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A Unified Theory with Applications to Geophysics, Celestial Mechanics, Astrophysics and Cosmology

Second Edition

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Preface to the Second Edition

The first edition of this book was published in 1987. Since that time new scientific and practical results in application of Jacobi dynamics were obtained. The results, which we include into the second edition, are as follows.

The oldest scientific problem of the origin and evolution of the Solar System bodies on the basis of the Jacobi's oscillating dynamics has found its resolution. We discovered by calculation that the present day orbital velocity of each planet is equal to the fist cosmic velocity of the contracting Protosun having its radius reduced to the semi-major axis of the planet's orbit. And also the orbital velocities of the planet's moons are equal to the first cosmic velocities of the corresponding protoplanets having their radiuses successively reduced to the semi-major axes of the moon's orbits. It looks like the protoplanets and their protosatellites were created from the separating upper shells of their contracting parents. The Sun itself and the small bodies like the comets, asteroids, meteors and so on, have the same history of creation.

The discovered mechanism of the Solar System body creation proves to be the physical basis of the Kepler's laws and the inverse square distance law of the outer force field distribution of a self-gravitating body. It also proves the idea that the 'heavens power of gravity' of body's orbital motion has the electromagnetic nature and is induced by the parental body.

We succeeded in understanding physical meaning of the Jacobi's virial equation obtained from the Newton's equations of motion written for a system of n interacting mass-particles. It was found that in his n-body problem, Jacobi, while transforming the initial equations, has converted the Newton's forces and moments of the interacting particles into their volumetric values, namely, into the energy and oscillating moment of inertia. Doing like this, Jacobi has changed the force as a measure of the particles interaction by energy of their interaction. From physical viewpoint Jacobi's approach opens the way to search the common nature for all the known physical models of the matter interaction. In order to prove this idea, we show in the book that the Jacobi's virial equation, besides the Newton's equations, is also derived from the equations of Euler, Hamilton, Einstein and quantum mechanics. This allowed us to put the subtitle of the book as a 'Unified theory of the interacting matter motion'.

We discuss in the book the main scientific discovery, made by the geodetic artificial satellite study that the Earth and the Moon do not stay in hydrostatic equilibrium. It puzzles us because the hydrostatics, up to now, is accepted as the physical basis in dynamics of stars, planets and other celestial bodies. In addition to this, we found that the Earth has no equilibrium between the kinetic and potential energy ($K/U \approx 1/300$). This is because the existing theories in dynamics do not take into account the kinetic energy of the body's interacting mass particles. It makes incorrect formulation of the problem of the equilibrium state. The conclusion is made that the conditions of the dynamical equilibrium state, as an alternative to the hydrostatics, in celestial mechanics, stellar dynamics and astrophysics should be introduced.

We found that the above fundamental effects are well explained by the theory of Jacobi dynamics which considers just self-gravitating, but not hydrostatically state systems and where relationship between the energy and the polar moment of inertia is the basic physical effect. In this connection this problem and physical meaning of the Jacobi's virial equation in the new Chapter 2 are discussed. In order to demonstrate all new effects in dynamics of stars, planets and satellites, including creation and evolution of the Solar system, Chaps. 6, 7, and 8 were revised and updated by applying the above discoveries. Also, the new Chap. "The Nature of Electromagnetic Field of a Celestial Body and Mechanism of Its Energy Generation" is introduced. The former Chaps. 1, 2, 3, 4, and 5 under new numbers were as a whole preserved. The second edition was prepared by V.I. Ferronsky.

V.I. Ferronsky

Preface to the First Edition

This book sets forth and builds upon the fundamentals of the dynamics of natural systems in formulating the problem presented by Jacobi in his famous lecture series "Vorlesung über Dynamik" (Jacobi 1884).

In the dynamics of systems described by models of discrete and continuous media, the many-body problem is usually solved in some approximation, or the behavior of the medium is studied at each point of the space it occupies. Such an approach requires the system of equations of motion to be written in terms of space co-ordinates and velocities, in which case the requirements of an internal observer for a detailed description of the processes are satisfied.

In the dynamics discussed here we study the time behavior of the fundamental characteristics of the physical system, i.e. the Jacobi function (polar moment of inertia) and energy (potential, kinetic and total), which are functions of mass density distribution, and the structure of a system. This approach satisfies the requirements of an external observer. It is designed to solve the problem of global dynamics and the evolution of natural systems in which the motion of the system's individual elements written in space co-ordinates and velocities is of no interest. It is important to note that an integral approach is made to internal and external interactions of a system which results in radiation and absorption of energy. This effect constitutes the basic content of global dynamics and the evolution of natural systems.

From the standpoint of methodology, the integral approach has an important advantage. In this approach the integral character of the principle of least action – the basic philosophical principle of mechanics and physics – is fully realized. It is achieved by using a canonical pair consisting of the Jacobi function and frequency in writing the basic equation of global dynamics. The practical use of this pair in Jacobi's virial equation made it possible to farther generalize the forms of motion and to show that the non-linear oscillations of a system is such a generalization.

We note that the ten well-known integrals of motion in the many-body problem in its classical formulation should be regarded as historically the earliest equations of the integral type. These integrals, however, reflects not the specific nature of a system under consideration but the general properties of space and time, i.e. homogeneity of space and time and isotropicity of space.

The first non-trivial equation of dynamics in terms of the integral characteristics of a system is Jacobi's virial equation, which describes changes in the moment of inertia (Jacobi function) as a function of time. The next step in this direction was taken by Chandrasekhar (1969). He used and developed for solution of problems in mechanics the method of moments, so called in analogy to the method well known in mathematical physics. However, the problem of non-trivial solution of the non-linearized equations in terms of integral characteristics was not solved in either of these cases.

Our work began in 1974. As a result, a number of articles on the theory of virial oscillations of celestial bodies were published in the journal Celestial Mechanics and other periodicals (Ferronsky et al. 1978, 1979a–c, 1981, 1982, 1984, 1985). The theory was based on solution of Jacobi's virial equation for conservative and dissipative systems.

To solve Jacobi's initial equation, we first used the heuristically found relationship between the potential energy and moment of inertia of a system, which was expressed in terms of the product of the corresponding form factors. It was found that the product depends little on the law of distribution of mass density for a wide range of formal, non-physical systems. It was then demonstrated that in the asymptotic limit of simultaneous collision of the particles constituting a system, the observed constancy of the product of form factors remained valid without any restrictions within the framework of the Newton and Coulomb interaction laws. The invariant found was also demonstrated to be valid for the widely used relativistic and non-relativistic physical models of natural systems. It enabled us to derive from Jacobi's equation a simple form of the equation of virial oscillations with one unknown function and to find its rigorous solution. The equation obtained describes the dynamics of a wide class of physical systems ranging from empty space-time and collapsing stars to the atom. Thus, it was established that the theory of virial oscillations of celestial bodies was valid far beyond the limits of celestial mechanics based on Newton's law of equations.

The work was done on concepts of Professor V.I. Ferronsky and under his supervision. Chaps. 1 and 7 were written by V.I. Ferronsky and S.A. Denisik; Chaps. 2, 3, 4, 5, 6 and 8 were written by S.V. Ferronsky.

This book presents a systematic description of our research work. It is intended for researchers, teachers and students engaged in theoretical and experimental research in the various branches of astronomy (astrophysics, celestial and stellar mechanics and radiophysics), geophysics (physics of the Earth, atmosphere and oceans), planetology and cosmogony, and for students and postgraduates of classical, statistical, quantum and relativistic mechanics and hydrodynamics.

It is our pleasant duty to express sincere gratitude to Professor G.N. Duboshin of the M.V. Lomonosov Moscow State University for his constant support and encouragement. We are indebted to Professor E.P. Aksenov, Director of the Sternberg Institute of Astronomy, Moscow, who organized helpful discussions of our work at a number of seminars. We also wish to express our gratitude to Dr. L. Osipkov from Leningrad University and Drs J. Schmidt, A. Lorenz, M. Mehta and T. Akity from the Division of Research and Laboratories, International Atomic Energy Agency, who read several chapters of the manuscript and made editorial contributions, and to Miriam Lewis, who edited the final manuscript. We also wish to thank Renate B. Blamhofer for assistance in preparing the book. We are particularly indebted to the International Atomic Energy Agency and its Division of Publications for support and assistance in preparation of the camera-ready manuscript.

The authors

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Chapter 1
Introduction: General Principles and Approaches in Dynamics

In 1842–1843, when Jacobi was a professor at Königsberg University, he delivered a special series of lectures on dynamics. The lectures were devoted to the dynamics of a system of *n* mass points, whose motion depends only on the distances between them and is independent of velocities. In this connection, by deriving the law of conservation of energy from the equations of motion of mass points for a conservative system, where the force function is a homogeneous function of space co-ordinates, Jacobi gave this law an unusual form and a new content. In transforming the equations of motion, he introduces an expression for the system's center of mass. Then, following Lagrange, he separated the motion of the center of mass from the relative motion of the mass points. Making the center of mass coincide with the origin of the co-ordinate system, he obtained the following equation (Jacobi 1884):

$$\frac{d^2}{dt^2}\left(\sum m_i r_i^2\right) = -(2k + 4)U + 4E, \tag{1.1}$$

where m_i is the mass point i; $r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$ the distance between the points and the center of mass; k the degree of homogeneity of the force function; U the system's potential energy; and E its total energy.

When $k = -1$, which corresponds to the interaction of mass points according to Newton's law, and denoting

$$\frac{1}{2}\sum m_i r_i^2 = \Phi,$$

Jacobi obtained

$$\ddot{\Phi} = U + 2T = 2E - U, \tag{1.2}$$

where Φ is the Jacobi function (the polar moment of inertia).

This is the Jacobi's generalized (nonaveraged) virial equation. In the Russian scientific literature it is known as the Lagrange–Jacobi equation since Jacobi, in

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deriving it, used Lagrange's method of separating the motion of the center mass from the relative motion of mass points.

In the right hand side of the virial equation there is a classical expression of the virial theorem, i.e. relation between the potential and kinetic energy. In the case of constancy of its left hand side, when motion of the system happens with a constant velocity, the equation acquires conditions of hydrostatic equilibrium of a system in the outer force field. The left hand side of the equation, i.e. the second derivative with respect to the Jacobi function expresses oscillation of the polar moment of inertia of the system, which, in fact, is kinetic energy of the inner volumetric torques (oscillations) of the interacted mass points moving in accordance with the Kepler's laws.

Jacobi hasn't paid attention to physics of his equation, which expresses kinetic energy of the interacted volumetric particles in the form of oscillation of the polar moment of inertia. He used the equation for a quantitative analysis of stability of the Solar System and noted that the system's potential and kinetic energies should always oscillate within certain limits. In the contemporary literature of celestial mechanics and analytical dynamics the Jacobi's virial equation is used for the same purposes (Whittaker 1937; Duboshin 1975). Since this equation contains two independent variables, it found no any other practical applications.

Our starting point for the farther development of Jacobi dynamics was the formal and physical relationship that we found between the Jacobi function and the potential energy in Eq. 1.2. This functional relationship enabled us to reduce it to an equation with one unknown function and to obtain a rigorous solution. It then appeared, as Jacobi had assumed, that the potential and kinetic energies and the moment of inertia of a conservative system oscillating periodically according to the law of conic sections. From the standpoint of physics, this means that as a result of interaction (collision) of its component parts or particles the system manifests its potential energy in the form of radiation. In a conservative system, the kinetic energy radiated during the period of pulsation is absorbed by the system itself and again converted into potential energy. In a dissipative system, a part of the energy is radiated during each period of pulsation from its surface and is lost irretrievably, entering into interaction with external systems. In this connection Eq. 1.2 is valid for study of dynamics of systems in classical mechanics and the dynamics of continuous media as well as in molecular physics, quantum mechanics and electrodynamics.

We now turn to some principles of and approaches to dynamics related to the results presented here in order to determine their place in solution of the general problem.

1.1 Principle of Mutual Reversibility

The world by its nature is unique and we want to represent it in terms of unified laws. For this purpose, out of the variety of observed natural objects and phenomena, we try to identify those which are most general and which constitute the basis of the existence of the material world. The main obstacles to this end are, however, the limited possibilities and subjective perceptions of the other world. Therefore, the observed outer multitude of natural objects and phenomena continue to be described by the different branches of physics.

In describing nature, physical objects and phenomena are represented in terms of geometrical entities – scalars, vectors and tensors – and correspondingly the representation of some sets are given in terms of others. Moreover, by generalizing the results of experiments, physical laws are established and expressed in mathematical models after excluding in advance subjective effects which are unavoidable in experiments. The results of generalization of experiments should not, of course, depend on the selection of the frame of reference of the observer, who is located somewhere and is moving somehow. Nor should they depend on the choice of the system of co-ordinates in his frame of reference, which is unlimited. In mathematical language this means that in generalizing experiments all assertions should be unique for any observer – or that laws should be covariant with respect to the corresponding classes of representation of sets of mathematical objects.

Vectorial quantities (co-ordinates, momenta, moments, etc.) and the functions representing them, which remain vectorial in character, suffer from one substantial defect: their components depend on the choice of the frame of reference and the co-ordinate system. Scalar functions, however, which remain invariant during the transformation of co-ordinates, are free from this defect. Therefore, we want to find the representation of these scalars. It is then possible to present vectorial representations in the form of gradients of the corresponding scalar functions. However, the transformation inverse to a given transformation remains a transformation which should have its own scalar function. Since a direct representation has no advantages over an inverse representation, its function should by its nature be symmetrical with the initial one. Then the equation of mutual reversibility of representations in the symmetric form can be written

$$L(r) + H(p) = rp, \tag{1.3}$$

where r and p are vectors from the first and second set of vectors, respectively; L and H are scalar functions defined in the set of vectors r and p, respectively.

Here the direct representation will be determined by the scalar function H as $r \Rightarrow H_p(p)$, where $H_p(p)$ is the gradient of function H(p) with respect to p. The inverse representation is determined by the scalar function L as $p \Rightarrow L_r(r)$, where $L_r(r)$ is the gradient of the function L(r) with respect to r. The product of the vector r and p will be a scalar quantity equal to

$$rp = \sum_{i=1}^{n} r_i p_i.$$

Functions of L and H will in this case have non-zero Hessians or matrices of the second derivatives of functions with respect to the components of their vector arguments. This condition is the condition of mutual reversibility of representations which, in other words, can be expressed in the following manner. The Hessian of scalar function is a Jacobian of vector function of representations of r on p or p on r. The condition of uniqueness of representations is a non-zero Jacobian of representations in the whole region of definition of the function.

We have so far mentioned two different sets of vectors of two representations. To represent the unity of the world, we have to make a generalization. For this purpose both the sets under consideration are combined into one, and the representation becomes an automorphism representing the vectors of a given set by vectors of the same set. Hence, if before combination of the sets the representation has transformed vector X into vector Y, it should, by transforming vector Y, transform the latter into vector X. This is called an involute transformation. When applied twice, it becomes an identical transformation of representations. The quantities which are mutually transformed into each other during involution are called associated quantities.

Anticipating what we are going to say later, we note that this property of representations is reflected in the general theory of relativity in the Einstein equations. There the fundamental tensors of Ricci (contraction of the tensor of Riemann curvature of its spur) and Einstein are mutually associated.

The principle of mutual reversibility described above is the most general for representing nature, and does not depend on nature's internal structure. In its most general term it reflects that the structure of nature is independent of who observes it and how.

The corresponding transformations of the generalized co-ordinates of observed objects will be canonical transformations. Following Wintner (1941), we should point out that we are not writing about any specific types of equations of motion: the basic general properties of transformation are established before these equations are chosen.

During the implementation of this program, which was of course carried out in a more complicated manner in practice, two mutually inverse matrices could be constructed, expressed ultimately in terms of the Jacobian transformation matrix. Historically, these matrices were found by Lagrange and Poisson and are called Lagrange brackets and Poisson brackets. We emphasize that the main idea of the Lagrange and Poisson brackets consists in obtaining the relations which are covariant during the transformation of co-ordinates, i.e. relations which do not change their form during the substitution of variables.

Derivation of the brackets can be started by writing the integral invariant of the canonical transformation, the expression for which, according to the Poincaré theorem (Goldstein 1980), will be

$$J = \iint_{s} \sum_{i} dq_i dp_i, \tag{1.4}$$

where q_i and p_i are canonical variables and s is some surface.

Here the integral invariant J is a scalar whose dimension cannot be obtained from general considerations. We shall show that according to experimental data this scalar has the dimension of action.

The Poincaré theorem for two pairs of canonical variables p, q and P, Q is written in the form

$$\iint_s \sum_i dq_i dp_i = \iint_s \sum_i dQ_i dP_i. \tag{1.5}$$

If both pairs of variables p, q and P, Q are expressed in terms of a third pair u, v on a surface s, we can write

$$dq_i dp_i = \frac{\partial(q_i, p_i)}{\partial(u, v)} dudv,$$

$$dQ_i dP_i = \frac{\partial(Q_i, P_i)}{\partial(u, v)} dudv.$$

Here all quantities are defined on the arbitrary surface *s*. The quantities dqdp, dQdP, dudv are by definition areas on this surface. The quantities

$$\frac{\partial(q_i, p_i)}{\partial(u, v)} \text{ and } \frac{\partial(Q_i, P_i)}{\partial(u, v)}$$

are the Jacobians of transformation from one co-ordinate system into another and serve for conversion of scales.

Now the Poincaré equation can be rewritten in the form

$$\iint_s \sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} dudv = \iint_s \sum_i \frac{\partial(Q_i, P_i)}{\partial(u, v)} dudv. \tag{1.6}$$

The expressions on the left- and right-hand sides of the equation are reduced to common differentials. But since the region of integration is arbitrary, the equation of the sums of the Jacobians follows, i.e.

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} = \sum_i \frac{\partial(Q_i, P_i)}{\partial(u, v)},$$

or, in the expanded form

$$\sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right) = \sum_i \left(\frac{\partial Q_i}{\partial u} \frac{\partial P}{\partial v} - \frac{\partial P_i}{\partial u} \frac{\partial Q}{\partial v} \right). \tag{1.7}$$

Equation 1.7 is the Lagrange brackets, which are denoted by $\{qp\}_{uv}$. This equation shows that the Lagrange brackets are the invariants of canonical transformations.

If as canonical variables (u,v) we successively take the pairs (q_i, q_j), (p_i, p_j), (p_i, q_j), we obtain the system of fundamental Lagrange brackets:

$$\{q_i, q_i\} = 0, \quad \{pi, pi\} = 0, \quad \{pi, qi\} = \delta ijC, \tag{1.8}$$

which are not dependent on the choice of canonical pair.

The mutually inverse matrix for the Lagrange brackets is called Poisson brackets, which are written in the form

$$[uv]_{p,q} = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i} \frac{\partial u}{\partial p_i} \right). \tag{1.9}$$

The expression (1.9) is also a canonical invariant conjugate to the Lagrange brackets, and the following relation is valid for it:

$$\sum \{u_i u_j\}[u_i u_k] = \delta_{ik},$$

which holds even for non-canonical transformations.

It is important to emphasize here that if both these brackets were derived from some earlier-existing mechanics, they would still axiomatically form the basis for constructing at list the mechanics from which they were derived. From the mathematical standpoint, these brackets are the so-called skew forms, with the property of anti-symmetry. In our case, they take the form

$$\{uv\} = -\{vu\}, \ [uv] = -[vu].$$

For skew forms the mathematical apparatus of outer calculus or the method of Cartan external forms has been developed, in which the operations of differentiation are reduced to calculations performed along the contour of the region. Hence is the name ‘outer calculus’.

For the Poisson as for the Lagrange brackets, the system of fundamental brackets is written in the form

$$[q_i, q_i] = 0, \quad [p_i, p_j] = 0, \quad [p_i, q_j] = \delta_{ij}. \tag{1.10}$$

In quantum mechanics the Poisson brackets form the basis for directly writing the equations of motion. There the commutator of two quantities multiplied by i/ħ (where ħ is the Planck’s constant) corresponds to the brackets.

In classical mechanics we can write the equations of motion in the same form. If, for example, the Hamiltonian H is taken as one of the quantities in brackets, we obtain

$$[q_i H] = \frac{\partial H}{\partial p_i} = \dot{q}_i,$$

$$[p_i H] = \frac{\partial H}{\partial q_i} = \dot{p}_i.$$

Within the brackets, the Jacobi identity

$$[u[vw]]+[v[wu]]+[w[uw]] = 0$$

well known also in the method of external forms since Grossman's time, will likewise be valid.

Thus, the principle of mutual reversibility forms the methodological basis for the objective representation of nature. The practical method of covariant transformation of co-ordinates with the help of mutually invertable matrices or the Lagrange and Poisson brackets is based on this principle. The entire mathematical apparatus of classical mechanics, statistical physics and quantum mechanics is, in its turn, built on the brackets as axioms.

1.2 Action and Integral Canonical Pairs

It was Kepler who, in his laws of motion along conic sections with a constant velocity of sweep over surfaces, established the law of conservation of angular momentum and thereby established the specific role of a quantity with the dimension of action. The variational principles of constructing mechanics based on the special role of this quantity were also developed quite a long time ago by Fermat and Maupertuis. It is again to Kepler, and to his conic sections as orbits of motion of celestial bodies, that we owe the second order of the corresponding differential equation describing this motion. Although, as Bondi noted, the special role of acceleration for the Earth's motion and its relation to the position of the Sun could be established simply by observation, it was sufficient to look in the direction of the vector of acceleration of the Earth in order to see that the Sun is always in this direction. Thus, it was established even than that the co-ordinates of position of a body in space and its momenta were the necessary and complete set of parameters for describing the motion of a system. Here a given pair of vectors is specified for each element constituting the system so that there are many such pairs for the whole system.

The next important development in the formation of dynamics was associated with the works of Clairaut and Legendre. The Clairaut equation is an example of an involute system which, with the help of the general Legendre transform, establishes the symmetry between the co-ordinates and momenta. The same Legendre transform converts the co-ordinate function into the momentum function and vice verse. The function performing this transformation is the Hamiltonian function of the system whose integral over a time interval has the dimension of action. The relationship between these functions resemble those between evolvent and evolute in mathematics. Here even their geometrical structure is preserved, which is seen in the shape of the family of enveloping lines reflecting the integral properties of the system from the standpoint of the external observer. It is true that the analytical apparatus of variational calculus then leads to the solution of a system of more equations with partial derivatives, and the integral character of the relations is lost. The reason here is that one wants to obtain a detailed description of the system, i.e. a description of the motion of all its elements, but not a change of structure of a system as a whole.

Clairaut obtained his equation by studying the motion of the Moon, so that we can consider his results to be based on observations of a natural system and therefore verified. It is interesting to note that the Clairaut equation is of second order since the first derivative is the second power. It resembles an equation with two branches of an equation combining two independent equations, and is similar to our first integral of the Jacobi virial equation. One branch of this equation is the envelope of the right-hand family of semi-tangents and the other that of the left-hand family. Thus, the idea of canonical conjugation of equations appeared in the early works of astronomers long before it was fully developed in mechanics.

Further developed of mechanics took place in the direction of its axiomatization and minimization of the number of initial postulates, where advances were made. The main achievement was the reduction of the fundamentals of mechanics to one general integral principle: the principle of least action. The fundamental nature of this principle can be traced in both mechanics and physics. For example, in quantum mechanics the Planck constant with the dimension of action is expressed in terms of the Coulomb electric charge and the velocity of light. Combining these quantities gives the expression known as the fine structure constant:

$$\frac{\hbar c}{e^2} = \frac{1}{\alpha} = 137,$$

where α is a dimensionless constant.

In this fundamental relation, which is apparently simple but absolute in character, gravitation, electrodynamics and quantum physics are connected through Planck's constant or the quantum of action.

Just as in the idea of variational calculus, the notion that there is some minimum value characterizing the true course of events in nature has existed since ancient times. However, it was only in 1662 that Fermat clearly formulated the principle of the shortest path in geometrical optics, from which follow the laws of refraction of light. This principle consists in requiring the value of the integral of reciprocal velocity to be minimal with respect to the trajectory of light between two points. A little later, in 1669, Leibniz in his treatise on the problems of dynamics introduced the concept of action, which is expressed in terms of the product of mass, velocity and path length. In other words, since path is the product of velocity and time, the Leibniz action function is written:

$$S = \int_{t_1}^{t_2} 2T dt$$

where T is the kinetic energy.

Descartes also suggested the quantity $m\dot{q}dq$ (where q is a co-ordinate) as an elementary action. However, since $m\dot{q}dq = m\dot{q}\dot{q}dq = 2Tdt,$ his action turned out to be equivalent to the Leibniz action.

The principle of least action was developed further by Maupertius, Bernoulli and Euler. Generalizing the results of Euler's studies, Lagrange then extended the principle to an arbitrary system of n mass points m interacting arbitrarily and being situated in a field of central forces which are proportional to arbitrary powers of distances. In this case, the motion of the system was determined by the requirement of the lowest or the highest value of the sum

$$S = \sum_{i=1}^{n} m_i \int v_i dr.$$

These studies were most closely connected with the development of variational calculus, where the following two principles are of primary importance in dynamics. The principle of virtual displacement states that a mechanical system is in equilibrium only when the total infinitesimal work done by active forces during any possible displacement of the system from a given position equals to zero:

$$\sum_i F_i \delta q_i = 0,$$

where F_i is the active force and δq_i the possible or virtual displacement which is consistent at a given instant with the constraints imposed on the system.

The d'Alembert–Lagrange principle states that for the real motion of a system, the total elementary work done by active forces and forces of inertia during any possible displacements at any instant equals zero:

$$\sum_i (F_i - m_i \ddot{q}_i) \delta q_i = 0.$$

On the basis of these principles, Hamilton derived his principle of least action, which he further developed and generalized.

It is obvious that

$$\ddot{q}_i \delta q_i = \frac{d}{dt}(\dot{q}_i \delta q_i) - \dot{q}_i \frac{d}{dt}(\delta q_i) = \frac{d}{dt}(\dot{q}_i \delta q_i) - \dot{q}_i\ \delta(\dot{q}_i) = \frac{d}{dt}(\dot{q}_i \delta q_i) - \frac{1}{2}\ \delta(\dot{q}_i)^2.$$

Then

$$\sum_i (F_i - m_i \ddot{q}_i)\ \delta q_i = 0 => \sum_i F_i\ \delta q_i = \sum_i m_i \ddot{q}_i\ \delta q_i$$
$$= \frac{d}{dt}\left(\sum_i m_i \dot{q}_i\ \delta q_i\right) = \delta T + \delta A$$

or

$$\frac{d}{dt}\left(\sum_i m_i \dot{q}_i \ \delta q_i\right) = \delta T + \ \delta A, \tag{1.11}$$

where T and A are the virtual work done by the reactions of constraints and external forces, respectively.

If we integrate both the parts of Eq. 1.11 over t from t_1 to t_2,, in which $\delta q_i = 0$, we obtain

$$\int_{t_1}^{t_2} \delta(T + A)dt = 0.$$

For a conservative system $\delta A = -\,\delta U$ (where U is potential) and

$$\int_{t_1}^{t_2} \delta A dt = -\int_{t_1}^{t_2} \delta U dt = -\,\delta \int_{t_1}^{t_2} U dt,$$

where $\delta t = \delta dt$ since time does not vary.

Finally, the expression for Hamilton's principle of least action will take the form

$$\delta \int_{t_i}^{t_2} L dt = \delta \int_{t_1}^{t_2} (T - U)dt = 0, \tag{1.12}$$

where L is the Lagrangian (free energy in thermodynamics).

Jacobi totally excluded time from the principle of least action. Since

$$T = E - U = \frac{1}{2} \frac{\sum_i m_i (dq_i)^2}{(dt)^2}$$

and

$$dt = \sqrt{\frac{\sum_i m_i (dq_i)^2}{2(E - U)}}$$

the principle itself can be written in the form

$$\delta \int_{t_1}^{t_2} \sqrt{2(E - U)} \sqrt{\sum_i m_i (dq_i)^2} = 0. \tag{1.13}$$

It must be emphasized that the principle of least action differs from other variational principles by its integral character. It does not consider the differential properties of motion (velocity at a given point and so on) characterizing it at each point but the properties which characterize motion in a finite segment measured by the integral over path. Hence it is possible in principle not to include the co-ordinates of a point in formulating a problem of dynamics. That is why the principle of least action characterizes motion independently of the choice of a particular coordinate system.

All the basic equations of motion are derived from the principle of least action, which is postulated as the basic principle. Thus, Lagrange equations of motion are derived from the Euler-Lagrange variational principle. Hamilton's equations are obtained directly from the principle of least action. Here, a frame of reference and a co-ordinate system associated with it are always chosen. Incorrect choice of co-ordinate system and frame of reference may result in equations which we are unable to solve. The problem of choosing the best frame of reference and co-ordinate system and the interest in methods of transforming co-ordinates follows from here. This has its own history, as described below.

The concept of relativity of motion was introduced in mechanics as long ago as Galilei's time, if we disregard ancient scientists. It reflected the need in astronomy to compare the results of observations made at different places and times. This is obvious in astronomy since the observers themselves are moving in relation to the objects observed and in relation to each other. For the results of observation to have any practical value, we should be able to convert them from one observation system to another.

In mathematical models of the phenomena under investigation, transformations of the appropriate quantities correspond to such a conversion. In mathematics, however, unlike physics, conversions can be introduced most arbitrarily and at one's discretion. Therefore, from all conceivable transformations it is necessary to select those, the methods of invariants have been developed; these are quantities which do not change their values for some classes of transformation. At a more abstract level, transformations are considered which retain the mathematical form of the equations describing the laws of motion of mechanical systems, or covariant transformations. These transformations, as applied to mechanics and to the principle of least action, are canonical transformations. As we have already pointed out, canonical transformations do not depend on the equations of dynamics and can be studied without them. Therein lies their fundamental strength.

We have already referred to the selection of groups of transformations of co-ordinates or covariant transformations, which do not change the form of the equations. In this case, the quantities enter into these equations as functions of their variables can and generally do change during transformation. Here the question may selects a class of covariant transformations. All kind of transformations could be tried out until the best result is obtained. In physics all defined quantities have a specific tensor rank – scalars, vectors, higher-rank tensor, spinors, etc. This is an experimental fact. The laws of physics are written in the form of equations and it is natural that all terms entering equations should have identical tensor dimensions.

We cannot, for example, add a scalar and a vector and so on. Hence the requirement of covariance. Returning to mechanics, we recall that the basic variables here are the co-ordinates of momentum space. The latter space is abstract; we do not perceive it by our sense organs. Therefore, the greatest general transformation which does not go beyond the specific transformation is only an arbitrary function of the co-ordinates of configuration space and time. These, known as point transformations for, are covariant transformations for the Lagrangian equations. However, transformations which are usually the most general as functions of all conceivable co-ordinates are covariant transformations for the Hamilton equations and are called canonical.

The so-called generating function of transformation is introduced into the theory of transformation of co-ordinates. It is interesting to note that for canonical transformations the action is such a function.

We seek the generating function to solve the problem of finding a transformation leading to a reference and co-ordinate system which is such that in it the equation of motion will have the simplest form. For this, it is required that in the transformed equations the Hamiltonian should generally be a constant and then, together with the Hamiltonian, the particle co-ordinate system should automatically also be constant. The system in such a co-ordinate system obtained after transformation should be in equilibrium. There can obviously be no simpler form. Then we obtain the equation for determining the generation function, known as the Hamilton-Jacobi equation, in partial derivatives. It is also well known because it is the classical limit of the Schrödinger quantum equation. Since this generating function is action, we again note that all the main branches of physics converge on the concept of action.

We now briefly recall the main postulates of the theory of canonical transformations and the derivation of the Hamilton-Jacobi equation.

By φ and ψ we denote the transformation functions of the co-ordinates of configuration space and momenta q and p so that

$$q_0 = \varphi(q, p, t), \quad p_0 = \psi(q, p, t), \tag{1.14}$$

where q_0, p_0 and q, p are the co-ordinates and momenta before and after their transformation, respectively.

In the theory of canonical transformations, the theorem about the existence of the conditions of canonicity of transformations is proven. A necessary and sufficient condition for the canonicity of the transformations under consideration is the condition of existence of function F(q, p, t), which is such that the following equality will be satisfied:

$$\sum_i \psi_i \delta\varphi_{ji} - C \sum_i p_i \delta q_i = -\,\delta F(q, p, t), \tag{1.15}$$

where the variation is performed for fixed time; C is some constant; and F(q, p, t) the generating function of transformation.

We shall not give the proof of this theorem. We only note that, for practical purposes, a very simple and convenient method of verifying whether a randomly chosen transformation is canonical is derived from this theorem. For this purpose, it is sufficient to verify the validity of the following three equations:

$$\left\{q_i q_j\right\} = 0, \quad \left\{p_i p_j\right\} = 0, \quad \left\{p_i q_j\right\} = \delta_{ij} C,$$

where {. . ..} are the Lagrange brackets.

If the verification gives a positive result, the constant C or the valence of transformation is calculated at once. Then the new Hamiltonian will be expressed in terms of the old one in the following manner:

$$H = CH_0 + \frac{\partial F}{\partial t} + \sum_i \psi \frac{\partial \varphi_i}{\partial t}, \tag{1.16}$$

where the variables p_0 and q_0 are expressed in terms of the new p and q with respect to transformations inverse to φ and ψ.

Now if we do what we earlier wanted to do and require that the new Hamiltonian be identically constant, p and q being constant, we shall obtain an equation for generating a function S (according to Jacobi, it is action):

$$\frac{\partial S}{\partial t} + CH_0 = 0.$$

The quantities p and q entering into the formula for the old Hamiltonian can be expressed in terms of the function S by the formula

$$\frac{\partial S}{\partial q_i} = Cp_i.$$

If we introduce the relation $S_n = S/C$, the equation will take the form

$$\frac{\partial S_n}{\partial t} + H\left(\frac{\partial S_n}{\partial q}, t\right) = 0. \tag{1.17}$$

It is the first-order Hamilton-Jacobi equation in partial derivatives. After its solution we obtain the equations

$$\frac{\partial S}{\partial q_i} = p_i,$$

$$\frac{\partial S}{\partial \alpha_i} = -\beta_i,$$

where α, β are the constants we chose at first which determine in the old co-ordinates the motions of the system in implicit form. These relations are integrals of the initial Hamiltonian system of equations.

It is as the theory of canonical equations and their canonical transformations that modern dynamics is developing. It is true – and we once again emphasize this – that it is developing mainly on the basis of detailed description of the motion of a system from an internal observer's standpoint. Even when the method of outer forms is used, these forms relate to individual elements constituting the system. So for the whole system many forms are considered. However, the integral invariants proposed by Poincaré (1965) and developed by Cartan are used rarely. In approaching the dynamics of a system from an external observer's standpoint, integral invariants should be the main tool of the researcher.

We now come back to the general equation of the principle of mutual reversibility of representations (1.3):

$$L(r) + H(p) = rp.$$

We note once more that in it the canonical pair r and p have the dimension of the scalar function H and L determining the transformation by its gradient. As shown above, the scalar function itself has the dimension of action. Hence the main canonical pairs in mechanics will be action-angle, energy-time and Jacobi function-frequency. The powers with which time enters (t^o, t^{+1}, t^{-1}) in the case of these pairs will be 0, +1, –1, respectively. Higher powers will not be fundamental because of the Jacobi identity. Proof can be work of Misner and co-workers (1973).

The pair used most extensively in mechanics is action-angle. It gives the simplest description of a system and forms the basis for using the methods of perturbation theory in mechanics.

The energy-time pair enables us to take into account processes with discontinuous functions of state such as, for example, collisions and phase transitions.

However, neither of these pairs which use the most general properties of space-time symmetry takes into account the specific effects of interaction. The Jacobi function-frequency pair for systems with a high degree of symmetry retains the merits of both those pairs and, in addition, takes into account the form of the law of interaction of particles constituting a system. From the principle of mutual reversibility one can already see the merits of this canonical pair, which is the basis of the Jacobi equation (1.2).

We can refer to many literature sources where historical interest in this equation has been noted. For example, Singh, in 1963, in his "Classical Dynamics", called Eq. 1.3 a striking result. Even Jacobi himself, in his "Vorlesung über Dynamik" (1884), considered his equation to be exceptionally interesting.

We give another, most recent, example from the general theory of relativity, which points to the importance of the Jacobi function-frequency canonical pair. In this theory, as also in geodesy, the equation of derivation of geodesics is given in the Riemannian form:

$$\nabla_{\vec{u}}\nabla_{\vec{u}}\vec{n} + R(\vec{n},\vec{u})\vec{u} = 0,$$

where $\nabla_{\vec{u}}$ is the covariant derivative along the tangent; $R(\vec{n}\vec{u})$ is the Riemannian curvature operator and $\vec{n}$ and $\vec{u}$ are the unit vectors of the normal and the tangent to the geodesic, respectively.

In the first term of this equation, differentiation operators are applied to the vector $\vec{n}$ and the curvature operator is applied to $\vec{u}$. However, when the Jacobi curvature tensor

$$J^{\mu}_{\nu\alpha\beta} = \frac{1}{2}\left(R^{\mu}_{\alpha\nu\beta} + R^{\mu}_{\beta\nu\alpha}\right),$$

where $R^{\mu}_{\alpha\nu\beta}$is the Riemannian curvature tensor, was used, the equation of derivation was written in the symmetric form

$$\nabla_{\vec{u}}\nabla_{\vec{u}}\vec{n} + J(\vec{n},\vec{u})\vec{u} = 0,$$

where $J(\vec{n},\vec{u})$ is the Jacobi curvature operator.

The last equation can also be written as one operator:

$$[\nabla_{\vec{u}}\nabla_{\vec{u}}\vec{n} + J(\vec{n},\vec{u})]\vec{u} = 0,$$

in which case we obtain a zero eigensolution problem.

It was these factors that aroused our special interest in the Jacobi function-frequency canonical pair, which had not previously been utilized.

1.3 Integral Characteristics in the Study of Dynamics of Natural Systems

Traditionally two approaches are used to solve the problems of dynamics. In the first approach the object under study is regarded as a system of mass points interacting in accordance with specific laws. In the second it is represented as a continuum model in which the interactions are expressed in terms of volume fields of forces also acting in accordance with physical laws.

In both cases the mathematical description of the dynamics of the object is based on the latter's fundamental integral characteristics – mass and energy – which have a definite physical interpretation. Let us examine how these characteristics are used and consider, first of all, the concept of energy of an object in the different approaches to its dynamics.

In the mechanics of a system of mass points, energy, E, as we know, can be expressed mathematically as follows:

$$E = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + U(q_1, q_2, \ldots, q_n), \tag{1.18}$$

where q_i and p_i are, respectively, the co-ordinates and moments of the mass points m_i, and U is potential energy.

The expression written in this form is, strictly speaking, a function of 2n arguments p_i and q_i which has a wide range of values. Energy means only those values of this function which, upon substitution into Eq. 1.18 of the definite arguments p_i and q_i as function of time, are the solutions of the corresponding equations of motion. However, for brevity, the same term 'energy' is commonly used to express also its corresponding mathematical form and its magnitude by a narrower set of values of arguments. The fact that the concept 'energy' is used in many senses does not, however, cause ambiguities in describing the dynamics of a system in co-ordinates and velocities.

The variational approach is generally used in constructing the equations of mechanics of mass points. For this purpose, the Lagrangian L with arguments q_i, $\dot{q}_i$, t are introduced. Here, for brevity, $\dot{q}_i$ is called the velocity of the i-th particle. Strictly speaking, this should be so understood that, if $q_i(t)$ as a function of time is the solution of the corresponding equation of motion, $\dot{q}_i$ is the time derivative of this function. Consequently, the differentiation of the function should be performed only after the solution of the equation of motion has been found.

Thereafter, the procedure of variational calculus is used to find the class of functions $q_i(t)$ satisfying the principle of least action. For this purpose, the functional S is written and its extremum is found:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt. \tag{1.19}$$

We write the variation of action S:

$$\delta S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt.$$

Integrating in parts the second terms of the sum, we obtain

$$\int_{t_1}^{t_2} \frac{\partial L}{\dot{q}_i \partial} \delta \left(\frac{\partial q_i}{\partial t} \right) dt = \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt.$$

The first term on the right-hand side of the equality is zero since, according to the condition, we have:

$$\partial q_i |_{t_1} = \partial q_i |_{t_2} = 0.$$

Then

$$\delta S = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\delta L}{\delta q_i} - \frac{\delta}{\delta t} \frac{\delta L}{\delta q_i} \; \delta q_i dt \right).$$

Since the variations δq_i are independent, we obtain the Lagrangian system of equations

$$\frac{\delta L}{\delta q_i} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i} \right) = 0. \tag{1.20}$$

Thus we have obtained the system of equations describing the motion of each point of the system.

The function S in Eq. 1.19 has the dimension of action between fixed points on the time axis. The function L is thus the time density of action, and the density of the Lagrangian L (per unit volume) introduced into the modern field theories, including quantum theory, finally symmetrize the functional S as the density of action in four-dimensional space-time.

However, the variational principle enables the equations for the search functions $q_i(t)$ to be found in the general case for an arbitrarily defined form of the Lagrangian. Therefore, the form of the Lagrangian is not determined from the variational principle. Moreover, even the set of its arguments is not determined. In this connection, the functional is varied with the Lagrangian. $L(q, \dot{q}, \ddot{q}, \ldots, t)$.

In this case, the principle of least action is applied and equations for determining the co-ordinates q_i are obtained. There are, of course, certain restrictions on the form of the Lagrangian, but none whatever on the number of arguments from the standpoint of mathematics. Here differential equations of an arbitrarily high order may be obtained. The type of Lagrangian, and especially the number of its arguments, are chosen from historical tradition from Kepler to Newton, who made their choice from a careful analysis of empirical observations and generalization of their results.

In such an approach we come back to where we started, i.e. to Newton's equations, while the general nature of the principle of least action seems to be unused. As for Newton's system of equations to describe the motion of mass points, the system of equations to describe the motion of mass points, the system of differential equations for three or more masses is unsolvable. Thus, there arose the many-body problem which has had no satisfactory solution even until now, while the mechanics of mass points, lacking a solution of the fundamental problem, reached a dead end.

The main difficulty encountered by Newtonian mechanics in going over to the integral approach was evidently the fact that the non-localizability of the energy of interaction of mass points m_1 and m_2, proportional to $m_1 m_2/r^2$ (where r is distance between masses) cannot in any reasonable way be related to any region or point in space, whereas, according to Einstein, energy is proportional to some mass.

However, it is known where this mass is. According to the equivalence principle, inertia is inherent in every mass, including mass defect occurring during their interaction.

There was only one thing to do – to study the behavior of natural objects in terms of integral (volumetric) characteristic and to write the virial equations with their help. The main reason to do this is that the gravitational interaction of any quantity of mass produces volumetric forces because any size of mass has volume. Jacobi very easily obtained his equation (1.2) from Newton's equations, noting its exceptionally interesting properties bud did not solve this equation in terms of integral (volumetric) characteristics. A century later, in his fundamental and complete work, Wintner (1941) frankly said that it was impossible in principle to solve this equation in terms of integral characteristics. Here a negative role was apparently played by the concept of non-localizability of potential energy in the mechanics of mass points. Nobody seemed to want to approach the problem from the standpoint of the external observer, where energy can be measured experimentally, since the problem had already been formulated from the internal observer's point of view. It was considered that it should in any case be solved in the style in which it had been formulated, without substituting one problem for another.Jacobi, however, who well understood the weak points of deriving the equations of motion of mass points, suggested his variant of the principle of least action, which was written in the form

$$\delta \int_{t_1}^{t_2} \sqrt{(E - U) ds} = 0, \tag{1.21}$$

where E and U are the total and potential energies and ds is a Riemannian linear element. The geometric interpretation of this equation is as follows. The trajectory is the geodesic of the configuration space in which the Riemannian linear element ds is specified, the latter being equal to $ds^2 = m_{ik} dq_i dq_k$, where m_{ik} is a coefficient written in the form

$$L(q, \dot{q}, t) = \frac{1}{2} m_{ik} \dot{q}_i \dot{q}_k - U.$$

The Hertz-Gauss principle or the principle of least curvature follows at once from the Jacobi principle since the geodesic is a curve of least curvature. This principle reduces the many-body problem to the motion of one body along the geodesic in a complex multidimensional configuration space. Moreover, it determines there the trajectory of motion but not the law of motion along this trajectory. This obviously explains why the given principle did not fully solve the problem. However, it is important to note that attempts were made to go over to the one-body problem.

We now consider how mechanics developed within the framework of the continuum model. In continuum mechanics there are fields of forces, and the energy squared over these fields and the mass associated with it are localized over the whole space. Electromagnetic interactions in relation to energy are localized by introducing the concept of pressure. However, in mechanics, pressure is introduced axiomatically and its internal structure is not determined. It is interesting to note that pressure has the same dimension as the volume density of energy, and that, since energy is the time density of action, pressure is the density of action which is four-dimensional in space-time as is the Lagrangian density.

After introducing the distributed quantities, we can carry out integration over the volume of the system and obtain the system's integral parameters, which have a clear physical sense since the corresponding physical quantities can be measured. It is possible for this reason that the virial equations in cosmology are (and were) easily and clearly obtained from the Einstein equations. However, there were more integral parameters than equations here, and the so-called equations of state were therefore used in order to close the system of equations of dynamics. These equations directly connect the integral (or integrable, where is sufficient) quantities: pressure and density (omitting co-ordinates and velocities). The equations of state themselves are taken either from experiment or from auxiliary theories (simplified, as in the molecular kinetic theory, or partly rigorous, like the Pauly principle in quantum mechanics). In any case, these relations exist and demonstrate the validity of the assertion that the virial equations can be solved in terms of integral characteristics.

It was not always the case that after the virial equations were written they were solved in terms of integral characteristics. Since there were more characteristics then equations and since additional relations (of the type of equations of state in continuum mechanics) cannot be found immediately, some authors did not stick to the initially chosen integral direction and reverted to the ordinary differential approach. An example is the work of Chandrasekhar (1969) referred to earlier, where he developed the method of moments. After obtaining the system of virial equations, he calculated for their solution the variations of the integral characteristics for small shifts (or deformation) in order to linearize the virial equations. Consequently, while calculating the variations, he abandoned the integral approach and expanded the domain of definition of functions which expresses the corresponding integral parameter. This work deserves serious attention, and we shall return to it when we discuss the problem of moments.

It is interesting to dwell on the integral approach used in thermodynamics. We have referred earlier to equations of state and noted that they were obtained both experimentally and with the use of special theories. Let us consider one such example. In the theory of non-ideal gases the following formula for the equation of state for a gas is introduced by means of molecular kinetic theory methods:

$$P = \frac{NT_0}{V}\left(1+\frac{NC_1(T_0)}{V}+\frac{N^2C_2(T_0)}{V^2}+\ldots\right).$$

Here P is the pressure of the gas; T_0 its temperature; V/N the specific volume per molecule and $C_i(T_0)$ coefficients independent of pressure.

Each term here corresponds to a particular type of interaction of molecules: the first term describes an ideal gas; the second defines pair interactions; the third the three-body interactions of molecules, and so on. The coefficients $C_i(T_0)$ in this theory are called virial coefficients. In statistical physics, the following parametrically defined equations of state are derived for their determination:

$$P = T_0 \sum_{n=1}^{\infty} \left(\frac{J_n}{n!} \xi^n \right),$$

$$\frac{N}{V} = \sum_{n=1}^{n} \left(\frac{J_n}{(n-1)!} \xi^n \right),$$

where ξ is a parameter which should be excluded; and J_n is a function dependent only on temperature and calculated for n-fold interaction of molecules from their given interaction function.

It should be noted that, at zero temperature, pressure in classical physics should become zero. The velocity of sound should also become zero. The energy of electromagnetic interaction of charges takes into account only the random motion of these charges. The Madelung energy (the energy of formation of an atomic lattice) determines the purely Coulomb interactions of charges and does not depend on temperature.

We have so far discussed the problem of using the integral characteristics of a system, which determine its behavior in time. It would be interesting to take a look at the properties of the integral invariants derived at different times. As we have already pointed out, the canonical representations retain some integral invariants and this is the basic property. Among the forms which they retain, the most important are Poincaré relative invariant and the forms associated with it.

The Poincaré integral invariant is

$$\oint pdq - Hdt$$

and the relative integral invariant is $\oint pdq$(where p and q are vectors).

It should be borne in mind that the invariant has the dimension of action and its form

$$pdq - Hdt = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 - Hdt,$$

taking into account that H/C is p_4 and that $cdt = dq_4$, shows that it is simply a scalar pseudo-Euclidian product of two four-vectors $p(p_1, p_2, p_3, p_4)$ and $q(q_1, q_2, q_3, q_4)$ – four-momentum and four-radius of the particle vector. Such, for example, is the eikonal determining the phase of a plane traveling wave:

$$kr - \omega t = k_1 X + k_2 Y + k_3 Z - k_4 ct,$$

which represents the scalar product (pseudo-Euclidian) of two four-vectors: $k(k_1, k_2, k_3, k_4 = \omega/c)$ – wave four-vector and four-radius of vector q. In the given case, this contraction is dimensionless. The Jacobi tensor, whose three-spur is the Jacobi function, can be contracted once or twice with the wave four-vector. During the first contraction we obtain a vector with the dimension of action so that the Jacobi function-frequency pair is a canonical pair, while the second contraction dives the Einstein function $E = mc^2$ (scalar). The Liouville theorem forming the basis of statistical physics is associated with the Poincaré invariant from which it follows.

The integral invariants considered above are regarded in mathematics as external forms of different orders and, as has been pointed out, they have been studied for a long time by means of the Cartan external forms, so that a well-developed mathematical apparatus for their study already exists.

From this brief description it can be seen that attempts have been made for a long time to approach the description of the dynamics of natural systems in terms of integral (volumetric) characteristics by means of the construction of virial equations. It is only required to introduce some concepts associated with the conditions of the external observer. In particular, the integral characteristics from the external observer's standpoint are functions of time only and consequently they cannot be subjected to partial differential operations. From the physical point of view this means that these characteristics should be measured directly (even if only in principle) and not calculated by formulae. For example, the potential energy in Eq. 1.18 can be emitted during the time of the system's formation. Mass as an integral characteristic of the system should also be determined from measurements performed externally. For example, the mass of the Sun is determined from the parameters of motion of planets around it. In atomic physics the eigenmoment of the atom can be measured, and this is easier to do than adding its eigen- and orbital moments, which is a mathematical problem.

In the general theory of relativity, co-ordinates near singularities are introduced similarly from the external observer's standpoint. The length of the circle surrounding the singularity, divided by 2π, is introduced instead of the Euclidian radius-vector. In this case, the chosen approach is a question of principle and not of convenience.

We note that for application of the integral approach in dynamics we already have the well-developed mathematical apparatus of external differentiation, discussed above, which is also used to study the contours surrounding the region externally. The distinctive feature of this calculus is that it has only the first differentials of function, the higher ones being absent. The parallels between the methods in physics and mathematics from the external observer's standpoint are obvious.

Another new concept is that natural systems from the external observer's standpoint are not conservative since they emit energy. The emitted energy is an integral effect of all possible types of interaction. However, a specific type of interaction in the system appears in the corresponding part of the emitted energy spectrum.

Finally, we introduce the concept of hierarchy of systems and sub-systems. We note that it is not simply and not only a concept for study of motion in nature since the very fact of the existence of such a hierarchy is empirical. For example, the chain: universe – galaxies – stars – planets – satellites – shells – plates – polycrystals – molecules – atoms – nucleons – electrons, etc., really exists so that this approach to the study of motion mirrors the structure of nature.

In studying motion inside a particular system, we should confine ourselves to the study of the influence exerted on the bodies in our system by the nearest neighbors only. We thus declare our system to be closed. Indeed, all laws of conservation are written for closed systems. It is nevertheless possible to describe with some accuracy the motion, say, of celestial bodies even prognostically (ephemeris) on the basis of mechanics with second-order equations of motion (here we are not going to take into account the co-ordinates-momenta parity). What may be the reason for this? Even the stability of such systems (admittedly according to Lyapunov and not according to Duboshin, i.e. with respect to variation of only the initial conditions and not interactions) is also studied to second-order time derivatives. From electrostatics we know both theoretically and experimentally (Coulomb's law and the Faraday cage) that a closed shell consisting of charges shifting freely in it will, if given sufficient time, distribute these charges over itself in such a way that the influence of charges external to the shell on the space inside the shell will be fully balanced by its own charge distribution. It is important to note that the potential inside the shell is not important since only its spatial gradient has an influence. It is known that this potential satisfies a second-order differential equation, the Poisson equation, so that Coulomb's law from the external observer's standpoint can be formulated as the principle of possibility of shielding the inside space of the shell from external actions. It was found that this approach automatically gives Coulomb's law: the law of inverse squares. Any deviation from two in the exponent of this law deprives the shell of such a property.

Coulomb's law is a static law. If it is to operate in the way indicated, the charge in the external actions must be sufficiently slow in comparison with the time needed for charge redistribution. Newton's law of universal gravitation has the same form as Coulomb's law. If we consider that the time of propagation of gravitational waves is the same as that of electromagnetic waves – this is just what the Einstein law states – and if we take into account that the structure of the universe is hierarchical, we have an analogy with electrostatics. The slowness of charges in the external actions of the shell (the latter being the system nearest to us and enveloping our system, e.g., for the Solar System it is our Galaxy) is ensured by the great distance of these outer bodies. Thus, there appears to be justification for the possible rejection of the influence of almost should be taken into account. The principle of shielding will itself give the criterion for the choice of almost-closed system.

1.3.1 Method of Moments: Specific Features in Integral Approach and First Moments

As we know, it is possible in mathematics under certain conditions to reconstruct a continuous function by means of its moments with the help of a generating function. Here each of the moments is an integral characteristic of the function and consequently, as applied to a physical system, is the external parameter of the function. The parameter can be measured in the appropriate manner.

However, although the function can be reconstructed by means of moments, we require for this purpose all moments, which are infinitely large in number. Moreover, in the detailed description of a system from the internal observer's standpoint, it is not possible to comprehend the results owing to the large number of the system's degrees of freedom even if number is finite. For this reason alone, one resorts to describing a system by the moments of the function which is being sought. Because of the artificial break in the infinite chain of moments, such a description is not only incomplete but also inaccurate. The results obtained in this manner require verification by experiment in order to demonstrate that the errors are not too large. Errors do occur since the aim is to obtain a full description of the motion of the whole system, i.e. the motion of each of its elements. Generally speaking, each moment is in itself a characteristic of the system, indeed of the system as a whole, and is a fully accurate quantity which is not determined with an error. For example, the first moment of mass density determines the position of the center of mass of a system and the zero moment determines its total mass. Both these characteristics are its accurate integral characteristics and will not be charged by whether or not we find still higher moments of the system. Since in the integral approach we are searching for canonical equations, the corresponding description of the system's behavior will be accurate.

In statistical physics the break in the chain of linked equations occurs after the introduction of some approximate relationship between the partial distribution functions. Its sense lies in the distortion of the law of interaction of the elements of the whole, since without introducing any law of interaction it is not possible to obtain a full description of the system from the internal observer's standpoint. This distortion is, of course, towards simplification of the model. The first two moments are the most important. They characterize the average value over the total distribution and its dispersion. The true distribution here is replaced by a Gaussian distribution with the same average value and dispersion. Then the virial theorem is used in order to choose the chain of linked equations, and the method of closure is order to choose is to average the corresponding equation over time.

We found a way to close this chain for high-symmetry systems, not by an approximate method but by a rigorous one. This will be discussed later. Here it is sufficient to note that in the most fundamental general theory of relativity the metric is determined quite rigorously either by one (Schwarzschild) or by two (Kerr) moments, and this is natural for their symmetry because the metric considered is external to the sources.

We now recall the basic postulates of the theory of moments in mathematics and make some comments on them from the standpoint of the integral approach considered here to the solution of the problems of dynamics of systems.

In mathematics there is the concept of characteristic function for a given function F(x) which is determined by the expression

$$\varphi(t) = \int_{\Omega} e^{itx} dF(x),$$

where Ω is the domain of definition of the function and F and t belong to this domain.

If the function F(x) has a derivative f(x) with a finite number of discontinuities of the second kind then

$$\varphi(t) = \int_{\Omega} e^{itx} f(x)dx,$$

where φ(x) is simply the Fourier transform of the function f(x).

The given relation is single-valued and unique; it is a theorem of mathematics. So is the inversion formula which enables us, with the help of the characteristic function, to reconstruct the function itself

$$F(b) - F(a) = \lim_{c\to\infty} \frac{1}{2n} \int_{-c}^{+c} \frac{e^{-ita} - e^{-1tb}}{it} \varphi(t)dt.$$

it is simply the inversion of the Fourier transform.

In the theorem of moments, a function is reconstructed with the help of its moments. Here it seems that this can be done, although in the general case the solution is non-unique. However, under some sufficient conditions of the form

$$\lim_{n\to\infty} 1 \frac{\sqrt[n]{\mu_n}}{n} < \infty,$$

where μ_n is the n-th moment of function f, the solution will be unique. In this case, the problem is solved by the following procedure. At first the function

$$\varphi(e) = \sum_{n=0}^{\infty} \left[\frac{(it)^n}{n!} \mu_n \right].$$

is calculated with the help of specified moments. This function will be the characteristic function f(x), or, in the more general case, of function F(x). Here, two important points are usually to be noted: (a) if the function to be reconstructed is given in a finite interval, the solution of the problem is generally unique; (b) the condition on sufficiency for uniqueness in the case of an infinite domain of definition can be satisfied only for particular moments. We further note that this condition itself can easily be fulfilled in physics so that we can consider that the problem of inversion of moments in physics is always solved uniquely.

However, we are speaking about all moments of the function, and they are infinitely large in number. In practice, the method of moments is used to find an approximate and not an accurate solution since, as a rule, one rarely succeeds in finding a general form of solution of the moment equation.

The moment method is used in the following manner. The initial equation for the function being sought in integrated successively over the whole domain of definition after prior multiplication by successive power of the argument. In the general case, we obtain an infinite chain on linked algebraic equations. If this infinite system of equations can be solved in the general form, we obtain all the moments and using the above formulae we find what we were looking for. As a rule, however, this cannot be done because of mathematical difficulties. Then we proceed as follows. We artificially break the chain of equations, so that the number of equations will be smaller than the number of unknowns. To solve such a system, we artificially supplement it by an equation constructed on the basis of some considerations, and solve the shortened system obtained. It is considered that this solution will be chosen closer to the truth.

