxed{6.141\text{ m}}$$

10.9 GYROSCOPIC ATTITUDE CONTROL

Momentum exchange systems ("gyros") are used to control the attitude of a spacecraft without throwing consumable mass overboard, as occurs with the use of thruster jets. A momentum exchange system is illustrated schematically in Figure 10.20. n flywheels, labeled 1, 2, 3, etc., are attached to the body of the spacecraft at

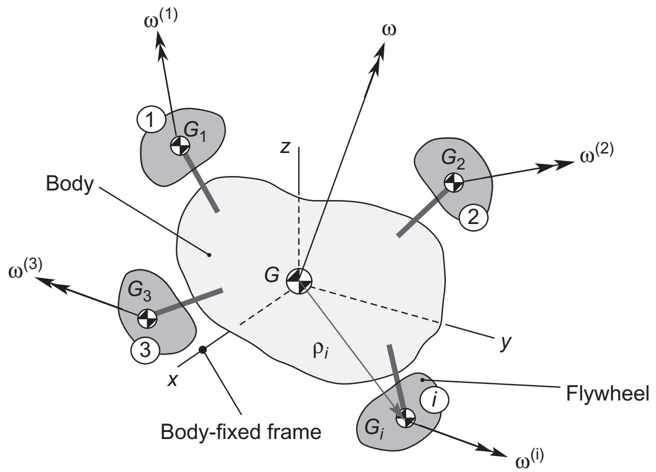


FIGURE 10.20

Several attitude control flywheels, each with their own angular velocity, attached to the body of a spacecraft.

various locations. The mass of flywheel i is m_i. The mass of the body of the spacecraft is m_0. The total mass of the entire system—the "vehicle"—is m,

$$m = m_0 + \sum_{i=1}^{n} m_i$$

The vehicle's center of mass is G, through which pass the three axes xyz of the vehicle's body-fixed frame. The center of mass G_i of each flywheel is connected rigidly to the spacecraft, but the wheel, driven by electric motors, rotates more or less independently, depending on the type of gyro. The body of the spacecraft has an angular velocity $\boldsymbol{\omega}$. The angular velocity of the ith flywheel is $\boldsymbol{\omega}^{(i)}$, and differs from that of the body of the spacecraft unless the gyro is "caged." A caged gyro has no spin relative to the spacecraft, in which case $\boldsymbol{\omega}^{(i)} = \boldsymbol{\omega}$.

According to Equation 9.39b, the angular momentum of the body itself relative to G is

$$\{\mathbf{H}_G^{(body)}\} = [\mathbf{I}_G^{(body)}]\{\boldsymbol{\omega}\} \tag{10.131}$$

where $[\mathbf{I}_G^{(body)}]$ is the moment of inertia tensor of the body about G and $\boldsymbol{\omega}$ is the angular velocity of the body.

Equation 9.27 gives the angular momentum of flywheel i relative to G as

$$\mathbf{H}_G^{(i)} = \mathbf{H}_{G_i}^{(i)} + \boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i \tag{10.132}$$

$\mathbf{H}_{G_i}^{(i)}$ is the angular momentum vector of the flywheel i about its own center of mass. Its components in the body frame are found from the expression

$$\left\{\mathbf{H}_{G_i}^{(i)}\right\} = \left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}^{(i)}\} \tag{10.133}$$

where $\left[\mathbf{I}_{G_i}^{(i)}\right]$ is the moment of inertia tensor of the flywheel about its own center of mass G_i, relative to axes that are parallel to the body-fixed *xyz* axes. Since a momentum wheel might be one that pivots on gimbals relative to the body frame, the inertia tensor $\left[\mathbf{I}_{G_i}^{(i)}\right]$ may be time dependent. The vector $\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i$ in Equation 10.132 is the angular momentum of the concentrated mass m_i of the flywheel about the *system* center of mass G. According to Equation 9.59 the components of $\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i$ in the body frame are given by

$$\{\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i\} = \left[\mathbf{I}_{m_G}^{(i)}\right]\{\boldsymbol{\omega}\} \tag{10.134}$$

where $\left[\mathbf{I}_{m_G}^{(i)}\right]$, the moment of inertia tensor of the point mass m_i about G, is given by Equation (9.44). Using Equations (10.133) and (10.134), Equation (10.132) can be written as

$$\{\mathbf{H}_G^{(i)}\} = \left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}^{(i)}\} + \left[\mathbf{I}_{m_G}^{(i)}\right]\{\boldsymbol{\omega}\} \tag{10.135}$$

The total angular momentum of the system in Figure 10.20 about G is that of the body plus all of the n flywheels.

$$\mathbf{H}_G = \mathbf{H}_G^{(body)} + \sum_{i=1}^{n}\mathbf{H}_G^{(i)}$$

Substituting Equations 10.131 and 10.135, we obtain

$$\{\mathbf{H}_G\} = [\mathbf{I}_G^{(body)}]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left(\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}^{(i)}\} + \left[\mathbf{I}_{m_G}^{(i)}\right]\{\boldsymbol{\omega}\}\right)$$

or

$$\{\mathbf{H}_G\} = \left(\left[\mathbf{I}_G^{(body)}\right] + \sum_{i=1}^{n}\left[\mathbf{I}_{m_G}^{(i)}\right]\right)\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}^{(i)}\} \tag{10.136}$$

Let

$$[\mathbf{I}_G^{(v)}] = [\mathbf{I}_G^{(body)}] + \sum_{i=1}^{n}\left[\mathbf{I}_{m_G}^{(i)}\right] \tag{10.137}$$

$[\mathbf{I}_G^{(v)}]$ is the time-independent total moment of inertia of the vehicle v, that is, that of the body plus the concentrated masses of all of the flywheels. Thus,

$$\{\mathbf{H}_G\} = [\mathbf{I}_G^{(v)}]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}^{(i)}\} \tag{10.138}$$

If $\boldsymbol{\omega}_{rel}^{(i)}$ is the angular velocity of the ith flywheel relative to the spacecraft, then its inertial angular velocity $\boldsymbol{\omega}^{(i)}$ is given by Equation 9.5,

$$\boldsymbol{\omega}^{(i)} = \boldsymbol{\omega} + \boldsymbol{\omega}_{\text{rel}}^{(i)} \tag{10.139}$$

where $\boldsymbol{\omega}$ is the inertial angular velocity of the spacecraft body. Substituting Equation 10.139 into 10.138 yields

$$\{\mathbf{H}_G\} = [\mathbf{I}_G^{(v)}]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega} + \boldsymbol{\omega}_{\text{rel}}^{(i)}\}$$

or

$$\{\mathbf{H}_G\} = \left([\mathbf{I}_G^{(v)}] + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\right)\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\} \tag{10.140}$$

An alternative form of this expression may be obtained by substituting Equation 10.137:

$$\begin{aligned}\{\mathbf{H}_G\} &= \left([\mathbf{I}_G^{(body)}] + \sum_{i=1}^{n}\left[\mathbf{I}_{m_G}^{(i)}\right] + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\right)\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\} \\ &= \left([\mathbf{I}_G^{(body)}] + \sum_{i=1}^{n}\left(\left[\mathbf{I}_{G_i}^{(i)}\right] + \left[\mathbf{I}_{m_G}^{(i)}\right]\right)\right)\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\}\end{aligned} \tag{10.141}$$

But, according to the parallel axis theorem (Equation 9.61),

$$[\mathbf{I}_G^{(i)}] = \left[\mathbf{I}_{G_i}^{(i)}\right] + \left[\mathbf{I}_{m_G}^{(i)}\right]$$

where $[\mathbf{I}_G^{(i)}]$ is the moment of inertia of the ith flywheel around the center of mass of the body of the spacecraft. Hence, we can write Equation 10.141 as

$$\{\mathbf{H}_G\} = \left([\mathbf{I}_G^{(body)}] + \sum_{i=1}^{n}[\mathbf{I}_G^{(i)}]\right)\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\} \tag{10.142}$$

The equation of motion of the system is given by Equations 9.30 and 1.56,

$$\left.\mathbf{M}_{G_{\text{net}}}\right)_{\text{external}} = \left.\frac{d\mathbf{H}_G}{dt}\right)_{\text{rel}} + \boldsymbol{\omega}\times\mathbf{H}_G \tag{10.143}$$

If $\left.\mathbf{M}_{G_{\text{net}}}\right)_{\text{external}} = \mathbf{0}$, then $\mathbf{H}_G$ = constant.

Example 10.9

A disk is attached to a plate at their common center of mass. Between the two is a motor mounted on the plate which drives the disk into rotation relative to the plate. The system rotates freely in the xy plane in gravity-free space. The moments of inertia of the plate and the disk about the z axis through G are I_p and I_w, respectively. Determine the change in the relative angular velocity ω_{rel} of the disk required to cause a given change in the inertial angular velocity ω of the plate.

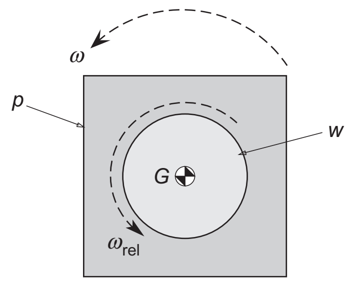


FIGURE 10.21

Plate and disk attached at their common center of mass G.

Solution

The plate plays the role of the body of a spacecraft and the disk is a momentum wheel. At any given time, the angular momentum of the system about G is given by Equation 10.142,

$$H_G = (I_p + I_w)\omega + I_w\omega_{rel}$$

At a later time (denoted by primes), after the torquing motor is activated, the angular momentum is

$$H'_G = (I_p + I_w)\omega' + I_w\omega'_{rel}$$

Since the torque is internal to the system, we have conservation of angular momentum, $H'_G = H_G$, which means

$$(I_p + I_w)\omega' + I_w\omega'_{rel} = (I_p + I_w)\omega + I_w\omega_{rel}$$

Rearranging terms we get

$$I_w(\omega'_{rel} - \omega_{rel}) = -(I_p + I_w)(\omega' - \omega)$$

Letting $\Delta\omega = \omega' - \omega$, this can be written

$$\boxed{\Delta\omega_{rel} = -\left(1 + \frac{I_p}{I_w}\right)\Delta\omega}$$

The change $\Delta\omega_{rel}$ in the relative rotational velocity of the disk is due to the torque applied to the disk at G by the motor mounted on the plate. An equal torque in the opposite direction is applied to the plate, producing the angular velocity change $\Delta\omega$ opposite in direction to $\Delta\omega_{rel}$.

Notice that if $I_p >> I_w$, which is true in an actual spacecraft, then the change in angular velocity of the momentum wheel must be very much larger than the required change in angular velocity of the body of the spacecraft.

Example 10.10

Use Equation 10.142 to obtain the equations of motion of a torque-free, axisymmetric dual-spin satellite, such as the one shown in Figure 10.22.

Solution

In this case we have only one "reaction wheel," namely, the platform p. The "body" is the rotor r. In Equation 10.142 we make the following substitutions ($\leftarrow$ means "is replaced by"):

$$\begin{aligned}
\boldsymbol{\omega} &\leftarrow \boldsymbol{\omega}^{(r)} \\
\boldsymbol{\omega}_{\text{rel}}^{(i)} &\leftarrow \boldsymbol{\omega}_{\text{rel}}^{(p)} \\
[\mathbf{I}_G^{(body)}] &\leftarrow [\mathbf{I}_G^{(r)}] \\
\sum_{i=1}^{n}\left[\mathbf{I}_G^{(i)}\right] &\leftarrow \left[\mathbf{I}_G^{(p)}\right] \\
\sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\} &\leftarrow \left[\mathbf{I}_{G_p}^{(p)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(p)}\}
\end{aligned}$$

so that Equation 10.142 becomes

$$\{\mathbf{H}_G\} = ([\mathbf{I}_G^{(r)}] + [\mathbf{I}_G^{(p)}])\{\boldsymbol{\omega}^{(r)}\} + \left[\mathbf{I}_{G_p}^{(p)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(p)}\} \tag{a}$$

Since $\left.\mathbf{M}_{G_{\text{net}}}\right)_{\text{external}} = \mathbf{0}$, Equation 10.143 yields

$$([\mathbf{I}_G^{(r)}] + [\mathbf{I}_G^{(p)}])\{\dot{\boldsymbol{\omega}}^{(r)}\} + \left[\mathbf{I}_{G_p}^{(p)}\right]\{\dot{\boldsymbol{\omega}}_{\text{rel}}^{(p)}\} + \{\boldsymbol{\omega}^{(r)}\} \times \left(([\mathbf{I}_G^{(r)}] + [\mathbf{I}_G^{(p)}])\{\boldsymbol{\omega}^{(r)}\} + \left[\mathbf{I}_{G_p}^{(p)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(p)}\}\right) = \{\mathbf{0}\} \tag{b}$$

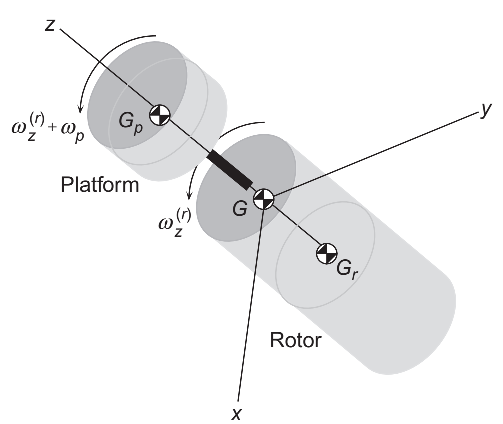


FIGURE 10.22

Dual-spin spacecraft.

The components of the matrices and vectors in (b) relative to the principal xyz body frame axes attached to the rotor are

$$[\mathbf{I}_G^{(r)}] = \begin{bmatrix} A_r & 0 & 0 \\ 0 & A_r & 0 \\ 0 & 0 & C_r \end{bmatrix} \qquad [\mathbf{I}_G^{(p)}] = \begin{bmatrix} A_p & 0 & 0 \\ 0 & A_p & 0 \\ 0 & 0 & C_p \end{bmatrix} \qquad \left[\mathbf{I}_{G_p}^{(p)}\right] = \begin{bmatrix} \overline{A}_p & 0 & 0 \\ 0 & \overline{A}_p & 0 \\ 0 & 0 & C_p \end{bmatrix} \tag{c}$$

and

$$\{\boldsymbol{\omega}^{(r)}\} = \begin{Bmatrix} \omega_x^{(r)} \\ \omega_y^{(r)} \\ \omega_z^{(r)} \end{Bmatrix} \qquad \{\boldsymbol{\omega}_{\text{rel}}^{(p)}\} = \begin{Bmatrix} 0 \\ 0 \\ \omega_p \end{Bmatrix} \tag{d}$$

A_r, C_r, A_p and C_p are the rotor and platform principal moments of inertia about the vehicle center of mass G, whereas $\overline{A}_p$ is the moment of inertia of the platform about its own center of mass. We also used the fact that $\overline{C}_p = C_p$, which of course is due to the fact that G and G_p both lie on the z axis. This notation is nearly identical to that employed in our consideration of the stability of dual-spin satellites in Section 10.4 (wherein $\omega_r = \omega_z^{(r)}$ and $\boldsymbol{\omega}_\perp = \omega_x^{(r)}\hat{\mathbf{i}} + \omega_y^{(r)}\hat{\mathbf{j}}$). Substituting (c) and (d) into each of the four terms in (b), we get

$$([\mathbf{I}_G^{(r)}] + [\mathbf{I}_G^{(p)}])\{\dot{\boldsymbol{\omega}}^{(r)}\} = \begin{bmatrix} A_r + A_p & 0 & 0 \\ 0 & A_r + A_p & 0 \\ 0 & 0 & C_r + C_p \end{bmatrix} \begin{Bmatrix} \dot{\omega}_x^{(r)} \\ \dot{\omega}_y^{(r)} \\ \dot{\omega}_z^{(r)} \end{Bmatrix} = \begin{Bmatrix} (A_r + A_p)\dot{\omega}_x^{(r)} \\ (A_r + A_p)\dot{\omega}_y^{(r)} \\ (C_r + C_p)\dot{\omega}_z^{(r)} \end{Bmatrix} \tag{e}$$

$$\{\boldsymbol{\omega}^{(r)}\} \times ([\mathbf{I}_G^{(r)}] + [\mathbf{I}_G^{(p)}])\{\boldsymbol{\omega}^{(r)}\} = \begin{Bmatrix} \omega_x^{(r)} \\ \omega_y^{(r)} \\ \omega_z^{(r)} \end{Bmatrix} \times \begin{Bmatrix} (A_r + A_p)\omega_x^{(r)} \\ (A_r + A_p)\omega_y^{(r)} \\ (C_r + C_p)\omega_z^{(r)} \end{Bmatrix} = \begin{Bmatrix} [(C_p - A_p) + (C_r - A_r)]\omega_y^{(r)}\omega_z^{(r)} \\ [(A_p - C_p) + (A_r - C_r)]\omega_x^{(r)}\omega_z^{(r)} \\ 0 \end{Bmatrix} \tag{f}$$

$$\left[\mathbf{I}_{G_p}^{(p)}\right]\left\{\dot{\boldsymbol{\omega}}_{\text{rel}}^{(p)}\right\} = \begin{bmatrix} \overline{A}_p & 0 & 0 \\ 0 & \overline{A}_p & 0 \\ 0 & 0 & C_p \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\omega}_p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ C_p\dot{\omega}_p \end{Bmatrix} \tag{g}$$

$$\{\boldsymbol{\omega}^{(r)}\} \times \left[\mathbf{I}_{G_p}^{(p)}\right]\left\{\boldsymbol{\omega}^{(p)}{}_{\text{rel}}\right\} = \begin{Bmatrix} \omega_x^{(r)} \\ \omega_y^{(r)} \\ \omega_z^{(r)} \end{Bmatrix} \times \begin{bmatrix} \overline{A}_p & 0 & 0 \\ 0 & \overline{A}_p & 0 \\ 0 & 0 & C_p \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega_p \end{Bmatrix} = \begin{Bmatrix} C_p\omega_y^{(r)}\omega_p \\ -C_p\omega_x^{(r)}\omega_p \\ 0 \end{Bmatrix} \tag{h}$$

With these four expressions, (b) becomes

$$\begin{Bmatrix} (A_r + A_p)\dot{\omega}_x^{(r)} \\ (A_r + A_p)\dot{\omega}_y^{(r)} \\ (C_r + C_p)\dot{\omega}_z^{(r)} \end{Bmatrix} + \begin{Bmatrix} [(C_p - A_p) + (C_r - A_r)]\omega_y^{(r)}\omega_z^{(r)} \\ [(A_p - C_p) + (A_r - C_r)]\omega_x^{(r)}\omega_z^{(r)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ C_p\dot{\omega}_p \end{Bmatrix} + \begin{Bmatrix} C_p\omega_y^{(r)}\omega_p \\ -C_p\omega_x^{(r)}\omega_p \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{i}$$

Combining the four vectors on the left-hand side, and then extracting the three components of the vector equation finally yields the three equations of motion of the dual-spin satellite in the body frame,

$$\boxed{\begin{aligned} A\dot{\omega}_x^{(r)} + (C - A)\omega_y^{(r)}\omega_z^{(r)} + C_p\omega_y^{(r)}\omega_p &= 0 \\ A\dot{\omega}_y^{(r)} + (A - C)\omega_x^{(r)}\omega_z^{(r)} - C_p\omega_x^{(r)}\omega_p &= 0 \\ C\dot{\omega}_z^{(r)} + C_p\dot{\omega}_p &= 0 \end{aligned}} \tag{j}$$

where A and C are the combined transverse and axial moments of inertia of the dual-spin vehicle about its center of mass,

$$A = A_r + A_p \qquad C = C_r + C_p \tag{k}$$

The three equations (j) involve four unknowns, $\omega_x^{(r)}$, $\omega_y^{(r)}$, $\omega_z^{(r)}$ and ω_p. A fourth equation is required to account for the means of providing the relative velocity ω_p between the platform and the rotor. Friction in the axle bearing between the platform and the rotor would eventually cause ω_p to go to zero, as pointed out in Section 10.4. We may assume that the electric motor in the bearing acts to keep ω_p constant at a specified value, so that $\dot{\omega}_p = 0$. Then Equation $(j)_3$ implies that $\omega_z^{(r)} = \text{constant}$ as well. Thus, ω_p and $\omega_z^{(r)}$ are removed from our list of unknowns, leaving $\omega_x^{(r)}$ and $\omega_y^{(r)}$ to be governed by the first two equations in (j). Note that we actually employed $(j)_3$ in the solution of Example 10.9 above.

Example 10.11

A spacecraft has three identical momentum wheels with their spin axes aligned with the vehicle's principal body axes. The spin axes of momentum wheels 1, 2 and 3 are aligned with the x, y and z axes, respectively. The inertia tensors of the rotationally symmetric momentum wheels about their centers of mass are, therefore,

$$\left[\mathbf{I}_{G_1}^{(1)}\right] = \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix} \qquad \left[\mathbf{I}_{G_2}^{(2)}\right] = \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J \end{bmatrix} \qquad \left[\mathbf{I}_{G_3}^{(3)}\right] = \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix} \tag{a}$$

The spacecraft moment of inertia tensor about the vehicle center of mass is

$$[\mathbf{I}_G^{(v)}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{b}$$

Calculate the spin accelerations of the momentum wheels in the presence of external torque.

Solution

For $n = 3$, Equation 10.140 becomes

$$\{\mathbf{H}_G\} = \left([\mathbf{I}_G^{(v)}] + \left[\mathbf{I}_{G_1}^{(1)}\right] + \left[\mathbf{I}_{G_2}^{(2)}\right] + \left[\mathbf{I}_{G_3}^{(3)}\right]\right)\{\boldsymbol{\omega}\} + \left[\mathbf{I}_{G_1}^{(1)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(1)}\} + \left[\mathbf{I}_{G_2}^{(2)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(2)}\} + \left[\mathbf{I}_{G_3}^{(3)}\right]\{\boldsymbol{\omega}_{\text{rel}}^{(3)}\} \tag{c}$$

The absolute angular velocity $\boldsymbol{\omega}$ of the spacecraft and the angular velocities $\boldsymbol{\omega}_{\text{rel}}^{(1)}$, $\boldsymbol{\omega}_{\text{rel}}^{(2)}$, $\boldsymbol{\omega}_{\text{rel}}^{(3)}$ of the three flywheels relative to the spacecraft are

$$\{\boldsymbol{\omega}\} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \qquad \{\boldsymbol{\omega}^{(1)}\}_{\text{rel}} = \begin{Bmatrix} \omega^{(1)} \\ 0 \\ 0 \end{Bmatrix} \qquad \{\boldsymbol{\omega}^{(2)}\}_{\text{rel}} = \begin{Bmatrix} 0 \\ \omega^{(2)} \\ 0 \end{Bmatrix} \qquad \{\boldsymbol{\omega}^{(3)}\}_{\text{rel}} = \begin{Bmatrix} 0 \\ 0 \\ \omega^{(3)} \end{Bmatrix} \tag{d}$$

Substituting (a), (b) and (d) into (c) yields

$$\{\mathbf{H}_G\} = \left(\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix} + \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix}\right)\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} + \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}\begin{Bmatrix} \omega^{(1)} \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J \end{bmatrix}\begin{Bmatrix} 0 \\ \omega^{(2)} \\ 0 \end{Bmatrix} + \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \omega^{(3)} \end{Bmatrix}$$

or

$$\{\mathbf{H}_G\} = \begin{bmatrix} A + I + 2J & 0 & 0 \\ 0 & B + I + 2J & 0 \\ 0 & 0 & C + I + 2J \end{bmatrix}\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} + \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{Bmatrix} \omega^{(1)} \\ \omega^{(2)} \\ \omega^{(3)} \end{Bmatrix} \tag{e}$$

Substituting this expression for $\{\mathbf{H}_G\}$ into Equation 10.143, we get

$$\begin{aligned} &\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{Bmatrix} \dot\omega^{(1)} \\ \dot\omega^{(2)} \\ \dot\omega^{(3)} \end{Bmatrix} + \begin{bmatrix} A + I + 2J & 0 & 0 \\ 0 & B + I + 2J & 0 \\ 0 & 0 & C + I + 2J \end{bmatrix}\begin{Bmatrix} \dot\omega_x \\ \dot\omega_y \\ \dot\omega_z \end{Bmatrix} \\ &+ \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \times \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{Bmatrix} \omega^{(1)} \\ \omega^{(2)} \\ \omega^{(3)} \end{Bmatrix} + \begin{bmatrix} A + I + 2J & 0 & 0 \\ 0 & B + I + 2J & 0 \\ 0 & 0 & C + I + 2J \end{bmatrix}\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}\right) = \begin{Bmatrix} M_{G_x} \\ M_{G_y} \\ M_{G_z} \end{Bmatrix} \end{aligned} \tag{f}$$

Expanding and collecting terms yields the time rates of change of the flywheel spins (relative to the spacecraft) in terms of the spacecraft's absolute angular velocity components,

$$\boxed{\begin{aligned} \dot\omega^{(1)} &= \frac{M_{G_x}}{I} + \frac{B - C}{I}\omega_y\omega_z - \left(1 + \frac{A}{I} + 2\frac{J}{I}\right)\dot\omega_x + \omega^{(2)}\omega_z - \omega^{(3)}\omega_y \\ \dot\omega^{(2)} &= \frac{M_{G_y}}{I} + \frac{C - A}{I}\omega_x\omega_z - \left(1 + \frac{B}{I} + 2\frac{J}{I}\right)\dot\omega_y + \omega^{(3)}\omega_x - \omega^{(1)}\omega_z \\ \dot\omega^{(3)} &= \frac{M_{G_z}}{I} + \frac{A - B}{I}\omega_x\omega_y - \left(1 + \frac{C}{I} + 2\frac{J}{I}\right)\dot\omega_z + \omega^{(1)}\omega_y - \omega^{(2)}\omega_x \end{aligned}} \tag{g}$$

Example 10.12

The communications satellite is in a circular earth orbit of period T. The body z axis always points towards the earth, so the angular velocity about the body y axis is $2\pi/T$. The angular velocities about the body x and z axes are zero. The attitude control system consists of three momentum wheels 1, 2 and 3 aligned with the principal x, y and z axes of the satellite. Variable torque is applied to each wheel by its own electric motor. At time $t = 0$ the angular velocities of the three wheels relative to the spacecraft are all zero. A small, constant environmental torque $\mathbf{M}_0$ acts on the spacecraft. Determine the axial torques $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$ that the three motors must exert on their wheels so that the angular velocity $\boldsymbol{\omega}$ of the satellite will remain constant. The moment of inertia of each reaction wheel about its spin axis is I.

The absolute angular velocity of the xyz frame is given by

$$\boldsymbol{\omega} = \omega_0 \hat{\mathbf{j}} \tag{a}$$

where $\omega_0 = 2\pi/T$, a constant. At any instant, the absolute angular velocities of the three reaction wheels are, accordingly,

$$\begin{aligned} \boldsymbol{\omega}^{(1)} &= \omega^{(1)}\hat{\mathbf{i}} + \omega_0\hat{\mathbf{j}} \\ \boldsymbol{\omega}^{(2)} &= [\omega^{(2)} + \omega_0]\hat{\mathbf{j}} \\ \boldsymbol{\omega}^{(3)} &= \omega_0\hat{\mathbf{j}} + \omega^{(3)}\hat{\mathbf{k}} \end{aligned} \tag{b}$$

From (a) it is clear that $\omega_x = \omega_z = \dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Therefore, Equations (g) of Example 10.11 become, for the case at hand,

$$\begin{aligned} \dot{\omega}^{(1)} &= \frac{M_{Gx}}{I} + \frac{B-C}{I}\omega_0(0) - \left(1 + \frac{A}{I} + 2\frac{J}{I}\right)(0) + \omega^{(2)}(0) - \omega^{(3)}\omega_o \\ \dot{\omega}^{(2)} &= \frac{M_{Gy}}{I} + \frac{C-A}{I}(0)(0) - \left(1 + \frac{B}{I} + 2\frac{J}{I}\right)(0) + \omega^{(3)}(0) - \omega^{(1)}(0) \\ \dot{\omega}^{(3)} &= \frac{M_{Gz}}{I} + \frac{A-B}{I}(0)\omega_0 - \left(1 + \frac{C}{I} + 2\frac{J}{I}\right)(0) + \omega^{(1)}\omega_o - \omega^{(2)}(0) \end{aligned}$$

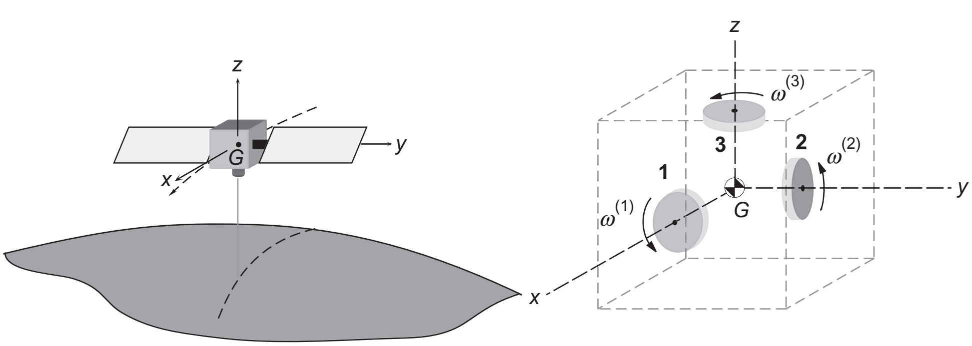


FIGURE 10.23

Three-axis stabilized satellite.

which reduce to the following set of three first-order differential equations,

$$\begin{aligned}\dot{\omega}^{(1)} + \omega_0\omega^{(3)} &= \frac{M_{Gx}}{I}\\ \dot{\omega}^{(2)} &= \frac{M_{Gy}}{I}\\ \dot{\omega}^{(3)} - \omega_0\omega^{(1)} &= \frac{M_{Gz}}{I}\end{aligned} \tag{c}$$

Equation $(c)_2$ implies that $\omega^{(2)} = M_{Gy}t/I +$ constant, and since $\omega^{(2)} = 0$ at $t = 0$, this means that for time thereafter,

$$\omega^{(2)} = \frac{M_{Gy}}{I}t \tag{d}$$

Differentiating $(c)_3$ with respect to t and solving for $\dot{\omega}^{(1)}$ yields $\dot{\omega}^{(1)} = \ddot{\omega}^{(3)}/\omega_o$. Substituting this result into $(c)_1$ we get

$$\ddot{\omega}^{(3)} + \omega_0{}^2\omega^{(3)} = \frac{\omega_0 M_{Gx}}{I}$$

The well-known solution of this differential equation is

$$\omega^{(3)} = a\cos\omega_0 t + b\sin\omega_0 t + \frac{M_{Gx}}{I\omega_0}$$

where a and b are constants of integration. According to the problem statement, $\omega^{(3)} = 0$ when $t = 0$. This initial condition requires $a = -M_{Gx}/\omega_0 I$, so that

$$\omega^{(3)} = b\sin\omega_0 t + \frac{M_{Gx}}{I\omega_0}(1 - \cos\omega_0 t) \tag{e}$$

From this we obtain $\dot{\omega}^{(3)} = b\omega_0\cos\omega_0 t + \left(M_{Gx}/I\right)\sin\omega_0 t$, which, when substituted into $(c)_3$ yields

$$\omega^{(1)} = b\cos\omega_0 t + \frac{M_{Gx}}{I\omega_0}\sin\omega_o t - \frac{M_{Gz}}{I\omega_0} \tag{f}$$

Since $\omega^{(1)} = 0$ at $t = 0$, this implies $b = M_{Gz}/\omega_0 \mathrm{I}$. In summary, therefore, the angular velocities of wheels 1, 2 and 3 relative to the satellite are

$$\omega^{(1)} = \frac{M_{Gx}}{I\omega_0}\sin\omega_0 t + \frac{M_{Gz}}{I\omega_0}(\cos\omega_0 t - 1) \tag{g$_1$}$$

$$\omega^{(2)} = \frac{M_{Gy}}{I}t \tag{g$_2$}$$

$$\omega^{(3)} = \frac{M_{Gz}}{I\omega_0}\sin\omega_0 t + \frac{M_{Gx}}{I\omega_0}(1 - \cos\omega_0 t) \tag{g$_3$}$$