We note the following, however. As will be seen from the above scheme of reconstruction of a function by means of moments, when truncating the series of moments we shell obtain a polynomial as the characteristics function. The polynomial has a finite number of poles, and consequently even an approximate solution will not be just inaccurate; it will not be specified in the whole domain of the function sought.

Let us give an obvious example. We approximate a sine curve by a third-order polynomial. The region where any approximation of the sine curve to a cubic parabola is at all possible is that between the extreme roots of the third-order equation, as shown in Fig. 1.1.

This is not the main point. The main point is that from the standpoint of mathematics, in calculating moments by this method, integration is performed over the whole domain of definition of the function, i.e. over all its arguments. If we speak of physics, this means integration over space and time co-ordinates. In this particular case, the moments will be constant or, more accurately, functions of the parameters of the problem. This is the case for which all the rigorous theorems of the theory of moments have been established in mathematics. In the physical applications, however, the problem is approached differently, so there is a new formulation of the problem itself. In the physical literature, integration over time is not performed and the moments (now these are spatial moments) are functions of time. The moment equations, which are obtained from the basic equation by multiplying by the successive power of space co-ordinates followed by integration over the whole configuration space, are ordinary differential equations. We further note that, in the case of multi-dimensional configuration space (and this is precisely the general case), multiplication by the corresponding n-th power of co-ordinates denotes multiplication by the product of the powers of all spatial arguments, the sum of all exponents of which is equal to n with all

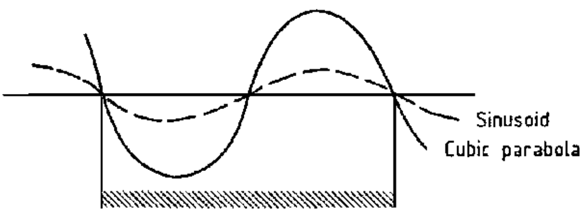

Fig. 1.1 Domain of approximation of the sinusoid to the cubic parabola

such possible combinations, so that ultimately we shell obtain not one moment of power $n(\mu_n)$ but a whole matrix of such values for each n, i.e. a tensor of the n-th rank. Since this is only a matter of mathematical technique, later in this chapter we shell understand by 'moment' the spur of the corresponding matrix, denoting it by μ_n to avoid submerging the physical essence in less substantial details.

The method of moments was developed generally in order to solve problems where insurmountable or almost insurmountable mathematical difficulties were encountered and was not intended for problems which are in principle unsolvable e.g. those in which the number of unknown functions exceeds that of equations for their determination. In the latter case, the method of moments alone cannot help.

In this connection, let us see how Chandrasekhar (1969) evaluates the method. He writes that the virial method is essentially the method of moments and that virial equations of different orders are in fact nothing more than the moments of the corresponding hydrodynamic equations. By moments of equations he means spatial moments, as he makes clear when he says that all these quantities (including volume, V) will, generally speaking, also be functions of time, t_o. So virial equations are differential equations, and it is fortunate that they are only ordinary differential equations and not partial derivatives.

The quantities (moments) entering into these virial equations are in fact integral quantities if we regard them as unknown functions of time (which they are) and do not try to linearize them. Unfortunately, there are more such functions then equations and, possibly, therefore even Chandrasekhar does not see in the virila equations anything new other then that the method of obtaining these equations is identical with the method of moments, and this is obvious from his statements quoted above.

To elucidate our idea, we return to Newton's equations for a system of gravitational mass points of hydrodynamic equations since, is this case, we can more clearly see the role of the integral parameters. Newton's system of equations in the above case is a system of equations of the many-body problem. It is defined accurately and should in principle be solved in the sense that the number of equations and the number of functions sought coincide and that the equations are ordinary differential equations. However, we are unable to solve them for lack of mathematical methods.

But when these equations are replaced by those of spatial moments, it is found (Jacobi) that these equations contain only the integral characteristics of a many-body system: Jacobi function, total energy (system mass) and potential energy (mass defect of the system). They all appear as unknown functions of time. Taking into account the law of conservation of energy, there is only one equation lacking for solving the system (for finding these functions). We are not concerned here with where the equation is to be taken from – this will be dealt with in due course – but we now return with new ideas to the equations of hydrodynamics.

A number of questions arise at ones. The first is what are we seeking as unknown functions in the equations of hydrodynamics before they are converted into the moment form? It is the density fluid for a compressible fluid and the velocity vector at each of its points. But this is indeed an infinite amount of information. Even if it were possible to obtain this information by some unknown means, what more is to be done with it? It appears, just as in the theory of probability, for example, or in quantum mechanics, that this information is used in an integral form. We are seeking the probability density or the probability amplitude in order to find the average values of the finite number of parameters of the system, the so-called measurable or observable quantities. The functions we are seeking cannot be used in their general form even if they existed (we mean precisely the general, i.e. infinite-valued, solution). Then, in reply to the question of how to solve those equations, we pose another question: what do the authors of those equations expect to obtain? They do not in fact consider them unsolvable. It is clear, as they say in mathematical physics, that solving an equation means obtaining it in the form of a finite algebraic combination of either so-called elementary functions or a convergent (although asymptotic) series of such functions, so that in practice, where the criterion is accuracy, one can confine oneself to a finite number of such functions. On the other hand, it may mean that one wants to reduce the solution to quadratures where the methods of finite approximation can also be used.

The main conclusion is that the solution is expected to be obtained in the form of a finite and not an infinite number of 'images'. For example, if as the solution to this kind of problem for a gaseous star we obtain the equation of its rotation (oscillation), one parameter – the angular velocity – is one 'image'. If a baroclinic rotation is obtained, one law – the change in angular velocity (frequency of oscillation) along the radius of the star, given by some elementary function or by their combination or at least by a polynomial approximation – also represents an image, while this time is an optical 'image'. We may also mention, for example, honeycomb convection, where there are more parameters then one to describe motion. But when, in practice, we encounter the phenomenon of turbulence, where pulsations are infinitely large in number, it is clear that they cannot be obtained from the equations of hydrodynamics. Thus, even the authors of the hydrodynamic equations do not expect to obtain from their equations arbitrarily general solutions, and this indicates from the very beginning that these equations contain too much information even in their formulation. It is also apparent that the method of moments can be stronger than the initial equations themselves. This will be the case if the initial equations do not simply possess excess information but also contain the entire integral information corresponding to the given physical theory. This information will then appear, whereas the equations will be accurate equations of the physical problem. Chandrasekhar evidently feels this. On the page where the statements referred to earlier appear, he points out that the advantage of these moment equations lies in the fact that the lowest-order equations often allow a simple physical interpretation. Unfortunately, he immediately makes a reservation to the effect that the purpose of using the moment equations. Later on, for specific problems of equilibrium and stability of configurations of fluids, he himself demonstrates how the method of moments (of finite number) gives a complete solution of these problems. This is done, in principle, by bypassing solution of the problem of obtaining full information on the distribution of density and motion in a system. Moreover, he even states clearly elsewhere in the book that, for solution of the Dirichlet problem, the second-order virial equations contain the necessary and sufficient conditions, and the virial equations therefore give all the necessary information for the study of equilibrium and stability of the permissible ellipsoidal figures.

The actual problems of science and technology set specific tasks for specific purposes rather than general speculative problems. For example, in the examples quoted above, it is necessary to elucidate the conditions of equilibrium and stability of real objects and not to speculate about finding some distribution function. In this connection, it is clear that the virial method will always work.

It is interesting to consider where the classical physicists use the specific formulation of a problem. The Dirichlet problem was first solved fully by Riemann, who used it in selecting the law of dependence on the velocity field at a particular time on the initial Lagrangian co-ordinates of the corresponding points. He selected this law by intuition (which he could afford to do) and expressed it as a linear matrix function with a matrix of coefficients of transformation dependent only on time. Later Likhnerovich and Chandrasekhar specially studied this law in the Dirichlet problem and established its compatibility with free boundary conditions, demonstrating the uniqueness of the selection. Thus, in this case, the specific formulation of the problem (the problem of equilibrium and stability of a given form of the equilibrium figure) manifested itself in requiring certain boundary conditions. These conditions, in their turn, determined the selection of the form of representation to describe internal motion in the system under study, and this form was the linear matrix combination of Lagrangian co-ordinates as the permissible velocity field of the fluid elements of the body

In this specific case it is possible to observe even at the classical level the mechanism of the action of limitation which follows from the specific formulation of the problem. In the general case, this is of course not required, nor is it always possible, and we go over directly to virial equations and their solutions, which are the solutions of the specific problem.

In summary, we emphasize once again that the sense of the virial equations in physics is not the same as that of the method of moments in mathematics. In the latter case, the purpose is to reconstruct the initial function but it is not important what sort of function it is.

In physics, on the other hand, the function which generates moments is the distribution function and it therefore has a precise role. If it is known, that is fine, since with its help we can find other functions of measurable or observable quantities. If it is not known, but if there is way of doing it, that is fine too. The virial equations of provide the means of doing without the intermediate step – the distribution function – but for this purpose, they must of course be correct. That means the moment should be taken for the most general and universal equation such as the Einstein equation in the general theory of relativity.

In concluding this chapter we note that scientific models describing nature are being developed continually in the direction of greater generality and simplicity. Galilei did not take into consideration the need for external causes for free motion; he pointed out that free motion was the natural state of matter. Einstein went further. He introduced motion along the geodesic, thereby excluding gravitation as a force. His motion along the geodesic is now the new form of free motion.

Acceleration of a body occurs only when it is prevented from moving along the geodesic. So, nature has electro-weak and strong interactions.

As to the oscillatory motion of matter, it is also universal. In the absence of forces of friction it also does not need an external force in order to exist since it is a natural form of motion. If we take into account the results obtained recently for empty space, where the oscillatory regime of metric changes also exists (Belinsky et al. 1970), it will be clear that the oscillatory regime of motion is the most general and natural property of nature. Energy is important for free motion, which is described by equations based on the energy-time canonical pair. The action-angle canonical pair is used to describe rotational motion. The Jacobi function-frequency canonical pair is appropriate for describing oscillatory motion. Hence the interest in the Jacobi equation (1.2), which makes practical use of this canonical pair.

Chapter 2
Recent Observations and Understanding Physical Meaning of Jacobi's Virial Equation

Fundamentals of all the planetary and Solar System sciences are tested first of all by the laws of the Earth movement, where the confidence limit to the laws can be checked by observation. More over, all the sense of human being is connected with this planet. As far as the techniques and instruments for observation were developed, then geodesists, astronomers and geophysicists have noticed that in the planet's inertial rotation some irregularities and deviations relative to the accepted standard parameters and hydrostatic state conditions have appeared. Those irregularities that are often called as inaccuracies, number of which is counted by more than dozen, finally were incorporated into two problems, namely, variation of the angular velocity in the daily, monthly, annually and secular time scale, and variation in the poles motion in the same time scales. Just after the problems became obvious and have not find resolution in the frame work of the accepted physical and theoretical conceptions of celestial mechanics the latter lost interest in the problems of the Earth dynamics. In this connection the well known German theoreticians in dynamics, Klein and Sommerfeld, stated that the Earth mechanics appear to be more complicated than the celestial mechanics and represents "*some confused labyrinths of geophysics*" (Klein and Sommerfeld 1903). In order to study irregular velocity of the Earth rotation and the pole motion numerous projects of observation and regular monitoring were organized by the planetary network. As it was always in such cases, the cause of the observed effects was searched in the effects of perturbations coming from the Moon and the Sun, and also in the influence of dynamical effects of the own shells like the atmosphere, the oceans and the liquid core, existence of which is considered by many researchers.

2.1 Dynamical Effects Discovered by Space Study

Artificial satellites, which made a start of space study in the second part of the twentieth century, opened a new page in space sciences. It was determined that the ultimate goal of this scientific program should be an answer to the question of

V.I. Ferronsky et al., *Jacobi Dynamics*, Astrophysics and Space Science Library 369,
DOI 10.1007/978-94-007-0498-5_2,

the Solar system origin. Investigation of the near Earth cosmic space for solution of geodetic and geophysical problems was in the beginning initiated.

The first geodetic satellites for studying dynamic parameters of the planet were launched almost 50 years ago. They gathered vast amounts of data that significantly improved our knowledge of the inner structure and dynamics of the Earth. They made it a real possibility to evaluate experimentally the correctness of basic physical ideas and hypotheses in astronomy, astrophysics, geophysics, geodesy and geology, and to compare theoretical calculations with observations. Success in this direction was achieved in a short period of time.

On the basis of satellite orbit measurements, the zonal, sectorial and tesseral harmonics of gravitational moments in expansion of the gravitational potential by a spherical function, up to tens, twenties and higher degrees were calculated. The calculations have resulted in an important discovery having far-reaching effects. The obtained results proved the long-held assumption of geophysicists that the Earth does not stay in hydrostatic equilibrium, which, in fact, is the basic principle of the theories of dynamics, figure and inner structure of the planet. The same conclusion was made about the Moon.

This conclusion means that the physical conception of hydrostatic equilibrium state of the Earth which was applied for construction of model of the outer and central force field does not satisfy the observed dynamic effects of gravitational interaction of mass particles and should be revised. But the state of scientific knowledge of this phenomenon has been found to be not ready to cope with such a situation. The story of the condition of hydrostatic equilibrium of the planet begins with Newton's consideration, in his famous work *Philosophiae Naturalis Principia Mathematica*, of the Earth's oblateness problem. The investigation based on hydrostatics was further developed by French astronomer and mathematician Clairaut. Later on the hypothesis of hydrostatic equilibrium was extended to all celestial bodies including stars. The authority of Newton was always so high that any other theories for solution of the problem in dynamics and celestial body structure were never proposed. But in current times the problem has arisen of the cause of the discrepancy between theory and observation and a moment has come to take over this crisis in the study of fundamentals of the Earth sciences. A situation like this happened at the beginning of the twentieth century when radioactive and roentgen radiation was discovered and the corpuscular-wave nature of light was proved. This was the starting point for development of quantum mechanics. We seem now to have a similar situation with respect to the planets motion.

We found a still more serious discrepancy related to the Earth hydrostatic equilibrium, which is as follows (Ferronsky and Ferronsky 2007). It is known that the planet's potential energy is almost 300 times more than the kinetic one represented by the body's rotation. This ratio between the potential and kinetic energy contradicts the requirement of the virial theorem according to which the potential energy of a body in the outer uniform force field should be twice as much of the kinetic one.

From point of view of the observed potential energy the Earth's angular velocity should be about 17 times as much as it is. However, the planet has remained for a long time in an equilibrium state. In fact, the Earth appears to have been deprived of its kinetic energy. Some of the other planets, such like Mars, Jupiter, Saturn, Uranus and Neptune, exhibit the same behavior. But for the Mercury, Venus, our Moon and the Sun, the equilibrium states of which are also accepted as hydrostatic, the potential energy exceeds their kinetic energy by 10^4 times. A logical explanation comes to mind that there is some hidden form of motion of the body's interacting mass particles, together with their respective kinetic energy, which has not been taken previously into account. It is known that the hydrostatic equilibrium condition of a body, being stay in the outer force field, satisfies the requirement of the Clausius virial theorem. The same requirement follows also from the Eulerian equations for a liquid-filled uniform sphere. The virial theorem gives an averaged relationship between the potential and kinetic energies of a body. A periodic component of the energy change there during the corresponding time interval is accepted as a constant value and eliminated from consideration. From this evidence it was not difficult to guess that the hidden form of motion and the source of needed kinetic energy of the Earth and the planets including the Moon and the Sun might be found in that eliminated periodic component. In the problem considered by Newton, that component was absent because of his concept of the central gravitational force field, the total sum of which is equal to zero.

Taking into account the relationship between the Earth's gravitational moments and the gravitational potential observed by the satellites, we came back to derivation of the virial theorem in classical mechanics (see below) and obtained its generalized form of the relationship between the energy and the polar moment of inertia of a body. Doing so, we obtained the equation of dynamical equilibrium of a body in its own force field where the hydrostatic equilibrium is a particular case of a uniform body in its outer force field. The equation establishes a relationship between the potential and kinetic energies of a body by means of energy of oscillation of the polar moment of inertia in the form of the energy conservation law. An analytical expression of the derived new form of the virial theorem, based on the Newton's laws of motion, appeared to be the Jacobi's virial equation. In this case the earlier lost of kinetic energy is found by taking into account the oscillating motion of the interacting mass particles, the integral effect of which is expressed through oscillation of the polar moment of inertia. That effect fits the relationship between the potential and kinetic energies in the classical virial theorem. At the same time a novel physical conception about gravitation and electromagnetic interaction is appeared and mechanism of the energy generation becomes clear. The nature of the gravity forces as a derivative of the body's inner energy appears to be discovered.

The obtained, on the basis of Jacobi dynamics, results related to the problem of the Earth are as follows. We found that the new effect, which creates dynamics of the Earth, is its own inner force field. Earlier, the sum of the inner forces and their moments being effected by the outer central force field were considered as equal to zero. We find that the mass forces of interaction being volumetric ones created the inner force field which appears to be the field of power (energy) pressure. That field, according to its definition, can not be equal to zero. The resultant of the field pressure appears to be a space envelope. The envelope has a spherical shape for a sphere and an elliptic shape for an ellipsoid. It was found that dynamic effects of the body's force field occur in oscillation and rotation of the shells according to Kepler's laws. A body that has a uniform mass density distribution realizes all its kinetic energy of the motion in the form of so-called virial oscillations. It was assumed, earlier, that wave properties of this nature, like oscillations for mass particles in mechanics of bodies, are unessential. We found that virial oscillations of a body initiated by the force field of its own interacting mass particles represent the main part of its kinetic energy. Theories based on hydrostatics ignore that energy. But, as it was noted above, in this case the potential energy of the Earth and other celestial bodies by two or more orders exceeds their kinetic energy represented only in the form of axial rotation of the mass. Such an unusual effect has a simple physical explanation. Still in the beginning of the last century French physicist Louis de Broglie expressed an assumption, proved later on, that any micro-particle including electron, proton, atom and molecule, acquires particle-wave properties. The relationship, discovered by the artificial satellites between changes of the Earth's gravitational potential and the moment of inertia, shows that interaction of the planet's masses takes place on their elementary particle levels. It means that the main form of motion of the interacting mass particles is their oscillation. Continuous 'trembling' of the planet's gravitational field, detected by satellites as the gravitational moments change, is another fact proving the de Broglie idea and extending it to the gravitational interaction of celestial body masses.

The dynamical approach to solve the problem under consideration allowed the authors to expand the body's potential energy on its normal, tangential and dissipative components. The differential equations that determine the main body's dynamical parameters, namely its oscillation and rotation, were written. A rigorous solution of the equations was considered on the basis for bodies with spherical and axial symmetry. The solutions of problems relating to rotation, oscillation, obliquity and oblateness of a body's orbit and itself was considered on the basis of the general solution of dynamics of a self-gravitating body in its own force field. It was found that precession and wobbling of the Earth and irregularity of its rotation depends on effects of the polar and equatorial oblateness and the separate rotation of the planet's, the Sun's and the Moon's shells. The induced outer force field of a body follows rotation of the resultant envelope of the shells, but with some delay because of the finite velocity of the energy propagation in the induced outer force field. Also the problems of inner structure of the Earth, the nature of the planet's electromagnetic field and mechanism of the energy generation were considered. The presented theory is applicable not only to the planets and satellites, but also to the stars, where hydrostatic equilibrium is considered as an equation of state. Finally, the theory opens a way to understand the physics of gravitation as the internal power (energy) pressure which occurs at mass interaction on the level of the molecules, atoms and nuclei and elementary mass particles.

The obtained new results which have common relation to all celestial bodies are presented in the corresponding chapters and sections of the book.

2.2 Interpretation of Satellite Orbits and Failure of Hydrostatic Equilibrium of the Earth and the Moon

We recall briefly the conditions of the Earth hydrostatic equilibrium. By definition the hydrostatics is a branch of the hydromechanics, which studies the equilibrium of a liquid and gas and the effects of a stationary liquid on immersed bodies relative to the chosen reference system. For a liquid equilibrated relative to a rigid body, when its velocity of motion is equal to zero and the field of densities is steady the equation of state follows from the Eulerian and Navier–Stokes equations in the form (Sedov 1970)

$$\mathrm{grad}\,\mathrm{p} = \rho F, \tag{2.1}$$

where p is the pressure; ρ is the density; F is the mass force.

In the Cartesian system of reference Eq. 2.1 is written as

$$\begin{aligned} \frac{\partial \mathrm{p}}{\partial \mathrm{x}} &= \rho \mathrm{F_x}, \\ \frac{\partial \mathrm{p}}{\partial \mathrm{y}} &= \rho \mathrm{F_y}, \\ \frac{\partial \mathrm{p}}{\partial \mathrm{z}} &= \rho \mathrm{F_z} \end{aligned} \tag{2.2}$$

If the outer mass forces are absent, i.e. $\mathrm{F_x = F_y = F_z = 0}$, then

$$\mathrm{grad\ p} = 0.$$

In this case, in accordance with the Pascal's law the pressure in all liquid points will be the same.

For the uniform incompressible liquid, when $\rho = \mathrm{const}$, its equilibrium can be only in the potential field of the outer forces. For general case of incompressible liquid and potential field of the outer forces from (2.1) one has

$$\mathrm{dp} = \rho \mathrm{dU}, \tag{2.3}$$

where U is the forces potential.

It follows from Eq. 2.3, that for an equilibrated liquid in the potential force field its density and pressure appear to be a function only of the potential U.

For a gravity force field, when in the steady-state liquid only these forces act, one has

$$\mathrm{F_x = F_y = 0, \quad F_z = -g, \quad U = -gz + const \quad and \quad p = p(z), \rho = \rho(z).}$$

Here the surfaces of constant pressure and density appear as horizontal planes. Then Eq. 2.3 is written in the form

$$\frac{dp}{dz} = -\rho g < 0. \tag{2.4}$$

It means that with elevation the pressure falls and with depth it grows. From here it follows that

$$p - p_0 = -\int_{z_0}^{z} \rho g dz = -\rho g(z - z_0) \tag{2.5}$$

where g is the acceleration of the gravity force.

If a spherical vessel is filled in by incompressible liquid and rotates around its vertical axis with constant angular velocity ω, then for determination of the equilibrated free surface of the liquid in Eq. 2.2 the centrifugal inertial forces should be introduced in the form

$$\begin{aligned} \frac{\partial p}{\partial x} &= \rho\,\omega^2 x, \\ \frac{\partial p}{\partial x} &= \rho\,\omega^2 y, \\ \frac{\partial p}{\partial x} &= -\rho g \end{aligned} \tag{2.6}$$

From here, for the rotating body with radius $r^2 = x^2 + y^2$, one finds

$$p = -\rho g z + \frac{\rho \omega^2 r^2}{2} + C. \tag{2.7}$$

For the points on the free surface $r = 0$, $z = z_o$ one has $p = p_o$. Then

$$C = p_0 + \rho g z_0, \tag{2.8}$$

$$p = p_0 + \rho g(z_0 - z) + \frac{\rho \omega r^2}{2}. \tag{2.9}$$

The equation of the liquid free surface, where $p = p_o$, has a paraboloidal shape

$$z - z_0 = \frac{\omega^2 r^2}{2g}. \tag{2.10}$$

Above relations determine the principal physical conditions and equations of hydrostatic equilibrium of a liquid. They remain a basis for the modern dynamics and theory of the Earth figure. The attempt to harmonize these conditions with the planet's motion conditions has failed, which was proved by observation. It will be shown below that the main obstacle for such harmonization is rejection of the planet's inner force field without which the hydrostatics is unable to provide the equilibrium between the body's interacted forces as Newton's third law requires. The Earth is a self-gravitating body. Its matter moves in the own force field which is generated by elementary mass particle interaction. The mass density distribution, rotation and oscillation of the body's shells result from the inner force field. And the orbital motion of the planet is controlled by interaction of the outer force fields of the planet and the Sun.

Let us look for more specific effects determining the absence of the Earth hydrostatic equilibrium and more realistic conditions of its equilibrium based on the results of the Earth's satellite orbit motion.

The initial factual material for the problem study is presented by the observed orbit elements of the geodetic satellites which move on perturbed Kepler's orbits. The satellite motion is fixed by means of observational stations located within zones of a visual height range of 1,000–2,500 km, which is optimal for the planet's gravity field study. It was found that the satellite's perturbed motion at such a close distance from the Earth's surface is connected with the non-uniform distribution of mass density, the consequences of which are the non-spherical shape in the figure and the corresponding non-uniform distribution of the outer gravity field around the planet. These non-uniformities cause corresponding changes in trajectories of the satellite's motion, which are fixed by tracking stations. Thus, distribution of the Earth's mass density determines an adequate equipotential trajectory in the planet's gravity field, which follows the satellite. The main goal of the geodetic satellites launched under different angles relative to the equatorial plane is in measurement of all deviations in the trajectory from the unperturbed Kepler's orbit.

The satellite orbits data for solving the Earth's oblateness problem are interpreted on the basis of the known (in celestial mechanics) theory of expansion of the gravity potential of a body, the structure and the shape of which do not much differ from the uniform sphere. The expression of the expansion, by spherical functions, recommended by the International Union of Astronomy, is the following equation (Grushinsky 1976):

$$U(r,\varphi,\lambda) = \frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n\left(\frac{R_e}{r}\right)^n P_n(\sin\varphi) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n P_{nm}(\sin\varphi)(C_{nm}\cos m\,\lambda + S_{nm}\sin m\,\lambda)\right], \quad (2.11)$$

where r, φ and λ are the heliocentric polar co-ordinates of an observation point; G is the gravity constant; M and R_e are the mass and the mean equatorial radius of the Earth; P_n is the Legendre polynomial of n order; $P_{nm}(\sin\varphi)$ is the associated spherical functions; J_n, C_{nm}, S_{nm} are the dimensionless constants characterizing the Earth's shape and gravity field.

The first terms of Eq. 2.11 determine the zero approximation of Newton's potential for a uniform sphere. The constants J_n, C_{nm}, S_{nm} represent the dimensionless gravitational moments, which are determined through analyzing the satellite orbits. The values J_n express the zonal moments, and C_{nm} and S_{nm} are the tesseral moments. In the case of hydrostatic equilibrium of the Earth as a body of rotation, in the expression of the gravitational potential (2.11) only the even *n*-zonal moments J_n are rapidly decreased with growth, and the odd zonal and all tesseral moments turn into zero, i.e.

$$U = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\ \theta) - \sum_{n=3}^{\infty} J_n\left(\frac{R_e}{r}\right)^{n+1} P_n(\cos\ \theta)\right], \tag{2.12}$$

where θ is the angle of the polar distance from the Earth's pole.

Here the constant J_2 represents the zonal gravitational moment, which characterizes the axial planet's oblateness and makes the main contribution to correction of the unperturbed potential. That constant determines the dimensionless coefficient of the moment of inertia relative to the polar axis and equal to

$$J_2 = \frac{C - A}{MR_e^2}, \tag{2.13}$$

where C and A are the Earth's moments of inertia with respect to the polar and equatorial axes accordingly, and R_e is the equatorial radius.

For expansion by spherical functions of the Earth's gravity forces potential, the rotation of which is taken to be under action of the outer inertial forces, but not of its own force field, the centrifugal force potential is introduced into Eq. 2.12. Then for the hydrostatic condition with the even zonal moments J_n one has

$$W = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\theta) - \sum_{n=3}^{\infty} J_n\left(\frac{R_e}{r}\right)^{n+1} P_n(\cos\theta)\right] + \frac{\omega^2 r^2}{3}[1 - P_2(\cos\theta)], \tag{2.14}$$

where W is the potential of the body of rotation; $\omega^2 r$ is the centrifugal force. The first two terms and the term of the centrifugal force in Eq. 2.14 express the normal potential of the gravity force

$$W = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\theta)\right] + \frac{\omega^2 r^2}{3}[1 - P_2(\cos\theta)]. \tag{2.15}$$

The potential (2.15) corresponds to the spheroid's surface which within oblateness coincides with the ellipsoid of rotation. Rewriting term $P_2(\cos\theta)$ in this equation through the sinus of the heliocentric latitude and the angular velocity – through the geodynamic parameter q, one can find the relationship of the Earth's oblateness ε with the dynamic constant J_2. Then the equation of the dynamic oblateness ε is obtained in the form (Grushinsky 1976; Melchior 1972)

$$\varepsilon = \frac{3}{2} J_2 + \frac{q}{2}, \tag{2.16}$$

where the geodynamic parameter q is the ratio of the centrifugal force to the gravity force at the equator

$$q = \frac{\omega^2 R}{GM/R^2}. \tag{2.17}$$

Dynamic parameter J_2, found by satellite observation in addition to the oblateness calculation, is used for determination of a mean value of the Earth's moment of inertia. For this purpose the constant of the planet's free precession is also used, which represents one more observed parameter expressing the ratio of the moments of inertia in the form:

$$H = \frac{C - A}{C}. \tag{2.18}$$

This is the theoretical base for interpretation of the satellite observations. But its practical application gave very contradictory results (Grushinsky 1976; Melchior 1972; Zharkov 1978). In particular, the zonal gravitation moment calculated by means of observation was found to be $J_2 = 0.0010827$, from where the polar oblateness $\varepsilon = 1/298.25$ appeared to be short of the expected value and equal to 1/297.3. The all zonal moments J_n, starting from J_3, which relate to the secular perturbation of the orbit, were close to constant value and equal, by an order of magnitude, to the square of the oblateness i.e., $\sim(1/300)^2$ and slowly decreasing with an increase of n. The tesseral moments C_{nm} and S_{nm} appeared to be not equal to zero, expressing the short-term nutational perturbations of the orbit. In the case of hydrostatic equilibrium of the Earth at the found value of J_2, the polar oblateness ε should be equal to 1/299.25. On this basis the conclusion was made that the Earth does not stay in hydrostatic equilibrium. The planet's deviation from the hydrostatic equilibrium evidenced that there is a bulge in the planet's equatorial region with amplitude of about 70 m. It means that the Earth body is forced by normal and tangential forces which develop corresponding stresses and deformations. Finally, by the measured tesseral and sectorial harmonics, it was directly confirmed that the Earth has an asymmetric shape with reference to the axis of rotation and to the equatorial plane.

Because the Earth does not stay in hydrostatic equilibrium, then the above described initial physical fundamentals for interpretation of the satellite observations should be recognized as incorrect and the related physical concepts cannot explain the real picture of the planet's dynamics.

The question is raised of how to interpret the obtained actual data and where the truth should be sought. First of all we should verify correctness of the oblateness interpretation and the conclusion about the Earth's equatorial bulge. It is known from observation that the Earth is a triaxial body (Grushinsky 1976). Theoretical application of the triaxial Earth model was not considered because it contradicts the hydrostatic equilibrium hypothesis. But after it was found that the hydrostatic equilibrium is absent, the alternative with the triaxial Earth should be considered first.

Let us analyze Eq. 2.16. It is known from the observation data, that the constant of the centrifugal oblateness q is equal to

$$q = \frac{\omega_3^2}{GM/R^3} = \left(\frac{1}{17.01}\right)^2 = \frac{1}{289.37}. \tag{2.19}$$

Determine a difference between the centrifugal oblateness constant q and the polar oblateness ε' found by the satellite orbits, assuming that the desired value has a relationship with the perturbation caused by the equatorial ellipsoid

$$\begin{aligned}\varepsilon' &= \frac{a-c}{a} - \frac{b-c}{a} = \frac{a-b}{a} = \frac{1}{289.37} - \frac{1}{298.25} = \frac{1}{9720}\\ &= 1.713\left(\frac{1}{289.37}\right)^2,\end{aligned} \tag{2.20}$$

where a, b and c are the semi-axes of the triaxial Earth.

The differences between the major and minor equatorial semi-axes can be found from Eq. 2.20. If the major semi-axis is taken in accordance with recommendation of the International Union of Geodesy and Geophysics as a = 6,378,160 m, then the minor equatorial semi-axis *b* can be equal to:

$$a - b = 6378160/9720 = 656\ \text{m}; \quad b = 6377504\,\text{m}.$$

There is a reason now to assume, that the value of equatorial oblateness $\varepsilon' = 1/9{,}720$ is a component in all the zonal gravitation moments J_n, related to the secular perturbations of the satellite orbits including J_2. They are perturbed both by the polar and the equatorial oblateness of the Earth. Experimental result of dependence of the satellite precession of equinoxes on the orbit angle to the equatorial plane proves the above statement. This effect ought to be expected because it was known long ago from observation that the Earth is a triaxial body. If our conclusion is true, then there is no ground for discussion about the equatorial bulge. And also the problem of the hydrostatic equilibrium is closed automatically because in this case the Earth is not a figure of rotation; and the nature of the observing fact of rotation of the Earth should be looked for rather in the action of its own inner force field but not in the effects of the inertial forces. As to the nature of the Earth's oblateness, then for its explanation later on the effects of perturbation arising during separation of the Earth's shells by mass density differentiation and separation of the Earth itself from the Protosun will be considered. In particular, the effect of heredity in creation of the body's oblateness is evidenced by the ratio of kinetic energy of the Sun and the Moon expressed through the ratio of square frequencies of oscillation ε'' of their polar moments of inertia, which is close to the planet's equatorial oblateness:

$$\varepsilon'' = \frac{\omega_c^2}{\omega_\pi^2} = \frac{\left(10^{-4}\right)^2}{\left(0.96576 \cdot 10^{-2}\right)^2} = \left(\frac{1}{96.576}\right)^2 = 1.73\left(\frac{1}{289.3}\right)^2,$$

where $\omega_c = 10^{-4}\ s^{-1}$ and $\omega_л = 0.96576 \cdot 10^{-2}\ s^{-1}$ are the frequencies of oscillation of the Sun's and the Moon's polar moment of inertia correspondingly.

By observation the Moon is also a triaxial body. In addition, the retrograde motion of the nodes of the Earth, the Moon and the artificial satellites is registered and is explained by rotation of the bodies' orbits. Later on it will be shown, that the above remarkable phenomenon is explained by rotation of the body's inner masses together with their gravity fields, the periods of which are equal to the periods of the precession of their oblique axes. The observed body rotation is valid only for the upper shells, which were separated during mass density differentiation in their own force fields and stay in that field in a suspended state of equilibrium.

The most prominent effect, which was discovered by investigation of the geodetic satellite orbits, is the fact of a physical relationship between the Earth's mean (polar) moment of inertia and the induced outer gravity field. That fact without exaggeration can be called a fundamental contribution to understanding the nature of the planet's self-gravity. The planet's moment of inertia is an integral characteristic of the mass density distribution. Calculation of the gravitational moments based on measurement of elements of the satellite orbits is the main content of satellite geodesy and geophysics. Short-periodic perturbations of the gravity field fixed at revolution of a satellite around the Earth, the period of which is small compared to the planet's period, provides evidence about oscillation of the moment of inertia or, to be more correct, about oscillating motion of the interacting mass particles. It will be shown, that oscillating motion of the interacting particles forms the main part of a body's kinetic energy and the moment of inertia itself is the periodically changing value.

Oscillation of the Earth's moment of inertia and also the gravitational field is fixed not only during the study by artificial satellites. Both parameters have also been registered by surface seismic investigations. Consider briefly the main points of these observations.

The study of the Earth's eigenoscillation started with Poisson's work on oscillation of an elastic sphere, which was considered in the framework of the theory of elasticity. In the beginning of the twentieth century Poisson's solution was generalized by Love in the framework of the problem solution of a gravitating uniform sphere of the Earth's mass and size. The calculated values of periods of oscillation were found to be within the limit of some minutes to 1 h.

In the middle of the twentieth century during the powerful earthquakes in 1952 and 1960 in Chile and Kamchatka an American team of geophysicists headed by Beneoff, using advanced seismographs and gravimeters, reliably succeeded in recording the an entire series of oscillations with periods from 8.4 min up to 57 min. Those oscillations in the form of seismograms have represented the dynamical effects of the interior of the planet as an elastic body, and the gravimetric records have shown the "tremor" of the inner gravitational field (Zharkov 1978). In fact, the effect of the simultaneous action of the potential and kinetic energy in the Earth's interior was fixed by the above experiments.

About 1,000 harmonics of different frequencies were derived by expansion of the line spectrum of the Earth's oscillation. These harmonics appear to be integral characteristics of the density, elastic properties and effects of the gravity field, i.e. of the potential and kinetic energy of separate volumetric parts of the non-uniform planet. As a result two general modes of the Earth's oscillations were found by the above spectral analysis, namely, spherical with a vector of radial direction and torsion with a vector perpendicular to the radius.

From the point of view of the existing conception about the planet's hydrostatic equilibrium, the nature of the observed oscillations was considered to be a property of the gravitating non-uniform (regarding density) body in which the pulsed load of the earthquake excites elementary integral effects in the form of elastic gravity quanta (Zharkov 1978). Considering the observed dynamical effects of earthquakes, geophysicists came close to a conclusion about the nature of the oscillating processes in the Earth's interior. But the conclusion itself still has not been expressed because it continues to relate to the position of the planet's hydrostatic equilibrium.

2.3 Imbalance Between the Earth's Potential and Kinetic Energy

We discovered the most likely serious cause, for which even formulation of the problem of the Earth's dynamics based on the hydrostatic equilibrium is incorrect. The point is that the ratio of kinetic to potential energy of the planet is equal to ~1/300, i.e. the same as its oblateness. Such a ratio does not satisfy the fundamental condition of the virial theorem, the equation of which expresses the hydrostatic equilibrium condition. According to that condition the considered energies' ratio should be equal to 1/2. Taking into account that kinetic energy of the Earth is presented by the planet's inertial rotation, then assuming it to be a rigid body rotating with the observed angular velocity $\omega_r = 7.29 \cdot 10^{-5}\ s^{-1}$, the mass $M = 6 \cdot 10^{24}$ kg, and the radius $R = 6.37 \cdot 10^6$ m, the energy is equal to:

$$\begin{aligned} T_e &= 0.6MR^2\ \omega_r^2 = 0.6 \cdot 6 \cdot 10^{24} \cdot \left(6.37 \cdot 10^6\right)^2 \left(7.29 \cdot 10^{-5}\right)^2 \\ &= 7.76 \cdot 10^{29} J = 7.76 \cdot 10^{36} erg. \end{aligned}$$

The potential energy of the Earth at the same parameters is

$$\begin{aligned} U_e &= 0.6 \cdot GM^2/R \ = 0.6 \cdot 6.67 \cdot 10^{-11} \cdot \left(6 \cdot 10^{24}\right)^2/6.37 \cdot 10^6 \\ &= 2.26 \cdot 10^{32} J \ = 2.26 \cdot 10^{39} erg. \end{aligned}$$

The ratio of the kinetic and potential energy comprises

$$\frac{T_e}{U_e} = \frac{7.76 \cdot 10^{29}}{2.26 \cdot 10^{32}} = \frac{1}{291}.$$

One can see that the ratio is close to the planet's oblateness. It does not satisfy the virial theorem and does not correspond to any condition of equilibrium of a really existing natural system because, in accordance with the third Newton's law, equality between the acting and the reacting forces should be satisfied. The other planets, the Sun and the Moon, the hydrostatic equilibrium for which is also accepted as a fundamental condition, stay in an analogous situation. Since the Earth in reality exists in equilibrium and its orbital motion strictly satisfies the ratio of the energies, then the question arises where the kinetic energy of the planet's own motion has disappeared. Otherwise the virial theorem for the Earth is not valid. Moreover, if one takes into account that the energy of inertial rotation does not belong to the body, then the Earth and other celestial bodies equilibrium problem appears to be out of discussion.

Thus, we came to the problem of the Earth equilibrium from two positions. From one side, the planet by observation does not stay in hydrostatic equilibrium, and from the other side, it does not stay in general mechanical equilibrium because there is no reaction forces to counteract to the acting potential forces. The answer to both questions is given below while deriving an equation of the dynamical equilibrium of the planet by means of generalization of the classical virial theorem.

2.4 Generalization of Classical Virial Theorem

The main methodological question arises: in what kind state of equilibrium the Earth exists? The answer to the question results from the generalized virial theorem for a self-gravitating body, i.e. the body which itself generates the energy for its own motion by interaction of the constituent particles having innate moments. The guiding effect which we use here is the observed by artificial satellite functional relationship between changes in the outer gravity field of the Earth and its mean (polar) moment of inertia. The deep physical meaning of this relationship is as follows. The observed planet's polar moment of inertia is an integral (volumetric) parameter, which represents not fixed interacted mass particles, but expresses changes in their motion under the inner body's energy. The Clausius' virial theorem represents relationship between the potential and kinetic energy in averaged form for a non-interacted (ideal) gaseous cloud of particles or a uniform body which stay in the outer force field. In order to generalize the theorem for a uniform and non-uniform body staying in the own force field we introduce there the volumetric moments of interacted particles, taking into account their volumetric nature. Moreover, the interacted mass particles of a continuous medium generate volumetric forces (pressure or capacity of energy) and volumetric moments, which, in fact, produce the motion in the form of oscillation and rotation of matter. The oscillating form of motion of the Earth and other celestial bodies is the dominating part of their kinetic energy which up to now has not been taken into account. We wish to fill in this gap in dynamics of celestial bodies.

The classic virial theorem is the analytical expression of the hydrostatic equilibrium condition and follows from Newton's and the Euler's equations of motion. Let us recall its derivation in accordance with classical mechanics (Goldstein 1980).

Consider a system of mass points, the location of which is determined by the radius vector $\mathbf{r}_i$ and the force $\mathbf{F}_i$ including the constraints. Then equations of motion of the mass points through their moments $\mathbf{p}_i$ can be written in the form

$$\dot{\mathrm{p}}_\mathrm{i} = \mathrm{F}_\mathrm{i}, \tag{2.21}$$

The value of the moment of momentum is

$$\mathrm{Q} = \sum_\mathrm{i} \mathrm{p}_\mathrm{i} \cdot \mathrm{r}_\mathrm{i},$$

where the summation is done for all masses of the system. The derivative with respect to time from that value is

$$\frac{\mathrm{dQ}}{\mathrm{dt}} = \sum_\mathrm{i} \dot{\mathrm{r}}_\mathrm{i} \cdot \mathrm{p}_\mathrm{i} + \sum_\mathrm{i} \dot{\mathrm{p}}_\mathrm{i} \cdot \mathrm{r}_\mathrm{i}. \tag{2.22}$$

The first term in the right hand side of (2.22) is reduced to the form

$$\sum_\mathrm{i} \dot{\mathrm{r}}_\mathrm{i} \cdot \mathrm{p}_\mathrm{i} = \sum_\mathrm{i} \mathrm{m}_\mathrm{i} \cdot \dot{\mathrm{r}}_\mathrm{i} \cdot \dot{\mathrm{r}}_\mathrm{i} = \sum_\mathrm{i} \mathrm{m}_\mathrm{i} \mathrm{v}_\mathrm{i}^2 = 2\mathrm{T},$$

where T is the kinetic energy of particle motion under action of the forces $\mathbf{F}_i$. The second term in the Eq. 2.22 is

$$\sum_\mathrm{i} \dot{\mathrm{p}}_\mathrm{i} \cdot \mathrm{r}_\mathrm{i} = \sum_\mathrm{i} \mathrm{F}_\mathrm{i} \cdot \mathrm{r}_\mathrm{i}.$$

Now Eq. 2.22 can be written as

$$\frac{\mathrm{d}}{\mathrm{dt}} \sum_\mathrm{i} \mathrm{p}_\mathrm{i} \cdot \mathrm{r}_\mathrm{i} = 2\mathrm{T} + \sum_\mathrm{i} \mathrm{F}_\mathrm{i} \cdot \mathrm{r}_\mathrm{i}. \tag{2.23}$$

The mean values in (2.23) within the time interval τ are found by their integration from 0 to t and division by τ:

$$\frac{1}{r}\int_0^t \frac{dQ}{dt}dt = \overline{\frac{dQ}{dt}} = \overline{2T} + \overline{\sum_l F_l \cdot r_l}$$

or

$$\overline{2T} + \overline{\sum_i F_i \cdot r_i} = \frac{1}{\tau}[Q(\tau) - Q(0)]. \quad (2.24)$$

For the system, in which the co-ordinates of mass point motion are repeated through the period τ, the right hand side of Eq. 2.24 after its averaging is equal to zero. If the period is too large, then the right hand side becomes a very small quantity. Then, the expression (2.24) in the averaged form gives the following relation

$$-\overline{\sum_i F_i \cdot r_i} = 2T, \quad (2.25)$$

or in mechanics it is written in the form

$$2T = -U$$

Equation 2.25 is known as the virial theorem, and its left hand side is called the virial of Clausius (German *virial* is from the Latin *vires* which means forces). The virial theorem is a fundamental relation between the potential and kinetic energy and is valid for a vide range of natural systems, the motion of which is provided by action of different physical interactions of their constituent particles. Clausius proved the theorem in 1870 when he solved the problem of work of the Carnot thermal machine, where the final effect of the water vapor pressure (the potential energy) was connected with the kinetic energy of the piston motion. The water vapor was considered as a perfect gas. And the mechanism of the potential energy (the pressure) generation at the coal burning in the firebox was not considered and was not taken into account.

The starting point in the above-presented derivation of virial theorem in mechanics is the moment of the mass point system, the nature of which is not considered both in mechanics and by Clausius. By Newton's definition the moment "*is the measure of that determined proportionally to the velocity and the mass*". The nature of the moment by his definition is "*the innate force of the matter*". By his understanding that force is an inertial force, i.e. the motion of a mass continues with a constant velocity.

The observed (by satellites) relationship between the potential and the kinetic energy of the gravitation field and the Earth's moment of inertia evidences, that the kinetic energy of the interacted mass particle motion, which is expressed as a volumetric effect of the planet's moment of inertia, is not taken into account. The evidence of that was given in the previous Sect. 2.3 in the quantitative calculation of a ratio between the kinetic and potential energies, equal to ~1/300.

In order to correct the contradiction, the kinetic energy of motion of the interacted particles should be taken into account in the derived virial theorem. Because of any mass has volume the momentum **p** should be written in volumetric form:

$$p_i = \sum_i m_i \dot{r}_i. \tag{2.26}$$

Now the volumetric moment of momentum acquires the wave nature and is presented as

$$Q = \sum_i p_i \cdot r_i = \sum_i m_i \cdot \dot{r}_i \cdot r_i = \frac{d}{dt}\left(\sum_i \frac{m_i r_i^2}{2}\right) = \frac{1}{2}\dot{I}_p \tag{2.27}$$

where I_p is the polar moment of inertia of the system of interacted particles (for the sphere it is equal to 3/2 of the axial moment).

The derivative from that value with respect to time is

$$\frac{dQ}{dt} = \frac{1}{2}\ddot{I}_p = \sum_i \dot{r}_i \cdot p_i + \sum_i \dot{p}_i \cdot r_i. \tag{2.28}$$

The first term in the right hand part of (2.28) remains without change

$$\sum_i \dot{r}_i \cdot p_i = \sum_i m_i \cdot \dot{r}_i \cdot \dot{r}_i = \sum_i m_i v_i^2 = 2T. \tag{2.29}$$

The second term represents the potential energy of the system

$$\sum_i \dot{p}_i \cdot r_i = \sum_i F_i \cdot r_i = U. \tag{2.30}$$

Equation 2.28 is written now in the form

$$\frac{1}{2}\ddot{I}_p = 2T + U. \tag{2.31}$$

Expression (2.31) represents a generalized equation of the virial theorem for a mass particle system interacted by the Newton's law. Here in the left hand side of (2.31) the ignored up to now inner kinetic energy of interaction of the mass particles appears. Solution of Eq. 2.31 gives a variation of the polar moment of inertia within the period τ. For a conservative (uniform with respect to density) system averaged expression (2.28) by integration from 0 to t within time interval τ gives

$$\frac{1}{T}\int_{0}^{t}\frac{dQ}{dt}dt=\overline{\frac{dQ}{dt}}=2\overline{T}+\overline{U}=\dot{I}_{p}. \tag{2.32}$$

Equation 2.32 at $\ddot{I}_p = 0$ (conservative system) gives $\dot{I}_p = E = \text{const.}$, where E is the total system's energy. It means that the interacted mass particles of the system move. with constant velocity. In the case of dissipative system, Eq. 2.32 is not equal to zero and the interacted mass particles move with acceleration. Now the ratio between the potential and kinetic energy has a value in accordance with the Eq. 2.31. Kinetic energy of the interacted mass particles in the form of oscillation of the polar moment of inertia in that equation is taken into account. And now in the frame of the law of energy conservation the ratio of the potential to kinetic energy of a celestial body has a correct value.

Expression (2.31) appears to be an equation of dynamical equilibrium of a self-gravitating body (star, planet, satellite). The hydrostatic equilibrium is absent here because the interacted particles are continuously moving by use of inner energy. Integral effect of the moving particles is fixed by the satellite orbits in the form of changing zonal, sectorial and tesseral gravitational moments. For derivation of the generalized virial theorem we used the potential energy generated by interacted particles of the initial moment (2.26). The initial moments form the inner, or "*innate*" by Newton's definition, energy of the body which has an inherited origin.

Thus, we obtained a differential equation of the second order (2.31) which describes the body dynamics and its dynamical equilibrium.

The virial Eq. 2.31 was obtained by Jacobi already one and a half century ago from the Newton's equations of motion in the form (Jacobi 1884)

$$\ddot{\phi} = U + 2T \tag{2.33}$$

wher где Φ is the Jacobi's function (the polar moment of inertia).

Jacobi has not considered physical meaning of his equation. He assumed that because of two independent variables Φ and U in the equation it can not be resolved.

We succeeded to find an empirical relationship between the two variables and obtained at first an approximate and later on rigorous solution of the equation (Ferronsky et al. 1978, 1987; Ferronsky 2005). The relationship is proved by means of the satellite observation.

Let us tray to explain the cause of discrepancy between the geometric (static) and dynamic oblateness of the Earth. The reason is as follows. The planet's moment of inertia (polar or axial) has changing in time value. The polar moment of inertia of a self-gravitating body has a functional relation with the potential energy, the generation of which results by interaction of the mass particles in regime of periodic oscillations. The hydrostatic equilibrium of a body does not express the real dynamic processes because of loss of energy of the interacting particle oscillation. Because of that it was not possible to understand the nature of the energy. The main part of the body's kinetic energy of the body's oscillation was also lost. As to the rotational motion of the body shells, it appears only in the case of the non-uniform distribution of the mass density. The contribution of rotation to the total body's kinetic energy covers its very small part.

The cause of the accepted incorrect ratio between the Earth potential and kinetic energy lies in the body's hydrostatic equilibrium. Clairaut's equation (1.20), derived for the planet's hydrostatic equilibrium state and applied to determine the geometric oblateness, because of the above discussed reason, has no functional relationship between the force function and the moment of inertia. Therefore for the Earth dynamics problem the equation gives only a first approximation. In formulation of the Earth oblateness problem, Clairaut accepted the Newton's model of action of the centripetal forces from the surface of the planet to its geometric center. In such a physical conception the total effect of the inner force field becomes equal to zero. Below in Sect. 2.5 it will be shown, that the force field of the continuous body's interacted masses represents volumetric pressure, but not a field of vector forces. That is the cause, why the accepted postulate related to the planet's inertial rotation is physically incorrect. It was proved in electrodynamics that the force is not acceptable to be a measure of particle interaction.

The question is raised about how was it happened, that geodynamic problems and first of all the problem of stability of the Earth motion up to now were solved without knowing the planet's kinetic energy. The probable explanation of that seems to lie in the history of the development of science. In Kepler's problem and in the Newton's two body problem solution the transition from the averaged parameters of motion to the real conditions is provided through the mean and the eccentric anomalies, which by geometric procedures indirectly take into account the above energy of motion. In the Earth figure problem this procedure of Kepler is not applicable. Therefore, the so called "inaccuracies" in the Earth motion appear to be the regular dynamic effects of a self-gravitating body, and the hydrostatic model in the problem is irrelevant. The hydrostatic model was accepted by Newton for the other problem, where just this model allowed discovery and formulation the general laws of the planets motion around the Sun. The Newton's centripetal forces in principle satisfy the Kepler's condition when the distance between bodies is mach more than their size accepted as mass points. Such model gives a first approximation in the problem solution. Kepler's laws express the real picture of the planets and satellites motion around their parent bodies in averaged parameters. All the deviations of those averaged values related to the outer perturbations are not considered as it was done in the Clausius' virial theorem for the perfect gas.

Newton solved the two body problem, which has been already formulated by Kepler. The solution was based on the heliocentric world system of Copernicus, on the Galilean laws of inertia and free fall in the outer force field and on Kepler's laws of the planet's motion in the central force field considered as a geometric plane task. The goal of Newton's problem was to find the force by which the planet's motion is resulted. His centripetal attraction and the inertial forces in the two body problem satisfy Kepler's laws.

As it was mentioned, Newton understood the physical meaning of his centripetal or attractive forces as a pressure, which is accepted now like a force field. But by his opinion, for mathematical solutions the force is a more convenient instrument. And in the two body problem the force-pressure is acting from the center (of point) to the outer space.

It is worth to discuss briefly the Newton's preference given to the force but not to the pressure. In mechanics the term "*mass point*" is understood as a geometric point of space, which has no dimension but possesses a finite mass. In physics a small amount of mass is called by the term "*particle*", which has a finite value of size and mass. But very often physicists use models of particles, which have neither size nor mass. A body model like mass point has been known since ancient times. It is simple and convenient for mathematical operations. The point is an irreplaceable geometric symbol of a reference point. The physical point, which defines inert mass of a volumetric body, is also suitable for operations. But the interacted and physically active mass point creates a problem. For instance, in the field theory the point value is taken to denote the charge, the meaning of which is not better understood than is the gravity force. But it is considered often there, that the point model for mathematical presentation of charges is not suitable because operations with it lead to zero and infinite values. Then for resolution of the situation the concept of charge density is introduced. The charge is presented as an integral of density for the taken volume and by this way the solving problem is resolved.

The point model in the two-body problem allowed reduction of it to the one-body problem and for a spherical body of uniform density to write the main seven integrals of motion. In the case when a body has a finite size, then not the forces but the pressure becomes an effect of the body particle interaction. The interacted body's mass particles form a volumetric gravitational field of pressure, the strength of which is proportional to the density of each elementary volume of the mass. In the case of a uniform body, the gravitational pressure should be also uniform within the whole volume. The outer gravitational pressure of the uniform body should be also uniform at the given radius. The non-uniform body has a non-uniform gravitational pressure of both inner and outer field, which has been observed in studying the real Earth field. Interaction of mass particles results in their collision, which leads to oscillation of the whole body system. In general if the mass density value is higher then the frequency of body oscillation has also higher vale.

It was known from the theory of elasticity, that in order to calculate the stress and the deformation of a beam from a continuous load, the latter can be replaced by the equivalent lumped force. In that case the found solution will be approximate because the beam's stress and deformation will be different. The question is what degree of approximation of the solution and what kind of the error is expected. Volumetric forces are not summed up by means of the parallelogram rule. Volumetric forces by their nature can not be reduced for application either to a point, or to a resultant vector value. Their actions are directed to the 4π space and they form inner and outer force field. The force field by its action is proportional to action of the energy. This is because the force is the derivative of the energy.

The centrifugal and Coriolis' forces are also proved to be inertial forces as a consequence of inertial rotation of the body. And the Archimedes force has not found its physical explanation, but it became an observational fact of hydrostatic equilibrium of a body mass immersed into a liquid.

Such is the short story of appearance and development of the hydrostatic equilibrium of the Earth in the outer uniform gravity field. The force of gravity of a body mass is an integral value. In this connection Newton's postulate about the gravity center as a geometric point should be considered as a model for presentation of two interacted bodies, when their mutual distance is much more of the body size. It is shown in the next section, that the reduced physical, but not geometrical, gravity center of a volumetric body is represented by an envelope of the figure, which draws averaged value of radial density distribution of the body.

The problem of dynamics of the Earth as a self-gravitating body, including the figure problem in its formulation and solution needs for a higher degree of approximation. Generalized virial theorem (2.31) satisfies the condition of the Earth dynamical equilibrium state and creates a physical and theoretical basis for farther development of theory. It follows from the theorem that hydrostatic equilibrium state there is the particular case of the dynamics. Solution of problem of the Earth dynamics based on the equation of dynamical equilibrium appears to be the next natural and logistic step from the hydrostatic equilibrium model to a more realistic model without loss of the previous preference.

Below we consider the problem of "decentralization" of the own force field for a self-gravitating body.

2.5 Reduction of Inner Gravitational Field of a Body to the Resultant Envelope of Pressure

As an example, consider the Earth as a self-gravitating sphere with uniform and one-dimensional interacting media. The motion of the Earth proceeds both in its own and in the Sun's force fields. It's known from theoretical mechanics that any motion of a body can be represented by a translation motion of its mass center, rotation around that center and motion of the body mass related to its changes in the shape and structure (Duboshin 1975). In the two-body problem the last two effects are neglected due to their smallness.

In order to study the Earth motion in the own force field the translational (orbital) motion relative to the fixed point (the Sun) should be separated from the two other components of motion. After that one can consider the rotation around the geometric center of the Earth masses under action of the own force field and changes in the shape and structure (oscillation). Such separation is required only for the moment of inertia, which depends on what frame of reference is selected. The force function

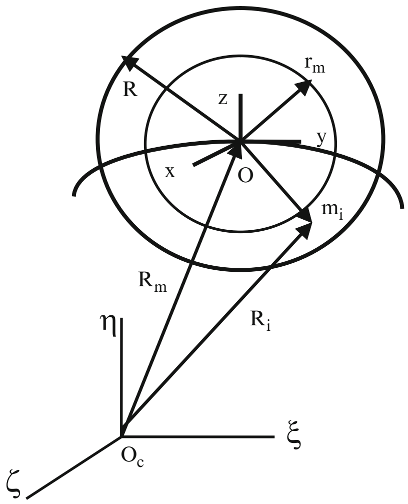

Fig. 2.1 Body motion in own force field

depends on a distance between the interacted masses and does not depend on selection of a frame of reference (Duboshin 1975). The moment of inertia of the Earth relative to the solar reference frame should be split into two parts. The first is the moment of the body mass center relative to the same frame of reference and the second – moment of inertia of the planet's mass relative to the own mass center.

So, set up the absolute Cartesian coordinates $O_c\xi\eta\zeta$ with the origin in the center of the Sun and transfer it to the system Oxyz with the origin in the geometrical center of the Earth's mass (Fig. 2.1).

Then, the moment of inertia of the Earth in the solar frame of reference is

$$I_c = \sum m_i R_i^2, \tag{2.34}$$

where m_i is the Earth mass of particle; R_i is its distance from the origin in the same frame.

The Lagrange's method is applied to separate the moment of inertia (2.34). The method is based on his algebraic identity

$$\left(\sum_{1\leqslant i\leqslant n} a_i^2\right)\left(\sum_{1\leqslant i\leqslant n} b_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} a_i b_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n}\left(a_i b_j - b_i a_j\right)^2, \tag{2.35}$$

where a_i and b_i are whichever values; *n* is any positive number.

Jacobi in his "*Vorlesungen über Dynamik*" was the first who performed the mathematical transformation for separation of the moment of inertia of n interacting mass points into two algebraic sums (Jacobi 1884; Duboshin 1975; Ferronsky et al. 1987). It was shown that if we denote (Fig. 2.1)

$$\xi_i = x_i + A; \quad \eta_i = y + B; \quad \zeta_i = z + C;$$

$$\sum m_i = M; \quad \sum m_i \ \xi_i = MA; \quad \sum m_i \ \eta_i = MB; \quad m_i \ \zeta_i = MC, \tag{2.36}$$

where A, B, C are the coordinates of the mass center in the solar frame of reference. Then, using identity (2.35), one has

$$\sum m_i r_i^2 = \sum m_i \ \xi_i^2 + \sum m_i \ \eta_i^2 + \sum m_i \ \zeta_i^2 = \sum m_i x_i^2 + 2A \sum m_i x_i A^2 \sum m_i$$
$$+ \sum m_i y_i^2 + 2B \sum m_i y_i + B^2 \sum m_i + \sum m_i z_i^2 + 2C \sum m_i z_i + C^2 \sum m_i.$$

Since

$$MA = \sum m_i \ \xi_i = \sum m_i x_i + \sum m_i A = \sum m_i x_i + MA,$$

then $\sum m_i x_i = 0$, and also $\sum m_i y_i = 0$, $\sum m_i z_i = 0$. Now, the moment of inertia (2.34) acquires the form

$$\sum m_i R_i^2 = M\left(A^2 + B^2 + C^2\right) + \sum m_i \left(x_i^2 + y_i^2 + z_i^2\right), \tag{2.37}$$

where

$$M\left(A^2 + B^2 + C^2\right) = \ M{R^2}_m, \tag{2.38}$$

$$\sum m_i \left({x_i}^2 + {y_i}^2 + {z_i}^2\right) = M \ {r_m}^2, \tag{2.39}$$

M is the Earth's mass; R_m and r_m are the radii of inertia of the Earth in the Sun's and the Earth's frame of reference.

Thus, we separated the moment of inertia of the Earth, rotating around the Sun in the inertial frame of reference, into two algebraic terms. The first one (2.38) is the Earth's moment of inertia in the solar reference system $O_c\xi\eta\zeta$. The second term (2.39) presents the moment of inertia of the Earth in the own frame of reference Oxyz. The Earth mass here is distributed over the spherical surface with the reduced radius of inertia r_m. In literature the geometrical center of mass O in the Earth reference system is erroneously identified with the center of inertia and center of gravity of the planet.

For farther consideration of the problem of the Earth's dynamics we accept the polar frame of reference with its origin in center O. Then expression (2.39) for the Earth polar moment of inertia I_p acquires the form

$$I_p = \sum m_i \left({x_i}^2 + {y_i}^2 + {z_i}^2\right) = \sum m_i {r_i}^2 = \ M \ {r_m}^2 \tag{2.40}$$

Now the reduced radius of inertia r_m, which draws a spherical surface, is

$$ {r_m}^2 = \frac{\sum m_i r_i^2}{M}. \tag{2.41}$$

Here $M = \sum m_i$ is the Earth's mass relative to own frame of reference.

Taking into account the spherical symmetry of the uniform and one-dimensional Earth, we consider the sphere as a concentric spherical shell with the mass $dm(r) = 4\pi\, r^2\rho(r)dr$. Then the expression (2.41) in the polar reference system can be rewritten in the form

$$r_m{}^2 = \frac{1}{M}\int_0^R r^2 4\,\pi r^2\rho(r)dr = \frac{4\pi R^2}{MR^2}\int_0^R r^4\rho(r)dr, \tag{2.42}$$

or

$$\frac{4\pi r_m^2}{4\,\pi R^2} = \frac{4\pi\int_0^R r^4\rho(r)dr}{MR^2} = \frac{\beta^2 MR^2}{MR^2} = \beta^2, \tag{2.43}$$

from where

$$r_m{}^2 = b^2R^2,$$

where $\rho(r)$ is the law of radial density distribution; R is the radius of the sphere; β^2 is the dimensionless coefficient of the reduced spheroid (ellipsoid) of inertia β^2MR^2.

The value of β^2 depends on the density distribution $\rho(r)$ and is changed within the limits of $1 \geq \beta^2 > 0$. Earlier (Ferronsky et al. 1987) it was defined as a structural form-factor of the polar moment of inertia.

Analogously, the reduced radius of gravity r_g, expressed as a ratio of the potential energy of interaction of the spherical shells with density $\rho(r)$ to the potential energy of interaction of the body mass distributed over the shell with radius R. The potential energy of the sphere is written as

$$U = 4\,\pi G\int_0^R r\rho(r)m(r)dr = \alpha^2\frac{GM^2}{R}, \tag{2.44}$$

from where

$$\alpha^2 = \frac{4\pi G\int_0^R r\rho(r)m(r)dr}{\frac{GM^2}{R}} = \frac{r_g^2}{R^2}, \tag{2.45}$$

where in expressions (2.44) and (2.45) $m(r) = 4\,\pi\int_0^r r^2\rho(r)dr$.

The value of β^2 depends on the density distribution $\rho(r)$ and is changed within the limits of $1 \geq \alpha^2 > 0$. Earlier (Ferronsky et al. 1987) it was defined as a structural form-factor of the force function.

Table 2.1 Numerical values of form factors α^2 and β^2 for radial distribution of mass density and for polytropic models

Density distribution law	α^2	$\beta_\perp{}^2$	β^2
Radial distribution of mass density			
$\rho(r) = \rho_0$	0.6	0.4	0.6
$\rho(r) = \rho_0(1-r/R)$	0.74	0.27	0.4
$\rho(r) = \rho_0(1-r^2/R^2)$,	0.71	0.29	0.42
$\rho(r) = \rho_0 \exp(1–kr/R)$	0.16 k	$8/k^2$	$12/k^2$
$\rho(r) = \rho_0 \exp(1-kr^2/R^2)$	$\sqrt{\frac{k}{2\pi}}$	1/k	1.5/k
$\rho(r) = \rho_0\, \delta(1-r/R)$	0.5	0.67	1.0
Politrope models			
0	0.6	0.4	0.6
1	0.75	0.26	0.38
1.5	0.87	0.20	0.30
2	1.0	0.15	0.23
3	1.5	0.08	0.12
3.5	2.0	0.045	0.07

Numerical values of the dimensionless form-factors α^2 and β^2 for a number of density distribution laws $\rho(r)$ are given in Table 2.1. The numerical calculations, done by integration of the numerators in Eqs. 2.43–2.45 for the polar moment of inertia and the force function, can be found in the paper (Ferronsky et al. 1978). Note, that value of the polar I_p and axial I_a moments of inertia of one dimensional sphere are related as $I_p = 3/2I_a$.

It follows from Table 2.1 that for a uniform sphere with $\rho(r) = \text{const}$ its reduced radius of inertia coincides with the radius of gravity. Here both dimensionless structural coefficients α^2 and β^2 are equal to 3/5, and the moments of gravitational and inertial forces are equilibrated and because of that the rotation of the mass is absent (Fig. 2.2a).

Thus

$$\frac{r_m^2}{R^2} = \frac{r_g^2}{R^2} = \frac{3}{5}, \tag{2.46}$$

from where

$$r_m = r_g = \sqrt{3/5R^2} = 0.7745966R. \tag{2.47}$$

For a non-uniform sphere at $\rho(r) \neq \mathit{const}$ from Eqs. 2.43–2.45 one has

$$0 < \frac{4\pi r_m^2}{4\;\pi R^2} < \frac{3}{5} < \frac{4\pi r_g^2}{4\;\pi R^2} < 1. \tag{2.48}$$

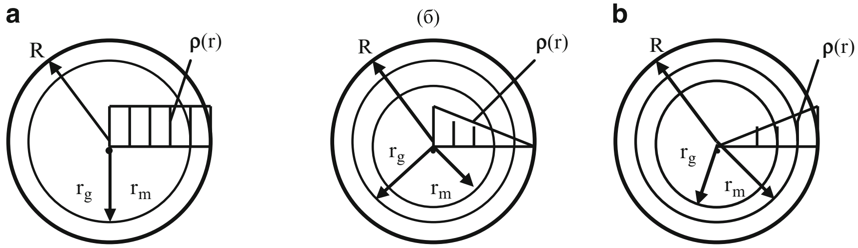

Fig. 2.2 Radius of inertia and radius of gravity for uniform (**a**) and non-uniform sphere with density increased to the center (**b**) and from the center (**c**)

It follows from inequality (2.48) and Table 2.1 that in comparison with the uniform sphere, the reduced radius of inertia of the non-uniform body decreases and the reduced gravity radius increases (Fig. 2.2b). Because of $r_m \neq r_g$ and $r_m < 0.77R < r_g$ the torque appears as a result of an imbalance between gravitational and inertial volumetric forces of the shells. Then from Eq. 2.48 it follows that

$$I_m = I_{mo} - \delta r_{mt} \quad \text{and} \quad r_g = r_{go} + \delta r_{gt} \tag{2.49}$$

where subscripts 0 and t relate to the uniform and non-uniform sphere.

In accordance with (2.48) and (2.49) rotation of shells of a one-dimensional body should be hinged-like and asynchronous. In the case of increasing mass density towards the body surface, then the signs in (2.48) and (2.49) are reversed (Fig. 2.2c). This remark is important because the direction of rotation of a self-gravitating body is function of its mass density distribution.

The main conclusion from the above consideration is that the inner force field of a self-gravitating body is reduced to a closed envelope (spheroid, ellipsoid or more complicated curve) of gravitational pressure, but not to a resulting force passing through the geometric center of the masses. In the case of a uniform body the envelopes have a spherical shape and both gravitational and inertial radii coincide. For a non-uniform body the radius of inertia does not coincide with the radius of gravity, the reduced envelope is closed but has non-spherical (ellipsoidal or any other) shape. Analytical solutions done below justify the above said.

So, we accept the force pressure as an effect of mass particles interaction which is the matter's property to do the work in the form of matter motion.

It follows from this Chapter that physical meaning of the Jacobi's virial equation consists in description of motion of a body (material system) by action of its own force field. This field is formed by the energy of the body's interacting elementary particles and expressed through oscillation of the polar moment of inertia. To the contrary of the failed hydrostatic equilibrium, Jacobi's equation describes the motion in the volumetric forces (energy) and in the volumetric moments (oscillations). The energy here is accepted as the measure of the matter interaction.

We now proceed to derivation of Jacobi's virial equation for the well known physical models of natural systems.

Chapter 3
Derivation of Jacobi's Virial Equation for Description of Dynamics of Natural Systems

Let us begin by deriving Jacobi's virial equation from the equations of Newton, Euler, Hamilton, Einstein and also from equations of quantum mechanics. By doing so we will show that Jacobi's virial equation appears to be a unified instrument for the description of dynamics of natural systems using volumetric (integral) characteristics in the framework of the various physical models of the matter interaction employed. The assumptions under which this equation is derived put only one restriction on the potential energy function to be homogeneous in the co-ordinates. But it will be seen that even this single restriction does not have to be always obligatory. The limitations following from any concrete physical model used for describing dynamics of systems in classical mechanics, hydrodynamics, statistical physics, or the theory of relativity, become unimportant.

We have defined the classical virial theorem for a system moving in the outer uniform force field, which determines the relationship between mean values of the potential and kinetic energy within a certain period of time to be the averaged virial theorem. To the contrary, the virial theorem for a system moving in its own force field and establishing a relationship between the potential and kinetic energy of the oscillating polar moment of inertia, is defined as the generalized (non-averaged) virial theorem or the equation of dynamical equilibrium of a body.

We come to the conclusion that the physical basis of hydrostatic equilibrium does not satisfy the demands of general equilibrium of a body motion. As it was shown in the previous chapter, hydrostatic equilibrium, expressed by the averaged virial theorem, does not take into account kinetic energy of the interacting mass particles of a self-gravitating body and does not provide fundamentals for study of its dynamics. The dynamic equilibrium state based on the generalized virial theorem ensures study of the following physical and dynamical problems:

– Body's shell oscillation and rotation;
– Interpretation of satellite data with respect to the body's precession, nutation and pole wobbling, non-tidal variation in the body's angular velocity, geopotential, sea level changes etc.;

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- The Sun, the Earth and the Moon perturbation effects based on analysis of the dynamical equilibrium of the interacting outer force fields of the bodies;
- Relationship between the gravitational and electromagnetic interaction of the mass particles and the nature of the gravity forces;
- Other dynamical effects arising from action of the inner force field, which earlier was not taken into account.

The theory presented in this book can be applied to study the body which, by its structure, presents a system that includes gaseous, liquid and solid shells. For this purpose derivation of the equation of Jacobi's virial equation from the equations of Newton, Euler, Hamilton, Einstein and also from the equations of quantum mechanics is presented. In this part of the work we justify physical applicability of the above fundamental equation for study of the dynamics and structure of stars, planets, satellites and their shells. The main idea of derivation by introduction of volumetric forces and moments into the transformed equations, as was done in Eqs. 2.27–2.28, is to show that the effect of matter interaction in nature is unique, namely, the motion by energy.

3.1 Derivation of Jacobi's Virial Equation from Newtonian Equations of Motion

Throughout this section the term 'system' is defined as an ensemble of material mass points m_i ($i = 1, 2, 3, \ldots, n$) which interact by Newton's law of universal attraction. This physical model of a natural system forms the basis for a number of branches of physics, such as classical mechanics, celestial mechanics, and stellar dynamics.

We shall not present the traditional introduction in which the main postulates are formulated; we shall simply state the problem (see, for example, Landau and Lifshitz 1973a). We start by writing the equations of motion of the system in some absolute Cartesian co-ordinates ξ, η, ζ. In accordance with the conditions imposed, the mass point m_i is not affected by any force from the other n–1 points except that of gravitational attraction. The projections of this force on the axes of the selected co-ordinates ξ, η, ζ can be written (Fig. 3.1):

$$\Xi_i = Gm_i \sum_{1 \leqslant j \leqslant n, i \neq j} \frac{m_j(\xi_j - \xi_i)}{\Delta_{ij}^3},$$

$$H_i = Gm_i \sum_{1 \leqslant j \leqslant n, i \neq j} \frac{m_j(\eta_j - \eta_i)}{\Delta_{ij}^3}, \tag{3.1}$$

$$Z_i = Gm_i \sum_{1 \leqslant j \leqslant n, i \neq j} \frac{m_j(\zeta_j - \zeta_i)}{\Delta_{ij}^3},$$

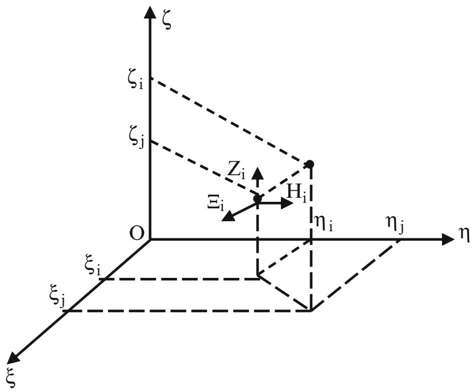

Fig. 3.1 Absolute Cartesian co-ordinate system Oξ η ζ

where G is the gravitational constant and

$$\Delta_{ji} = \sqrt{(\xi_j - \xi_i)^2 \left(\eta_j - \eta_i\right)^2 + (\zeta_j - \zeta_i)^2}$$

is the reciprocal distance between points i and j of the system.

It is easy to check that the forces affect the i-th material point of the system and are determined by the scalar function U, which is called the potential energy function of the system, and is given by

$$U = -G \sum_{1 \leqslant i < j \leqslant n} \frac{m_i m_j}{\Delta_{ij}}. \tag{3.2}$$

Now Eqs. 3.1 can be rewritten in the form

$$\Xi_i = -\frac{\partial U}{\partial \xi_i},$$

$$H_i = -\frac{\partial U}{\partial \eta_i},$$

$$Z_i = -\frac{\partial U}{\partial \zeta_i}.$$

Then Newton's equations of motion for the i-th point of the system take the form

$$\begin{aligned} m_i \ddot{\xi}_i &= \Xi_i, \\ m_i \ddot{\eta}_i &= H_i, \\ m_i \ddot{\zeta}_i &= Z_i, \end{aligned} \tag{3.3}$$

or

$$m_i\ddot{\xi}_i = -\frac{\partial U}{\partial \xi_i},$$

$$m_i\ddot{\eta}_i = -\frac{\partial U}{\partial \eta_i}, \tag{3.4}$$

$$m_i\ddot{\zeta}_i = -\frac{\partial U}{\partial \zeta_i},$$

where dots over co-ordinate symbols mean derivatives with respect to time.

The motion of a system is described by Eqs. 3.3 and 3.4 and is completely determined by the initial data. In classical mechanics, the values of projections ξ_{i0}, η_{i0}, ζ_{i0} and velocities $\dot{\xi}_{i0}$, $\dot{\eta}_{i0}$, $\dot{\zeta}_{i0}$ at the initial moment of time $t = t_0$ may be known from the initial data .

The study of motion of a system of *n* material points affected by self-forces of attraction forms the essence of the classical many-body problem. In the general case, ten classical integrals of motion are known for such a system, and they are obtained directly from the equations of motion.

Summing all the equations (3.3) for each co-ordinate separately, it is easy to be convinced of the correctness of the expressions:

$$\sum_{1\leqslant i\leqslant n} \Xi_i = 0,$$

$$\sum_{1\leqslant i\leqslant n} H_i = 0,$$

$$\sum_{1\leqslant i\leqslant n} Z_i = 0.$$

From those equations it follows that

$$\sum_{1\leqslant i\leqslant n} m_i\ddot{\xi}_i = 0,$$

$$\sum_{1\leqslant i\leqslant n} m_i\ddot{\eta}_i = 0, \tag{3.5}$$

$$\sum_{1\leqslant i\leqslant n} m_i\ddot{\zeta}_i = 0.$$

Equations 3.5, appearing as a sequence of equations of motion, can be successively integrated twice. As a result, the first six integrals of motion are obtained:

$$\sum_{1\leqslant i\leqslant n} m_i\dot{\xi}_i = a_1,$$

$$\sum_{1\leqslant i\leqslant n} m_i\dot{\eta}_i = a_2,$$

$$\sum_{1\leqslant i\leqslant n} m_i\dot{\zeta}_i = a_3.$$

$$\sum_{1\leqslant i\leqslant n} m_i(\xi_i - \dot{\xi}_i t) = b_1, \tag{3.6}$$

$$\sum_{1\leqslant i\leqslant n} m_i(\eta_i - \dot{\eta}_i t) = b_2,$$

$$\sum_{1\leqslant i\leqslant n} m_i(\zeta_i - \dot{\zeta}_i t) = b_3,$$

where a_1, a_2, $a_{3,}$ $в_1$, $в_2$, $в_3$ are integration constants.

These integrals are called integrals of motion of the center of mass. The integration constants a_1, a_2, $a_{3,}$ $в_1$, $в_2$, $в_3$ can be determined from the initial data by substituting their values at the initial moment of time for the values of all the co-ordinates and velocities.

Let us obtain one more group of first integrals. To do this, the second of Eqs. 3.3 can be multiplied by $-\zeta_i$, and the third третье by η_i . Then all expressions obtained should be added and summed over the index i. In the same way, the first of Eqs. 3.3 should be multiplied by ζ_i, and the third by $-\xi_i$ added and summed over index i . Finally, the second of Eqs. 3.3 should be multiplied by ξ_i, and the first by $-\eta_i$ added and summed over index i . It is easy to show directly that the right-hand sides of the expressions obtained are equal to zero:

$$\sum_{1\leqslant i\leqslant n} (Z_i\eta_i - H_i\zeta_i) = 0,$$

$$\sum_{1\leqslant i\leqslant n} (\Xi_i\zeta_i - Z_i\xi_i) = 0,$$

$$\sum_{1\leqslant i\leqslant n} (H_i\xi_i - \Xi_i\eta_i) = 0.$$

Consequently their left-hand sides are also equal to zero:

$$\sum_{1\leqslant i\leqslant n} m_i(\ddot{\zeta}_i\eta_i - \ddot{\eta}_i\zeta_i) = 0,$$

$$\sum_{1\leqslant i\leqslant n} m_i(\ddot{\xi}_i\zeta_i - \ddot{\zeta}_i\xi_i) = 0, \tag{3.7}$$

$$\sum_{1\leqslant i\leqslant n} m_i(\ddot{\eta}_i\xi_i - \ddot{\xi}_i\eta_i) = 0.$$

Integrating Eqs. 3.7 over time, three more first integrals can be obtained:

$$\begin{aligned}\sum_{1\leqslant i\leqslant n} m_i(\dot{\zeta}_i\eta_i - \dot{\eta}_i\zeta_i) &= c_1,\\ \sum_{1\leqslant i\leqslant n} m_i(\dot{\xi}_i\zeta_i - \dot{\zeta}_i\xi_i) &= c_2,\\ \sum_{1\leqslant i\leqslant n} m_i(\dot{\eta}_i\xi_i - \dot{\xi}_i\eta_i) &= c_3.\end{aligned} \tag{3.8}$$

The integrals (3.8) are called area integrals or integrals of moments of momentum. Three integration constants c_1, c_2, c_3 are also determined from the initial data by changing over from the values of all the co-ordinates and velocities to their values at the initial moment of time.

The last of the classical integrals can be obtained by multiplying the three Eqs. 3.4 by $\dot{\xi}_i$, $\dot{\eta}_i$ and $\dot{\zeta}_i$ respectively, and adding and summing all the expressions obtained. As a result, the following equation is obtained:

$$\sum_{1\leqslant i\leqslant n} m_i\left(\ddot{\xi}_i\dot{\xi}_i+\ddot{\eta}_i\dot{\eta}_i+\ddot{\zeta}_i\dot{\zeta}_i\right) = -\sum_{1\leqslant i\leqslant n}\left(\frac{\partial U}{\partial \xi_i}\dot{\xi}_i + \frac{\partial U}{\partial \eta_i}\dot{\eta}_i + \frac{\partial U}{\partial \zeta_i}\dot{\zeta}_i\right). \tag{3.9}$$

It is not difficult to see that the right-hand side of Eq. 3.9 is the complete differential over time of the potential energy function U of the system as a whole. The left-hand side of the same equation is also the complete differential of some function T called the kinetic energy function of the system, and equal to

$$T = \frac{1}{2}\sum_{1\leqslant i\leqslant n} m_i\left(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2\right). \tag{3.10}$$

Equation (3.9) can then be written finally in the form

$$\frac{d}{dt}(T) = -\frac{d}{dt}(U),$$

from which, after integration, one finds that

$$E = T + U, \tag{3.11}$$

where E is the integration constant, determined from the initial conditions.

Equation 3.11 is called the integral of motion or the integral of living (kinetic) forces.

To derive the equation of dynamic equilibrium, or Jacobi's virial equation, each of the equations (3.4) should be multiplied by ξ_i, η_i and ζ_i respectively; then, after summing all the expressions, one can obtain

$$\sum_{1\leqslant i\leqslant n} m_i(\xi_i\xi_i + \eta_i\eta_i + \zeta_i\zeta_i) = -\sum_{1\leqslant i\leqslant n}\left(\xi_i\frac{\partial U}{\partial \xi_i} + \eta_i\frac{\partial U}{\partial \eta_i} + \zeta_i\frac{\partial U}{\partial \zeta_i}\right). \tag{3.12}$$

We can take farther advantage of the obvious identities:

$$m_i \xi_i \ddot{\xi}_i = \frac{1}{2} \frac{d^2}{dt^2} \left(m_i \xi_i^2 \right) - m_i \dot{\xi}_i^2,$$

$$m_i \eta_i \ddot{\eta}_i = \frac{1}{2} \frac{d^2}{dt^2} \left(m_i \eta_i^2 \right) - m_i \dot{\eta}_i^2,$$

$$m_i \zeta_i \ddot{\zeta}_i = \frac{1}{2} \frac{d^2}{dt^2} \left(m_i \zeta_i^2 \right) - m_i \dot{\zeta}_i^2$$

from the Eulerian theorem concerning the homogenous functions. For the interaction of the system points, according to Newton's law of universal attraction, the degree of homogeneity of the potential energy function of the system is equal to –1, and hence

$$-\sum_{1 \leqslant i \leqslant n} \left(\xi_i \frac{\partial U}{\partial \xi_i} + \eta_i \frac{\partial U}{\partial \eta_i} + \zeta_i \frac{\partial U}{\partial \zeta_i} \right) = U.$$

Substituting the above expressions into the right- and left-hand side of Eq. 3.12, one obtains

$$\frac{d^2}{dt^2} \left[\frac{1}{2} \sum_{1 \leqslant i \leqslant n} m_i \left(\xi_i^2 + \eta_i^2 + \zeta_i^2 \right) \right] - 2 \sum_{1 \leqslant i \leqslant n} \frac{1}{2} m_i \left(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2 \right) = U.$$

For a system of material points we now introduce the Jacobi function expressed through the moment of inertia of the system and presented in the form

$$\ddot{\Phi} = \frac{1}{2} \sum_{1 \leqslant i \leqslant n} m_i \left(\xi_i^2 + \eta_i^2 + \zeta_i^2 \right).$$

Then taking into account (3.1), the previous equation can be rewritten in a very simple form as follows:

$$\ddot{\Phi} = 2E - U. \tag{3.13}$$

This is the equation of dynamic equilibrium or Jacobi's virial equation describing both the dynamics of a system and its dynamic equilibrium using integral (volumetric) characteristics Φ and U or T.

Let us derive now another form of Jacobi's virial equation where the translational moment of the center of mass of the system is separated and all the characteristics depend only on the relative distance between the mass points of the system. For this purpose the Lagrangian identity can be used:

$$\left(\sum_{1\leqslant i\leqslant n} a_i^2\right)\left(\sum_{1\leqslant i\leqslant n} b_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} a_i b_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n}\left(a_i b_j - b_i a_j\right)^2, \quad (3.14)$$

where a_i and b_i may acquire any values and n is any positive number.

Let us now put $a_i = \sqrt{m_i}$, and b_i equal to $\sqrt{m_i}\xi_i$, $\sqrt{m_i}\ \eta_i$ and $\sqrt{m_i}\zeta_i$ respectively. Then three identities can be obtained from (3.7):

$$\left(\sum_{1\leqslant i\leqslant n} m_i\right)\left(\sum_{1\leqslant i\leqslant n} m_i\xi_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} m_i\xi_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j\left(\xi_j - \xi_i\right)^2,$$

$$\left(\sum_{1\leqslant i\leqslant n} m_i\right)\left(\sum_{1\leqslant i\leqslant n} m_i\eta_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} m_i\eta_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j\left(\eta_j - \eta_i\right)^2,$$

$$\left(\sum_{1\leqslant i\leqslant n} m_i\right)\left(\sum_{1\leqslant i\leqslant n} m_i\zeta_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} m_i\zeta_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j\left(\zeta_j - \zeta_i\right)^2.$$

In summing up one finds

$$2m\Phi = \left(\sum_{1\leqslant i\leqslant n} m_i\xi_i\right)^2 + \left(\sum_{1\leqslant i\leqslant n} m_i\eta_i\right)^2 + \left(\sum_{1\leqslant i\leqslant n} m_i\zeta_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j \Delta_{ij}^2.$$

Using now Eqs. 3.6, the last equality can be rewritten in the form

$$2m\Phi = \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j \Delta_{ij}^2 + (a_1 t + b_1)^2 + (a_2 t + b_2)^2 + (a_3 t + b_3)^2, \quad (3.15)$$

where

$$m = \sum_{1\leqslant i\leqslant n} m_i$$

is the total mass of the system.

Let us put

$$\Phi_0 = \frac{1}{4m}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j \Delta_{ij}^2.$$

The value Φ_0 does not depend on the choice of the co-ordinate system and coincides with the value of the Jacobi function in the barycentric co-ordinate system. Moreover, from Eq. 3.15 it follows that

$$\ddot{\Phi} = \ddot{\Phi}_0 + \frac{a_1^2 + a_2^2 + a_3^2}{m}.$$

Excluding the value Φ from Jacobi's equation (3.13) with the help of the last equality, the same equation can be obtained in the barycentric co-ordinate system:

$$\ddot{\Phi}_0 = 2E_0 - U, \tag{3.16}$$

where $E_0 = T_0 + U_0$ is the total energy of the system in the barycentric co-ordinate system equal to

$$E_0 = E - \frac{a_1^2 + a_2^2 + a_3^2}{2m}.$$

We can now show that the value of E_0 does not depend on the choice of the co-ordinate system. For this purpose we can again use the Lagrangian identity (3.14). In this case $a_i = \sqrt{m_i}$, and $b_i = \sqrt{m_i}\dot{\xi}_i$, $\sqrt{m_i}\dot{\eta}_i$ and $\sqrt{m_i}\dot{\zeta}_i$. Then the following three identities can be justified:

$$\left(\sum_{1\leqslant i\leqslant n} m_i\right)\left(\sum_{1\leqslant i\leqslant n} m_i\dot{\xi}_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} m_i\dot{\xi}_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j\left(\dot{\xi}_j - \dot{\xi}_i\right)^2,$$

$$\left(\sum_{1\leqslant i\leqslant n} m_i\right)\left(\sum_{1\leqslant i\leqslant n} m_i\dot{\eta}_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} m_i\dot{\eta}_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j\left(\dot{\eta}_j - \dot{\eta}_i\right)^2,$$

$$\left(\sum_{1\leqslant i\leqslant n} m_i\right)\left(\sum_{1\leqslant i\leqslant n} m_i\dot{\zeta}_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} m_i\dot{\zeta}_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j\left(\dot{\zeta}_j - \dot{\zeta}_i\right)^2.$$

After summing and using (3.16) one obtains

$$2mT = \left(a_1^2 + a_2^2 + a_3^2\right) + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j$$
$$\left[\left(\dot{\xi}_i - \dot{\xi}_j\right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j\right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j\right)^2\right]$$

or

$$T = \frac{\left(a_1^2 + a_2^2 + a_3^2\right)}{2m} + \frac{1}{2m}$$
$$\times\left\{\frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n} m_i m_j\left[\left(\dot{\xi}_i - \dot{\xi}_j\right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j\right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j\right)^2\right]\right\}. \tag{3.17}$$

Here the second term on the right-hand side of Eq. 3.17 coincides with the expression for the kinetic energy T_o of a system.

Substituting (3.17) into an expression for E_0, one obtains

$$E_0 = T_0 + U = \frac{1}{2m} \sum_{1 \leqslant i < j \leqslant n} m_i m_j \left[\left(\dot{\xi}_i - \dot{\xi}_j \right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j \right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j \right)^2 \right] - G \sum_{1 \leqslant i < j \leqslant n} \frac{m_i m_j}{\Delta_{ij}}. \tag{3.18}$$

Thus, the total energy of the system E_0 depends only on the distance between the points of the system and on the velocity changes of these distances. But Jacobi's equation (3.16) appears to be invariant with respect to the choice of the co-ordinate system.

We can show now that the requirement of homogeneity of the potential energy function for deriving Jacobi's virial equation is not always obligatory. For this purpose we consider two examples.

3.2 Derivation of a Generalized Jacobi's Virial Equation for Dissipative Systems

Let us derive Jacobi's virial equation for a non-conservative system. We consider a system of *n* material points, the motion of which is determined by the force of their mutual gravitation interaction and the friction force. It is well known that the friction force always appears in the course of evolution of any natural system. It is also known that there is no universal law describing the friction force (Bogolubov and Mitropolsky 1974). The only general statement is that the friction force acts in the direction opposite to the vector of velocity of a considered mass point.

Consider as an example the simplest law of Newtonian friction when its force is proportional to the velocity of motion of the mass:

$$\begin{aligned} \Xi_f &= -k m_i \dot{\xi}_i, \\ H_f &= -k m_i \dot{\eta}_i, \\ Z_f &= -k m_i \dot{\zeta}_i, \end{aligned} \tag{3.19}$$

where $\dot{\xi}_i$, $\dot{\eta}_i$, $\dot{\zeta}_i$ are the components of the radius-vector of the velocity of the i-th mass point in the barycentric co-ordinate system; k is a constant independent of i; $k > 0$.

Sometimes the friction force is independent of the velocity of the mass point. There are also some other laws describing the friction force.

We derive the equation of dynamical equilibrium for a system of n material points using the equations of motion (3.4) and taking into account the friction force expressed by Eqs. 3.19:

$$
\begin{aligned}
m_i\ddot{\xi}_i &= -\frac{\partial U}{\partial \xi_i} - km_i\dot{\xi}_i,\\
m_i\ddot{\eta}_i &= -\frac{\partial U}{\partial \eta_i} - km_i\dot{\eta}_i,\\
m_i\ddot{\zeta}_i &= -\frac{\partial U}{\partial \zeta_i} - km_i\dot{\zeta}_i,
\end{aligned}
\tag{3.20}
$$

where the value of the system's potential energy is determined by Eq. 3.2.

Multiplying each of Eqs. 3.20 by ξ_i, η_i and ζ_i, respectively, and summing through all i, one obtains

$$
\begin{aligned}
\sum_{1\leqslant i\leqslant n} m_i\left(\xi_i\ddot{\xi}_i + \eta_i\ddot{\eta}_i + \zeta_i\ddot{\zeta}_i\right) = &-\sum_{1\leqslant i\leqslant n}\left(\frac{\partial U}{\partial \xi_i}\xi_i + \frac{\partial U}{\partial \eta_i}\eta_i + \frac{\partial U}{\partial \zeta_i}\zeta_i\right)\\
&-\sum_{1\leqslant i\leqslant n} km_i\left(\xi_i\dot{\xi}_i + \eta_i\dot{\eta}_i + \zeta_i\dot{\zeta}_i\right).
\end{aligned}
\tag{3.21}
$$

Transforming the right- and left-hand sides of Eq. 3.21 in the same way as in deriving Eq. 3.13, one obtains

$$
\ddot{\Phi} - 2T = U - k\dot{\Phi}
$$

or (3.22)

$$
\ddot{\Phi} = 2E - U - k\dot{\Phi}.
$$

Let us show that the total energy E of the system is a monotonically decreasing function of time. For this purpose we multiply each of the equations (3.20) by the vectors $\dot{\xi}_i$, $\dot{\eta}_i$, $\dot{\zeta}_i$, respectively, and sum over all from 1 to n, which results in

$$
\begin{aligned}
\sum_{1\leqslant i\leqslant n} m_i\left(\ddot{\xi}_i\dot{\xi}_i + \ddot{\eta}_i\dot{\eta}_i + \ddot{\zeta}_i\dot{\zeta}_i\right) = &-\sum_{1\leqslant i\leqslant n}\left(\frac{\partial U}{\partial \xi_i}\dot{\xi}_i + \frac{\partial U}{\partial \eta_i}\dot{\eta}_i + \frac{\partial U}{\partial \zeta_i}\dot{\zeta}_i\right)\\
&-k\sum_{1\leqslant i\leqslant n} m_i\left(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2\right).
\end{aligned}
$$

The last expression can be rewritten in the form

$$\frac{d}{dt}(T) = -\frac{d}{dt}(U) - 2kT$$

or

$$dE = -2kTdt. \tag{3.23}$$

Since the kinetic energy T of the system is always greater than zero, $dE \leq 0$, i.e. the total energy of a gravitating system is a monotonically decreasing function of time. Thus the expression for the total energy E(t) of the system can be written as

$$E(t) = E_0 - 2k \int_{t_0}^{t} T(t)dt = E_0[1 + q(t)],$$

where q(t) is a monotonically increasing function of time.

Finally, the equation of dynamical equilibrium for a non-conservative system takes the form

$$\ddot{\Phi} = 2E_0[1 + q(t)] - U - k\dot{\Phi}. \tag{3.24}$$

The second example where the requirement of homogeneity of the potential energy function for deriving Jacobi's virial equation is not obligatory is as follows. We derive Jacobi's virial equation for a system whose mass points interact mutually in accordance with Newton's law and move without friction in a spherical homogenous cloud whose density ρ_o is constant in time. Let, also, the geometric center of the cloud coincide with the center of mass of the considered system. The equations of motion for such a system can be written in the form:

$$\begin{aligned} m_i\frac{d^2\xi_i}{dt^2} &= -\frac{4}{3}\pi G\rho_0 m_i\xi_i - \frac{\partial U}{\partial \xi_i}, \\ m_i\frac{d^2\eta_i}{dt^2} &= -\frac{4}{3}\pi G\rho_0 m_i\eta_i - \frac{\partial U}{\partial \eta_i}, \\ m_i\frac{d^2\zeta_i}{dt^2} &= -\frac{4}{3}\pi G\rho_0 m_i\zeta_i - \frac{\partial U}{\partial \zeta_i}, \end{aligned} \tag{3.25}$$

where i =1, 2, . . ., n.

It is obvious that the above system of equations possesses the ten first integrals of motion and that Jacobi's virial equation, written in the form

$$\frac{d^2\Phi}{dt^2} = 2E - U - \frac{8}{3}\pi G\rho_0\Phi. \tag{3.26}$$

is valid for it.

The equation in the form (3.26) was first obtained by Duboshin et al. (1971). Equations 3.24 and 3.26 can be written in a more general form:

$$\ddot{\Phi} = 2E - U + X\big(t, \Phi, \dot{\Phi}\big), \tag{3.27}$$

where $X\big(t, \Phi, \dot{\Phi}\big)$ is a given function of time t, the Jacobi function Φ and first derivative $\dot{\Phi}$. Moreover, we can call Eq. 3.27 a generalized equation of dynamical equilibrium.

The examples considered above justify the statement that for conditions of homogeneity of the potential energy function, required for the derivation of Jacobi's virial equation, is not always necessary. This condition is required for description of dynamics of conservative systems but not for dissipative systems or for systems in which motion is restricted by some other conditions.

3.3 Derivation of Jacobi's Virial Equation from Eulerian Equations

We now derive Jacobi's virial equation by transforming of the hydrodynamic or continuum model of a physical system. As is well known, the hydrodynamic approach to solving problems of dynamics is based on the system of differential equations of motion supplement, in the simplest case, by the equations of state and continuity, and by the appropriate assumptions concerning boundary conditions and perturbations affecting the system.

In this section, we understand by the term 'system' some given mass M of ideal gas localized in space by a finite volume V and restricted by a closed surface S. Let the gas in the system move by the forces of mutual gravitational interaction and of baric gradient. In addition, we accept the pressure within the volume to be isotropic and equal to zero on the surface S bordering the volume V. Then for a system in some Cartesian inertial co-ordinate system ξ, η, ζ, the Eulerian equations can be written in the form

$$\begin{aligned}
&\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial}{\partial \xi}u + \rho v\frac{\partial}{\partial \eta}u + \rho w\frac{\partial}{\partial \zeta}u = -\frac{\partial p}{\partial \xi} + \rho\frac{\partial U_G}{\partial \xi},\\
&\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial}{\partial \xi}v + \rho v\frac{\partial}{\partial \eta}v + \rho w\frac{\partial}{\partial \zeta}v = -\frac{\partial p}{\partial \eta} + \rho\frac{\partial U_G}{\partial \eta},\\
&\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial}{\partial \xi}w + \rho v\frac{\partial}{\partial \eta}w + \rho w\frac{\partial}{\partial \zeta}w = -\frac{\partial p}{\partial \zeta} + \rho\frac{\partial U_G}{\partial \zeta},
\end{aligned} \tag{3.28}$$

where $\rho(\xi, \eta, \zeta, t)$ is the gas density; u, v, w are components of the velocity vector $\bar{v}(\xi, \eta, \zeta, t)$ in a given point of space; $p(\xi, \eta, \zeta, t)$ is the gas pressure; U_G is Newton's potential in a given point of space.

The value U_G is given by

$$U_G = G \int\limits_{(V)} \frac{\rho(x, y, z, t)}{\Delta} dxdydz, \tag{3.29}$$

where G is the gravity constant; $\Delta = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ is the distance between system points.

The potential energy of the gravitational interaction of material points of the system is linked to the Newtonian potential (3.29) by the relation

$$U = -\frac{1}{2} \int\limits_{(V)} U_G \rho(\xi, \eta, \zeta, t) d\xi d\eta d\zeta.$$

To supplement the system of equations of motion we write the equation of continuity:

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial \xi}(\rho u) + \frac{\partial}{\partial \eta}(\rho v) + \frac{\partial}{\partial \zeta}(\rho w) = 0 \tag{3.30}$$

and the equation of state

$$p = f(\rho) \tag{3.31}$$

assuming at the same time that the processes occurring in the system are barotropic.

Let us obtain the ten classical integrals for the system whose motion is described by Eqs. 3.28.

We derive the integrals of the motion of the center of mass by integrating each of the equations (3.28) with respect to all the volume filled by the system. Integrating the first equation, we obtain

$$\int\limits_{(V)} \rho \frac{du}{dt} d\xi d\eta d\zeta + \int\limits_{(V)} \rho \left(u \frac{du}{d\xi} + v \frac{du}{d\eta} + w \frac{du}{d\zeta} \right) d\xi d\eta d\zeta$$
$$= -\int\limits_{(V)} \frac{dp}{d\xi} d\xi d\zeta \eta d + G \int\limits_{(V)} \rho(\xi, \eta, \zeta, t) \left[\int\limits_{(V)} \rho(x,y,z,t) \frac{x - \xi}{\Delta^3} dxdydz \right] d\xi d\eta d\zeta \tag{3.32}$$

The second term in the right-hand side of Eq. 3.32 disappears because of the symmetry of the integral expression with respect to x and ξ. In accordance with the Gauss-Ostrogradsky theorem the first term in the right-hand side of Eq. 3.32 terns to zero. In fact

$$\int_{(V)} \frac{dp}{d\xi} d\xi d\eta d\zeta = \int_{(S)} p d\eta d\zeta = 0 \tag{3.33}$$

as pressure p on the border of the considered system is equal to zero owing to the absence of outer effects.

Bearing in mind the possibility of passing to a Lagrangian co-ordinate system, and taking into account the law of the conservation of mass $dm = \rho dV = \rho_0 dV_0 = dm_0$, we get

$$\int_{(V)} \rho \frac{du}{dt} d\xi d\eta d\zeta + \int_{(V)} \rho \left(u \frac{du}{d\xi} + v \frac{du}{d\eta} + w \frac{du}{d\zeta} \right) d\xi d\eta d\zeta$$
$$= \int_{(V)} \rho \frac{du}{dt} dV = \int_{(V_0)} \rho_0 \frac{du}{dt} dV_0 = \frac{d}{dt} \int_{(V_0)} u \rho_0 dV_0 = \frac{d}{dt} \int_{(V)} \rho u dV,$$

where V_o and ρ_o are the volume and the density in the initial moment of time t_0 .

Finally, Eq. 3.32 can be rewritten as

$$\frac{d}{dt} \int_{(V)} \rho\, u dV = 0. \tag{3.34}$$

Integrating (3.34) with respect to time and writing analogous expressions for two other equations of the system (3.28), we obtain the first three integrals of motion:

$$\begin{aligned} &\int_{(V)} \rho\, u dV = a_1, \\ &\int_{(V)} \rho\, v dV = a_2, \\ &\int_{(V)} \rho\, w dV = a_3. \end{aligned} \tag{3.35}$$

Equations 3.35 represent the law of conservation of the system moments. Integration constants a_1, a_2, a_3 can be obtained from the initial conditions.

We consider the first equation of the system (3.35) using again the law of conservation of mass. Then it is obvious that

$$\int_{(V)} u\,\rho dV = \int_{(V)} \frac{d\xi}{dt}\rho dV = \int_{(V_0)} \frac{d\xi}{dt}\rho_0 dV_0 = \frac{d}{dt}\int_{(V_0)} \xi\rho_0 dV_0 = \frac{d}{dt}\int_{(V)} \xi\rho dV = a_1. \tag{3.36}$$

Analogous expressions can be written for the two other equations (3.35). Integrating them with respect to time, we obtain integrals of motion of the center of mass of the system in the form

$$\begin{aligned} \int_{(V)} \xi\rho dV &= a_1 t + b_1, \\ \int_{(V)} \eta\rho dV &= a_2 t + b_2, \\ \int_{(V)} \zeta\rho dV &= a_3 t + b_3. \end{aligned} \tag{3.37}$$

We now derive three integrals of the moment of momentum of motion. For this purpose we multiply the second of Eqs. 3.28 by $-\zeta$, the third by η, and then sum and integrate the resulting expressions with respect to volume V occupied by the system. We obtain

$$\int_{(V)} \rho\left(\eta\frac{dw}{dt} - \zeta\frac{dv}{dt}\right)dV = -\int_{(V)} \left(\eta\frac{\partial p}{\partial \zeta} - \zeta\frac{\partial p}{\partial \eta}\right)dV + \int_{(V)} \rho\left(\eta\frac{\partial U_G}{\partial \zeta} - \zeta\frac{\partial U_G}{\partial \eta}\right)dV. \tag{3.38}$$

Analogously, multiplying the first of Eqs. 3.28 by ζ, the third by $-\xi$, then summing and integrating with respect to volume V, we obtain

$$\int_{(V)} \rho\left(\zeta\frac{du}{dt} - \xi\frac{dw}{dt}\right)dV = -\int_{(V)} \left(\zeta\frac{\partial p}{\partial \xi} - \xi\frac{\partial p}{\partial \zeta}\right)dV + \int_{(V)} \rho\left(\zeta\frac{\partial U_G}{\partial \xi} - \xi\frac{\partial U_G}{\partial \zeta}\right)dV. \tag{3.39}$$

Multiplying the second of Eqs. 3.28 by ξ, the first by –η, and summing and integrating as above, the third equality can be written

$$\int_{(V)} \rho\left(\xi\frac{d\,v}{dt}-\eta\frac{du}{dt}\right)dV=-\int_{(V)}\left(\xi\frac{\partial p}{\partial\eta}-\eta\frac{\partial p}{\partial\xi}\right)dV+\int_{(V)}\rho\left(\xi\frac{\partial U_G}{\partial\eta}-\eta\frac{\partial U_G}{\partial\xi}\right)dV. \tag{3.40}$$

We write the second integral in the right-hand side of Eq. 3.38 in the form

$$\int_{(V)} \rho\left(\eta\frac{dw}{dt}-\zeta\frac{dv}{dt}\right)dV=G\int_{(V)}\rho(\xi,\eta,\zeta,t)\eta d\xi d\eta d\zeta\int_{(V)}\rho(x,y,z,t)\frac{z-\zeta}{\Delta^3}dx,dy,dz$$
$$-G\int_{(V)}\rho(\xi,\eta,\zeta,t)\zeta d\xi d\eta d\zeta\int_{(V)}\rho(x,y,z,t)\frac{y-\zeta}{\Delta^3}dx,dy,dz.$$

The integral is equal to zero owing to the asymmetry expressed by the integral expressions with respect to z, ζ and y, η. Because the pressure at the border of the domain S is equal to zero, the first term in the right-hand side of Eq. 3.38 is also equal to zero. Actually,

$$\int_{(V)}\left(\eta\frac{\partial p}{\partial\zeta}-\zeta\frac{\partial p}{\partial\eta}\right)dV=\int_{(V)}\left[\frac{d}{d\eta}(\xi p)-\frac{d}{d\xi}(\eta p)\right]dV=\int_{(V)}[\xi p d\xi d\zeta-\eta p d\eta d\zeta]=0.$$

Taking into account the law of mass conservation, the left-hand side of Eq. 3.38 in the Lagrange co-ordinate system can be rewritten as

$$\int_{(V)}\rho\left(\eta\frac{dw}{dt}-\zeta\frac{dv}{dt}\right)dV=\int_{(V)}p\frac{d}{dt}(\eta w-\zeta v)dV$$
$$=\frac{d}{dt}\int_{(V)}p(\eta w-\zeta v)dV=0 \tag{3.41}$$

Integrating this equation with respect to time, the first of the three integrals is obtained:

$$\int_{(V)}p(\eta w-\zeta v)dV=C_1.$$

The other two integrals can be obtained analogously. Thus the system of integrals of the moment of momentum has the form

$$\int\limits_{(V)} p(\eta w - \zeta v)dV = C_1,$$

$$\int\limits_{(V)} p(\zeta u - \xi w)dV = C_2, \tag{3.42}$$

$$\int\limits_{(V)} p(\xi v - \eta u)dV = C_3.$$

To derive the tenth integral of motion representing the law of energy conservation, we multiply each of the system of equations (3.28) by u, v, and w accordingly, and then sum and integrate the equality obtained with respect to the system volume

$$\begin{aligned}\int\limits_{(V)} \rho\left(\frac{du}{dt}u + \frac{dv}{dt}v + \frac{dw}{dt}w\right)dV = &-\int\limits_{(V)}\left(\frac{\partial p}{\partial \xi}u + \frac{\partial p}{\partial \eta}v + \frac{dp}{d\zeta}w\right)dV \\ &+ \int\limits_{(V)} \rho(\xi,\eta,\zeta,t)\left(\frac{\partial U_G}{\partial \xi}u + -\frac{\partial U_G}{\partial \eta}v + \frac{\partial U_G}{\partial \zeta}w\right)dV.\end{aligned} \tag{3.43}$$

Applying the law of mass conservation for an elementary volume, it can easily be seen that the left-hand side of Eq. 3.43 expresses the change of the velocity of kinetic energy of the system:

$$\int\limits_{(V)} \rho\left(\frac{du}{dt}u + \frac{dv}{dt}v + \frac{dw}{dt}w\right)dV = \frac{d}{dt}\left[\frac{1}{2}\int\limits_{(V)}\left(u^2 + v + w^2\right)dV\right] = \frac{d}{dt}(T).$$

The first integral in the right-hand side of Eq. 3.43 can be transferred into

$$-\int\limits_{(V)}\left(\frac{\partial p}{\partial \xi}u + \frac{\partial p}{\partial \eta}v + \frac{dp}{d\zeta}w\right)dV = 3\frac{d}{dt}\int\limits_{(V)} pdV$$

and gives the change of velocity of the internal energy of the system.

The second integral in the right-hand side of the same equation expresses the velocity of the potential energy change:

$$\begin{aligned}&\int\limits_{(V)} \rho(\xi,\eta,\zeta,t)d\xi d\eta d\zeta\left(\frac{\partial U_G}{\partial \xi}\frac{d\xi}{dt} + -\frac{\partial U_G}{\partial \eta}\frac{d\eta}{dt} + \frac{\partial U_G}{\partial \zeta}\frac{d\zeta}{dt}\right) \\ &= \frac{d}{dt}\left[-\frac{1}{2}\int\limits_{(V)} \rho(\xi,\eta,\zeta,t)d\xi d\eta d\zeta\, U_G\right] = -\frac{d}{dt}(U).\end{aligned}$$

Finally, the law of energy conservation can be written in the form

$$T + U + W = E = \text{const.} \tag{3.44}$$

where W is the internal energy of the system.

We now derive Jacobi's virial equation for a system described by Eqs. 3.28–3.31. For this purpose we multiply each of Eqs. 3.28 by ξ, η and ζ respectively, summing and integrating the resulting expressions with respect to the volume of the system:

$$\begin{aligned}\int\limits_{(V)} \rho\left(\frac{du}{dt}\xi + \frac{dv}{dt}\eta + \frac{dw}{dt}\zeta\right)dV = &- \int\limits_{(V)} \left(\frac{\partial p}{\partial \xi}\xi + \frac{\partial p}{\partial \eta}\eta + \frac{dp}{d\zeta}\zeta\right)dV \\ &+ \int\limits_{(V)} \rho\left(\frac{\partial U_G}{\partial \xi}\xi + -\frac{\partial U_G}{\partial \eta}\eta + \frac{\partial U_G}{\partial \zeta}\zeta\right)dV\end{aligned} \tag{3.45}$$

Using the obtained identities considered in the previous section, we have

$$\frac{du}{dt}\xi = \frac{1}{2}\frac{d^2}{dt^2}\left(\xi^2\right) - u^2,$$

$$\frac{dv}{dt}\eta = \frac{1}{2}\frac{d^2}{dt^2}\left(\eta^2\right) - v^2,$$

$$\frac{dw}{dt}\zeta = \frac{1}{2}\frac{d^2}{dt^2}\left(\zeta^2\right) - w^2.$$

Taking into account the law of conservation of mass for elementary volume, we transform the left-hand side of Eq. 3.45 as follows:

$$\begin{aligned}\int\limits_{(V)} \rho\left(\frac{du}{dt}\xi + \frac{dv}{dt}\eta + \frac{dw}{dt}\zeta\right)dV = &\frac{1}{2}\int\limits_{(V)} \rho\frac{d^2}{dt^2}\left(\xi^2 + \eta^2 + \zeta^2\right)dV \\ &- \int\limits_{(V)} \rho\left(u^2 + v^2 + w^2\right)dV = \ddot{\Phi} - 2T,\end{aligned} \tag{3.46}$$

where

$$\Phi = \frac{1}{2}\int\limits_{(V)} \rho\left(\xi^2 + \eta^2 + \zeta^2\right)dV$$

is the Jacobi function and

$$T = \frac{1}{2} \int_{(V)} \rho(u^2 + v^2 + w^2) dV$$

is the kinetic energy of the system.

We now transform the first integral in the right-hand side of Eq. 3.45. Using the Gauss-Ostrogradsky theorem and the equality with zero pressure at the border of the system, we can write

$$-\int_{(V)} \left(\frac{\partial p}{\partial \xi}\xi + \frac{\partial p}{\partial \eta}\eta + \frac{dp}{d\zeta}\zeta \right) dV = -\int_{(V)} \left[\frac{\partial}{\partial \xi}(p\xi) + \frac{\partial}{\partial \eta}(p\eta) + \frac{d}{d\zeta}(p\zeta) \right] dV$$
$$+ 3\int_{(V)} p dV = 3\int_{(V)} p dV. \qquad (3.47)$$

The obtained equation expresses the doubled internal energy of the system.

The second integral in the right-hand side of Eq. 3.45 is equal to the potential energy of the gravitational interaction of mass particles within the system

$$\int_{(V)} \rho \left(\frac{\partial U_G}{\partial \xi}\xi + -\frac{\partial U_G}{\partial \eta}\eta + \frac{\partial U_G}{\partial \zeta}\zeta \right) dV = U. \qquad (3.48)$$

Substituting Eqs. 3.46, 3.47 and 3.48 into (3.45), Jacobi's virial equation is obtained in the form

$$\ddot{\Phi} - 2T = 3\int_{(V)} p dV + U. \qquad (3.49)$$

Taking into account the law of conservation of energy (3.44), we rewrite Eq. 3.49 in a form which will be used farther:

$$\ddot{\Phi} = 2E - U, \qquad (3.50)$$

where $E = T + U + W$ is the total energy of the system.

3.4 Derivation of Jacobi's Virial Equation from Hamiltonian Equations

Let the system of material points be described by Hamiltonian equations of motion. Let also the considered system consist of n material points with masses m_i. Its generalized co-ordinates and moments are q_i and $p_i = m_i(dq_i/dt)$. Hamiltonian equations for such a system can be written as

$$\dot{p}_i = -\frac{\partial H}{dq_i},$$

$$\dot{q}_i = \frac{\partial H}{dp_i}, \tag{3.51}$$

where H (p, q) is the Hamiltonian; i = 1, 2, . . ., n.

Using values q_i and p_i, we can construct the moment of momentum

$$\sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{n} m_i q_i \dot{q}_i = \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{m_i q_i^2}{2}\right).$$

Now the Jacobi function may be introduced

$$\sum_{i=1}^{n} p_i q_i = \dot{\Phi}. \tag{3.52}$$

Differentiating Eq. 3.52 with respect to time, Jacobi's virial equation is obtained in the form

$$\ddot{\Phi} = \sum_{i=1}^{n} \dot{p}_i q_i + \sum_{i=1}^{n} p_i \dot{q}_i. \tag{3.53}$$

Substituting expressions for $\dot{p}_i$ and $\dot{q}_i$ taken from the Hamiltonian equations (3.51) into the right-hand side of (3.52), we obtain Jacobi's virial equation written in Hamiltonian form:

$$\ddot{\Phi} = \sum_{i=1}^{n} \left(-\frac{\partial H}{\partial q_i} q_i + \frac{\partial H}{\partial p_i} p_i\right). \tag{3.54}$$

The Hamiltonian of the system of material points interacting according to the law of the inverse squares of distance is a homogeneous function in terms of moments p_i with a degree of homogeneity of the function equal to 2, and in terms of co-ordinates q_i with a degree of homogeneity equal to –1. It follows from this

$$H(p, q) = T(p) + U(q)$$

and hence

$$\sum_{i=1}^{n} P_i \frac{\partial H}{\partial q_i} = 2T.$$

$$\sum_{i=1}^{n} q_i \frac{\partial H}{\partial q_i} = -U$$

Taking these relationships into account, Eq. 3.54 acquires the usual form of Jacobi's virial equation (3.50) for the system of mass points interacting according to the law of inverse squares of distance.

Equation 3.54 is more general then Eq. 3.50. The use of generalized co-ordinates and moments as independent variables permits us to obtain the solution of Jacobi's virial equation, taking into account gravitational and electromagnetic perturbations as well as quantum effects, both in the framework of classical physics and in terms of the Hamiltonian written in an operator form. In the general case, Eq. 3.54 can be reduced to (3.50) as the potential energy of interaction of the system's points is a homogenous function of its co-ordinates.