The angular momenta of the reaction wheels are

$$\begin{aligned}\mathbf{H}_{G_1}^{(1)} &= I_x^{(1)}\omega_x^{(1)}\hat{\mathbf{i}} + I_y^{(1)}\omega_y^{(1)}\hat{\mathbf{j}} + I_z^{(1)}\omega_z^{(1)}\hat{\mathbf{k}}\\ \mathbf{H}_{G_2}^{(2)} &= I_x^{(2)}\omega_x^{(2)}\hat{\mathbf{i}} + I_y^{(2)}\omega_y^{(2)}\hat{\mathbf{j}} + I_z^{(2)}\omega_z^{(2)}\hat{\mathbf{k}}\\ \mathbf{H}_{G_3}^{(3)} &= I_x^{(3)}\omega_x^{(3)}\hat{\mathbf{i}} + I_y^{(3)}\omega_y^{(3)}\hat{\mathbf{j}} + I_z^{(3)}\omega_z^{(3)}\hat{\mathbf{k}}\end{aligned} \tag{h}$$

According to (b), the components of the flywheels' angular velocities are

$$\begin{array}{lll}\omega_x^{(1)} = \omega^{(1)} & \omega_y^{(1)} = \omega_0 & \omega_z^{(1)} = 0\\ \omega_x^{(2)} = 0 & \omega_y^{(2)} = \omega^{(2)} + \omega_0 & \omega_z^{(2)} = 0\\ \omega_x^{(3)} = 0 & \omega_y^{(3)} = \omega_0 & \omega_z^{(3)} = \omega^{(3)}\end{array}$$

Furthermore, $I_x^{(1)} = I_y^{(2)} = I_z^{(3)} = I$, so that (h) becomes

$$\begin{aligned}\mathbf{H}_{G_1}^{(1)} &= I\omega^{(1)}\hat{\mathbf{i}} + I_y^{(1)}\omega_0\hat{\mathbf{j}}\\ \mathbf{H}_{G_2}^{(2)} &= I(\omega^{(2)} + \omega_0)\hat{\mathbf{j}}\\ \mathbf{H}_{G_3}^{(3)} &= I_y^{(3)}\omega_0\hat{\mathbf{j}} + I\omega^{(3)}\hat{\mathbf{k}}\end{aligned} \tag{i}$$

Substituting (g) into these expressions yields the angular momenta of the wheels as a function of time,

$$\begin{aligned}\mathbf{H}_{G_1}^{(1)} &= \left[\frac{M_{G_x}}{\omega_0}\sin\omega_0 t + \frac{M_{G_z}}{\omega_0}(\cos\omega_0 t - 1)\right]\hat{\mathbf{i}} + I_y^{(1)}\omega_0\hat{\mathbf{j}}\\ \mathbf{H}_{G_2}^{(2)} &= (M_{G_y}t + I\omega_0)\hat{\mathbf{j}}\\ \mathbf{H}_{G_3}^{(3)} &= I_y^{(3)}\omega_0\hat{\mathbf{j}} + \left[\frac{M_{G_z}}{\omega_0}\sin\omega_0 t + \frac{M_{G_x}}{\omega_0}(1 - \cos\omega_0 t)\right]\hat{\mathbf{k}}\end{aligned} \tag{j}$$

The torque on the reaction wheels is found by applying Euler's equation to each one. Thus, for wheel 1

$$\begin{aligned}\mathbf{M}_{G_1\,\text{net}} &= \left.\frac{d\mathbf{H}_{G_1}^{(1)}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega}\times\mathbf{H}_{G_1}^{(1)}\\ &= (M_{Gx}\cos\omega_0 t - M_{Gz}\sin\omega_0 t)\hat{\mathbf{i}} + \left[M_{Gz}(1 - \cos\omega_0 t) - M_{Gx}\sin\omega_0 t\right]\hat{\mathbf{k}}\end{aligned}$$

Since the axis of wheel 1 is in the x direction, the torque is the x component of this moment (the z component being a gyroscopic bending moment),

$$\boxed{C^{(1)} = M_{Gx}\cos\omega_0 t - M_{Gz}\sin\omega_0 t}$$

For wheel 2,

$$\mathbf{M}_{G_2\,\text{net}} = \left.\frac{d\mathbf{H}_{G_2}^{(2)}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_{G_2}^{(2)} = M_{Gy}\hat{\mathbf{j}}$$

Thus,

$$\boxed{C^{(2)} = M_{Gy}}$$

Finally, for wheel 3

$$\begin{aligned}\mathbf{M}_{G_3\,\text{net}} &= \left.\frac{d\mathbf{H}_{G_3}^{(3)}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_{G_3}^{(3)} \\ &= \left[M_{G_x}(1 - \cos\omega_0 t) + M_{G_z}\sin\omega_0 t\right]\hat{\mathbf{i}} + (M_{G_x}\sin\omega_0 t + M_{G_z}\cos\omega_0 t)\hat{\mathbf{k}}\end{aligned}$$

For this wheel, the torque direction is the z axis, so

$$\boxed{C^{(3)} = M_{G_x}\sin\omega_0 t + M_{G_z}\cos\omega_0 t}$$

The external torques on the spacecraft of the previous example may be due to thruster misalignment or they may arise from environmental effects such as gravity gradients, solar radiation pressure or interaction with the earth's magnetic field. The example assumed that these torques were constant, which is the simplest means of introducing their effects, but they actually vary with time. In any case, their magnitudes are extremely small, typically less than $10^{-3}\text{N}\cdot\text{m}$ for ordinary-sized, unmanned spacecraft. Equation (g^2) of the example reveals that a small torque normal to the satellite's orbital plane will cause the angular velocity of momentum wheel 2 to slowly but constantly increase. Over a long enough period of time, the angular velocity of the gyro might approach its design limits, whereupon it is said to be saturated. At that point, attitude jets on the satellite would have to be fired to produce a torque around the y axis while the wheel is "caged," i.e., its angular velocity reduced to zero or to its nonzero bias value. Finally, note that if all of the external torques were zero, none of the momentum wheels in the example would be required. The constant angular velocity $\boldsymbol{\omega} = (2\pi/T)\hat{\mathbf{j}}$ of the vehicle, once initiated, would continue unabated.

So far we have dealt with momentum wheels, which are characterized by the fact that their axes are rigidly aligned with the principal axes of the spacecraft, as shown in Figure 10.24. The speed of the electrically-driven wheels is varied to produce the required rotation rates of the vehicle in response to external torques. Depending on the spacecraft, the nominal speed of a momentum wheel may be from zero to several thousand rpm.

Momentum wheels that are free to pivot on one or more gimbals are called ***control moment gyros***. Figure 10.25 illustrates a double-gimbaled control moment gyro. These gyros spin at several thousand rpm. The motor-driven speed of the flywheel is constant, and moments are exerted on the vehicle when torquers (electric motors) tilt the wheel about a gimbal axis. The torque direction is normal to the gimbal axis.

Set $n = 1$ in Equation 10.140 and replace i with w (representing "wheel") to obtain

$$\{\mathbf{H}_G\} = [\mathbf{I}_G^{(v)}]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_{G_w}^{(w)}\right]\left(\{\boldsymbol{\omega}\} + \{\boldsymbol{\omega}_{\text{rel}}^{(w)}\}\right) \tag{10.144}$$

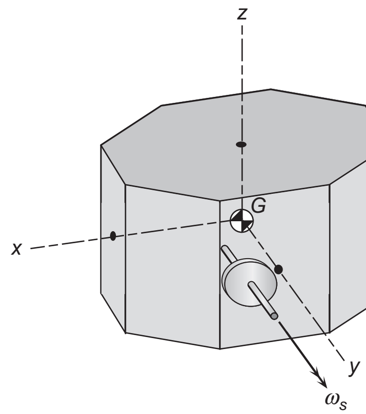


FIGURE 10.24

Momentum wheel aligned with a principal body axis.

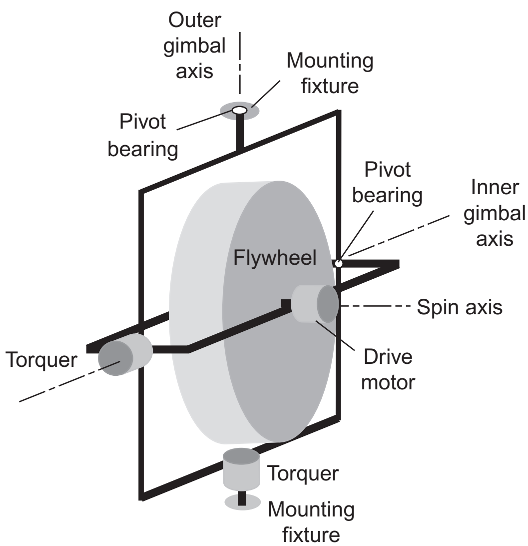


FIGURE 10.25

Two-gimbal control moment gyro.

The relative angular velocity of the rotor is

$$\boldsymbol{\omega}_{\text{rel}}^{(w)} = \boldsymbol{\omega}_p + \boldsymbol{\omega}_n + \boldsymbol{\omega}_s \tag{10.145}$$

where $\boldsymbol{\omega}_p$, $\boldsymbol{\omega}_n$ and $\boldsymbol{\omega}_s$ are the precession, nutation and spin rates of the gyro relative to the vehicle. Substituting Equation 10.145 into 10.144 yields

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)}\right]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_{G_w}^{(w)}\right](\{\boldsymbol{\omega}\} + \{\boldsymbol{\omega}_p\} + \{\boldsymbol{\omega}_n\} + \{\boldsymbol{\omega}_s\}) \tag{10.146}$$

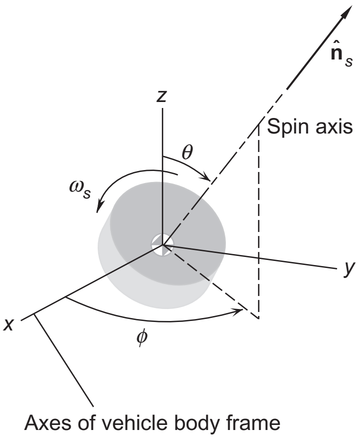


FIGURE 10.26

Inclination angles of the spin vector of a gyro.

The spin rate of the gyro is three or more orders of magnitude greater than any of the other rates. That is, under conditions in which a control moment gyro is designed to operate,

$$\|\boldsymbol{\omega}_s\| \gg \|\boldsymbol{\omega}\| \qquad \|\boldsymbol{\omega}_s\| \gg \|\boldsymbol{\omega}_p\| \qquad \|\boldsymbol{\omega}_s\| \gg \|\boldsymbol{\omega}_n\|$$

Therefore,

$$\{\mathbf{H}_G\} \approx \left[\mathbf{I}_G^{(v)}\right]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_{G_w}^{(w)}\right]\{\boldsymbol{\omega}_s\} \tag{10.147}$$

Since the spin axis of a gyro is an axis of symmetry, about which the moment of inertia is $C^{(w)}$, this can be written

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)}\right]\{\omega\} + C^{(w)}\omega_s\{\hat{\mathbf{n}}_s\} \tag{10.148}$$

where

$$[\mathbf{I}_G^{(v)}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

and $\hat{\mathbf{n}}_s$ is the unit vector along the spin axis, as illustrated in Figure 10.26. Relative to the body frame axes of the spacecraft, the components of $\hat{\mathbf{n}}_s$ appear as follows,

$$\hat{\mathbf{n}}_s = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{10.149}$$

Thus, Equation 10.148 becomes

$$\{\mathbf{H}_G\} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} + C^{(w)}\omega_s\begin{Bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{Bmatrix} = \begin{Bmatrix} A\omega_x + C^{(w)}\omega_s\sin\theta\cos\phi \\ B\omega_y + C^{(w)}\omega_s\sin\theta\sin\phi \\ C\omega_z + C^{(w)}\omega_s\cos\theta \end{Bmatrix} \tag{10.150}$$

It follows that

$$\frac{d}{dt}\{\mathbf{H}_G\} = \frac{d}{dt}\begin{Bmatrix} A\omega_x + C^{(w)}\omega_s \sin\theta\cos\phi \\ B\omega_y + C^{(w)}\omega_s \sin\theta\sin\phi \\ C\omega_z + C^{(w)}\omega_s \cos\theta \end{Bmatrix} + \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \times \begin{Bmatrix} A\omega_x + C^{(w)}\omega_s \sin\theta\cos\phi \\ B\omega_y + C^{(w)}\omega_s \sin\theta\sin\phi \\ C\omega_z + C^{(w)}\omega_s \cos\theta \end{Bmatrix} \tag{10.151}$$

Expanding the right hand side, collecting terms and setting the result equal to the net external moment, we find

$$\begin{aligned} &A\dot{\omega}_x + C^{(w)}\omega_s\dot{\theta}\cos\phi\cos\theta - C^{(w)}\omega_s\dot{\phi}\sin\phi\sin\theta + C^{(w)}\dot{\omega}_s \cos\phi\sin\theta \\ &\quad + (C^{(w)}\omega_s \cos\theta + C\omega_z)\omega_y - (C^{(w)}\omega_s \sin\phi\sin\theta + B\omega_y)\omega_z = M_{G_x} \end{aligned} \tag{10.152a}$$

$$\begin{aligned} &B\dot{\omega}_y + C^{(w)}\omega_s\dot{\theta}\sin\phi\cos\theta + C^{(w)}\omega_s\dot{\phi}\cos\phi\sin\theta + C^{(w)}\dot{\omega}_s \sin\phi\sin\theta \\ &\quad - (C^{(w)}\omega_s \cos\theta + C\omega_z)\omega_x + (C^{(w)}\omega_s \cos\phi\sin\theta + A\omega_x)\omega_z = M_{G_y} \end{aligned} \tag{10.152b}$$

$$\begin{aligned} &C\dot{\omega}_z - C^{(w)}\omega_s\dot{\theta}\sin\theta + C^{(w)}\dot{\omega}_s \cos\theta \\ &\quad - (C^{(w)}\omega_s \cos\phi\sin\theta + A\omega_x)\omega_y + (C^{(w)}\omega_s \sin\phi\sin\theta + B\omega_y)\omega_x = M_{G_z} \end{aligned} \tag{10.152c}$$

Additional gyros are accounted for by adding the spin inertia, spin rate and inclination angles for each one into Equations 1.37.

Example 10.13

A satellite is in torque-free motion ($\mathbf{M}_{G_{\text{net}}} = \mathbf{0}$). A nongimbaled gyro (momentum wheel) is aligned with the vehicle's x axis and is spinning at the rate ω_{s_0}. The spacecraft angular velocity is $\boldsymbol{\omega} = \boldsymbol{\omega}_x\hat{\mathbf{i}}$. If the spin of the gyro is increased at the rate $\dot{\omega}_s$, find the angular acceleration of the spacecraft.

Solution

Using Figure 10.26 as a guide, we set $\phi = 0$ and $\phi = 90°$ to align the spin axis with the x axis. Since there is no gimbaling, $\dot{\theta} = \dot{\phi} = 0$. Equations 10.152 then yield

$$\begin{aligned} A\dot{\omega}_x + C^{(w)}\dot{\omega}_s &= 0 \\ B\dot{\omega}_y &= 0 \\ C\dot{\omega}_z &= 0 \end{aligned}$$

Clearly, the angular velocities around the y and z axes remain zero, whereas

$$\boxed{\dot{\omega}_x = -\frac{C^{(w)}}{A}\dot{\omega}_s}$$

Thus, a change in the vehicle's roll rate around the x axis can be initiated by accelerating the momentum wheel in the opposite direction. See Example 10.9.

Example 10.14

A satellite is in torque-free motion. A control moment gyro, spinning at the constant rate ω_s, is gimbaled about the spacecraft y and z axes, with $\phi = 0$ and $\phi = 90°$ (cf. Figure 10.26). The spacecraft angular velocity is $\boldsymbol{\omega} = \omega_z\hat{\mathbf{k}}$. If the spin axis of the gyro, initially along the x direction, is rotated around the y axis at the rate $\dot{\theta}$, what is the resulting angular acceleration of the spacecraft?

Solution

Substituting $\omega_x = \omega_y = \dot{\omega}_s = \phi = 0$ and $\theta = 90°$ into Equations 10.152 gives

$$\begin{aligned} A\dot{\omega}_x &= 0 \\ B\dot{\omega}_y + C^{(w)}\omega_s(\omega_z + \dot{\phi}) &= 0 \\ C\dot{\omega}_z - C^{(w)}\omega_S\dot{\theta} &= 0 \end{aligned}$$

Thus, the components of vehicle angular acceleration are

$$\boxed{\dot{\omega}_x = 0 \qquad \dot{\omega}_y = -\frac{C^{(w)}}{B}\omega_s(\omega_z + \dot{\phi}) \qquad \dot{\omega}_z = \frac{C^{(w)}}{C}\omega_s\dot{\theta}}$$

We see that pitching the gyro at the rate $\dot{\theta}$ around the vehicle y axis alters only ω_z, leaving ω_x unchanged. However, to keep $\omega_y = 0$ clearly requires $\dot{\phi} = -\omega_z$. In other words, for the control moment gyro to control the angular velocity about only one vehicle axis, it must therefore be able to precess around that axis (the z axis in this case). That is why the control moment gyro must have two gimbals.

10.10 GRAVITY GRADIENT STABILIZATION

Consider a satellite in circular orbit, as shown in Figure 10.27. Let $\mathbf{r}$ be the position vector of a mass element dm relative to the center of attraction, $\mathbf{r}_0$ the position vector of the center of mass G, and $\boldsymbol{\rho}$ the position of dm relative to G. The force of gravity on dm is

$$d\mathbf{F}_g = -G\frac{Mdm}{r^3}\mathbf{r} = -\mu\frac{\mathbf{r}}{r^3}dm \tag{10.153}$$

where M is the mass of the central body, and $\mu = GM$. The net moment of the gravitational force around G is

$$\mathbf{M}_{G\text{net}} = \int_m \rho \times d\mathbf{F}_g\, dm \tag{10.154}$$

Since $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$, and

$$\begin{aligned} \mathbf{r}_0 &= R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}} \\ \boldsymbol{\rho} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \end{aligned} \tag{10.155}$$

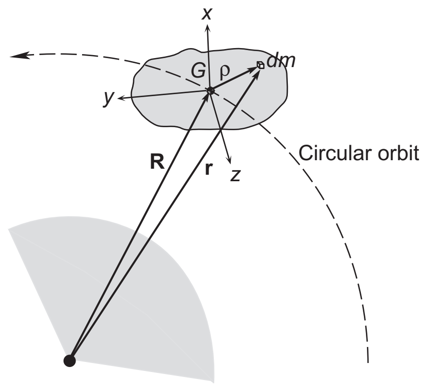


FIGURE 10.27

Rigid satellite in a circular orbit. *xyz* is the principal body frame.

we have

$$\boldsymbol{\rho} \times d\mathbf{F}_g = -\mu\frac{dm}{r^3}\boldsymbol{\rho} \times (\mathbf{R} + \boldsymbol{\rho}) = -\mu\frac{dm}{r^3}\boldsymbol{\rho} \times \mathbf{R} = -\mu\frac{dm}{r^3}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ R_x & R_y & R_z \end{vmatrix}$$

Thus,

$$\boldsymbol{\rho} \times d\mathbf{F}_g = -\mu\frac{dm}{r^3}(R_z y - R_y z)\hat{\mathbf{i}} - \mu\frac{dm}{r^3}(R_x z - R_z x)\hat{\mathbf{j}} - \mu\frac{dm}{r^3}(R_y x - R_x y)\hat{\mathbf{k}}$$

Substituting this back into Equation 10.154 yields

$$\mathbf{M}_{G_{\text{net}}} = \left(-\mu R_z\int_m \frac{y}{r^3}dm + \mu R_y\int_m \frac{z}{r^3}dm\right)\hat{\mathbf{i}} + \left(-\mu R_x\int_m \frac{z}{r^3}dm + \mu R_z\int_m \frac{x}{r^3}dm\right)\hat{\mathbf{j}} + \left(-\mu R_y\int_m \frac{x}{r^3}dm + \mu R_x\int_m \frac{y}{r^3}dm\right)\hat{\mathbf{k}}$$

or

$$\begin{aligned} M_{G_{\text{net}\ x}} &= -\mu R_z\int_m \frac{y}{r^3}dm + \mu R_y\int_m \frac{z}{r^3}dm \\ M_{G_{\text{net}\ y}} &= -\mu R_x\int_m \frac{z}{r^3}dm + \mu R_z\int_m \frac{x}{r^3}dm \\ M_{G_{\text{net}\ z}} &= -\mu R_y\int_m \frac{x}{r^3}dm + \mu R_x\int_m \frac{y}{r^3}dm \end{aligned} \qquad (10.156)$$

Now, since $\|\boldsymbol{\rho}\| \ll \|\mathbf{R}\|$, it follows from Equation 7.20 that

$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5}\mathbf{R} \cdot \boldsymbol{\rho}$$

or

$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5}(R_x x + R_y y + R_z z)$$

Therefore,

$$\int_m \frac{x}{r^3}dm = \frac{1}{R^3}\int_m x\,dm - \frac{3R_x}{R^5}\int_m x^2 dm - \frac{3R_x}{R^5}\int_m xydm - \frac{3R_x}{R^5}\int_m xzdm$$

But the center of mass lies at the origin of the *xyz* axes, which are principal moment of inertia directions. That means

$$\int_m x\,dm = \int_m xy\,dm = \int_m xz\,dm = 0$$

so that

$$\int_m \frac{x}{r^3}dm = -\frac{3R_x}{R^5}\int_m x^2 dm \tag{10.157}$$

In a similar fashion, we can show that

$$\int_m \frac{y}{r^3}dm = -\frac{3R_y}{R^5}\int_m y^2 dm \tag{10.158}$$

and

$$\int_m \frac{z}{r^3}dm = -\frac{3R_z}{R^5}\int_m z^2 dm \tag{10.159}$$

Substituting these last three expressions into Equations 10.156 leads to

$$\begin{aligned} M_{G_{\text{net}}x} &= \frac{3\mu R_y R_z}{R^5}\left(\int_m y^2 dm - \int_m z^2 dm\right) \\ M_{G_{\text{net}}y} &= \frac{3\mu R_x R_z}{R^5}\left(\int_m z^2 dm - \int_m x^2 dm\right) \\ M_{G_{\text{net}}z} &= \frac{3\mu R_x R_y}{R^5}\left(\int_m x^2 dm - \int_m y^2 dm\right) \end{aligned} \tag{10.160}$$

From Section 9.5 we recall that the moments of inertia are defined as

$$A = \int_m y^2 dm + \int_m z^2 dm \qquad B = \int_m x^2 dm + \int_m z^2 dm \qquad C = \int_m x^2 dm + \int_m y^2 dm \tag{10.161}$$

from which we may write

$$B - A = \int_m x^2 dm - \int_m y^2 dm \qquad A - C = \int_m z^2 dm - \int_m x^2 dm \qquad C - B = \int_m y^2 dm - \int_m z^2 dm$$

It follows that Equations 10.160 reduce to

$$\begin{aligned} M_{G_{\text{net}}\,x} &= \frac{3\mu R_y R_z}{R^5}(C - B) \\ M_{G_{\text{net}}\,y} &= \frac{3\mu R_x R_z}{R^5}(A - C) \\ M_{G_{\text{net}}\,z} &= \frac{3\mu R_x R_y}{R^5}(B - A) \end{aligned} \tag{10.162}$$

These are the components, in the spacecraft body frame, of the gravitational torque produced by the variation of the earth's gravitational field over the volume of the spacecraft. To get an idea of these torque magnitudes, note first of all that R_x/R, R_y/R and R_z/R are the direction cosines of the position vector of the center of mass, so that their magnitudes do not exceed 1. For a satellite in a low earth orbit of radius 6700 km, $3\mu/R^3 \cong 4 \times 10^{-6}\,\text{s}^{-2}$, which is therefore the maximum order of magnitude of the coefficients of the inertia terms in Equation 10.162. The moments of inertia of the space shuttle are on the order of $10^6\,\text{kg}\cdot\text{m}^2$, so the gravitational torques on that large vehicle are on the order of 1 N·m.

Substituting Equations 10.162 into Euler's equations of motion (Equations 9.72b), we get

$$\begin{aligned} A\dot{\omega}_x + (C - B)\omega_y\omega_z &= \frac{3\mu R_y R_z}{R^5}(C - B) \\ B\dot{\omega}_y + (A - C)\omega_z\omega_x &= \frac{3\mu R_x R_z}{R^5}(A - C) \\ C\dot{\omega}_z + (B - A)\omega_x\omega_y &= \frac{3\mu R_x R_y}{R^5}(B - A) \end{aligned} \tag{10.163}$$

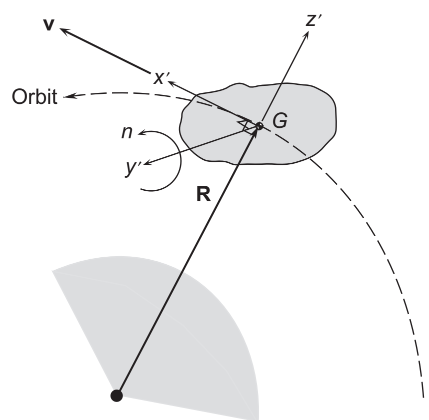


FIGURE 10.28

Orbital reference frame $x'y'z'$ attached to the center of mass of the satellite.

Now consider the local vertical/local horizontal orbital reference frame shown in Figure 10.28. It is actually the Clohessy-Wiltshire frame of Chapter 7, with the axes relabeled. The z' axis points radially outward from the center of the earth, the x' axis is in the direction of the local horizon, and the y' axis completes the right-handed triad by pointing in the direction of the orbit normal. This frame rotates around the y' axis with an angular velocity equal to the mean motion n of the circular orbit. Suppose we align the satellite's principal body frame axes xyz with $x'y'z'$, respectively. When the body x axis is aligned with the x' direction, it is called is the roll axis. The body y axis, when aligned with the y' direction, is the pitch axis. The body z axis, pointing outward from the earth in the z' direction, is the yaw axis. These directions are illustrated in Figure 10.29. With the spacecraft aligned in this way, the body frame components of the inertial angular velocity $\boldsymbol{\omega}$ are $\omega_x = \omega_z = 0$ and $\omega_y = n$. The components of the position vector $\mathbf{R}$ are $R_x = R_y = 0$ and $R_z = R$. Substituting this data into Equations 10.163 yields

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

That is, the spacecraft will orbit the planet with its principal axes remaining aligned with the orbital frame. If this motion is stable under the influence of gravity alone, without the use of thrusters, gyros or other devices, then the spacecraft is gravity-gradient stabilized. We need to assess the stability of this motion so we can determine how to orient a spacecraft to take advantage of this type of passive attitude stabilization.

Let the body frame xyz be slightly misaligned with the orbital reference frame, so that the yaw, pitch and roll angles between the xyz axes and the $x'y'z'$ axes, respectively, are very small, as suggested in Figure 10.29. The absolute angular velocity $\boldsymbol{\omega}$ of the spacecraft is the angular velocity $\boldsymbol{\omega}_{\text{rel}}$ relative to the orbital reference frame plus the inertial angular velocity $\boldsymbol{\Omega}$ of the $x'y'z'$ frame,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{rel}} + \boldsymbol{\Omega}$$

The components of $\boldsymbol{\omega}_{\text{rel}}$ in the body frame are found using the yaw, pitch and roll relations, Equations 9.129. In so doing, it must be kept in mind that all angles and rates are assumed to be so small that their

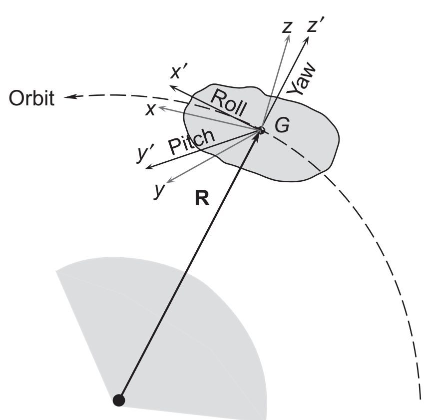


FIGURE 10.29

Satellite body frame slightly misaligned with the orbital frame $x'y'z'$.

squares and products may be neglected. Recalling that $\sin\alpha = \alpha$ and $\cos\alpha = 1$ when $\alpha << 1$, we therefore obtain

$$\omega_{x_{\text{rel}}} = \omega_{\text{roll}} - \omega_{\text{yaw}} \overbrace{\sin\theta_{\text{pitch}}}^{=\theta_{\text{pitch}}} = \dot{\psi}_{\text{roll}} - \overbrace{\dot{\phi}_{\text{yaw}}\theta_{\text{pitch}}}^{\text{neglect product}} = \dot{\psi}_{\text{roll}} \tag{10.164}$$

$$\omega_{y_{\text{rel}}} = \omega_{\text{yaw}} \overbrace{\cos\theta_{\text{pitch}}}^{=1} \overbrace{\sin\psi_{\text{roll}}}^{=\psi_{\text{roll}}} + \omega_{\text{pitch}} \overbrace{\cos\psi_{\text{roll}}}^{=1} = \overbrace{\dot{\phi}_{\text{yaw}}\psi_{\text{roll}}}^{\text{neglect product}} + \dot{\theta}_{\text{pitch}} = \dot{\theta}_{\text{pitch}} \tag{10.165}$$

$$\omega_{z_{\text{rel}}} = \omega_{\text{yaw}} \overbrace{\cos\theta_{\text{pitch}}}^{=1} \overbrace{\cos\psi_{\text{roll}}}^{=1} - \omega_{\text{pitch}} \overbrace{\sin\psi_{\text{roll}}}^{=\psi_{\text{roll}}} = \dot{\phi}_{\text{yaw}} - \overbrace{\dot{\theta}_{\text{pitch}}\psi_{\text{roll}}}^{\text{neglect product}} = \dot{\phi}_{\text{yaw}} \tag{10.166}$$

The orbital frame's angular velocity is the mean motion n of the circular orbit, so that

$$\mathbf{\Omega} = n\hat{\mathbf{j}}'$$

To obtain the orbital frame's angular velocity components along the body frame, we must use the transformation rule

$$\{\mathbf{\Omega}\}_x = [\mathbf{Q}]_{x'x}\{\mathbf{\Omega}\}_{x'} \tag{10.167}$$

where $[\mathbf{Q}]_{x'x}$ is given by Equation 9.119. (Keep in mind that $x'y'z'$ are playing the role of XYZ in Figure 9.26.) Using the small angle approximations in Equation 9.119 leads to

$$[\mathbf{Q}]_{x'x} = \begin{bmatrix} 1 & \phi_{\text{yaw}} & -\theta_{\text{pitch}} \\ -\phi_{\text{yaw}} & 1 & \psi_{\text{roll}} \\ \theta_{\text{pitch}} & -\psi_{\text{roll}} & 1 \end{bmatrix}$$

With this, Equation 10.167 becomes

$$\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{bmatrix} 1 & \phi_{\text{yaw}} & -\theta_{\text{pitch}} \\ -\phi_{\text{yaw}} & 1 & \psi_{\text{roll}} \\ \theta_{\text{pitch}} & -\psi_{\text{roll}} & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ n \\ 0 \end{Bmatrix} = \begin{Bmatrix} n\phi_{\text{yaw}} \\ n \\ -n\psi_{\text{roll}} \end{Bmatrix}$$

Now we can calculate the components of the satellite's inertial angular velocity along the body frame axes,

$$\begin{aligned} \omega_x &= \omega_{x\text{rel}} + \Omega_x = \dot{\psi}_{\text{roll}} + n\phi_{\text{yaw}} \\ \omega_y &= \omega_{y\text{rel}} + \Omega_y = \dot{\theta}_{\text{pitch}} + n \\ \omega_z &= \omega_{z\text{rel}} + \Omega_z = \dot{\phi}_{\text{yaw}} - n\psi_{\text{roll}} \end{aligned} \tag{10.168}$$

Differentiating these with respect to time, remembering that n is constant for a circular orbit, gives the components of inertial angular acceleration in the body frame,

$$\begin{aligned}\dot{\omega}_x &= \ddot{\psi}_{\text{roll}} + n\dot{\phi}_{\text{yaw}}\\ \dot{\omega}_y &= \ddot{\theta}_{\text{pitch}}\\ \dot{\omega}_z &= \ddot{\phi}_{\text{yaw}} - n\dot{\psi}_{\text{roll}}\end{aligned} \tag{10.169}$$

The position vector of the satellite's center of mass lies along the z' axis of the orbital frame,

$$\mathbf{R} = R\hat{\mathbf{k}}'$$

To obtain the components of $\mathbf{R}$ in the body frame we once again use the transformation matrix $[\mathbf{Q}]_{x'x}$

$$\begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} = \begin{bmatrix} 1 & \phi_{\text{yaw}} & -\theta_{\text{pitch}} \\ -\phi_{\text{yaw}} & 1 & \psi_{\text{roll}} \\ \theta_{\text{pitch}} & -\psi_{\text{roll}} & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ R \end{Bmatrix} = \begin{Bmatrix} -R\theta_{\text{pitch}} \\ R\psi_{\text{roll}} \\ R \end{Bmatrix} \tag{10.170}$$

Substituting Equations 10.168, 10.169 and 10.170, together with $n = \sqrt{\mu/R^3}$, into Equations 10.163, and setting

$$A = I_{\text{roll}} \qquad B = I_{pitch} \qquad C = I_{yaw} \tag{10.171}$$

yields

$$\begin{aligned} I_{\text{roll}}(\ddot{\psi}_{\text{roll}} + n\dot{\phi}_{\text{yaw}}) + (I_{\text{yaw}} - I_{\text{pitch}})(\dot{\theta}_{\text{pitch}} + n)(\dot{\phi}_{\text{yaw}} - n\psi_{roll}) &= 3(I_{\text{yaw}} - I_{\text{pitch}})n^2\psi_{\text{roll}} \\ I_{\text{pitch}}\ddot{\theta}_{\text{pitch}} + (I_{\text{roll}} - I_{\text{yaw}})(\dot{\psi}_{\text{roll}} + n\phi_{\text{yaw}})(\dot{\phi}_{\text{yaw}} - n\psi_{\text{roll}}) &= -3(I_{\text{roll}} - I_{\text{yaw}})n^2\theta_{\text{pitch}} \\ I_{\text{yaw}}(\ddot{\phi}_{\text{yaw}} - n\dot{\psi}_{\text{roll}}) + (I_{\text{pitch}} - I_{\text{roll}})(\dot{\theta}_{\text{pitch}} + n)(\dot{\psi}_{\text{roll}} + n\phi_{\text{yaw}}) &= -3(I_{\text{pitch}} - I_{\text{roll}})n^2\theta_{\text{pitch}}\psi_{\text{roll}} \end{aligned}$$

Expanding terms and retaining terms at most linear in all angular quantities and their rates yields

$$I_{\text{yaw}}\ddot{\phi}_{\text{yaw}} + (I_{\text{pitch}} - I_{\text{roll}})n^2\phi_{\text{yaw}} + (I_{\text{pitch}} - I_{\text{roll}} - I_{\text{yaw}})n\dot{\psi}_{\text{roll}} = 0 \tag{10.172}$$

$$I_{\text{roll}}\ddot{\psi}_{\text{roll}} + (I_{\text{roll}} - I_{\text{pitch}} + I_{\text{yaw}})n\dot{\phi}_{\text{yaw}} + 4(I_{\text{pitch}} - I_{\text{yaw}})n^2\psi_{\text{roll}} = 0 \tag{10.173}$$

$$I_{\text{pitch}}\ddot{\theta}_{\text{pitch}} + 3(I_{\text{roll}} - I_{\text{yaw}})n^2\theta_{\text{pitch}} = 0 \tag{10.174}$$

These are the differential equations governing the influence of gravity gradient torques on the small angles and rates of misalignment of the body frame with the orbital frame.