3.5 Derivation of Jacobi's Virial Equation in Quantum Mechanics

It is known that in quantum mechanics some physical value L by definition takes the linear Hermitian operator$\hat{L}$. Any physical state of the system take the normalized wave function ψ . The physical value of L can take the only eigenvalues of the operator $\hat{L}$. The mathematical expectation $\hat{L}$ of the value L at state ψ is determined by the diagonal matrix element

$$\overline{L} = <\psi|\hat{L}|\psi>. \tag{3.55}$$

The matrix element of the operators of the Cartesian co-ordinates $\hat{x}_i$ and the Cartesian components of the conjugated moments $\hat{p}_k$ calculated within wave functions f and g of the system satisfy Hamilton's equations of classical mechanics:

$$\frac{d}{dt}<f|\hat{p}_i|g> = -<f|\frac{\partial \hat{H}}{d\hat{x}_i}|g>, \tag{3.56}$$

$$\frac{d}{dt}<f|\hat{x}_i|g> = -<f|\frac{\partial \hat{H}}{\partial \hat{p}_i}|g>, \tag{3.57}$$

where $\hat{H}$ is the operator which corresponds to the classical Hamiltonian.

Operators $\hat{p}_i$ and $\hat{x}_k$ satisfy the commutation relations

$$\begin{aligned} &[\hat{p}_i, \hat{x}_k] = i\hbar\ \delta_{ik}, \\ &[\hat{p}_i, \hat{p}_k] = 0, \\ &[\hat{x}_i, \hat{x}_k] = 0, \end{aligned} \tag{3.58}$$

where ħ is Planck's constant; δ_{ik} is the Kronecker's symbol; $\delta_{ik} = 1$ at $i = k$ and $\delta_{ik} = 0$ at $i \neq k$.

Operator components of momentum $\hat{p}_i$ for the functions whose arguments are Cartesian co-ordinates $\hat{x}_i$ have the form

$$\hat{p}_i = i\hbar \frac{\partial}{\partial x_i} \tag{3.59}$$

and reverse vector

$$\hat{p} = -i\hbar\nabla.$$

The derivative taken from the operator with respect to time does not depend explicitly on time; it is defined by the relation

$$\hat{L} = -\frac{i}{\hbar}\left[\hat{L}, \hat{H}\right] \tag{3.60}$$

where $\hat{H}$ is the Hamiltonian operator that can be obtained from the Hamiltonian of classical mechanics in accordance with the correspondence principle.

We have already noted that in the classical many-body problem the translational motion of the center of mass can be separated from the relative motion of the mass points if only the inertial forces affect the system. We can show that in quantum mechanics the same separation is possible.

The Hamiltonian operator of a system of n particles which is not affected by external forces in co-ordinates is

$$\hat{H} = -\frac{\hbar^2}{2}\sum_{i=1}^{n}\frac{\nabla_i^2}{m_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{i=1}^{n} U_{ik}(x_i - x_k, y_i - y_k, z_i - z_k). \tag{3.61}$$

Let us replace in (3.61) the three *n* co-ordinates x_i, y_i, z_i by co-ordinates X, Y, Z of the center of mass and by co-ordinates $\xi_\lambda, \eta_\lambda, \zeta_\lambda$, which determine the position of a particle λ (λ = 1, 2, . . ., n – 1) relative to particle n. We obtain

$$X = \frac{1}{M}\sum_{i=1}^{n} m_i x_i,$$

$$M = \sum_{i=1}^{n} m_i, \tag{3.62}$$

$$\xi_\lambda = x_\lambda - x_n,$$

where λ = 1, 2, . . ., n – 1.

Analogously the corresponding relations for Y, Z, η_λ, ζ_λ are obtained.
It is easy to obtain from (3.62) the following operator relations:

$$\frac{d}{dx_p} = \frac{m_p}{M}\frac{\partial}{\partial X} + \frac{\partial}{\partial \xi_p}, \quad p = 1, 2, \ldots, n-1,$$

$$\frac{\partial}{\partial X_n} = \frac{m_n}{M}\frac{\partial}{\partial X} - \sum_{\lambda=1}^{n-1}\frac{\partial}{\partial \xi_\lambda},$$

$$\begin{aligned}\sum_{\lambda=1}^{n-1}\frac{1}{\partial x_i}\frac{\partial^2}{\partial x_i^2} &= \sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\left(\frac{m_\lambda^2}{M^2}\frac{\partial^2}{\partial X^2} + 2\frac{m_\lambda}{M}\frac{\partial^2}{\partial X\partial\xi_\lambda} + \frac{\partial^2}{\partial \xi_\lambda^2}\right)\\ &\quad + \frac{1}{m_n}\left(\frac{m_n^2}{M^2}\frac{\partial^2}{\partial X^2} - 2\frac{m_n}{M}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial X\partial\xi_\lambda} + \sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial\xi_\mu\partial\xi_\lambda}\right)\\ &= \frac{1}{m_n}\frac{\partial^2}{\partial X^2} + \left(\sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\frac{\partial^2}{\partial\xi_\lambda^2} + \frac{1}{m_n}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial\xi_\mu\partial\xi_\lambda}\right),\end{aligned}$$

where summing on the Greek index is provided from 1 до n – 1. It is seen that all the combined derivatives $\partial^2/\partial X\partial\xi_\lambda$ were mutually reduced and do not enter into the final expression. This allows the Hamiltonian to be separated into two parts:

$$H = H_o + H_r$$

where, in the right-hand side, the first term

$$H_o = \frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial Z^2}\right)$$

describes the motion of the center of mass, and the second term

$$H_r = -\frac{\hbar^2}{2}\left(\sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\nabla_\lambda^2 + \frac{1}{m_n}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\nabla_\lambda\nabla\ \mu\right) + U \tag{3.63}$$

describes the relative motion of the particles.
The potential energy in (3.63), which is

$$U = \frac{1}{2}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}U_{\lambda\mu}(\xi_\lambda - \xi_\mu, \eta_\lambda - \eta_\mu, \zeta_\lambda - \zeta_\mu) + \sum_{\lambda-1}^{n-1}U_{\lambda\mu}(\xi_\lambda, \eta_\lambda, \zeta_\lambda), \tag{3.64}$$

also certainly does not depend on the co-ordinates of the center of mass.

Now the Schrödinger's equation

$$(H_o + H_r)\ \psi = E\ \psi \tag{3.65}$$

permits the separation of variables.

Assuming $\psi = \phi(X, Y, Z)$ and $(\xi_\lambda, \eta_\lambda, \zeta_\lambda,)$, we obtain

$$-\frac{\hbar^2}{2V}\nabla^2\phi = E_0\phi, \tag{3.66}$$

$$H_r u = E_r u, \tag{3.67}$$

$$E_0 + E_r = E. \tag{3.68}$$

The solution of Eq. 3.66 has the form of a plane wave:

$$\phi = e^{ik\overline{R}}, \tag{3.69}$$

$$E_0 = \frac{\hbar^2 k^2}{2M},$$

where R is a vector with co-ordinates X, Y, Z.

The result obtained is in full accordance with the classical law of the conservation of motion of the center of mass. This means that the center of mass of the system moves like a material point with mass m and momentum $\hbar\overline{k}$. The mode of relative motion of the particles is determined by Eq. 3.67, which does not depend on the motion of the center of mass.

The existence in the right-hand side of Eq. 3.63 of the third term restricts further factorization of the function $u(\xi_\lambda, \eta_\lambda, \zeta_\lambda)$. Only in the two-body problem, where $n = 2$ and at $\lambda = \mu = 1$, a part of the Hamiltonian connected with the relative motion simplified and takes the form

$$H_r = -\frac{\hbar^2}{2}\left(\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2\right) + U_{12}(\xi_1, \eta_1, \zeta_1) \tag{3.70}$$

It seems that choosing the corresponding system of co-ordinates can lead us to an approach for separating the motion of the center of mass to the many-body problem.

Introducing into Eq. 3.70 the reduced mass m*, which is determined as in classical mechanics by the relation

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{m*}, \tag{3.71}$$

and omitting indices in the notation for relative co-ordinates and potential energy U_{12}, we come to

$$-\frac{\hbar^2}{2m*}\nabla^2 u + U(\xi, \eta, \zeta)u = E_r u. \quad (3.72)$$

This is Schrödinger's equation for the equivalent one-particle problem.

Considering the hydrogen atom in the framework of the one-particle problem, it is assumed that the nucleus is in ground state. In accordance with Eq. 3.72, the normalized mass of the nucleus and electron m* should be introduced. No changes which account for the effect of the nucleus on the relative motion should be introduced. Because of the nucleus, mass *m* is much heavier than electron mass m_e*; instead of Eq. 3.71 we can use its approximation

$$m^* = m\left(1 - \frac{m}{M}\right).$$

Comparing, for example, the frequency of the red line H_α ($n = 3$–$n' = 2$) in the spectrum of a hydrogen atom:

$$\omega(H_\alpha) = \frac{5}{36}\frac{m_H^* e^4}{2\hbar^2 h}$$

with the frequency of the corresponding line in the spectrum of a deuterium atom:

$$\omega(D_\alpha) = \frac{5}{36}\frac{m_D^* e^4}{2\hbar^2 h},$$

and taking into account that $m_D \approx 2m_H$, for the difference of frequencies, we obtain

$$\omega(D_\alpha) - \omega(H_\alpha) = \frac{m_D^* - m_H^*}{m_H^*}\,\omega(H_\alpha) \approx \frac{m}{2M_H}\,\omega(H_\alpha).$$

This difference is not difficult to observe experimentally. At wavelength 6,563 Å it is equal to 4.12 cm^{-1}. Heavy hydrogen was discovered in 1932 by Urey, Brickwedde and Murphy, who observed a weak satellite D_α in the line H_α of the spectrum of natural hydrogen. This proves the practical significance of even the first integrals of motion.

We now show that the virial theorem is valid for any quantum mechanical system of particles retained by Coulomb (outer) forces:

$$2\overline{T} + U = 0.$$

We prove this by means of scale transformation of the co-ordinates keeping unchanged normalization of wave functions of a system.

The wave function of a many-particle system with masses m_i and electron charge e_i satisfies the Schrödinger's equation:

$$-\frac{\hbar^2}{2}\sum_{i=1}^{n-1}\frac{1}{m_i}\nabla_i^2\,\psi+\frac{1}{2}\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\frac{e_i e_k}{r_{ik}}\psi=E\psi \tag{3.73}$$

and the normalization condition

$$\int d\,\tau_1\ldots\int\ \psi^*\ \psi d\,\tau_n=1. \tag{3.74}$$

The mean values of the kinetic and potential energies of a system at stage ψ are determined by the expressions

$$T=-\frac{\hbar^2}{2}\sum_{i=1}^{n-1}\frac{1}{m_i}\int d\,\tau_1\ldots.\int\ \psi^*\nabla_i^2\ \psi d\,\tau_n, \tag{3.75}$$

$$U=\frac{1}{2}\sum_{i=1}^{n-1}\sum_{i=1}^{n-1}e_i e_k\int d\,\tau_1\ldots\int d\,\tau_n\frac{\psi^*\psi}{r_{ik}}d\,\tau_n. \tag{3.76}$$

The scale transformation

$$\bar{r}_i^{'}=\ \lambda\bar{r}_i, \tag{3.77}$$

keeps in force the condition (3.74) and means that the wave function

$$\psi(\bar{r}_i,\ldots,\bar{r}_n) \tag{3.78}$$

is replaced by the function

$$\psi_\lambda=\lambda^{3n/2}\ \psi(\ \lambda\bar{r}_i,\ldots,\ \lambda\bar{r}_n) \tag{3.79}$$

Substituting (3.79) into Eqs. 3.76 and 3.75 and passing to new variables of integration (3.77), and taking into account that

$$\nabla_i^2=\lambda^2\nabla_i^{\prime 2},$$

$$\frac{1}{r_{ik}}=\lambda\frac{1}{r'_{ik}},$$

instead of the true value of the energy, $\overline{E}=\overline{T}+\overline{U}$, we obtain

$$\overline{E}(\ \lambda)=\lambda^2\overline{T}+\ \lambda\overline{U}. \tag{3.80}$$

Equation 3.80 should have a minimum value in the case when the function which is the solution of the Schrödinger's equation is taken from the family of functions (3.79), i.e. when $\lambda = 1$. So, at $\lambda = 1$ the expression

$$\frac{\partial \overline{\mathrm{E}}(\lambda)}{\partial \lambda} = 2\,\lambda^2 \overline{\mathrm{T}} + \overline{\mathrm{U}}$$

should turn into zero, and thus

$$2\overline{\mathrm{T}} + \overline{\mathrm{U}} = 0,$$

which is what we want to prove.

We now derive Jacobi's virial equation for a particle in the inner force field with the potential U(q) and fulfilling the condition

$$-\,\mathrm{q}\nabla \mathrm{U}(\mathrm{q}) = \mathrm{U} \tag{3.81}$$

using the quantum mechanical principle of correspondence. We shall also show that in quantum mechanics Jacobi's virial equation has the same form and contents as in classical mechanics, the only difference being that its terms are corresponding operators.

In the simplest case the Hamiltonian of a particle is written

$$\hat{\mathrm{H}} = -\frac{\hbar^2}{2\mathrm{m}}\nabla^2 + \hat{\mathrm{U}}, \tag{3.82}$$

and its Jacobi function is

$$\hat{\Phi} = \frac{1}{2}\mathrm{m}\hat{\mathrm{q}}^2. \tag{3.83}$$

It is clear that the following relations are valid:

$$\nabla\hat{\Phi} = \mathrm{m}\hat{\mathrm{q}},$$

$$\nabla^2\hat{\Phi} = \mathrm{m}.$$

Following the definition of the derivative with respect to time from the operator of the Jacobi function of a particle (3.60), we can write $\hat{U}$

$$\dot{\hat{\Phi}} = -\frac{1}{\hbar}\left[\hat{\Phi}, \hat{\mathrm{H}}\right],$$

where, after corresponding simplification, quantum mechanical Poisson brackets can be reduced to the form

$$\left[\hat{\Phi}, \hat{\mathrm{H}}\right] = \frac{\hbar^2}{2\mathrm{m}}\left\{\nabla^2\hat{\Phi} + 2\left(\nabla\hat{\Phi}\right)\nabla\right\} = \frac{\hbar^2}{2\mathrm{m}}(\mathrm{m} + 2\mathrm{m}\hat{\mathrm{q}}\nabla). \tag{3.84}$$

The second derivative with respect to time from the operator of the Jacobi function is:

$$\ddot{\hat{\Phi}} = -\frac{1}{\hbar^2}\left\{\left[\hat{\Phi}, \hat{\mathrm{H}}\right], \hat{\mathrm{H}}\right\}. \tag{3.85}$$

Substituting the corresponding value of $\left[\hat{\Phi}, \hat{H}\right]$ and $\hat{H}$ from (3.84) and (3.82) into the right-hand side of (3.85), we obtain

$$\ddot{\hat{\Phi}} = -\frac{\hbar^2}{2\mathrm{m}}\frac{1}{\hbar^2}\left[(\mathrm{m} + 2\mathrm{m}\hat{q}\nabla), \left(-\frac{\hbar^2}{2\mathrm{m}}\nabla^2 + \hat{\mathrm{U}}\right)\right]. \tag{3.86}$$

After simple transformation, the right-hand side of (3.86) will be

$$\ddot{\hat{\Phi}} - \frac{1}{2\mathrm{m}}\left\{2\hbar^2\nabla^2 + 2\mathrm{m}\hat{q}\left(\nabla\hat{\mathrm{U}}\right)\right\} = -\frac{2\hbar^2}{2\mathrm{m}}\nabla^2 + \hat{\mathrm{U}}, \tag{3.87}$$

where, in writing this expression in the right-hand side, we used condition (3.81).

Add and subtract the operator from the right-hand side of Eq. 3.87 and, following the definition of the Hamiltonian of the system (3.82), we obtain the quantum mechanical Jacobi virial equation (equation of dynamical equilibrium of the system), which has the form

$$\ddot{\hat{\Phi}} = 2\hat{\mathrm{H}} - \hat{\mathrm{U}}. \tag{3.88}$$

From Eq. 3.88, by averaging with respect to time, we obtain the quantum mechanical analogue of the classical virial theorem (equation of hydrostatic equilibrium of the system). In accordance with this theorem the following relation is kept for a particle performing finite motion in space

$$2\hat{H} = \overline{\hat{U}}. \tag{3.89}$$

Analogously, one can derive Jacobi's virial equation and the classical virial theorem for a many-particle system, the interaction potential for which depends on distance between any particle pair and is a homogeneous function of the coordinates. In particular, Jacobi's virial equation for Coulomb interactions will have the form of Eq. 3.88.

3.6 General Covariant Form of Jacobi's Virial Equation

Jacobi's initial equation

$$\ddot{\Phi} = 2E - U,$$

which was derived in the framework of Newtonian mechanics and is correct for the system of material points interacting according to Newton and Coulomb laws, includes two scalar functions Φ and U relates to each other by a differential relation. We draw attention to the fact that neither function, in its structure, depends explicitly on the motion of the particles constituting the body. The Jacobi function Φ is defined by integrating the integrand $\rho(r)r^2$ over the volume (where ρ(r) is the mass density and r is the radius vector of the material point) and is independent in explicit form of the particle velocities. The potential energy U also represents the integral of m(r)dm(r)/r over the volume (where m(r) is the mass of the sphere's shell of radius r*;* dm(r) is the shell's mass) independent of the motion of the particles for the same reason.

Let us derive Jacobi's equation from Einstein's equation written in the form

$$\Delta G = 3\pi T, \tag{3.90}$$

where ΔG and T are the Einstein tensor and energy-momentum tensor accordingly.

In fact, since the covariant divergence of Einstein's tensor is equal to zero, we consider the covariant divergence of the energy-momentum tensor T only of substance and fields (not gravitational). Moreover, the ordinary divergence of the sum of the tensor T and pseudotensor t of the energy-momentum of the gravitational field can be substituted for the covariant divergence of the tensor T. This ordinary divergence leads to the existence of the considered quantities.

Let us define the sum of the tensor T and pseudotensor t through T_{ij} and derive Jacobi's equation in this notation.

The equation for ordinary divergence of the sum $T_{ij} = (T + t)_{ij}$ can be written

$$T_{0k,k} - T_{00,0} = 0, \tag{3.91}$$

$$T_{jk,k} - T_{j0,0} = 0. \tag{3.92}$$

We multiply Eq. 3.92 by x^j and integrate over the whole space (assuming the existence of a synchronous co-ordinate system). Integrating by parts, neglecting the surface integrals (they vanish at infinity), and transforming to symmetrical form with respect to indices, we obtain

$$\int T_{ij} dV = \frac{1}{2}\left[\int \left(T_{i0}x^j + T_{j0}x^i\right) dV\right] = 0, \tag{3.93}$$

where i, j are spatial indices.

Similarly, multiplying (3.91) by x^i x^j and integrating over the whole space, it follows that

$$\left[\int T_{00}x^i x^j dV\right]_{,0} = -\int \left(T_{i0}x^j + T_{j0}x^i\right)dV. \tag{3.94}$$

From (3.93) and (3.94) we finally get

$$\int T_{ijdV} = \tfrac{1}{2}\left[\int T_{00}x^i x^j\right]_{,0,0}. \tag{3.95}$$

It is worth recalling that T_{00} also includes the gravitational defect of the mass due to the pseudotensor t by definition

The integral $\int T_{00}x^i x^j dV$ represents the generalization of the Jacobi function $\Phi = \frac{1}{2}\int \rho r^2 dV$ introduced earlier, if we take the spur (also commonly known as the trace) of Eq. 3.95. Let us clarify this operation.

In Eq. 3.95 the spur is taken by the spatial co-ordinates. It is therefore necessary either to represent the total zero spur by four indices, as happens in the case of a transverse electromagnetic field, or to represent the relationship between the reduced spur with three indices and the total spur, as happens in the case of the energy-momentum tensor of matter.

Special care should be taken while representing the spur of the pseudotensor of the energy-momentum t. Consider the post-Newtonian approximation. In this approximation, assuming the value of 2u to be –-g_{oo} –1, the components of the pseudotensor t are written in the form

$$t^{00} = -\frac{7}{8\,\pi}u_{,j,i},$$

$$t^{ij} = -\frac{1}{4\,\pi}\left(u_{,j,i} - \frac{1}{2}\ \delta_{ij}u_{,k}u_{,k}\right),$$

so that

$$S_p t = t^{00} + S_p\left(t^{ij}\right) = -\frac{1}{\pi}u_{,i}u_{,j} = \frac{1}{7}t^{00},$$

$$S_p\left(t^{ij}\right) = \frac{6}{7}t^{00}.$$

The spur in the left-hand side of Eq. 3.95 can therefore be reduced to the energy of the Coulomb field, the total energy of the transverse electromagnetic field and the gravitational energy (when it can be separated, i.e. post-Newtonian approximation).

Finally, it follows in this case that the scalar form of Jacobi's equation holds:

$$\Phi_{,0,0} = mc^2, \tag{3.96}$$

where *m* is the mass, accounting for the baryon defect of the mass and the total energy of the electromagnetic radiation. We do not take into account the radiation of the gravitational waves.

The result obtained by Tolman for the spherical mass distribution (Tolman 1969) is of interest:

$$m = 4\,\pi \int \hat{\varepsilon} r^2 dr, \tag{3.97}$$

where r is the radius and $\hat{\varepsilon}$ is the energy density.

The integral (3.97) acquires a form which is also valid in the case of flat space-time. This result can be explained as follows. The curvature of space-time is exactly compensated by the mass defect. This probably explains the fact that Jacobi's virial equation, derived from Newton's equations of motion which are valid in the case of non-relativistic approximation for a weak gravitational field, becomes more universal than the equations from which it was derived.

We shall not study the general tensor of Jacobi's virial equation, since in the framework of the assumed symmetry for the considered problems we are interested only in the scalar form of the equation as applied to electromagnetic interactions. As follows from the above remarks, in this case Jacobi's equation remains unchanged and the energy of the free electromagnetic field is accounted for in the term defining the total energy of the system. Total energy enters into Jacobi's equation without the electromagnetic energy irradiated up to the considered moment of time. Moreover, for the initial moment of time we take the moment of system formation. This irradiated energy appears also to be responsible for the growth of the gravitational mass defect in the system, as was mentioned above.

3.7 Relativistic Analogue of Jacobi's Virial Equation

Let us derive Jacobi's virial equation for asymptotically flat space-time. We write the expression of a 4-moment of momentum of a particle:

$$p^i x_i, \tag{3.98}$$

where $p^i = mcu^i$ is the 4-momentum of the particle; c is the velocity of light; $u^i = dx^i/ds$ is the 4- velocity; x^i is the 4-co-ordinate of the particle; s is the interval of events, and i is the running index with values 0, 1, 2, 3.

In asymptotically flat space-time we write

$$\frac{d}{ds}\left(p^i x_i\right) = mc\frac{d}{ds}\left(u^i x_i\right) = mc\frac{d^2}{ds^2}\left(\frac{x^i x_i}{2}\right). \tag{3.99}$$

Since

$$x^i x_i = c^2 t^2 - r^2 \text{and} \ \frac{d}{ds} = \frac{\gamma}{c}\frac{d}{dt},$$

where $\gamma = 1/\sqrt{1-(v^2/c^2)}$, and r is the radius of mass particle.

Then we continue transformation of the Eq. 3.99:

$$mc\frac{d^2}{ds^2}\left(\frac{x^i x_i}{2}\right) = mc\frac{\gamma^2}{c^2}\frac{d^2}{dt^2}\left(\frac{c^2t^2 - r^2}{2}\right) = mc\ \gamma^2 - \frac{\gamma^2}{c^2}\frac{d^2}{dt^2}\left(\frac{mr^2}{2}\right),$$

and finally

$$\frac{d}{ds}\left(p^i x_i\right) = mc\ \gamma^2 - \frac{\gamma^2}{c}\ddot{\Phi}, \tag{3.100}$$

where

$$\frac{d^2}{dt^2}\left(\frac{mr^2}{2}\right) = \ddot{\Phi}.$$

is the Jacobi function.

On the other hand, we have

$$\frac{d}{ds}\left(p^i x_i\right) = mc\frac{d}{ds}\left(u^i x_i\right) = mcu^i u_i + mc\frac{du^i}{ds}x_i. \tag{3.101}$$

Using the identity $u_i u^i \equiv 1$ and the geodetic equation

$$\frac{du^i}{ds} = -\Gamma^i_{kl}u^k u^l,$$

where

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{km}}{\partial x^l} + \frac{\partial g_{lm}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^m}\right).$$

are the Christoffel's symbols, and the equation (3.101) will be rewritten as

$$\frac{d}{ds}\left(p^i x_i\right) = mc - mcx_i\Gamma^i_{kl}u^k u^l. \tag{3.102}$$

The metric tensor g_{ik} for a weak stationary gravitational field is

$$g_{ik} = \eta_{ik} + \xi_{ik}, \tag{3.103}$$

where in our notation η_{ik} is the Lorentz tensor with signature (+,–,–,–).

For the Schwarzschild metric tensor ξ_{ik} we write

$$\xi_{00} = -\frac{r_g}{r}; \xi_{11} = -\frac{1}{1 - r_g/r} + 1 \approx -\frac{r_g}{r}; \xi_{ik} = 0 \text{ if } i \neq k \text{ and } i \neq 0, 1. \quad (3.104)$$

Here $r_g = 2GV/c^2$ is the Schwarzschild gravitational radius of the mass m'.

Now we can rewrite the second term in the right-hand side of Eq. 3.102, using (3.103) and (3.104)

$$\begin{aligned} mcx_i\Gamma^i_{kl}u^k u^l &= mcx^m u^k u^l\left(\frac{\partial \xi_{km}}{\partial x^m} - \frac{1}{2}\frac{\partial \xi_{kl}}{\partial x^m}\right) \\ &= mc\left(x^0u^0u^1\frac{\partial \xi_{00}}{\partial x^1} + x^1u^1u^1\frac{\partial \xi_{11}}{\partial x^1} - \frac{1}{2}x^1u^0u^0\frac{\partial \xi_{00}}{\partial x^1} - x^1u^1u^1\frac{\partial \xi_{11}}{\partial x^1}\right). \end{aligned} \quad (3.105)$$

But $u^1 \ll u^0 = \gamma$ and $x^1 = r$.

We therefore obtain for Eq. 3.105

$$\begin{aligned} mcx_i\Gamma^i_{kl}u^k u^l &= -\frac{mc}{2}x^1u^0u^0\frac{\partial \xi_{00}}{\partial x^1} \\ &= -\frac{mc}{2}r\,\gamma^2\frac{r_g}{r^2} = \frac{mc}{2}\,\gamma^2\frac{2Gm'}{c^2 r} = -\frac{\gamma^2}{c}\frac{Gm'm}{r} \end{aligned} \quad (3.106)$$

Finally, we see that

$$\frac{d}{ds}\left(p^i x_i\right) = mc - \frac{\gamma^2}{c}U, \quad (3.107)$$

where U is the potential energy of the mass in the gravitational field of the mass m'.

Identification of the expression $(d/ds)(p^i x_i)$ obtained from Eqs. 3.100 and 3.107 gives

$$mc\,\gamma^2 - \frac{\gamma^2}{c}\ddot{\Phi} = mc - \frac{\gamma^2}{c}U. \quad (3.108)$$

It is easy to see that

$$mc\left(\gamma^2 - 1\right) = mc\left(\frac{1}{1 - v^2/c^2} - 1\right) = mc\frac{v^2}{c^2}\frac{1}{1 - v^2/c^2} = \frac{\gamma^2}{c}mv^2 = \frac{\gamma^2}{c}2T.$$

We then obtain

$$\frac{\gamma^2}{c}\ddot{\Phi} = \frac{\gamma^2}{c}U + \frac{\gamma^2}{c}2T,$$

which gives

$$\ddot{\Phi} = \mathrm{U} + 2\mathrm{T},$$

or

$$\ddot{\Phi} = 2\mathrm{E} - \mathrm{U}, \tag{3.109}$$

where T is the kinetic energy of the particle m and $\mathrm{E} = \mathrm{U} + \mathrm{T}$ is its total energy.

Equations 3.109 are known as classical Jacobi's virial equations, and the expression (3.102) represents its relativistic analogue for asymptotically flat space-time.

3.8 Universality of Jacobi's Virial Equation for Description of Dynamics of Natural Systems

It follows from this derivation of Jacobi's virial equation that it appears to be a universal mathematical expression for consideration of the dynamics of celestial bodies described by equations of motion for a wide range of existing physical models. The derived equation represents not only formal mathematical transformation of the initial equations of motion. Physical quintessence of mathematical transformation of the equations of motion involves change of the vector forces and moment of momentums by the volumetric forces or pressure and the oscillation of the interacted mass particles (inner energy) expressed through the energy of oscillation of the polar moment of inertia of a body. Here the potential (kinetic) energy and the polar moment of inertia of a body have a functional relationship and within the period of oscillation are inversely changed by the same law. Moreover, as it was demonstrated in Sect. 2.2 of Chap. 2. and will be shown in Chap. 6, the virial oscillations of a body represent the main part of the body's kinetic energy, which is lost in the hydrostatic equilibrium model. The change of the vector forces and moment of momentums by the force pressure and the oscillation of the interacting mass particles disclose the physical meaning of the gravitation and mechanism of generation of the gravitational and electromagnetic energy and their common nature, which is considered in Chap. 6. The most important advantage given by Jacobi's virial equation, is its independence from the choice of the co-ordinate system, transformation of which, as a rule, creates many mathematical difficulties.

By averaging for a uniform system the generalized virial equation $\ddot{\Phi} = 2\mathrm{E} - \mathrm{U}$, when the first derivative over the polar moment of inertia Φ acquires constant value, it becomes the classical virial theorem $2\mathrm{E} = \mathrm{U}$, or $-\mathrm{U} = 2\ \mathrm{T}$, which expresses the condition of the hydrostatic equilibrium being in the outer force field and without kinetic energy of oscillations of the interacting particles.

The starting point for derivation of the virial theorem is the particle momentum. By Newton's definition this value "*is a certain measure determined proportionally to the velocity and the mass*". This value is defined or it is found experimentally. All the other force parameters are obtained by transformation of the initial momentum and those actions are explained by physical interaction of the mass particles, which are the carrier of the momentum. In fact, we recognize the momentum to be "*innate*", according to Newton's terminology, value, i.e. the hereditary value. Under the "*innate*" value Newton understood "both the resistance and the pressure of the mass" and finally the effect acquires its status of the inertial force. But the essence does not change, because the momentum appears together with the mass. Thus, the circle of the philosophical speculations is locked by the momentum, i.e. by the mass and its oscillation. All other attributes of the motion are formed by mathematical transformations.

One more mainly physical problem that was solved in derivation in this chapter of Jacobi's virial equation by mathematical transformations is an understanding of the nature and dynamical effects of the gravitational interaction of mass particles for a continuous body. Contrary to the interaction of two bodies presented by mass points, when a dynamical effect is developed in the orbital motion by vector force and angular momentum, the dynamical effect of the interacting mass particles of the continuous body is developed in the form of volumetric pressure and volumetric oscillation. The integral effect of the mass interaction is expressed by oscillation of the polar moment of inertia. In the next chapter we consider solution of Jacobi's virial equation.

Chapter 4
Solution of Jacobi's Virial Equation for Conservative Systems

In Chap. 3 we derived Jacobi's virial equation of dynamical equilibrium in the framework of various physical models which are used for describing the dynamics of natural systems. We showed that, instead of the traditional description of a system in co-ordinates and velocities, the problem of dynamics can be studied from the position of an external observer. In this case the system as a whole is described by a compact and elegant equation and is characterized by integral (volumetric) parameters. Such a description of the integral equation does not depend on the choice of the frame of reference. The external observer can estimate by observations only some moments of distribution of mass density, i.e. total mass and energy of a system, which are its integral characteristics. Moreover, in order to solve the problem of a body's motion in the framework of its dynamical equilibrium, we invoked the relationship between its force function and the polar moment of inertia, which is the source of motion. This relationship reveals the nature of the gravitational energy. We also succeeded in reanimating the lost kinetic energy and obtaining both an equation of dynamics and an equation of dynamical equilibrium in the form of the oscillating motion during each period of time and within the whole duration of the system's evolution.

The problem is now to find the general solution of Jacobi's virial equation relative to oscillation and rotation of a body and to apply the solution to study its dynamics. This application is valid for studying the Sun, the Earth, the Moon and other celestial bodies.

In this chapter we show that Jacobi's virial equation provides first of all a solution for the models of natural systems, which have explicit solutions in the framework of the classical many-body problem. We shall give parallel solutions for both the classical and dynamical approaches, and in doing so we shall show that, with the dynamical approach, the solution acquires a new physical meaning. We shall also consider a general case of the solution of Jacobi's virial equation for conservative and dissipative systems.

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4.1 Solution of Jacobi's Virial Equation in Classical Mechanics

The many-body problem is known to be the key problem in classical mechanics and especially in celestial mechanics. A particular example of this is the unperturbed problem of Keplerian motion, when the system consists of only two material points interacting by Newtonian law. The explicit solution of the problem of unperturbed Keplerian motion permits the many-body problem to be solved with some approximation by varying arbitrary constants. In this case the problem of dynamics, for example that of the Solar System, is transferred into the solution of the problem of dynamics of nine pairs of bodies in each of which one body is always the Sun and the second is each of the nine planets forming the system. Considering each planet-sun sub-system, the influence of the other eight planets of the system is taken into account by introducing the perturbation function. By the virial approach we can obtain for the Sun one characteristic period of circulation with respect to the center of mass of the system which will not coincide with any period of the planets. The dynamical approach evidences that the planet's orbital motion is performed by the central body, i.e. by the Sun, by the energy of its outer force field or by the field of the pressure. Each planet interacts with the solar force field by the energy of its own outer force field. The planet's orbit is the certain curve of its equilibrium motion which results from the two interacting fields of pressure. The planet's own oscillation and rotation perform by action of the inner fields of pressure.

Following these brief physical comments on the dynamical equilibrium motion of a planet, we now present two methods of solving the Keplerian problem: the classical and the integral.

4.1.1 The Classical Approach

The traditional way of solving the unperturbed Keplerian problem is excellently described in the university courses for celestial mechanics found in (Duboshin 1978). Here we present only the principle ideas. The method consists in transforming the two-body problem described by the system of equations (3.10) into the one-body problem using six integrals of motion of the center of mass (3.13). The system of equations obtained is sixth order and expresses the change of barycentric co-ordinates of one point with respect to the center of mass of the system as a whole. Let us write it in the form

$$\ddot{x} = -\frac{\mu x}{r^3},$$

$$\ddot{y} = -\frac{\mu y}{r^3}, \tag{4.1}$$

$$\ddot{z} = -\frac{\mu z}{r^3},$$

where μ is the constant depending on the number of the point and for which the second point is equal to

$$\mu = \frac{G m_1^3}{(m_1 + m_2)^2}.$$

We then pass on from that Cartesian system of co-ordinates OXYZ to orbital ξ η ζ, using first integrals of the system of equations (4.1). Those are three integrals of the area,

$$\begin{aligned} y\dot{z} - z\dot{y} &= c_1, \\ z\dot{x} - x\dot{z} &= c_2, \\ x\dot{y} - y\dot{x} &= c_3, \end{aligned} \tag{4.2}$$

the energy integral,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{2\mu}{r} + h, \tag{4.3}$$

and the Laplacian integrals,

$$\begin{aligned} -\frac{\mu x}{r} + c_3\dot{y} - c_2\dot{z} &= f_1, \\ -\frac{\mu y}{r} + c_1\dot{z} - c_3\dot{x} &= f_2, \\ -\frac{\mu z}{r} + c_2\dot{x} - c_1\dot{y} &= f_3. \end{aligned} \tag{4.4}$$

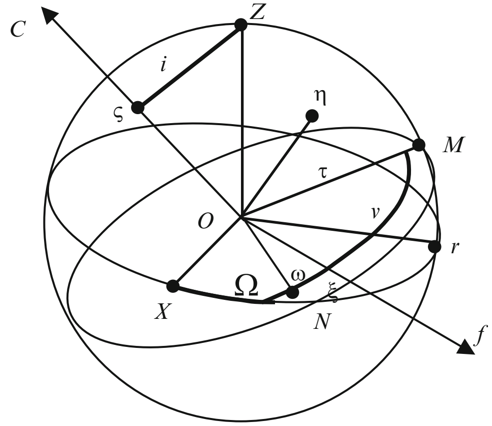
As these seven integrals are not independent, we conclude that they cannot form a general solution of the system (4.1). In fact there are two relations for these integrals:

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0,$$

$$f_1^2 + f_2^2 + f_3^2 = \mu^2 + h\left(c_1^2 + c_2^2 + c_3^2\right),$$

showing that only five of them are independent. But the last integral needed can be found by simple quadrature. Using these integrals we can pass on to the system of orbital co-ordinates Oξηζ using the transformation relations (see Fig. 4.1):

Fig. 4.1 Transition from Cartesian co-ordinate system OXYZ to orbital Oξηζ



$$\xi = \frac{f_1}{f}x + \frac{f_2}{f}y + \frac{f_3}{f}z,$$

$$\eta = \frac{C_2f_3 - C_3f_2}{Cf}x + \frac{C_3f_1 - C_1f_2}{Cf}y + \frac{C_1f_2 - C_3f_1}{Cf}z, \qquad (4.5)$$

$$\zeta = \frac{C_1}{C}x + \frac{C_2}{C}y + \frac{C_3}{C}z.$$

The equation of the curve along which the point moves in accordance with (4.1) has the simplest form in the system of initial co-ordinates. The equation is

$$\zeta = 0,$$

$$\mu r = C^2 - f\xi. \qquad (4.6)$$

Finally, introducing the polar orbital co-ordinates r and v, which are related to the rectangular orbital co-ordinates ξ and η by the expressions (see Fig. 4.2)

$$\xi = r\cos v$$

and

$$\eta = r\sin v,$$

and using the integral of areas

$$r^2v = C,$$

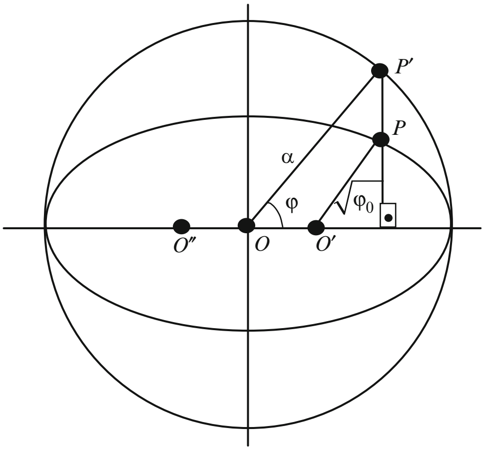


Fig. 4.2 Relationship between the polar and the rectangular co-ordinates

we come to the equation

$$C(t-r)=\left(\frac{C^2}{\mu}\right)^2\int_0^v\frac{dv}{\left(1+\frac{f}{\mu}\cos v\right)^2}. \tag{4.7}$$

The solution of Eq. 4.7 gives the change of function v with respect to time. Repetition of the transformation in the reverse order leads to solution of the problem. In doing this, we obtain the expression for the change of co-ordinates of the material point with respect to the initial data $\xi_{10},\ \eta_{10},\ \zeta_{10},\ \xi_{20},\ \eta_{20},\ \zeta_{20}, \dot{\xi}_{10}, \dot{\eta}_{10}, \dot{\zeta}_{10}, \dot{\xi}_{20}, \dot{\eta}_{20}, \dot{\zeta}_{20}$. It is remarkable that if the total energy (4.3) has negative value, then the solution of Eq. 4.7 leads to the Keplerian equation

$$E'-e\sin E'=n(t-\ \tau), \tag{4.8}$$

where the function v is related to the variable E′ by the expression

$$\mathrm{tg}\frac{v}{2}=\sqrt{\frac{1+e}{1-e}}\mathrm{tg}\frac{E}{2},$$

and

$$e=\frac{f}{\mu},\quad n=\frac{\sqrt{\mu}}{a^{3/2}},\quad p=\frac{C^2}{\mu}=a\left(1-e^2\right).$$

Because energy by definition is the property to do work (motion) and can be only a positive value, then the physical meaning of negative total energy which defines the elliptic orbit of a body moving in the central field of the two-body problem should be revealed. In the presented solution of the two-body problem, the left-hand side of the energy integral (4.3) expresses the kinetic energy and the right-hand side means the potential energy of the mass interaction. The integral of energy (4.3) as a whole, in the co-ordinates and in the velocities, represents the averaged virial theorem, where the potential energy has formally a negative value. Here the physical meaning of the total energy determination consists in comparison of magnitude of the potential and kinetic energy. A negative value of the total energy means that the potential energy exceeds the kinetic one by that value. As it follows from analysis of the inner force field of a self-gravitating body presented in Chap. 2, the potential energy exceeds the kinetic one only in the case of non-uniform distribution of the mass density and cannot be less than it. In the case of equality of both energies the total potential energy is realized into oscillating motion. The excited part of the potential energy is used for rotation of the masses and in the dissipation. The last case is discussed in Chap. 7.

4.1.2 The Dynamic Approach

Let us consider the solution of the problem of unperturbed motion of two material points on the basis of Jacobi's virial equation which in accordance with Eq. 3.16 is written in the form

$$\ddot{\Phi}_0 = 2E_0 - U,$$

where $E_0 = T_0 + U = \text{const}$ is the total energy of the system in a barycentric co-ordinate system;

The Jacobi function Φ_0 is expressed by (3.15):

$$\Phi_0 = \frac{m_1 m_2}{2(m_1 + m_2)}\left[(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2\right],$$

and the potential energy U in accordance with (3.2) is

$$U = \frac{G m_1 m_2}{\sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2}}.$$

It is easy to see that between the Jacobi function Φ_0 and the potential energy U exists the relationship

$$|U|\sqrt{\Phi_0} = \frac{G(m_1 m_2)^{3/2}}{\sqrt{2(m_1 + m_2)}} = G^{1/2} m \mu^{3/2} = B = \text{const}, \quad (4.9)$$

where μ is the generalized mass of the two bodies; m is the total mass of the system; B is a constant value.

The relationship (4.9) is remarkable because it is independent of the initial data, i.e. of its co-ordinates and velocities. Being an integral characteristic of the system and dependent only on the total mass and the generalized mass of the two points, the relationship permits Jacobi's virial equation to be transformed to an equation with one variable as follows:

$$\ddot{\Phi}_0 = 2E_0 + \frac{B}{\sqrt{\Phi_0}}. \tag{4.10}$$

We consider the solution of Eq. 4.10 for the case when total energy E_0 has negative value. Introducing $A = -2E_0 > 0$, Eq. 4.10 can be rewritten:

$$\ddot{\Phi}_0 = -A + \frac{B}{\sqrt{\Phi_0}}. \tag{4.11}$$

We apply the method of change of variable for solution of Eq. 4.11 and show that partial solution of two linear equations (Ferronsky et al. 1984a):

$$\left(\sqrt{\Phi_0}\right)'' + \sqrt{\Phi_0} = \frac{B}{A}, \tag{4.12}$$

$$t'' + t = \frac{4B\lambda}{(2A)^{3/2}}, \tag{4.13}$$

which include only two integration constants, is also the solution of Eq. 4.11.

We now introduce the independent variable λ into Eqs. 4.12 and 4.13, where primes denote differentiation with respect to λ. Note that time here is not an independent variable. This allows us to search for the solution of two linear equations instead of solving one non-linear equation. The solution of Eqs. 4.12 and 4.13 can be written in the form

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)], \tag{4.14}$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)]. \tag{4.15}$$

Let us prove that the partial solution (4.14) and (4.15) differential Eqs. 4.12 and 4.13 is the solution of Eq. 4.10 that is sought. For this purpose we express the first and second derivatives of the function $\sqrt{\Phi_0}$ with respect to λ through corresponding derivatives with respect to time using Eq. 4.15. From (4.15) it follows that

$$\frac{dt}{d\lambda} = \frac{4B}{(2A)^{3/2}}[1 - \varepsilon\cos(\lambda - \psi)]. \tag{4.16}$$

We can replace the right-hand side of the obtained relationship by $\sqrt{\Phi_0}$ from (4.14)

$$\frac{dt}{d\lambda} = \sqrt{\Phi_0}\sqrt{\frac{2}{A}}. \tag{4.17}$$

Transforming the derivative from $\sqrt{\Phi_0}$ with respect to λ into the form

$$\frac{d\sqrt{\Phi_0}}{d\lambda} = \frac{d\sqrt{\Phi_0}}{dt}\frac{dt}{d\lambda} = \frac{\dot{\Phi}_0}{2\sqrt{\Phi_0}}\frac{dt}{d\lambda}$$

and taking into account (4.17), we can write

$$\left(\sqrt{\Phi_0}\right)' = \frac{\dot{\Phi}_0}{\sqrt{2A}}.$$

The second derivative can be written analogously:

$$\left(\sqrt{\Phi_0}\right)'' = \frac{dt}{d\lambda}\frac{d}{dt}\left(\frac{\dot{\Phi}_0}{\sqrt{2A}}\right) = \frac{\ddot{\Phi}_0}{\sqrt{2A}}\sqrt{\Phi_0}\sqrt{\frac{2}{A}} = \frac{\ddot{\Phi}_0\sqrt{\Phi_0}}{A}. \tag{4.18}$$

Putting Eq. 4.18 into (4.12), we obtain

$$\frac{\ddot{\Phi}_0\sqrt{\Phi_0}}{A} + \sqrt{\Phi_0} = \frac{B}{A}.$$

Dividing the above expression by $\sqrt{\Phi_0}/A$, we can finally write

$$\ddot{\Phi}_0 = -A + \frac{B}{\sqrt{\Phi_0}}.$$

This shows that the partial solution of the two linear differential Eqs. 4.12 and 4.13 appears to be the solution of the non-linear Eq. 4.11.

4.2 Solution of the N-Body Problem in the Framework of Conservative System

After solving Jacobi's virial equation for the unperturbed two-body problem, we come to dynamics of a system of n material particles where $n \to \infty$.

Let us assume that an external observer studying the dynamics of a system of *n* particles in the framework of classical mechanics has the following information. He has the mass of the system, its total and potential energy and can determine the Jacobi function and its first derivative with respect to time in any arbitrary moment. Then he can use Jacobi's virial Eq. 4.9 and, making only the assumption needed for its solution that $|U|\sqrt{\Phi_0} = B = \text{const}$, may predict the dynamics of the system, i.e. the dynamics of its integral characteristics at any moment of time. The assumption $|U|\sqrt{\Phi_0} = \text{const}$will be considered separately in Chap. 6.

If the total energy E_0 of the system has negative value, the external observer can immediately write the solution of the problem of the Jacobi function change with respect to time in the form of (4.14) and (4.15):

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)],$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)],$$

where $A = -2E_0$; ε and ψ are constants depending on the initial values of the Jacobi function Φ_0 and its first derivative $\dot{\Phi}_0$ at the moment of time t_0 .

Let us obtain the values of constants ε and ψ, in explicit form expressed through the values Φ_0 and $\dot{\Phi}_0$ at the initial moment of time t_0. For convenience we introduce a new independent variable φ, connected to λ by the relationship соотношением $\lambda - \psi = \varphi$. Then Eqs. 4.14 and 4.15 can be rewritten:

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos\varphi], \tag{4.19}$$

$$t - \frac{4B}{(2A)^{3/2}}\psi = \frac{4B}{(2A)^{3/2}}[\varphi - \varepsilon\sin\varphi]. \tag{4.20}$$

Using Eq. 4.19 we write the expression for φ:

$$\varphi = \arccos\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} \tag{4.21}$$

and taking into account the equality

$$\frac{d\sqrt{\Phi_0}}{d\lambda} = \frac{d\sqrt{\Phi_0}}{d\varphi},$$

substitute Eq. 4.21 into the expression

$$\frac{\dot{\Phi}_0}{\sqrt{2A}} = \frac{B}{A}\varepsilon\sin\varphi.$$

The last equation can be rewritten finally in the form

$$\frac{\dot{\Phi}_0}{\sqrt{2A}} = \frac{B}{A}\ \varepsilon\sqrt{1 - \left(\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}. \tag{4.22}$$

Equation 4.22 allows us to determine the first constant of integration ε as a function of the initial data Φ_0 and $\dot{\Phi}_0$ at $t = t_0$. Solving Eq. 4.22 with respect to ε after simple algebraic transformation, we obtain

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)}|_{t=t_0} = \text{const.} \tag{4.23}$$

The second constant of integration ψ can be expressed through the initial data after solving Eq. 4.20 with respect to ψ and change of value φ by its expression from Eq. 4.21. Defining

$$t - \frac{4B}{(2A)^{3/2}}\ \psi = \ \tau,$$

we obtain

$$-\tau = \left\{\frac{4B}{(2A)^{3/2}}\left[\arccos\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} - \ \varepsilon\sqrt{1 - \left(\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}\right] - t\right\}\Big|_{t\,=\,t_0} = \text{const.} \tag{4.24}$$

The physical meaning of the integration constants ε, τ, and the parameter $T_v = 8\pi B/(2A)^{3/2}$ can be understood after the definitions

$$T_v = \frac{8\pi B}{(2A)^{3/2}},$$

$$n = \frac{2\pi}{T_v} = \frac{(2A)^{3/2}}{4B},$$

$$a = \frac{B}{A}$$

and rewriting Eqs. 4.19 and 4.20 in the form

$$\sqrt{\Phi_0} = a(1 - \ \varepsilon\cos\varphi), \tag{4.25}$$

$$M = \ \varphi - \ \varepsilon\sin\varphi, \tag{4.26}$$

where $M = n(t\ -\ \ \tau)$.

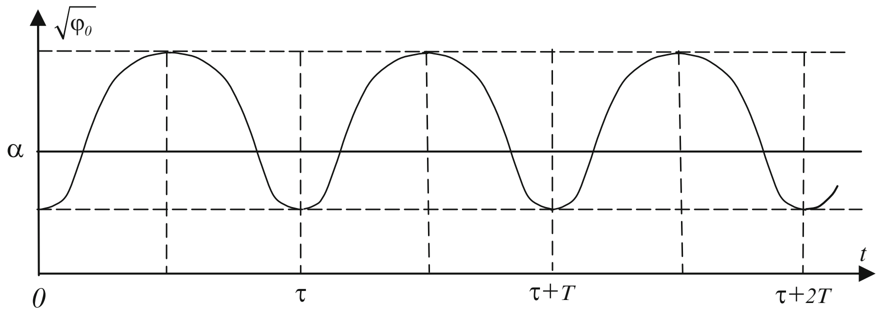


Fig. 4.3 Changes of the Jacobi function over time

The value $\sqrt{\Phi_0}$ draws an ellipse during the period of time $T_v = 8\pi B/(2A)^{3/2}$ (see Fig.4.3). The ellipse is characterized by a semi-major axis *a* equal to B/A and by the eccentricity ε which is defined by expression (4.23). In the case considered, where $E_0 < 0$, the value ε is changed in time from 0 to 1. The value τ characterizes the moment of time when the ellipse passes the pericentre.

Let us obtain explicit expressions with respect to time for the functions $\sqrt{\Phi_0}$, Φ_0 and $\dot{\Phi}_0$. For this purpose we write Eq. 4.24 in the form of a Lagrangian:

$$F(\varphi) = \varphi - \varepsilon \sin \varphi - M = 0. \tag{4.27}$$

It is known (Duboshin 1978) that by application of Lagrangian formulas we can write in the form of a series the expressions for the root of the Lagrange Eq. 4.27 and for the arbitrary function f which is dependent φ:

$$\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{d^{k-1}}{dM^{k-1}} \left[\sin^k M\right] = M + \varepsilon \sin M + \frac{\varepsilon^2}{1 \cdot 2} \frac{d}{dM} \left[\sin^2 M\right] + \cdots, \tag{4.28}$$

$$\begin{aligned} f(\varphi) = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{d^{k-1}}{dM^{k-1}} \left[f'(M)\sin^k M\right] = f(M) + \varepsilon f'(M) \sin M \\ + \frac{\varepsilon^2}{1 \cdot 2} \frac{d}{dM} \left[f(M)\sin^2 M\right] + \dots . \end{aligned} \tag{4.29}$$

Using Eq. 4.29, we write expressions for cosφ, $\cos^2\varphi$ and sinφ in the form of a Lagrangian series of parameter ε power:

$$\begin{aligned} \cos \varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{d^{k-1}}{dM^{k-1}} \left[(-1) \sin M \sin^k M\right] = \cos M + \varepsilon(-1) \sin M \sin(M) \\ + \frac{\varepsilon^2}{1 \cdot 2} \frac{d}{dM} \left[(-1) \sin M \sin^2 M\right] + \cdots = \cos M - \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cos 2M \\ - \frac{3}{4} \varepsilon^3 \cos M + \frac{3}{8} \varepsilon^2 \cos 3M + \cdots \end{aligned} \tag{4.30}$$

$$\cos^2\varphi = \sum_{k=0}^{\infty}\frac{\varepsilon^{k-1}}{k!}\frac{d^{k-1}}{dM^{k-1}}\left[(-2)\sin M\cos M\sin^k M\right] = \cos^2 M$$
$$+\ \varepsilon(-2)\sin M\cos M\sin M + \frac{\varepsilon^2}{1\cdot 2}\frac{d}{dM}\left[(-2)\sin M\cos M\sin^2 M\right] + \cdots$$
$$= \cos^2 M - 2\ \varepsilon\sin^2 M\cos M + \frac{\varepsilon^2}{2}(-2)\left(3\sin^2 M\cos^2 M - \sin^4 M\right) + \cdots \tag{4.31}$$

$$\sin\varphi = \sum_{k=0}^{\infty}\frac{\varepsilon^{k-1}}{k!}\frac{d^{k-1}}{dM^{k-1}}[\cos M\sin M] = \sin M + \ \varepsilon\cos M\sin M$$
$$+ \frac{\varepsilon^2}{1\cdot 2}\frac{d}{dM}\left[\cos M\sin^2 M\right] + \ldots = \sin M + \ \varepsilon\cos M\sin M \tag{4.32}$$
$$+ \frac{\varepsilon^2}{1\cdot 2}\left[2\sin M\cos^2 M - \sin^3 M\right] + \cdots$$

We write the expressions for $\sqrt{\Phi_0}$, Φ_0, $\dot{\Phi}_0$ using Eqs. 4.25 and 4.26 in the form

$$\sqrt{\Phi_0} = a(1 - \ \varepsilon\cos\varphi), \tag{4.33}$$

$$\Phi_0 = a^2\left(1 - 2\ \varepsilon\cos\varphi + \ \varepsilon^2\cos^2\varphi\right), \tag{4.34}$$

$$\dot{\Phi}_0 = \sqrt{\frac{2}{A}}\,\varepsilon B\sin\varphi. \tag{4.35}$$

Substituting into (4.33)–(4.35) the expressions for $\cos\varphi$, $\cos^2\varphi$ and $\sin\varphi$ in the form of the Lagrangian series (4.30)–(4.32) we obtain

$$\sqrt{\Phi_0} = \frac{B}{A}\left[1 + \frac{\varepsilon^2}{2} + \left(-\ \varepsilon + \ \frac{3}{8}\ \varepsilon^3\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{3}{8}\ \varepsilon^3\cos 3M + \cdots\right], \tag{4.36}$$

$$\Phi_0 = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\ \varepsilon^2 + \left(-2\ \varepsilon + \frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + \cdots\right], \tag{4.37}$$

$$\dot{\Phi}_0 = \sqrt{\frac{2}{A}}\,\varepsilon B\left[\sin M + \frac{1}{2}\ \varepsilon\sin 2M + \frac{\varepsilon^2}{2}\sin M\left(2\cos^2 M - \sin^2 M\right) + \ldots\right]. \tag{4.38}$$

The series of Eqs. 4.36–4.38 obtained are put in order of increased power of parameter ε and are absolutely convergent at any value of M in the case when the parameter ε satisfies the condition

$$\varepsilon < \bar{\varepsilon} = 0{,}6627\ldots, \tag{4.39}$$

where $\bar{\varepsilon}$ is the Laplace limit.

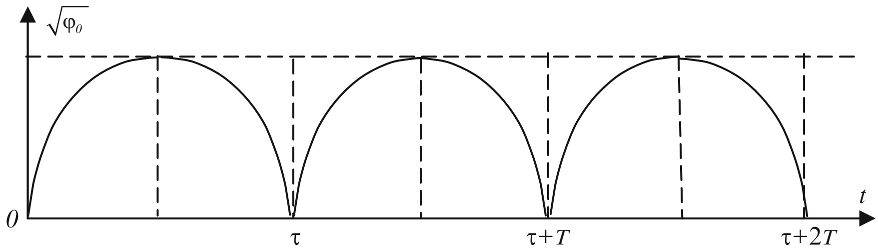


Fig. 4.4 Changes of the value $\sqrt{\Phi_0}$ in time at $\varepsilon = 1$

In some cases it is convenient to expand the values $\sqrt{\Phi_0}$, Φ_0, $\dot{\Phi}_0$ in the form of a Fourier series, using conventional methods (see, for example, Duboshin 1978). Figure 4.4 demonstrates the changes of $\sqrt{\Phi_0}$ in time at $\varepsilon = 1$.

It is also possible to consider the case solution of Jacobi's virial equation for $E_0 = 0$ and $E_0 > 0$. The reader can find here without difficulty a full analogy of these results as well as the solution of the two-body problem.

4.3 Solution of Jacobi's Virial Equation in Hydrodynamics

Let us consider the solution of the problem of the dynamics of a homogeneous isotropic gravitating sphere in the framework of traditional hydrodynamics and the virial approach we have developed.

4.3.1 The Hydrodynamic Approach

The sphere is assumed to have radius R_0 and be filled by an ideal gas with ρ_0. We assume that at the initial time the field of velocities which has the only component is described by equation

$$u = H_0 r, \tag{4.40}$$

where u is the radial component of the velocity of the sphere's matter at the distance r from the center of mass; H is independent of the quantity r and equal to H_0 at time t_0.

We also assume that the motion of the matter of the sphere goes on only under action of the forces of mutual gravitational interaction between the sphere particles. In this case the influence of the pressure gradient is not taken into account, assuming that the matter of the sphere is sufficiently diffused. Then the symmetric spherical shells will move only under forces of gravitational attraction and will not coincide. In this case the mass of the matter of any sphere shell will keep its constant value and the condition (4.40) will be satisfied at any moment of time, and constant H should be dependent on time.

Under those conditions the Eulerian system of equations (3.28) can be written in the form

$$\rho\frac{\partial u}{\partial t}+\ \rho(u\nabla)u=\ \rho\frac{\partial U_G}{\partial r}, \tag{4.41}$$

where $\rho(t)$ is the density of the matter of the sphere at the moment of time t; u is the radial component of the velocity of matter at distance r from the sphere's center; U_G is the Newtonian potential for the considered point of the sphere.

The expression for the Newtonian potential U_G (3.29) can be written as follows:

$$U_G = G\frac{4}{3}\pi\rho r^2, \tag{4.42}$$

and the continuity equation will be

$$\frac{\partial \rho}{\partial t}+\ \rho\frac{\partial U}{\partial r}=0. \tag{4.43}$$

Within the framework of the traditional approach, the problem is to define the sphere radius R and the value of the constant H at any moment of time, if the radius R_0, density ρ_0 and the value of the constant H_0 at the initial moment of time t_0 are given. If we know the values H(t) and R(t), we can then obtain the field of velocities of the matter within the sphere which is defined by Eq. 4.40, and also the density ρ of matter at any moment of time, using the relationship

$$\frac{4}{3}\ \pi R_0^3\ \rho_0=\frac{4}{3}\ \pi R^3\rho=\text{const}=m.$$

Hence the formulated problem is reduced to identification of the law of motion of a particle which is on the surface of the sphere and within the field of attraction of the entire sphere mass $m = 4/3\ \pi\ \rho_0 {R_0}^3$.

The equation of motion for a particle on the surface of the sphere, which follows from Eq. 4.41 after transforming the Eulerian co-ordinates into a Lagrangian, has the form

$$\frac{d^2R}{dt^2}=-G\frac{m}{R^2}. \tag{4.44}$$

It is necessary to determine the law of change of R(t), resolving Eq. 4.44 at the initial data:

$$R(t_0)=R_0, \tag{4.45}$$

$$\left.\frac{dR}{dt}\right|_{t=t_o}=H_0R_0.$$

We reduce the order of Eq. 4.44. To do so we multiply it by dR/dt:

$$\frac{dR}{dt}\frac{d^2R}{dt^2} = -\frac{dR}{dt}\frac{Gm}{R^2}$$

and integrate with respect to time:

$$\int_{t_0}^{t} \frac{1}{2}\frac{d}{dt}\left(\dot{R}\right)^2 = \int_{t_0}^{t} \frac{d}{dt}\left(\frac{Gm}{R}\right)dt.$$

After integration we obtain

$$\frac{1}{2}\dot{R}^2 - \frac{1}{2}\dot{R}_0^2 = \frac{Gm}{R} - \frac{Gm}{R_0}$$

or

$$\frac{1}{2}\dot{R}^2 = \frac{Gm}{R} + k, \tag{4.46}$$

where the constant k is determined as

$$\begin{aligned} k &= \frac{1}{2}\dot{R}_0^2 - \frac{Gm}{R_0} = \frac{1}{2}H_0^2R_0^2 - G\frac{4\pi}{3}\rho_0\frac{R_0^3}{R_0} \\ &= \frac{1}{2}H_0^2R_0^2\left[1 - \frac{8\pi}{3}\frac{G\rho_0}{H_0^2}\right] = \frac{1}{2}H_0^2R_0^2[1-\Omega] = \text{const} \end{aligned} \tag{4.47}$$

Here the quantity $\Omega = \rho_o/\rho_{cr}$, where $\rho_{cr} = 3{H_0}^2/8\pi G$.

Note that Eq. 4.46 obtained after reduction of the order of the initial Eq. 4.44 is in its substance the energy conservation law. Equation 4.46 permits the variables to be divided and can be rewritten in the form

$$\int_{R_0}^{R} \frac{dR}{\sqrt{\frac{2Gm}{R} + 2k}} = \int_{t_0}^{t} dt. \tag{4.48}$$

The plus sign before the root is chosen assuming that the sphere at the initial time is expanding, i.e., $H_0 > 0$.

The differential Eq. 4.46 has three different solutions at $k = 0$, $k > 0$ and $k < 0$ depending on the sign of the constant k, which is in its turn defined by the value of the parameter Ω at the initial moment of time. First we consider the case when $k = 0$ which relates, by analogy with the Keplerian problem, to the parabolic model at k = 0. Equation 4.46 is easily integrated and for the expression case, i.e.,$\dot{R}>0$, we obtain

$$\dot{R}^2 = \frac{2Gm}{R},$$
$$\dot{R} = \frac{(2Gm)^{1/2}}{R^{1/2}},$$

from which it follows that

$$R^{1/2}dR = (2Gm)^{1/2}dt$$

or

$$\frac{2}{3}R^{3/2} = (2Gm)^{1/2}t + \text{const.} \tag{4.49}$$

We choose as initial counting time t = 0, the moment when R = 0. In this case the integration constant disappears:

$$R = \left(\frac{9}{2}Gm\right)^{1/3} t^{2/3}. \tag{4.50}$$

The density of the matter changes in accordance with the law

$$\rho(t) = \frac{m}{\frac{4}{3}\pi R^3} = \frac{1}{6\pi G t^2}, \tag{4.51}$$

and the quantity H(t), as a consequence of (4.50), has the form

$$H(t) = \frac{\dot{R}}{R} = \frac{2}{3}\frac{1}{t}. \tag{4.52}$$

For the case when k > 0, which corresponds to so-called hyperbolic motion, the solution of Eq. 4.46 can be written in parametric form (Zeldovich and Novikov 1967)

$$R = \frac{Gm}{2k}(\text{ch}\eta - 1), \tag{4.53}$$

$$t = \frac{Gm}{(2k)^{3/2}}(\{\text{ch}\eta - \eta),$$

where the constants of integration in (4.53) have been chosen so that t = 0, η = 0 at R = 0.

Finally we consider the case when $k < 0$, which corresponds to elliptic motion. At $k < 0$ the expansion of the sphere cannot continue for unlimited time and the expansion phase should be changed by attraction of the sphere.

The explicit solution of Eq. 4.46 at $k < 0$ can be written in parametric form (Zeldovich and Novikov 1967)

$$R = \frac{Gm}{2|k|}(1 - \mathrm{ch}\eta), \tag{4.54}$$

$$t = \frac{Gm}{(2|k|)^{3/2}}(\eta - \mathrm{sh}\eta).$$

The maximum radius of the sphere is determined from Eq 4.46 on the condition $dR/dt = 0$ and equals

$$R_{max} = \frac{Gm}{|E|}. \tag{4.55}$$

The time needed for expansion of the sphere from $R_0 = 0$ at $t_0 = 0$ to R_{max} is

$$t_{max} = \frac{\pi Gm}{(2|k|)^{3/2}}. \tag{4.56}$$

So the sphere should make periodic pulsations with period T_p equal to

$$T_p = \frac{2\pi Gm}{(2|k|)^{3/2}}. \tag{4.57}$$

The considered solution has important cosmologic applications.

4.3.2 The Virial Approach

We shall limit ourselves by formal consideration of the same problem in the framework of the condition of the dynamical equilibrium of a self-gravitating body based on the solution of Jacobi's virial equation, which we discussed earlier.

As shown in Chap. 3, Jacobi's virial equation (3.50), derived from Eulerian equations (3.28), is valid for the considered gravitating sphere. It was written in the form

$$\ddot{\Phi} = 2E - U, \tag{4.58}$$

where Φ is the Jacobi function for a homogeneous isotropic sphere and is defined by

$$\Phi = \frac{1}{2}\int_0^R 4\,\pi r^2 \rho r^2 dr = \frac{2\pi\rho R^5}{5} = \frac{3}{10} mR^2. \tag{4.59}$$

The potential gravitational energy of the matter of the sphere is expressed as

$$U = -4\pi G\int_0^R r\rho(r)m(r)dr = -\frac{16\pi^2}{15}G\rho^2R^2 = -\frac{3}{5}G\frac{m^2}{R}. \tag{4.60}$$

The total energy of the sphere E will be equal to the sum of the potential U and kinetic T energies.

The kinetic energy T is expressed as

$$T = \frac{1}{2}\int_0^R 4\ \pi u^2\rho r^2 dr = \frac{1}{2}\int_0^R 4\ \pi H^2 r^2 \rho r^2 dr = \frac{4\pi\rho H^2R^5}{10} = \frac{3}{10}mH^2R^2. \tag{4.61}$$

For a homogeneous isotropic gravitating sphere, the constancy of the relationship between the Jacobi function (4.59) and the potential energy (4.60) can be written:

$$|U|\sqrt{\Phi} = B = \frac{3}{5}G\frac{m^2}{R}\sqrt{\frac{3}{10}mR^2} = \frac{1}{\sqrt{2}}\left(\frac{3}{5}\right)^{3/2}Gm^{3/2}, \tag{4.62}$$

where B has constant value because of the conservation law of mass m of the considered sphere.

The total energy E of the sphere also has a constant value:

$$E = T + U = \frac{A}{2}. \tag{4.63}$$

Then, if the total energy of the sphere has a negative value, Jacobi's virial equation can be written in the form:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}. \tag{4.64}$$

Let us consider the conditions under which the total energy of the system will have a negative value. For this purpose we write it explicitly:

$$E = T + U = -\frac{16}{15}\ \pi^2G\rho^2R^5 + \frac{2\pi\rho H^2R^5}{5} = \frac{2}{5}\pi\rho H^2R^5\left[1 - \frac{8\pi G\rho}{3H^2}\right]. \tag{4.65}$$

It is clear from Eq. 4.65 that the total energy E has a negative value, when $\rho > \rho_c$, where $\rho_c = 3H^2/8\pi G$.

The general solution of Eq. 4.64 has the form of Eqs. 4.14 and 4.15:

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \ \varepsilon\cos(\ \lambda - \ \psi),] \tag{4.66}$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon \sin(\lambda - \psi)], \tag{4.67}$$

where ε and ψ are constants dependent on the initial values of the Jacobi function Φ_0 and its first derivative $\dot{\Phi}_0$ at the moment of time t_o. The constants ε and ψ are determined by Eqs. 4.23 and 4.24 accordingly.

If we express all the constants in Eq. 4.23:

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)}\Big|_{t = t_0} = \text{const.} \tag{4.68}$$

through mass m of the system, it is not difficult to see that

$$-\dot{\Phi}_0^2 + 4B\sqrt{\Phi_0} - 2A\Phi_0 = 0.$$

Then the constant ε will be equal to zero. Hence the solutions (4.28) and (4.29) coincide with the solution (4.54), which was obtained in the framework of the traditional hydrodynamic approach. In this case the period of eigenpulsations of the Jacobi function (the polar moment of inertia) of the sphere $T = 8\pi R/(2A)^{3/2}$ will be equal to the period of change of its radius $T_p = 2\pi Gm/(2|k|)^{3/2}$ obtained from Eq. 4.54.

4.4 The Hydrogen Atom as a Quantum Mechanical Analogue of the Two-Body Problem

Let us consider the problem concerning the energy spectrum of the hydrogen atom, which is a unique example of the complete conformity of the analytical solution with experimental results. The problem consists of a study of all the forms of motion using the postulates of quantum mechanics and based on the solution of Jacobi's virial equation.

The classical Hamiltonian in the two-body problem is written as

$$H = \frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} + U(|\bar{r}_1 - \bar{r}_2|), \tag{4.69}$$

where

$$\bar{p}_1 = \frac{\partial H}{\partial \dot{\vec{r}}_1} = m_1 \dot{\bar{r}}_1,$$

$$\bar{p}_2 = \frac{\partial H}{\partial \dot{\vec{r}}_2} = m_2 \dot{\bar{r}}_2,$$

which after separation of the center of mass can be transformed into the form

$$H = \frac{\bar{P}^2}{2M} + \frac{\bar{p}^2}{2m} + U(r), \tag{4.70}$$

where $r = |\bar{r}_1 - \bar{r}_2|$ is the distance between two particles and

$$P = M\dot{\bar{R}}; \quad p = m\dot{\bar{r}}; \quad M = m_1 + m_2;$$

$$R = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1 + m_2}; \quad m = \frac{m_1 m_2}{m_1 + m_2}.$$

We obtain the Hamiltonian operator for the quantum mechanical two-body problem through changing the pulses and radii by the corresponding operators with the communication relations

$$[\hat{p}_i, \hat{p}_0] = -i\hbar\ \delta_{ik},$$

$$[\hat{p}_i, \hat{r}_k] = -i\hbar\ \delta_{ik}.$$

Then

$$\hat{H} = -\frac{\hbar^2}{2M}\Delta_R - \frac{\hbar^2}{2m}\Delta_r + \hat{U}(r).$$

The wave functionu$(\bar{r}_1,\bar{r}_2) = \varphi(\overline{R})\psi(r)$, which satisfies the Schrödinger equation

$$\hat{H}u = \varepsilon u,$$

describes the motion of the inertia center (the free motion of the particle of mass m_c is described by the function $\varphi(R)$ and the motion of the particle of mass m in the U(r) is described by the wave function $\Psi(r)$). Subsequently we consider only the wave function of the motion of particle m.

The Schrödinger equation

$$\Delta\Psi + \frac{2m}{\hbar^2}[E - U(r)]\Psi = 0$$

written here for the stationary state in a central symmetrical field in spherical coordinates, has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Psi}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2\Psi}{\partial\varphi^2}\right] + \frac{2m}{\hbar^2}[E - U(r)]\ \psi = 0. \tag{4.71}$$

Using the Laplacian operator $\hat{\ell}^2$:

$$\hat{\ell}^2 = \left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2}{\partial\varphi^2}\right],$$

we obtain

$$\frac{\hbar^2}{2m}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\hat{\ell}^2}{r^2}\Psi\right] + U(r)\Psi = E\Psi.$$

The operators $\hat{\ell}^2$ and $\hat{\ell}_z$ ($\hat{\ell}_z = -i\partial/\partial\varphi$) commutate with the Hamiltonian $\hat{H}(r)$ and therefore there are common eigenfunctions of the operators $\hat{H}$, $\hat{\ell}^2$ и $\hat{\ell}_z$. We consider only such solutions of Schrödinger equations. This condition determines the dependence of the function Ψ on the angles

$$\Psi(r, \Theta, \varphi) = R(r)Y_{\ell k}(\Theta, \varphi),$$

where the quantity $Y_{\ell k}(\Theta, \varphi)$ is determined by the expression

$$Y_{\ell k}(\Theta, \varphi) = \frac{1}{\sqrt{2\ \pi}}e^{ik\varphi}(-1)^k i^\ell\sqrt{\frac{(2\ell+1)(\ell-k)!}{2(\ell+k)!}}P_\ell^k(\cos\Theta),$$

and $P_\ell^k(\cos\Theta)$ is the associated Legendre polynomial, which is

$$P_\ell^k(\cos\Theta) = \frac{1}{2^\ell\ell!}\sin^k\Theta\frac{d^{r+\ell}}{d\cos\Theta^{r+\ell}}\left(\cos^2\Theta - 1\right)^\ell.$$

Since

$$\hat{\ell}^2 Y_{\ell k} = \ell(\ell+1)Y_{\ell k},$$

we obtain for the radial part of the wave function R(r)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R + \frac{2m}{\hbar^2}[E - U(r)]R = 0. \tag{4.72}$$

Equation 4.72 does not contain the value $\ell_z = m$, i.e. at the given ℓ the energy level E corresponds to $2\ell + 1$ states differing by the value ℓ_z.
The operator

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{d\Psi}{dr}\right).$$

is equivalent to the expression

$$\frac{1}{r}\frac{d^2}{dr^2}(rR)$$

and thus it is convenient to make the change of variables, assuming that

$$X(r) = rR(r).$$

So that Eq. 4.71 can be rewritten in the form

$$\frac{d^2X}{dr^2} - \frac{\ell(\ell+1)}{r^2}X + \frac{2m}{\hbar^2}[E - U(r)]X = 0. \tag{4.73}$$

We now consider the demand following from the boundary conditions and related to the behavior of the wave function X(r). At $r \to 0$ and the potentials satisfying the condition

$$\lim U(r)r^2 = 0, \tag{4.74}$$

$$r \to 0;$$

only the first two terms play an important role in Eq. 4.73. $X(r) \sim r^v$ and we obtain

$$v(v-1) = \ell(\ell+1).$$

This equation has roots $v_1 = \ell + 1$ and $v_2 = -\ell$.

The requirement of normalization of the wave function is incompatible with the values $v = -\ell$ at $\ell \neq 0$ because the normalization integral

$$\int_0^\infty |X_r^2(r)dr|$$

will be divergent for the discrete spectrum, and the condition

$$\int \Psi(\lambda, \xi)\Psi(\lambda, X)d\lambda = \delta(X - \xi)$$

does not hold for the continuous spectrum.

At $\ell = 0$ the boundary conditions are determined by the demand for the finiteness of the mean value of the kinetic energy which is satisfied only at $v = 1$. So, when the condition (4.74) is satisfied, then the wave function of a particle is everywhere finite and at any ℓ

$$X(0) = 0.$$

Let us consider the energy spectrum and the wave function of the bounded states of a system of two charges. The bounded states exist only in the case of the attracted particles. Such a system defines the properties of the hydrogen atom and hydrogen-like ions.

The equation for the radial wave function is

$$\frac{d^2R}{dt^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R + \frac{2m}{\hbar^2}\left(E + \frac{\alpha}{r}\right)R = 0, \tag{4.75}$$

where $\alpha = Ze^2$ is constant, characterizing the potential; e is the electron charge; Z is the whole number equal to the nucleus charge in the charge units.

The constants e^2, m and $\hbar$ allow us to construct the value with the dimension of length

$$a_0 = \frac{\hbar^2}{me^2} = 0.529 \cdot 10^{-8}\text{cm}$$

known as the Bohr radius, and the time

$$t_0 = \frac{\hbar^3}{me^4} = 0.242 \cdot 10^{-11}\text{s}.$$

These quantities define the typical space and time scale for describing a system, and it is therefore convenient to use these units as the basic system of atomic units. Equation 4.75 in atomic units (at $Z = 1$) takes the form

$$\frac{d^2R}{dt^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R + 2\left(E + \frac{1}{r}\right)R = 0. \tag{4.76}$$

At $E < 0$ the motion is finite and the energy spectrum is discrete. We need the solutions (4.76) quadratically integrable with r^2. Let us introduce the specification

$$n = \frac{1}{\sqrt{-2E}} \qquad \rho = \frac{2r}{n}.$$

Equation 4.76 can be written as

$$\frac{d^2R}{dt^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[\frac{n}{\rho} - \frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2}\right]R = 0. \tag{4.77}$$

We find the asymptotic forms of the radial function R(r). At $\rho \to \infty$ and omitting the terms $\sim \rho^{-1}$ and $\sim \rho^{-2}$ in (4.77), we obtain

$$\frac{d^2R}{d\rho^2} = \frac{R}{4}.$$

Therefore at high values of ρ, $R \propto e^{\pm\rho/2}$. The normalization demand is satisfied only by $R(\rho) \propto e^{-\rho/2}$. The asymptotic forms at $r \to 0$ have already been determined.

Substituting

$$R(\rho) = \rho^{\ell} e^{-\rho/2}\, \omega(\rho),$$

Equation 4.77 is reduced to the form

$$\rho \frac{d^2\omega}{d\rho^2} + (2\ell + 2 - \rho)\frac{d\omega}{d\rho} + (n - \ell - 1)\,\omega = 0. \tag{4.78}$$

To solve this equation in the limit of $\rho = 0$, we substitute $\omega(\rho)$ in the form of a power series

$$\omega(\rho) = 1 + \frac{(0 - v)}{(0 + \lambda)}\rho + \frac{(0 - v)(1 - v)}{(0 + \lambda)(1 + \lambda)}\frac{\rho^2}{2!} + \frac{(0 - v)(1 - v)(2 - v)}{(0 + \lambda)(1 + \lambda)(2 + \lambda)}\frac{\rho^3}{3!} + \ldots, \tag{4.79}$$

where $\lambda = 2\ell + 2$ and $-v = -n + \ell + 1$.

At $\rho \to \infty$, the function $\omega(\rho)$ should increase, but not faster than the limiting power ρ. Then $\omega(\rho)$ has to be a polynomial of v power. So, $-n + \ell + 1 = -k$, and $n = \ell + 1 + k\ (k = 0, 1, 2, \ldots.)$ at a given value of ℓ. Hence, using the definition for n, we can find the expression for the energy spectrum

$$E_n = -\frac{1}{2n^2}. \tag{4.80}$$

The number n is called the principal quantum number. In general units it has the form

$$E = -Z^2 \frac{me^4}{2\hbar^2 n^2}. \tag{4.81}$$

This formula was obtained by Bohr in 1913 on the basis of the old quantum theory, by Pauli in 1926 from matrix mechanics, and by Schrödinger in 1926 by solving the differential equations.

Let us solve the problem of the spectrum of the hydrogen atom using the equation of dynamical equilibrium of the system. In Chap. 3 we obtained Jacobi's virial equation for a quantum mechanical system of particles whose interaction is defined by the potential being a homogeneous function of the co-ordinates. This equation in the operator form is

$$\ddot{\hat{\Phi}} = 2\hat{H} - \hat{U}v \tag{4.82}$$

where $\hat{\Phi}$ is the operator of the Jacobi function, which, for the hydrogen atom, is written

$$\hat{\Phi} = \frac{1}{2} m\hat{r}^2, \tag{4.83}$$

The Hamiltonian operator is

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta_r + \hat{U} \tag{4.84}$$

and the operator of the function of the potential energy for the hydrogen atom is

$$\hat{U} = -\frac{e^2}{r}. \tag{4.85}$$

We solve the problem with respect to the eigenvalues of Eq. 4.82, using the main idea of quantum mechanics. For this we use the Schrödinger equation

$$\hat{H}\Psi = E\Psi$$

and rewrite Eq. 4.82 in the form

$$\ddot{\hat{\Phi}} = 2E - \hat{U}v \tag{4.86}$$

This equation includes two (unknown in the general case) operator functions $\hat{\Phi}$ and $\hat{U}$. In the case of the interaction, the potential is determined by the relation (4.85), and we can use a combination of the operators $\hat{\Phi}$ and $\hat{U}$ in the form

$$|\hat{U}|\sqrt{\hat{\Phi}} = \frac{e^2 m^{1/2}}{\sqrt{2}} = B. \tag{4.87}$$

We now transform (4.86) into the form which was considered in classical mechanics:

$$\ddot{\hat{\Phi}} = 2E + \frac{B}{\sqrt{\hat{\Phi}}}. \tag{4.88}$$

Equation 4.88 is a consequence of Eq. 4.86 when the Schrödinger equation and the relationship (4.87) are satisfied. Its solution for the bounded state, i.e. when total energy E is determined in parametric form, can be written

$$\sqrt{\Phi} = \frac{B}{2|E|}(1 - \varepsilon\cos\varphi), \tag{4.89}$$

$$\varphi - \varepsilon \sin \varphi = \mathrm{M}, \tag{4.90}$$

where the parameter M is defined by the relation

$$\mathrm{M} = \frac{(4|\mathrm{E}|)^{3/2}}{4\mathrm{B}}(\mathrm{t} - \tau), \tag{4.91}$$

where ε and τ are integration constants and where

$$\varepsilon = \sqrt{1 - \frac{\mathrm{AC}}{2\mathrm{B}^2}},$$

$$C = -\dot{\hat{\Phi}}_0^2 + 4\mathrm{B}\sqrt{\hat{\Phi}_0} - 2\mathrm{A}\hat{\Phi}_0.$$

Moreover, the solution can be written in the form of Fourier and Lagrange series. Thus, the expression (4.37) describes the expansion of the operator $\hat{\Phi}$ into a Lagrange series including the accuracy of ε^3, and has the form

$$\hat{\Phi}_0 = \frac{\mathrm{B}^2}{\mathrm{A}^2}\left[1 + \frac{3}{2}\ \varepsilon^2 + \left(-2\ \varepsilon + \frac{\varepsilon^3}{4}\right)\cos \mathrm{M} - \frac{\varepsilon^2}{2}\cos 2\mathrm{M} - \frac{\varepsilon^3}{4}\cos 3\mathrm{M} + \ldots.\right] \tag{4.92}$$

Using the general expression for the mean values of the observed quantities in quantum mechanics

$$<\Psi|\hat{\Phi}|\Psi> = \bar{\Phi}\nu$$

and taking into account that the mean value of the Jacobi function of the hydrogen atom should be different from zero, we find that our system has multiple eigenfrequencies $\nu_n = \mathrm{n}\nu_o$ with respect to the basic ν_o which corresponds to the period

$$\mathrm{T}_\mathrm{v} = \frac{8\pi\mathrm{B}}{(4|\mathrm{E}|)^{3/2}}. \tag{4.93}$$

In accordance with the expression

$$\mathrm{E}_n = \hbar\ \omega_\mathrm{n} = \frac{\hbar 2\pi\mathrm{n}}{\mathrm{T}_0} \tag{4.94}$$

each of these frequencies corresponds to the energy level $\mathrm{E_n}$ of the hydrogen atom. We substitute the expression (4.93) for $\mathrm{T_v}$ into Eq. 4.94 and resolve it in relation to $\mathrm{E_n}$:

$$|\mathrm{E_n}| = \frac{\hbar 2\pi\mathrm{n}(4|\mathrm{E_n}|)^{3/2}}{8\ \pi\mathrm{B}} = \frac{\hbar\mathrm{n}(4|\mathrm{E_n}|)^{3/2}}{\frac{4\mathrm{e}^2\mathrm{m}^{1/2}}{\sqrt{2}}} = \frac{\hbar\mathrm{n}2\sqrt{2|\mathrm{E_n}|}}{\mathrm{e}^2\mathrm{m}^{1/2}}. \tag{4.95}$$

The expression obtained by Bohr follows from (4.95):

$$E_n = \frac{e^4 m}{2\hbar^2 n^2}. \tag{4.96}$$

This equation solves the problem.

4.5 Solution of a Virial Equation in the Theory of Relativity (Static Approach)

We consider now the solution of Jacobi's virial equation in the framework of the theory of relativity, showing its equivalence to Schwarzschild's solution.

Let us write down the known expression for the radius of curvature of space-time as a function of mass density:

$$\frac{1}{R^2} = \frac{8\pi}{3}\frac{G\rho}{c^2}, \tag{4.97}$$

where R is the curvature radius; ρ is the mass density; G is the gravitational constant and c is the velocity of light.

Equation 4.97 can also be rewritten in the form

$$\rho R^2 = \frac{3}{8\pi}\frac{c^2}{G}. \tag{4.98}$$

If the product ρR^2 in Eq. 4.98 is the Jacobi function ($\Phi = \rho R^2$ is the density of the Jacobi function) then, from (4.98):

$$\Phi = \frac{3}{8\pi}\frac{c^2}{G}. \tag{4.99}$$

and it follows that the Jacobi function is a fundamental constant for the Universe. (In general relativity, the spatial distance does not remain invariant. Therefore, instead of this the Gaussian curvature is used, which has the dimension of the universe distance and is the invariant or, more precisely, the covariant.)

The constancy of the Jacobi function in this case reflects the smoothness of the description of motion in general relativity. The oscillations relative to this smooth motion described by Jacobi's equation are the gravitational waves and horizons, in particular the collapse and all types of singularity up to the process of condensation of matter in galaxies, stars etc.

Now we can show that Schwarzschild's solution in general relativity is equivalent to the solution of Jacobi's equation when $\ddot{\Phi} = 0$. Let us write the expression for the energy-momentum tensor

$$T_i^k = (\rho + p)u_i u^k + p\delta_i^k. \tag{4.100}$$

In the corresponding co-ordinate system, we obtain

$$u^i = \left(0, 0, 0, \frac{1}{\sqrt{-g_{00}}}\right), \tag{4.101}$$

where $\rho = \rho(r)$ and $p = p(r)$.

The independent field equations are written

$$G_1^1 = T_1^1, \quad G_0^0 = T_0^0, \tag{4.102}$$

$$R^{-2} = \frac{1}{3} G\rho c^2.$$

The expression for the metric is written in the form

$$ds^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2(d\Omega)^2 - \left\{A - B\sqrt{1 - \frac{r^2}{R^2}}\right\}^2 c^2 r^2, \tag{4.103}$$

where

$$\frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2(d\Omega)^2.$$

is the spatial element.

In this case the expression for the volume occupied by the system is written

$$V = \int_0^r \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin\Theta}{\sqrt{1 - \frac{r^2}{R^2}}} dr d\Theta d\Psi = \frac{4\pi R^3}{3}\left[\arcsin\frac{r}{R} - \frac{r}{R}\sqrt{1 - \frac{r^2}{R^2}}\right] \tag{4.104}$$

It can be easily verified that the right-hand side of Eq. 4.104 coincides with solution (4.14) and (4.15) of the equation of virial oscillations (4.11) at $\ddot{\Phi} = 0$, i.e.,

$$\arcsin x - x\sqrt{1 - x^2} = \arccos\left(\frac{\frac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{AC}{2B^2}}}\right) - \sqrt{1 - \frac{AC}{2B^2}} \times \sqrt{1 - \left(\frac{\frac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{AC}{2B^2}}}\right)^2}. \tag{4.105}$$

In fact, Eq. 4.105 is satisfied for

$$x = \frac{\frac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{AC}{2B^2}}} \text{ and } x = \sqrt{1 - \frac{AC}{2B^2}},$$

i.e.

$$\frac{A}{B}\sqrt{\Phi} - 1 = 1 - \frac{AC}{2B^2}, \text{ or } \frac{AC}{2B^2} + \frac{A\sqrt{\Phi}}{B} = 2.$$

At $\ddot{\Phi} = 0$, the parameter of virial oscillations,

$$e = \sqrt{1 - \frac{AC}{2B^2}} \text{ and } \sqrt{\Phi} = \frac{B}{A},$$

so the last condition is satisfied.

Schwarzschild's solution is rigorous and unique for Einstein's equation for a static model of a system with spherical symmetry.

Since this solution coincides with the solution of virial oscillations at the same conditions, the solutions (4.14) and (4.15) of Eq. 4.11, obtained in this chapter, should be considered rigorous. Thus we can conclude that the constancy of the product $U\sqrt{\Phi}$ in the framework of the static system model is proven. In Chap. 6 we will come back to this condition and will obtain another proof of the same very important relationship which is applied for study of the Earth's dynamics.

Chapter 5
Perturbed Virial Oscillations of a System

In the previous chapter we have considered a number of cases of explicitly solved problems in mechanics and physics for the dynamics of an n-body system and shown that all those classical problems have also explicit solution in the framework of the virial approach. But in the latter case, the solutions acquire a new physical meaning because the dynamics of a system is considered with respect to new parameters, i.e. its Jacobi function (polar moment of inertia) and potential (kinetic) energy. In fact, the solution of the problem in terms of co-ordinates and velocities specifies the changes in location of a system or its constituents in space. The solution, with respect to the Jacobi function and the potential energy, identifies the evolutionary processes of the structure or redistribution of the mass density of the system. Moreover, the main difference of the two approaches is that the classical problem considers motion of a body in the outer central force field. The virial approach considers motion of a body both in the outer and in its own force field applying, instead of linear forces and moments, the volumetric forces (pressure) and moments (oscillations).

It appears from the cases considered that the existence of the relationship between the potential energy of a system and its Jacobi function written in the form

$$U\sqrt{\Phi} = B = \text{const.} \tag{5.1}$$

is the necessary condition for the resolution of Jacobi's equation.

This is the only case when the scalar equation

$$\ddot{\Phi} = 2E - U$$

is transferred into a non-linear differential equation with one variable in the form

$$\ddot{\Phi} = 2E + \frac{B}{\sqrt{\Phi}}. \tag{5.2}$$

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It was shown in Chap. 4 that if the total energy of a system $E_o = -A/2 < 0$, then the general solution for Eq. 5.2 can be written as

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)] \tag{5.3}$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)],$$

where ε and ψ are integration constants, the values of which are determined from initial data using Eqs 4.23 and 4.24.

Equation 5.2 was called the equation of virial oscillation because its solution discovers a new physical effect – periodical non-linear change of the Jacobi function and hence the potential energy of a system around their mean values determined by the generalized virial theorem. Thus, in addition to the static effects determined by the hydrostatic equilibrium, in the study of dynamics of a system the effects, determined by a condition of dynamical equilibrium expressed by the Jacobi function, are introduced.

The equation of virial oscillations (5.2) reflects physics of motion of the interacting mass particles of a body or masses of bodies themselves by the inverse square law. Its application opens the way to study the nature and the mechanism of generation of the body's energy, which performs its motion, and to search the law of change for the system's configuration, i.e. a mutual change location of particles or the law of redistribution of the mass density for the system's matter during its oscillations. This problem was considered earlier in our work (Ferronsky et al. 1987). We continue its study in the next chapter.

As described in Chap. 4, cases of solution of Eq. 5.2 relate to unperturbed conservative systems. But in reality, in nature all systems are affected by internal and external perturbations which, from a physical point of view, are developed in the form of dissipation or absorption of energy. In this connection, as shown in Chap. 3 in the right–hand side of the equation of virial oscillations (5.2), an additional term appears which is proportional to the Jacobi function Φ (indicating the presence of gravitational background or the existence of interaction between the system particles in accordance with Hook's law) and its first derivative $\dot{\Phi}$ depending on time t (indicating the existence of energy dissipation). All these and other possible cases can be formally described by a generalized equation of virial oscillations (3.27):

$$\ddot{\Phi} = 2E + \frac{B}{\sqrt{\Phi}} + X(t,\Phi,\dot{\Phi}), \tag{5.4}$$

where $X(t, \Phi, \dot{\Phi})$ is the perturbation function, the value of which is small in comparison with the term $B/\sqrt{\Phi} \neq \text{const}$.

In this chapter we consider general as well as some specific approaches to the solution of Eq. 5.4 in the framework of different physical models of a system.

5.1 Analytical Solution of a Generalized Equation of Virial Oscillations

The equation of perturbed virial oscillations is generalized in the form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} + X\left(t, \Phi, \dot{\Phi}\right), \tag{5.5}$$

where $A = -2E$; B is constant; $X\left(t, \Phi, \dot{\Phi}\right)$ is the perturbation function which we assume is given and dependent in general cases on time t, the Jacobi function Φ and its first derivative $\dot{\Phi}$.

We consider two ways for analytical construction of the solution of Eq. 5.5. In addition, let the function $X\left(t, \Phi, \dot{\Phi}\right)$ in Eq. 5.5 depend on some small parameter e in relation to which the function can be expanded into absolutely convergent power series of the form

$$X\left(t, \Phi, \dot{\Phi}\right) = \sum_{r=1}^{\infty} e^k X^k\left(t, \Phi, \dot{\Phi}\right). \tag{5.6}$$

Let the series be convergent in some time interval t absolute for all values of e which are satisfied to condition $|e| < \bar{e}$. Then Eq. 5.5 can be rewritten in the form

$$X\left(t, \Phi, \dot{\Phi}\right) = -A + \frac{B}{\sqrt{\Phi}} \sum_{r=1}^{\infty} e^k X^k\left(t, \Phi, \dot{\Phi}\right). \tag{5.7}$$

We look for the solution of Eq. 5.7 also in the form of the power series of parameter e. For this purpose we write the function $\Phi(t)$ in the form of a power series, the coefficients of which are unknown:

$$\Phi(t) = \sum_{k=0}^{\infty} e^k \Phi^{(k)}(t). \tag{5.8}$$

Putting (5.8) into (5.7), the task can be reduced to the determination of such functions $\Phi^{(k)}(t)$ which identically satisfy Eq. 5.7. In this case, the coefficient $\Phi^{(o)}(t)$ becomes the solution of the unperturbed oscillation equation (5.2), which can be obtained from (5.7) by putting $e = 0$.

One can consider the series (5.8) as a Taylor series expansion in order to determine all the other coefficients $\Phi^{(k)}(t)$, i.e.,

$$\Phi^{(k)} = \frac{1}{k!}\left(\frac{d^k\Phi}{de^k}\right)\Big|_{e=0},$$

$$\dot{\Phi}^{(k)} = \frac{1}{k!}\left(\frac{d^k\dot{\Phi}}{de^k}\right)\Big|_{e=0}. \tag{5.9}$$

Accepting the series (5.8) for introduction into Eq. 5.7, it becomes identical with respect to the parameter *e*. Thus we have justified the differentiation of the identity with respect to the parameter *e* several times assuming that the identity remains after repeated differentiation.

We next obtain

$$\frac{d^2}{dt^2}\left(\frac{d\Phi}{de}\right) = -\frac{1}{2}\frac{B}{\Phi^{3/2}}\left(\frac{d\Phi}{de}\right) + \sum_{k=1}^{\infty} k e^{k-1} X^{(k)} + \sum_{k=1}^{\infty} e^k \left(\frac{dX^{(k)}}{de}\right), \tag{5.10}$$

where $dX^{(k)}/de$ is the total derivative of the function $X^{(k)}$ with respect to parameter e, expressed by

$$\frac{dX^{(k)}}{de} = \frac{\partial X^{(k)}}{\partial \Phi}\left(\frac{d\Phi}{de}\right) + \frac{\partial X^{(k)}}{\partial \dot{\Phi}}\left(\frac{d\dot{\Phi}}{de}\right).$$

Now let $e = 0$ in (5.10). Then by taking into account (5.8) and (5.9), we obtain

$$\frac{d^2\Phi^{(1)}}{dt^2} + p_1\Phi^{(1)} = X_1, \tag{5.11}$$

where

$$p_1 = \frac{1}{2}\frac{B}{\Phi^{3/2}}\Big|_{e=0} = \frac{1}{2}\frac{B}{\Phi^{(0)3/2}},$$

$$X_1 = X^1\left(t, \Phi^{(0)}, \dot{\Phi}^{(0)}\right)$$

are known functions of time, since the solution of the equation in the zero approximation (unperturbed oscillation equation (5.3)) is known.

Carrying out differentiation of Eq. 5.7 with respect to parameter e for the second, third and so on $(k-1)$ times, and assuming after each differentiation that $e = 0$, we will step by step obtain equations determining second, third and so on approximations. It is possible to show that in each succeeding approximation the equation will have the same form and the same coefficient p_1 as in Eq. 5.11. If so, the equation determining the functions $\Phi^{(k)}$ and $\dot{\Phi}^{(k)}$ has the form

$$\frac{d^2\Phi^{(k)}}{dt^2} + p_1\Phi^{(k)} = X_k\left(t, \Phi^{(0)}, \dot{\Phi}^{(0)}, \ldots, \Phi^{(k-1)}, \dot{\Phi}^{(k-1)}\right), \tag{5.12}$$

where the function X_k depends on $\Phi^{(0)}$, $\dot{\Phi}^{(0)}$,... $\Phi^{(k-1)}$, $\dot{\Phi}^{(k-1)}$, which were determined earlier and are the functions of t and unknown functions $\Phi^{(0)}$ and $\dot{\Phi}^{(0)}$.

It is known that there is no general way of obtaining a solution for any linear differential equation with variable coefficients, but in our case we can use the following theorem of Poincaré (Duboshin 1975). Let the general solution of the unperturbed virial oscillation equation be determined by the function $\Phi^{(0)} = f(t, C_1, C_2)$, where C_1 and C_2 are, for instance, arbitrary constants ε and Ψ in the solution (5.3) of Eq. 5.2. Then, Poincaré's theorem confirms that the function determined by the equalities

$$\Phi_1 = \frac{\partial f}{\partial C_1},$$
$$\Phi_2 = \frac{\partial f}{\partial C_2}$$

satisfies the linear homogeneous differential equation reduced by omission of the right-hand side of Eq. 5.12.

Thus, the general solution of the linear homogeneous equation

$$\frac{d^2\Phi^{(k)}}{dt^2} + p_1\Phi^{(k)} = 0$$

has the form

$$\Phi_1 C_1^{(k)} + \Phi_2 C_2^{(k)} = \Phi^{(k)} \tag{5.13}$$

and the general solution of Eq. 5.12 can be obtained by the method of variation of arbitrary constants, i.e. assuming that $C_2^{(k)}$ are functions of time. Then, using the key idea of the method of variation of arbitrary constants, we obtain a system of two equations:

$$\dot{C}_1^{(k)}\Phi_1 + \dot{C}_2^{(k)}\Phi_2 = 0,_k. \tag{5.14}$$
$$\dot{C}_1^{(k)}\dot{\Phi}_1 + \dot{C}_2^{(k)}\dot{\Phi}_2 = X_k.$$

Solving this system with respect to $\dot{C}_1^{(k)}$ and $\dot{C}_2^{(k)}$ and integrating the expression obtained, we write the general solution of Eq. 5.12 as follows:

$$\Phi^{(k)}(t) = \Phi_2 \int\limits_{t_0}^{t} \frac{\Phi_1 X_k dt}{\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1} - \Phi_1 \int\limits_{t_0}^{t} \frac{\Phi_2 X_k dt}{\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1},$$

where

$$\Phi_1 = \frac{\partial f(t, C_1, C_2)}{\partial C_1}$$

and

$$\Phi_2 = \frac{\partial f(t, C_1, C_2)}{\partial C_2}.$$

Thus, we can determine any coefficient of the series (5.8), reducing Eq. 5.7 into an identity, and therefore write the general solution of Eq. 5.5 in the form

$$\Phi = \sum_{k=0}^{\infty} e^k \Phi^{(k)} = \sum_{k=0}^{\infty} e^k \left[\Phi_2 \int_{t_0}^{t} \frac{\Phi_1 X_k dt}{\Phi_1 \dot{\Phi}_2 - \Phi_2 \dot{\Phi}_1} - \Phi_1 \int_{t_0}^{t} \frac{\Phi_2 X_k dt}{\Phi_1 \dot{\Phi}_2 - \Phi_2 \dot{\Phi}_1} \right]. \quad (5.15)$$

Let us consider the second way of approximate integration of the perturbed virial equation (5.5), based on Picard's method (Duboshin 1975). It is convenient to apply this method of integrating the equations which was obtained using the Lagrange method of variation of arbitrary constants.

We assume that the first integrals (4.23) and (4.24)

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)}, \quad (5.16)$$

$$-\tau = \left\{ \frac{4B}{(2A)^{3/2}} \left[\arccos \frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} - \varepsilon \sqrt{1 - \left(\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} \right)^2} \right] - t \right\} \quad (5.17)$$

of the unperturbed virial oscillation equation (5.2) are also the first integrals of the perturbed oscillation equation (5.5). But constants ε and τ are now unknown functions of time. Let us derive differential equations which are satisfied by these functions, using the first integrals (5.16) and (5.17). For convenience, we replace the integration constant ε by C, using the expression

$$\varepsilon = \sqrt{1 - \frac{AC}{2B^2}}.$$

Now we rewrite Eq. 5.16 in the form

$$C = -\dot{\Phi}_0^2 + 4B\sqrt{\Phi_0} - 2A\Phi_0. \quad (5.18)$$

Then using the main idea of the Lagrange method, after variation of the first integrals (5.17) and (5.18) and replacement of $\ddot{\Phi}$ by

$$\left(-A + \frac{B}{\sqrt{\Phi}} + X\left(t, \Phi, \dot{\Phi}\right) \right)$$

we write

$$\dot{C} = -2\dot{\Phi} X\left(t, \Phi, \dot{\Phi}\right), \quad (5.19)$$

$$\dot{\tau} = \Psi(\Phi, C)\dot{C} = -2\dot{\Phi} X\left(t, \Phi, \dot{\Phi}\right)\Psi(\Phi, C), \quad (5.20)$$

where

$$\Psi(\Phi, C) = -\frac{4B}{(2A)^{3/2}}\frac{d}{dC}\left[\arccos\frac{1-\frac{A}{B}\sqrt{\Phi}}{\sqrt{1-\frac{AC}{2B^2}}} - \sqrt{1-\frac{AC}{2B^2}}\sqrt{1-\left(\frac{1-\frac{A}{B}\sqrt{\Phi}}{\sqrt{1-\frac{AC}{2B^2}}}\right)^2}\right].$$

We now express Φ and $\dot{\Phi}$ in explicit form through C, τ and t using, for example, the Lagrangian series (4.36) and (4.37)

$$\Phi(t) = \frac{B^2}{A^2}\left[1+\frac{3}{2}\varepsilon^2+\left(-2\varepsilon+\frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + \ldots\right], \tag{5.21}$$

$$\dot{\Phi}(t) = \sqrt{\frac{2}{A}}\,\varepsilon B\left[\sin M + \frac{1}{2}\varepsilon\sin 2M + \frac{\varepsilon^2}{2}\sin M\left(2\cos^2 M - \sin^2 M\right) + \ldots\right]. \tag{5.22}$$

Thus, taking into account Eqs. 5.21 and 5.22 for the functions Φ and $\dot{\Phi}$, Eqs. 5.19 and 5.20 can be rewritten as

$$\frac{dC}{dt} = F_1(t, C, \tau), \tag{5.23}$$

$$\frac{d\tau}{dt} = F_2(t, C, \tau).$$

To solve the system of differential equations (5.23), we use Picard's successive approximation method, obtained in the k-th approximation expressions for $C^{(k)}$ and $\tau^{(k)}$ in the form

$$C^{(k)} = C^{(0)} + \int_{t_0}^{t} F_1\left(t, C^{(k-1)},\ \tau^{(k-1)}\right)dt, \tag{5.24}$$

$$\tau^{(k)} = \tau^{(0)} + \int_{t_0}^{t} F_2\left(t, C^{(k-1)},\ \tau^{(k-1)}\right)dt, \tag{5.25}$$

where $C^{(0)}$ and $\tau^{(0)}$ are the values of arbitrary constants C and τ at initial time t_0, and $k = 1, 2, \ldots$.

Then, in the limit of $k \to \infty$, we obtain the solution of the system (5.23):

$$C = \lim_{k\to\infty} C^{(k)},$$

$$\tau = \lim_{k\to\infty} \tau^{(k)}. \tag{5.26}$$

Consider now two possible cases of the perturbation function behavior. First, assume that the perturbation function X does not depend explicitly on time. Then, since it is possible to expand functions Φ and $\dot{\Phi}$ into a Fourier series in terms of sine and cosine of argument M, the right-hand sides of the system (5.23) can also be expanded into a Fourier series in terms of sine and cosine of M.

Finally we obtain

$$\frac{dC}{dt} = \left[A_0 + \sum_{k=1}^{\infty} (A_k \cos kM + B_k \sin kM) \right], \tag{5.27}$$

$$\frac{d\tau}{dt} = \left[a_0 + \sum_{k=1}^{\infty} (a_k \cos kM + b_k \sin kM) \right], \tag{5.28}$$

where A_0, A_k, B_k, a_0, a_k, b_k are the corresponding coefficients of the Fourier series which are

$$A_0 = \frac{2}{\pi} \int_0^{2\pi} F_1(M, C) dM,$$

$$A_k = \frac{2}{\pi} \int_0^{2\pi} F_1(M,C) \cos kM dM,$$

$$B_k = \frac{2}{\pi} \int_0^{2\pi} F_1(M,C) \sin kM dM,$$

$$a_0 = \frac{2}{\pi} \int_0^{2\pi} F_2(M,C) dM,$$

$$a_k = \frac{2}{\pi} \int_0^{2\pi} F_2(M,C) \cos kM dM,$$

$$b_k = \frac{2}{\pi} \int_0^{2\pi} F_2(M,C) \sin kM dM.$$

Following Picard's method, in order to solve Eqs. 5.27 and 5.28 in the first approximation, we introduce into the right-hand side of the equations the values of arbitrary constants C and τ corresponding to the initial time t_0 . Then we obtain

$$C^{(1)}(t) = C^{(0)} + A_0^{(0)}(t - t_0) + \sum_{k=1}^{\infty} \frac{1}{kn} \left\{ A_k^{(0)}[\sin kM - \sin kM_0] + B_k^{(0)}[\cos kM - \cos kM_0] \right\} \quad (5.29)$$

$$\tau^{(1)}(t) = \tau^{(0)} + a_0^{(0)}(t - t_0) + \sum_{k=1}^{\infty} \frac{1}{kn} \left\{ a_k^{(0)}[\sin kM - \sin kM_0] + b_k^{(0)}[\cos kM - \cos kM_0] \right\} \quad (5.30)$$

Thus, when the function X does not depend explicitly on time t, solutions (5.29) and (5.30) of Eq. 5.5 have three analytically different parts. The first is a constant term, depending on the initial values of the arbitrary constants. It is usually called the constant term of perturbation of the first order. The second part is a function monotonically increasing in time. It is called the secular term of a perturbation of the first order. The third part consists of an infinite set of trigonometric terms. All of them are periodic functions of M and consequently of time t. This is called periodic perturbation

Similarly, we can obtain solutions in the second, third etc., orders. Here we limit our consideration only within the first order of perturbation theory. In practice, few terms of the periodic perturbation can be taken into account and the solution obtained becomes effective only for a short period of time.

When the perturbation function X is a periodic function of some argument M',

$$M' = n'(t - \tau'),$$

the right-hand side of the system of Eqs. 5.23 are periodic functions of the two independent arguments M and M'. Therefore, they can be expanded into a double Fourier series in terms of sine and cosine of the linear combination of arguments M and M'. Then in the first approximation of perturbation theory we obtain the following system of equations:

$$\frac{dC^{(1)}}{dt} = A_{00}^{(0)} + \sum_{k',k=-\infty}^{\infty} \left[A_{k,k'}^{(0)} \cos(kM + k'M') + B_{k,k'}^{(0)} \sin(kM + k'M') \right], \quad (5.31)$$

$$\frac{d\tau^{(1)}}{dt} = a_{00}^{(0)} + \sum_{k',k=-\infty}^{\infty} \left[a_{k,k'}^{(0)} \cos(kM + k'M') + b_{k,k'}^{(0)} \sin(kM + k'M') \right]. \quad (5.32)$$

Integrating equations (5.31) and (5.32) with respect to time, we obtain a solution of the system:

$$
\begin{aligned}
C^{(1)}(t) = C^{(0)} + A_{00}^{(0)}(t - t_0) + \sum_{k',k=-\infty}^{\infty} \frac{1}{kn + k'n'} \Big\{ & A_{k,k'}^{(0)}[\cos(kM + k'M') \\
& - \cos(kM_0 + k'M'_0)] + B_{k,k'}^{(0)}[\sin(kM + k'M') \\
& - \sin(kM_0 + k'M'_0)] \Big\}
\end{aligned}
\tag{5.33}
$$

$$
\begin{aligned}
\tau^{(1)}(t) = \tau^{(0)} + a_{00}^{(0)}(t - t_0) + \sum_{k',k=-\infty}^{\infty} \frac{1}{kn + k'n'} \Big\{ & a_{k,k'}^{(0)}[\cos(kM + k'M') - \\
& - \cos(kM_0 + k'M'_0)] + b_{k,k'}^{(0)}[\sin(kM + k'M') \\
& - \sin(kM_0 + k'M'_0)] \Big\}
\end{aligned}
\tag{5.34}
$$

Equations 5.33 and 5.34 have the same analytical structure as (5.29) and (5.30). At the same time, in this case, the periodic part of the perturbation can be divided into two groups, depending on the value of the divisor $kn + k'n'$. If the values of k and k' are such that the divisor is sufficiently large, then period $T_{k,k'} = 2\pi/(kn + k'n')$ of the corresponding inequality will be rather small. Such inequalities are called short-periodic. Their amplitudes are also rather small, and they can play a role only within short periods of time.

If the values of k and k' are such that the divisor $kn + k'n'$ is sufficiently small but unequal to zero, then the period of the corresponding inequality will become large. The amplitude of such terms could also be large and play a role within large periods of time. Such terms form series of long-periodic inequalities. In the case of such k and k', when $kn + k'n' = 0$, the corresponding terms are independent of t and change the value of the secular term in the solutions (5.33) and (5.34).

5.2 Solution of the Virial Equation for a Dissipative System

In Chap. 3 we derived Jacobi's virial equation for a non-conservative system in the form

$$
\ddot{\Phi} = 2E_0[1 + q(t)] - U - k\dot{\Phi}. \tag{5.35}
$$

At $k \ll 1, t \gg t_o, |U|\sqrt{\Phi} = B = \text{const}, 2E_o = - A_o$, and when the magnitude of the term $k\dot{\Phi}$ is sufficiently small, Eq. 5.35 can be rewritten in a parametric form

$$\ddot{\Phi} = -A_0[1 + q(t)] + \frac{B}{\sqrt{\Phi}}, \tag{5.36}$$

where q(t) is a monotonically increasing function of time due to dissipation of energy during ‘smooth’ evolution of a system within a time interval $t \in [0, \tau]$.

Using the theorem of continuous solution depending on the parameter, we write the solution of Eq. 5.36 as follows:

$$\begin{aligned}&-\arccos W + \arccos W_0 - \sqrt{1 - \frac{A_0[1+q(t)]C}{2B^2}}\sqrt{1-W^2}\\&+\sqrt{1-\frac{A_0C}{2B^2}}\sqrt{1-W_0^2} = \sqrt{\frac{(2A_0[1+q(t)])^{3/2}}{4B}}(t-t_0),\end{aligned} \tag{5.37}$$

$$\begin{aligned}&\arccos W - \arccos W_0 + \sqrt{1 - \frac{A_0[1+q(t)]C}{2B^2}}\sqrt{1-W^2}\\&-\sqrt{1-\frac{A_0C}{2B^2}}\sqrt{1-W_0^2} = \sqrt{\frac{(2A_0[1+q(t)])^{3/2}}{4B}}(t-t_0),\end{aligned} \tag{5.38}$$

where

$$W = \frac{\frac{A_0[1+q(t)]}{B}\sqrt{\Phi}-1}{\sqrt{1-\frac{A_0[1+q(t)]C}{2B^2}}}; \quad W_0 = \frac{\frac{A_0}{B}\sqrt{\Phi}-1}{\sqrt{1-\frac{A_0C}{2B^2}}};$$

$$A_0[1+q(t)]>0; \quad C<\frac{2B^2}{A_0[1+q(t)]};$$

$$|-A_0[1+q(t)]\sqrt{\Phi}+B|<B\sqrt{1-\frac{A_0[1+q(t)]C}{2B^2}};$$

$$C = -2A_0\Phi_0 + 4B\sqrt{\Phi_0} - \dot{\Phi}_0^2.$$

Equations of discriminant curves which bound oscillations of the Jacobi function Φ by analogy with the case of the conservative system can be written as

$$\sqrt{\Phi_1} = \frac{B}{A_0[1+q(t)]}\left[1+\sqrt{1-\frac{A_0[1+q(t)]C}{2B^2}}\right], \quad t \in [0, \ \tau], \tag{5.39}$$

$$\sqrt{\Phi_2} = \frac{B}{A_0[1+q(t)]}\left[1-\sqrt{1-\frac{A_0[1+q(t)]C}{2B^2}}\right], \quad t \in [0, \ \tau]. \tag{5.40}$$

It is obvious that the solution of Jacobi's virial equation for a non-conservative system is quasi-periodic with period

$$T_v(q) = \frac{8\pi B}{(2A_0[1 + q(t)])^{3/2}}, \tag{5.41}$$

and an amplitude of Jacobi function oscillations

$$\Delta\sqrt{\Phi} = \frac{B}{A_0[1 + q(t)]}\left(1 - \frac{A_0[1 + q(t)]C}{2B^2}\right)^{1/2}. \tag{5.42}$$

As q(t) is a monotonically and continuously increasing parameter confined in time, the period and the amplitude of the oscillations will gradually decrease and tend to zero in the time limit.

In Fig. 5.1a the integral curves (5.37) and (5.38) and the discriminant curves (5.39) and (5.40) are shown in a general case when $0 < C < 2B^2/A_0$. At the point O_b, the integral and discriminant curves tend to coincide and the value of the amplitude of the Jacobi function (polar moment of inertia) oscillations of the system goes to zero.

When $C = 0$ (Fig. 5.1b) the discriminant line (5.39) coincides with the axis of abscissa, $\Phi_2 = 0$. In the accepted case of constancy of the system mass, the point O_b, where the integral and discriminant curves coincide, will be reached in the time limit $t \to \infty$.

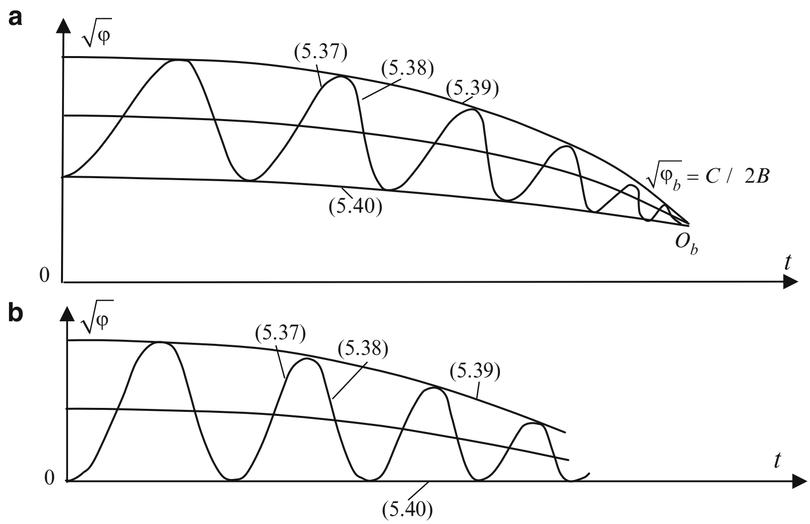

Fig. 5.1 Virial oscillations of the Jacobi function in time for a non-conservative system (**a**) and for the general (Wintner's) case (**b**)

Where $2B^2/A_o \to C$ and $C < 0$, the solutions (5.37)–(5.39), (4.40) could be complex so the processes considered are not physical.

We note that, by analogy with the case for a conservative system, considered in Chap. 4, we can show here that the asymptotic relations (4.30)–(4.32) for the solutions (5.37) and (5.38) of Jacobi's Eq. 5.36 in the points of contact of the discriminant line $\Phi_2 = 0$, are justified. In the points of contact for the integral curves (5.37) and (5.38) and the discriminant curves (5.39) and (5.40) for which Φ_1 and Φ_2 are not equal to zero, the following asymptotic relations are also justified:

$$\left(\sqrt{\Phi_1} - \sqrt{\Phi}\right) \propto (t' - t)^2, \tag{5.43}$$

$$\left(\sqrt{\Phi} - \sqrt{\Phi_2}\right) \propto (t - t')^2, \tag{5.44}$$

where t' is time of a tangency point for the corresponding integral curve of the discriminant lines $\Phi_{1,2}$ when $\Phi_{1,2} \neq 0$.