Equation 10.174, governing the pitching motion around the y' axis, is not coupled to the other two equations. We make the classical assumption that the solution is of the form

$$\theta_{pitch} = Pe^{pt} \tag{10.175}$$

where P and p are constants. P is the amplitude of the small disturbance that initiates the pitching motion. Substituting Equation 10.175 into Equation 10.174 yields $[I_{\text{pitch}}p^2 + 3(I_{\text{roll}} - I_{\text{yaw}})n^2]Pe^{pt} = 0$ for all t, which implies that the bracketed term must vanish, and that means p must have either of the two values

$$p_{1,2} = \pm i\sqrt{3\frac{(I_{\text{roll}} - I_{\text{yaw}})n^2}{I_{\text{pitch}}}} \qquad (i = \sqrt{-1})$$

Thus,

$$\theta_{pitch} = P_1 e^{p_1 t} + P_2 e^{p_2 t}$$

yields the stable, small-amplitude, steady-state harmonic oscillator solution only if p_1 and p_2 are imaginary, that is, if

$$I_{\text{roll}} > I_{\text{yaw}} \qquad \text{For stability in pitch} \tag{10.176}$$

The stable pitch oscillation frequency is

$$\omega_{f\,\text{pitch}} = n\sqrt{3\frac{(I_{\text{roll}} - I_{\text{yaw}})}{I_{\text{pitch}}}} \tag{10.177}$$

(If $I_{\text{yaw}} > I_{\text{roll}}$, then p_1 and p_2 are both real, one positive, the other negative. The positive root causes $\theta_{pitch} \to \infty$, which is the undesirable, unstable case.)

Let us now turn our attention to Equations 10.172 and 10.173, which govern yaw and roll motion under gravity gradient torque. Again, we assume the solution is exponential in form,

$$\phi_{\text{yaw}} = Ye^{qt} \quad \psi_{roll} = Re^{qt} \tag{10.178}$$

Substituting these into Equations 10.172 and 10.173 yields

$$[(I_{\text{pitch}} - I_{\text{roll}})n^2 + I_{\text{yaw}}q^2]Y + (I_{\text{pitch}} - I_{\text{roll}} - I_{\text{yaw}})nqR = 0$$
$$(I_{\text{roll}} - I_{\text{pitch}} + I_{\text{yaw}})nqY + [4(I_{\text{pitch}} - I_{\text{yaw}})n^2 + I_{\text{roll}}q^2]R = 0$$

In the interest of simplification, we can factor I_{yaw} out of the first equation and I_{roll} out of the second one to get

$$\begin{aligned}\left(\frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}}n^2 + q^2\right)Y + \left(\frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}} - 1\right)nqR &= 0\\ \left(1 - \frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}}\right)nqY + \left(4\frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}}n^2 + q^2\right)R &= 0\end{aligned} \tag{10.179}$$

Let

$$k_Y = \frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}} \qquad k_R = \frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}} \tag{10.180}$$

It is easy to show from Equations 10.161, 10.171 and 10.180 that

$$k_Y = \frac{\left(\int_m x^2 dm \Big/ \int_m y^2 dm\right) - 1}{\left(\int_m x^2 dm \Big/ \int_m y^2 dm\right) + 1} \qquad k_R = \frac{\left(\int_m z^2 dm \Big/ \int_m y^2 dm\right) - 1}{\left(\int_m z^2 dm \Big/ \int_m y^2 dm\right) + 1}$$

which means

$$|k_Y| < 1 \qquad |k_R| < 1$$

Using the definitions in Equation 10.180, we can write Equations 10.179 more compactly as

$$\begin{aligned}(k_Y n^2 + q^2)Y + (k_Y - 1)nqR &= 0\\ (1 - k_R)nqY + (4k_R n^2 + q^2)R &= 0\end{aligned}$$

or, using matrix notation,

$$\begin{bmatrix} k_Y n^2 + q^2 & (k_Y - 1)nq \\ (1 - k_R)nq & 4k_R n^2 + q^2 \end{bmatrix} \begin{Bmatrix} Y \\ R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.181}$$

In order to avoid the trivial solution ($Y = R = 0$), the determinant of the coefficient matrix must be zero. Expanding the determinant and collecting terms yields the characteristic equation for q,

$$q^4 + bn^2q^2 + cn^4 = 0 \tag{10.182}$$

where

$$b = 3k_R + k_Y k_R + 1 \qquad c = 4k_Y k_R \tag{10.183}$$

This quartic equation has four roots which, when substituted back into Equation 10.178, yield

$$\begin{aligned}\phi_{\text{yaw}} &= Y_1 e^{q_1 t} + Y_2 e^{q_2 t} + Y_3 e^{q_3 t} + Y_4 e^{q_4 t}\\ \psi_{roll} &= R_1 e^{q_1 t} + R_2 e^{q_2 t} + R_3 e^{q_3 t} + R_4 e^{q_4 t}\end{aligned}$$

In order for these solutions to remain finite in time, the roots $q_1, \ldots, q_4$ must be negative (solution decays to zero) or imaginary (steady oscillation at initial small amplitude).

To reduce Equation 10.182 to a quadratic equation, let us introduce a new variable λ and write,

$$q = \pm n\sqrt{\lambda} \tag{10.184}$$

Then Equation 10.182 becomes

$$\lambda^2 + b\lambda + c = 0 \tag{10.185}$$

the familiar solution of which is

$$\lambda_1 = -\frac{1}{2}\left(b + \sqrt{b^2 - 4c}\right) \qquad \lambda_2 = -\frac{1}{2}\left(b - \sqrt{b^2 - 4c}\right) \tag{10.186}$$

To guarantee that q in Equation 10.184 does not take a positive value, we must require that λ be real and negative (so q will be imaginary). For λ to be real requires that $b > 2\sqrt{c}$, or

$$3k_R + k_Y k_R + 1 > 4\sqrt{k_Y k_R} \tag{10.187}$$

For λ to be negative requires $b^2 > b^2 - 4c$, which will be true if $c > 0$; that is,

$$k_Y k_R > 0 \tag{10.188}$$

Equations 10.187 and 10.188 are the conditions required for yaw and roll stability under gravity gradient torques, to which we must add Equation 10.176 for pitch stability. Observe that we can solve Equations 10.180 to obtain

$$I_{\text{yaw}} = \frac{1 - k_R}{1 - k_Y k_R} I_{\text{pitch}} \qquad I_{\text{roll}} = \frac{1 - k_Y}{1 - k_Y k_R} I_{\text{pitch}}$$

By means of these relationships, the pitch stability criterion, $I_{\text{roll}}/I_{\text{yaw}} > 1$, becomes

$$\frac{1 - k_Y}{1 - k_R} > 1$$

In view of the fact that $|k_R| < 1$, this means

$$k_Y < k_R \tag{10.189}$$

Figure 10.30 shows those regions I and II on the $k_Y - k_R$ plane in which all three stability criteria (Equations 10.187, 10.188 and 10.189) are simultaneously satisfied, along with the requirement that the three moments of inertia I_{pitch}, I_{roll} and I_{yaw} are positive.

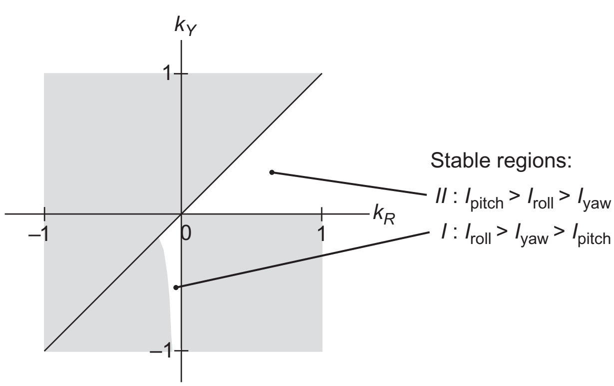


FIGURE 10.30

Regions in which the values of k_Y and k_R yield neutral stability in yaw, pitch and roll of a gravity gradient satellite.

In the small sliver of region *I*, $k_Y < 0$ and $k_R < 0$; therefore, according to Equations 10.180, $I_{yaw} > I_{pitch}$ and $I_{roll} > I_{pitch}$, which together with Equation 10.176, yield $I_{roll} > I_{yaw} > I_{pitch}$. Remember that the gravity gradient spacecraft is slowly "spinning" about the minor pitch axis (normal to the orbit plane) at an angular velocity equal to the mean motion of the orbit. So this criterion makes the spacecraft a "minor axis spinner," the roll axis (flight direction) being the major axis of inertia. With energy dissipation, we know this orientation is not stable in the long run. On the other hand, in region *II*, k_Y and k_R are both positive, so that Equations 10.180 imply $I_{pitch} > I_{yaw}$ and $I_{pitch} > I_{roll}$. Thus, along with the pitch criterion ($I_{roll} > I_{yaw}$), we have $I_{pitch} > I_{roll} > I_{yaw}$. In this, the preferred, configuration, the gravity gradient spacecraft is a "major axis spinner" about the pitch axis, and the minor yaw axis is the minor axis of inertia. It turns out that all of the known gravity-gradient stabilized moons of the solar system, like the earth's, whose "captured" rate of rotation equals the orbital period, are major axis spinners.

In Equation 10.177 we presented the frequency of the gravity gradient pitch oscillation. For completeness we should also point out that the coupled yaw and roll motions have two oscillation frequencies, which are obtained from Equations 10.184 and 10.186,

$$\left.\omega_{f_{\text{yaw-roll}}}\right)_{1,2} = n\sqrt{\frac{1}{2}\left(b \pm \sqrt{b^2 - 4c}\right)} \tag{10.190}$$

Recall that *b* and *c* are found in Equation 10.183.

We have assumed throughout this discussion that the orbit of the gravity gradient satellite is circular. Kaplan (1976) shows that the effect of a small eccentricity turns up only in the pitching motion. In particular, the natural oscillation expressed by Equation 10.176 is augmented by a forced oscillation term,

$$\theta_{pitch} = P_1 e^{p_1 t} + P_2 e^{p_2 t} + \frac{2e \sin nt}{3\left(\dfrac{I_{roll} - I_{yaw}}{I_{pitch}}\right) - 1} \tag{10.191}$$

where *e* is the (small) eccentricity of the orbit. From this we see that there is a pitch resonance. When $(I_{roll} - I_{yaw})/I_{pitch}$ approaches 1/3, the amplitude of the last term grows without bound.

Example 10.15

The uniform, monolithic 10,000 kg slab, having the dimensions shown in Figure 10.31, is in a circular LEO. Determine the orientation of the satellite in its orbit for gravity gradient stabilization, and compute the periods of the pitch and yaw/roll oscillations in terms of the orbital period *T*.

According to Figure 9.9c, the principal moments of inertia around the xyz axes through the center of mass are

$$A = \frac{10{,}000}{12}(1^2 + 9^2) = 68{,}333\,\text{kg}\cdot\text{m}^2$$

$$B = \frac{10{,}000}{12}(3^2 + 9^2) = 75{,}000\,\text{kg}\cdot\text{m}^2$$

$$C = \frac{10{,}000}{12}(3^2 + 1^2) = 8333.3\,\text{kg}\cdot\text{m}^2$$

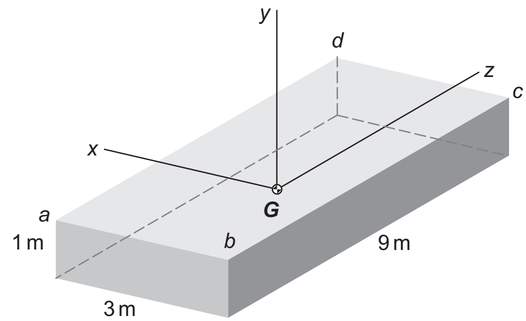


FIGURE 10.31

Parallelepiped satellite.

Solution

Let us first determine whether we can stabilize this object as a minor axis spinner. In that case,

$$I_{\text{pitch}} = C = 8333.3\,\text{kg}\cdot\text{m}^2 \qquad I_{\text{yaw}} = A = 68{,}333\,\text{kg}\cdot\text{m}^2 \qquad I_{\text{roll}} = B = 75{,}000\,\text{kg}\cdot\text{m}^2$$

Since $I_{\text{roll}} > I_{\text{yaw}}$, the satellite would be stable in pitch. To check yaw/roll stability, we first compute

$$k_Y = \frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}} = -0.97561 \qquad k_R = \frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}} = -0.8000$$

We see that $k_Y k_R > 0$, which is one of the two requirements. The other one is found in Equation 10.187, but in this case

$$1 + 3k_R + k_Y k_R - 4\sqrt{k_Y k_R} = -4.1533 < 0$$

so that condition is not met. Hence, the object cannot be gravity gradient stabilized as a minor axis spinner.

As a major axis spinner, we must have

$$I_{\text{pitch}} = B = 75{,}000\,\text{kg}\cdot\text{m}^2 \qquad I_{\text{yaw}} = C = 8333.3\,\text{kg}\cdot\text{m}^2 \qquad I_{\text{roll}} = A = 68{,}333\,\text{kg}\cdot\text{m}^2$$

Then $I_{\text{roll}} > I_{\text{yaw}}$, so the pitch stability condition is satisfied. Furthermore, since

$$k_Y = \frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}} = 0.8000 \qquad k_R = \frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}} = 0.97561$$

we have

$$\begin{aligned} k_Y k_R &= 0.7805 > 0 \\ 1 + 3k_R + k_Y k_R - 4\sqrt{k_Y k_R} &= 1.1735 > 0 \end{aligned}$$

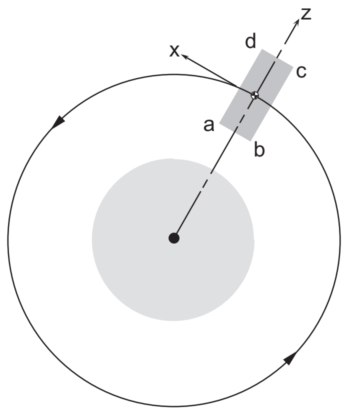


FIGURE 10.32

Orientation of the parallelepiped for gravity gradient stabilization.

which means the two criteria for stability in the yaw and roll modes are met. The satellite should therefore be orbited as shown in Figure 10.32, with its minor axis aligned with the radial from the earth's center, the plane *abcd* lying in the orbital plane, and the body *x* axis aligned with the local horizon.

According to Equation 10.177, the frequency of the pitch oscillation is

$$\omega_{f\,\mathrm{pitch}} = n\sqrt{3\frac{I_{\mathrm{roll}} - I_{\mathrm{yaw}}}{I_{\mathrm{pitch}}}}$$

$$= n\sqrt{3\frac{68{,}333 - 8333.3}{75{,}000}} = 1.5492n$$

where n is the mean motion. Hence, the period of this oscillation, in terms of that of the orbit, is

$$T_{\mathrm{pitch}} = \frac{2\pi}{\omega_{f\,\mathrm{pitch}}} = 0.6455\frac{2\pi}{n} = \boxed{0.6455T}$$

For the yaw/roll frequencies, we use Equation 10.190,

$$\omega_{f\,\mathrm{yaw/roll}}\Big)_1 = n\sqrt{\frac{1}{2}\left(b + \sqrt{b^2 - 4c}\right)}$$

where

$$b = 1 + 3k_R + k_Y k_R = 4.7073 \quad \text{and} \quad c = 4k_Y k_R = 3.122$$

Thus,

$$\omega_{f\,\mathrm{yaw/roll}}\Big)_1 = 2.3015n$$

Likewise,

$$\omega_{f\,\text{yaw/roll}}\Big)_2 = \sqrt{\frac{1}{2}\left(b - \sqrt{b^2 - 4c}\right)} = 1.977n$$

From these we obtain

$$T_{\text{yaw/roll}_1} = \boxed{0.5058T} \qquad T_{\text{yaw/roll}_2} = \boxed{0.4345T}$$

Finally, observe that

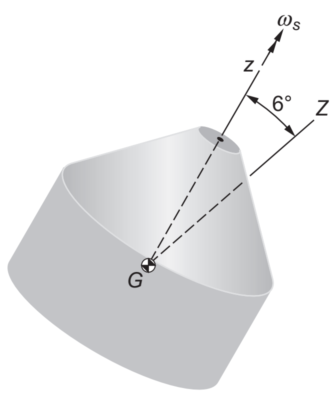
$$\frac{I_{\text{roll}} - I_{\text{yaw}}}{I_{\text{pitch}}} = 0.8$$

so that we are far from the pitch resonance condition that exists if the orbit has a small eccentricity.

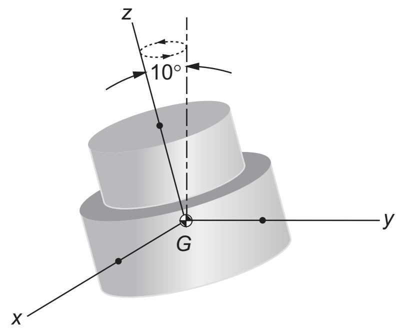
PROBLEMS

Section 10.2

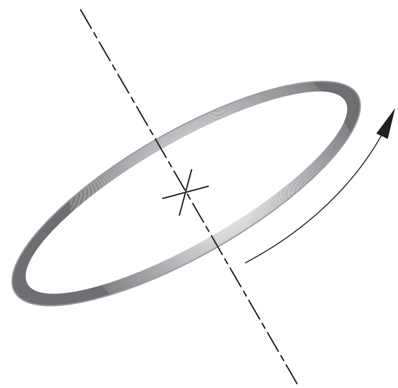
10.1 The axisymmetric satellite has axial and transverse mass moments of inertia about axes through the mass center G of $C = 1200\,\text{kg-m}^2$ and $A = 2600\,\text{kg-m}^2$, respectively. If it is spinning at $\omega_s = 6\,\text{rad/s}$ when it is launched, determine its angular momentum. Precession occurs about the inertial Z axis.
{Ans.: $\|\mathbf{H}_G\| = 13{,}450\ \text{kg}\cdot\text{m}^2/\text{sec}$}



10.2 A spacecraft is symmetrical about its body-fixed z-axis. Its principal mass moments of inertia are $A = B = 300\,\text{kg-m}^2$ and $C = 500\,\text{kg-m}^2$. The z axis sweeps out a cone with a total vertex angle of 10° as it precesses around the angular momentum vector. If the spin velocity is 6 rad/s, compute the period of precession.
{Ans.: 0.417 s}



10.3 A thin ring tossed into the air with a spin velocity of ω_s has a very small nutation angle θ (in radians). What is the precession rate ω_p?
{Ans.: $\omega_p = 2\omega_s(1 + \theta^2/2)$, retrograde}



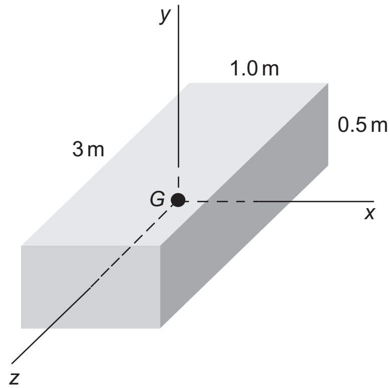
10.4 For an axisymmetric rigid satellite,

$$[\mathbf{I}_G] = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 5000 \end{bmatrix} \text{kg} \cdot \text{m}^2$$

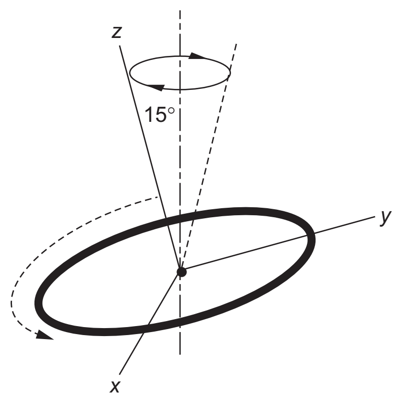
It is spinning about the body z-axis in torque-free motion, precessing around the angular momentum vector **H** at the rate of 2 rad/sec. Calculate the magnitude of **H**.
{Ans.: $2000\,\text{kg} \cdot \text{m}^2/\text{s}$}

10.5 At a given instant the box-shaped 500 kg satellite (in torque-free motion) has an absolute angular velocity $\boldsymbol{\omega} = 0.01\hat{\mathbf{i}} - 0.03\hat{\mathbf{j}} + 0.02\hat{\mathbf{k}}$ (rad/s). Its moments of inertia about the principal body axes

xyz are $A = 385.4\,\text{kg}\cdot\text{m}^2$, $B = 416.7\,\text{kg}\cdot\text{m}^2$ and $C = 52.08\,\text{kg}\cdot\text{m}^2$, respectively. Calculate the magnitude of its absolute angular acceleration.
{Ans. 6.167×10^{-4} rad/s^2}



10.6 An 8 kg thin ring in torque-free motion is spinning with an angular velocity of 30 rad/s and a constant nutation angle of 15°. Calculate the rotational kinetic energy if $A = B = 0.36\,\text{kg-m}^2$, $C = 0.72\,\text{kg-m}^2$.
{Ans.: 370.5 J}

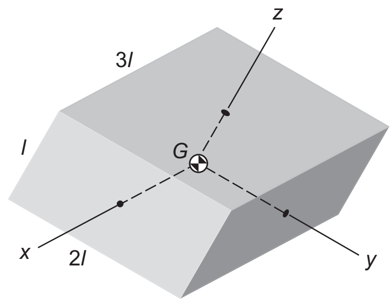


10.7 The rectangular block has an angular velocity $\boldsymbol{\omega} = 1.5\omega_0\hat{\mathbf{i}} + 0.8\omega_0\hat{\mathbf{j}} + 0.6\omega_0\hat{\mathbf{k}}$, where ω_0 has units of rad/s.

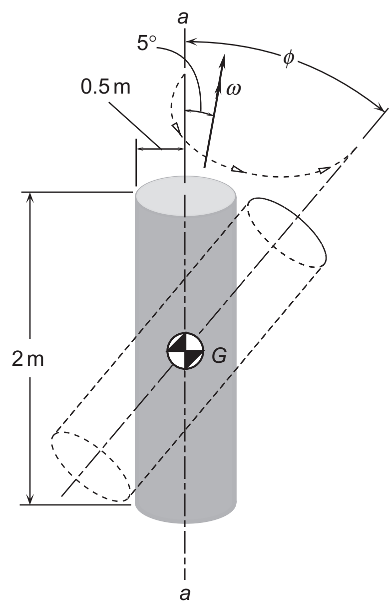
(a) Determine the angular velocity ω of the block if it spins around the body z axis with the same rotational kinetic energy.

(b) Determine the angular velocity ω of the block if it spins around the body z axis with the same angular momentum.

{Ans.: (a) $\omega = 1.31\omega_0$ (b) $\omega = 1.04\omega_0$}



10.8 The solid right-circular cylinder of mass 500 kg is set into torque-free motion with its symmetry axis initially aligned with the fixed spatial line *a-a*. Due to an injection error, the vehicle's angular velocity vector **ω** is misaligned 5° (the wobble angle) from the symmetry axis. Calculate the maximum angle ϕ between fixed line *a-a* and the axis of the cylinder.
{Ans.: 30.96°}

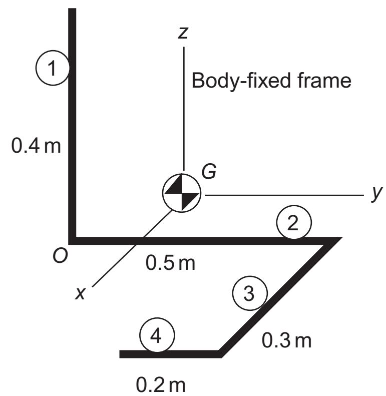


Section 10.3

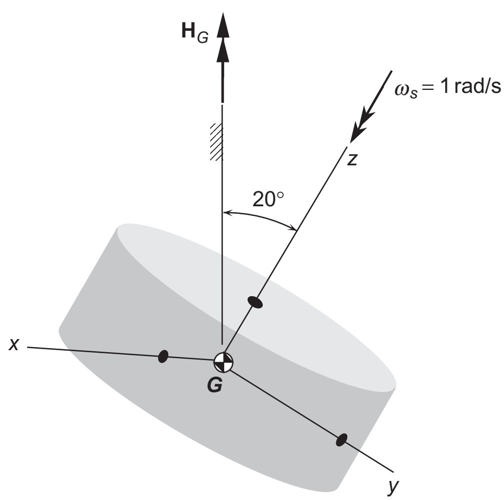
10.9 For a rigid axisymmetric satellite, the mass moment of inertia about its long axis is 1000 kg · m^2, and the moment of inertia about transverse axes through the center of mass is 5000 kg · m^2. It is spinning about the minor principal body axis in torque-free motion at 6 rad/s with the angular velocity lined up with the angular momentum vector **H**. Over time, the energy degrades due to internal effects and the

satellite is eventually spinning about a major principal body axis with the angular velocity lined up with the angular momentum vector **H**. Calculate the change in rotational kinetic energy between the two states.
{Ans.: −14.4 kJ}

10.10 Let the object in Example 9.11 be a highly dissipative torque-free satellite, whose angular velocity at the instant shown is $\boldsymbol{\omega} = 10\hat{\mathbf{i}}$ rad/s. Calculate the decrease in kinetic energy after it becomes, as eventually it must, a major axis spinner.
{Ans.: –0.487 J}

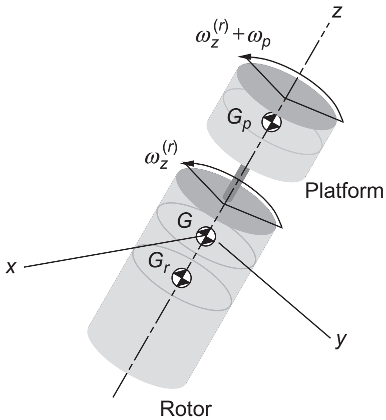


10.11 The dissipative torque-free cylindrical satellite has the initial spin state shown. $A = B = 320\,\text{kg}\cdot\text{m}^2$ and $C = 560\,\text{kg}\cdot\text{m}^2$. Calculate the magnitude of the angular velocity when it reaches its stable spin state.
{Ans.: −1.419 rad/s}



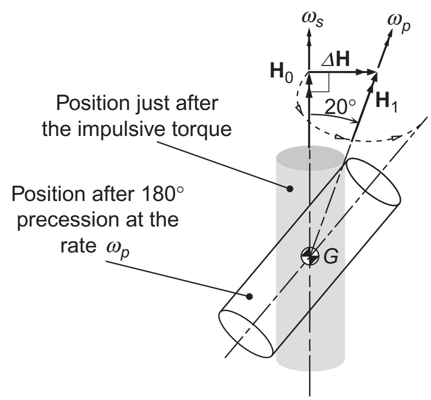
Section 10.4

10.12 For a nonprecessing, dual-spin satellite, $C_r = 1000\,\text{kg}\cdot\text{m}^2$ and $C_p = 500\,\text{kg}\cdot\text{m}^2$. The angular velocity of the rotor is $3\hat{\mathbf{k}}$ rad/s and the angular velocity of the platform relative to the rotor is $1\hat{\mathbf{k}}$ rad/s. If the relative angular velocity of the platform is reduced to $0.5\hat{\mathbf{k}}$ rad/s, what is the new angular velocity of the rotor?
{Ans.: 3.17 rad/s}



Section 10.6

10.13 For a rigid axisymmetric satellite, the mass moment of inertia about its long axis is $1000\,\text{kg}\cdot\text{m}^2$, and the moment of inertia about transverse axes through the center of mass is $5000\,\text{kg}\cdot\text{m}^2$. It is initially spinning about the minor principal body axis in torque-free motion at $\omega_s = 0.1$ rad/s, with the angular velocity lined up with the angular momentum vector $\mathbf{H}_0$. A pair of thrusters exert an external impulsive torque on the satellite, causing an instantaneous change $\Delta\mathbf{H}$ of angular momentum in the direction normal to $\mathbf{H}_0$, so that the new angular momentum is $\mathbf{H}_1$, at an angle of 20° to $\mathbf{H}_0$, as shown in the figure. How long does it take the satellite to precess ("cone") through an angle of 180° around $\mathbf{H}_1$?
{Ans.: 147.6 s}

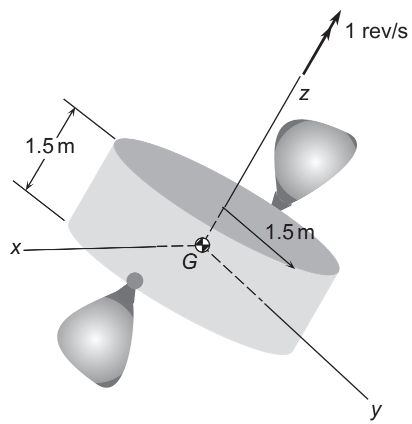


Section 10.7

10.14 A satellite is spinning at 0.01 rev/s. The moment of inertia of the satellite about the spin axis is 2000 kg-m^2. Paired thrusters are located at a distance of 1.5 m from the spin axis. They deliver their thrust in pulses, each thruster producing an impulse of 15 N-s per pulse. At what rate will the satellite be spinning after 30 pulses?
{Ans.: 0.0637 rev/s}

10.15 A satellite has moments of inertia $A = 2000\,\text{kg} \cdot \text{m}^2$, $B = 4000\,\text{kg} \cdot \text{m}^2$, and $C = 6000\,\text{kg} \cdot \text{m}^2$ about its principal body axes *xyz*. Its angular velocity is $\boldsymbol{\omega} = 0.1\hat{\mathbf{i}} + 0.3\hat{\mathbf{j}} + 0.5\hat{\mathbf{k}}$ (rad/s). If thrusters cause the angular momentum vector to undergo the change $\Delta\mathbf{H}_G = 50\hat{\mathbf{i}} - 100\hat{\mathbf{j}} + 300\hat{\mathbf{k}}$ (kg · m^2/s), what is the magnitude of the new angular velocity?
{Ans.: 1.045 rad/s}

10.16 The body-fixed *xyz* axes are principal axes of inertia passing through the center of mass of the 300 kg cylindrical satellite, which is spinning at 1 revolution per second about the *z* axis. What impulsive torque about the *y* axis must the thrusters impart to cause the satellite to precess at 5 revolutions per second?
{Ans.: 6740 N-m-s}



Section 10.8

10.17 A satellite is to be despun by means of a tangential-release yo-yo mechanism consisting of two masses, 3 kg each, wound around the mid plane of the satellite. The satellite is spinning around its axis of symmetry with an angular velocity $\omega_s = 5$ rad/sec. The radius of the cylindrical satellite is 1.5 m and the moment of inertia about the spin axis is $C = 300\,\text{kg} \cdot \text{m}^2$.
(a) Find the cord length and the deployment time to reduce the spin rate to 1 rad/s.
(b) Find the cord length and time to reduce the spin rate to zero.
{Ans.: (a) $l = 5.902$ m, $t = 0.787$ s; (b) $l = 7.228$ m, $t = 0.964$ s.}

10.18 A cylindrical satellite of radius 1 m is initially spinning about the axis of symmetry at the rate of two revolutions per second with a nutation angle of 15°. The principal moments of inertia are,

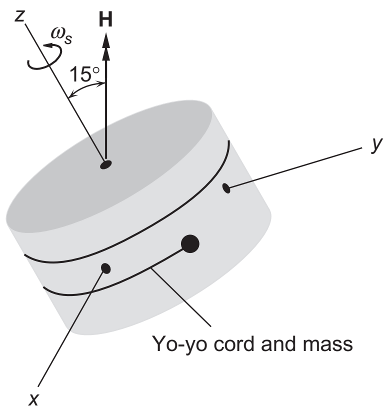
$A = B = 30\,\text{kg}\cdot\text{m}^2 C = 60\,\text{kg}\cdot\text{m}^2$. An energy dissipation device is built into the satellite, so that it eventually ends up in pure spin around the z axis.