5.3 Solution of the Virial Equation for a System with Friction

Let us consider the solution of Jacobi's virial equation for conservative systems, but let the relationship between its potential energy and the Jacobi function be as follows:

$$U\sqrt{\Phi} = B + k\dot{\Phi}. \tag{5.45}$$

In this case, the equation of virial oscillations (5.2) can be written

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} - k\frac{\dot{\Phi}}{\sqrt{\Phi}}. \tag{5.46}$$

The term $-k\dot{\Phi}/\sqrt{\Phi}$ in (5.46) plays the role of perturbation function, reflecting the effect of internal friction of the matter while the system is oscillating.

In principle, Eq. 5.46 can be solved using the above perturbation theory methods. However, we can show that a particular solution exists for the system of two differential equations of the second order, which satisfies Eq. 5.46. These differential equations are as follows:

$$\left(\sqrt{\Phi}\right)'' + \sqrt{\frac{2}{A}}k\left(\sqrt{\Phi}\right)' + \sqrt{\Phi} = \frac{B}{A}, \tag{5.47}$$

$$t'' + \sqrt{\frac{2}{A}}kt' + t = \frac{4B}{(2A)^{3/2}}\lambda. \tag{5.48}$$

In Eqs. 5.47 and 5.48 we introduced a new variable λ, so the primes at Φ and t mean differentiation with respect to λ. Note also that time t here is not an independent variable. This allows us to transfer the non-linear equation into two linear equations. The partial solution of Eqs. 5.47 and 5.48 containing two integration constants is

$$\sqrt{\Phi} = \frac{B}{A}\left[1 - \varepsilon e^{-r/2\sqrt{2/A}\lambda}\cos\left(\sqrt{\frac{4A - 2k^2}{4A}}\lambda + \psi + \tau\right)\right], \tag{5.49}$$

$$t = \frac{4B}{(2A)}\left[\lambda - \varepsilon e^{-r/2\sqrt{2/A}\lambda}\sin\left(\sqrt{\frac{4A - 2k^2}{4A}}\lambda + \psi\right)\right] - \frac{4B}{(2A^{3/2})}\sqrt{\frac{2}{A}}k, \tag{5.50}$$

where ε and ψ are arbitrary constants and

$$\tau = \operatorname{arctg}\sqrt{\frac{2}{A}}k\left(\frac{4A - 2k^2}{4A}\right)^{-1/2}.$$

To show that Eqs. 5.49 and 5.50 of the two linear differential equations (5.47) and 5.48 are also general solutions of (5.46), let us do as follows.

Differentiating (5.50) with respect to λ, we obtain

$$t' = \sqrt{\frac{2}{A}}\sqrt{\Phi}. \tag{5.51}$$

We write the derivative from function $\sqrt{\Phi}$ with respect to λ using Eq. 5.51 in the form

$$\left(\sqrt{\Phi}\right)' = \frac{\dot{\Phi}}{\sqrt{2A}}. \tag{5.52}$$

Then the second derivative from $\sqrt{\Phi}$ with respect to λ can be obtained analogously

$$\left(\sqrt{\Phi}\right)'' = \frac{\ddot{\Phi}}{\sqrt{2A}}t' = \frac{\ddot{\Phi}\sqrt{\Phi}}{A}. \tag{5.53}$$

Substituting Eqs. 5.52 and 5.53 for $\left(\sqrt{\Phi}\right)'$ and $\left(\sqrt{\Phi}\right)''$ into Eq. 5.47, we obtain

$$\frac{\ddot{\Phi}\sqrt{\Phi}}{A} + \sqrt{\frac{2}{A}}k\frac{\dot{\Phi}}{\sqrt{\Phi}} + \sqrt{\Phi} = \frac{B}{A}. \tag{5.54}$$

Dividing Eq. 5.54 by $\sqrt{\Phi}/\mathrm{A}$ we have

$$\ddot{\Phi} + \mathrm{k}\frac{\dot{\Phi}}{\sqrt{\Phi}} + \mathrm{A} = \frac{\mathrm{B}}{\sqrt{\Phi}},$$

which is in fact our Eq. 5.46. This means that Eqs. 5.49 and 5.50 are the general solution of Eq. 5.46.

Note that Eq. 5.50 differs in general from Kepler's equation both by the exponential factor before the sine function and by the constant term in the right-hand side of Eq. 5.50. In addition, it follows from Eq. 5.49 that the period of virial oscillations of the Jacobi function depends on the parameter k. Therefore, when λ changes its value by $2\,\pi/\left[\sqrt{(4\mathrm{A} - 2\mathrm{k}^2)/4\mathrm{A}}\right]$ the value of $\sqrt{\Phi}$ remains unchanged (we neglect the changes of the amplitude of virial oscillations due to existence of the exponential factor) assuming that

$$\frac{\mathrm{k}}{2}\sqrt{\frac{2}{\mathrm{A}}}2\,\pi \Bigg/ \sqrt{\frac{4\mathrm{A} - 2\mathrm{k}^2}{4\mathrm{A}}} << 1.$$

It follows from Eq. 5.50 that time t changes by the relationship of $\mathrm{T} = 8\,\pi\mathrm{B}/(2\mathrm{A})^{3/2}\sqrt{(4\mathrm{A} - 2\mathrm{k}^2)/4\mathrm{A}}$ defining the period of the damping virial oscillations. Therefore, from solutions (5.49) and (5.50) of Eq. 5.46 it follows that if during the evolution of the system the value $\mathrm{U}\sqrt{\Phi}$ varies only slightly around the constant, this leads to damping of the virial oscillations of the integral characteristics of the system around their averaged virial theorem value..

In conclusion we have to note that derivation of the equation of dynamical equilibrium and its solution for conservative and dissipative systems shows that dynamics of celestial bodies in their own force field puts forward a wide class of geophysical, astrophysical and geodetic problems which can be solved by the methods of celestial mechanics and with the new physical concepts we have introduced.

Chapter 6
Relationship Between Jacobi Function and Potential Energy

In the previous chapters we have considered the general approach to the formulation and solution of global dynamics problems for a self-gravitating system in terms of volumetric (integral) characteristics. For this purpose we have transformed Jacobi's virial equations for conservative and non-conservative systems:

$$\ddot{\Phi} = 2E - U, \tag{6.1}$$

$$\ddot{\Phi} = 2E - U + X(t, \Phi, \dot{\Phi}) \tag{6.2}$$

into equations of virial oscillations in the form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}, \tag{6.3}$$

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} + X(t, \Phi, \dot{\Phi}). \tag{6.4}$$

The transfer from Eqs. 6.1 and 6.2 to Eqs. 6.3 and 6.4 has been made by using the following relationship between the Jacobi function and potential energy:

$$U\sqrt{\Phi} = B = \text{const}. \tag{6.5}$$

As shown in Chap. 4, the validity of the relationship (6.5) for explicitly solved cases of the many-body problem in mechanics and physics is an obvious fact. Consequently, for example, in the case of two-body problem which represents the conservative system, the solutions of Eq. 6.3 will be analogous to Keplerian equations of conic sections according to which the Jacobi function (or potential energy) changes with time. In the same manner the solution of the generalized equation of virial oscillations (6.4) in celestial mechanics will correspond to the solution for the periodic motion in the two-body problem obtained by perturbation theory methods.

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The validity of Eq. 6.5 for a many-body system in a general case is not obvious despite the fact that both volumetric integral characteristics considered are functions of the distribution of mass density of a system.

In this chapter we consider in detail the main physical aspect of the relationship between the Jacobi function and the potential energy of a system

6.1 Asymptotic Limit of Simultaneous Collision of Mass Points for a Conservative System

We take advantage of the results presented by Wintner (1941) in order to study the many-body problem. From such a study it follows that a conservative system of n mass points of arbitrary configuration interacting according to Newton's law, the following statement is valid.

If the motion of the material points of a system of arbitrary configuration has the consequence that all of them tend to simultaneous collision then the relationship $U\sqrt{\Phi}$ approaches a constant value. This result obtained by Wintner supplements the general properties of conservative systems of material points interacting according to Newton's law when their number remains constant all the time. The condition of constancy of the number of mass points of a system is equivalent to that of the distance $\Delta_{ij} = |r_i - r_j|$ between any pair of points at any moment of time and should be $\Delta_{ij}>0$, where r_i and r_j denote the 3-vectors of the co-ordinates of mass points in the barycentric co-ordinate system.

For such a system, from the analysis of Jacobi's virial equation (6.1) and the expression for the Jacobi function Φ,

$$\Phi = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i m_j \Delta_{ij}^2 \tag{6.6}$$

for kinetic energy T

$$T = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i m_j (\dot{r}_i - \dot{r}_j)^2 \tag{6.7}$$

and for potential energy U

$$U = -G \sum_{1 \le i < j \le n} \frac{m_i m_j}{\Delta_{ij}} \tag{6.8}$$

three inequalities were obtained which produce restrictions on the Jacobi function (or potential energy) and its derivatives. These inequalities can be written in the form

$$|\dddot{\Phi}| \leq \eta(|\ddot{\Phi}| + 2|E|)^{5/2} \tag{6.9}$$

$$(\ddot{\Phi} - 2E)\Phi^{1/2} \geq \mu > 0 \tag{6.10}$$

$$\ddot{\Phi} - E - \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi} \geq \frac{M^2}{4\Phi} \tag{6.11}$$

where constants

$$\eta = \frac{\sqrt{2m}}{G}\sum_{1\leq i<j\leq n}(m_i m_j)^{-3/2}>0,$$

$$\mu = \frac{G}{\sqrt{2m}}\sum_{1\leq i<j\leq n}(m_i m_j)^{3/2}>0,$$

$$M^2 = C_1^2 + C_2^2 + C_3^2,$$

$$m = \sum_{i=1} m_i$$

and m_i is the mass of the i-th point; E = T + U is the total energy; C_1, C_2, C_3, are projections of the angular momentum M on the axes.

The third inequality (6.11) is more complicated than the others as is contains, besides the constant E, which is the total energy of the system, the value M of the constant angular momentum.

It has been shown by Wintner (1941) that if the motion of material points of an arbitrary configuration system provides their simultaneous collision, then the system possesses zero angular momentum and a simultaneous collision will occur in the finite interval of time. In addition, the behavior of the Jacobi function in the vicinity of the time moment t_o of simultaneous collision is defined by the following asymptotics:

$$\Phi \propto (t - t_0)^{4/3}, \tag{6.12}$$

$$\Phi \propto (t - t_0)^{1/3}, \tag{6.13}$$

$$\Phi \propto (t - t_0)^{-2/3}. \tag{6.14}$$

Following Wintner (1941), we introduce the definition of a central configuration which is needed for further consideration of the problem. If the positions of the material points in the system are such that the following relation is satisfied:

$$U_\eta = \sigma m_i r_i \tag{6.15}$$

then the configuration of the system is called central.

Here, in Eq. 6.15

$$\sigma = -\frac{U}{2\Phi}.$$

The definition (6.15) of the central configuration can be rewritten in equivalent form:

$$(U^2\Phi)_\eta = 0. \tag{6.16}$$

As proved by Wintner (1941), the important relation follows from asymptotics (6.12)–(6.14) at $t \rightarrow t_o$:

$$(U^2\Phi)_\eta \rightarrow 0 \tag{6.17}$$

which, together with the definition for the central configuration, leads to the following theorem:

Any arbitrary configuration of material points in the asymptotic time limit of simultaneous collisions of all the mass points tends to the central configuration.

It follows from this that

$$\lim_{t\rightarrow t_0} |U|\sqrt{\Phi} = \text{const.} \tag{6.18}$$

This theorem justifies the transformation of Jacobi's virial equation (6.1) and (6.2) into equation of virial oscillations (6.3) and (6.4) within the framework of Newton's law of interaction of material points of a conservative system.

6.2 Asymptotic Limit of Simultaneous Collision of Mass Points for Non-conservative Systems

The model of a conservative system permits a limited number of problems to be solved. In reality all natural systems are non-conservative. Study of the dynamics of such systems is the main object of the problem of evolution.

It is well known from the observations described in the general course of physics by Kittel et al. (1965) that the gravitating systems in nature are contracting while losing part of their total energy through friction and electromagnetic radiation. From the kinematics point of view this gravitational contraction is equivalent to the simultaneous collision of all n mass points of the system. We consider below the validity of the theorem expressed by Eq. 6.18 for non-conservative systems.

Let the motion of a system of n mass points occur by means of the gravitational interaction and Newtonian friction of the mass points. Then Jacobi's virial equation, in agreement with (3.22), can be written as

$$\ddot{\Phi} = 2E(t) - U(t) - k\dot{\Phi}, \tag{6.19}$$

where E(t) is the value of the total energy of the system at the moment of time t. From analyses of the equations of motion resulting in (3.23) it follows that

$$E(t) = E_0 - 2k\int_{t_0}^{t} T(t)dt = E_0[1 + q(t)],$$

where E_0 is the value of the total energy of the system at the initial moment of time t_o; q(t) is a monotonically increasing function of time.

We also accept the condition of the constancy of the number of mass particles in the system, from which it follows that the distance between any pairs of points $\Delta_{ij}>0$ and the following relation is correct:

$$\left|\frac{d}{dt}\Delta_{ij}\right| \le |\dot{r}_i - \dot{r}_j|$$

In the framework of this essentially important condition which forbids paired, threefold and higher-fold collisions, we obtain three inequalities analogous to (6.9)–(6.11). The inequalities are valid at any stage of the system's evolution and place restrictions on the Jacobi function and its derivatives.

From expression (6.8) for the potential energy of the system, the following inequalities can be written:

$$|U| = \left|G\sum_{1\le i<j\le n}\frac{m_i m_j}{\Delta_{ij}^2}\dot{\Delta}_{ij}\right| \le G\sum_{1\le i<j\le n}\frac{m_i m_j}{\Delta_{ij}^2}|\dot{r}_i - \dot{r}_j| \tag{6.20}$$

and

$$\frac{Gm_i m_j}{\Delta_{ij}} < -U,$$

where r_i and r_j are 3-vectors of co-ordinates of mass points in the barycentric co-ordinate system.

Substituting the last inequality into (6.20) we obtain

$$|\dot{U}| \le \frac{U^2}{G}\sum_{1\le i<j\le n}\frac{|\dot{r}_i - \dot{r}_j|}{m_i m_j}.$$

Since $m_i m_j (\dot{r}_i - r_j) \leq 2mT$, and assuming

$$\eta = \frac{1}{G} \sum_{1 \leq i<j \leq n} \frac{m^{1/2}}{(m_i m_j)^{3/2}},$$

we obtain

$$|\dot{U}| \leq U^2 \eta (2T)^{1/2}. \tag{6.21}$$

Then using Eq. 6.19 in the form

$$U = 2E_0[1 + q(t)] - \ddot{\Phi} - k\dot{\Phi} \tag{6.22}$$

and the law of conservation of energy for a dissipative system

$$U + T = E_0[1 + q(t)] \tag{6.23}$$

We rewrite the inequality (6.21) in the form

$$\begin{aligned} |\dot{U}| &\leq \left\{2|E_0|[1 + q(t)] + |\ddot{\Phi}\} + k|\dot{\Phi}|\right\}^2 \eta\sqrt{2}\left\{2|E_0|[1 + q(t)] + |\ddot{\Phi}\} + \ k|\dot{\Phi}|\right\}^{1/2} \\ &= \sqrt{2}\ \eta\left\{2|E_0|[1 + q(t)] + |\ddot{\Phi}\} + k|\dot{\Phi}\right\}^{5/2} \end{aligned} \tag{6.24}$$

Differentiating (6.22) with respect to time and substituting this into (6.24), we finally obtain the first inequality:

$$|\dddot{\Phi} + k\ddot{\Phi} - 2E_0\dot{q}(t)| \leq \sqrt{2}\ \eta\left\{2|E_0|[1 + q(t)] + |\ddot{\Phi}| + k|\dot{\Phi}|\right\}^{5/2}. \tag{6.25}$$

In the same way, it follows from (6.6) that

$$\Phi^{1/2} \geq \frac{1}{(2m)^{1/2}} (m_i m_j)^{1/2} \Delta_{ij}.$$

Then

$$\frac{\Phi^{1/2} m_i m_j}{\Delta_{ij}} \geq (2m)^{1/2} (m_i m_j)^{1/2}.$$

By virtue of (6.4) and (6.8),

$$\ddot{\Phi} + k\dot{\Phi} - 2E[1 - q(t)] = G \sum_{1 \leq i<j \leq n} \frac{m_i m_j}{\Delta E}.$$

The second inequality has the form

$$\ddot{\Phi} + k\dot{\Phi} - 2E[1 + q(t)]\Phi^{1/2} \geq \mu > 0 \tag{6.26}$$

where

$$\mu = \frac{G}{(2m)^{1/2}} \sum_{1 \leq i < j \leq n} (m_i m_j)^{3/2}.$$

Now let us derive the third inequality following from the Cauchy-Bunjakowski inequality, which is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Since

$$r_i^2 = |r_i|^2 \quad \text{and} \quad |(r_i \cdot \dot{r}_i)| = (|r_i| \cdot |\dot{r}_i|)$$

and from the definition of the Jacobi function, one obtains

$$\Phi = \sum_{i=1}^{n} m_i(|r_i| \cdot |\dot{r}_i|).$$

Applying the Cauchy-Bunjakowski inequality to this expression at

$$a_i = m_i^{1/2}|r| \quad \text{and} \quad b_i = m_i^{1/2}|r|$$

we can write

$$\dot{\Phi}^2 \leq 2\Phi \sum_{i=1}^{n} m_i |\dot{r}_i{}^2| = 2\Phi \sum_{i=1}^{n} \frac{m_i (r_i \cdot \dot{r}_i)^2}{r_i^2}.$$

Assuming

$$a_i = m_i^{1/2}|r_i|, \quad A_i = \frac{m_i^{1/2}[r_i X \dot{r}_i]}{|r_i|},$$

the vector of the angular momentum M is

$$M = \sum_{i=1}^{n} a_i A_i.$$

Then in a similar may we write

$$M^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} A_i^2\right) \equiv 2\Phi \sum_{i=1}^{n} \frac{m_i[r_i X \dot{r}_i]}{r_i^2}.$$

The addition of the last two inequalities yields

$$\dot{\Phi}^2 + M^2 \le 2\Phi \sum_{i=1}^{n} \frac{m_i\left\{(r_i \cdot \dot{r}_i)^2 + [r_i X \dot{r}_i]^2\right\}}{r_i^2}.$$

But since

$$\left\{(r_i \cdot \dot{r}_i)^2 + [r_i X \dot{r}_i]^2\right\} = r_i^2 \cdot \dot{r}_i^2,$$

we have

$$\dot{\Phi}^2 + M^2 \le 2\Phi \sum_{i=1}^{n} m_i \dot{r}_i.$$

As Jacobi's equation can be written in the form

$$\ddot{\Phi} + k\dot{\Phi} - E_0[1 + q(t)] = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2,$$

then after substitution of this into the right-hand side of the last inequality, we obtain

$$\dot{\Phi}^2 + M^2 \le 4\Phi\left\{\ddot{\Phi} + k\dot{\Phi} - E_0[1 + q(t)]\right\}.$$

Hence the third inequality can be written

$$\ddot{\Phi} + k\dot{\Phi} - E_0[1 + q(t)] - \frac{\dot{\Phi}^2}{4\Phi} \ge \frac{M^2}{4\Phi}. \tag{6.27}$$

Let us now analyze the behavior of the Jacobi function Φ and its derivatives. For this purpose we introduce the auxiliary function $Q = Q(t)$, equal to

$$Q = k\dot{\Phi}\,\Phi^{1/2} - E_0[1 + q(t)]\Phi^{1/2} + \frac{1/4\dot{\Phi}^2 + 1/4M}{\Phi^{1/2}}, \tag{6.28}$$

where $\Phi^{1/2} > 0$.

Then differentiating (6.28) and using

$$\frac{d}{dt}\left(\Phi^{1/2}\right) = \frac{\dot{\Phi}}{2\Phi^{1/2}},$$

we obtain

$$\dot{Q} = \frac{1}{2}\frac{\dot{\Phi}}{\Phi^{1/2}}\left\{\ddot{\Phi} + k\dot{\Phi} - E_0[1 + q(t)] - \frac{1}{4}\frac{M^2}{\Phi} - \frac{1}{4}\frac{\dot{\Phi}^2}{4\Phi}\right\} + \Phi^{1/2}\left[k\ddot{\Phi} - E_0\dot{q}(t)\right]$$

where $\dot{q}(t)>0$ and, in agreement with (6.27),

$$\left\{\ddot{\Phi} + k\dot{\Phi} - E_0[1 + q(t)] - \frac{1}{4}\frac{M^2}{\Phi} - \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi}\right\} \geq 0.$$

Let t_0 be the time of simultaneous collision of all the particles of the system. Then for $t \to t_o$ $(t \to t_o)$, $\Phi \to 0$. Let us show that the necessary condition for existence of such t_0 for which $\Phi \to 0$ (if $t \to t_o$) is that the constant angular momentum M must be zero.

Note that if, for $t \to t_o$, $\Phi \to 0$, then all mutual $\Delta_{ij} = \left|r_i - r_j\right|$ also tend to zero, and the potential energy $U \to -\infty$.

Since

$$\ddot{\Phi} = 2E_0[1 + q(t)] - U - k\dot{\Phi},$$

where $E_0 = \text{const}$, $|\dot{\Phi}| \to \infty$, $|q(t)|$, $|\dot{q}(t)| < \infty$, then, for $t \to t_o$, $\ddot{\Phi} \to \infty$. Thus for t sufficiently close to t_0 we have $\ddot{\Phi} > 0$ and therefore the derivative $\dot{\Phi}$ increases and does not change its sign. Since $\Phi > 0$ and $\Phi \to 0$, then Φ is monotonically decreasing function. It therefore follows from the expression for $\dot{Q}$ that the function Q in (6.28) for t sufficiently close to t_0 must decrease and its time limit for $t \to t_o$ might be $-\infty$, but cannot be $+\infty$. Moreover, it follows from the above statement that for $t \to t_0$ the limit of function (6.28) is

$$\lim_{t\to t_0} Q = \lim_{t\to t_0} \frac{1}{4}\frac{\dot{\Phi}^2 + M^2}{\Phi^{1/2}}, \tag{6.29}$$

but since $\Phi^{1/2} > 0$, the time limit (6.29) must be finite and non-negative. Hence for $t \to t_o$ and $\Phi \to 0$ the value $M^2/\Phi^{1/2}$ must remain limited. Therefore, since $M^2 = \text{const}$, then $M \equiv 0$ and proof is completed.

The above analysis shows that, at $t \to t_o$, $\ddot{\Phi} \to \infty$, and it therefore follows from (6.25) that

$$|\dddot{\Phi} = 2E_0\dot{q}(t) + k\ddot{\Phi}| \leq \text{const}\left(|\ddot{\Phi}| + k|\dot{\Phi}|\right)^{5/2}. \tag{6.30}$$

Using the second inequality (6.26), it can be shown that if t_o is the time moment of simultaneous collision of all the particles of the system, then as $\Phi^{1/2} > 0$ at $t \to t_{o,}$ the ratio $\dot{\Phi}/\Phi^{1/2}$ tends to a finite and positive limit.

In fact, as has been shown above, the limit (6.29) of the function (6.28) for $t \to t_o$ has a finite value. Since M = 0

$$\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}$$

will also be finite and non-negative. Let us show that this limit cannot be equal to zero.

Since for $t \to t_{o,}$ $M = 0$, $\Phi^{1/2} \to 0$, then the function (6.28) and its limit (6.29) may be written in the form

$$Q = k\dot{\Phi}\Phi^{1/2} - E_0[1 - q(t)]\Phi^{1/2} + \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi^{1/2}}, \tag{6.31}$$

$$\mu_0 = \frac{1}{4}\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}, \tag{6.32}$$

where

$$\mu_0 = \lim_{t-t_0} Q.$$

From (6.31) we find that

$$2Q\Phi^{1/2} = k\dot{\Phi}\Phi - 2E_0[1 - q(t)]\Phi + \frac{1}{2}\dot{\Phi}^2.$$

Hence

$$\frac{d}{dt}\left(2Q\Phi^{1/2}\right) = \ddot{\Phi}\dot{\Phi} + k\ddot{\Phi}\Phi + 2k\dot{\Phi}^2 - 2E_0[1 - q(t)]\dot{\Phi} - 2E_0\Phi\dot{q}.$$

Let us catty out the integration between the limit t_o and $\bar{t}$ of the last relation where t_o has a fixed value and $\bar{t} \to t_o$. We take into account that

$$\lim_{t-t_0} \Phi^{1/2} = 0,$$

$$\mu_0 = \frac{1}{4}\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}} < \infty.$$

Then we write

$$2Q\Phi^{1/2} = \int_{t_0}^{\bar{t}} \left\{ \left[\ddot{\Phi} - 2E_0(1 - q(t)) + 2k\dot{\Phi}\right]\dot{\Phi} + \left[2k\ddot{\Phi} - 2E_0\dot{q}\right]\Phi \right\} dt.$$

As shown above, the derivative Φ retains its sign in the sufficiently small neighborhood of point t_o. Since $\Phi \geq 0$ and $q > 0$, the positive constant μ in the inequality (6.26) will be such that in the sufficiently small neighborhood of t_o we have

$$2|Q|\Phi^{1/2} \geq \int_{t_0}^{\bar{t}} \left\{ \frac{\mu}{\Phi^{1/2}}\dot{\Phi} + \left[2k\ddot{\Phi} - 2E_0\dot{q}\right]\Phi \right\} dt.$$

The first integral to the right of this inequality being equal to $2\mu\Phi^{1/2}$, and $\Phi^{1/2} \to 0$ with $t \to t_o$, then, in the sufficient small neighborhood of t_o, we have

$$2|Q|\Phi^{1/2} \geq 2\mu\Phi^{1/2} \quad \text{or} \quad |Q| \geq \mu.$$

Since $\mu > 0$, and taking into account the existence of the time limit (6.32), we have finished the proof of correctness of the inequality

$$\lim_{t \to t_0} \left(\frac{\dot{\Phi}}{\Phi^{1/2}} \right) > 0.$$

The above analysis allows us to obtain the following asymptotic relations for the Jacobi function when $t \to t_o$.

Since the limit

$$\mu_0 = \frac{1}{4} \lim_{t \to t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}$$

has a non-zero value, the function $\Phi = \Phi(t) > 0$ tends to zero as $t \to t_o$ in such a way that, in neighborhood of t_o, it is proportional to $(t - t_o.)^{4/3}$ with a coefficient of proportionality of $((9/4)\mu_o)^{2/3}$, and one can differentiate this asymptotic relation with respect to t. Hence the following asymptotic relations are satisfied:

$$\Phi \propto \left(3/2\mu_o^{1/2}\right)^{4/3}(t - t_0)^{4/3}, \tag{6.33}$$

$$\Phi \propto \left(12\mu_o^2\right)^{1/3}(t - t_0)^{1/3}. \tag{6.34}$$

In fact (6.34) follows from (6.33) not only from groundless differentiation, but actually from (6.33), if (6.32) is taken into account. The asymptotic relation (6.33) itself follows from (6.32), if we write the last relation in the form

$$\pm\frac{dt}{d\Phi}\propto\frac{1}{2}\ \mu_0^{-1/2}\Phi^{-1/4}$$

and then integrate it between the limits $\Phi = 0$ and $\Phi > 0$, but sufficiently close to $\Phi = 0$. Integration (but not differentiation) of such an asymptotic relation is always as allowed procedure and hence the asymptotic relations (6.33) and (6.34) are satisfied.

Let us show that besides (6.32), (6.33) and (6.34), the following asymptotic relations are also available:

$$\mu_0 = \pm\lim_{t\to t_0}\Phi^{1/2}\ddot{\Phi}, \tag{6.35}$$

$$\Phi\propto\left(2/3\mu_o^{1/2}\right)^{2/3}(t-t_0)^{-2/3}. \tag{6.36}$$

To prove relation (6.35), we multiply (6.27) by $\Phi^{1/2}$. Assuming for $t \to t_o$ and $\Phi^{1/2} \to 0$, $|E_0|(1 + q(t)) < \infty$, $M \equiv 0$ and using (6.32), we find that the lower limit $\underline{\lim}\,\Phi^{1/2}\ddot{\Phi} \geq \mu_0$. Since (6.35) is equivalent to (6.36), this asymptotic relation will be proved, if the upper limit $\overline{\lim}\,\Phi^{1/2}\ddot{\Phi} \leq \mu_0$.

For the proof we assume $F = \left(\dot{\Phi}\right)^3$, so that

$$\ddot{F} = 6\dot{\Phi}\ddot{\Phi}^2 + 3\dot{\Phi}^2\dddot{\Phi}.$$

Then, with the aid of (6.30)

$$\left|\dddot{\Phi} - 2E_0\dot{q} + k\ddot{\Phi}\right| \leq \text{const}\left(|\ddot{\Phi}| + k|\dot{\Phi}|\right)^{5/2}$$

and expressing $\dot{\Phi}$ and $\ddot{\Phi}$ through the function $\dot{F} = \Phi^3$ and $\dot{F} = 3\dot{\Phi}^2\ddot{\Phi}$, we find

$$\left|\ddot{F} + 6\dot{q}(t)F^{2/3}\right| < \text{const}\frac{\dot{F}^2 + \left(|\dot{F}|\right)^{5/2}}{|F|}.$$

On the right-hand side of this inequality, we find from (6.34) where $\dot{\Phi} = F^{1/3}$ that for $t \to t_o$

$$\left|\ddot{F} + 6\dot{q}(t)F^{2/3}\right| < \text{const}\frac{\dot{F}^2 + \left(|\dot{F}|\right)^{5/2}}{t - t_0}. \tag{6.37}$$

Finally, if v_o is a positive constant equal to $(12\mu_0)^2$, then for $t \to t_o$

$$F \propto v_0(t \to t_0), \tag{6.38}$$

$$\underline{\lim}\dot{F} \geq v_0. \tag{6.39}$$

In fact, $F = \dot{\Phi}^3$ then (6.38) is equivalent to (6.34). At the same time, by virtue of the relation $v_o = (12\mu_0)^2$, $F = \dot{\Phi}^3$, $\dot{F} = 3\dot{\Phi}^2\ddot{\Phi}$ and (6.32), the inequality (6.39) is another form of the inequality $\underline{\lim}\ \Phi^{1/2}\ddot{\Phi} \geq \mu_0$ which we have already proved. Therefore, we are bound to prove the inequality which can be written in the form $\overline{\lim}\,\dot{\Phi} \leq \mu_0$ by analogy with (6.39). Hence we must prove that the asymptotic relations (6.38) and (6.39) with the aid of the 'Trauberian condition' (6.37), yields the inequality $\lim \dot{F} \leq v_o$ which denotes that $F \to v_o$. From this inequality and from (6.39) the existence of the succession of time intervals follows:

$$t_1^I < t < t_1^{II}, \ldots, t_k^I < t < {}_k^{II}$$

which tends to t_0 as $k \to \infty$ in such a way that whenever $t_k^I < t < {}_k^{II}$

$$0 < \upsilon_0 < p = \dot{F}\left(t_k^I\right) < \dot{F}(t) < \dot{F}\left(t_k^{II}\right) < q \tag{6.40}$$

where p and q are some fixed numbers which are chosen between the limits $\underline{\lim}\dot{F}$, $\overline{\lim}\,\dot{F}$ $(\leq \infty)$ of the conditions function $\lim \dot{F}(t)$. It is obvious that we can assume that $t_o = 0$. If we accept const = const $(p^2 + p^{5/2})$, then for any t in any of the time intervals $t_k^I < t < {}_k^{II}$, by virtue of (6.37) and (6.40), we find that the following inequality holds:

$$\left|\ddot{F}(t) + 6\dot{q}(t)F^{2/3}(t)\right| < \frac{\text{const}}{|t|}.$$

Since t tends to $t_o = 0$, increasing or decreasing, all t_k^I and t_k^{II} lie on the same side of $t_o = 0$. Integration of the inequality (6.40) between the limits t_k^I and t_k^{II} yields

$$\left|\dot{F}\left(t_k^{II}\right) - \dot{F}\left(t_k^I\right) + \int_{t_k^I}^{t_k^{II}} 6\dot{q}(t)F^{2/3}(t)dt\right| < \text{const}\ \log\left|\frac{t_k^{II}}{t_k^I}\right|.$$

By virtue of (6.40) the difference $\dot{F}\left(t_k^{II}\right) - \dot{F}\left(t_k^I\right)$ is equal to a positive constant $p - q$ and

$$\int_{t_k^I}^{t_k^{II}} 6\dot{q}(t)F^{2/3}(t)dt > 0.$$

Hence the limit $\log|t_k^{II}/t_k^{I}|$, as $k \to \infty$, is greater than a certain positive number. For this reason, when $k \to \infty$, there exists a certain positive number λ which satisfies the relation

$$\frac{t_k^{II}}{t_k^{I}} > \lambda > 0. \tag{6.41}$$

Then with the aid of (6.38) it follows that

$$\frac{|F(t_k^{I})|}{|t_k^{I}|} \to v_0 \quad \text{and} \quad \frac{|F(t_k^{II})|}{|t_k^{II}|} \to v_0$$

since

$$t_k^{I} \to t_0, \quad t_k^{II} \to t_0, \quad t_0 = 0, \quad v = 0.$$

On the other hand, if k is sufficiently large, the following inequality is valid:

$$\frac{|F(t_k^{II})|}{|t_k^{II}|}\left|\frac{|t_k^{II}|}{t_k^{I}}\right| - \frac{|F(t_k^{I})|}{|t_k^{II}|}\left|\frac{|t_k^{II}|}{F|t_k^{II}|}\right| > p\left|\frac{|t_k^{II}|}{|t_k^{I}|} - 1\right|. \tag{6.42}$$

In fact, all t_k^{I} and t_k^{II} lie on the same side of t_o, and then

$$\left||t_k^{II}| - |t_k^{I}|\right| = t_k^{II} - t_k^{I}.$$

Since $t_k^{I} \to t_0$ and $t_k^{II} \to t_0$, then for sufficiently large k all $F(t_k^{I})$, $F(t_k^{II})$ have the same sign. Hence, (6.42) can be written in the form

$$\left||F(t_k^{II})| - |F(t_k^{I})|\right| > p\left||t_k^{II}| - |t_k^{I}|\right|$$

and is equivalent to the inequality

$$|F(t_k^{II}) - F(t_k^{I})| > p|t_k^{II} - t_k^{I}|.$$

The validity of the last inequality is obvious, since by virtue of (6.40) for $t_k^{I} < t < {}_k^{II}$ we have $\dot{F}(t) > p > o$. Therefore, inequality (6.42) also holds.

From (6.42) in the limit $k \to \infty$ and with the aid of (6.41) where $\nu_o > 0$, we obtain the following inequality:

$$\nu_0|\lambda - \nu_0\nu_0^{-1}| = p|\lambda - 1|.$$

Finally, by virtue of (6.41)

$$|\lambda - \nu_0\nu_0^{-1}| = |\lambda - 1| > 0,$$

and hence $\nu_o \geq p$. On the other hand, by virtue of (6.40) $p \geq \nu_o$. The observed contradiction that the supposition we made at the beginning ($\overline{\lim}\,\dot{F} >> \nu_0$) is false. Thus we have proved the validity of the increase inequality $\overline{\lim}\,\dot{F} \leq \nu_0$ and this completes the proof of the relations (6.35) and (6.36).

Let us now show that if the motion of n points with masses m_i in the time limit $t \to t_o$ produces their simultaneous collision, then the configuration of these n particles tends to central configuration (6.15) as $t \to t_o$. In the proof, we shell use the asymptotic relations (6.33), (6.34) and (6.36) and the Tauberian lemma, which states that if the function g(u) has continuous derivatives $\dot{g}(u)$ and $\ddot{g}(u)$ for $u \to \infty$ and tends, as $u \to \infty$, to a finite limit and $\ddot{g}(u)$ <const, then $\dot{g}(u) \to 0$.

There is no loss of generality in assuming that $t \to t_o \to 0$, so that $t \to t_o > 0$. Then the asymptotic relations (6.33), (6.34) and (6.36) are simply equivalent to

$$t^{-4/3}\Phi \to \ \mu_1 > 0, \tag{6.43}$$

$$t\left(t^{-4\dot{/}3}\Phi\right) \to 0, \tag{6.44}$$

$$t\left(t^{-4\ddot{/}3}\Phi\right) \to 0 \tag{6.45}$$

where

$$\mu_1 = \left(\frac{3}{2}\ \mu_0^{1/2}\right)^{4/3} \quad \text{and} \quad t \to 0.$$

Since

$$\Phi = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2,$$

it follows from (6.43) that the time limit $t \to 0$ all n mass particles collide at the origin of the barycentric co-ordinate system OXYZ in such a way that, for sufficiently small t, the linear dimensions of the configuration will be proportional to $t^{2/3}$. For this reason we eliminate this factor $t^{2/3}$ simply by multiplying the unit of length by the factor $t^{-2/3}$. Then we consider, instead of the values:

$$r_i, \Delta_{ij} = \left|r_i - r_j\right|,$$

$$\Phi = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2, \tag{6.46}$$

$$U = -G\sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\Delta_{ij}}, \tag{6.47}$$

the corresponding values:

$$\bar{r} = t^{-2/32} r_i, \quad \bar{\Delta}_{ij} = \left|\bar{r}_i - \bar{r}_j\right| = t^{-2/3}\Delta_{ij},$$

$$\bar{\Phi} = t^{-4/3}\Phi = \frac{1}{2}\sum_{i=1}^{n} m_i \bar{r}_i^2,$$

$$U = t^{2/3}U = -G\sum_{1\leq i<j\leq n}\frac{m_i m_j}{\Delta_{ij}}.$$

The procedure is permissible since the definition of the central configuration is an invariant relative scale transformation of all the co-ordinates $r_i \rightarrow \delta r_i$ where δ is an arbitrary non-zero factor. Then, the relation (6.15) is invalid for the fixed $t \neq 0$, but

$$\left(\bar{\Phi}\,\bar{U}^2\right)_{\bar{r}_i} = 0 \tag{6.48}$$

where $I = 1, 2, \ldots, n$ in the same limit $t \rightarrow 0$.

The proof of this theorem, the mathematically precise formulation of which is expressed by (6.48), has several stages.

First, we show that in the time limit $t \rightarrow 0$

$$\frac{4}{9}\bar{\Phi} + U \rightarrow 0 \tag{6.49}$$

and

$$\bar{\Delta}_{ij} > \text{const} > 0. \tag{6.50}$$

Let us introduce a time transformation, changing t to $\bar{t} = -\ln t$ in such a way to have $\bar{t} \rightarrow \infty$ for $t \rightarrow 0$. Let this transformation be

$$t = e^{-\bar{t}}. \tag{6.51}$$

Then, if the arbitrary function f depends on time t, we have

$$t\frac{df}{dt} = -\frac{df}{dt}, \tag{6.52}$$

$$t^2\frac{d^2f}{dt^2} = \frac{d^2f}{dt^2} + \frac{df}{dt}. \tag{6.53}$$

With the aid of (6.51)–(6.53) we rewrite the equation of motion

$$m_i \ddot{r}_i = -U_\eta$$

in the form

$$m_i\left(\ddot{\bar{r}}_i - \frac{1}{3}\dot{\bar{r}}_i - \frac{2}{9}\bar{r}_i\right) = -\overline{U}_{r_i} - k\dot{\bar{r}}_i \tag{6.54}$$

where derivatives are written with respect to $\bar{t}$ and $\overline{U}_{\bar{r}_i} = t^{4/3}U_{\bar{r}_i}$.

Similarly, let us rewrite the energy conservation law and Jacobi's virial equation in the form

$$\frac{1}{2}\sum_{i=1}^{n} m_i\left(\dot{\bar{r}}_i - \frac{2}{3}\bar{r}_i\right)^2 + \overline{U} = E_0[1 + q(t)]e^{-2/3\bar{t}}, \tag{6.55}$$

$$\ddot{\overline{\Phi}} - \frac{5}{3}\dot{\overline{\Phi}} - \frac{4}{9}\overline{\Phi} = -U + 2E_0[1 + q(t)]e^{-2/3\bar{t}}. \tag{6.56}$$

Assuming $f = \Phi$ in (6.52) and (6.53), we obtain relations which are valid in the time limit $\bar{t} \to \infty$ and similar to (6.43)–(6.45) as $t \to 0$:

$$\overline{\Phi} \to \mu_1 > 0, \tag{6.57}$$

$$\dot{\overline{\Phi}} \to 0, \tag{6.58}$$

$$\ddot{\overline{\Phi}} \to 0. \tag{6.59}$$

In the limit $\bar{t} \to \infty$ from (6.56), where $E_0(1 + q(t))$ is finite, with the aid of (6.57)–(6.59), it follows that (6.49) is valid. Moreover, it is obvious from (6.49) to (6.57) that the potential energy U tends to a finite value and hence (6.50) follows from (6.47).

Secondly, let us show that the time limit $\bar{t} \to +\infty$ $(t \to 0)$:

$$\dot{\bar{r}} \to 0, \tag{6.60}$$

$$\ddot{\bar{r}} < \text{const}, \tag{6.61}$$

$$\dddot{\bar{r}} < \text{const}. \tag{6.62}$$

Note that (6.46) yields

$$\dot{\overline{\Phi}} = \sum_{i=1}^{n} m_i \dot{\bar{r}}_i \bar{r}_i. \tag{6.63}$$

Then in the time limit $\bar{t} \to \infty$ and with the aid of (6.49) and (6.63), we obtain

$$\sum_{i=1}^{n} m_i \dot{\bar{r}}_i^2 \to 0$$

which gives (6.60). Furthermore,

$$\bar{r} < \text{const}, \tag{6.64}$$

$$\left|\overline{U}_{r_i}\right| < \text{const}. \tag{6.65}$$

In fact Eq. 6.64 follows from (6.57) by virtue of Eq. 6.46. At the same time, Eq. 6.65 follows from (6.47) to (6.50). Equation 6.56 follows from (6.54), (6.60), (6.64) and (6.65). Finally, by differentiating (6.56) with respect to $\bar{t}$ and then using (6.60) and (6.61), it is easy to see that for the proof of (6.62) it is sufficient to show the boundedness of the second derivatives of the functions $\overline{U}(\bar{r}_1, r_2, \ldots, r_n)$ in the time limit $t \to \infty$. But the boundedness of these derivatives follows obviously from (6.47), (6.50) and (6.64).

Finally, in accordance with (6.60) and (6.62), the Tauberian lemma is valid if we consider the function $g(u) = \dot{\bar{r}}_i$, where $u = \bar{t}$. Hence, not only $\dot{\bar{r}}_i \to 0$, but $\ddot{\bar{r}}_i \to 0$.

It follows therefore from (6.54) that

$$\frac{2}{9} m_i \bar{r}_i - U_{\bar{r}_i} \to 0.$$

Then by virtue of (6.46)

$$\frac{2}{9} \overline{\Phi}_{\bar{\eta}} - U_{\bar{r}_i} \to 0.$$

From the last expression, with the aid of (6.49) and (6.57), it follows that

$$\left(\overline{\Phi}\,\overline{U}^2\right)_{\bar{r}_i} = \overline{\Phi}_{\bar{r}_i} \overline{U}^2_{\bar{r}_i} + 2\bar{\Phi}\overline{U}\overline{U}_{\bar{r}_i} \to 0$$

and therefore

$$\left(\Phi U^2\right)_{\eta} \to 0$$

as $t \to 0$.

The last expression completes the proof of the theorem that an arbitrary non-conservative system tends to central configuration in the asymptotic limit of simultaneous collision of all its particles.

6.3 Asymptotic Limit of Simultaneous Collision of Charged Particles of a System

The following analysis is given for a system consisting of a large number of charged material particles. The particles considered are positively charged nuclei of atoms and electrons.

The objective is to prove the statement that the arbitrary configuration of a system of charged particles interacting according to an inverse law (i.e. gravitational or Coulomb) in the asymptotic time limit of simultaneous collision of all the particles (for $t \rightarrow t_o$) tends to a central configuration.

Using the definition of central configuration (6.15), (Wintner 1941), and assuming its uniqueness, the statement to be proved can be written in the form

$$\lim_{t \rightarrow t_0} \left(|U_\Sigma| \sqrt{\Phi} \right) = \text{const} \tag{6.66}$$

where $U_\Sigma = U + U_c$ is the potential energy of the system, which is equal to the sum of the gravitational potential energy of Coulomb interactions.

Using Wintner's method (Wintner 1941), we have previously studied the asymptotic time limit of (6.66) for conservative and non-conservative systems whose particles are interacting according to the law of gravitation. Since the relationship (6.66) is linear as a function of potential energy, we have to prove it only for Coulomb interactions of system particles. The proof given below for a non-conservative system is also based on Wintner's method, modified for the case of charged particles.

So, for a non-conservative system of n particles interacting according to the Coulomb law, let us write down in an inertial barycentric co-ordinate system the Jacobi function, functions of the potential and kinetic energies as well as the energy conservation law and Jabobi's virial equation as follows:

$$\Phi = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i m_j \Delta_{ij}^2, \tag{6.67}$$

$$T = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i m_j (\dot{r}_i - \dot{r}_j)^2, \tag{6.68}$$

$$U = -G \sum_{1 \le i < j \le n} \frac{q_i q_j}{\Delta_{ij}}, \tag{6.69}$$

$$E = E(t) = E_0 - E_\gamma = T + U_c, \tag{6.70}$$

$$\ddot{\Phi} = 2E(t) - E_c \tag{6.71}$$

where $q_i = eZ_i$ is the charge of i-th particle with mass m_i; $Z_i = -1, 1, +2, \ldots,$ $N \le n$; m is the total mass of the system; $E_\gamma < \infty$; $\dot{E}\gamma < \infty$, i.e. the total energy and the luminosity of the system at any time t are functions, monotonically bounded from above.

The proof of the relationship (6.66) can easily be obtained from the asymptotic expressions for Jacobi function and its first and second derivatives as

$$\Phi \propto (t - t_0)^{4/3}, \tag{6.72}$$

$$\dot{\Phi} \propto (t - t_0)^{1/3}, \tag{6.73}$$

$$\ddot{\Phi} \propto (t - t_0)^{-2/3}. \tag{6.74}$$

where $t \to t_o$, and t_o is the moment of simultaneous collision of the charged particles of the system.

From the expressions (6.72)–(6.74), the limit (6.66), which we are proving, follows from exact repetition of Wintner's arguments (Wintner 1941). However, Eqs. 6.72–6.74 follows from the existence of the limits

$$\lim_{t - t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}} = \mu_0 = \text{const} > 0, \tag{6.75}$$

$$\lim_{t \to t_0} \ddot{\Phi}\Phi^{1/2} = \eta_0 = \text{const} > 0. \tag{6.76}$$

The limits (6.75) and (6.76) may be obtained in future from analysis of the Jacobi function in the neighborhood of t_o, using the auxiliary function

$$Q = -(E - E_\gamma)\Phi^{1/2} + \frac{1}{4}\frac{\dot{\Phi}^2 + M^2}{\Phi^{1/2}}$$

and the three inequalities, correct in the most general case, i.e. not especially in the close neighborhood of the point of simultaneous collision of particles. These inequalities are

$$\left|\ddot{\Phi} + 2E_\gamma\right| \leq \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right)^{5/2}\eta_0, \tag{6.77}$$

$$\left[\ddot{\Phi} - 2(E - E_\gamma)\Phi^{1/2}\right] \geq \eta_0 > 0, \tag{6.78}$$

$$\ddot{\Phi} - E + E\gamma - \frac{\dot{\Phi}^2}{4\Phi} \geq \frac{M^2}{4\Phi} \tag{6.79}$$

where M is the angular moment of the system.

Let us prove inequalities (6.77)–(6.79) for a system of particles interacting according to Coulomb law.

To prove the inequality (6.77), it is essential that the absolute value of the total potential energy of the system of particles is less than the absolute value of the energy of mutual interactions of any pair of charged particles, i.e.

$$\frac{q_i q_j}{\Delta_{ij}} \leq |U_c|. \tag{6.80}$$

Since

$$\left|\dot{U}_c\right| = \left|\sum_{1\le i<j\le n} \frac{q_i q_j}{\Delta_{ij}^2}\frac{d}{dt}\Delta_{ij}\right| \le \sum_{1\le i<j\le n} \frac{\left|q_i q_j\right|}{\Delta_{ij}^2}\left|\dot{r}_i - \dot{r}_j\right|,$$

and

$$\frac{1}{\Delta_{ij}} \le \frac{|U_c|}{\left|q_i q_j\right|^2},$$

then

$$\left|\dot{U}_c\right| \le |U_c|^2 \sum_{1\le i<j\le n} \frac{|\dot{r}_i - \dot{r}_j|}{\left|q_i q_j\right|}.$$

Analogously, since

$$m_i m_j\left|\dot{r}_i - \dot{r}_j\right|^2 \le 2mT$$

and

$$m_i \ge \frac{|q_i|}{e}\mu_e,$$

then

$$2mT \ge \frac{\left|q_i q_j\right|}{e^2}\mu_e^2\left|\dot{r}_i - \dot{r}_j\right|^2$$

and therefore

$$\left|\dot{U}_c\right| \le \left|\dot{U}_c\right|^2 T^{1/2}\frac{(2m)^{1/2}}{\mu_e}\sum_{1\le i<j\le n}\frac{1}{\left|q_i q_j\right|^{3/2}}$$

where μ_e is the electron mass.

From Jacobi's equation and the law of conservation of energy it follows that

$$\left|\dot{U}_c\right| = \left|\dddot{\Phi} + 2\dot{E}_\gamma\right|,$$

$$|U_c| \le \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right),$$

$$|T| \le \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right),$$

and finally we obtain the first inequality:

$$\left|\dddot{\Phi} + 2\dot{E}_\gamma\right| \ \le \ \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right)^{5/2}\eta_0,$$

$$\eta_0 = \frac{(2m)^{1/2}}{\mu_c} e \sum_{1 \le i < j \le n} \frac{1}{\left(q_i q_j\right)^{3/2}} > 0.$$

The second inequality (6.78) may be derived from Jacobi's equation:

$$\ddot{\Phi} - 2\left(E - E_\gamma\right) = -U_c = -\sum_{1 \le i < j \le n} \frac{q_i q_j}{\Delta_{ij}} = |U_c| \ge \frac{\left|q_i q_j\right|}{\Delta_{ij}}$$

and the inequality following from the definition of the Jacobi function:

$$2m\Phi \ge m_i m_j \Delta_{ij},$$

$$\frac{1}{\Delta_{ij}} \ge \frac{\left(m_i m_j\right)^{1/2}}{(2m)^{1/2}\Phi^{1/2}}.$$

Thus, finally we have

$$\left[\ddot{\Phi} - 2\left(E - E_\gamma\right)\right]\Phi^{1/2} \ge \frac{\left|q_i q_j\right|\left(m_i m_j\right)^{1/2}}{(2m)^{1/2}} = \mu_0 > 0.$$

The derivation of the third inequality (6.79) is based on the Cauchy-Bunjakowski inequality:

$$\left(\sum_{1 \le i \le n}^{n} a_i b_i\right)^2 \ \le \ \left(\sum_{1 \le i \le n}^{n} a_i^2\right)\left(\sum_{1 \le i \le n}^{n} b_i^2\right).$$

Substituting into it

$$a_i = m_i^{1/2}|r_i|, \quad b_i = m_i^{1/2}\frac{d}{dt}|r_i|$$

we have

$$\dot{\Phi} = \sum_{1 \le i \le n} m_i |r_i| \frac{d}{dt}|r_i|,$$

$$\left(\dot{\Phi}\right)^2 \le 2\Phi \sum_{1\le i\le n} m_i \frac{\left(r_i \frac{d}{dt} r_i\right)^2}{|r_i|^2}.$$

Substituting as before

$$a_i = m_i^{1/2}|r_i|, \quad b_i = m_i^{1/2}\frac{[r_i\dot{r}_i]}{|r_i|},$$

we obtain

$$M^2 \le 2\Phi \sum_{1\le i\le n} \frac{m_i|r_i\dot{r}_i|^2}{|r_i|^2}$$

where M is the angular momentum of the system equal to

$$M = \sum_{1\le i\le n} m_i[r_i\dot{r}_i].$$

Summing up the two inequalities just obtained, we have

$$\left(\dot{\Phi}\right)^2 + M^2 \le 2\Phi \sum_{1\le i\le n} \frac{m_i}{|r_i|^2}\left\{(r_i\dot{r}_i)^2 + [r_i\dot{r}_i]^2\right\} = 2\Phi \sum_{1\le i\le n} m_i(\dot{r}_i)^2 = 4T\Phi$$
$$= 4\Phi\left[\ddot{\Phi} - \left(E - E_\gamma\right)\right].$$

We finally obtain an expression for the third inequality (6.79):

$$\ddot{\Phi} - E + E_\gamma - \frac{\dot{\Phi}^2}{4\Phi} \ge \frac{M^2}{4\Phi}.$$

This ends the proof of the expression (6.66) for the Coulomb interactions of charged particles of the system in the asymptotic time limit of their simultaneous collision.

6.4 Relationship Between Jacobi Function and Potential Energy for a System with High Symmetry

If the value

$$|U|\sqrt{\Phi} = B \tag{6.81}$$

does not change for different mass density distribution laws and configurations of the system, the problem of its dynamics would be solved in the framework of an integral (volumetric) approach and a dynamic equilibrium model. In this case, using Jacobi's virial equation,

$$\ddot{\Phi} = 2E - U \tag{6.82}$$

we transfer it into the equation of virial oscillations:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} \tag{6.83}$$

and obtain three different forms of solutions: elliptic, parabolic or hyperbolic. The form of the solution depends on the total energy which can respectively be negative, equal to zero or positive. The problem is solved with respect to the changes in the time of the simplest moment of distribution of mass density $\rho(\bar{x})$ of the system.

However, even for a system with spherical symmetry and fixed mass, the value of (6.81) changes for different laws of distribution of the mass density ρ(r) (where r is the radius of the shell with density ρ(r); $r \in [0,R]$). In this connection, transformation of Eq. 6.82 into 6.83 is possible only after special study, which is described below.

We pay special attention to the systems with high symmetry, namely spherical and elliptical. This is because most of the natural systems from galaxies to atoms possess such a symmetry. We consider below the conditions which allow us to transform Eq. 6.82 into 6.83 for systems with spherical and elliptical symmetry.

6.4.1 Systems with Spherical Symmetry

Let us begin by consideration the value of Eq. 6.81 for a spherical system. It is convenient to start such a study after rewriting the expressions for the Jacobi function and the potential energy in the form

$$\Phi = \frac{1}{2}\beta^2 mR^2, \tag{6.84}$$

$$U = -\alpha^2 \frac{Gm^2}{R}, \tag{6.85}$$

where α^2 and β^2 are dimensionless form-factors independent of radius R and mass m of the spherical system (See Sect. 2.6).

We now rewrite (6.81), using (6.84) and (6.85), as

$$B = \alpha^2 \beta G m^{5/2}. \tag{6.86}$$

Use of form factors α2 and β2 allows us to show that the parameter B in (6.81) does not depend on radius of the spherical system. The product of α2 and β depends on mass density distribution law ρ(r) and does not depend on the total mass of the system. Hence the problem of the study of the changes of parameter B in (6.81) for an arbitrary spherical system is reduced to consideration of the dependence of the product of the α2 and β form factors on the mass density distribution law for the sphere with radius unity and mass unity. Let us consider such a sphere and calculate the value

$$a = \alpha^2 \beta \tag{6.87}$$

For the arbitrarily given law of density distribution ρ(k), k ∈ [0, 1], satisfying the condition

$$\int_{(V)} \rho(k) dV(k) = 1.$$

The volume of the sphere with radius unity is

$$V = \iiint_{(V)} dx, dy, dz = \int_0^1 k^2 dk \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{4}{3}\pi.$$

The volume of the sphere with radius k is

$$V(k) = \int_0^k k'^2 dk' \int_o^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{4}{3}\pi k^3. \tag{6.88}$$

The volume of the spherical shell with radius k and thickness dk is

$$dV(k) = k^2 dk \int_o^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi k^2 dk. \tag{6.89}$$

The mass of the spherical shell with radius k and thickness dk is

$$dm(k) = 4\pi\rho(k)k^2 dk.$$

The mass of the sphere with radius k is

$$m(k) = 4\pi \int_0^k \rho\left(k'\right)\left(k'\right)^2 dk'. \tag{6.90}$$

The mass of the sphere as a whole is

$$m = 4\pi \int_0^1 \rho(k)k^2 dk = 1. \tag{6.91}$$

The polar moment of inertia of the shell with radius k and thickness dk is

$$dI(k) = k^2 dm(k) = 4\pi\rho(k)k^4 dk.$$

The Jacobi function of the sphere is

$$\Phi = \frac{4\pi}{2} \int_0^1 \rho(k)k^4 dk. \tag{6.92}$$

We can write the expression for the form factor β from (6.84) using (6.91) and (6.92):

$$\beta = \sqrt{\frac{\Phi}{\frac{1}{2}m}} = \frac{\sqrt{\int_0^1 \rho(k)k^4 dk}}{\int_o^1 \rho(k)k^2 dk}. \tag{6.93}$$

The potential energy of the shell with radius k and thickness dk in the gravitational field of the sphere of radius k is

$$dU(k) = -G\frac{m(k)dm(k)}{k} = -G\frac{16\pi^2\rho(k)k^2 dk \int_o^k \rho(k')(k')^2 dk'}{k}.$$

The potential energy of the sphere as a whole is

$$U = -16\,\pi^2 G \int_0^1 \rho(k)k dk \int_0^k \rho(k')(k')^2 dk'. \tag{6.94}$$

We can write the expression for the form factor α using (6.85), (6.91) and (6.94) as

$$\alpha^2 = -\frac{U}{Gm^2} \frac{\int_0^1 \rho(k)k dk \int_o^k \rho(k')(k')^2 dk'}{\left(\int_o^1 \rho(k)k^2 dk\right)^2}. \tag{6.95}$$

Finally, the product of form factors α^2 and β represents the functional of the function of mass density distribution $\rho(k)$:

$$a = \alpha^2\beta = \frac{\int_0^1 k\rho(k)dk \int_o^k \rho(k')(k')^2 dk' \sqrt{\int_0^1 \rho(k)k^4 dk}}{\left(\int_o^1 \rho(k)k^2 dk\right)^{5/2}}. \tag{6.96}$$

The values of the form factors α^2 and β^2 and of their product $\alpha^2\beta$ for different formal laws of mass density distribution are given in Table 6.1. The numerical calculations of this table can be found in our paper (Ferronsky et al. 1978).

It can be seen from Table 6.1 that the form factor β changes from 0 to 1: $\beta \in [0, 1]$. It reaches the value of unity in the case when the entire mass of the sphere is distributed within its outer shell (at $k = 1$). The minimal value of the form factor β must be when the entire mass concentrates in the centre of the sphere (at $k = 0$). But if we do not place any strong restrictions on the function $\rho(k)$, i.e. in the general case, nothing can be said about the changing interval of the value $a = \alpha^2\beta$ (6.85). It is only possible to note that $a = \alpha^2\beta$ always has a positive value. From Table 6.1 it can also be assumed that the value of a is more then $(3/5)^{3/2} \approx 0.46$, which corresponds to the homogeneous distribution of the mass density within the sphere. It is known also from Chap. 4 that the homogeneous sphere, while contracting under gravitational forces, conserves its homogeneity up to the moment of simultaneous collision of all its particles.

The sphere expands and then (the time is reversible in classical physics) becomes homogeneous again. So, in accordance with the definitions given in the previous section, the homogeneous sphere appears to be the central configuration. Applying the main idea of the central configuration theorem discussed above in the general case, we assume the following qualitative picture of the evolution of a heterogeneous spherical system. During the contraction of the system the $\alpha^2\beta$ decreases and tends to the quantity $(3/5)^{3/2}$, reaching this value at the moment

Table 6.1 Numerical values of form factors α and β and their product $\alpha^2\beta$ for various formal laws of radial mass density distribution of the spherical system

Law of mass density distribution $\rho(k)$, $k \in [0, 1]$	α^2	$\beta^2\perp$	β^2	$\alpha^2\beta$
$\rho(r) = \rho_0$	0.6	0.4	0.6	0.46
$\rho(r) = \rho_0(1-k)$	0.728	0.27	0.4	0.47
$\rho(r) = \rho_0(1-k^2)$	0.7142	0.29	0.42	0.46
$\rho(r) = \rho_0(1-k)^n$	$\frac{(5T+8)(T+3)^2}{8(2n+3)(2n+5)}$	$\frac{8}{(n+4)(n+5)}$	$\frac{12}{(n+4)(n+5)}$	At $n \to \infty$, 0.54
$\rho(r) = \rho_0 k^n$	$\frac{n+3}{2n+5}$	$\frac{2n+9}{6n+15}$	$\frac{n+3}{2n+5}0$	At $n \to \infty$,0,5
$\rho(r) = \rho_0\,\delta(1-k)$	0.5	0.67	1.0	0.5

of simultaneous collision of all the particles. If the expansion starts before the moment of simultaneous collision of the matter (at the neighborhood of singularity), then the value of $\alpha^2\beta$ again increases. Thus, there is a case of perturbed virial oscillations of the system. This case is known in the literature as 'stormy relaxation' of a gaseous sphere and is described quantitatively by the following equation of change of value of $|U|\sqrt{\Phi}$ (Ferronsky 1984):

$$U\sqrt{\Phi} = B - k\Phi$$

where $B = \text{const}$, and k is also constant.

This law of change of value of $|U|\sqrt{\Phi}$ will be considered in detail in Chap. 8, which is devoted to astrophysics applications. Here we only note that mechanism that drives the matter of a system towards simultaneous collision is the loss of energy through radiation. So, for conservative systems, the equation of virial oscillations has the form:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} - \frac{k\dot{\Phi}}{\sqrt{\Phi}}.$$

The term $k\dot{\Phi}/\sqrt{\Phi}$ is part of the perturbation function. It does not lead to the loss of total energy of the system, and we can call it internal friction.

6.4.2 *Polytropic Gas Sphere Model*

The laws of mass density distribution in the previous section were considered formally, neglecting the requirement of hydrodynamic stability of the system. However, it is well known that for the many really existing celestial gas bodies, a polytropic model in the central domain, is a good one.

Let us study the value of the form factors α and β and their product $\alpha^2\beta$ for the polytropic gas sphere model at various quantities of polytropic index. The equation of state for a gas sphere is

$$\frac{dp(k)}{dk} = -G\frac{m(k)\rho(k)}{k^2}, \tag{6.97}$$

where p(k) is the gas pressure; $\rho(k)$ is the mass density of the gas, and G is the gravitational constant.

Using Eq. 6.97 we can rewrite it for the sphere with radius k and mass m in the form

$$\frac{1}{k}\frac{d}{dk}\left|\frac{k^2}{\rho(k)}\frac{dp(k)}{dk}\right| = -4\pi G\rho(k). \tag{6.98}$$

This is one of the basic equations in the theory of the internal structure of the stars used up to now.

It is assumed that for polytropic models, the two independent characteristics in Eq. 6.98, namely pressure p(k) and mass density ρ(k), are linked by the relationship

$$p(k) = C\rho^{b}(k), \tag{6.99}$$

where C and b are constants.

From (6.99) it follows that

$$\frac{1}{\rho(k)}\frac{dp(k)}{dk} = C\frac{b}{b-1}\frac{d\rho^{b-1}(k)}{dk}. \tag{6.100}$$

Substituting (6.100) into (6.98) and introducing specification

$$\rho^{b-1}(k) = u(k), \quad n = \frac{1}{b-1} \tag{6.101}$$

we obtain

$$C(1+n)\frac{1}{k^2}\frac{d}{dk}\left|k^2\frac{du(k)}{dk}\right| = 4\pi G u^n(k). \tag{6.102}$$

Equation 6.102 can be simplified if dimensionless variables $\Theta(x) = u(x)/u_o$ and $x = \lambda k$ are introduced. Here u_o is the value u(k) in the center of the sphere, i.e. at $k = 0$. The coefficient λ is selected with the condition that, after substitution of the function Θ(x) into (6.102), all the constants should be cancelled. Then the following relationship for λ can be obtained:

$$C(1+n)\lambda^2 = 4\pi G u_0^{n-1} \tag{6.103}$$

and Eq. 6.102, known as the Emden equation, takes the form:

$$\frac{1}{x^2}\frac{d}{dx}\left|x^2\frac{d\Theta(x)}{dx}\right| = -\Theta^n(x). \tag{6.104}$$

It is obvious that for $x = 0$ the function Θ(x), known as the Emden function, should satisfy two conditions:

$$\Theta(x)|_{x=0} = 1, \quad \frac{d\Theta(x)}{dx}|_{x=0} = 0. \tag{6.105}$$

We now obtain the expression for the form factor α^2 for a sphere with polytropic index n. For this purpose we write the expression of potential energy in the form

$$U = -G\int \frac{m(k)dm(k)}{k}.$$

Using Eq. 6.97 for the gas sphere and the expression for dm(k), we rewrite (6.105) as follows:

$$U = \int \frac{k}{\rho(k)}\frac{dp(k)}{dk}dm(k) = 4\pi\int k^3 dp(k). \tag{6.106}$$

After integration by parts of the right-hand side of (6.105) we obtain

$$U = -12\,\pi\int_0^1 k^2 p(k)dk. \tag{6.107}$$

On the other hand, (6.105) can be rewritten in the form

$$U = -\frac{G}{2}\int \frac{dm^2(k)}{k}.$$

Integrating the right-hand side of the last relationship by parts, we obtain

$$U = -\frac{G}{2}\frac{m^2(k)}{k}\Big|_{k=0}^{k=1} - \frac{G}{2}\int \frac{m^2(k)dk}{k^2}. \tag{6.108}$$

The integral in the right-hand side of (6.108) is transformed with the help of (6.97) as follows

$$-\frac{G}{2}\int \frac{m^2(k)dk}{k^2} = \frac{1}{2}\int \frac{m(k)}{\rho(k)}\frac{dp(k)}{dk}dk.$$

Thus, using (6.100), we obtain

$$-\frac{G}{2}\int \frac{m^2(k)dk}{k^2} = \frac{1}{2}\int m(k)C\frac{b}{b-1}d\rho^{b-1}(k)$$

and, integrating by parts, we have

$$-\frac{G}{2}\int\frac{m^2(k)dk}{k^2}=\frac{1}{2}C\frac{b}{b-1}\rho^{b-1}(k)m(k)\Big|_{k=0}^{k=1}-\frac{1}{2}\int C\frac{b}{b-1}\rho^{b-1}(k)4\,\pi k^2\,\rho(k)dk$$

$$=-\frac{1}{2}\int(n+1)4\pi k^2 p(k)dk. \tag{6.109}$$

Substituting (6.109) into (6.108), we obtain the second expression for the potential energy:

$$U=-\frac{G}{2}-\frac{4\pi(n+1)}{2}\int_0^1 k^2\rho(k)dk, \tag{6.110}$$

where the condition $m(1) = 1$ has been taken into account.

Solving the system of Eqs. 6.110 and 6.107 with respect to U, we find that

$$U=-G\frac{3}{5-n}$$

and hence

$$\alpha^2=\frac{3}{5-n}. \tag{6.111}$$

Now we derive the expression for the form factor β. For this purpose we write the Jacobi function expression for a polytropic sphere:

$$\Phi=\frac{4\pi}{2}\int_0^1 k^4\rho(k)dk=\frac{4\pi}{2}\int_0^{x_1}\frac{\Theta^n(x)x^4dx}{\lambda^5}, \tag{6.112}$$

where x_1 is the first root of the equation $\Theta(x) = 0$.

Let us specify

$$\nu=\int_0^{x_1}\Theta^n(x)x^4dx$$

and, taking into account (6.103), we write

$$C(1+n)\lambda^2=4\pi G u_0^{n-1}$$

Then

$$\Phi = \frac{4\pi\nu}{2}\frac{u_0^n}{\lambda^5} = \frac{4\pi\nu}{2}\frac{[C(1+n)]n/n-1}{(4\pi G)n/n-1}\ \lambda^{(5-3n)/n-1}. \tag{6.113}$$

Now we obtain the second expression for the Jacobi function using the condition of Eq. 6.99 at the border surface of the sphere, i.e. at $k = 1$. Then

$$\frac{1}{\rho(k)}\frac{dp(k)}{dk}|_{k=1} = -\frac{Gm(k)}{k^2}|_{k=1} \tag{6.114}$$

and

$$m(k)k^2\Big|_{k=1} = -\frac{k^4}{G}\frac{1}{\rho(k)}\frac{dp(k)}{dk}|_{k=1}.$$

The left-hand side of Eq. 6.114, taking into account (6.100) and (6.101), is

$$\frac{1}{\rho(k)}\frac{dp(k)}{dk}|_{k=1} = C\frac{b}{b-1}\frac{d\rho^{b-1}(k)}{dk} = C(n-1)\frac{du(k)}{dk}. \tag{6.115}$$

Finally we obtain

$$\Phi = \frac{1}{2}\beta^2 m(k)k^2|_{k=1} = -\frac{1}{2}\ \beta^2\frac{C(n+1)}{G}k^4\frac{du(k)}{dk}|_{k=1}$$
$$= -\frac{1}{2}\beta^2\frac{C(n+1)}{G}u_0\frac{x^4}{\lambda^3}\frac{d\Theta(x)}{dk}|_{x=x_1}.$$

Or when using (6.103),

$$\Phi = \frac{1}{2}\pi\beta^2\frac{C(1+n)^{n/n-1}}{(4\ \pi G)^{n/n-1}}\ \lambda^{(5-3n)/n-1}\left|x^4\frac{d\Theta(x)}{dk}\right|\Big|_{x=x_1}. \tag{6.116}$$

Dividing (6.116) by (6.113), we obtain

$$\beta = \sqrt{\frac{\nu}{\left[-x^4\frac{d\Theta(x)}{dx}\right]|_{x=x_1}}}. \tag{6.117}$$

We calculated the values of α^2 and β and their product $\alpha^2\beta$ using the data for ν, x_1, and

$$\frac{-x^2 d\Theta(x)}{dx}|_{x=x_1}$$

Table 6.2 Numerical values of form factors α and β and their product αβ for different values of polytropic index n

Index n	α^2	x_1	$-x^2\frac{B\Theta(x)}{dx}$	ν	β	$\alpha^2\beta$
0	0.6	2.42	4.9	17.63	0.77	0.46
1	0.75	3.14	3.14	12.15	0.62	0.465
1.5	0.87	3.63	2.71	11.12	0.55	0.475
2	1.0	4.35	2.41	10.61	0.48	0.482
3	1.5	6.89	2.01	10.85	0.34	0.502
3.5	2.0	9.53	1.89	11.74	0.26	0.52

at different polytropic index values, taken from Chandrasekhar (1939). The calculated data are shown in Table 6.2. It is interesting to note that in the framework of the really existing physical laws of mass density distribution ρ(k), the quantity $\alpha^2\beta$ changes within the narrow limits despite the fact that each of the form factors α and β varies almost three times more the variation of the polytropic index from 0 to 3,5.

6.4.3 System with Elliptical Symmetry

We have shown in the previous section that the property of the central configurations consisting in the constancy of the product $\alpha^2\beta$ holds for system with spherical symmetry.

Now we prove that this property holds for elliptical symmetry with an ellipsoidal mass distribution. Moreover, we show that among all the configurations only ellipsoidal mass distribution possess this property of central configurations.

Let us write the equation of the general ellipsoid with semi-axes a, b, c:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c_2}=1, \tag{6.118}$$

where x, y, z are the Cartesian co-ordinates of the surface of this ellipsoid.

The equation of a set of similar ellipsoidal shells of this ellipsoid with the ellipsoidal mass distribution ρ(x) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c_2}=k^2, \tag{6.119}$$

where $k \in [0, 1]$ is a parameter of the homogeneous ellipsoidal shell.

The gravitational potential inside this ellipsoidal shell is equal to a constant at an arbitrary point (x, y, z)

$$F(x,y,z) = -\frac{Gm_s}{2}\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \tag{6.120}$$

where m_s is the mass of the shell; u is a parameter of integration.

We write down the form factor α_e of the potential energy U of this ellipsoid as

$$\alpha_e^2 = -\frac{au}{Gm^2}, \tag{6.121}$$

where a is semi-major axis in the equatorial plane; m is total mass

The volume of an ellipsoid bounded by the surface (6.119) with the parameter k is

$$V(k) = \frac{4}{3}\pi abck^3. \tag{6.122}$$

The volume of the thin shell bounded by ellipsoidal surfaces with the parameters k and k + dk is

$$dV(k) = 4\pi abck^2 dk. \tag{6.123}$$

The mass of this shell is expressed as

$$dm_s(k) = 4\pi abck^2 \rho(k)dk. \tag{6.124}$$

Then the total mass of the ellipsoid is

$$m = 4\pi abc \int_0^1 k^2 \rho(k)dk. \tag{6.125}$$

The mass of an ellipsoid bounded by the surface with the parameter k is

$$m(k) = 4\pi abc \int_o^k \left(k'\right)^2 \rho(k')dk'. \tag{6.126}$$

Using the reciprocation theorem (Duboshin 1975), we write the potential energy of the ellipsoid in the form

$$U = -\int_0^1 m(k)dF(k). \tag{6.127}$$

The gravitational potential inside the thin shell bounded by elliptical surface with parameters k and k + dk (6.120) is

$$dF(k) = 2\pi Gabck\rho(k)dk \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \tag{6.128}$$

Now we write the expression for the form factor α_e using the corresponding values of U and m as

$$a_e = -\frac{aU}{Gm^2} = \frac{a}{2}\frac{\int_0^1 k\rho(k)dk\int_0^k (k')^2\rho(k')dk'}{\left[\int_0^1 k^2\rho(k)dk\right]^2}\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$= \alpha\frac{a}{2}\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad (6.129)$$

where α is the potential energy form factor corresponding to the radial mass distribution law $\rho(k)$.

It is easy to see from Eq. 6.129 that when $a = b$ we obtain the value of the form factor α_e for the ellipsoid of rotation

$$\alpha_e^2 = \alpha^2\frac{\arcsin e}{e}. \quad (6.130)$$

Since

$$e = \sqrt{\frac{a^2-c^2}{a^2}} \in |0, 1|$$

then

$$\alpha_e^2 \in \left[\alpha^2, \frac{\pi}{2}\alpha^2\right].$$

When $a > b > c$, Eq. 6.129 be (Janke et al. 1960)

$$\alpha_e^2 = \alpha^2\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$= \alpha^2\frac{a}{\sqrt{a^2-c^2}}F\left(\arcsin\sqrt{\frac{a^2-c^2}{a^2}}, \sqrt{\frac{a^2-b^2}{a^2-c^2}}\right).$$

Denoting

$$\arcsin\sqrt{\frac{a^2-c^2}{a^2}} = \arcsin e_1 = \varphi \quad \text{and} \quad \sqrt{\frac{a^2-b^2}{a^2-c^2}} = \frac{e_2}{e_1} = \sin\ \alpha^2 = f,$$

we obtain

$$\alpha_e^2 = \alpha^2 \frac{F(\varphi, f)}{\sin\varphi}, \tag{6.131}$$

where F(φ, f) is an incomplete elliptical integral of the first degree in the normal Legendre form. If $e_1 < 0.999$ and $0 < e_2 < e_1$, the function $F(\varphi, f) \cdot \sin^{-1}\varphi \in [1.000; 3.999]$ (Janke et al. 1960). When the arguments φ and f increase, the function $F(\varphi, f) \cdot \sin^{-1}\varphi$ also increases continuously.

Let us now consider the form factor β, which may be written

$$\beta = \left[\frac{\Phi}{ma^2}\right]^{1/2}. \tag{6.132}$$

Obviously β can be obtained by corresponding integration over the parameter $k \in [0, 1]$, if one writes the Jacobi function for the homogeneous thin shell bounded by the surfaces within the parameters k and k + dk and with mass distribution ρ(k) in the integrand.

Since the Jacobi function for a homogeneous ellipsoid with mass density ρ_o is

$$\Phi = \frac{2}{15}\pi abc\rho_0\left(a^2+b^2+c^2\right) \tag{6.133}$$

the Jacobi function for a thin ellipsoid shell may be written

$$d\Phi(k) = \frac{2}{3}\pi abc\rho(k)k^4dk\left(a^2+b^2+c^2\right). \tag{6.134}$$

Consequently, the Jacobi function Φ of the ellipsoid is equal to

$$\Phi = \frac{2}{3}\pi abc\left(a^2+b^2+c^2\right)\int_0^1 \rho(k)k^4dk. \tag{6.135}$$

Finally, using (6.135) and (6.125), Eq. 6.132 for the form factor β will be

$$\beta_e = \left[\frac{a^2+b^2+c^2}{3a^2}\,\frac{\int_0^1 \rho(k)k^4dk}{\int_0^1 \rho(k)k^2dk}\right]^{1/2} = \beta\left[\frac{a^2+b^2+c^2}{3a^2}\right]^{1/2}, \tag{6.136}$$

where β is a form factor of the Jacobi function of the system with radial mass distribution ρ(k) and the expression

$$\left[\frac{a^2+b^2+c^2}{3a^2}\right]^{1/2} \in \left[\frac{1}{\sqrt{3}}, 1\right].$$

So the value a_e is equal to

$$a_e = \alpha_e^2\beta_e = a\frac{F(\varphi, f)}{\sin\varphi}\left[\frac{a^2+b^2+c^2}{3a^2}\right]^{1/2}. \quad (6.137)$$

Now it can be shown that the property (6.137) of the product $\alpha^2\beta$ constancy is possessed only by systems with elliptical symmetry and ellipsoidal mass density distribution. This means that for such systems the form factors α and β may be expressed as a product of corresponding for factors of the sphere and terms depending on the form of the boundary surface.

For this proof we consider an arbitrary system with a similar law of mass distribution ρ(k), k ∈ |0, 1| and the boundary surface S. Then, since we consider only one-dimensional ρ(k), mass density will be constant on any surface with a fixed parameter k and similar to S. The area of this surface is

$$S'(k) = Sk^2. \quad (6.138)$$

If the volume of the body is equal to V, then the volume of the part of the body bounded by the surface S′(k) is

$$V'(k) = Vk^3 \quad (6.139)$$

And its mass is

$$m(k) = V\int_0^1 k^2\rho(k)dk. \quad (6.140)$$

Let us introduce the Cartesian co-ordinate system OXYZ with an origin coinciding with the center of similarity. Let us denote by h in the equatorial plane OXY the longest distance from the center of similarity to the boundary and assume that the form factor α_e^2 of the body can be expressed as a product of the form factors of the potential energy α for the radial mass density distribution law and some term Δ(S) depending on the form of the boundary surface

$$\alpha_e = -\frac{Uh}{Gm^2} = \alpha\Delta(S) = \frac{\int_0^1 k\rho(k)dk\int_0^k (k')^2\rho(k')dk'}{\left(\int_0^1 k^2\ \rho(k)dk\right)^2}\Delta(S). \quad (6.141)$$

From Eq. 6.141 we can obtain the potential energy in the for

$$U = -\frac{Gm^2}{h}\alpha\Delta(S) = -\frac{GV}{h}\int_0^1 k\rho(k)dk\Delta(S). \tag{6.142}$$

Since the term G, V, H, Δ(S) do not depend on the parameter k, let us put them into the integrand and denote

$$\frac{GV}{h}k\rho(k)\Delta(S)dk = F(k).$$

Then Eq. 6.142 may be written

$$U = -\int_0^1 m(k)dF(k). \tag{6.143}$$

Comparing Eqs. 6.143 and 6.127, one can see that Eq. 6.143 an equation for the reciprocation theorem, whose validity is based on the constancy of the gravitational potential dF(k) inside the thin shell bounded by the similar and similarly situated surfaces with parameters k and k + dk. But as shown in the work of Dive (1931), where one can find rigorous proof of the reverse Newton theorem, only ellipsoidal shells possess such a property. Therefore the body with the one-dimensional mass distribution law ρ(k) for which the form factor α_e is equal to the product of the form factor of the sphere and some term depending on the form of the boundary surface Δ(S) must satisfy the equation of the ellipsoid (6.118).

6.4.4 System with Charged Particles

We shell now show, with the help of a model solution, that for the Coulomb interactions of charged particles, constituting a system, Eq. 6.5 holds with the same conditions as for the previous models discussed above.

The derivation of the expression for the potential energy of the Coulomb interactions of a celestial body is based on the concept of an atom following, for example, from the Tomas-Fermi model (Flügge 1971). In our problem this approach does not result in limited conclusions since the expression for the potential energy, which we shell write, will be correct within a constant factor.

Let us consider a one-component, ionized quasineutral and gravitating gaseous cloud with spherically symmetrical mass distribution and radius of sphere R. We shell not consider the problem of its stability, assuming that the potential energy of interaction of charged particles is represented by the Coulomb energy. Therefore, in order to prove the relationship (6.5), it is necessary to obtain the energy of the Coulomb interactions of positively charged ions with their electron clouds.

Assume that each ion of the gaseous cloud has mass number A_i and order number Z, and that the function $\rho(r)$ expresses the law of mass distribution inside the gaseous cloud. The mass of the ion will be $A_i\, m_p$ (where $m_p = 1.66 \times 10^{-24}$ g is the mass of the proton) and its total charge will be +Ze (where $e = 4.8 \times 10^{-10}$ GCSE is elementary charge). Then let the total charge of the electron cloud, which is equal to $-Ze$, be distributed around the ion in the spherically symmetrical volume of radius r_i with charge density $q_e(r_e)$, $r_e \in [0, 1]$. Radius r_i of the effective volume of the ion may be expressed through the mass density distribution $\rho(x)$ by

$$\frac{4}{3}\pi r_i^3 = \frac{A_i m_p}{\rho(r)}. \tag{6.144}$$

Then

$$r_i = \sqrt[3]{\frac{3A_i m_p}{4\pi\rho(r)}}. \tag{6.145}$$

Let us calculate the Coulomb energy V_c per ion, using Eq. 6.145. Assuming that the charge distribution law in the effective volume of radius r_i is given, we may write U_c in the form

$$U_c' = U_c^{(+)} + U_c^{(-)}, \tag{6.146}$$

where $U_c^{(-)}$ is the potential energy of the Coulomb repulsion of electrons inside the effective volume of radius r_i;$U_c^{(+)}$ is the potential energy of attraction of the electron cloud to positive ion.

Let us assume that the charge distribution law inside the electron cloud is $q_e(r_e)$. Then normalization of the electron charge of the cloud surrounding the ion may be written

$$-Ze = \int_0^{r_1} 4\pi q_e(r_e) r_e^2 dr_e. \tag{6.147}$$

From the Eq. 6.147 we may obtain the normalization constant q_0, which will depend on the given law of charge distribution, as

$$q_0 = -\frac{Ze}{4\pi \int_0^{r_1} r_e f(r_e) dr_e.} \tag{6.148}$$

Now it is easy to obtain expressions for $U_c^{(-)}$ and $U_c^{(+)}$ in the form

$$U_c^{(-)} = (4\pi)^2 q_0^2 \int_0^{r_1} r_e f(r_e) dr_e \int_0^{r_e} (r'_e)^2 f(r'_e) dr'_e, \tag{6.149}$$

$$U_c^{(+)} = 4\pi Z e q_0 \int_0^{r_1} r_e f(r_e) dr_e. \tag{6.150}$$

Finally, Eq. 6.146 for the potential energy U_c corresponding to one ion may be rewritten using Eqs. 6.148–6.150 in the form

$$U'_c = -e^2 Z \left\{ \frac{\int_0^{r_1} r_e f(r_e) dr_e}{\int_0^{r_1} r_e^2 f(r_e) dr_e} - \frac{\int_0^{r_1} r_e f(r_e) dr_e \int_0^{r_e} (r'_e)^2 f(r'_e) dr'_e}{\left(\int_0^{r_1} r_e^2 f(r_e) dr_e \right)^2} \right\}. \tag{6.151}$$

It is easy to see in the right-hand side of Eq. 6.151 that the expression enclosed in brackets determines the inverse value of some effective diameter of the electron cloud, which may be expressed through the form factor α_i of the ion and the radius r_i, i.e.

$$\frac{\int_0^{r_1} r_e f(r_e) dr_e}{\int_0^{r_1} r_e^2 f(r_e) dr_e} - \frac{\int_0^{r_1} r_e f(r_e) dr_e \int_0^{r_e} (r'_e)^2 f(r'_e) dr'_e}{\left(\int_0^{r_1} r_e^2 f(r_e) dr_e \right)^2} = -\frac{d_i}{r_i}. \tag{6.152}$$

Thus, Eq. 6.151, using (6.152), yields

$$-U'_c = \alpha_i \frac{e^2 Z}{r_i}. \tag{6.153}$$

The numerical values of the form factor α_i, depending on the charge distribution $q_e(r_e)$ inside the electron cloud, are given in Table 6.3; they were calculated in our work (Ferronsky et al. 1981).

Using Eq. 6.153, the total energy of the Coulomb interaction of particles may be written

$$-U_c = 4\pi \int_0^R \frac{\rho(r)}{A_i m_p} U'_c r^2 dr = \frac{3\alpha_i^2 e^2 Z^2}{R} \int_0^R R r^2 \left(\frac{4\pi\rho(r)}{3A_i m_p} \right)^{4/3} dr. \tag{6.154}$$

Table 6.3 Numerical values of form factors α for different radial charge distribution of the electron cloud around the ion

The law of charge distribution[a]	α_i^2
$q_e(r_e) = q_o = \text{const}$	$\frac{9}{10}$
$q_e(r_e) = q_o\left(1 - \frac{r_e}{r_i}\right)$	$\frac{44}{35}$
$q_e(r_e) = q_o\left(1 - \frac{r_e}{r_i}\right)^n$	$\frac{(n+3)(11n^2+41n+36)}{8(2n+3)(2n+5)}$
$q_e(r_e) = q_o\left(\frac{r_e}{r_i}\right)$	$\frac{16}{21}$
$q_e(r_e) = q_0\left(\frac{r_e}{r_i}\right)^n$	$\frac{(n+3)^2}{(n+2)(2n+5)}$
The same for $n \to \infty$	$\alpha_{ii}^2 \to \frac{1}{2}$

[a] Here q_o is the charge value in the center of the sphere; t_e is the parameter of radius, $r_e \in [0, 1]$; n is an arbitrary number, n = 0, 1, 2, ...

Introducing in Eq. 6.154 the form factor of the Coulomb energy α_c^2, depending on the mass distribution in the gaseous cloud and on the charge distribution inside the effective volume of the ion, we obtain

$$-U_c = \alpha_c^2 \frac{e^2 Z}{R}\left(\frac{m}{A_i m_p}\right)^{4/3}, \tag{6.155}$$

where

$$\alpha_c^2 = \frac{3\alpha_c^2 \int_0^R \left[\left(\frac{4\pi}{3}\rho(r)\right)\right]^{4/3} Rr^2 dr}{m^{4/3}},$$

$$m = \sum_{i=1}^{n} m_i = 4\pi \int_0^R r^2 \rho(r) dr,$$

Since the total number of ions N in the gaseous cloud is equal to

$$N = \frac{m}{A_i m_p}$$

and the relation between the radius of the cloud and the radius of the ion may be obtained from the relationship of the corresponding volumes

$$\frac{4}{3}\pi R^3 = N\frac{4}{3}\pi r_i^3,$$

then Eq. 6.155 may be rewritten in the form

$$-U'_c = \alpha_c^2 \frac{N^{4/3} e^2 Z^2}{R} = \alpha_c^2 N \frac{e^2 Z^2}{r_i}. \tag{6.156}$$

Hence, the form factor entering the expression of the potential energy of the Coulomb interaction acquires a simple meaning. It may be represented as a ratio of the average radius of all spherical volumes per ion to the average effective distance between electrons, disposed on some spherical shell of radius r_i, i.e.

$$\alpha_c^2 = \frac{r_i}{2r_{ei}}. \tag{6.157}$$

Let us now examine by means of numerical data the relationship (6.5), assuming different law of mass distribution. The expression for Jacobi function of the system, which we have previously derived (Ferronsky et al. 1978), is

$$\Phi = \frac{4\pi}{3} \int_0^R r^4 \rho(r) dr = \beta^2 m R^2. \tag{6.158}$$

Thus, Eq. 6.5, using (6.156) and (6.158), may be written

$$-U_c \sqrt{\Phi} = \alpha_c^2 N^{4/3} \frac{Z^2 e^2}{R} \sqrt{\frac{\beta^2 m R^2}{2}} = \frac{1}{\sqrt{2}}\ \alpha_c^2\ \beta N^{4/3} m^{1/2} e^2 Z^2. \tag{6.159}$$

Since we have assumed that the mass of the system and its ion composition are constants, examination of Eq. 6.5 will be equivalent to analysis of the product of the form factors α_c and β. Equation 6.5 holds if

$$\alpha_c^2\ \beta = \frac{r_i}{2r_{ei}} \approx \text{const}. \tag{6.160}$$

The results of the numerical calculations of the form factors α_c and β for different mass distribution in the cloud are shown in Table 6.4, and calculations were carried out in our work (Ferronsky et al. 1981). The values of the form factor α_i of the ion, the numerical value of which depends on the choice of charge distribution $q_e(r_e)$, are shown in Table 6.3.

In the Table 6.4 the numerical values of the form facto α_c^2 and the product of the form factors $\alpha_c^2\ \beta$ are given for the case of homogeneous distribution of the electron charge around ion, i.e. when $q_e(r_e) = \text{const}$. From Table 6.4 it follows that for different laws of mass distribution, when the mass increases to the center, the product of form factors α_c^2 and β remains constant, and therefore Eq. 6.5 holds, with the same comments as were made previously.

Table 6.4 Numerical values of the form factors α_c^2 and β product for different laws of radial mass distribution

The law of mass distribution[a]	$\frac{\alpha_c^2}{\alpha_i^2}$	α_c^2 for $\alpha_i^2 = 0.9$[b]	β	$\alpha_c^2\beta$ for $\alpha_i^2 = 0.9$[b]
$\rho(r) = \text{const}$	1	0.9	0.6324	0.5692
$\rho(r) = \rho_0(1 - r/R)$	1.1303	1.0173	0.5163	0.5253
$\rho(r) = \rho_0(1 - r/R)^2$	1.3331	1.1998	0.4364	0.5236
$\rho(r) = \rho_0(1 - r/R)^3$	1.5510	1.3959	0.3779	0.5276
$\rho(r) = \rho_0(1 - r/R)^n$	$\frac{27}{\sqrt[3]{6}}\frac{[(n+1)(n+2)(n+3)]^{4/3}}{(4n+3)(4n+6)(4n+9)}$	$\frac{243}{10\sqrt[3]{6}}\frac{[(n+1)(n+2)(n+3)]^{4/3}}{(4n+3)(4n+6)(4n+9)}$	$\sqrt{\frac{8}{(n+4)(n+5)}}$	for $n \to \infty$, 0.5909
$\rho(r) = \rho_0(r/R)$	1.0159	0.9143	2/3	0.6095
$\rho(r) = \rho_0(r/R)^2$	1.0461	0.9415	0.6900	0.6497
$\rho(r) = \rho_0(r/R)^n$	$\frac{27}{10\sqrt[3]{3}}\frac{(n+3)^{4/3}}{(4n+9)}$	$\frac{81}{100\sqrt[3]{3}}\frac{(n+3)^{4/3}}{(4n+9)}$	$\sqrt{\frac{2n+3}{3n+5}}$	for $n \to \infty, \to \infty$
$\rho(r) = \rho_0 e^{-K(r/R)}$	0.2321 k	0.2089K	$2\sqrt{2}/k$	0.5909
$\rho(r) = \rho_0 e^{-K(r/R)^2}$	$0.5907k^{1/2}$	$0.5316K^{1/2}$	$(1/k)^{1/2}$	0.5316

[a]Here, ρ_0 is the normalization constant; r is the parameter of the radius $r \in [0, R]$; n and k are arbitrary numbers, $n = 0, 1, 2, \ldots$

[b]The value α_i corresponds to the homogeneous charge distribution in the electron cloud, surrounding the ion $q_e(r_e) = \text{constant}$ (see Table 6.3)

From Eq. 6.157 it follows, however, that the form factor of the Coulomb energy α_c becomes infinite, when the volume occupied by the ions tends to zero. Correspondingly, the Coulomb energy in this case will also tend to infinity. In Table 6.4 there are two laws of mass distribution for which the last condition holds. They are $\rho(r) = \rho_o(1 - (r/R)^n)$ for $n \to \infty$. When the particles of the system are gathering at the shell of the finite radius, the energy of the Coulomb interaction tends to infinity whereas the energy of gravitational interaction has a finite value. When the mass distribution is $\rho(r) = \rho_0(1 - (r/R)^n)$, the form factor of gravitational and Coulomb energies are both finite. But the form factors of the Jacobi function of the system in this case tends to zero, a circumstance which provides the constancy of the product of the form factors α_c^2 and β. This difference might play a decisive role in the evolution of the system.

In conclusion, we note that the results of the study on the relationship between the Jacobi function and the potential energy allows us to consider that the transfer from Jacobi's equation (6.1) and (6.2) into the equations of virial oscillations (6.3) and (6.4) is form the point of view of physics justified. This justification has been achieved in the framework of Newton and Coulomb interactions of the particles of the system. At the same time, the observed deviations of the value of parameter B in (6.5) form some constant quantity can be accounted for by small perturbation when studying the evolution of a heterogeneous system.

Chapter 7
Applications in Celestial Mechanics and Geodynamics

In the previous chapters we presented physical and theoretical fundamentals of the unified theory for study of unperturbed and perturbed motion of a self-gravitating celestial body which generates energy by interaction of its constituent particles. The theory is based on the functional relationship between the polar moment of inertia and the energy (potential, kinetic and total) of the natural conservative and dissipative system in the form of Jacobi's virial equation or the generalized virial theorem The remarkable property of Jacobi's virial equation is its ability to be simultaneously both the equation of dynamical equilibrium and equation of motion. As it was shown in Chap. 3, the equation is valid for all the known physical models of the matter interaction describing dynamics of natural systems. The functional relationship between the potential energy and the polar moment of inertia was revealed by analyzing of orbits of the artificial satellites. The potential energy, which is generated by interaction of the elementary mass particles, is the force function of the body, i.e. the active component of its motion. This energy induces the inner and outer force field of the body. The kinetic energy appears to be the reactive component of the force field which is developed in the form of motion of the same body's mass particles and the body as a whole. Thus, Jacobi dynamics of a body is based on its own inner forces, which are developed by interaction of the elementary particle. The induced inner and outer force field of the body is the main recoded dynamical effect of the interacted elementary particles.

Now we would like to apply the obtained results of the general solution of Jacobi's virial equation presented in Chaps. 3, 4 and 5 to study dynamics of the Sun, the Earth and the Moon, which represent celestial bodies of the Solar System, and to obtain some quantitative data concerning the concrete elements of their non-perturbed and perturbed motion. As in celestial mechanics, under non-perturbed motion we understand the motion under action of the body's own force field generated by interaction of its own masses. The perturbation motion of a body is considered to be the effects developed by the outer force fields of the Sun, the Earth and the Moon. Interplanetary perturbations are not considered here because of their indirect effect and this is a specific problem. The study of dynamics of a body in its own force field starts first of all with investigation of the basic modes, namely, the oscillation and rotation of the shells, which are developed by interaction of their own masses. The energy which is emitted from the body's surface forms the outer

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force field of the body. Dynamical equilibrium between the Sun and the Earth and between the Earth and the Moon is achieved by interaction of their outer force fields and guaranties stability of their orbital motion. The effect of differential rotation of shells and changes in the slope of the body axis of rotation is also examined. Herein, the observed rotation of the planet as a rigid body in reality is not proved. The effect of perturbation of the upper shell and the integral effect of rotation of all other shells is discussed as well.

In order to find a quantitative solution of the task it is necessary to have data about the mass, radius, moment of inertia and radial density distribution of the planet's mass. We have sufficiently reliable data about the mass and radius for the Sun, the Earth and the Moon. The reliable mean value of the moment of inertia was found now by artificial satellites only for the Earth. The radial density distribution data appears to be unreliable even for the Earth. This is because the existing methodology of interpretation of seismic data by the Williamson-Adams equation is based on the planet's hydrostatic equilibrium and needs to be reconsidered. That is why we have to find such law of density distribution which satisfies to the experimentally found moment of inertia and the condition of the observed differentiation of masses on the shells with respect to density.

We start the study dynamics of a body with its own oscillations. After that we move to determine the properties of the interacting masses which have not been earlier considered and which are needed. Among them are the structure of the potential and kinetic energy, the nature of the Archimedes' and Coriolis' forces and the electromagnetic component of the body's potential energy. They are the basis for consideration of body's dynamical effects. Finally, we turn to solution of problem of the body shells rotation, the nature of precession, nutation, and the orbit plane dynamics.

7.1 The Problem of Eigenoscillations of a Celestial Body

In order to demonstrate application of the new theory to study of a body dynamics we consider both traditional hydrostatic and proposed dynamic approaches being applied to the Earth. Applying the volumetric forces and volumetric moments we show in this solution that the eigenoscillations of a body is a natural kinetic integral effect of its interacting particles.

7.1.1 Hydrostatic Approach

We consider first the problem of radial oscillations of a gravitating elastic sphere within the framework of the traditional hydrostatic equilibrium approach which is used to study eigenoscillations of a celestial body

The equation of motion of a deformable body in the presence of the mass forces of the outer uniform force field is written in the form (Landau and Lifshitz 1954)

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho F_i = \rho \frac{\partial^2 u}{\partial t^2}, \tag{7.1}$$

where ρF_i is the i-th component of the mass force; u_i is the i-th component of the displacement vector; σ_{ik} is the stress vector; ρ is the mass density of the sphere. We write the vector components of the mass force in the spherical system of co-ordinates:

$$\rho F_0 = 0,$$

$$\rho F_\lambda = 0,$$

$$\rho F_r = -G\,\rho \frac{m(r)}{r^2},$$

where m(r) is the body's mass within the sphere with radius r.

Because of radial deformation of the sphere the only radial component of the displacement vector differs from zero, i.e.

$$u_r = 0,$$

$$u_\theta = 0,$$

$$\frac{\partial^2 u_k}{\partial r^2} = \frac{\partial^2 r}{\partial t^2}.$$

For isotropic media and for small deformations the stress tensor σ_{ik} and the deformation tensor u_{ik} have the linear relationship, according to Hooke's law:

$$\sigma_{ik} = k u_{ik}\ \delta_{ik} + 2\,\mu\left(u_{ik} - \frac{1}{3}\ \delta_{ik} u_{ik}\right) = \lambda u_{ik} + 2\,\mu u_{ik}, \tag{7.2}$$

where k is the displacement modulus; μ is the shear modulus; $\lambda = k - 2/3\ \mu$ is the Lamé constant; $\sigma_{ik} = 0$ at $i \neq k$ and $\sigma_{ik} = 1$ at $i = k$.

In the case of radial deformations, the components of the deformation tensor are equal to

$$u_{rr} = \frac{du_r}{dr},$$

$$u_{\theta\theta} = u_{\phi\phi} \frac{u_r}{r}, \tag{7.3}$$

$$u_{\theta\phi} = u_{\lambda r} = u_{r\theta}.$$

The components of the stress tensor are:

$$\sigma_{rr} = (\lambda + 2\mu)\frac{du_r}{dr} + 2\lambda\frac{u_r}{r},$$
$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \lambda\frac{du_r}{dr} + (2\lambda + \mu)\frac{u_r}{r}, \qquad (7.4)$$
$$\sigma_{\theta\phi} = \sigma_{\phi r} = \sigma_{r\theta} = 0.$$

The general equation (7.1) of motion now takes the form

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}\right) + \rho F_i = \rho\frac{d^2u_r}{dt^2}. \qquad (7.5)$$

Putting Eq. 7.4 into Eq. 7.5, we obtain an equation describing the radial displacement of matter in the sphere:

$$(\lambda + 2\mu)\left(\frac{d^2u_r}{dr^2} + \frac{2}{r}\frac{du_r}{r^2}\right) + \left(\frac{du_r}{r^2} + 2\frac{u_r}{r}\right)\frac{d\lambda}{dr} + 2\frac{d\mu}{dr}\frac{du_r}{dr} + \rho F_r = \rho\frac{d^2u_r}{dt^2}. \qquad (7.6)$$

Equation 7.6 is used to study the problem of the radial oscillations of the body in the traditional hydrostatic approach. This equation is solved at boundary conditions of uniform radial displacement of matter or at uniform pressure over a spherical surface enveloping the body.

The normal stress at the sphere's surface which is formed by the outer layer of the body with radius r is

$$T_r = \overline{\Sigma}\cdot\frac{r}{r}. \qquad (7.7)$$

where $\overline{\Sigma}$ is the stress tensor, the components of which are

$$\left|t_r, t_\theta, t_\varphi\right| = |100|\begin{vmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\varphi\varphi} \end{vmatrix} \qquad (7.8)$$

Here only $t_r = \sigma_{rr} \neq 0$, but tensor T_r is the purely normal stress, which is

$$T_r = \left|(\lambda + 2\mu)\frac{du_r}{dr} + 2\lambda\frac{u_r}{r}\right|\frac{\bar{r}}{r}. \qquad (7.9)$$

Let us consider a uniform sphere with $\lambda = \text{const}$ and $\mu = \text{const}$. Then

$$\frac{d}{dt}\left(\frac{du_r}{dr} + 2\frac{u_r}{r}\right) = \frac{d\theta}{dr} = 0, \qquad (7.10)$$

where $\theta = u_{rr} + u_{\theta\theta} + u_{\varphi\varphi}$ is the dilation of the body.

The general solution of Eq. 7.10 is

$$u_r = Ar + \frac{B}{r^2}. \tag{7.11}$$

Constants A and B can be defined from the following boundary conditions: at the centre of the body ($r = 0$) the displacement $u_r = 0$ and the value $B = 0$; on the surface of the sphere radius $r = a$ and $T_r = -p$.Then

$$\left|(\lambda + 2\mu)\frac{du_r}{dr} + 2\lambda\frac{u_r}{r}\right|_{r=A} = -p \tag{7.12}$$

from which it follows that

$$A = -\frac{p}{3\lambda + 2\mu}.$$

Now the general solution (7.11) takes the form

$$u_r = -\frac{pr}{3\lambda + 2\mu} \tag{7.13}$$

$$\theta = -\frac{3}{3\lambda + 2\mu} = -\frac{p}{\lambda + 2/3\,\mu} = -\frac{p}{k}.$$

Substituting the solution (7.13) into (7.4), we obtain the expression for the components of the stress tensor:

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p. \tag{7.14}$$

It is seen from (7.14) that the value of the stress components is reduced to the constant hydrostatic pressure of the body matter.

To solve the problem of the eigenoscillations of a uniform spherical body, we assume

$$u_r = U(r)e^{i\omega t}, \tag{7.15}$$

where ω is the eigenoscillation frequency of the sphere.

Substituting (7.15) into Eq. 7.6, we obtain

$$(\lambda + 2\mu)\left(\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} - \frac{2U}{r^2}\right) + \rho\,\omega^2 U + \frac{4}{3}\pi r G\rho^2 = 0. \tag{7.16}$$

We introduce the new variable *x*:

$$x = \sqrt{\frac{p}{\lambda + 2\,\mu}}\,\omega r = hr. \tag{7.17}$$

Then Eq. 7.16 can be rewritten as

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} - \frac{2U}{dr} + \frac{x^2}{r^2}U = 0.$$

or

$$\frac{d^2U}{dx^2}\left(\frac{dx}{dr}\right)^2 + \frac{2}{r}\frac{dU}{dx}\left(\frac{dx}{dr}\right) - \frac{2U}{r^2} + \frac{x^2}{r^2}U = 0.$$

Considering that

$$\frac{dx}{dr} = \sqrt{\frac{p}{x + 2\,\mu}}\,\omega = \frac{x}{r},$$

we obtain

$$\frac{d^2U}{dx^2} + \frac{2}{r}\frac{dU}{dx} - \frac{2}{x^2}U = 0. \tag{7.18}$$

Equation 7.18 is known as the Riccati equation. Its solution is

$$U(r) = \frac{d}{dx}\left(\frac{A\sin x + B\cos x}{x}\right) \tag{7.19}$$

The arbitrary integration constants A and B can be found from the boundary conditions. In the centre of the body ($r = 0$, $x = 0$) we have $U(r) = 0$. Hence, $B = 0$. Moreover, the outer surface of the body should be in equilibrium. This means that the surface pressure should be equal to zero:$(\sigma_{rr})|_{r=A} = 0$and

$$(\lambda + 2\,\mu)\frac{dU}{dt}e^{i\omega t} + 2\,\lambda\frac{U}{r}e^{i\omega t} = 0,$$

or

$$(\lambda + 2\,\mu)\frac{d}{dx}\left|\frac{d}{dx}\left(A\frac{\sin x}{x}\right)\right| + 2\frac{\lambda}{r}\frac{d}{dx}\left(A\frac{\sin x}{x}\right) = 0. \tag{7.20}$$

Differentiating this, we have

$$-(\lambda + 2\,\mu)x^2\mathrm{sinx} - 4\,\mu\mathrm{xcosx} + 4\,\mu\mathrm{sinx} = 0. \tag{7.21}$$

After dividing (7.21) by 4μ sin x, we obtain

$$\mathrm{xctgx} = 1 - \frac{\lambda + 2\mu}{4\,\mu}x^2.$$

The value x at the outer surface is equal to ah. Finally, we obtain the equation for determining the eigenoscillation values of the body:

$$\mathrm{ahctgah} = 1 - \frac{\lambda + 2\mu}{4\,\mu}a^2h^2. \tag{7.22}$$

7.1.2 Dynamic Approach

Now we consider the problem of the eigenoscillations of the Earth as a uniform body on the basis of its dynamical equilibrium, i.e. under an action of its own internal force field or within the framework of the dynamic approach. For this purpose we use Eq. 7.5 assuming that the internal pressure in the body is isotropic, i.e.

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = -p.$$

Then we have from (7.5)

$$\rho\frac{\partial^2 r}{\partial t^2} = -\frac{\partial p}{\partial r} + \rho F_r \tag{7.23}$$

Now let us derive Jacobi's virial equation from Eq. 7.23 for the spherical symmetric model of the body. For this purpose we multiply the right-hand sides of Eq. 7.23 by $4\pi r^3 dr$ and integrate it with respect to dr from 0 to R:

$$\int_0^R 4\,\pi r^3\ \rho(r)\frac{\partial^2 r}{\partial t^2}dr = -\int_0^R \frac{\partial p}{\partial r}4\,\pi r^3 dr - 4\,\pi G\int_0^R r\ \rho(r)m(r)dr \tag{7.24}$$

The left-hand side of Eq. 7.24 gives

$$\int_0^R 4\ \pi r^3\ \rho(r)\frac{\partial^2 r}{\partial t^2}dr = \frac{\partial^2}{\partial t^2}\left[\frac{1}{2}\int_0^R 4\ \pi r^4\ \rho(r)dr - \int_0^R r\ \pi r^2\left(\frac{\partial r}{\partial t}\right)^2\ \rho(r)dr\right]$$
$$= \ddot{\Phi} - 2T, \tag{7.25}$$

where Φ is the Jacobi function of the body and T is the kinetic energy of the displacements of the matter.

The first term in the right-hand side of (7.24) is equal to

$$\int_0^R \frac{dp}{dr}\frac{dr}{d\rho} d\rho 4\pi r^3 = -\int_{\rho_0}^{\rho_s} \frac{dp}{d\rho} 4\,\pi r^3 d\rho, \tag{7.26}$$

where ρ_o and ρ_s are the mass densities in the centre and on the surface of the body respectively.

Taking into account that, within the framework of the model of an elastic medium, the system reaches its mechanical equilibrium faster than its thermal equilibrium, we assume that the entropy of the system is equal to the constant value, and therefore we write

$$\frac{dp}{d\rho} = \left(\frac{dp}{d\rho}\right)_s = \frac{k}{\rho} = c_s^2, \tag{7.27}$$

where c_s is the velocity of sound in elastic media and $c_s^2 = v_p^2 - 4/3\, v_g^2$; $v_p = \sqrt{(k + 4/3\mu)/\rho}$ is the velocity of the longitudinal waves in an elastic medium; $v_g = \sqrt{\mu/\rho}$ is the velocity of the transverse waves in the elastic medium.

Finally, Eq. 7.26 can be rewritten in the form

$$-\int_0^R \frac{\partial p}{\partial r} 4\,\pi r^3 dr = -\int_0^R c_s^2 4\,\pi r^3 dr \tag{7.28}$$

If the velocity of sound does not depend on the radius of the body, then (7.28) is

$$\begin{aligned} -\int_0^R \frac{\partial p}{\partial r} 4\pi r^3 dr = -\int_0^R c_s^2 4\pi r^3 d\rho = -\rho(r) c_s^2 4\pi^3 \Big|_0^R \\ + \int_0^R 3x 4\pi r^3 \rho(r) c_s^2 dr = 3c_s^2 m, \end{aligned} \tag{7.29}$$

where $\rho(R) = 0,\ \rho(0) \neq \infty$.

In the general case when the velocity of sound depends on the radius, the expression for the energy of elastic deformations can be written by the expression for the velocity of sound as a mean value through the mass body, i.e.,

$$2E_e = -\int_0^R \frac{\partial p}{\partial r} 4\pi r^3 dr = 3m\bar{c}_s^2, \tag{7.30}$$

where

$$\bar{c}_s^2 = -\frac{4\pi}{3m}\int_0^R c_c^2(r)r^3 d\rho(r). \tag{7.31}$$

The phenomenological parameter c_s takes into account the different aggregative states of the substance of a body, i.e. gaseous, liquid, solid, and plasma (Ferronsky et al. 1981a).

The second term in the right-hand side of Eq. 7.24 is the potential energy of the sphere:

$$4\,\pi G\int_0^R r\rho(r)m(r)dr = U. \tag{7.32}$$

Finally, Eq. 7.24 can be rewritten in the form of the Jacobi virial equation:

$$\ddot{\Phi} = 2T + 3m\bar{c}_s^2 + U = 2E - U, \tag{7.33}$$

where $E = T + 3/2m\bar{c}_s^2 + U$ is the total energy of the body.

One may see now that Eq. 7.33 represents the generalized virial equation (2.31) obtained from Euler's equation of motion for a deformable body (7.1) by means of transformation of the kinetic energy of the mass interaction (7.25) through the polar moment of inertia (the Jacobi function Φ). The polar moment of inertia in (7.33) has a functional relation with the potential energy (7.32). Here the polar moment of inertia physically represents the body's structure and shows its changes with a change in the potential energy.

Averaging Jacobi's virial equation (7.33) with respect to a sufficiently long period of time gives the classical averaged virial theorem of the body and expresses the condition of its hydrostatic equilibrium in the outer force field without taking into account kinetic energy of the interacted masses:

$$2E = U,$$

or

$$-U = 3m\overline{c_s^2}. \tag{7.34}$$

Equation 7.34 follows also from the condition of hydrostatic equilibrium of the sphere matter when the left-hand side of the equation of motion (7.23) is equal to zero:

$$\frac{\partial p}{\partial r} = \rho F_r.$$

In accordance with (7.30), the left-hand side of this equation represents a double value of the total body's energy, the matter of which stays in hydrostatic equilibrium in the outer uniform force field. The right-hand side of the equation determines the potential energy of the matter interaction.

It follows from Eq. 7.34 that the velocity of sound in the elastic media determines not only the potential energy of the mass interaction, but also the velocity of propagation of the potential energy flux in the media. This relationship between the potential energy and the sound velocity is used now in seismic studies for determination of mass density. We also will apply it for interpretation of the radial density distribution of the Earth.

To obtain the equation of virial oscillations from (7.33), we accept the following assumptions. We assume that the total energy E has a constant value and the relationship between the Jacobi function Φ and the gravitational potential energy U is held in the form

$$\left|U\right|\sqrt{\Phi} = B = \text{const} \tag{7.35}$$

It follows from Eqs. 2.31 and 2.32 and Table 1 of Chap. 2 that, for a density-uniform sphere, the expression (7.35) is strict. Then, for the uniform body Eq. 7.33 with the help of (7.35) can be written in the form of a non-linear differential equation of the second order with respect to the variable Φ:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} \tag{7.36}$$

where$A = -2E = |U|$

Expression (7.36) is the equation of virial oscillations of the uniform Earth. As it is shown in Chap. 4, the solution of Eq. 7.36 represents the periodic change of polar moment of inertia, i.e. oscillation of the interacting mass particles, and synchronously with this change of the potential (kinetic) energy. The solution of (7.36) is described by Eqs. 4.14 and 4.15 and we rewrite them in the same form:

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)], \tag{7.37}$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)] \tag{7.38}$$

Here ε and ψ are the integration constants depending on the initial conditions of the Jacobi function Φ_o and its first derivative at the first moment of time t_o .

Equations 7.37 and 7.38 and the integration constants after corresponding generalization were obtained in Chap. 4 in the explicit form:

$$\sqrt{\Phi_0} = a[1 - \varepsilon\cos\varphi], \tag{7.39}$$

$$\omega = \frac{2\pi}{T_V} = \frac{(2A)^{3/2}}{4B} = \frac{(2 \cdot 2E)^{3/2}}{4U\sqrt{\Phi}} = \sqrt{\frac{GM}{r^3}} = \sqrt{\frac{4}{3}\pi G\rho}, \tag{7.40}$$

$$a = \frac{B}{A}, \tag{7.41}$$

$$M_c = \varphi - \varepsilon \sin\varphi, \tag{7.42}$$

where $M_c = n(t - \tau)$; T_v is the period of the virial oscillations; ω is the frequency of oscillations; ρ is the mass density of the body.

Physical meaning of Eqs. 7.39–7.42 is expressed by Kepler's motion laws, in particular, Eq. 7.39 describes the first and second laws, and Eq. 7.40 describes its third law. And now, due to the functional relationship between the potential energy and the Jacobi function (polar moment of inertia), the Kepler laws express dynamic but not static equilibrium of the planet. In fact, the Jacobi function $\sqrt{\Phi_0}$, which represents the polar moment of inertia, traces a second-order curve with period of oscillations T_v and frequency ω. The curve of the uniform body is the circle having the semi-major axis a and the eccentricity ε. The expression (7.42) represents the Kepler equation. Equations 7.39–7.42 as a whole describe an oscillating motion of the body in accordance with Kepler's laws. But their significance grows owing to the involved volumetric polar moment of inertia and its relationship with the potential energy of the body determining its dynamical effects. Figure 5.3 demonstrates the graphic picture of this effect and Eqs. 4.36–4.38 show the effect in the explicit form by the Lagrange series:

$$\sqrt{\Phi}_0 = \frac{B}{A}\left[1 + \frac{\varepsilon^2}{2} + \left(-\varepsilon + \frac{3}{8}\varepsilon^3\right)\cos M_c - \frac{\varepsilon^2}{2}\cos 2M_c - \frac{3}{8}\varepsilon^3\cos 3M_c + \ldots\right] \tag{7.43}$$

$$\Phi_0 = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M_c - \frac{\varepsilon^2}{2}\cos 2M_c - \frac{\varepsilon^3}{4}\cos 3M_c + \ldots\right] \tag{7.44}$$

Note that because the polar moment of inertia of the body has a functional relationship with the potential energy, Eqs. 7.43 and 7.44 and Fig. 5.3 express also the effect of a change of the potential energy in time. This fact is important for understanding of the nature and mechanism of the mass particle interaction, the result of which is generation of the potential energy. Later on we will discuss this problem in more details.

Quantitative values of the parameters of the body's virial oscillations are considered together with solution of the problem of oscillation and rotation of the non-uniform body. In order to do this the potential and kinetic energies of a non-uniform body need to be expanded and some other effects of its non-uniform structure have to be understood. It was noted in Section 2.5 of Chap. 2 that the non-uniformities play an important role in dynamical processes of bodies. Let us start consideration of the effect of the non-uniformities with separation of the potential and kinetic energies.

7.2 Structure of Potential and Kinetic Energies of a Non-uniform Body

In fact, all the celestial bodies of the Solar System, including the Sun, are non-uniform creatures. They have a shell structure and the shells themselves are also non-uniform elements of a body. It was shown in Section 2.2 of Chap. 2 that according to the artificial satellite data all the measured gravitational moments of the Earth, including tesseral ones, have significant values. In geophysics this fact is interpreted as a deviation of the Earth from the hydrostatic equilibrium and attendance of the tangential forces which are continuously developed inside the body. From the point of view of the planet's dynamical equilibrium, the fact of the measured zonal and tesseral gravitational moments is a direct evidence of permanent development of the normal and tangential volumetric forces which are the components of the inner gravitational force field. In order to identify the above effects the inner force field of the body should be accordingly separated.