(a) Calculate the final spin rate about the z axis.

(b) Calculate the loss of kinetic energy.

(c) A tangential release yo-yo despin device is also included in the satellite. If the two yo-yo masses are each 7 kg, what cord length is required to completely despin the satellite? Is it wrapped in the proper direction in the figure?

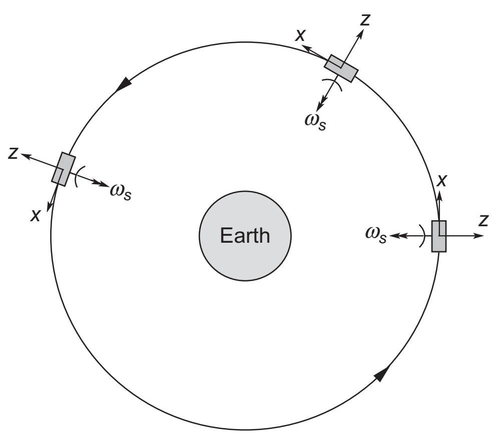
{Ans.: (a) 2.071 rad/s; (b) 8.62 J; (c) 2.3 m}



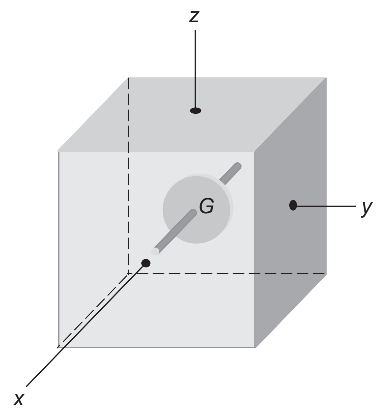
Section 10.9

10.19 A communications satellite is in a GEO (geostationary equatorial orbit) with a period of 24 hours. The spin rate ω_s about its axis of symmetry is 1 revolution per minute, and the moment of inertia about the spin axis is $550\,\text{kg}\cdot\text{m}^2$. The moment of inertia about transverse axes through the mass center G is $225\,\text{kg}\cdot\text{m}^2$. If the spin axis is initially pointed towards the earth, calculate the magnitude and direction of the applied torque $\mathbf{M}_G$ required to keep the spin axis pointed always towards the earth.

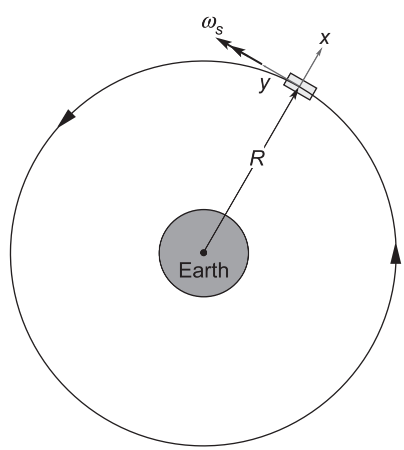
{Ans.: $0.00420\,\text{N}\cdot\text{m}$, about the negative x-axis}



10.20 The moments of inertia of a satellite about its principal body axes *xyz* are $A = 1000\,\text{kg}\cdot\text{m}^2$, $B = 600\,\text{kg}\cdot\text{m}^2$, and $C = 500\,\text{kg}\cdot\text{m}^2$, respectively. The moments of inertia of a momentum wheel at the center of mass of the satellite and aligned with the *x* axis are $I_x = 20\,\text{kg}$ and $I_y = I_z = 6\,\text{kg}\cdot\text{m}^2$. The absolute angular velocity of the satellite with the momentum wheel locked is $\boldsymbol{\omega}_0 = 0.1\hat{\mathbf{i}} + 0.05\hat{\mathbf{j}}$ (rad/s). Calculate the angular velocity ω_f of the momentum wheel (relative to the satellite) required to reduce the *x*-component of the absolute angular velocity of the satellite to 0.003 rad/s.
{Ans.: 4.95 rad/s}



10.21 A solid circular cylindrical satellite of radius 1 m, length 4 m and mass 250 kg is in a circular earth orbit with period 90 minutes. The cylinder is spinning at 0.001 radians per second (no precession) around its axis, which is aligned with the *y* axis of the Clohessy-Wiltshire frame. Calculate the magnitude of the external torque required to maintain this attitude.
{Ans.: $-0.00014544\hat{\mathbf{i}}$ (N-m)}



Section 10.10

10.22 A satellite has principal moments of inertia $A = 300\,\text{kg} \cdot \text{m}^2$, $B = 400\,\text{kg} \cdot \text{m}^2$, $C = 500\,\text{kg} \cdot \text{m}^2$. Determine the permissible orientations in a circular orbit for gravity gradient stabilization. Specify which axes may be aligned in the pitch, roll and yaw directions. (Recall that, relative to a Clohessy-Wiltshire frame at the center of mass of the satellite, yaw is about the x-axis (outward radial from earth's center); roll is about the y-axis (velocity vector); pitch is about the z-axis (normal to orbital plane).

List of Key Terms

body cone
control moment gyros
energy sink analysis
Euler's equations for torque-free motion
Euler's equations for torque-free motion with rotational symmetry
major axis spinners
minor axis spinners
oblate
prolate
prograde precession
retrograde precession
Routh-Hurwitz stability criteria
space cone
wobble angle

CHAPTER

Rocket vehicle dynamics

11

Chapter outline

11.1 Introduction 655
11.2 Equations of motion 656
11.3 The thrust equation 658
11.4 Rocket performance 660
11.5 Restricted staging in field-free space 667
11.6 Optimal staging 678

11.1 INTRODUCTION

In previous chapters we have made frequent reference to delta-v maneuvers of spacecraft. These require a propulsion system of some sort whose job it is to throw vehicle mass (in the form of propellants) overboard. Newton's balance of momentum principle dictates that when mass is ejected from a system in one direction, the mass left behind must acquire a velocity in the opposite direction. The familiar and oft-quoted example is the rapid release of air from an inflated toy balloon. Another is that of a diver leaping off a small boat at rest in the water, causing the boat to acquire a motion of its own. The unfortunate astronaut who becomes separated from his ship in the vacuum of space cannot with any amount of flailing of arms and legs "swim" back to safety. If he has tools or other expendable objects of equipment, accurately throwing them in the direction opposite to his spacecraft may do the trick. Spewing compressed gas from a tank attached to his back through to a nozzle pointed away from the spacecraft would be a better solution.

The purpose of a rocket motor is to use the chemical energy of solid or liquid propellants to steadily and rapidly produce a large quantity of hot, high-pressure gas, which is then expanded and accelerated through a nozzle. This large mass of combustion products flowing out of the nozzle at supersonic speed possesses a lot of momentum and, leaving the vehicle behind, causes the vehicle itself to acquire a momentum in the opposite direction. This is represented as the action of the force we know as thrust. The design and analysis of rocket propulsion systems is well beyond our scope.

This chapter contains a necessarily brief introduction to some of the fundamentals of rocket vehicle dynamics. The equations of motion of a launch vehicle in a gravity turn trajectory are presented first. This is followed by a simple development of the thrust equation, which brings in the concept of specific impulse. The thrust equation and the equations of motion are then combined to produce the rocket equation, which relates delta-v to propellant expenditure and specific impulse. The sounding rocket provides an important but relatively simple application of the concepts introduced to this point. After a computer simulation of a gravity-turn trajectory, the chapter concludes with an elementary consideration of multistage launch vehicles.

Those seeking a more detailed introduction to the subject of rockets and rocket performance will find the texts by Wiesel (1997) and Hale (1994), as well as references cited therein, useful.

11.2 EQUATIONS OF MOTION

Figure 11.1 illustrates the trajectory of a satellite launch vehicle and the forces acting on it during the powered ascent. Rockets at the base of the booster produce the thrust **T**, which acts along the vehicle's axis in the direction of the velocity vector **v**. The aerodynamic drag force **D** is directed opposite to the velocity, as shown. Its magnitude is given by

$$D = qAC_D \tag{11.1}$$

where $q = 1/2\rho v^2$ is the ***dynamic pressure***, in which ρ is the density of the atmosphere and v is the speed, i.e., the magnitude of **v**. A is the ***frontal area*** of the vehicle and C_D is the ***drag coefficient***. C_D depends on the speed and the external geometry of the rocket. The force of gravity on the booster is $m\mathbf{g}$, where m is its mass and **g** is the local gravitational acceleration, pointing towards the center of the earth. As discussed in Section 1.3, at any point of the trajectory, the velocity **v** defines the direction of the unit tangent $\hat{\mathbf{u}}_t$ to the path. The unit normal $\hat{\mathbf{u}}_n$ is perpendicular to **v** and points towards the center of curvature C. The distance of point C from the path is ρ (not to be confused with density). ρ is the radius of curvature.

In Figure 11.1 the vehicle and its flight path are shown relative to the earth. In the interest of simplicity we will ignore the earth's spin and write the equations of motion relative to a nonrotating earth. The small acceleration terms required to account for the earth's rotation can be added for a more refined analysis. Let us resolve Newton's second law, $\mathbf{F}_{net} = m\mathbf{a}$, into components along the path directions $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$. Recall from Section 1.3 that the acceleration along the path is

$$a_t = \frac{dv}{dt} \tag{11.2}$$

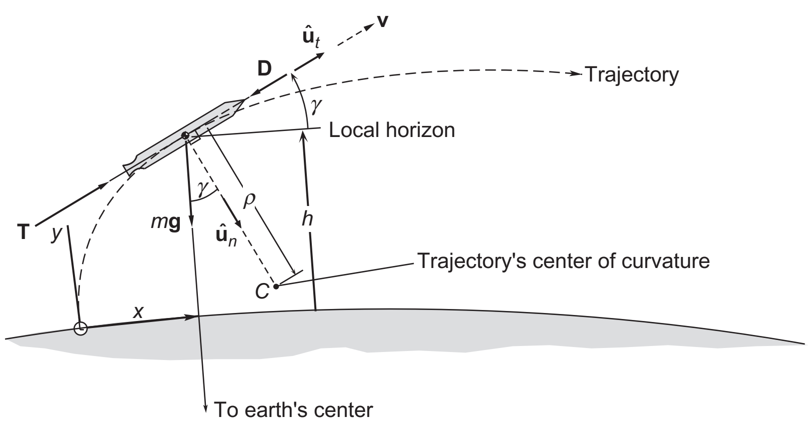


FIGURE 11.1

Launch vehicle boost trajectory. γ is the flight path angle.

and the normal acceleration is $a_n = v^2/\rho$ (where ρ is the radius of curvature). It was shown in Example 1.8 (Equation 1.37) that for flight over a flat surface, $v/\rho = -d\gamma/dt$, in which case the normal acceleration can be expressed in terms of the flight path angle as

$$a_n = -v\frac{d\gamma}{dt}$$

To account for the curvature of the earth, as was done in Section 1.7, one can use polar coordinates with origin at the earth's center to show that a term must be added to this expression, so that it becomes

$$a_n = -v\frac{d\gamma}{dt} + \frac{v^2}{R_E + h}\cos\gamma \tag{11.3}$$

where R_E is the radius of the earth and h (instead of z as in previous chapters) is the altitude of the rocket. Thus, in the direction of $\hat{\mathbf{u}}_t$ Newton's second law requires

$$T - D - mg\sin\gamma = ma_t \tag{11.4}$$

whereas in the $\hat{\mathbf{u}}_n$ direction

$$mg\cos\gamma = ma_n \tag{11.5}$$

After substituting Equations 11.2 and 11.3, these latter two expressions may be written

$$\frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g\sin\gamma \tag{11.6}$$

$$v\frac{d\gamma}{dt} = -\left(g - \frac{v^2}{R_E + h}\right)\cos\gamma \tag{11.7}$$

To these we must add the equations for downrange distance x and altitude h,

$$\frac{dx}{dt} = \frac{R_E}{R_E + h}v\cos\gamma \qquad \frac{dh}{dt} = v\sin\gamma \tag{11.8}$$

Recall that the variation of g with altitude is given by Equation 1.36. Numerical methods must be used to solve Equations 11.6, 11.7 and 11.8. To do so, one must account for the variation of the thrust, booster mass, atmospheric density, the drag coefficient, and the acceleration of gravity. Of course, the vehicle mass continuously decreases as propellants are consumed to produce the thrust, which we shall discuss in the following section.

The free body diagram in Figure 11.1 does not include a lifting force, which, if the vehicle were an airplane, would act normal to the velocity vector. Launch vehicles are designed to be strong in lengthwise compression, like a column. To save weight they are, unlike an airplane, made relatively weak in bending, shear and torsion, which are the kinds of loads induced by lifting surfaces. Transverse lifting loads are held closely to zero during powered ascent through the atmosphere by maintaining zero angle of attack, that is, by keeping the axis of the booster aligned with its velocity vector (the relative wind). Pitching maneuvers are done early in the launch, soon after the rocket clears the launch tower, when its speed is still low. At the high speeds acquired within a minute or so after launch, the slightest angle of attack can produce destructive transverse loads in the vehicle. The space shuttle orbiter has wings so it can act as a glider after reentry into the atmosphere. However, the launch configuration of the orbiter is such that its wings are at the zero-lift angle of attack throughout the ascent.

Satellite launch vehicles take off vertically and, at injection into orbit, must be flying parallel to the earth's surface. During the initial phase of the ascent, the rocket builds up speed on a nearly vertical trajectory taking it above the dense lower layers of the atmosphere. While it transitions the thinner upper atmosphere, the trajectory bends over, trading vertical speed for horizontal speed so the rocket can achieve orbital perigee velocity at burnout. The gradual transition from vertical to horizontal flight, illustrated in Figure 11.1 is caused by the force of gravity, and it is called a ***gravity turn trajectory***.

At lift off the rocket is vertical, and the flight path angle γ is 90°. After clearing the tower and gaining speed, vernier thrusters or gimbaling of the main engines produce a small, programmed pitchover, establishing an initial flight path angle γ_o, slightly less than 90°. Thereafter, γ will continue to decrease at a rate dictated by Equation 11.7. (For example, if $\gamma = 85°$, $v = 110\,\text{m/s}$ (250 mph), and $h = 2\,\text{km}$, then $d\gamma/dt = -0.44\,\text{deg/s}$.) As the speed v of the vehicle increases, the coefficient of $\cos\gamma$ in Equation 11.7 decreases, which means the rate of change of the flight path angle becomes increasingly smaller, tending towards zero as the booster approaches orbital speed, $v_{\text{circular orbit}} = \sqrt{g(R+h)}$. Ideally, the vehicle is flying horizontally ($\gamma = 0$) at that point.

The gravity turn trajectory is just one example of a practical trajectory, tailored for satellite boosters. On the other hand, sounding rockets fly straight up from launch through burnout. Rocket-powered guided missiles must execute high-speed pitch and yaw maneuvers as they careen towards moving targets, and require a rugged structure to withstand the accompanying side loads.

11.3 THE THRUST EQUATION

To discuss rocket performance requires an expression for the thrust T in Equation 11.6. It can be obtained by a simple one-dimensional momentum analysis. Figure 11.2(a) shows a system consisting of a rocket and its propellants. The exterior of the rocket is surrounded by the static pressure p_a of the atmosphere everywhere except at the rocket nozzle exit where the pressure is p_e. p_e acts over the nozzle exit area A_e. The value of p_e depends on the design of the nozzle. For simplicity, we assume no other forces act on the system. At time t the mass of the system is m and the absolute velocity in its axial direction is v. The propellants combine chemically in the rocket's combustion chamber, and during the small time interval Δt a small mass Δm of combustion products is forced out of the nozzle, to the left. Because of this expulsion, the velocity of the rocket changes by the small amount Δv, to the right. The absolute velocity of Δm is v_e, assumed to be to the left. According to Newton's second law of motion,

(momentum of the system at $t + \Delta t$) − (momentum of the system at t) = net external impulse

or

$$[(m - \Delta m)(v + \Delta v)\hat{\mathbf{i}} + \Delta m(-v_e\hat{\mathbf{i}})] - mv\hat{\mathbf{i}} = (p_e - p_a)A_e\Delta t\hat{\mathbf{i}} \tag{11.9}$$

Let $\dot{m}_e$ (a positive quantity) be the rate at which exhaust mass flows across the nozzle exit plane. The mass m of the rocket decreases at the rate dm/dt, and conservation of mass requires the decrease of mass to equal the mass flow rate out of the nozzle. Thus,

$$\frac{dm}{dt} = -\dot{m}_e \tag{11.10}$$

Assuming $\dot{m}_e$ is constant, the vehicle mass as a function of time (from $t = 0$) may therefore be written

$$m(t) = m_o - \dot{m}_e t \tag{11.11}$$

where m_o is the initial mass of the vehicle. Since Δm is the mass which flows out in the time interval Δt, we have

$$\Delta m = \dot{m}_e \Delta t \tag{11.12}$$

Let us substitute this expression into Equation 11.9 to obtain

$$[(m - \dot{m}_e \Delta t)(v + \Delta v)\hat{\mathbf{i}} + \dot{m}_e \Delta t(-v_e \hat{\mathbf{i}})] - mv\hat{\mathbf{i}} = (p_e - p_a)A_e \Delta t \hat{\mathbf{i}}$$

Collecting terms, we get

$$m\Delta v\hat{\mathbf{i}} - \dot{m}_e \Delta t(v_e + v)\hat{\mathbf{i}} - \dot{m}_e \Delta t \Delta v\hat{\mathbf{i}} = (p_e - p_a)A_e \Delta t \hat{\mathbf{i}}$$

Dividing through by Δt, taking the limit as $\Delta t \to 0$, and canceling the common unit vector lead to the equation

$$m\frac{dv}{dt} - \dot{m}_e c_a = (p_e - p_a)A_e \tag{11.13}$$

where c_a is the speed of the exhaust relative to the rocket,

$$c_a = v_e + v \tag{11.14}$$

Rearranging terms, Equation 11.13 may be written

$$\dot{m}_e c_a + (p_e - p_a)A_e = m\frac{dv}{dt} \tag{11.15}$$

The left-hand side of this equation is the unbalanced force responsible for the acceleration dv/dt of the system in Figure 11.2. This unbalanced force is the thrust T,

$$T = \dot{m}_e c_a + (p_e - p_a)A_e \tag{11.16}$$

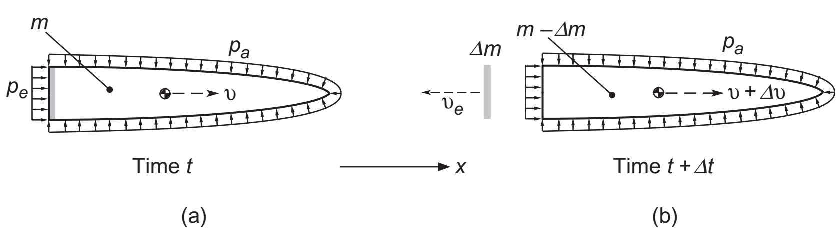


FIGURE 11.2

(a) System of rocket and propellant at time t. (b) The system an instant later, after ejection of a small element Δm of combustion products.

where $\dot{m}_e c_a$ is the jet thrust and $(p_e - p_a)A_e$ is the pressure thrust. We can write Equation 11.16 as

$$T = \dot{m}_e \left[c_a + \frac{(p_e - p_a)A_e}{\dot{m}_e} \right] \tag{11.17}$$

The term in brackets is called the ***effective exhaust velocity*** c,

$$c = c_a + \frac{(p_e - p_a)A_e}{\dot{m}_e} \tag{11.18}$$

In terms of the effective exhaust velocity, the thrust may be expressed simply as

$$T = \dot{m}_e c \tag{11.19}$$

The ***specific impulse*** I_{sp} is defined as the thrust per sea-level weight rate (per second) of propellant consumption. That is,

$$I_{sp} = \frac{T}{\dot{m}_e g_o} \tag{11.20}$$

where g_0 is the standard sea level acceleration of gravity. The unit of specific impulse is force ÷ (force/second) or seconds. Together, Equations 11.19 and 11.20 imply that

$$c = I_{sp} g_o \tag{11.21}$$

Obviously, one can infer the jet velocity directly from the specific impulse. Specific impulse is an important performance parameter for a given rocket engine and propellant combination. However, large specific impulse equates to large thrust only if the mass flow rate is large, which is true of chemical rocket engines. The specific impulse of chemical rockets typically lies in the range 200–300 s for solid fuels and 250–450 s for liquid fuels. Ion propulsion systems have very high specific impulse ($>10^4$ s), but their very low mass flow rates produce much smaller thrust than chemical rockets.

11.4 ROCKET PERFORMANCE

From Equations 11.10 and 11.20 we have

$$T = -I_{sp} g_o \frac{dm}{dt} \tag{11.22}$$

or

$$\frac{dm}{dt} = -\frac{T}{I_{sp} g_o}$$

If the thrust and specific impulse are constant, then the integral of this expression over the burn time Δt is

$$\Delta m = -\frac{T}{I_{sp} g_o} \Delta t$$

from which we obtain

$$\Delta t = \frac{I_{sp} g_o}{T}(m_o - m_f) = \frac{I_{sp} g_o}{T} m_o \left(1 - \frac{m_f}{m_o}\right) \tag{11.23}$$

where m_o and m_f are the mass of the vehicle at the beginning and end of the burn, respectively. The ***mass ratio*** is defined as the ratio of the initial mass to final mass,

$$n = \frac{m_o}{m_f} \tag{11.24}$$

Clearly, the mass ratio is always greater than unity. In terms of the mass ratio, Equation 11.23 may be written

$$\Delta t = \frac{n-1}{n} \frac{I_{sp}}{T/m_o g_o} \tag{11.25}$$

T/mg_o is the ***thrust-to-weight ratio***. The thrust-to-weight ratio for a launch vehicle at lift off is typically in the range 1.3 to 2.

Substituting Equation 11.22 into Equation 11.6, we get

$$\frac{dv}{dt} = -I_{sp} g_o \frac{dm/dt}{m} - \frac{D}{m} - g \sin\gamma$$

Integrating with respect to time, from t_o to t_f, yields

$$\Delta v = I_{sp} g_o \ln\frac{m_o}{m_f} - \Delta v_D - \Delta v_G \tag{11.26}$$

where the drag loss Δv_D and the gravity loss Δv_g are given by the integrals

$$\Delta v_D = \int_{t_o}^{t_f} \frac{D}{m} dt \qquad \Delta v_G = \int_{t_o}^{t_f} g \sin\gamma \, dt \tag{11.27}$$

Since the drag D, acceleration of gravity g, and flight path angle γ are unknown functions of time, these integrals cannot be computed. (Equations 11.6 through 11.8, together with 11.3, must be solved numerically to obtain $v(t)$ and $\gamma(t)$; but then Δv would follow from those results.) Equation 11.26 can be used for rough estimates where previous data and experience provide a basis for choosing conservative values of Δv_D and Δv_G. Obviously, if drag can be neglected, then $\Delta v_D = 0$. This would be a good approximation for the last stage of a satellite booster, for which it can also be said that $\Delta v_G = 0$, since $\gamma \cong 0°$ when the satellite is injected into orbit.

Sounding rockets are launched vertically and fly straight up to their maximum altitude before falling back to earth, usually by parachute. Their purpose is to measure remote portions of the earth's atmosphere. ("Sound" in this context means to measure or investigate.) If for a sounding rocket $\gamma = 90°$, then $\Delta v_G \approx g_o(t_f - t_0)$, since g is within 90% of g_o out to 300 km altitude.

Example 11.1

A sounding rocket of initial mass m_o and mass m_f after all propellant is consumed is launched vertically ($\gamma = 90°$). The propellant mass flow rate $\dot{m}_e$ is constant.

(a) Neglecting drag and the variation of gravity with altitude, calculate the maximum height h attained by the rocket.
(b) For what flow rate is the greatest altitude reached?

Solution

The vehicle mass as a function of time, up to burnout, is

$$m = m_o - \dot{m}_e t \tag{a}$$

At burnout, $m = m_f$, so the burnout time t_{bo} is

$$t_{bo} = \frac{m_o - m_f}{\dot{m}_e} \tag{b}$$

The drag loss is assumed zero, and the gravity loss is

$$\Delta v_G = \int_0^{t_{bo}} g_o \sin(90°)\, dt = g_o t$$

Recalling that $I_{sp} g_o = c$ and using (a), it follows from Equation 11.26 that, up to burnout, the velocity as a function of time is

$$v = c \ln \frac{m_o}{m_o - \dot{m}_e t} - g_o t \tag{c}$$

Since $dh/dt = v$, the altitude as a function of time is

$$h = \int_0^t v\, dt = \int_0^t \left(c \ln \frac{m_o}{m_o - \dot{m}_e t} - g_o t \right) dt = \frac{c}{\dot{m}_e} \left[(m_o - \dot{m}_e t) \ln \frac{m_o - \dot{m}_e t}{m_o} + \dot{m}_e t \right] - \frac{1}{2} g_o t^2 \tag{d}$$

The height at burnout h_{bo} is found by substituting (b) into this expression,

$$h_{bo} = \frac{c}{\dot{m}_e} \left(m_f \ln \frac{m_f}{m_o} + m_o - m_f \right) - \frac{1}{2} \left(\frac{m_o - m_f}{\dot{m}_e} \right)^2 g_o \tag{e}$$

Likewise, the burnout velocity is obtained by substituting (b) into (c),

$$v_{bo} = c \ln \frac{m_o}{m_f} - \frac{g_o}{\dot{m}_e} (m_o - m_f) \tag{f}$$

After burnout, the rocket coasts upward with the constant downward acceleration of gravity,

$$v = v_{bo} - g_o(t - t_{bo})$$
$$h = h_{bo} + v_{bo}(t - t_{bo}) - \frac{1}{2}g_o(t - t_{bo})^2$$

Substituting (b), (e) and (f) into these expressions yields, for $t > t_{bo}$,

$$v = c\ln\frac{m_o}{m_f} - g_o t$$
$$h = \frac{c}{\dot{m}_e}\left(m_o \ln\frac{m_f}{m_o} + m_o - m_f\right) + ct\ln\frac{m_o}{m_f} - \frac{1}{2}g_0 t^2 \tag{g}$$

The maximum height h_{max} is reached when $v = 0$,

$$c\ln\frac{m_o}{m_f} - g_o t_{max} = 0 \Rightarrow t_{max} = \frac{c}{g_o}\ln\frac{m_o}{m_f} \tag{h}$$

Substituting t_{max} into (g) leads to our result,

$$\boxed{h_{max} = \frac{1}{2}\frac{c^2}{g_o}\ln^2 n - \frac{cm_o}{\dot{m}_e}\frac{n\ln n - (n-1)}{n}} \tag{i}$$

where n is the mass ratio ($n > 1$). Since $n\ln n$ is greater than $n - 1$, it follows that the second term in this expression is positive. Hence, h_{max} can be increased by increasing the mass flow rate $\dot{m}_e$. In fact,

$$\boxed{\text{the greatest height is achieved when } \dot{m}_e \to \infty}$$

In that extreme, all of the propellant is expended at once, like a mortar shell.

Example 11.2

The data for a single stage rocket are as follows:

Launch mass: $m_o = 68{,}000\,\text{kg}$
Mass ratio: $n = 15$
Specific impulse: $I_{sp} = 390\,\text{s}$
Thrust: $T = 933.91\,\text{kN}$

It is launched into a vertical trajectory, like a sounding rocket. Neglecting drag and assuming that the gravitational acceleration is constant at its sea-level value $g_o = 9.81\,\text{m/s}^2$, calculate

(a) The time until burnout.
(b) The burnout altitude.
(c) The burnout velocity.
(d) The maximum altitude reached.

Solution

(a) From Example 11.1(b) the burnout time t_{bo} is

$$t_{bo} = \frac{m_o - m_f}{\dot{m}_e} \tag{a}$$

The burnout mass m_f is obtained from Equation 11.24,

$$m_f = \frac{m_o}{n} = \frac{68{,}000}{15} = 4533.3 \text{ kg} \tag{b}$$

The propellant mass flow rate $\dot{m}_e$ is given by Equation 11.20,

$$\dot{m}_e = \frac{T}{I_{sp} g_o} = \frac{933{,}913}{390 \cdot 9.81} = 244.10 \text{ kg/s} \tag{c}$$

Substituting (b), (c) and $m_o = 68{,}000$ kg into (a) yields the burnout time,

$$t_{bo} = \frac{68{,}000 - 4533.3}{244.10} = \boxed{260.0 \text{ s}}$$

(b) The burnout altitude is given by Example 11.1(e)

$$h_{bo} = \frac{c}{\dot{m}_e}\left(m_f \ln\frac{m_f}{m_o} + m_o - m_f\right) - \frac{1}{2}\left(\frac{m_o - m_f}{\dot{m}_e}\right)^2 g_o \tag{d}$$

The exhaust velocity c is found in Equation 11.21,

$$c = I_{sp} g_o = 390 \cdot 9.81 = 3825.9 \text{ m/s} \tag{e}$$

Substituting (b), (c) and (e), along with $m_o = 68{,}000$ kg and $g_o = 9.81 \text{ m/s}^2$ into (d), we get

$$h_{bo} = \frac{3825.9}{244.1}\left(4533.3 \ln\frac{4533.3}{68{,}000} + 68{,}000 - 4533.3\right) - \frac{1}{2}\left(\frac{68{,}000 - 4533.3}{244.1}\right)^2 \cdot 9.81$$

$$\boxed{h_{bo} = 470.74 \text{ km}}$$

(c) From Example 11.1(f) we find

$$v_{bo} = c \ln\frac{m_o}{m_f} - \frac{g_o}{\dot{m}_e}(m_o - m_f) = 3825.9 \ln\frac{68{,}000}{4533.3} - \frac{9.81}{244.1}(68{,}000 - 4533.3)$$

$$\boxed{v_{bo} = 7.810 \text{ km/s}}$$

(d) To find h_{max}, where the speed of the rocket falls to zero, we use Example 11.1(i),

$$h_{max} = \frac{1}{2}\frac{c^2}{g_o}\ln^2 n - \frac{cm_o}{\dot{m}_e}\frac{n\ln n - (n-1)}{n} = \frac{1}{2}\frac{3825.9^2}{9.81}\ln^2 15 - \frac{3825.9 \cdot 68{,}000}{244.1}\frac{15\ln 15 - (15-1)}{15}$$

$$\boxed{h_{max} = 3579.7 \text{ km}}$$

Notice that the rocket coasts to a height more than seven times the burnout altitude.

We can employ the integration schemes introduced in Section 1.8 to solve Equations 11.6 through 11.8 numerically. This permits a more accurate accounting of the effects of gravity and drag. It also yields the trajectory.

Example 11.3

The rocket in Example 11.2 has a diameter of 5 m. It is to be launched on a gravity turn trajectory. Pitchover begins at an altitude of 130 m with an initial flight path angle γ_o of 89.85°. What are the altitude h and speed v of the rocket at burnout ($t_{bo} = 260$ s)? What are the velocity losses due to drag and gravity (cf. Equations 11.27)?

Solution

The MATLAB® program *Example_11_03.m* in Appendix D.40 finds the speed v, the flight path angle γ, the altitude h and the downrange distance x as a function of time. It does so by using the ordinary differential equation solver *rkf_45.m* (Appendix D.4) to numerically integrate Equations 11.6 through 11.8, namely

$$\frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g\sin\gamma \tag{a}$$

$$\frac{d\gamma}{dt} = -\frac{1}{v}\left(g - \frac{v^2}{R_E + h}\right)\cos\gamma \tag{b}$$

$$\frac{dh}{dt} = v\sin\gamma \tag{c}$$

$$\frac{dx}{dt} = \frac{R_E}{R_E + h}v\cos\gamma \tag{d}$$

The variable mass m is given in terms of the initial mass $m_o = 68{,}000$ kg and the constant mass flow rate $\dot{m}_e$ by Equation 11.11,

$$m = m_o - \dot{m}_e t \tag{e}$$

The thrust $T = 933.913$ kN is assumed constant, and $\dot{m}_e$ is obtained from T and the specific impulse $I_{sp} = 390$ by means of Equation 11.20,

$$\dot{m}_e = \frac{T}{I_{sp} g_o} \tag{f}$$

The drag force D in (a) is given by Equation 11.1,

$$D = \frac{1}{2}\rho v^2 A C_D \tag{g}$$

The drag coefficient is assumed to have the constant value $C_D = 0.5$. The frontal area $A = \pi d^2/4$ is found from the rocket diameter $d = 5\,\text{m}$. The atmospheric density profile is assumed exponential,

$$\rho = \rho_o e^{-h/h_o} \tag{h}$$

where $\rho_o = 1.225\,\text{kg/m}^3$ is the sea level atmospheric density and $h_o = 7.5\,\text{km}$ is the ***scale height*** of the atmosphere. (The scale height is the altitude at which the density of the atmosphere is about 37% of its sea level value.)

Finally, the acceleration of gravity varies with altitude h according to Equation 1.36,

$$g = \frac{g_o}{(1 + h/R_E)^2} \qquad (R_E = 6378\ \text{km},\ g_o = 9.91\,\text{m/s}^2) \tag{i}$$

The drag loss and gravity loss are found by numerically integrating Equations 11.27.

Between lift off and pitchover, the flight path angle γ is held at 90°. Pitchover begins at the altitude $h_p = 130\,\text{m}$ with the flight path angle set at $\gamma_o = 89.85°$.

For the input data described above, the output of *Example_11_03.m* is as follows. The solution is very sensitive to the values of h_p and γ_o.

```
Initial flight path angle  =     89.85 deg
Pitchover altitude         =       130 m
Burn time                  =       260 s
Final speed                =   8.62116 km/s
Final flight path angle    =   8.80161 deg
Altitude                   =   133.211 km
Downrange distance         =   462.318 km
Drag loss                  =  0.298199 km/s
Gravity loss               =   1.44096 km/s
```

Thus, at burnout

Altitude	= 133.2 km
Speed	= 8.621 km/s

The speed losses are

Due to drag:	0.2982 km/s
Due to gravity:	1.441 km/s

Figure 11.3 shows the gravity-turn trajectory.

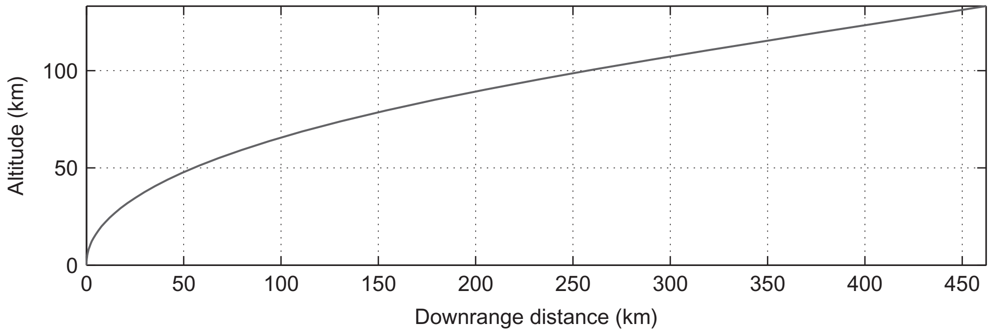


FIGURE 11.3

Gravity-turn trajectory for the data given in Examples 11.2 and 11.3.

11.5 RESTRICTED STAGING IN FIELD-FREE SPACE

In field-free space, we neglect drag and gravitational attraction. In that case, Equation 11.26 becomes

$$\Delta v = I_{sp} g_o \ln \frac{m_o}{m_f} \tag{11.28}$$

This is at best a poor approximation for high-thrust rockets, but it will suffice to shed some light on the rocket staging problem. Observe that we can solve this equation for the mass ratio to obtain

$$\frac{m_o}{m_f} = e^{\frac{\Delta v}{I_{sp} g_o}} \tag{11.29}$$

The amount of propellant expended to produce the velocity increment Δv is $m_o - m_f$. If we let $\Delta m = m_o - m_f$, then Equation 11.29 can be written as

$$\frac{\Delta m}{m_o} = 1 - e^{-\frac{\Delta v}{I_{sp} g_o}} \tag{11.30}$$

This relation is used to compute the propellant required to produce a given delta-v.

The gross mass m_o of a launch vehicle consists of the empty mass m_E, the propellant mass m_p and the payload mass m_{PL},

$$m_o = m_E + m_p + m_{PL} \tag{11.31}$$

The empty mass comprises the mass of the structure, the engines, fuel tanks, control systems, etc. m_E is also called the structural mass, although it embodies much more than just structure. Dividing Equation 11.31 through by m_o, we obtain

$$\pi_E + \pi_p + \pi_{PL} = 1 \tag{11.32}$$

where $\pi_E = m_E/m_o$, $\pi_p = m_p/m_o$ and $\pi_{PL} = m_{PL}/m_o$ are the structural fraction, propellant fraction and payload fraction, respectively. It is convenient to define the ***payload ratio***

$$\lambda = \frac{m_{PL}}{m_E + m_p} = \frac{m_{PL}}{m_o - m_{PL}} \tag{11.33}$$

and the ***structural ratio***

$$\varepsilon = \frac{m_E}{m_E + m_p} = \frac{m_E}{m_o - m_{PL}} \tag{11.34}$$

The mass ratio n was introduced in Equation 11.24. Assuming all of the propellant is consumed, that may now be written

$$n = \frac{m_E + m_p + m_{PL}}{m_E + m_{PL}} \tag{11.35}$$

λ, ε and n are not independent. From Equation 11.34 we have

$$m_E = \frac{\varepsilon}{1-\varepsilon} m_p \tag{11.36}$$

whereas Equation 11.33 gives

$$m_{PL} = \lambda(m_E + m_p) = \lambda\left(\frac{\varepsilon}{1-\varepsilon} m_p + m_p\right) = \frac{\lambda}{1-\varepsilon} m_p \tag{11.37}$$

Substituting Equations 11.36 and 11.37 into Equation 11.35 leads to

$$n = \frac{1+\lambda}{\varepsilon + \lambda} \tag{11.38}$$

Thus, given any two of the ratios λ, ε and n, we obtain the third from Equation 11.38. Using this relation in Equation 11.28 and setting Δv equal to the burnout speed v_{bo}, when the propellants have been used up, yields

$$v_{bo} = I_{sp} g_o \ln n = I_{sp} g_o \ln \frac{1+\lambda}{\varepsilon + \lambda} \tag{11.39}$$

This equation is plotted in Figure 11.4 for a range of structural ratios. Clearly, for a given empty mass, the greatest possible Δv occurs when the payload is zero. However, what we want to do is maximize the amount of payload while keeping the structural weight to a minimum. Of course, the mass of load-bearing structure, rocket motors, pumps, piping, etc., cannot be made arbitrarily small. Current materials technology places a lower limit on ε of about 0.1. For this value of the structural ratio and $\lambda = 0.05$, Equation 11.39 yields

$$v_{bo} = 1.94 I_{sp} g_o = 0.019 I_{sp} \text{ (km/s)}$$

The specific impulse of a typical chemical rocket is about 300 s, which in this case would provide $\Delta v = 5.7$ km/s. However, the circular orbital velocity at the earth's surface is 7.905 km/s. Therefore, this

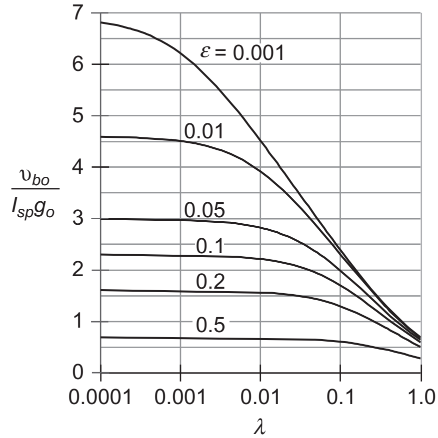


FIGURE 11.4

Dimensionless burnout speed versus payload ratio.

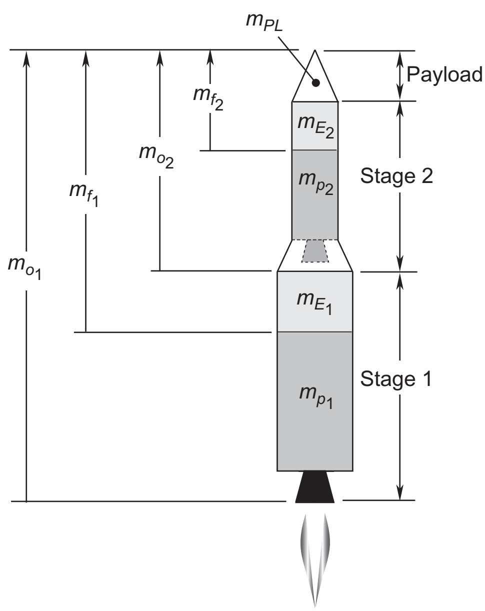


FIGURE 11.5

Tandem two-stage booster.

booster by itself could not orbit the payload. The minimum specific impulse required for a single stage to orbit would be 416 s. Only today's most advanced liquid hydrogen/liquid oxygen engines, e.g., the space shuttle main engines, have this kind of performance. Practicality and economics would likely dictate going the route of a multistage booster.

Figure 11.5 shows a series or tandem two-stage rocket configuration, with one stage sitting on top of the other. Each stage has its own engines and propellant tanks. The dividing line between the stages is where

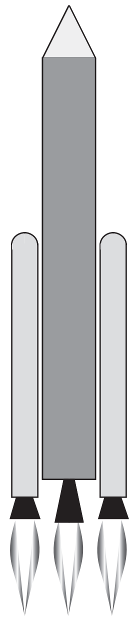


FIGURE 11.6

Parallel staging.

they separate during flight. The first stage drops off first, the second stage next, etc. The payload of an N stage rocket is actually stage $N + 1$. Indeed, satellites commonly carry their own propulsion systems into orbit. The payload of a given stage is everything above it. Therefore, as illustrated in Figure 11.5, the initial mass m_{o1} of stage 1 is that of the entire vehicle. After stage 1 expels all of its fuel, the mass m_{f1} that remains is stage 1's empty mass m_{E1} plus the mass of stage 2 and the payload. After separation of stage 1, the process continues likewise for stage 2, with m_{o2} being its initial mass.

Titan II, the launch vehicle for the Gemini program, had the two-stage, tandem configuration. So did the Saturn 1B, used to launch earth orbital flights early in the Apollo program, as well as to send crews to Skylab and an Apollo spacecraft to dock with a Russian Soyuz in 1975.

Figure 11.6 illustrates the concept of parallel staging. Two or more solid or liquid rockets are attached ("strapped on") to a core vehicle carrying the payload. Whereas in the tandem arrangement, the motors in a given stage cannot ignite until separation of the previous stage, all of the rockets ignite at once in the parallel-staged vehicle. The strap-on boosters fall away after they burn out early in the ascent. The space shuttle is the most obvious example of parallel staging. Its two solid rocket boosters are mounted on the external tank, which fuels the three "main" engines built into the orbiter. The solid rocket boosters and the external tank are cast off after they are depleted. In more common use is the combination of parallel and tandem staging, in which boosters are strapped to the first stage of a multistage stack. Examples include the Titan III and IV, Delta, Ariane, Soyuz, Proton, Zenith and H-1.

The venerable Atlas, used in many variants, including the early D model which launched the orbital flights of the Mercury program, had three main liquid-fuel engines at its base. They all fired simultaneously at launch, but several minutes into the flight, the outer two "boosters" dropped away, leaving the central sustainer engine to burn the rest of the way to orbit. Since the booster engines shared the sustainer's propellant tanks, the Atlas exhibited partial staging, and is sometimes referred to as a one and a half stage rocket.

We will for simplicity focus on tandem staging, although parallel-staged systems are handled in a similar way (Wiesel, 1997). Restricted staging involves the simple but unrealistic assumption of ***similar stages***. That is, each stage has the same specific impulse I_{sp}, the same structural ratio ε, and the same payload ratio λ. From Equation 11.38 it follows that the mass ratios n are identical, too. Let us investigate the effect of restricted staging on the final burnout speed v_{bo} for a given payload mass m_{PL} and overall payload fraction

$$\pi_{PL} = \frac{m_{PL}}{m_o} \tag{11.40}$$

where m_o is the total mass of the tandem-stacked vehicle.

For a single stage vehicle, the payload ratio is

$$\lambda = \frac{m_{PL}}{m_o - m_{PL}} = \frac{1}{\dfrac{m_o}{m_{PL}} - 1} = \frac{\pi_{PL}}{1 - \pi_{PL}} \tag{11.41}$$

so that, from Equation 11.38, the mass ratio is

$$n = \frac{1}{\pi_{PL}(1-\varepsilon) + \varepsilon} \tag{11.42}$$

According to Equation 11.39, the burnout speed is

$$v_{bo} = I_{sp} g_o \ln \frac{1}{\pi_{PL}(1-\varepsilon) + \varepsilon} \tag{11.43}$$

Let m_o be the total mass of the two-stage rocket of Figure 11.5, that is,

$$m_o = m_{o1} \tag{11.44}$$

The payload of stage 1 is the entire mass m_{o2} of stage 2. Thus, for stage 1 the payload ratio is

$$\lambda_1 = \frac{m_{o2}}{m_{o1} - m_{o2}} = \frac{m_{o2}}{m_o - m_{o2}} \tag{11.45}$$

The payload ratio of stage 2 is

$$\lambda_2 = \frac{m_{PL}}{m_{o2} - m_{PL}} \tag{11.46}$$

By virtue of the two stages being similar, $\lambda_1 = \lambda_2$, or

$$\frac{m_{o2}}{m_o - m_{o2}} = \frac{m_{PL}}{m_{o2} - m_{PL}}$$

Solving this equation for m_{o2} yields

$$m_{o2} = \sqrt{m_o}\sqrt{m_{PL}}$$

But $m_o = m_{PL}/\pi_{PL}$, so the gross mass of the second stage is

$$m_{o2} = \sqrt{\frac{1}{\pi_{PL}}}m_{PL} \tag{11.47}$$

Putting this back into Equation 11.45 (or 11.46), we obtain the common two-stage payload ratio $\lambda = \lambda_1 = \lambda_2$,

$$\lambda_{2-\text{stage}} = \frac{\pi_{PL}{}^{1/2}}{1 - \pi_{PL}{}^{1/2}} \tag{11.48}$$

This together with Equation 11.38 and the assumption that $\varepsilon_1 = \varepsilon_2 = \varepsilon$ leads to the common mass ratio for each stage,

$$n_{2-\text{stage}} = \frac{1}{\pi_{PL}{}^{1/2}(1-\varepsilon) + \varepsilon} \tag{11.49}$$

If stage 2 ignites immediately after burnout of stage 1, the final velocity of the two-stage vehicle is the sum of the burnout velocities of the individual stages,

$$v_{bo} = v_{bo1} + v_{bo2}$$

or

$$v_{bo_{2-\text{stage}}} = I_{sp}g_o \ln n_{2-\text{stage}} + I_{sp}g_o \ln n_{2-\text{stage}} = 2I_{sp}g_o \ln n_{2-\text{stage}}$$

so that, with Equation 11.49, we get

$$v_{bo_{2-\text{stage}}} = I_{sp}g_o \ln\left[\frac{1}{\pi_{PL}{}^{1/2}(1-\varepsilon) + \varepsilon}\right]^2 \tag{11.50}$$

The empty mass of each stage can be found in terms of the payload mass using the common structural ratio ε,

$$\frac{m_{E1}}{m_{o1} - m_{o2}} = \varepsilon \qquad \frac{m_{E2}}{m_{o2} - m_{PL}} = \varepsilon$$

Substituting Equations 11.40 and 11.44 together with 11.47 yields

$$m_{E1} = \frac{(1 - \pi_{PL}{}^{1/2})\varepsilon}{\pi_{PL}}m_{PL} \quad m_{E2} = \frac{(1 - \pi_{PL}{}^{1/2})\varepsilon}{\pi_{PL}{}^{\frac{1}{2}}}m_{PL} \tag{11.51}$$

Likewise, we can find the propellant mass for each stage from the expressions

$$m_{p1} = m_{o1} - (m_{E1} + m_{o2}) \quad m_{p2} = m_{o2} - (m_{E2} + m_{PL}) \tag{11.52}$$

Substituting Equations 11.40 and 11.44, together with 11.47 and 11.51, we get

$$m_{p1} = \frac{(1 - \pi_{PL}{}^{1/2})(1 - \varepsilon)}{\pi_{PL}} m_{PL} \qquad m_{p2} = \frac{(1 - \pi_{PL}{}^{1/2})(1 - \varepsilon)}{\pi_{PL}{}^{1/2}} m_{PL} \tag{11.53}$$

Example 11.4

The following data are given

$$\begin{aligned} m_{PL} &= 10{,}000 \text{ kg} \\ \pi_{\text{PL}} &= 0.05 \\ \varepsilon &= 0.15 \\ I_{sp} &= 350 \text{ s} \\ g_o &= 0.00981 \text{ km/s}^2 \end{aligned} \tag{a}$$

Calculate the payload velocity v_{bo} at burnout, the empty mass of the launch vehicle and the propellant mass for (a) a single stage and (b) a restricted, two-stage vehicle.

Solution

(a) From Equation 11.43 we find

$$v_{bo} = 350 \cdot 0.00981 \ln \frac{1}{0.05(1 + 0.15) + 0.15} = \boxed{5.657 \text{ km/s}}$$

Equation 11.40 yields the gross mass

$$m_o = \frac{10{,}000}{0.05} = 200{,}000 \text{ kg}$$

from which we obtain the empty mass using Equation 11.34,

$$m_E = \varepsilon(m_o - m_{PL}) = 0.15(200{,}000 - 10{,}000) = \boxed{28{,}500 \text{ kg}}$$

The mass of propellant is

$$m_p = m_o - m_E - m_{PL} = 200{,}000 - 28{,}500 - 10{,}000 = \boxed{161{,}500 \text{ kg}}$$

(b) For a restricted two-stage vehicle, the burnout speed is given by Equation 11.50,

$$v_{bo_{2-\text{stage}}} = 350 \cdot 0.00981 \ln \left[\frac{1}{0.05^{1/2}(1 - 0.15) + 0.15} \right]^2 = \boxed{7.407 \text{ km/s}}$$

The empty mass of each stage is found using Equations 11.51,

$$m_{E1} = \frac{(1-0.05^{1/2})\cdot 0.15}{0.05}\cdot 10{,}000 = \boxed{23{,}292\ \text{kg}}$$

$$m_{E2} = \frac{(1-0.05^{1/2})\cdot 0.15}{0.05^{1/2}}\cdot 10{,}000 = \boxed{5208\ \text{kg}}$$

For the propellant masses, we turn to Equations 11.53

$$m_{p1} = \frac{(1-0.05^{1/2})\cdot(1-0.15)}{0.05}\cdot 10{,}000 = \boxed{131{,}990\ \text{kg}}$$

$$m_{p2} = \frac{(1-0.05^{1/2})\cdot(1-0.15)}{0.05^{1/2}}\cdot 10{,}000 = \boxed{29{,}513\ \text{kg}}$$

The total empty mass, $m_E = m_{E1} + m_{E2}$, and the total propellant mass, $m_p = m_{p1} + m_{p2}$ are the same as for the single stage rocket. The mass of the second stage, including the payload, is 22.4% of the total vehicle mass.

Observe in the previous example that, although the total vehicle mass was unchanged, the burnout velocity increased 31% for the two-stage arrangement. The reason is that the second stage is lighter and can therefore be accelerated to a higher speed. Let us determine the velocity gain associated with adding another stage, as illustrated in Figure 11.7.

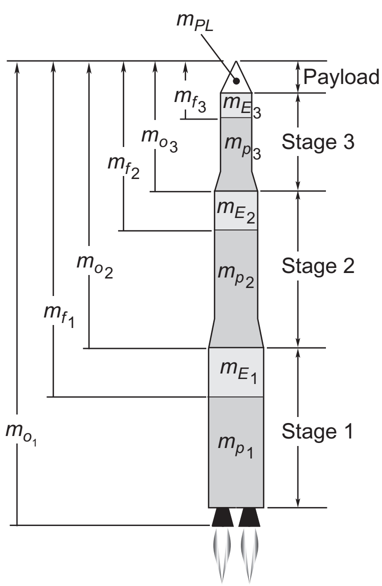


FIGURE 11.7

Tandem three-stage launch vehicle.

The payload ratios of the three stages are

$$\lambda_1 = \frac{m_{o2}}{m_{o1} - m_{o2}} \qquad \lambda_2 = \frac{m_{o3}}{m_{o2} - m_{o3}} \qquad \lambda_3 = \frac{m_{PL}}{m_{o3} - m_{PL}}$$

Since the stages are similar, these payload ratios are all the same. Setting $\lambda_1 = \lambda_2$ and recalling that $m_{o1} = m_o$, we find

$$m_{o2}{}^2 - m_{o3}m_o = 0$$

Similarly, $\lambda_1 = \lambda_3$ yields

$$m_{o2}m_{o3} - m_o m_{PL} = 0$$

These two equations imply that

$$m_{o2} = \frac{m_{PL}}{\pi_{PL}{}^{2/3}} \qquad m_{o3} = \frac{m_{PL}}{\pi_{PL}{}^{1/3}} \tag{11.54}$$

Substituting these results back into any one of the above expressions for λ_1, λ_2 or λ_3 yields the common payload ratio for the restricted three-stage rocket,

$$\lambda_{3-\text{stage}} = \frac{\pi_{PL}{}^{1/3}}{1 - \pi_{PL}{}^{1/3}}$$

With this result and Equation 11.38, we find the common mass ratio,

$$n_{3-\text{stage}} = \frac{1}{\pi_{PL}{}^{1/3}(1 - \varepsilon) + \varepsilon} \tag{11.55}$$

Since the payload burnout velocity is $v_{bo} = v_{bo1} + v_{bo2} + v_{bo3}$, we have

$$v_{bo3-\text{stage}} = 3I_{sp}g_o \ln n_{3-\text{stage}} = I_{sp}g_o \ln\left(\frac{1}{\pi_{PL}{}^{1/3}(1 - \varepsilon) + \varepsilon}\right)^3 \tag{11.56}$$

Because of the common structural ratio across each stage,

$$\frac{m_{E1}}{m_{o1} - m_{o2}} = \varepsilon \qquad \frac{m_{E2}}{m_{o2} - m_{o3}} = \varepsilon \qquad \frac{m_{E3}}{m_{o3} - m_{PL}} = \varepsilon$$

Substituting Equations 11.40 and 11.54 and solving the resultant expressions for the empty stage masses yields

$$m_{E1} = \frac{(1 - \pi_{PL}{}^{1/3})\varepsilon}{\pi_{PL}} m_{PL} \qquad m_{E2} = \frac{(1 - \pi_{PL}{}^{1/3})\varepsilon}{\pi_{PL}{}^{2/3}} m_{PL} \qquad m_{E3} = \frac{(1 - \pi_{PL}{}^{1/3})\varepsilon}{\pi_{PL}{}^{1/3}} m_{PL} \tag{11.57}$$

The stage propellant masses are

$$m_{p1} = m_{o1} - (m_{E1} + m_{o2}) \qquad m_{p2} = m_{o2} - (m_{E2} + m_{o3}) \qquad m_{p3} = m_{o3} - (m_{E3} + m_{PL})$$

Substituting Equations 11.40, 11.54 and 11.57 leads to

$$\begin{aligned} m_{p1} &= \frac{(1 - \pi_{PL}{}^{1/3})(1-\varepsilon)}{\pi_{PL}} m_{PL} \\ m_{p2} &= \frac{(1 - \pi_{PL}{}^{1/3})(1-\varepsilon)}{\pi_{PL}{}^{2/3}} m_{PL} \\ m_{p3} &= \frac{(1 - \pi_{PL}{}^{1/3})(1-\varepsilon)}{\pi_{PL}{}^{1/3}} m_{PL} \end{aligned} \tag{11.58}$$

Example 11.5

Repeat Example 11.4 for the restricted three-stage launch vehicle.

Solution

Equation 11.56 gives the burnout velocity for three stages,

$$v_{bo} = 350 \cdot 0.00981 \cdot \ln\left(\frac{1}{0.05^{1/3}(1-0.15)+0.15}\right)^3 = \boxed{7.928 \text{ km/s}}$$

Substituting $m_{PL} = 10{,}000\,\text{kg}$, $\pi_{\text{PL}} = 0.05$ and $\varepsilon = 0.15$ into Equations 11.57 and 11.58 yields

$$\boxed{\begin{array}{lll} m_{E1} = 18{,}948 \text{ kg} & m_{E2} = 6980 \text{ kg} & m_{E3} = 2572 \text{ kg} \\ m_{p1} = 107{,}370 \text{ kg} & m_{p2} = 39{,}556 \text{ kg} & m_{p3} = 14{,}573 \text{ kg} \end{array}}$$

Again, the total empty mass and total propellant mass are the same as for the single and two-stage vehicles. Notice that the velocity increase over the two-stage rocket is just 7%, which is much less than the advantage the two-stage had over the single stage vehicle.

Looking back over the velocity formulas for one, two and three stage vehicles (Equations 11.43, 11.50 and 11.56), we can induce that for a N-stage rocket,

$$v_{bo_{N-\text{stage}}} = I_{sp} g_o \ln\left(\frac{1}{\pi_{PL}{}^{1/N}(1-\varepsilon)+\varepsilon}\right)^N = I_{sp} g_o N \ln\left(\frac{1}{\pi_{PL}{}^{1/N}(1-\varepsilon)+\varepsilon}\right) \tag{11.59}$$

What happens as we let N become very large? First of all, it can be shown using Taylor series expansion that, for large N,

$$\pi_{PL}{}^{1/N} \approx 1 + \frac{1}{N}\ln \pi_{PL} \tag{11.60}$$

Substituting this into Equation 11.59, we find that

$$v_{bo_{N-\text{stage}}} \approx I_{sp} g_o N \ln\left[\frac{1}{1+\frac{1}{N}(1-\varepsilon)\ln \pi_{PL}}\right]$$

Since the term $(1/N)(1 - \varepsilon)\ln\pi_{PL}$ is arbitrarily small, we can use the fact that

$$1/(1+x) = 1 - x + x^2 - x^3 +$$

to write

$$\frac{1}{1+\frac{1}{N}(1-\varepsilon)\ln \pi_{PL}} \approx 1 - \frac{1}{N}(1-\varepsilon)\ln \pi_{PL}$$

which means

$$v_{bo_{N-\text{stage}}} \approx I_{sp} g_o N \ln\left[1 - \frac{1}{N}(1-\varepsilon)\ln \pi_{PL}\right]$$

Finally, since $\ln(1 - x) = -x - x^2/2 - x^3/3 - x^4/4 - \ldots$, we can write this as

$$v_{bo_{N-\text{stage}}} \approx I_{sp} g_o N\left[-\frac{1}{N}(1-\varepsilon)\ln \pi_{PL}\right]$$

Therefore, as N, the number of stages, tends towards infinity, the burnout velocity approaches

$$v_{bo\infty} = I_{sp} g_o (1-\varepsilon)\ln\frac{1}{\pi_{PL}} \tag{11.61}$$

Thus, no matter how many similar stages we use, for a given specific impulse, payload fraction and structural ratio, we cannot exceed this burnout speed. For example, using $I_{sp} = 350\,\text{s}$, $\pi_{PL} = 0.05$ and $\varepsilon = 0.15$ from the previous two examples yields $v_{bo\infty} = 8.743\,\text{km/s}$, which is only 10% greater than v_{bo} of a three-stage vehicle. The trend of v_{bo} towards this limiting value is illustrated by Figure 11.8.

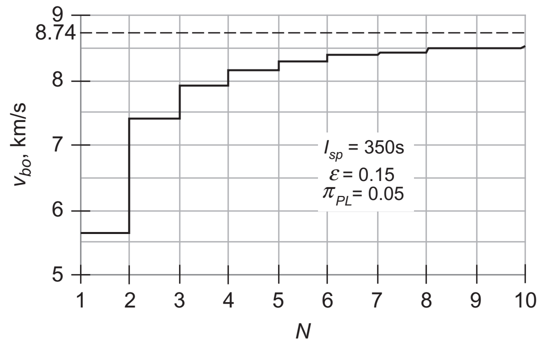


FIGURE 11.8

Burnout velocity versus number of stages (Equation 11.59).

Our simplified analysis does not take into account the added weight and complexity accompanying additional stages. Practical reality has limited the number of stages of actual launch vehicles to rarely more than three.

11.6 OPTIMAL STAGING

Let us now abandon the restrictive assumption that all stages of a tandem-stacked vehicle are similar. Instead, we will specify the specific impulse I_{sp_i} and structural ratio ε_i of each stage, and then seek the minimum-mass N-stage vehicle that will carry a given payload m_{PL} to a specified burnout velocity v_{bo}. To optimize the mass requires using the Lagrange multiplier method, which we shall briefly review.

11.6.1 Lagrange multiplier

Consider a bivariate function f on the xy plane. Then $z = f(x,y)$ is a surface lying above or below the plane, or both. $f(x,y)$ is stationary at a given point if it takes on local maximum or a local minimum, i.e., an extremum, at that point. For f to be stationary means $df = 0$; that is,

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \tag{11.62}$$

where dx and dy are independent and not necessarily zero. It follows that for an extremum to exist,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{11.63}$$

Now let $g(x,y) = 0$ be a curve in the xy plane. Let us find the points on the curve $g = 0$ at which f is stationary. That is, rather than searching the entire xy plane for extreme values of f, we confine our attention to the curve $g = 0$, which is therefore a constraint. Since $g = 0$, it follows that $dg = 0$, or

$$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \tag{11.64}$$

If Equations 11.62 and 11.64 are both valid at a given point, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\partial g/\partial x}{\partial g/\partial y}$$

That is,

$$\frac{\partial f/\partial x}{\partial g/\partial x} = \frac{\partial f/\partial y}{\partial g/\partial y} = -\eta$$

From this we obtain

$$\frac{\partial f}{\partial x} + \eta\frac{\partial g}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} + \eta\frac{\partial g}{\partial y} = 0$$

But these, together with the constraint $g(x,y) = 0$, are the very conditions required for the function

$$h(x, y, \eta) = f(x, y) + \eta g(x, y) \tag{11.65}$$

to have an extremum, namely,

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial f}{\partial x} + \eta \frac{\partial g}{\partial x} = 0 \\ \frac{\partial h}{\partial y} &= \frac{\partial f}{\partial y} + \eta \frac{\partial g}{\partial y} = 0 \\ \frac{\partial h}{\partial \eta} &= g = 0 \end{aligned} \tag{11.66}$$

η is the Lagrange multiplier. The procedure generalizes to functions of any number of variables.

One can determine mathematically whether the extremum is a maximum or a minimum by checking the sign of the second differential d^2h of the function h in Equation 11.65,

$$d^2h = \frac{\partial^2 h}{\partial x^2} dx^2 + 2 \frac{\partial^2 h}{\partial x \partial y} dxdy + \frac{\partial^2 h}{\partial y^2} dy^2 \tag{11.67}$$

If $d^2h < 0$ at the extremum for all dx and dy satisfying the constraint condition, Equation 11.64, then the extremum is a local maximum. Likewise, if $d^2h > 0$, then the extremum is a local minimum.

Example 11.6

(a) Find the extrema of the function $z = -x^2 - y^2$.

(b) Find the extrema of the same function under the constraint $y = 2x + 3$.

Solution

(a) To find the extrema we must use Equations 11.63. Since $\partial z/\partial x = -2x$ and $\partial z/\partial y = -2y$, it follows that $\partial z/\partial x = \partial z/\partial y = 0$ at $x = y = 0$, at which point $z = 0$. Since z is negative everywhere else (see Figure 11.9), it is clear that

$$\boxed{\text{the extreme value, } z = 0\text{, is the maximum value}}$$

(b) The constraint may be written $g = y - 2x - 3$. Clearly, $g = 0$. Multiply the constraint by the Lagrange multiplier η and add the result (zero!) to the function $-(x^2 + y^2)$ to obtain

$$h = -(x^2 + y^2) + \eta(y - 2x - 3)$$

This is a function of the three variables x, y and η. For it to be stationary, the partial derivatives with respect to all three of these variables must vanish. First we have

$$\frac{\partial h}{\partial x} = -2x - 2\eta$$

Setting this equal to zero yields

$$x = -\eta \tag{a}$$

Next,

$$\frac{\partial h}{\partial y} = -2y + \eta$$

For this to be zero means

$$y = \frac{\eta}{2} \tag{b}$$

Finally

$$\frac{\partial h}{\partial \eta} = y - 2x - 3$$

Setting this equal to zero gives us back the constraint condition,

$$y - 2x - 3 = 0 \tag{c}$$

Substituting (a) and (b) into (c) yields $\eta = 1.2$, from which (a) and (b) imply,

$$x = -1.2 \qquad y = 0.6 \tag{d}$$

These are the coordinates of the point on the line $y = 2x + 3$ at which $z = -x^2 - y^2$ is stationary. Using (d), we find that

$$\boxed{z = -1.8}$$

at this point.

Figure 11.9 is an illustration of this problem, and it shows that the computed extremum (a maximum, in the sense that small negative numbers exceed large negative numbers) is where the surface $z = -x^2 - y^2$ is closest to the line $y = 2x + 3$, as measured in the z-direction. Note that in this case, Equation 11.67 yields $d^2h = -2dx^2 - 2dy^2$, which is negative, confirming our conclusion that the extremum is a maximum.

Now let us return to the optimal staging problem. It is convenient to introduce the ***step mass*** m_i of the ith stage. The step mass is the empty mass plus the propellant mass of the stage, exclusive of all the other stages,

$$m_i = m_{Ei} + m_{pi} \tag{11.68}$$

The empty mass of stage i can be expressed in terms of its step mass and its structural ratio ε_i as follows,

$$m_{Ei} = \varepsilon_i(m_{E_i} + m_{pi}) = \varepsilon_i m_i \tag{11.69}$$

The total mass of the rocket excluding the payload is M, which is the sum of all of the step masses,

$$M = \sum_{i=1}^{N} m_i \tag{11.70}$$

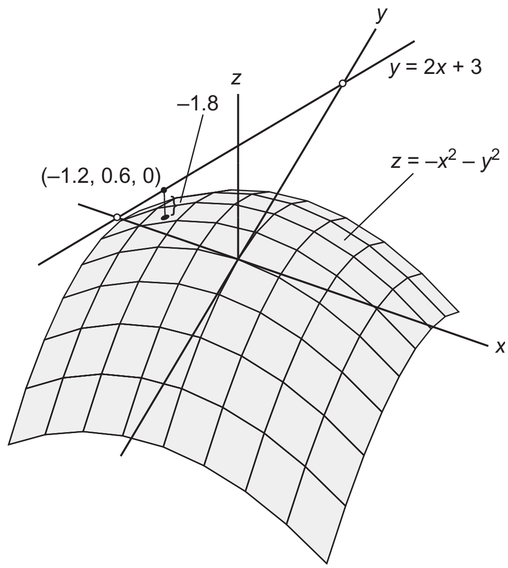


FIGURE 11.9

Location of the point on the line $y = 2x + 3$ at which the surface $z = -x^2 - y^2$ is closest to the xy plane.

Thus, recalling that m_o is the total mass of the vehicle, we have

$$m_o = M + m_{PL} \tag{11.71}$$

Our goal is to minimize m_o.

For simplicity, we will deal first with a two-stage rocket, and then generalize our results to N stages. For a two-stage vehicle, $m_o = m_1 + m_2 + m_{PL}$, so we can write,

$$\frac{m_o}{m_{PL}} = \frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} \frac{m_2 + m_{PL}}{m_{PL}} \tag{11.72}$$

The mass ratio of stage 1 is

$$n_1 = \frac{m_{o1}}{m_{E1} + m_2 + m_{PL}} = \frac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}} \tag{11.73}$$

where Equation 11.69 was used. Likewise, the mass ratio of stage 2 is

$$n_2 = \frac{m_{o2}}{\varepsilon_2 m_2 + m_{PL}} = \frac{m_2 + m_{PL}}{\varepsilon_2 m_2 + m_{PL}} \tag{11.74}$$

We can solve Equations 11.73 and 11.74 to obtain the step masses from the mass ratios,

$$
\begin{aligned}
m_2 &= \frac{n_2 - 1}{1 - n_2\varepsilon_2} m_{PL} \\
m_1 &= \frac{n_1 - 1}{1 - n_1\varepsilon_1}(m_2 + m_{PL})
\end{aligned} \tag{11.75}
$$

Now,

$$
\frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} = \frac{1 - \varepsilon_1}{1 - \varepsilon_1} \frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL} + (\varepsilon_1 m_1 - \varepsilon_1 m_1)} \frac{\dfrac{1}{\varepsilon_1 m_1 + m_2 + m_{PL}}}{\dfrac{1}{\varepsilon_1 m_1 + m_2 + m_{PL}}}
$$

These manipulations do not alter the ratio on the left. Carrying out the multiplications proceeds as follows,

$$
\begin{aligned}
\frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} &= \frac{(1 - \varepsilon_1)(m_1 + m_2 + m_{PL})}{\varepsilon_1 m_1 + m_2 + m_{PL} - \varepsilon_1(m_1 + m_2 + m_{PL})} \frac{\dfrac{1}{\varepsilon_1 m_1 + m_2 + m_{PL}}}{\dfrac{1}{\varepsilon_1 m_1 + m_2 + m_{PL}}} \\
&= \frac{(1 - \varepsilon_1)\dfrac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}}}{\dfrac{\varepsilon_1 m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}} - \varepsilon_1 \dfrac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}}}
\end{aligned}
$$

Finally, with the aid of Equation 11.73, this algebraic trickery reduces to

$$
\frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} = \frac{(1 - \varepsilon_1)n_1}{1 - \varepsilon_1 n_1} \tag{11.76}
$$

Likewise,

$$
\frac{m_2 + m_{PL}}{m_{PL}} = \frac{(1 - \varepsilon_2)n_2}{1 - \varepsilon_2 n_2} \tag{11.77}
$$

so that Equation 11.72 may be written in terms of the stage mass ratios instead of the step masses,

$$
\frac{m_o}{m_{PL}} = \frac{(1 - \varepsilon_1)n_1}{1 - \varepsilon_1 n_1} \frac{(1 - \varepsilon_2)n_2}{1 - \varepsilon_2 n_2} \tag{11.78}
$$

Taking the natural logarithm of both sides of this equation, we get

$$
\ln \frac{m_o}{m_{PL}} = \ln \frac{(1 - \varepsilon_1)n_1}{1 - \varepsilon_1 n_1} + \ln \frac{(1 - \varepsilon_2)n_2}{1 - \varepsilon_2 n_2}
$$

Expanding the logarithms on the right side leads to

$$\ln\frac{m_o}{m_{PL}} = \left[\ln(1-\varepsilon_1) + \ln n_1 - \ln(1-\varepsilon_1 n_1)\right] + \left[\ln(1-\varepsilon_2) + \ln n_2 - \ln(1-\varepsilon_2 n_2)\right] \tag{11.79}$$

Observe that for m_{PL} fixed, $\ln(m_o/m_{PL})$ is a monotonically increasing function of m_o,

$$\frac{d}{dm_o}\left(\ln\frac{m_o}{m_{PL}}\right) = \frac{1}{m_o} > 0$$

Therefore, $\ln(m_o/m_{PL})$ is stationary when m_o is.

From Equations 11.21 and 11.39, the burnout velocity of the two-stage rocket is

$$v_{bo} = v_{bo1} + v_{bo2} = c_1 \ln n_1 + c_2 \ln n_2 \tag{11.80}$$

which means that, given v_{bo}, our constraint equation is

$$v_{bo} - c_1 \ln n_1 - c_2 \ln n_2 = 0 \tag{11.81}$$

Introducing the Lagrange multiplier η, we combine Equations 11.79 and 11.81 to obtain

$$\begin{aligned} h = {} & \left[\ln(1-\varepsilon_1) + \ln n_1 - \ln(1-\varepsilon_1 n_1)\right] + \left[\ln(1-\varepsilon_2) + \ln n_2 - \ln(1-\varepsilon_2 n_2)\right] \\ & + \eta(v_{bo} - c_1 \ln n_1 - c_2 \ln n_2) \end{aligned} \tag{11.82}$$

Finding the values of n_1 and n_2 for which h is stationary will extremize $\ln(m_o/m_{PL})$ (and, hence, m_o) for the prescribed burnout velocity v_{bo}. h is stationary when $\partial h/\partial n_1 = \partial h/\partial n_2 = \partial h/\partial \eta = 0$. Thus,

$$\frac{\partial h}{\partial n_1} = \frac{1}{n_1} + \frac{\varepsilon_1}{1-\varepsilon_1 n_1} - \eta\frac{c_1}{n_1} = 0$$

$$\frac{\partial h}{\partial n_2} = \frac{1}{n_2} + \frac{\varepsilon_2}{1-\varepsilon_2 n_2} - \eta\frac{c_2}{n_2} = 0$$

$$\frac{\partial h}{\partial \eta} = v_{bo} - c_1 \ln n_1 - c_2 \ln n_2 = 0$$

These three equations yield, respectively,

$$n_1 = \frac{c_1\eta - 1}{c_1\varepsilon_1\eta} \qquad n_2 = \frac{c_2\eta - 1}{c_2\varepsilon_2\eta} \qquad v_{bo} = c_1 \ln n_1 + c_2 \ln n_2 \tag{11.83}$$

Substituting n_1 and n_2 into the expression for v_{bo}, we get

$$c_1 \ln\left(\frac{c_1\eta - 1}{c_1\varepsilon_1\eta}\right) + c_2 \ln\left(\frac{c_2\eta - 1}{c_2\varepsilon_2\eta}\right) = v_{bo} \tag{11.84}$$

This equation must be solved iteratively for η, after which η is substituted into Equations $11.83_{1,2}$ to obtain the stage mass ratios n_1 and n_2. These mass ratios are used in Equations 11.75 together with the assumed structural ratios, exhaust velocities, and payload mass to obtain the step masses of each stage.

We can now generalize the optimization procedure to an N-stage vehicle, for which Equation 11.82 becomes

$$h = \sum_{i=1}^{N}\left[\ln(1-\varepsilon_i) + \ln n_i - \ln(1-\varepsilon_i n_i)\right] - \eta\left(v_{bo} - \sum_{i=1}^{N} c_i \ln n_i\right) \tag{11.85}$$

At the outset, we know the required burnout velocity v_{bo}, the payload mass m_{PL}, and for every stage we have the structural ratio ε_i and the exhaust velocity c_i (i.e., the specific impulse). The first step is to solve for the Lagrange parameter η using Equation 11.84, which, for N stages is written

$$\sum_{i=1}^{N} c_i \ln\frac{c_i\eta - 1}{c_i\varepsilon_i\eta} = v_{bo}$$

Expanding the logarithm, this can be written

$$\sum_{i=1}^{N} c_i \ln(c_i\eta - 1) - \ln\eta\sum_{i=1}^{N} c_i - \sum_{i=1}^{N} c_i \ln c_i\varepsilon_i = v_{bo} \tag{11.86}$$

After solving this equation iteratively for η, we use that result to calculate the optimum mass ratio for each stage (cf. Equation 11.83),

$$n_i = \frac{c_i\eta - 1}{c_i\varepsilon_i\eta},\ i = 1,2,\ldots,N \tag{11.87}$$

Of course, each n_i must be greater than 1.

Referring to Equations 11.75, we next obtain the step masses of each stage, beginning with stage N and working our way down the stack to stage 1,

$$\begin{aligned}
m_N &= \frac{n_N - 1}{1 - n_N\varepsilon_N} m_{PL} \\
m_{N-1} &= \frac{n_{N-1} - 1}{1 - n_{N-1}\varepsilon_{N-1}}(m_N + m_{PL}) \\
m_{N-2} &= \frac{n_{N-2} - 1}{1 - n_{N-2}\varepsilon_{N-2}}(m_{N-1} + m_N + m_{PL}) \\
&\vdots \\
m_1 &= \frac{n_1 - 1}{1 - n_1\varepsilon_1}(m_2 + m_3 + \ldots m_{PL})
\end{aligned} \tag{11.88}$$

Having found each step mass, each empty stage mass is

$$m_{Ei} = \varepsilon_i m_i \tag{11.89}$$

and each stage propellant mass is

$$m_{pi} = m_i - m_{Ei} \tag{11.90}$$

For the function h in Equation 11.85, it is easily shown that

$$\frac{\partial^2 h}{\partial n_i \partial n_j} = 0,\ i, j = 1, \ldots N\ (i \neq j)$$

It follows that the second differential of h is

$$d^2 h = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 h}{\partial n_i \partial n_j} dn_i dn_j = \sum_{i=1}^{N} \frac{\partial^2 h}{\partial n_i^2}(dn_i)^2 \tag{11.91}$$

where it can be shown, again using Equation 11.85, that

$$\frac{\partial^2 h}{\partial n_i^2} = \frac{\eta c_i (\varepsilon_i n_i - 1)^2 + 2\varepsilon_i n_i - 1}{(\varepsilon_i n_i - 1)^2 n_i^2} \tag{11.92}$$

For h to be minimum at the mass ratios n_i given by Equation 11.87, it must be true that $d^2h > 0$. Equations 11.91 and 11.92 indicate that this will be the case if

$$\eta c_i (\varepsilon_i n_i - 1)^2 + 2\varepsilon_i n_i - 1 > 0, \quad i = 1, \ldots, N \tag{11.93}$$

Example 11.7

Find the optimal mass for a three-stage launch vehicle that is required to lift a 5000 kg payload to a speed of 10 km/s. For each stage, we are given that

Stage 1	$I_{sp1} = 400\ \text{s}\ (c_1 = 3.924\ \text{km/s})$	$\varepsilon_1 = 0.10$
Stage 2	$I_{sp2} = 350\ \text{s}\ (c_2 = 3.434\ \text{km/s})$	$\varepsilon_2 = 0.15$
Stage 3	$I_{sp3} = 300\ \text{s}\ (c_3 = 2.943\ \text{km/s})$	$\varepsilon_3 = 0.20$

Solution

Substituting this data into Equation 11.86, we get

$$3.924\ln(3.924\eta - 1) + 3.434\ln(3.434\eta - 1) + 2.943\ln(2.943\eta - 1) - 10.30\ln\eta + 7.5089 = 10$$

As can be checked by substitution, the iterative solution of this equation is

$$\eta = 0.4668$$

Substituting η into Equations 11.87 yields the optimum mass ratios,

$$n_1 = 4.541 \qquad n_2 = 2.507 \qquad n_3 = 1.361$$

For the step masses, we appeal to Equations 11.88 to obtain

$$m_1 = 165{,}700 \text{ kg} \qquad m_2 = 18{,}070 \text{ kg} \qquad m_3 = 2477 \text{ kg}$$

The total mass of the vehicle is

$$m_o = m_1 + m_2 + m_3 + m_{PL} = \boxed{191{,}200 \text{ kg}}$$

Using Equations 11.89 and 11.90, the empty masses and propellant masses are found to be

$$\begin{array}{lll} m_{E1} = 16{,}570 \text{ kg} & m_{E2} = 2710 \text{ k} & m_{E3} = 495.4 \text{ kg} \\ m_{p1} = 149{,}100 \text{ kg} & m_{p2} = 15{,}360 \text{ kg} & m_{p3} = 1982 \text{ kg} \end{array}$$

The payload ratios for each stage are

$$\lambda_1 = \frac{m_2 + m_3 + m_{PL}}{m_1} = 0.1542$$

$$\lambda_2 = \frac{m_3 + m_{PL}}{m_2} = 0.4139$$

$$\lambda_3 = \frac{m_{PL}}{m_3} = 2.018$$

The overall payload fraction is

$$\pi_{\text{PL}} = \frac{m_{PL}}{m_o} = \frac{5000}{191{,}200} = 0.0262$$

Finally, let us check Equation 11.93,

$$\eta c_1(\varepsilon_1 n_1 - 1)^2 + 2\varepsilon_1 n_1 - 1 = 0.4541$$
$$\eta c_2(\varepsilon_2 n_2 - 1)^2 + 2\varepsilon_2 n_2 - 1 = 0.3761$$
$$\eta c_3(\varepsilon_3 n_3 - 1)^2 + 2\varepsilon_3 n_3 - 1 = 0.2721$$

A positive number in every instance means we have indeed found a local minimum of the function in Equation 11.85.

PROBLEMS

Section 11.4

11.1 A two stage, solid-propellant sounding rocket has the following properties.
First stage: $m_0 = 249.5$ kg $\quad m_f = 170.1$ kg $\quad \dot{m}_e = 10.61$ kg/s $\quad I_{sp} = 235$ s
Second stage: $m_0 = 113.4$ kg $\quad m_f = 58.97$ kg $\quad \dot{m}_e = 4.053$ kg/s $\quad I_{sp} = 235$ s

Delay time between burnout of first stage and ignition of second stage: 3 seconds.
As a preliminary estimate, neglect drag and the variation of earth's gravity with altitude to calculate the maximum height reached by the second stage after burnout.
{Ans.: 322 km}

11.2 A two-stage launch vehicle has the following properties:
First stage: two solid propellant rockets. Each one has a total mass of 525,000 kg, 450,000 kg of which is propellant. $I_{sp} = 290$ s.
Second stage: two liquid rockets with $I_{sp} = 450$ s. Dry mass = 30,000 kg, propellant mass = 600,000 kg.
Calculate the payload mass to a 300 km orbit if launched due east from KSC. Let the total gravity and drag loss be 2 km/s.
{Ans.: 114,000 kg}

Section 11.5

11.3 Suppose a spacecraft in permanent orbit around the earth is to be used for delivering payloads from low earth orbit (LEO) to geostationary equatorial orbit (GEO). Before each flight from LEO, the spacecraft is refueled with propellant, which it uses up in its round trip to GEO. The outbound leg requires four times as much propellant as the inbound return leg. The delta-v for transfer from LEO to GEO is 4.22 km/s. The specific impulse of the propulsion system is 430 s. If the payload mass is 3500 kg, calculate the empty mass of the vehicle.
{Ans.: 2733 kg}

11.4 Consider a rocket comprising three similar stages (i.e., each stage has the same specific impulse, structural ratio and payload ratio). The common specific impulse is 310 s. The total mass of the vehicle is 150,000 kg, the total structural mass (empty mass) is 20,000 kg and the payload mass is 10,000 kg. Calculate
(a) The mass ratio n and the total Δv for the three-stage rocket.
{Ans. $n = 2.04$, $\Delta v = 6.50$ km/s}
(b) m_{p1}, m_{p2}, and m_{p3}.
(c) m_{E1}, m_{E2} and m_{E_3}.
(d) m_{o1}, m_{o2}, and m_{o3}.

Section 11.6

11.5 Find the extrema of the function $z = (x + y)^2$ subject to y and z lying on the circle $(x - 1)^2 + y^2 = 1$.
{Ans.: $z = 0.1716$ at $(x, y) = (0.2929, -0.7071)$, $z = 5.828$ at $(x, y) = (1.707, 0.7071)$ and $z = 0$ at $(x, y) = (0,0)$ and $(x, y) = (1, -1)$.}

11.6 A small two-stage vehicle is to propel a 10 kg payload to a speed of 6.2 km/s. The properties of the stages are:
First stage: $I_{sp} = 300$ s and $\varepsilon = 0.2$.
Second stage, $I_{sp} = 235$ s and $\varepsilon = 0.3$.
Estimate the optimum mass of the vehicle.
{Ans.: 1125 kg}

List of Key Terms

drag coefficient
dynamic pressure
effective exhaust velocity
frontal area
gravity turn trajectory
mass ratio
payload ratio
scale height
similar stages
sounding rockets
specific impulse
step mass
structural ratio
thrust-to-weight ratio

APPENDIX

Physical data

A

The following tables contain information that is commonly available and may be found in the literature and on the world wide web. See, for example, the *Astronomical Almanac* (U.S. Naval Observatory, 2008) and *National Space Science Data Center* (NASA Goddard Space Flight Center, 2003).

Table A.1 Astronomical Data for the Sun, the Planets and the Moon

Object	Radius (km)	Mass (kg)	Sidereal Rotation Period	Inclination of Equator to Orbit Plane	Semimajor Axis of Orbit (km)	Orbit Eccentricity	Inclination of Orbit to the Ecliptic Plane	Orbit Sidereal Period
Sun	696,000	1.989×10^{30}	25.38d	7.25°	—	—	—	—
Mercury	2440	330.2×10^{21}	58.65d	0.01°	57.91×10^{6}	0.2056	7.00°	87.97d
Venus	6052	4.869×10^{24}	243d*	177.4°	108.2×10^{6}	0.0067	3.39°	224.7d
Earth	6378	5.974×10^{24}	23.9345h	23.45°	149.6×10^{6}	0.0167	0.00°	365.256d
(Moon)	1737	73.48×10^{21}	27.32d	6.68°	384.4×10^{3}	0.0549	5.145°	27.322d
Mars	3396	641.9×10^{21}	24.62h	25.19°	227.9×10^{6}	0.0935	1.850°	1.881y
Jupiter	71,490	1.899×10^{27}	9.925h	3.13°	778.6×10^{6}	0.0489	1.304°	11.86y
Saturn	60,270	568.5×10^{24}	10.66h	26.73°	1.433×10^{9}	0.0565	2.485°	29.46y
Uranus	25,560	86.83×10^{24}	17.24h*	97.77°	2.872×10^{9}	0.0457	0.772°	84.01y
Neptune	24,760	102.4×10^{24}	16.11h	28.32°	4.495×10^{9}	0.0113	1.769°	164.8y
(Pluto)	1195	12.5×10^{21}	6.387d*	122.5°	5.870×10^{9}	0.2444	17.16°	247.7y

*Retrograde

Table A.2 Gravitational Parameter (μ) and Sphere of Influence (SOI) Radius for the Sun, the Planets and the Moon

Celestial body	μ (km^3/s^2)	SOI radius (km)
Sun	132,712,000,000	–
Mercury	22,030	112,000
Venus	324,900	616,000
Earth	398,600	925,000
Earth's moon	4903	66,100
Mars	42,828	577,000
Jupiter	126,686,000	48,200,000
Saturn	37,931,000	54,800,000
Uranus	5,794,000	51,800,000
Neptune	6,835,100	86,600,000
Pluto	830	3,080,000

Table A.3 Some Conversion Factors

1 ft = 0.3048 m
1 mile (mi) = 1.609 km
1 nautical mile (n mi) = 1.151 mi = 1.852 km
1 mi/h = 0.0004469 km/s
1 lb (mass) = 0.4536 kg
1 lb (force) = 4.448 N
1 psi = 6895 kPa

A road map

B

Figure B.1 is a road map through Chapters 1, 2 and 3. Those who from time to time feel they have lost their bearings may find it useful to refer to this flow chart, which shows how the various concepts and results are interrelated. The pivotal influence of Sir Isaac Newton is obvious. All of the equations of classical orbital mechanics (the two-body problem) are derived from those listed here.

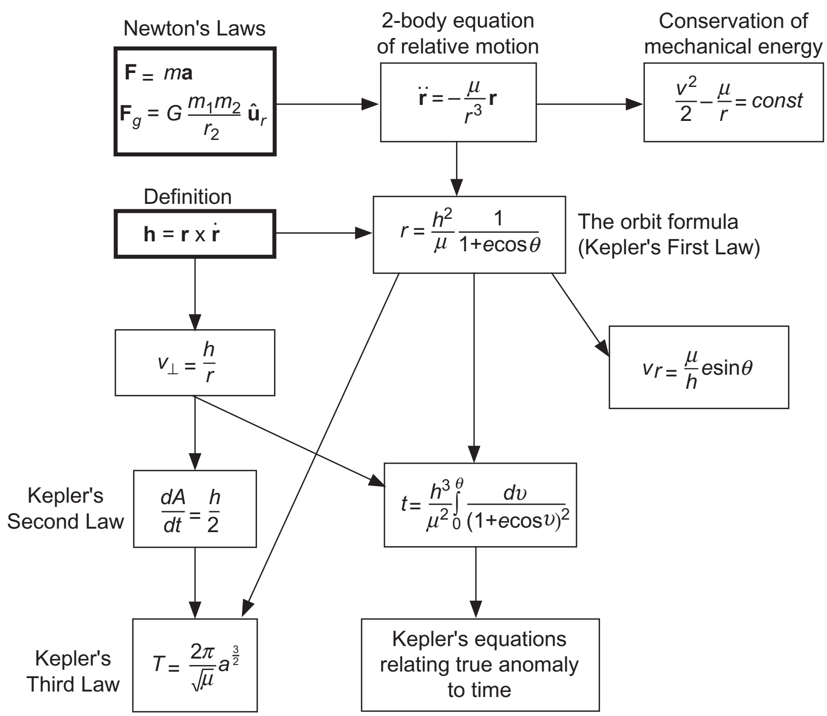


FIGURE B.1

Logic flow for the major outcomes of Chapters 1, 2 and 3.

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APPENDIX

Numerical integration of the *n*-body equations of motion

C

Without loss of generality we shall derive the equations of motion of the three-body system illustrated in Figure C.1. The equations of motion for *n* bodies can easily be generalized from those of a three-body system.

Each mass of a three-body system experiences the force of gravitational attraction from the other members of the system. As shown in Figure C.1, the forces exerted on body 1 by bodies 2 and 3 are $\mathbf{F}_{12}$ and $\mathbf{F}_{13}$, respectively. Likewise, body 2 experiences the forces $\mathbf{F}_{21}$ and $\mathbf{F}_{23}$ whereas the forces $\mathbf{F}_{31}$ and $\mathbf{F}_{32}$ act on body 3. These gravitational forces can be inferred from Equation 2.9:

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = \frac{Gm_1m_2(\mathbf{R}_2 - \mathbf{R}_1)}{\left\|\mathbf{R}_2 - \mathbf{R}_1\right\|^3} \tag{C.1a}$$

$$\mathbf{F}_{13} = -\mathbf{F}_{31} = \frac{Gm_1m_3(\mathbf{R}_3 - \mathbf{R}_1)}{\left\|\mathbf{R}_3 - \mathbf{R}_1\right\|^3} \tag{C.1b}$$

$$\mathbf{F}_{23} = -\mathbf{F}_{32} = \frac{Gm_2m_3(\mathbf{R}_3 - \mathbf{R}_2)}{\left\|\mathbf{R}_3 - \mathbf{R}_2\right\|^3} \tag{C.1c}$$

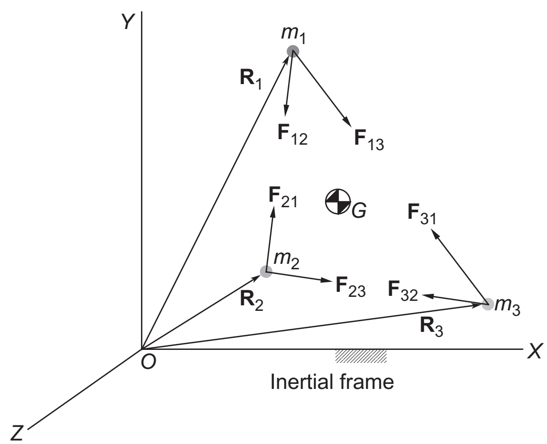

FIGURE C.1

Three-body problem.

Relative to an inertial frame of reference the accelerations of the bodies are

$$\mathbf{a}_i = \ddot{\mathbf{R}}_i \quad i = 1,2,3$$

where $\mathbf{R}_i$ is the absolute position vector of body i. The equation of motion of body 1 is

$$\mathbf{F}_{12} + \mathbf{F}_{13} = m_1\mathbf{a}_1$$

Substituting Equations C.1a and C.1b yields

$$\mathbf{a}_1 = \frac{Gm_2(\mathbf{R}_2 - \mathbf{R}_1)}{\|\mathbf{R}_2 - \mathbf{R}_1\|^3} + \frac{Gm_3(\mathbf{R}_3 - \mathbf{R}_1)}{\|\mathbf{R}_3 - \mathbf{R}_1\|^3} \tag{C.2a}$$

For bodies 2 and 3 we find in a similar fashion that

$$\mathbf{a}_2 = \frac{Gm_1(\mathbf{R}_1 - \mathbf{R}_2)}{\|\mathbf{R}_1 - \mathbf{R}_2\|^3} + \frac{Gm_3(\mathbf{R}_3 - \mathbf{R}_2)}{\|\mathbf{R}_3 - \mathbf{R}_2\|^3} \tag{C.2b}$$

$$\mathbf{a}_3 = \frac{Gm_1(\mathbf{R}_1 - \mathbf{R}_3)}{\|\mathbf{R}_1 - \mathbf{R}_3\|^3} + \frac{Gm_2(\mathbf{R}_2 - \mathbf{R}_3)}{\|\mathbf{R}_2 - \mathbf{R}_3\|^3} \tag{C.2c}$$

The velocities are related to the accelerations by

$$\frac{d\mathbf{v}_i}{dt} = \mathbf{a}_i \qquad i = 1,2,3 \tag{C.3}$$

and the position vectors are likewise related to the velocities,

$$\frac{d\mathbf{R}_i}{dt} = \mathbf{v}_i \qquad i = 1,2,3 \tag{C.4}$$

Equations C.2 though C.4 constitute a system of ordinary differential equations (ODEs) in the variable time.

Given the initial positions $\mathbf{R}_{i_0}$ and initial velocities $\mathbf{v}_{i_0}$, we must integrate Equations C.3 to find $\mathbf{v}_i$ as a function of time and substitute those results into Equations C.4 to obtain $\mathbf{R}_i$ as a function of time. The integrations must be done numerically.

To do this using MATLAB®, we first resolve all of the vectors into their three components along the *XYZ* axes of the inertial frame and write them as column vectors,

$$\mathbf{R}_1 = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} \quad \mathbf{R}_2 = \begin{Bmatrix} X_2 \\ Y_2 \\ Z_2 \end{Bmatrix} \quad \mathbf{R}_3 = \begin{Bmatrix} X_3 \\ Y_3 \\ Z_3 \end{Bmatrix} \tag{C.5}$$

$$\mathbf{v}_1 = \begin{Bmatrix} \dot{X}_1 \\ \dot{Y}_1 \\ \dot{Z}_1 \end{Bmatrix} \quad \mathbf{v}_2 = \begin{Bmatrix} \dot{X}_2 \\ \dot{Y}_2 \\ \dot{Z}_2 \end{Bmatrix} \quad \mathbf{v}_3 = \begin{Bmatrix} \dot{X}_3 \\ \dot{Y}_3 \\ \dot{Z}_3 \end{Bmatrix} \tag{C.6}$$

According to Equations C.2,

$$\mathbf{a}_1 = \begin{Bmatrix} \ddot{X}_1 \\ \ddot{Y}_1 \\ \ddot{Z}_1 \end{Bmatrix} = \begin{Bmatrix} \dfrac{Gm_2(X_2 - X_1)}{R_{12}{}^3} + \dfrac{Gm_3(X_3 - X_1)}{R_{13}{}^3} \\ \dfrac{Gm_2(Y_2 - Y_1)}{R_{12}{}^3} + \dfrac{Gm_3(Y_3 - Y_1)}{R_{13}{}^3} \\ \dfrac{Gm_2(Z_2 - Z_1)}{R_{12}{}^3} + \dfrac{Gm_3(Z_3 - Z_1)}{R_{13}{}^3} \end{Bmatrix} \tag{C.7a}$$

$$\mathbf{a}_2 = \begin{Bmatrix} \ddot{X}_2 \\ \ddot{Y}_2 \\ \ddot{Z}_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{Gm_1(X_1 - X_2)}{R_{12}{}^3} + \dfrac{Gm_3(X_3 - X_2)}{R_{13}{}^3} \\ \dfrac{Gm_1(Y_1 - Y_2)}{R_{12}{}^3} + \dfrac{Gm_3(Y_3 - Y_2)}{R_{13}{}^3} \\ \dfrac{Gm_1(Z_1 - Z_2)}{R_{12}{}^3} + \dfrac{Gm_3(Z_3 - Z_2)}{R_{13}{}^3} \end{Bmatrix} \tag{C.7b}$$

$$\mathbf{a}_3 = \begin{Bmatrix} \ddot{X}_3 \\ \ddot{Y}_3 \\ \ddot{Z}_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{Gm_1(X_1 - X_3)}{R_{12}{}^3} + \dfrac{Gm_2(X_2 - X_3)}{R_{13}{}^3} \\ \dfrac{Gm_1(Y_1 - Y_3)}{R_{12}{}^3} + \dfrac{Gm_2(Y_2 - Y_3)}{R_{13}{}^3} \\ \dfrac{Gm_1(Z_1 - Z_3)}{R_{12}{}^3} + \dfrac{Gm_2(Z_2 - Z_3)}{R_{13}{}^3} \end{Bmatrix} \tag{C.7c}$$

where

$$R_{12} = \|\mathbf{R}_2 - \mathbf{R}_1\| \quad R_{13} = \|\mathbf{R}_3 - \mathbf{R}_1\| \quad R_{23} = \|\mathbf{R}_3 - \mathbf{R}_2\| \tag{C.8}$$

Next, we form the 18-component column vector

$$\mathbf{y} = \lfloor \mathbf{R}_1 \quad \mathbf{R}_2 \quad \mathbf{R}_3 \quad \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \rfloor^T \tag{C.9}$$

The first derivatives of the components of this vector comprise the column vector

$$\dot{\mathbf{y}} = \mathbf{f} = \lfloor \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \rfloor^T \tag{C.10}$$

According to Equations C.7, the accelerations are functions of $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$. Hence, Equation C.10 is of the form

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{C.11}$$

given in Equation 1.95, although in this case time t does not appear explicitly. Equation C.11 can be employed in procedures such as Algorithms 1.1, 1.2 or 1.3 to obtain a numerical solution for $\mathbf{R}_1(t)$, $\mathbf{R}_2(t)$ and $\mathbf{R}_3(t)$. We shall choose MATLAB's *ode45* Runge-Kutta solver.

For simplicity, we will solve the three-body problem in the plane. That is, we will restrict ourselves to only the *XY* components of the vectors **R**, **v** and **a**. The reader can use these scripts as a starting point for investigating more complex *n*-body problems.

The MATLAB function *threebody.m* contains the subfunction *rates*, which computes the accelerations given above in Equation C.7. That information together with the initial conditions are passed to *ode45*, which integrates the system given by Equation C.11. Subfunction *plotit* plots the solutions. The results of this program were used to create Figures 2.4 and 2.5.

Function file `threebody.m`

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
function threebody
% ~~~~~~~~~~~~~~~~
%{
This program presents the graphical solution of the motion of three bodies in
the plane for data provided in the input definitions below.

MATLAB's ode45 Runge-Kutta solver is used.
 G                              - gravitational constant (km^3/kg/s^2)
 t0, tf                         - initial and final times (s)
 m1, m2, m3                     - masses of the three bodies (kg)
 m                              - total mass (kg)
 X1,Y1; X2,Y2; X3,Y3            - coordinates of the three masses (km)
 VX1,VY1; VX2,VY2; VX3,VY3      - velocity components of the three masses (km/s)
 XG, YG                         - coordinates of the center of mass (km)
 y0                             - column vector of the initial conditions
 t                              - column vector of times at which the solution
                                  was computed
 y                              - matrix, the columns of which contain the
                                  position and velocity components evaluated at
                                  the times t(:):
                                    y(:,1) , y(:, 2) = X1(:), Y1(:)
                                    y(:,3) , y(:, 4) = X2(:), Y2(:)
                                    y(:,5) , y(:, 6) = X3(:), Y3(:)

                                    y(:,7) , y(:, 8) = VX1(:), VY1(:)
                                    y(:,9) , y(:,10) = VX2(:), VY2(:)
                                    y(:,11), y(:,12) = VX3(:), VY3(:)

User M-functions required:none
User subfunctions required:rates, plotit
%}
% ---------------------------------------------------------------------

clear all
close all
clc

G = 6.67259e-20;
```

```
%...Input data:
m1 = 1.e29; m2 = 1.e29; m3 = 1.e29;

t0 = 0; tf = 67000;

X1 = 0;      Y1 = 0;
X2 = 300000; Y2 = 0;
X3 = 2*X2;   Y3 = 0;

VX1 = 0;     VY1 = 0;
VX2 = 250;   VY2 = 250;
VX3 = 0;     VY3 = 0;
%...End input data

m = m1 + m2 + m3;
y0 = [X1 Y1   X2 Y2   X3 Y3   VX1 VY1   VX2 VY2   VX3 VY3]';

%...Pass the initial conditions and time interval to ode45, which
%   calculates the position and velocity of each particle at discrete
%   times t, returning the solution in the column vector y. ode45 uses
%   the subfunction 'rates' below to evaluate the accelerations at each
%   integration time step.
[t,y] = ode45(@rates, [t0 tf], y0);

X1 = y(:,1); Y1 = y(:,2);
X2 = y(:,3); Y2 = y(:,4);
X3 = y(:,5); Y3 = y(:,6);

%...Locate the center of mass at each time step:
XG = []; YG = [];
for i = 1:length(t)
    XG = [XG; (m1*X1(i) + m2*X2(i) + m3*X3(i))/m];
    YG = [YG; (m1*Y1(i) + m2*Y2(i) + m3*Y3(i))/m];
end

%...Coordinates of each particle relative to the center of mass:
X1G = X1 - XG; Y1G = Y1 - YG;
X2G = X2 - XG; Y2G = Y2 - YG;
X3G = X3 - XG; Y3G = Y3 - YG;

plotit

return

% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
  function dydt = rates(t,y)
% ~~~~~~~~~~~~~~~~~~~~~~~~~
%{
  This function evaluates the acceleration of each member of a planar 3-body
  system at time t from their positions and velocities at that time.
```

```
 t                          - time (s)
 y                          - column vector containing the position and velocity
                              components of the three masses at time t
 R12                        - cube of the distance between m1 and m2 (km^3)
 R13                        - cube of the distance between m1 and m3 (km^3)
 R23                        - cube of the distance between m2 and m3 (km^3)
 AX1,AY1; AX2,AY2; AX3,AY3  - acceleration components of each mass (km/s^2)
 dydt                       - column vector containing the velocity and
                              acceleration components of the three masses at time t
%}
% ------------------------------------------------------------------------------

X1 = y( 1);
Y1 = y( 2);

X2 = y( 3);
Y2 = y( 4);

X3 = y( 5);
Y3 = y( 6);

VX1 = y( 7);
VY1 = y( 8);

VX2 = y( 9);
VY2 = y(10);

VX3 = y(11);
VY3 = y(12);

%...Equations C.8:
R12 = norm([X2 - X1, Y2 - Y1])^3;
R13 = norm([X3 - X1, Y3 - Y1])^3;
R23 = norm([X3 - X2, Y3 - Y2])^3;

%...Equations C.9:
AX1 = G*m2*(X2 - X1)/R12 + G*m3*(X3 - X1)/R13;
AY1 = G*m2*(Y2 - Y1)/R12 + G*m3*(Y3 - Y1)/R13;
AX2 = G*m1*(X1 - X2)/R12 + G*m3*(X3 - X2)/R23;
AY2 = G*m1*(Y1 - Y2)/R12 + G*m3*(Y3 - Y2)/R23;
AX3 = G*m1*(X1 - X3)/R13 + G*m2*(X2 - X3)/R23;
AY3 = G*m1*(Y1 - Y3)/R13 + G*m2*(Y2 - Y3)/R23;

dydt = [VX1 VY1 VX2 VY2 VX3 VY3 AX1 AY1 AX2 AY2 AX3 AY3]';
end %rates
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

% ~~~~~~~~~~~~~
function plotit
% -------------
```

```
%...Plot the motions relative to the inertial frame (Figure 2.4):
figure(1)
title('Figure 2.4: Motion relative to the inertial frame', ...
      'Fontweight', 'bold', 'FontSize', 12)

hold on
plot(XG, YG,   '--k', 'LineWidth', 0.25)
plot(X1, Y1,     'r', 'LineWidth', 0.5)
plot(X2, Y2,     'g', 'LineWidth', 0.75)
plot(X3, Y3,     'b', 'LineWidth', 1.00)
xlabel('X(km)'); ylabel('Y(km)')
grid on
axis('equal')

%...Plot the motions relative to the center of mass (Figure 2.5):
figure(2)
title('Figure 2.5: Motion relative to the center of mass', ...
      'Fontweight', 'bold', 'FontSize', 12)

hold on
plot(X1G, Y1G,    'r', 'LineWidth', 0.5)
plot(X2G, Y2G, '--g', 'LineWidth', 0.75)
plot(X3G, Y3G,    'b', 'LineWidth', 1.00)
xlabel('X(km)'); ylabel('Y(km)')
grid on
axis('equal')
end %plotit
% ~~~~~~~~~~~~~

end %threebody
```

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APPENDIX

MATLAB® algorithms

D

Appendix D can be found at http://www.elsevierdirect.com/companions/9780123747785

Appendix Outline

D.1 Introduction
D.2 Algorithm 1.1: numerical integration of a system of first order differential equations by choice of Runge-Kutta methods RK1, RK2, RK3 or RK4
D.3 Algorithm 1.2: numerical integration of a system of first order differential equations by Heun's predictor-corrector method
D.4 Algorithm 1.3: numerical integration of a system of first order differential equations by the Runge-Kutta-Fehlberg 4(5) method with adaptive step size control
D.5 Algorithm 2.1: numerical solution for the motion of two bodies relative to an inertial frame.
D.6 Algorithm 2.2: numerical solution for the motion of m_2 of relative to m_1
D.7 Calculation of the Lagrange coefficients f and g and their time derivatives in terms of change in true anomaly
D.8 Algorithm 2.3: calculation of the state vector given the initial state vector and the change in true anomaly
D.9 Algorithm 2.4: find the root of a function using the bisection method
D.10 MATLAB solution of Example 2.18
D.11 Algorithm 3.1: solution of Kepler's equation by Newton's method
D.12 Algorithm 3.2: solution of Kepler's equation for the hyperbola using Newton's method
D.13 Calculation of the Stumpff functions $S(z)$ and $C(z)$
D.14 Algorithm 3.3: solution of the universal Kepler's equation using Newton's method
D.15 Calculation of the Lagrange coefficients f and g and their time derivatives in terms of change in universal anomaly
D.16 Algorithm 3.4: calculation of the state vector given the initial state vector and the time lapse Δt
D.17 Algorithm 4.1: obtain right ascension and declination from the position vector
D.18 Algorithm 4.2: calculation of the orbital elements from the state vector
D.19 Calculation of $\tan^{-1}(y/x)$ to lie in the range 0 to 360°
D.20 Algorithm 4.3: obtain the classical Euler angle sequence from a DCM
D.21 Algorithm 4.4: obtain the yaw, pitch and roll angles from a DCM

D.22 Algorithm 4.5: calculation of the state vector from the orbital elements
D.23 Algorithm 4.6: calculate the ground track of a satellite from its orbital elements
D.24 Algorithm 5.1: Gibbs method of preliminary orbit determination
D.25 Algorithm 5.2: solution of Lambert's problem
D.26 Calculation of Julian day number at 0 hr UT
D.27 Algorithm 5.3: calculation of local sidereal time
D.28 Algorithm 5.4: calculation of the state vector from measurements of range, angular position and their rates
D.29 Algorithms 5.5 and 5.6: Gauss method of preliminary orbit determination with iterative improvement
D.30 Calculate the state vector at the end of a finite-time, constant thrust delta-v maneuver
D.31 Algorithm 7.1: Find the position, velocity and acceleration of *B* relative to *A*'s co-moving frame
D.32 Plot the position of one spacecraft relative to another
D.33 Solve the linearized equations of relative motion of a chaser relative to a target whose orbit is an ellipse
D.34 Convert the numerical designation of a month or a planet into its name
D.35 Algorithm 8.1: calculation of the state vector of a planet at a given epoch
D.36 Algorithm 8.2: calculation of the spacecraft trajectory from planet 1 to planet 2
D.37 Algorithm 9.1: Calculate the direction cosine matrix from the quaternion
D.38 Algorithm 9.2: Calculate the quaternion form the direction cosine matrix
D.39 Solution of the spinning top problem (Example 9.21)
D.40 Calculation of a gravity-turn trajectory

APPENDIX

Gravitational potential of a sphere

E

Figure E.1 shows a point mass m with Cartesian coordinates (x,y,z) as well as a system of N point masses $m_1, m_2, m_3, \ldots m_N$, The ith one of these particles has mass m_i, and coordinates (x_i,y_i,z_i). The total mass of the N particles is M,

$$M = \sum_{i=1}^{N} m_i \tag{E.1}$$

The position vector drawn from m_i to m is $\mathbf{r}_i$ and the unit vector in the direction of $\mathbf{r}_i$ is

$$\hat{\mathbf{u}}_i = \frac{\mathbf{r}_i}{r_i}$$

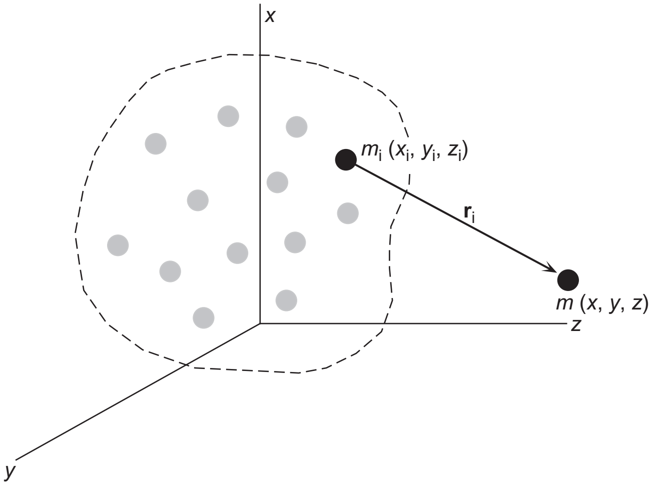

FIGURE E.1

A system of point masses and a neighboring test mass m.

The gravitational force exerted on m by m_i is opposite in direction to $\mathbf{r}_i$, and is given by

$$\mathbf{F}_i = -\frac{Gmm_i}{r_i^2}\hat{\mathbf{u}}_i = -\frac{Gmm}{r_i^3}$$

The potential energy of this force is

$$V_i = -G\frac{mm_i}{r_i} \tag{E.2}$$

The total gravitational potential energy of the system due to the gravitational attraction of all of the N particles is

$$V = \sum_{i=1}^{N} V_i \tag{E.3}$$

Therefore, the total force of gravity $\mathbf{F}$ on the mass m is

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right) \tag{E.4}$$

Consider the solid sphere of mass M and radius R_0 illustrated in Figure E.2. Instead of a discrete system as above, we have a continuum with mass density ρ. Each "particle" is a differential element $dM = \rho dv$ of the total mass M. Equation E.1 becomes

$$M = \iiint_v \rho\, dv \tag{E.5}$$

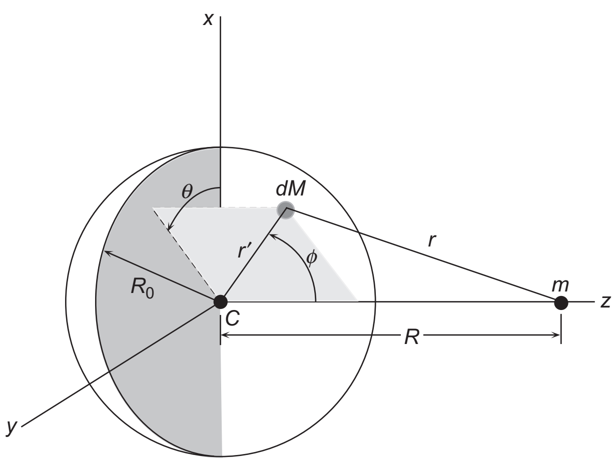

FIGURE E.2

Sphere with a spherically symmetric mass distribution.

where dv is the volume element and v is the total volume of the sphere. In this case, Equation E.2 becomes

$$dV = -G\frac{mdM}{r} = -Gm\frac{\rho dv}{r}$$

where r is the distance from the differential mass dM to the finite point mass m. Equation E.3 is replaced by

$$V = -Gm\iiint\limits_{v}\frac{\rho dv}{r} \tag{E.6}$$

Let the mass of the sphere have a spherically symmetric distribution, which means that the mass density ρ depends only on r', the distance from the center C of the sphere. An element of mass dM has spherical coordinates (r',θ,ϕ), where the angle θ is measured in the xy plane of a cartesian coordinate system with origin at C, as shown in Figure E.2. In spherical coordinates the volume element is

$$dv = r'^2 \sin\phi d\phi dr' d\theta \tag{E.7}$$

Therefore Equation E.5 becomes

$$M = \int_{\theta=0}^{2\pi}\int_{r'=0}^{R_0}\int_{\phi=0}^{\pi}\rho r'^2 \sin\phi\, d\phi dr' d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right)\left(\int_0^{R_0}\rho r'^2\, dr'\right) = (2\pi)(2)\left(\int_0^{R_0}\rho r'^2\, dr'\right)$$

so that the mass of the sphere is given by

$$M = 4\pi\int_{r'=0}^{R_0}\rho r'^2\, dr' \tag{E.8}$$

Substituting Equation E.7 into Equation E.6 yields

$$V = -Gm\int_{\theta=0}^{2\pi}\int_{r'=0}^{R_0}\int_{\phi=0}^{\pi}\frac{\rho r'^2 \sin\phi\, d\phi dr' d\theta}{r} = -2\pi Gm\left[\int_0^{R_0}\left(\int_0^{\pi}\frac{\sin\phi\, d\phi}{r}\right)\rho r'^2\, dr'\right] \tag{E.9}$$

The distance r is found by using the law of cosines,

$$r = (R^2 + r'^2 - 2r'R\cos\phi)^{\frac{1}{2}}$$

where R is the distance from the center of the sphere to the mass m. Differentiating this equation with respect to ϕ, holding r' constant, yields

$$\frac{dr}{d\phi} = \frac{1}{2}(R^2 + r'^2 - 2r'R\cos\phi)^{-\frac{1}{2}}(2r'R\sin\phi\, d\phi) = \frac{r'R\sin\phi}{r}$$

so that

$$\sin\phi\, d\phi = \frac{rdr}{r'R}$$

It follows that

$$\int_{\phi=0}^{\pi} \frac{\sin\phi\, d\phi}{r} = \frac{1}{r'R}\int_{R-r'}^{R+r'} dr = \frac{2}{R}$$

Substituting this result along with Equation E.8 into Equation E.9 yields

$$V = -\frac{GMm}{R}$$

We conclude that the gravitational potential energy, and hence (from Equation E.4) the gravitational force, of a sphere with a spherically symmetric mass distribution M is the same as that of a point mass M located at the center of the sphere.

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Index

A
Absolute acceleration
  angular, 402–8, 436–40, 484–6
  nutation dampers, 595
  point masses, 29–37
  rigid-body kinematics, 486–95
  two-body motion, 68
Absolute angular momentum, 497–501
Absolute angular velocity
  gyroscopic attitude control, 623
  nutation dampers, 594
  rigid-body dynamics, 486–91
  torque-free motion, 576–77
Absolute position vectors, 29–37
Absolute velocity
  close-proximity circular orbits, 421
  rigid-body kinematics, 485–92
  two-body motion, 68
  two-impulse maneuvers, 425
  vectors, 29–37
Acceleration
  see also absolute…; angular…; relative
  Coriolis, 30
  five-term, 30, 31
  gravitational, 18–19, 233
  gyroscopic attitude control, 521–5
  oblateness, 233–34
  point masses, 10–14, 15–17, 25–37
  preliminary orbit determination, 297
  relative motion and rendezvous, 391–94
  restricted three-body motion, 136
  rocket vehicle dynamics, 656
  three-body systems, 696
Adaptive step size, 41, 52, see Appendix D page 701
Advance of perigee, 236–37, 240
Aiming radius
  hyperbolic trajectories, 104–5, 108, 451
  planetary rendezvous, 451–52, 454
Altitude
  equation, 657
  gravity-gradient stabilization, 635
  perigee, 100, 312–13, 383–84
  preliminary orbit determination, 305, 312–13, 361–65
  rocket performance, 661, 662–5
  Sun-synchronous three dimensional orbits, 237
  two-body motion, 68–70
Amplitude, 586
Angles
  see also flight path…
  auxiliary, 174–75
  azimuth, 297, 314–15, see Appendix D page 701
  dihedral, 208, 355
  elevation, 287–9, 293, 297, see Appendix D page 701
  Euler's, 208–12, 538–48, 560
  to periapse, 451–2
  phase, 432–6
  preliminary orbit determination, 297
  of rotation, 347
  spin, 607
  tilt, 536
  turn, 105, 112, 452, 461, 465
  wobble, 579
Angular acceleration
  absolute, 490–3, 495–7, 524, 526, 544
  gyroscopic attitude control, 630–1
  point masses, 19–20, 25
  relative, 393, 487–88, 490
  rendezvous, 393
  rigid-body kinematics, 486–95
  satellite attitude dynamics, 582–4, 594, 612
  torque-free motion, 582–4
  yo-yo despin, 612
Angular momentum
  chase maneuvers, 350–53
  conservation of, 75
  double-gimbaled control moment gyros, 627–28
  Hohmann transfers, 321
  hyperbolic trajectories, 111–12, 141
  Lagrange coefficients, 117–29
  moments of inertia, 501–4, 507–8, 513–8
  orbit formulas, 74–82
  plane change maneuvers, 362–3
  planetary departure, 442–3
  planetary flyby, 460
  point masses, 21–22

Angular momentum (*Continued*)
preliminary orbit determination, 297–9, 322–25, 334–41
rigid-body dynamics, 498–508, 524, 533, 534
satellite attitude dynamics
coning maneuvers, 601–8
dual-spin spacecraft, 590–91
gyroscopic control, 615, 616–24, 627–31
nutation dampers, 596
thrusters, 605–9
torque-free motion, 584–86, 590, 596, 601
yo-yo despin, 608–9, 610
spinning tops, 537
three dimensional orbits, 199, 399
torque-free motion, 584–86, 590, 596, 601
two-body motion, 68–70, 74–5, 78–89
Angular position, 432, see Appendix D page 701
Angular velocity
close-proximity circular orbits, 419
Euler angles, 541–44
Euler's equations, 524–28
moments of inertia, 503–4
pitch, 558–60
point masses, 25–27, 32, 36–38
relative motion and rendezvous, 393, 396
rigid-body dynamics, 485–86, 534–38, 543–46
roll, 566–67
satellite attitude dynamics
dual-spin spacecraft, 590–92
gravity-gradient stabilization, 635–36
gyroscopic control, 615–16, 621, 623–25
thrusters, 605
torque-free motion, 574–84, 591, 649
yo-yo despin, 608–13
spinning tops, 533
two-body motion, 70
yaw, 549–50
Angular-impulse, 22–23, 500–1
Apoapse, 89, 355–57, 455
Apogee
kick, 323–4
radius, 100, 148, 240
towards the sun, 165–66
velocity, 95
Applied torque, 500–3
Approach trajectories, 451, 452, 460, 462
see also two-body motion
Apse lines, 260, 344–56
Arcseconds, 471
Areal velocity, 76
Argument of perigee
oblateness, 233–35, 239–42
orbital elements, 208, 239, 243
Arrival phase, 475–78
Astronomical units, 471, 472
Attitude dynamics *see* satellite…
Auxiliary angles, 174–78
Axial bearing loads, 548
Axial torques, 624–27
Axis of rotation, 200
Axisymmetric dual-spin, 589–90, 620
Axisymmetric tops, 533–38
Azimuth
angles, 297, 314–15
averaged radius, 96
plane change maneuvers, 359–60

B

bac-cab rule, 9
Bearing forces, 546–48
Bent rods, 520–23
Bessel functions, 169–70
Bi-elliptic Hohmann transfers, 328–32
Bias values, 627
Bisection method, 135–6, 137, 138, see Appendix D page 701
Bivariate functions, 678
Body cones, 580
Body frames, 26
Boundary value problem, 38
Burnout
Jacobi constant, 139–40
rocket vehicle dynamics, 662–5, 668–87
sensitivity analysis, 448–50

C

Capture orbits, 451–52, 454, 455–58
Capture radius, 454
Cartesian coordinates
elliptical orbits, 89
equation of a parabola, 101
hyperbolic trajectories, 105
rotation, 223–26
three dimensional orbits, 204–5
Cassini gravity assist maneuvers, 469
Celestial bodies, 201–3
Center of curvature, 13–15
Center of mass
inertial frames, 62–65
moving reference frames, 70–74

rigid-body dynamics
Euler angles, 538–40
Euler's equations, 524–30
moments of inertia, 504–5, 507–8, 510–12, 517–23
parallel axis theorem, 517–23
rotational motion, 500–1
translational motion, 495–97
two-body motion, 62–9, 70–4
Characteristic energy, 108
Chase maneuvers, 350–54, 400–21
Chasles' theorem, 485–86
Circular orbits
close-proximity relative motion, 419–21
Hohmann transfers, 325, 328
parking, 443, 479–80
position as a function of time, 156–57
rigid-body kinetic energy, 532–33
two-body motion, 70–71
Classical orbital elements, 199–200, 207, 261–63, 297–304
Classical Euler angle sequence, 224, see Appendix D page 701
Clohessy–Wiltshire (CW) frames
equations, 407–11, 417–18, 338, 340
gravity-gradient stabilization, 635–36
matrices, 410, 415, 417
Close-proximity circular orbits, 419–27
Co-moving reference frames, 392–9, 402–6
Coaxial elliptic orbits, 325, 338–50
Common apse lines, 338–43
Common focus, 343
Conics, 441–42, 475–81
Coning maneuvers, 603–4, 607
Conservation of…
angular momentum, 75–6
energy, 83, 608–11
momentum, 75–6, 608–9
Constant amplitude, 576
Continuous three dimensional bodies, 495–97
Control moment gyros, 627–31
Coordinate systems, 280–89
see also Cartesian…; topocentric…
polar, 90
Coordinate transformations
geocentric equatorial, 216–29, 242–4, 283–9
perifocal frames, 229–33
rotation, 200–3
three dimensional orbits, 199–200, 242–4
topocentric, 280–82
Coplanar orbits, 319, 339–40, 355
Cord lengths, 613–15
Cord unwind rates, 611–12
Coriolis acceleration, 30
Cosine vectors, 304, 306
Coupling coefficients, 43, see Appendix D page 701
Cross product, 7–8, see Appendix D page 701
Cruise phase, 429, 475–76
Curvature of the earth, 657–58
Curvilinear motion, 1–7, 11
CW *see* Clohessy–Wiltshire frames

D

Damped natural frequency, 46, see Appendix D page 701
damping coefficient, 45
Damping factor, 45, see Appendix D page 701
Dark side approaches, 462–63, 466
Declination
preliminary orbit determination, 297, 298, 305–6
state vectors, 204–8
three dimensional orbits, 199–203
Delta-H requirements, 603–4, 606–8
Delta-v requirements
bi-elliptic Hohmann transfers, 328–32
chase maneuvers, 350–54
Hohmann transfers, 321–28, 430–32
impulsive orbital maneuvers, 368–74, 411–19
interplanetary trajectories, 429–30, 479, 481–83
non-Hohmann transfers, 339–43, 430–32
phasing maneuvers, 332–37
plane change maneuvers, 355–58
planetary rendezvous, 451, 479–83
rocket vehicle dynamics, 655–56
two-impulse maneuvers, 425–27
Departure trajectories, 444–48, 479, 481–84
Despin mechanisms, 608–13
Diagonal moment of inertia matrices, 507–10
Dihedral angles, 208, 355
Direct ascent trajectories, 444
Direction angles, 4
Direction cosine matrix, 217, 219–26, see Appendix D page 701
Direction cosine vectors, 290–92, 304
Distances between planets, 473–5
Dot product, 5, see Appendix D page 701
Double-gimbaled control moment gyros, 627–30
Downrange equations, 657
Drag force, 656, 666
Dual-spin satellites, 621–2, 649
Dual-spin spacecraft, 589–93

E

Earth
  centered inertial frames, 33–7
  earth orbits, 84–85, 199, 319–21
  earth satellites, 199, 246–53
  earth-moon systems, 136–39
  earth's curvature, 656–57
  earth's gravitational parameter, 85
  earth's oblateness, 233–43
  earth's shadow, 165–7
  earth's sphere of influence, 437–41
  low earth orbits, 84, 319, 574
East longitude, 276–80, 281–86
East-North-Zenith (ENZ) frame, 284
Easterly launches, 359–60
Eccentric anomaly
  hyperbolic trajectories, 174–82
  Kepler's equation, 155–57, 168, see Appendix D page 701
  MATLAB algorithms, 163, 178, see Appendix D page 701
  oblateness, 241
  orbit equation, 171
  position as a function of time, 158–62, 171, 174–82, 183
Eccentricity
  chase maneuvers, 350–54
  elliptical orbits, 89–100
  hyperbolic trajectories, 150
  interplanetary trajectories, 442–43, 465, 467
  limiting values, 168–69
  non-Hohmann transfers, 341–5
  orbit formulas, 78–79
  orbital elements, 208–9, 210–11, 213
  plane change maneuvers, 366–67
  planetary departure, 443–44
  planetary ephemeris, 471, 472
  planetary flyby, 460
  planetary rendezvous, 452
  position as a function of time, 161, 168–9, 174–5
  preliminary orbit determination, 297
Ecliptic plane, 200
Effective exhaust velocity, 660
Eigenvalues, 513–17
Eigenvectors, 513–17
Elementary rotation matrix, 245
Elevation angles, 287–93, 297, 314–16, see Appendix D page 701
Elliptical orbits
  Hohmann transfers, 321–8
  non-Hohmann trajectories, 475–76
  position as a function of time, 157–72, 182–4
  two-body motion, 68–78
Empty masses, 667, 686
Energy
  circular orbits, 84
  conservation of, 83, 608–11
  dissipation, 589–96
  elliptical orbits, 89
  Hohmann transfers, 321
  hyperbolic trajectories, 108
  kinetic, 530–53, 586–91, 608–11
  law, 82–83
  non-Hohmann transfers, 340–41
  orbital elements, 208–9
  plane change maneuvers, 358
  position as a function of time, 183
  potential, 65–66, 704
  sinks, 590–93
  three dimensional orbits, 208–9
ENZ *see* East-North-Zenith
Ephemeris, 201–3, 470–75
Epochs, 398, see Appendix D page 701
Equations of motion
  double-gimbaled control moment gyros, 627
  dual-spin spacecraft, 589
  inertial frames, 62–66
  integration, 693–6
  interplanetary trajectories, 430–31
  linearization of relative motion, 400–2
  numerical integration, 693–6
  relative, 70–4, 400–6
  rocket vehicle dynamics, 656–58
  rotational, 497–501, 596–602
  satellite attitude dynamics, 574–82
  translational, 495–97
Equations of parabolas, 101
Equatorial frames
  see also geocentric…
  plane change maneuvers, 358–62, 363–64
  state vectors, 208–12
  three dimensional orbits, 199
  topocentric coordinates, 280–83, 285–88
Equilibrium points, 133–8
Escape velocity, 101, 143
Euler angle sequence, 224, see Appendix D page 701
Euler axis, 552–5
Euler, Leonhard, 24
Euler rotations, 552–69
Euler's angles, 208–9, 538–48, 582

Euler's equation
rigid-body dynamics, 524–30
satellite attitude dynamics, 582, 585–87, 613, 626
Euler principal angle, 555
Euler's method, 43, 48
Excess speed, 108–9
Excess velocity, 442–48, 451–58, 476–78
Exhaust, 660–64
Extremum, 679–81

F
Field-free space restricted staging, 667–78
Five-term acceleration, 30, 31
Flattening *see* oblateness
Flight path angles
elliptical orbits, 95
hyperbolic flyby, 464
Newton's law of gravitation, 17–18
non-Hohmann transfers, 339–40
parabolic trajectories, 101
rocket vehicle dynamics, 656–8
Flight time, 329–32
Floor, 168
Flow rates, 663
Fluids, 593–4
Flyby, 458–70
Flywheels, 615–31
Forces
see also gravitational…
bearing, 546–8
drag, 656, 666
gyroscopic, 546–8
lifting, 657
net, 21–3
nutation dampers, 595
point masses, 15–9
sphere of influence, 437–41
units of, 15–9
Free-fall, 16–8

G
Gauss's method of preliminary orbit determination, 297–312, see Appendix D page 701
GEO *see* geostationary equatorial orbits
Geocentric…
latitude, 281–2
orbits, 163–4, 532–3
position vectors, 256–63
right ascension-declination, 200–3
satellites, 341–3
Geocentric equatorial frames
coordinate transformations, 229–33, 285–6
MATLAB algorithms, 293, see Appendix D page 701
orbital elements, 208–15
perifocal frame, 229–33, 285–6
state vectors, 208–15, 233
topocentric transformations, 280–2
transformations, 229–33, 285–6
Geodetic latitude, 281–2
Geostationary equatorial orbits (GEO), 85–8
phasing maneuvers, 336–7
plane change maneuvers, 361–4, 372–3
Geosynchronous dual-spin communication satellites, 494
Gibb's method, 256–63, see Appendix D page 701
Gimbals, 492–4, 627–8
Gradient operator, 65
Gravitation
acceleration, 15–9, 233
attraction, 61–145
geocentric right ascension-declination, 201
point masses, 15–9
potential energy, 704, 706
restricted three-body motion, 131–2
satellite attitude dynamics, 631–2
sphere of influence, 437–41
Gravity assist maneuvers, 469
Gravity gradient stabilization, 631–44
Gravity turn trajectories, 665–7
Greenwich sidereal time, 276, 278, 280
Ground track, 244–9, see Appendix D page 701
Guided missiles, 658
Gyros
gyroscope equation, 537
gyroscopic attitude control, 615–31
gyroscopic forces, 546–8
gyroscopic moment, 537
motors, 528–30
rotors, 492–3, 528–30
satellite attitude dynamics, 574

H
Heliocentric trajectories, 441
approach velocity, 451
post-flyby, 458–70
speed, 446, 459
velocity, 442, 458–68
Heun's method, 44, 49–50, see Appendix D page 701
High-energy precession rates, 537–8

Hohmann transfers
bi-elliptic transfers, 328–32
common apse line, 338
interplanetary trajectories, 430–1, 475–81
non-Hohmann trajectories, 475–81
orbital maneuvers, 321–37, 338
phasing maneuvers, 332–7
plane change maneuvers, 355–67
planetary rendezvous, 451–2, 456
Horizon coordinate system, 284–9
Hyperbolas, 178, see Appendix D page 701
Hyperbolic trajectories
approach, 451–6, 481
departure, 442–8
excess velocity, 442–8, 451–69, 476–8
flyby, 458–70
position as a function of time, 174–82
rotations, 452–3
two-body motion, 104–13

I

Identity matrices, 218
Impulse
angular, 20–2, 500–1
coning maneuvers, 601–5
rendezvous maneuvers, 321–37, 411–8
rocket vehicle dynamics, 660, 665–7, 671–3, 678–85
Impulsive orbital maneuvers, 320–37
Inclination
double-gimbaled control moment gyros, 627
plane change maneuvers, 355–60
planetary ephemeris, 470–1
Sun-synchronous orbits, 237
three dimensional orbits, 208–9, 210
Inertia
see also moments of inertia
angular velocity, 129, 487–8, 617–8, 635–6
equations of two-body motion, 62–9
gravity-gradient stabilization, 633–5, 640–4
matrices, 508, 509, 513–9, 533
rigid-body dynamics, 501–23, 547
tensors, 508–16, 516, 523
torque-free motion stability, 586
velocity, 131, 293, 487–91, 635–7
Initial value problem, 38, 42
Interchange of the dot and cross, 10
Insertion points, 358–9
Integration, equations of motion, 693–9
Intercept trajectories, 350–4
Intermediate-axis spinners, 600–1
Interplanetary dual-spin spacecraft, 593
Interplanetary trajectories, 429–84
ephemeris, 470–5
flyby, 458–70
Hohmann transfers, 430–1
method of patched conics, 441–2
non-Hohmann, 475–81
patched conics, 441–2
planetary departure, 442–8
planetary ephemeris, 470–5
planetary flyby, 458–70
planetary rendezvous, 451–8
rendezvous, 432–6, 451–8
sensitivity analysis, 448–50
sphere of influence, 437–41
three dimensional orbits, 199
Iterations, 135, 136–7, 164–5, 178, 190, 312, see Appendix D page 701

J

Jacobi constant, 139–45
Julian centuries, 471–3
Julian days (JD), 276–9
numbers, 276–9, 471–3, see Appendix D page 701
Jupiter's right ascension, 286–7

K

Kepler, Johannes, 76
Kepler's equation
Bessel functions, 169–70
eccentric anomaly, 158–65, 175, see Appendix D page 701
hyperbola eccentric anomaly, 175–6, 178–82, see Appendix D page 701
hyperbolic trajectories, 176–82
MATLAB algorithms, 178–82, 188, see Appendix D page 701
Newton's method, see Appendix D page 701
position as a function of time, 76–9, 92–4, 160–76, 178–9, 182–9
universal variables, 182–94
Kepler's second law, 76
Kilograms, 19–24
Kinematics, 9–15, 486–95
Kinetic energy, 530–3, 586–8, 591, 608–11

L

Lagrange coefficients
MATLAB algorithms, see Appendix D page 701
position as a function of time, 191–4

preliminary orbit determination, 265–9, 298–300, 311
two-body motion, 117–29
Lagrange multiplier method, 678–86
Lagrange points, 133–9
Lambert's problem
chase maneuvers, 350–4
MATLAB algorithms, 270, see Appendix D page 701
patched conics, 475–81
preliminary orbit determination, 263–75, 616–22, see Appendix D page 701
Laplace limit, 168–9
Latitude, 87–8, 281–2, 286–7, 358–61
Latus rectum, 81–2, 366–7
Launch azimuth, 359–61
Launch vehicle boost trajectories, 656–8
Leading-side flyby, 458–9, 461–2
LEO *see* low-earth orbits
Libration points, 133–9
Lifting forces, 657
Limiting values, 168
Linear momentum, 498–9
Linearized equations of relative motion, 403
Local horizon, 81
Local sidereal time, 276, 278–80, see Appendix D page 701
Longitude of perihelion, 471
Low earth orbits (LEO), 84–4, 361–5
Low-energy precession rates, 536–7
Lunar trajectories, 442
LVLH frame, 392, 394, see Appendix D page 701

M

Major-axis spinners, 586–8, 641
Mars missions, 436
Mass
gravitational potential energy, 704–6
moments of inertia, 503
nutation dampers, 593–601
point masses, 15–9
ratios, 671–3, 682, 684–5, 686
rocket vehicle dynamics, 658–65
MATLAB algorithms, see Appendix D page 701
acceleration, 696
angular position, 293, see Appendix D page 701
chase maneuvers, 353–4
classical orbital elements, 261–3, see Appendix D page 701
eccentric anomaly, 163–5, 167–9, see Appendix D page 701
epochs, 398, see Appendix D page 701
Gauss's method of preliminary orbit determination, 304–12, see Appendix D page 701
geocentric equatorial position, 293, see Appendix D page 701
Gibbs method of preliminary orbit determination, 260–3, see Appendix D page 701
hyperbola eccentric anomaly, 178, see Appendix D page 701
Julian day number, 471, see Appendix D page 701
Kepler's equation, 178–82, 188, see Appendix D page 701
Lagrange coefficients, see Appendix D page 701
Lambert's problem, 270, see Appendix D page 701
local sidereal time, 279, see Appendix D page 701
month identity conversions, see Appendix D page 701
Newton's method, 163, 178, 188, see Appendix D page 701
non-Hohmann trajectories, 476
numerical designation conversions, see Appendix D page 701
orbital elements from the state vector, 209–12, see Appendix D page 701
planet identity conversions, see Appendix D page 701
planet state vector calculation, 471, see Appendix D page 701
planetary ephemeris, 471–2, see Appendix D page 701
position as a function of time, 163, 178, 188, 191, see Appendix D page 701
preliminary orbit determination, 260, 270, 279, 293, see Appendix D page 701
range, 293, see Appendix D page 701
sidereal time, 279, see Appendix D page 701
spacecraft trajectories, 476, see Appendix D page 701
sphere of influence, 476, see Appendix D page 701
state vectors, 66, 231, 293, 394, see Appendix D page 701
Stumpff functions, 185, see Appendix D page 701
three-body systems, 694–9
time lapse, see Appendix D page 701
transformation matrices, 232
universal anomaly, 188, see Appendix D page 701
universal Kepler's equation, 188, see Appendix D page 701
Universal Time, 476, see Appendix D page 701
Matrices
see also transformation…
Clohessy–Wiltshire frames, 409–10, 418, 421
diagonal, 403–5
direction cosines, 217–22, 224–9, 231, 232, 286–8
identity matrices, 218

Matrices (*Continued*)
inertia, 508, 509, 513–9, 533
moments of inertia, 508–17, 533
orthogonal, 218, 231, 508, 539
rotation, 549–52
unit, 218
Mean…
anomaly, 157–61, 167–9, 174, 209
distance, 94
longitude, 471
motion, 158, 241, 405, 407, 414, 417
Mercator projections, 244–5
Method of patched conics, 441–2, 475–6
Minor-axis spinners, 586–8, 641–2
Missiles, 658
Molniya orbit, 238–9
Moments, 410–14, 435–40, 454–6
Moments of inertia
gravity-gradient stabilization, 633–4
matrices, 508–17, 533
parallel axis theorem, 517–23
principal, 503, 515–7, 520–3, 524, 547
rigid-body dynamics, 501–23, 547
torque-free motion stability, 589
Momentum
see also angular…
absolute angular, 498–501
conservation of, 611–12
exchange systems, 615–31
linear, 498–9
rigid-body rotational motion, 501
rocket vehicle dynamics, 658–60
yo-yo despin, 608–15
Month identity conversions, see Appendix D page 701
Moon ephemeris, 203
Moving reference frames, 70–3, 421, 488, 528
Moving vectors, 24–8
Multi-stage vehicles, 672, 676, 677
Mutual gravitational attraction, 61–113
see also two-body motion

N

Natural frequency, 46
n body equations of motion, 693–9
Net forces, 19, 22–3, 36–7
Net moments, 498–501, 528–30
Newton's law of gravitation, 15–9, 437–40
Newton's laws of motion, 19–24, 547
Newton's method
Kepler's equation, see Appendix D page 701
MATLAB algorithms, 163, 178, 188, see Appendix D page 701
preliminary orbit determination, 268, 269, 271
roots, 162–3
universal Kepler's equation, 184–5, 188–9, see Appendix D page 701
Newton's second law of motion, 19–24, 547
Node regression, 236–8
Nodes, 42, 43
Noncoplanar orbits, 355–67
Non-Hohmann transfers, 338–43, 475–81
Non-rotating inertial frames, 32
Normal acceleration, 15, 18, 257
Numerical designation conversions, see Appendix D page 701
Numerical integration, equations of motion, 693–9
Nutation
dampers, 593–601
double-gimbaled control moment gyros, 627–8
rigid-body dynamics, 534–7
spinning tops, 538
torque-free motion, 574–5

O

Oblateness
preliminary orbit determination, 281
satellite attitude dynamics, 579–80, 593–4
spinner stability, 589
three dimensional orbits, 233–44
Obliquity of the ecliptic, 200
One-dimensional momentum analysis, 658–60
Optimal staging, 678–86
Orbit formulas, 74–82, 188
Orbit rotation, 366–7
Orbital elements
geocentric equatorial frame, 203–8
interplanetary trajectories, 471, 472, 476
non-Hohmann trajectories, 477
oblateness, 241–4
planet state vectors, 471, see Appendix D page 701
planetary flyby, 462–3
preliminary orbit determination, 260–3, 270–3, 293–6
state vectors, 208–15, 231, see Appendix D page 701
three dimensional orbits, 208–15
Orbital maneuvers, 319–90
apse line rotation, 343–50
bi-elliptic Hohmann transfers, 328–32
chase maneuvers, 350–4
common apse line, 338–43
Hohmann transfers, 321–8, 338

impulsive, 320–74
non-Hohmann transfers, 338–43
phasing maneuvers, 332–7
plane change, 355–67
two-impulse rendezvous, 411–8
Orbital parameters, 95, 96, 163, 173, 233, 235, 344, 346, 351, 365, 443
Orbiting Solar Observatory (OSO-1), 590
Ordinary differential equation, 38, 41, 46
Orientation
delta-v maneuver, 341–3, 348–50
gravity-gradient stabilization, 641–4
rigid-body dynamics, 538
Orthogonal transformation matrices, 393, 516–20, 526, 540–3
Orthogonal unit vectors, 13–5
Orthonormal basis vectors, 216
Osculating plane, 13
Overall payload fractions, 671, 686

P

Parabolic trajectories, 100–3, 172–3
Parallel axis theorem, 517–23
Parallelogram rule, 2–3
Parallel staging, 670
Parallelepipeds, 546–8, 642–3
Parameter of the orbit, 81
Parking orbits, 443–8, 479–80
Particles, 1, 15
Passive altitude stabilization, 635
Passive energy dissipation, 593–600
Patched conics, 441–2, 475–81
Payloads
masses, 667, 671–2
ratios, 668–9, 671, 675–8
velocity, 673–4
Periapse
angle to, 456–8
orbit formulas, 81
plane change maneuvers, 355–67
radius, 443–4, 452–6
speed, 444
time since, 155–7
two-body motion, 82, 88–90
Perifocal frame, 113–6, 229–33, 259–60
Perigee
advance, 236–8, 242
altitude, 95–8, 100, 270–3, 274–5
argument of, 208–9, 211–12, 231, 235–8, 239–43
location, 125–6, 446–8
orbit equation, 100–3
passage, 164–5, 179, 270–3
radius, 96, 99–100, 274–5
time since, 179, 208–9, 270–3, 352
time to, 273–5
towards the sun, 167–72
velocity, 96–7
Perihelion radius, 467, 469
Period of orbit
circular orbits, 84, 85
elliptical orbits, 92–3, 100
orbital elements, 214–5
rendezvous opportunities, 433, 436
restricted three-body motion, 130
Perturbations
gravitation, 200–1
oblateness, 235
sphere of influence, 439–40
torque-free motion stability, 586
Phase angles, 432–6
Phasing maneuvers, 332–7, 432
Physical data, 689–90
Pitch, 449–52, 635–44
Pitchover, 658
Pivots, 613, 627–8
Plane change maneuvers, 355–67
Planetary…
see also interplanetary trajectories
departure, 442–8
ephemeris, 470–1
flyby, 458–70
rendezvous, 451–8
Planets
geocentric right ascension-declination, 200–3
identity conversions, see Appendix D page 701
state vectors, see Appendix D page 701
Planning Hohmann transfers, 326
Point masses, 1–54
absolute vectors, 29–37
force, 15–9
gravitational potential energy, 704–6, see Appendix D page 701
kinematics, 10–5
mass, 15–9
moments of inertia, 503–4
moving vector time derivatives, 24–8
Newton's law of gravitation, 15–9
Newton's law of motion, 19–24
relative motion, 29–37
relative vectors, 29–37

Polar coordinates, 82
Position errors, 448–50
Position as a function of time, 155–94
circular orbits, 156–7
elliptical orbits, 157–72, 183–4
hyperbolic trajectories, 174–82
MATLAB algorithms, 163, 178, 188, 191, see Appendix D page 701
parabolic trajectories, 172–3
universal variables, 182–94
Position vectors
absolute, 29–37
equatorial frames, 229–30
geocentric, 229, 256–63
Gibb's method, 256–63
gravitational potential energy, 704
gravity-gradient stabilization, 631–2, 638
inertial frames, 62–6
Lagrange coefficients, 117–29, 191–4
MATLAB algorithms, 209–12, 231, 593, see Appendix D page 701
nutation dampers, 594
orbit formulas, 78–81
perifocal frame, 114–6
point masses, 2–7, 29–37
preliminary orbit determination, 256–63, 265, 270, 274, 280, 281, 283, 285, 287–8
restricted three-body motion, 131–2
rigid-body dynamics, 486–95, 497–501
satellite attitude dynamics, 594, 597, 605, 609, 631
three dimensional geocentric orbits, 204–6
two-body motion, 62–6, 70–3, 114–9
two-impulse maneuvers, 414, 416
yo-yo despin, 609
Post-flyby orbits, 458–70
Potential energy, 65, 704–6
Pound, 19
Powered ascent phase, 358
Pre-flyby ellipse, 463–4
Precession
double-gimbaled control moment gyros, 627
nutation dampers, 596
rigid-body dynamics, 536–8
satellite attitude dynamics, 578–80, 584, 596, 602–3
spinning tops, 534–8
thrusters, 607
torque-free motion, 578–81
Predictor-corrector method, 48–9, see Appendix D page 701
Preliminary orbit determination, 255–312
angle measurements, 289–97
Gauss's method, 297–312, see Appendix D page 701
Gibbs method, 256–63, see Appendix D page 701
Lagrange coefficients, 265–9, 298–300, 311, see Appendix D page 701
Lambert's problem, 263–75, 616–21, see Appendix D page 701
MATLAB algorithms, 260, 270, 279, 293, see Appendix D page 701
range measurements, 289–96
sidereal time, 275–80
topocentric coordinate systems, 280–9
Primed systems, 216, 218, 219, 508–9
Principal directions, 513–7, 520–3
Principal moments of inertia, 503, 514–7, 520–4, 547
Prograde…
coasting flights, 476–8
precession, 578–81
trajectories, 264–5
Prolate bodies, 579–80, 494
Propellant
field-free space restricted staging, 667–78
Lagrange multiplier method, 678–86
mass, 320–1, 373, 446–7
rocket vehicle dynamics, 658–60, 662–71, 673–6
thrust equation, 658–60
Propellers, 488–90
Propulsion, 84, 320, 358, 369, 372, 469, 655, 660, 670

Q

Quaternion, 552–61

R

r-bars, 392
Radar observations, 57, see Appendix D page 701
Radial distances, 85–8
Radial release, 613–15
Radius
aiming, 106, 113, 451, 453, 456, 457, 465
apoapse, 355
azimuth, 75, 76
capture, 454
earth's sphere of influence, 441, 476–77
gravitational potential energy, 704–6
periapse, 360–2, 369, 370, 372–3
perigee, 96, 99, 208–12
perihelion, 430, 431
true-anomaly-averaged, 94, 95–7
Radius of curvature, 13, 14, 15, 19, 656
Range measurements, 289–97, see Appendix D page 701

Rates of precession, 534–8, 540–1, 543
Rates of spin, 534–8, 540–1, 543
Rectilinear motion, 53, see Appendix D page 701
Rectilinear orbit, 147
Regulus, 201
Relative acceleration
angular, 526
point masses, 26, 30–1
relative motion and rendezvous, 392–5
rigid-body kinematics, 486–95
two-body motion, 70
Relative angular…
acceleration, 526
momentum, 74–6
velocity, 432–33
Relative linear momentum, 499
Relative motion, 391–427
Clohessy–Wiltshire equations, 407–11, 417–8
close-proximity circular orbits, 419–21
co-moving reference frames, 393–5, 398, 402–3
linearization of equations of relative motion, 400–5
point masses, 29–37
restricted three-body motion, 68, 141
two-impulse maneuvers, 411–8
Relative position
point masses, 29, 31, 33
preliminary orbit determination, 283
sphere of influence, 438
two-body motion, 70
Relative vectors, 29–37, 56
Relative velocity
Clohessy–Wiltshire equations, 408–9
close-proximity circular orbits, 419–21
point masses, 30, 31, 34
relative motion and rendezvous, 392–5
rigid-body kinematics, 486–95
two-body motion, 70
two-impulse maneuvers, 411, 414, 418
Rendezvous, 391–427
Clohessy–Wiltshire equations, 407–11, 417–8
close-proximity circular orbits, 419–21
co-moving reference frames, 393–5, 398, 402–3
equations of relative motion, 400–5
Hohmann transfers, 321–2
interplanetary trajectories, 432–6, 453–8
relative motion equations, 400–5
two-impulse maneuvers, 411–8
Restricted staging, 667–78
Restricted three-body motion, 129–45
Retrofire, 321, 326, 333
Retrograde orbits, 235, 359–60
Right ascension
oblateness, 233–44
planetary ephemeris, 471
preliminary orbit determination, 283–4, 286–7, 291–4
state vectors, 204–7
three dimensional orbits, 200–3
Rigid-body dynamics, 485–571
Chasles' theorem, 485
equations of rotational motion, 497–501
equations of translational motion, 495–7
Euler angles, 538–48
Euler's equations, 524–30
inertia, 501–23
kinematics, 486–95
kinetic energy, 530–33
moments of inertia, 501–23
moving vector time derivatives, 24–28
parallel axis theorem, 517–23
pitch, 549–52
plane change maneuvers, 366–7
roll, 549–52
rotation of the ellipse, 366–7
rotational motion, 497–501
satellite attitude dynamics, 584–9
spinning tops, 533–8
translational motion, 495–7
yaw, 549–52
Rocket equation, 655–6
Rocket vehicle dynamics, 655–87
equations of motion, 656–8
field-free space restricted staging, 667–78
impulsive orbital maneuvers, 320–1
Lagrange multiplier method, 678–86
motors, 320–1
optimal staging, 678–86
restricted staging, 667–78
rocket performance, 660–7
staging, 667–86
thrust equation, 658–60
Rods, 505–6, 519–22
Roll, 549–52, 635–44
Roots, 514, 515
Rotating platforms, 537, 546
Rotation
axis of, 200
Cartesian coordinate systems, 216–9
coordinate transformations, 216–9
geocentric equatorial frames, 229–31
matrices, 549–52

Rotation (*Continued*)
perifocal frames, 229–31
three dimensional orbits, 199, 216–9
true anomaly, 347–8
Rotational…
equations of motion, 497–501, 588–91
kinetic energy, 530–3, 586–8, 609–11
motion equations, 497–501, 588–91
Rotationally symmetric satellites, 574
Round-trip missions, 433–5
Routh–Hurwitz stability criteria, 600–1
Runge-Kutta method, 42–8, see Appendix D page 701
Runge-Kutta-Fehlberg method, 50–1, see Appendix D page 701

S

Satellite attitude dynamics, 573–644
axisymmetric dual-spin satellites, 589–93, 630–1
coning maneuvers, 601–5
control thrusters, 605–8
despin mechanisms, 608–15
dual-spin spacecraft, 589–93
gravity-gradient stabilization, 631–44
gyroscopic attitude control, 615–31
gyrostats, 589–93
nutation dampers, 593–601, 608
passive energy dissipaters, 593–601
rigid-body dynamics, 485
thrusters, 605–8
torque-free motion, 574–84, 584–9, 619–20, 630–1
yo-yo despin, 608–15
Satellites
dual-spin, 589–93, 630–1
earth, 199, 239–43
geocentric, 341–3
orientation, 641–3
Saturation, 627
Second order differential equations, 404–6
Second zonal harmonics, 234
Semi-latus rectum, 81
Semimajor axis
elliptical orbits, 89, 92
equation, 183–4
hyperbolic trajectories, 105
phasing maneuvers, 333
planetary ephemeris, 471, 472
three dimensional orbits, 208–9, 235–6
Semiminor axis equation, 183
Sensitivity analysis, 448–50
Series two-stage rockets, 669–74, 681, 683
SEZ *see* South-East-Zenith
Shafts on rotating platforms, 546–7
Sidereal time, 275–80, 293, see Appendix D page 701
Single stage rockets, 663–5
Single-spin stabilized spacecraft, 590–3
Slant ranges, 297–8, 301–3
Slug, 19
Sounding rockets, 656, 658, 661–5
South-East-Zenith (SEZ) frame, 284
Space cones, 580
Spacecraft trajectories, 475, see Appendix D page 701
Specific energy
circular orbits, 84
elliptical orbits, 92
Hohmann transfers, 321
hyperbolic trajectories, 108
non-Hohmann transfers, 340
three dimensional orbits, 208
Specific impulse
impulsive orbital maneuvers, 320–1
rocket vehicle dynamics, 658, 660, 663, 665, 668, 671–7
Speed
circular orbits, 84–5
elliptical orbits, 97
excess, 108
hyperbolic trajectories, 178, 179
parabolic trajectories, 100, 101
planetary departure, 442
yo-yo despin, 610, 614–5
Sphere of influence, 437–41, 442, see Appendix D page 701
Spheres, 703–6
Spherically symmetric distribution, 703–6
Spin
accelerations, 622–7
angles, 607
rates, 488, 491, 534–6
stabilized spacecraft, 573
Spinning rotors, 537
Spinning tops, 533–8
Stability
dual-spin spacecraft, 589–93
gravity-gradient stabilization, 635
nutation dampers, 593–4
spinning satellites, 573
torque-free motion, 584–9
Stable pitch oscillation frequency, 638
Stage, 42
Staging, 667–86

Stars, 201
State vectors
geocentric equatorial frame, 203–8, 229–30
MATLAB algorithms, 205–6, 231, 293, see Appendix D page 701
non-Hohmann trajectories, 475–6
orbital elements, 208–15, see Appendix D page 701
planetary ephemeris, 470–1
preliminary orbit determination, 289, 293–4, 297–8
three dimensional orbits, 208–15
two-impulse maneuvers, 413
Step mass, 680, 682, 684–6
Strap-on boosters, 670
Structural ratios, 667–8, 671, 672, 675, 677
Stumpff functions, 184–5, 191, 268, see Appendix D page 701
Sun-synchronous orbits, 237–40
Sunlit side approaches, 448, 462–3, 465–6, 467–8
Synodic period, 433, 436

T

Tandem two-stage rockets, 669–74
Tangential acceleration, 12
Tangential release, 614
Target vehicles, 400–21
Taylor series, 40–2
Tension, 613–15
Three dimensional curvilinear motion, 10–15
Three dimensional orbits, 199–249
celestial sphere, 201–4
coordinate transformations, 216–29
declination, 200–3
earth's oblateness, 233–44
geocentric equatorial frame, 203–8, 229–33
geocentric right ascension-declination, 200–3
oblateness, 233–44
orbital elements, 208–15
patched conics, 441–2
perifocal frame transformations, 229–33
right ascension, 200–3
state vectors, 203–15
Three-body systems, 68, 437–41, 693–9
Three-stage launch vehicles, 674, 676, 685
Thrust equation, 658–60
Thrust-to-weight ratio, 661
Thrusters, 605–8
Tilt angles, 536
Time
see also position as a function of time
dependent vectors, 26
derivatives
Lagrange coefficients, 118–9, 123, see Appendix D page 701
moving vectors, 24–8
relative motion, 29–37
Hohmann transfers, 326–7
lapse, see Appendix D page 701
manned Mars missions, 436
to perigee, 273, 275
satellite attitude dynamics, 603–5, 611–12
since periapse, 185
since perigee, 179, 208, 241, 270–3, 352
Titan II, 670
Topocentric coordinates, 280–9, 291–3
Torque
axial, 624–7
free motion, 574–89
rigid-body dynamics, 500–1
satellite attitude dynamics, 513–14, 521–5, 533
Trailing-side flyby, 458, 459, 460
Trajectory, 11, see Appendix D page 701
Transfer ellipses, 321–4, 430–1, 434
Transfer times, 265
Transformation matrices
MATLAB algorithms, 226
moments of inertia, 508–15
orthogonal, 393, 508–15, 538–9
pitch, 549–52
relative motion and rendezvous, 393
rigid-body dynamics, 508–15, 538–9
roll, 549–52
satellite attitude dynamics, 637
three dimensional orbits, 216–29
topocentric horizon system, 284–7
torque-free motion, 583–4
two-impulse maneuvers, 414–6
yaw, 549–52
Translational motion equations, 495–7
Transverse bearing loads, 548
True anomalies
averaged orbital radius, 94, 95–97
elliptical orbits, 161
hyperbolic flyby, 464–5
hyperbolic trajectories, 104, 112, 179, 188–91
Lagrange coefficients, 119–25
non-Hohmann transfers, 338–43
parabolic trajectories, 100–1, 184
plane change maneuvers, 369
position as a function of time, 155–9, 163–5, 171–4
preliminary orbit determination, 263–75

True anomalies (*Continued*)
rendezvous opportunities, 432
three dimensional orbits, 208–9, 212, 233, 240–1
time since periapse, 185
universal variables, 186–8
Turn angles, 105, 112, 452, 453, 459, 461, 465
Truncation error, 41, see Appendix D page 701
Two-body motion
angular momentum, 74–82
energy law, 82–3
equations of motion, 62–74
equations of relative motion, 70–74
hyperbolic trajectories, 104–13
inertial frame equations of motion, 62–9
Lagrange coefficients, 117–29
mutual gravitational attraction, 61–113, 129
orbit formulas, 74–82
parabolic trajectories, 100–3
perifocal frame, 113–6
restricted three-body motion, 129–45
three dimensional orbits, 204
Two-impulse maneuvers, 332–7, 411–8
Two-stage rockets, 669–74, 681, 683

U
unit binormal vector, 13, 14, 55
Unit matrices, 218
Unit normal vector, 13, 15
Unit tangent vector, 12, 14
Unit triads, 220
Unit vectors, 2, see Appendix D page 701
gravitational potential energy, 704
Lagrange coefficients, 117
moments of inertia, 502
point masses, 5–7, 32
three dimensional orbits, 216–29
Units of force, 19
Universal anomaly, 184–94
Universal gravitational constant, 15, 63, see Appendix D page 701
Universal Kepler's equation, 188–9, 310
Universal Time (UT), 276–9
Universal variables, 182–94
Unprimed systems, 216–9, 512
UT *see* Universal time

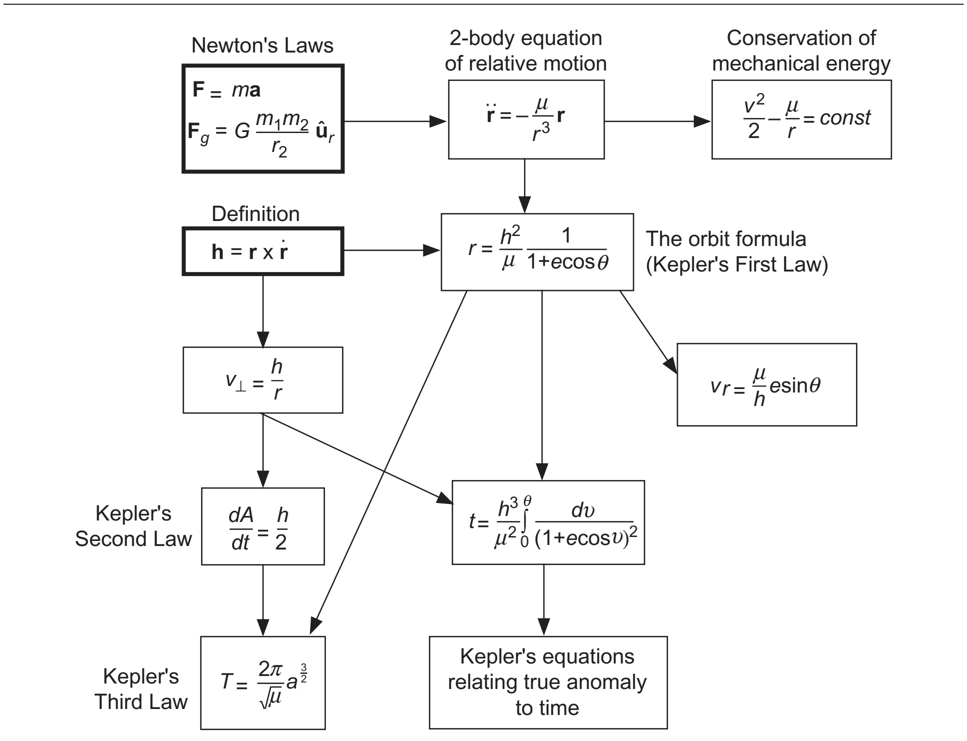
V
Vectors, 33–105
see also position…; state…; two-body motion; unit…; velocity…
absolute, 29–37
direction cosine, 292–3, 297
eigenvectors, 513–14
geocentric equatorial frame, 231
moving, 24–8
Lagrange coefficients, 117–29, 191
MATLAB algorithms, 209–13, 231, see Appendix D page 701
orthogonal unit, 13
orthonormal basis, 216
perifocal frame, 113–4
point masses, 2–10, 24–8, 29–37
preliminary orbit determination, 256–63, 265, 270, 274
relative, 29–37, 56
restricted three-body motion, 131
rotations, 357–8
satellite attitude dynamics, 574, 575, 608–9
time dependent, 26
time derivatives, 24–28
two-impulse maneuvers, 413
weight, 559
Velocity
see also delta-v…
errors, 448–50
escape, 101, 143
excess, 442–8, 451–8, 476–8
geocentric orbits, 200–3
Hohmann transfers, 321
non-Hohmann transfers, 338–40
plane change maneuvers, 355–9
relative motion and rendezvous, 391–5
rocket vehicle dynamics, 667–78
Vector triple product, 9
Venus ephemeris, 201–2
Venus flyby, 462–4
Vernal equinox, 200–4
Visible surface areas, 86–7

W
Wait time, 435–6
Weight, 15–19, 20, 43, 64
Weight vectors, 534
Wobble angles, 579

Y
Yaw, 549–52, 635–44
Yo-yo despin, 608–15
Yaw-pitch-roll sequence, 549–50, 557–8, see Appendix D page 701

Z
Zonal variation, 233–44

A road map



See Appendix B (p. 691) for more information.