The expressions (2.46)–(2.49) in Chap. 2 indicate that the force function and the polar moment of a non-uniform self-gravitating sphere can be expanded with respect to their components related to the uniform mean density mass and its non-uniformities. In accordance with the superposition principle these components are responsible for the normal and tangential dynamical effects of a non-uniform body. Such a separation of the potential energy and polar moment of inertia through their dimensionless form-factors α^2 and β^2 was done by Garcia Lambas et al. (Garcia et al. 1985) with our interpretation (Ferronsky et al. 1996). Taking into account that the observed satellite irregularities are caused by the non-uniform distribution of the mass density, an auxiliary function relative to the radial density distribution was introduced for the separation:

$$\Psi(s) = \int_0^s \frac{(\rho_r - \rho_0)}{\rho_0} x^2 dx, \qquad (7.45)$$

where $s = r/R$ is the ratio of the running radius to the radius of the sphere R; ρ_o is the mean density of the sphere of radius r ; ρ_r is the radial density; x is the running coordinate; the value $(\rho_r - \rho_o)$ satisfies $\int_0^R (\rho_r - \rho_0) r^2 dr = 0$ and the function $\Psi(1) = 0$.

The function $\Psi(s)$ expresses a radial change in the mass density of the non-uniform sphere relative to its mean value at the distance r/R. Now we can write expressions for the force function and the moment of inertia by using the structural form-factors α^2 and β^2 which were introduced in Section 2.5:

$$U = 4\pi G \int_0^R r\rho(r)m(r)dr = \alpha^2 \frac{Gm^2}{R}, \qquad (7.46)$$

$$I = 4\pi \int_0^R r^4 \rho(r) dr = \beta^2 mR^2 \tag{7.47}$$

By (7.45) we can do the corresponding change of variables in (7.46) and (7.47). As a result, the expressions for the potential energy U and polar moment of inertia I are found in the form of their components composed of their uniform and non-uniform constituents (Garcia Lambas et al. 1985; Ferronsky et al. 1996):

$$U = \alpha \frac{Gm^2}{R} = \left[\frac{3}{5} + 3\int_0^1 \psi x dx + \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 dx\right] \frac{Gm^2}{R}, \tag{7.48}$$

$$I = \beta^2 mR^2 = \left[\frac{3}{5} - 6\int_0^1 \psi x dx\right] mR^2 \tag{7.49}$$

It is known that the moment of inertia multiplied by the square of the frequency ω of the oscillation-rotational motion of the mass is the kinetic energy of the body. Then Eq. 6.49 can be rewritten

$$K = I\,\omega^2 = \beta^2 MR^2\,\omega^2 = \left[\frac{3}{5} - 6\int_0^1 \psi x dx\right] mR^2\omega^2. \tag{7.50}$$

Let us clarify the physical meaning of the terms in expressions (7.48) and (7.50) of the potential and kinetic energy

As it follows from (2.46) and Table 1, the first terms in (7.48) and (7.50), numerically equal to 3/5, represent α_0^2 and β_0^2 being the structural coefficients of the uniform sphere with radius r, the density of which is equal to its mean value. The ratio of the potential and kinetic energies of such a sphere corresponds to the condition of the body's dynamical equilibrium when its kinetic energy is realized in the form of oscillations.

The second terms of the expressions can be rewritten in the form

$$3\int_0^1 \psi x dx \equiv 3\int_0^1 \left(\frac{\psi}{x}\right) x^2 dx, \tag{7.51}$$

$$-6\int_0^1 \psi x dx \equiv -6\int_0^1 \left(\frac{\psi}{x}\right) x^2 dx \tag{7.52}$$

One can see that there are written here the additive parts of the potential and kinetic energies of the interacting masses of the non-uniformities of each sphere shell with the uniform sphere having a radius r of the sphere shell. Note that the structural coefficient β^2 of the kinetic energy is twice as high as the potential energy and has the minus sign. It is known from physics that interaction of mass particles, uniform and non-uniform with respect to density is accompanied by their elastic and inelastic scattering of energy and appearance of a tangential component in their trajectories of motion. In this particular case the second terms in Eqs. 7.48 and 7.50 express the tangential (torque) component of the potential and kinetic energy of the body. Moreover, the rotational component of the kinetic energy is twice as much as the potential one.

The third term of Eq. 7.48 can be rewritten as

$$\frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 dx \equiv \frac{9}{2}\int_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2 dx. \tag{7.53}$$

Here, there is another additive part of the potential energy of the interacting non-uniformities. It is the non-equilibrated part of the potential energy which does not have an appropriate part of the reactive kinetic energy and represents a dissipative component. Dissipative energy represents the electromagnetic energy which is emitted by the body and it determines the body's evolutionary effects. This energy forms the electromagnetic field of the body.

Thus, by expansion of the expression of the potential energy and the polar moment of inertia we obtained the components of both forms of energy which are responsible for oscillation and rotation of the non-uniform body. Applying the above results we can write separate conditions of the dynamical equilibrium for each form of the motion and separate virial equations of the dynamical equilibrium of their motion.

7.3 Equations of Dynamical Equilibrium of Oscillation and Rotation of a Body

Equations 7.48 and 7.50 can be written in the form

$$U = (\alpha_0^2 + \alpha_t^2 + \alpha_\gamma^2)\frac{Gm^2}{R}, \tag{7.54}$$

$$K = (\beta_0^2 - 2\beta_t^2)mR^2\omega^2, \tag{7.55}$$

where $\alpha_0^2 = \beta_0^2$; $2\alpha_t^2 = \beta_t^2$; the subscripts o, t, γ define the radial, tangential, and dissipative components of the considered values.

Because the potential and kinetic energies of the uniform body are equal ($\alpha_0^2 = \beta_0^2 = 3/5$) then from (7.48) and (7.50) one has

$$U_o = K_o, \tag{7.56}$$

$$E_o = U_o + K_o = 2U_o \tag{7.57}$$

In order to express dynamical equilibrium between the potential and kinetic energies of the non-uniform interacting masses we can write, from (7.48) and (7.50),

$$2U_t = K_t, \tag{7.58}$$

$$E_t = U_t + K_t = 3U_t, \tag{7.59}$$

where E_o, E_t, U_o, K_o, U_t, K_t are the total, potential and kinetic energies of oscillation and rotation accordingly. Note, that the energy is always a positive value.

Equations 7.56–7.59 present expressions for uniform and non-uniform components of an oscillating system which serves as the conditions of their dynamical equilibrium. Evidently, the potential energy U_γ of interaction between the non-uniformities, being irradiated from the body's outer shell, is irretrievably lost and provides a mechanism of body's evolution.

In accordance with classical mechanics, for the above-considered non-uniform gravitating body, being a dissipative system, the torque N is not equal to zero, the angular momentum L of the sphere is not a conservative parameter, and its energy is continuously spent during the motion, i.e.,

$$N = \frac{dL}{dt} > 0, \quad L \neq \text{const.}, \quad E \neq \text{const.} > 0$$

A system physically cannot be conservative if friction or other dissipation forces are present, because F·ds due to friction is always positive and an integral cannot vanish (Goldstein 1980), i.e.: $\oint F \cdot ds > 0$.

7.4 Equations of Oscillation and Rotation of a Body and Their Solution

After we have found that the resultant of the body's gravitational field is not equal to zero and the system's dynamical equilibrium is maintained by the virial relationship between the potential and kinetic energies, the equations of a self-gravitating body motion can be written.

Earlier (Ferronsky et al. 1987) we used the obtained virial equation (7.33) for describing and studying the motion of both uniform and non-uniform self-gravitating spheres. Jacobi (1884) derived it from Newton's equations of motion of n mass points and reduced the n-body problem to the particular case of the one-body task with two independent variables, namely, the force function U and the polar moment of inertia Φ, in the form

$$\ddot{\Phi} = 2E - U. \tag{7.60}$$

Equation 7.60 represents the energy conservation law and describes the system in scalar U and Φ volumetric characteristics. In Chap. 3 it was shown that Eq. 7.60 is also derived from Euler's equations for a continuous medium, and from the equations of Hamilton, Einstein, and quantum mechanics. Its time-averaged form gives the Clausius virial theorem for a system with outer source of forces. It was earlier mentioned that Clausius was deducing the theorem for application in thermodynamics and, in particular, as applied to assessing and designing of Carnot's machines. As the machines operate in the Earth's outer force field, Clausius introduced the coefficient 1/2 to the term of "living force" or kinetic energy, i.e.,

$$K = \frac{1}{2}\sum_{i} m_i v_i^2$$

As Jacobi has noted, the meaning of the introduced coefficient was to take into account only the kinetic energy generated by the machine, but not by the Earth's gravitational force. That was demonstrated, for instance, by the work of a steam hammer for driving piles. The machine raises the hammer, but it falls down under the action of the force of the Earth's gravity. That is why the coefficient 1/2 of the kinetic energy of a uniform self-gravitating body in Eqs. 7.48–7.50 has disappeared. In its own force field the body moves due to release of its own energy.

Earlier by means of relation $U\sqrt{\Phi} \approx$ const, an approximate solution of Eq. 7.60 for a non-uniform body was obtained (Ferronsky et al. 1987). Now, after expansion of the force function and polar moment of inertia, at $U_\gamma = 0$ and taking into account the conditions of the dynamical equilibrium (7.57) and (7.59), Eq. 7.60 can be written separately for the radial and tangential components in the form

$$\ddot{\Phi}_0 = \frac{1}{2}E_0 - U_0, \tag{7.61}$$

$$\ddot{\Phi}_t = \frac{1}{3}E_t - U_t. \tag{7.62}$$

Taking into account the functional relationship between the potential energy and the polar moment of inertia

$$|U|\sqrt{\Phi} = B = \text{const},$$

and taking into account that the structural coefficients $\alpha_0^2 = \beta_0^2$ and $2\alpha_t^2 = \beta_t^2$, both Eqs. 7.61 and 7.62 are reduced to an equation with one variable and have a rigorous solution:

$$\ddot{\Phi}_n = -A_n + \frac{B_n}{\sqrt{\Phi_n}}, \tag{7.63}$$

where A_n and B_n are the constant values and subscript n defines the non-uniform body.

The general solution of Eq. 7.62 is (4.14) and (4.15):

$$\sqrt{\Phi_n} = \frac{B_n}{A_n}[1 - \varepsilon\cos(\xi - \varphi)], \tag{7.64}$$

$$\omega = \frac{2\pi}{T_v} = \frac{4B}{(2A)^{3/2}}[\xi - \varepsilon\sin(\xi - \varphi)], \tag{7.65}$$

where ε and φ are, as previously, the integration constants depending on the initial values of Jacobi's function Φ_n and its first derivative Φ_n at the time moment t_0 (the time here is an independent variable); T_v is the period of virial oscillations; ω is the oscillation frequency; ξ is the auxiliary independent variable; $A_n = A_0 - 1/2\ E_0 > 0$; $B_n = B_0 = U_0\sqrt{\Phi_0}$ for radial oscillations; $A_n = A_t = -1/3E_t, > 0$; $B_n = B_t = U_t\sqrt{\Phi_t}$ for rotation of the body.

The expressions for the Jacobi function and its first derivative in an explicit form can be obtained after transforming them into the Lagrange series:

$$\sqrt{\Phi_n} = \frac{B}{A}\left[1 + \frac{\varepsilon^2}{2} + \left(-\varepsilon + \frac{3}{8}\varepsilon^3\right)\cos M_c - \frac{\varepsilon^2}{2}\cos 2M_c - \frac{3}{8}\varepsilon^3\cos 3M_c + \dots\right],$$

$$\Phi_n = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M_c - \frac{\varepsilon^2}{2}\cos 2M_c - \frac{\varepsilon^3}{4}\cos 3M_c + \dots\right], \tag{7.66}$$

$$\Phi_n = \sqrt{\frac{2}{A}}\,\varepsilon B\left[\sin M_c + \frac{1}{2}\varepsilon\sin 2M_c + \frac{\varepsilon^2}{2}\sin M_c(2\cos^2 M_c - \sin^2 M_c) + \dots\right].$$

Radial frequency of oscillation ω_{or} and angular velocity of rotation ω_{tr} of the shells of radius r can be rewritten from (7.40) as

$$\omega_{0r} = \frac{(2A_0)^{3/2}}{4B_0} = \sqrt{\frac{U_{or}}{J_{or}}} = \sqrt{\frac{\alpha_{or}^2 G m_r}{\beta_{or}^2 r^3}} = \sqrt{\frac{4}{3}\pi G\rho_{0r}}, \tag{7.67}$$

$$\omega_{tr} = \frac{(2A_t)^{3/2}}{4B_t} = \sqrt{\frac{2U_{tr}}{J_{tr}}} = \sqrt{\frac{2\alpha_{tr}^2 G m_r}{\beta_{tr}^2 r^3}} = \sqrt{\frac{4}{3}\pi G\rho_{0r} k_{er}}, \tag{7.68}$$

where U_{or} and U_{tr} are the radial and tangential components of the force function (potential energy); J_{or} and $J_{tr} = 2/3J_{or}$ are the polar and axial moment of inertia; $\rho_{0r} = \frac{1}{V_r}\int_{V_r}\rho(r)dV_r$; ρ(r) is the law of radial density distribution; ρ_{0r} is the mean density value of the sphere with a radius r; V_r is the sphere volume with a radius r; $2\alpha\ {}_{tr}{}^2 = \beta_{tr}{}^2$; k_{er} is the dimensionless coefficient of the energy dissipation or tidal friction of the shells equal to the shell's oblateness.

The relations (7.64)–(7.65) represent Kepler's laws of body rotation in dynamical equilibrium. In the case of uniform mass density distribution the frequency (7.57) of oscillation of the sphere's shells with radius r is $\omega_{or} = \omega_o = \text{const}$. It means that here all the shells are oscillating with the same frequency. Thus, it appears that only non-uniform bodies are rotating systems.

Rotation of each body's shell depends on the effect of the potential energy scattering at the interaction of masses of different density. As a result, a tangential component of energy appears which is defined by the coefficient k_{er}. In geodynamics the coefficient is known as the geodynamical parameter. Its value is equal to the ratio of the radial oscillation frequency and the angular velocity of a shell and can be obtained from Eqs. 7.67 and7.68), i.e.

$$k_e = \frac{\omega_t^2}{\omega_0^2} = \frac{\omega_t{}^2}{\frac{4}{3}\pi G\ \rho_0}. \tag{7.69}$$

It was found that in the general case of a three-axial (a, b, c) ellipsoid with the ellipsoidal law of density distribution, the dimensionless coefficient $k_e \in [0, 1]$ is equal to (Ferronsky et al. 1987)

$$k_r = \frac{F(\varphi, f)}{\sin\varphi} \Big/ \frac{a^2 + b^2 + c^2}{3a^2},$$

where$\varphi = \arcsin\sqrt{\frac{a^2-c^2}{a^2}}$, $f = \sqrt{\frac{a^2-b^2}{a^2-c^2}}$, and F(φ,f) is an incomplete elliptic integral of the first degree in the normal Legendre form.

Thus, in addition to the earlier obtained solution of radial oscillations (Ferronsky et al. 1987), now we have a solution of its rotation. It is seen from expression (7.67) that the shell oscillation do not depend on the phase state of the body's mass and are determined by its density.

It follows from Eqs. 7.67 and 7.68 that in order to obtain the frequency of oscillation and angular velocity of rotation of a non-uniform body, the law of radial density distribution should be revealed. This problem will be considered later on. But before that the problem of the nature of a body shells separation with respect to their density needs to be solved.

7.5 Application of Roche's Tidal Approach for Separation of the Body Shells

It is well known that celestial bodies have a quasi-spherical shell structure. This phenomenon has been confirmed by recording and interpretation of seismic longitudinal and transversal wave propagation during earthquakes. In order to understand the physics and mechanism of a body mass differentiation with respect to its density, we apply Roche's tidal dynamics.

Newton's theorem of gravitational interaction between a material point and a spherical layer states that the layer does not affect a point located inside the layer. On the contrary, the outside-located material point is affected by the spherical layer. Roche's tidal dynamics is based on the above theorem. His approach is as follows (Ferronsky et al. 1996).

There are two bodies of masses M and m interacting in accordance with Newton's law (Fig. 7.1a).

Let $M \gg m$ and $R \gg r$, where r is the radius of the body m, and R is the distance between the bodies M and m. Assuming that the mass of the body M is uniformly distributed within a sphere of radius R, we can write the accelerations of the points A and B of the body m as

$$q_A = \frac{GM}{(R-r)^2} - \frac{Gm}{r^2}, \quad q_B = \frac{GM}{(R+r)^2} + \frac{Gm}{r^2}.$$

The relative tidal acceleration of the points A and B is

$$\begin{aligned} q_{AB} &= G\left[\frac{M}{(R-r)^2} - \frac{M}{(R+r)^2} - \frac{2m}{r^2}\right] \\ &= \frac{4\pi}{3}G\left[\rho_M R^3 \frac{4Rr}{(R^2-r^2)^2} - 2\rho_m r\right] \approx \frac{8\pi}{3}Gr(2\rho_M - \rho_m). \end{aligned} \tag{7.70}$$

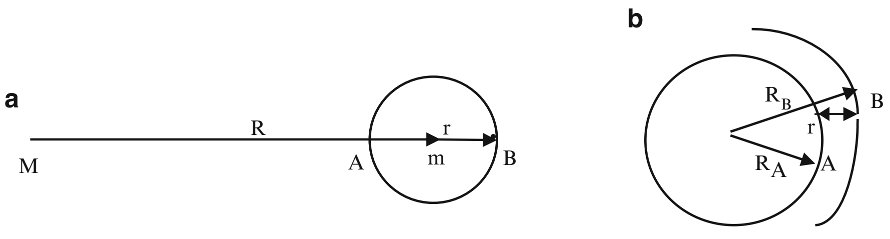

Fig. 7.1 The tidal gravitational stability of a sphere (**a**), and the sphere layer (**b**)

Here $\rho_M = M/\frac{4}{3}\pi R^3$ and $\rho_m = m/\frac{4}{3}\pi r^3$ are the mean density distributions for the spheres of radius R and r. Roche's criterion states that the body with mass m is stable against the tidal force disruption of the body M if the mean density of the body m is at least twice as high as that of the body M in the sphere with radius R. Roche considered the problem of the interaction between two spherical bodies without any interest in their creation history and in how the forces appeared. From the point of view of the origin of celestial bodies and of the interpretation of dynamical effects, we are interested in the tidal stability of separate envelopes of the same body. For this purpose we can apply Roche's tidal dynamics to study the stability of a non-uniform spherical envelope.

Let us assess the tidal stability of a spherical layer of radius R and thickness $r = R_B - R_A$ (Fig. 2.1b). The layer of mass m and mean density $\rho_m = m/4\pi {R_A}^2 r$ is affected at point A by the tidal force of the sphere of radius R_A. The mass of the sphere is M and mean density $\rho_M = M/\frac{4}{3}\pi R_A^3$ The tidal force in point B is generated by the sphere of radius R + r and mass M + m. Then the accelerations of the points A and B are

$$q_A = \frac{GM}{R_A^2}, \quad q_B = \frac{G(M+m)}{(R_A+r)^2}.$$

The relative tidal acceleration of the points A and B is

$$q_{AB} = GM\left[\frac{1}{R_A^2} - \frac{1}{(R_A+r)^2}\right] - \frac{Gm}{(R_A+r)^2}$$

$$= \left(\frac{8}{3}\pi G\rho_M - 4\pi G\rho_m\right)r = 4\pi Gr\left(\frac{2}{3}\rho_M - \rho_m\right), (R >> r). \tag{7.71}$$

Equations 7.70 and 7.71 give the possibility to understand the nature of a body shell separation including some other dynamical effects.

7.6 Physical Meaning of Archimedes and Coriolis Forces and Separation of the Earth's Shells

The Archimedes principle states: *The apparent loss in weight of a body totally or partially immersed in a liquid is equal to the weight of the liquid displaced.* We saw in the previous section that the principle is described by Eqs. 7.70 and 7.71 and the forces that sink down or push out the body or the shell are of a gravitational nature. In fact, in the case of $\rho_n = \rho_M$ the body immersed in a liquid (or in any other medium) is kept in place due to equilibrium between the forces of the body's weight and the forces of the liquid reaction. In the case of $\rho_n > \rho_M$ or $\rho_n < \rho_M$ the body is sinking or floating up depending on the resultant of the above forces. Thus, the Archimedes forces seem to have a gravity nature and are the radial component of the Earth's inner force field.

It is assumed that the Coriolis forces appeared as an effect of the body motion in the rotational system of co-ordinates relative to the inertial reference system. In this case rotation of the body is accepted as the inertial motion and the Coriolis forces appear to be the inertial ones. It follows from the solution of Eq. 7.62 that the Coriolis' forces appear to be the tangential component of body's inner force field, and the body rotation is caused by the moment of those forces that are relative to the three-dimensional centre of inertia which also does not coincide with the three-dimensional gravity centre.

In accordance with Eq. 7.71 of the tidal acceleration of an outer non-uniform spherical layer at $\rho_M \neq \rho_m$, the mechanism of the gravitational density differentiation of masses is revealed. If $\rho_M < \rho_m$, then the shell immerses (is attracted) up to the level where $\rho_M = \rho_m$. At $\rho_M > \rho_m$ the shell floats up to the level where $\rho_M = \rho_m$ and at $\rho_M > 2/3\rho_m$ the shell becomes a self-gravitating one. Thus, in the case when the density increases towards the sphere's center, which is the Earth's case, then each overlying stratum appears to be in a suspended state due to repulsion by the Archimedes forces which, in fact, are a radial component of the gravitational interaction forces.

The effect of the gravitational differentiation of masses explains the nature of creation of shell-structured celestial bodies and corresponding processes (for instance, the Earth's crust and its oceans, geotectonic, orogenic and seismic processes, including earthquakes). All these phenomena appear to be a consequence of the continuous gravitational differentiation in density of the planet's masses. We assume that creation of the Earth and the Solar system as a whole was resulted by this effect. For instance, the mean value of the Moon's density is less than 2/3 of the Earth's, i.e., $\rho_M < 2/3\rho_m$. If one assumes that this relation was maintained during the Moon's formation, then, in accordance with Eq. 7.71, this body separated at the earliest stage of the Earth's mass differentiation. Creation of the body from the separated shell should occur by means of the cyclonic eddy mechanism, which was proposed in due time by Descartes and which was unjustly rejected. If we take into account existence of the tangential forces in the non-uniform mass, then the above mechanism seems to be realistic.

7.7 Self-similarity Principle and the Radial Component of a Non-uniform Sphere

It follows from Eq. 7.71 that in the case of the uniform density distribution ($\rho_m = \rho_M$), all spherical layers of the gravitating sphere move to the centre with accelerations and velocities which are proportional to the distance from the centre. It means that such a sphere contracts without loss of its uniformity. This property of self-similarity of a dynamical system without any discrete scale is unique for a uniform body (Ferronsky et al. 1996).

A continuous system with a uniform density distribution is also ideal from the point of view of Roche's criterion of stability with respect to the tidal effect. That is why there is a deep physical meaning in separation of the first term of potential energy in expression (7.48). A uniform sphere is always similar in its structure in spite of the fact that it is a continuously contracting system. Here, we do not consider the Coulomb forces effect. For this case we have considered the specific proton and electron branches of the evolution of the body (see Chap. 8).

Note that in Newton's interpretation the potential energy has a non-additive category. It cannot be localized even in the simplest case of the interaction between two mass points. In our case of a gravitating sphere as a continuous body, for the interpretation of the additive component of the potential energy we can apply Hooke's concept. Namely, according to Hooke there is a linear relationship between the force and the caused displacement. Therefore the displacement is in square dependence on the potential energy. Hooke's energy belongs to the additive parameters. In the considered case of a gravitating sphere, the Newton force acting on each spherical layer is proportional to its distance from the centre. Thus, here from the physical point of view, the interpretations of Newton and Hooke are identical.

At the same time in the two approaches there is a principal difference even in the case of uniform distribution of the body density. According to Hooke the cause of displacement, relative to the system, is the action of the outer force. And if the total energy is equal to the potential energy, then equilibrium of the body is achieved. The potential energy plays here the role of elastic energy. The same uniform sphere with Newton's forces will be contracted. All the body's elementary shells will move without change of uniformity in the density distribution. But the first terms of Eqs. 7.48–7.50 show that the tidal effects of a uniform body restrict motion of the interacting shells towards the centre. In accordance with Newton's third law and the d'Alembert principle the attraction forces, under the action of which the shells move, should have equally and oppositely direct forces of Hooke's elastic counter-action. In the framework of the elastic gravitational interaction of shells, the dynamical equilibrium of a uniform sphere is achieved in the form of its elastic oscillations with equality between the potential and kinetic energy. The uniform sphere is dynamically stable relative to the tidal forces in all of its shells during the time of the system contraction. Because the potential and kinetic energies of a sphere are equal, then its total energy in the framework of the averaged virial theorem within one period of oscillation is accepted formally as equal to zero. Equality of the potential and kinetic energy of each shell means the equality of the centripetal (gravitational) and centrifugal (elastic constraint) accelerations. This guarantees the system remaining in dynamical equilibrium. On the contrary, all the spherical shells will be contracted towards the gravity centre which, in the case of the sphere, coincides with the inertia centre but does not coincide with the geometric centre of the masses. Because the gravitational forces are acting continuously, the elastic constraint forces of the body's shells are reacting also continuously.

The physical meaning of the self-gravitation of a continuous body consists in the permanent work which applies the energy of the interacting shell masses on one side and the energy of the elastic reaction of the same masses in the form of oscillating motion on the other side. At dynamical equilibrium the body's equality of potential and kinetic energy means that the shell motion should be restricted by the elastic oscillation amplitude of the system. Such an oscillation is similar to the standing wave which appears without transfer of energy into outer space. In this case the radial forces of the shell's elastic interactions along the outer boundary sphere should have a dynamical equilibrium with the forces of the outer gravitational field. This is the condition of the system to be held in the outer force field of the mother's body. Because of this, while studying the dynamics of a conservative system, its rejected outer force field should be replaced by the corresponding equilibrated forces as they do, for instance, in Hooke's theory of elasticity.

Thus, from the point of view of dynamical equilibrium the first terms in Eqs. 7.48–7.50 represent the energy which provides the field of the radial forces in a non-uniform sphere. Here, the potential energy of the uniform component plays the role of the active force function, and the kinetic energy is the function of the elastic constraint forces.

7.8 Charges-Like Motion of Non-uniformities and Tangential Component of the Force Function

Let us now discuss the tidal motion of non-uniformities due to their interactions with the uniform body. The potential and kinetic energies of these interactions are given by the second terms in Eqs. 7.48 and 7.50. In accordance with (7.71), the non-uniformity motion looks like the motion of electrical charges interacting on the background of a uniform sphere contraction. Spherical layers with densities exceeding those of the uniform body (positive anomalies) come together and move to the centre in elliptic trajectories. The layers with deficit of the density (negative anomalies) come together, but move from the centre on the parabolic path. Similar anomalies come together, but those with the opposite sign are dispersed with forces proportional to the layer radius. In general, the system tends to reach a uniform and equilibrium state by means of redistribution of its density up to the uniform limit. Both motions happen not relative to the empty space, but relative to the oscillating motion of the uniform sphere with a mean density. Separate consideration of motion of the uniform and non-uniform components of a heterogeneous sphere is justified by the superposition principle of the forces action which we keep here in mind. The considered motion of the non-uniformities looks like the motion of the positive and negative charges interacting on the background of the field of the uniformly dense sphere (Ferronsky et al. 1996). One can see here that in the case of gravitational interaction of mass particles of a continuous body, their motion is the consequence not only of mutual attraction, but also mutual repulsion by the same law $1/r^2$. In fact, in the case of a real natural non-uniform body it appears that the Newton and Coulomb laws are identical in details. Later on, while considering a body's by-density differentiated masses, the same picture of motion of the positive and negative anomalies will be seen.

If the sphere shells, in turn, include density non-uniformities, then by means of Roche's dynamics it is possible to show that the picture of the non-uniformity motion does not differ from that considered above.

In physics the process of interaction of particles with different masses without redistribution of their moments is called elastic scattering. The interaction process resulting in redistribution of their moments and change in the inner state or structure is called inelastic scattering. In classical mechanics while solving the problems of motion of the uniform conservative systems (like motion of the material point in the central field or motion of the rigid body), the effects of the energy scattering do not appear. In the problem of dynamics of the self-gravitating body, where interaction of the shells with different masses and densities are considered, the elastic and inelastic scattering of the energy becomes an evident fact following from consideration of the physical meaning of the expansion of the energy expressions in the form of (7.48) and (7.50). In particular, their second terms represent the potential and kinetic energies of gravitational interaction of masses having a non-uniform density with the uniform mass and express the effect of elastic scattering of density-different shells. Both terms differ only in the numeric coefficient and sign. The difference in the numerical coefficient evidences that the potential energy here is equal to half of the kinetic one ($U_t = 1/2K_t$). This part of the active and reactive force function characterizes the degree of the non-coincidence of the volumetric centre of inertia and that of the gravity centre of the system expressed by Eqs. 2.48 and 2.49. This effect is realized in the form of the angular momentum relative to the inertia centre.

Thus, we find that inelastic interaction of the non-uniformities with the uniform component of the system generates the tangential force field which is responsible for the system rotation. In other words, in the scalar force field of the by-density uniform body the vector component appears. In such a case, we can say that, by analogy with an electromagnetic field, in the gravitational scalar potential field of the non-uniform sphere U(R, t) the vector potential A(R, t) appears for which $U = \text{rot } A$ and the field U(R, t) will be solenoidal. In this field the conditions for vortex motion of the masses are born, where $\text{div } A = 0$. This vector field, which in electrodynamics is called solenoidal, can be represented by the sum of the potential and vector fields. The fields, in addition to the energy, acquire moments and have a discrete-wave structure. In our case the source of the wave effects appears to be the interaction between the elementary shells of the masses by means of which we can construct a continuous body with a high symmetry of forms and properties. The source of the discrete effects can be represented by the interacting structural components of the shells, namely, atoms, molecules and their aggregates. We shell continue discussion about the nature of the gravitational and electromagnetic energy in Chap. 8.

7.9 Radial Distribution of Mass Density and the Body's Inner Force Field

At present only the Earth has experimental data which allow to interpret them with respect to radial distribution of the body's mass density. Taking into account our consideration of dynamics of celestial bodies as self-gravitating systems we assume that formation of the Earth's mass density distribution is typical at least for all the planets and satellites.

The existent idea about the radial mass density distribution of the Earth is based on interpretation of transmission velocity of the longitudinal and transverse seismic waves. Figure 7.2 presents the classic curve of transmission velocities of the longitudinal and transverse seismic waves in the Earth plotted after generalization of numerous experimental data (Jeffreys 1970; Melchior 1972; Zharkov 1978).

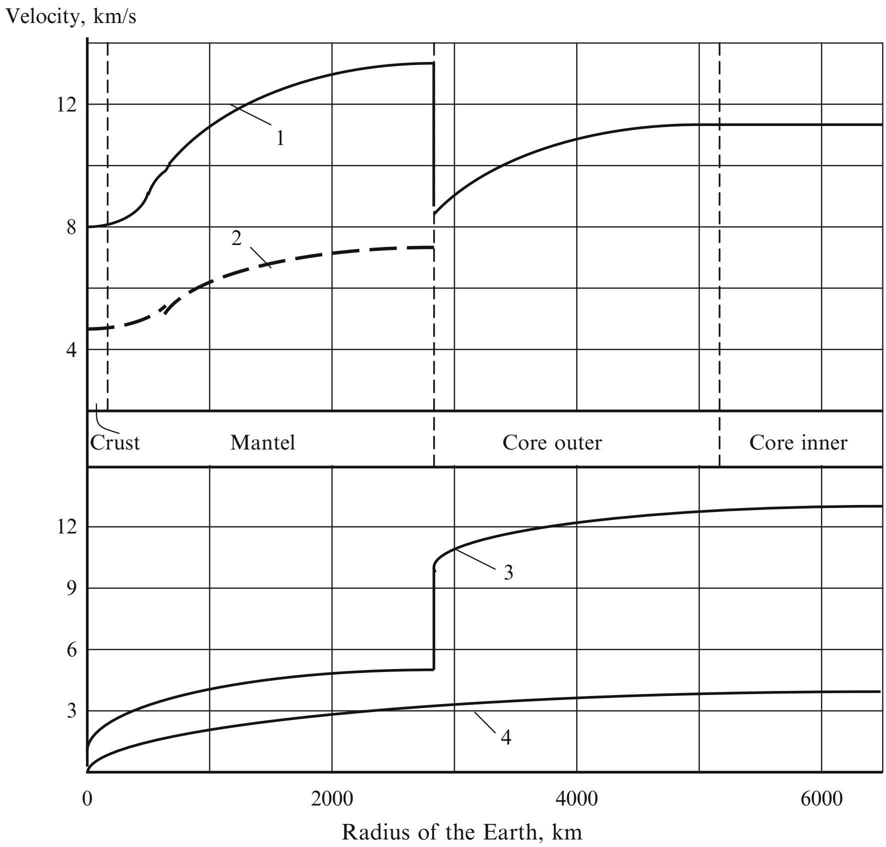

Fig. 7.2 Present-day interpretation of the curves of transmission velocities of longitudinal (1) and transverse (2) seismic waves, density (3), and hydrostatic pressure (4) in the Earth

The curves of the radial density and hydrostatic pressure distribution based on interpretation of the velocities of the longitudinal and transverse seismic waves are also shown.

The picture of the transmission velocities of the seismic waves was obtained by observations and therefore is realistic and correct. But interpretation of the obtained data was based on the idea of hydrostatic equilibrium of the Earth. It leads to incredibly high pressures in the core and high values of the mass density.

In accordance with Bullen's approach for interpretation of the seismic data, the density distribution is characterized by the following values (Bullen 1974; Melchior 1972; Zharkov 1978). The density of the crust rocks is 2.7–2.8 g/cm^3 and increases towards the centre by a certain curve up to ~13.0 g/cm^3 with jumps at the Mohorovičić-Guttenberg discontinuity, between the upper and lower mantle, and on the border of the outer core. Within the core the values of the transverse seismic waves are equal to zero. Despite the jump of the longitudinal seismic wave velocity at the outer core border dropping down, Bullen accepted that the density increases toward the center. It was done after his unsuccessful attempt to approximate the seismic data of the parabolic curve which gives a decrease of density in the core. Such a tendency is not consistent with the idea of iron core content. Bullen certainly had no idea that the radius of inertia and radius of gravity of the body do not coincide with its geometric centre of mass and, therefore, the maximum value of density is not located there. In accordance with our concept of the equilibrium condition of the planet and its dynamical parameters, the approach to interpretation of the seismic data related to the radial density and radial pressure distribution should be done on a new basis.

Now, when we accept the concept of dynamical equilibrium of the Earth and refuse its hydrostatic version, the basic idea to search for a solution of the problem seems to be the found relationship between the polar moment of inertia and the potential (kinetic) energy. The value of the structural form-factor of the Earth's mean axial moment of inertia $\beta^2{}_\perp = J_\perp/MR^2 = 0.3315$ found by artificial satellites (Zharkov 1978) should be taken as a starting point. The mean polar moment of inertia of the assumed spherical non-uniform planet is equal to $\beta^2 = (3/2)\beta^2{}_\perp =$ 0.49725. We accept this value for development of the methodology.

Let us take as a basis the found mechanism of the shell separation with respect to the mass density which was presented in Sects. 7.5–7.8. The conditions and mechanism of the shell separation into radial and tangential components of the inner force field (by the Archimedes and Coriolis forces) represent continually acting effects and create physics for the Earth's structure formation. These effects explain the jumps between the shells observed by seismic data density. We take also into account the effect, expressed by Eq. 7.30, according to which the velocity of the sound recorded by the transmission velocity of the longitudinal and transverse seismic waves quantitatively characterize the energy of the elastic deformation of the media and velocity of its transmission there.

Applying the conception of Sect. 7.8, we accept that the non-uniformities of the spherical shells come together and, after their density becomes lower than that of the mean density of the inner sphere, move from the center by the parabolic law because they interact according to the law $1/r^2$. So, we can find a probable law of the radial density distribution in the form

$$\rho(r) = \rho_0(ax^2 + bx + c), \tag{7.72}$$

where $x = r/R$ is the ratio of the running and the final radius of the planet; ρ_0 is the body's mean density; a, b, c are the numerical coefficients.

The numerical coefficients were selected for different densities for the upper shell and in such a way that the planet's total mass M would be constant, i.e.,

$$M = 4\pi \int_0^R r^2 \rho(r) dr = 4\pi \int_0^R r^2 \rho_0 \left(-a\frac{r^2}{R^2} + b\frac{r}{R} + c\right) dr$$
$$= \frac{4}{3}\pi R^3 \rho_0 \left(-\frac{3}{5}a + \frac{3}{4}b + c\right)$$

Here the term $(3/5)a + (3/4)b + c = 1$ in the right-hand side of the expression allows us to calculate and plot the distribution density curves in a dimensionless form.

We have selected three most typical parabolas (7.73) which satisfy the condition of equality of their moment of inertia, found by artificial satellite data, namely, the axial moment of inertia $J_\perp = \beta_\perp^2 mR^2 = 0.3315\, mR^2$ or the polar moment of inertia $J = \beta^2 mR^2 = 0.4973\, mR^2$. In addition, the first relation in (7.73) represents the straight line for which the surface mass density and that in the center correspond to the present-day version and to the form-factor $\beta_\perp^2$. The fifth straight line represents the uniform spherical planet. The curve equations with selected numerical coefficients a, b, and c are as follows:

$$\begin{aligned}
&1. \quad \rho(r) = \rho_0\left(-2\frac{r}{R} + 2.495\right), \quad a = 0,\ \rho_s = 2.73\ \Gamma/\text{CM}^3;\\
&2. \quad \rho(r) = \rho_0\left(-1.51\frac{r^2}{R^2} + 0.016\frac{r}{R} + 1.894\right),\ \rho_s = 2.08\ \Gamma/\text{CM}^3;\\
&3. \quad \rho(r) = \rho_0\left(-3.26\frac{r^2}{R^2} + 2.146\frac{r}{R} + 1.3465\right),\ \rho_s = 1.28\ \Gamma/\text{CM}^3;\\
&4. \quad \rho(r) = \rho_0\left(-5.24\frac{r^2}{R^2} + 5.132\frac{r}{R} + 0.295\right),\ \rho_s = 1.03224\ \Gamma/\text{CM}^3;\\
&5. \quad \rho(r) = \rho_0 = \text{const.}
\end{aligned} \tag{7.73}$$

Figure 7.3 shows all the curves of (7.73). They intersect the straight line 5 of the mean density in the common point which corresponds to the value $r/R = 0.61475$.

Using Eq. 7.73 and the found (by observations) form-factor $\beta_\perp^2 = 0.3315$, the main dynamical parameters were calculated for all four curves. The calculations

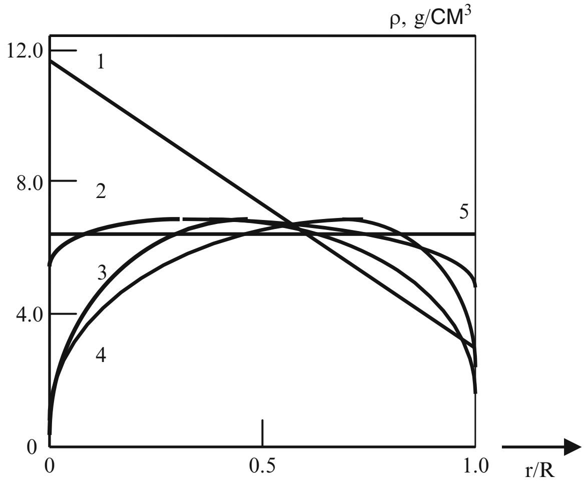

Fig. 7.3 Parabolic curves of radial density distribution calculated by Eq. 7.73

were done by the known formulae of the theory of attraction (Duboshin 1975) and taking into account the relations of (7.48) and (7.50) obtained in Sect. 7.2. These calculations are presented below for equation (4), as an example.

The potential energy of the non-uniform sphere with the density distribution law $\rho(r)$ is found from the equation:

$$U = 4\pi G \int_0^R r\rho(r)m(r)dr, \tag{7.74}$$

where

$$\rho(r) = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right),\ a = -5.24;\ в = 5.132;\ c = 0.295;$$

$$m(r) = 4\pi\int_0^r r^2\rho(r)dr = 4\ \pi\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)dr$$

$$= \frac{4}{3}\ \pi r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)$$

Then

$$U = 4\pi G\int_0^R r\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\frac{4}{3}\pi r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)dr$$

$$= \left(\frac{4}{3}\pi\rho_0\right)^2 GR^5\frac{R}{R}\left(\frac{1}{5}a^2 + \frac{81}{160}ab + \frac{9}{28}b^2 + \frac{24}{35}ac + \frac{7}{8}bc + \frac{3}{5}c^2\right)$$

$$= 0.660143\frac{GM^2}{R}$$

The form-factor of the potential energy is $\alpha^2 = {r_g}^2/R^2 = 0.660143$, and the reduced radius of gravity is $r_g = \sqrt{0.660143R^2} = 0.8124918R$.

In accordance with Eq. 7.48, the potential energy of the non-uniform sphere is expanded into the components

$$U = U_0 + U_t + U_\gamma. \tag{7.75}$$

The potential energy of the uniform sphere is equal to

$$U_0 = \frac{3}{5}\frac{GM^2}{R}. \tag{7.76}$$

The form-factors of the potential and kinetic energy are equal to $\alpha_0^2 = 0.6$ and $\beta_0^2 = 0.6$ accordingly.

In accordance with the second term of the right–hand side of Eq. 7.48, the tangential component of the non-uniform sphere is written as

$$U_t = -\frac{1}{2}4\pi G\int_0^R r\rho_t(r)m_0(r)dr, \tag{7.77}$$

where

$$\rho_t(r) = \rho(r) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right);$$

$$m_0(r) = 4\pi\int_0^r r^2\rho_0 dr = \frac{4}{3}\pi\rho_0 r^3.$$

The coefficient ½ in (7.77) is taken as the ratio of the second terms of the right–hand side of Eqs. 7.48 and 7.50, as in this particular case the tangential component of the potential energy is determined through the tangential component of the kinetic energy and is equal to half its value. Then

$$U_t = -\frac{1}{2}4\frac{4}{3}(\pi\rho_0)^2 G\int_0^R r^4\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)dr$$

$$= -\frac{1}{2}\frac{GM^2}{R}\left(\frac{3}{7}a + \frac{1}{2}b + \frac{3}{5}c - \frac{3}{5}\right) = 0.0513571\frac{GM^2}{R}$$

The form-factors of the tangential components of the potential and kinetic energy are equal to $\alpha_t^2 = 0.051357$ and $\beta_t^2 = 2 \cdot 0.051357 = 0.102714$ accordingly.

In accordance with the third term in the right–hand side of Eq. 7.48, the dissipative component of the potential energy of the non-uniform sphere is

$$U_\gamma = 4\pi G \int_0^R r\rho_t(r)m_t(r)dr, \tag{7.78}$$

where

$$\rho_t(r) = \rho(r) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right);$$

$$m_0(r) = 4\pi\int_0^r r^2\rho_t(r)dr = 4\pi\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)dr$$

$$= \frac{4}{3}\pi\rho_0 r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c - 1\right).$$

Then

$$U_\gamma = 4\frac{4}{3}(\pi\rho_0)^2 G\int_0^R r^4\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c - 1\right)dr$$

$$= \frac{GM^2}{R}\left(\frac{1}{5}a^2 + \frac{81}{160}ab + \frac{24}{35}ac - \frac{24}{35}a + \frac{9}{28}b^2 + \frac{7}{8}bc - \frac{7}{8}b + \frac{3}{5}c^2 - \frac{5}{6}c + \frac{3}{5}\right)$$

$$= 0.008786\frac{GM^2}{R} \tag{7.79}$$

So, the value of the form-factor of the dissipative component is $\alpha_\gamma^2 = 0.008786$.

The radial distribution of the potential energy for interaction of a test mass point with the non-uniform sphere is

$$U(r) = \frac{4\pi G}{r}\int_0^r r^2\rho(r)dr + 4\pi G\int_r^R r\rho(r)dr = \frac{4\pi G}{r}\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)dr$$

$$+4\pi G\int_r^R r\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)dr = \frac{GMm_1}{R}\left(-\frac{3}{20}a\frac{r^4}{R^4} - \frac{1}{4}b\frac{r^3}{R^3} - \frac{1}{2}c\frac{r^2}{R^2} + \frac{3}{4a} + b + \frac{3}{2}c\right)$$

$$= \frac{GMm_1}{R}\left(0.786\frac{r^4}{R^4} - 1.283\frac{r^3}{R^3} - 0.1475\frac{r^2}{R^2} + 1.6445\right). \tag{7.80}$$

At r/R = 0, then $\alpha_u^2(r) = 1.6445$; and at r/R = 1 then $\alpha_u^2(r) = 1$.

The radial distribution of the interaction force of the test mass point with the non-uniform sphere is

$$\begin{aligned} q(r) &= -\frac{4\pi G}{r^2}\int_0^r r^2\rho(r)dr = -\frac{4\pi G}{r^2}\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2}+b\frac{r}{R}+c\right)dr \\ &= -\frac{GMm_1}{R^2}\left(\frac{3}{5}a\frac{r^3}{R^3}+\frac{3}{4}b\frac{r^2}{R^2}+c\frac{r}{R}\right) \\ &= -\frac{GMm_1}{R^2}\left(-3.144\frac{r^3}{R^3}+3.849\frac{r^2}{R^2}+0.295\frac{r}{R}\right). \end{aligned} \tag{7.81}$$

At $r/R = 0$ then $\alpha_q^2(r) = 0$; and at $r/R = 1$ then $\alpha_q^2(r) = 1$.

Table 7.1 demonstrates the results of the calculated dynamical parameters for all the density curves (7.73) and Fig. 7.4 shows the curves of radial distribution of the potential energy and gravity force for the test mass point.

We wish to evaluate all four curves of mass density distribution in order to recognize which one is closer to the real Earth. In this case we keep in mind that the observed density jumps can be obtained for any curve by approximation of its continuous section with the mean value for each shell.

Figure 7.4 shows that the radial density values are substantially different for each curve. It refers, first of all, to the surface and centre of the body. At the same time Table 7.1 demonstrates the complete identity of the dynamical parameters of all the non-uniform spheres. It means that a fixed value of the polar moment of inertia permits us to have a multiplicity of curves of the radial density distribution with identical dynamical parameters of the body. The found property of the non-

Table 7.1 Physical and dynamical parameters of the Earth for the density distribution presented by Eq. 7.73

Equation no	1	2	3	4
ρ_s, g/cm^3	2.76	2.08	1.65	1.03224
ρ_c, g/cm^3	13.8	10.455	6.315	1.6284
ρ_{max}, g/cm^3/km	13.8/0	10.455/0	8.26/2096	8.57/3122
$\beta^2{}_\perp$	0.3315	0.3315	0.3315	0.3315238
β^2	0.49725	0.49725	0.49725	0.49725858
$\beta_t{}^2$	0.10275	0.10275	0.102752	0.102714
α^2	0.660737	0.660737	0.660737	0.660143
$\alpha_t{}^2$	0.051371	0.051371	0.0513714	0.0513571
$\alpha_\gamma{}^2$	0.009366	0.009366	0.009366	0.0087859
r_g, km	5178.6	5178.7	5178.6	5176.4
r_m, km	4492.6	4492.6	4492.6	4492.7

ρ_s, ρ_c, ρ_{max} are the density on the sphere's surface, in the centre, and maximal accordingly

$\beta^2{}_\perp$, β^2, $\beta_t{}^2$ are the form-factors of the axial, polar, and tangential components of the radius of inertia accordingly

α^2, $\alpha_t{}^2$, $\alpha_\gamma{}^2$ are the form-factors of the radial, tangential, and dissipative components of the force function accordingly

r_g, r_m are the radiuses of the gravity and inertia

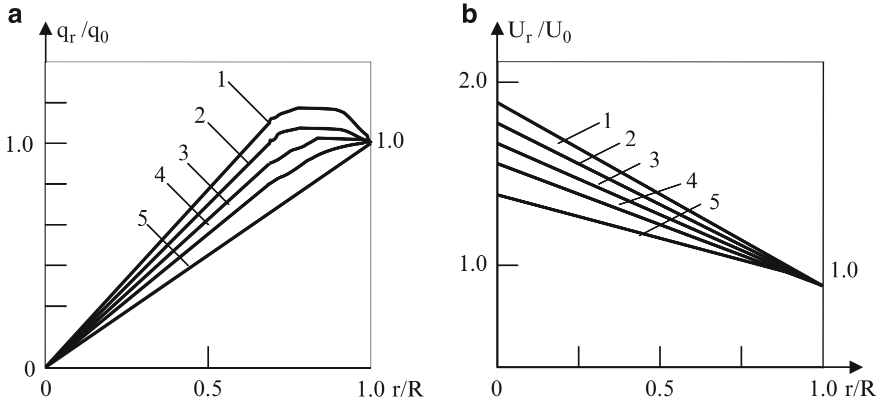

Fig. 7.4 The curves of the radial distribution of the potential energy (**a**) and gravity force (**b**) for the mass point test done by Eqs. 7.74 and 7.81

uniform self-gravitating sphere proves the rigor of the discovered functional relationship between the potential (kinetic) energy and the polar moment of inertia of the sphere. This property, in turn, is explained by the energy conservation law of a body during its motion and evolution in the form of the dynamical equilibrium equation or generalized virial theorem.

If we accept the conditions of the mass density separation presented in Sects. 7.5–7.8, then the range of curves of the density distribution gives a principal picture of its evolutionary redistribution and can be applied for reconstruction of the Earth's history. It follows from Eq. 7.71 that the density value of each overlying shell of the created Earth should be higher than the mean density of the inner mass. Otherwise, such a shell can not be retained and should be dispersed by the tidal forces. It follows from this that the planet's formation process should be strictly operated by the dynamical laws of motion in the form of the virial oscillations and accompanied by differentiation of the non-uniform shells. The model of a cyclonic vortex which was proposed by Descartes is the most acceptable from the point of view of the considered ideas of planets' and satellites' creation from a common nebula. This problem needs a separate consideration. We only note here that from the presented curves of radial density distribution the parabola (4) more closely reflects the present-day planet's evolution as fixed by observations. In this case location of the Earth's reduced inertia radius falls on the lower mantle and the reduced gravity radius – on the upper mantle. The density maximum falls also on the lower mantle. Its value is found by ordinary means, namely, by taking the derivative from the density distribution law as equated to zero. From here ρ_{max} = 8.57 g/cm^3 is found to be at a distance of r = 3,122 km. It means that the density maximum comes close to the border of the outer core where, as seismic observations show, the main density jump occurs. Curve (4) corrects the values of the radial density distribution in the mantle and changes its earlier interpretation in the outer and inner core. Because of zero values of the transverse velocities the matter of the inner core has

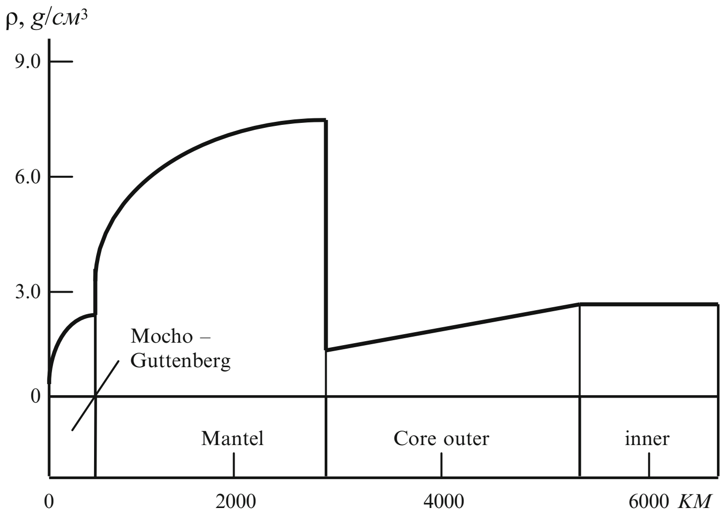

Fig. 7.5 Radial density distribution of the Earth by the authors' interpretation

a uniform density structure and, from the point of view of the equilibrium state, seems to be in a gaseous state at a pressure of 1–2 atmospheres. Taking into account the location of the maximum density value, there is a reason to assume that the outer core matter stays in the liquid or supercritical gaseous stage. In any case, the density and pressure of the inner and outer core are much lower and should have values corresponding to the seismic wave velocities. On the basis of the equation of mass density differentiation (7.71) we interpret the density jumps observed (by seismic data) near-by the Mohorovičić-Guttenberg and at the outer core borders as the borders of the shell's dynamical equilibrium. A shell which is found over that border appears in a suspended state due to action of the radial component of the gravitational pressure developed by the denser underlying shell. While the thickness of the suspended shell is growing it acquires its own equilibrium pressure (iceberg effect). The extremely high pressures in the Earth's interior, which follow from the hydrostatic equilibrium conditions, are impossible in its own force field.

The concept discussed above in relation to the Earth's density distribution is illustrated in Fig. 7.5 .

7.10 Oscillation Frequency and Angular Velocity of the Earth's Shell Rotation

In order to determine numerical values of frequency of the virial oscillations and the angular velocities, which are the main dynamical parameters of the Earth's shells, we accept equation (4) of the density distribution (7.73) as the first approximation. All further relevant calculations can be made by applying this equation.

We know the mean values of the planet's density $\rho_0 = 5.519$ g/cm^3 and angular velocity of the upper shell $\omega_t = 7.29 \cdot 10^{-5}$ s^{-1}. Applying these values, the frequency and period of the virial oscillations, and the coefficient k_e of the tangential component of the inner forces, can be found. In accordance with Eq. 7.67 the frequency of the upper shell is equal to

$$\omega_0(\mathrm{r}) = \sqrt{\frac{4}{3}\pi \mathrm{G}\rho_0(\mathrm{r})} = \sqrt{\frac{4}{3} \cdot 3.14 \cdot 6.67 \cdot 10^{-8} \cdot 5.519} = 1.24 \cdot 10^{-3}\,\mathrm{s}^{-1}.$$

The period of oscillation is found from the expression

$$\mathrm{T}_\omega = \frac{2\pi}{\omega_0(\mathrm{r})} = \frac{6.28}{1.24 \cdot 10^{-3}} = 5060.4\,\mathrm{s} = 1.405\,\mathrm{h}.$$

The product of the found frequency and the Earth's radius gives the value of the planet's first cosmic velocity, the mean value of which is

$$v_1 = \omega_0(\mathrm{r})\mathrm{r}_e = (1.24 \cdot 10^{-3}) \cdot 6370 = 7.9\,\mathrm{km/s}.$$

Unlike the usual expression for the first cosmic velocity in the form of $v_1 = \sqrt{\mathrm{GM/r}}$, we used here the physical condition of the dynamical equilibrium at the Earth's surface between the inner gravitational pressure of interacting masses and the outer background pressure including atmospheric pressure.

Given below our own observation data on the near-surface atmospheric pressure and temperature oscillations at the near-surface layer and the results of the spectral analysis prove the above theoretical calculations of the planet's frequency of virial oscillations.

Now, applying the known mean value of the Earth's angular velocity $\omega_t = 7.29 \cdot 10^{-5}$ s^{-1} and the known value of the frequency of virial oscillations for the upper shell $\omega_o = 1.24 \cdot 10^{-3}$ s^{-1} by Eq. 7.69 the coefficient k_e can be found

$$\mathrm{k}_e = \frac{\omega_t^2}{\omega_0^2} = \frac{\left(7.29 \cdot 10^{-5}\right)^2}{\left(1.24 \cdot 10^{-3}\right)^2} = \frac{1}{289.33} = 0.003456.$$

The coefficient k_e is known in geodynamics as a parameter that shows the ratio between the centrifugal force at the Earth's equator and the acceleration of the gravity force there equal to $k_e = 1/289.37$ (Melchior 1972). The parameter is used to study the Earth's figure based on the Clairaut hydrostatic theory.

7.10.1 Thickness of the Upper Earth's Rotating Shell

It is known that the value of the mean linear velocity of the upper planet's shell is $v_e = 0.465$ km/c. We can find the thickness h_e at which the velocity v_e corresponds to the found frequency of radial oscillations of the shell $\omega_o = 1.24 \cdot 10^{-3}$ s^{-1}

$$\mathrm{h}_e = \frac{\mathrm{v}}{\omega_o(\mathrm{r})} = \frac{0.465}{1.24 \cdot 10^{-3}} = 375\ \mathrm{Km}. \tag{7.82}$$

Such is the thickness of the upper shell of the Earth which is rotating by forces in its own force field. It is assumed that the shell is found in the solid state. In reality it is known that the rigid shell has a thickness less than 50 km. The remaining more than 300 km-thick part of the shell has a viscous-plastic consistency, the density of which increases with depth. The border of the shell has a decreased density because of the melted substance due to high friction and saturation by a gaseous component. The border plays a role of some sort of spherical hinge. Because the density of the Earth's crust is lower than that of the underlying matter, then it occurs in the suspended state. During the oscillating motion the crust shells are affected by the alternating-sign acceleration and the inertial isostatic effects.

7.10.2 Oscillation of the Earth's Shells

Let us obtain the expression of virial oscillations for the Earth's other shells by applying expression (4) of (7.73) for the radial density distribution. Write Eq. 7.67

$$\omega_0(r) = \sqrt{\frac{4}{3}\pi G\rho_0(r)},$$

Where

$$\rho_0(\mathrm{r}) = \frac{\mathrm{m}_0(\mathrm{r})}{\frac{4}{3}\pi \mathrm{r}^3} = \frac{4\pi\int\limits_0^r \mathrm{r}^2\rho(\mathrm{r})\mathrm{dr}}{\frac{4}{3}\pi \mathrm{r}^3} = \frac{\frac{4}{3}\pi \mathrm{r}^3\rho_0\left(\frac{3}{5}\mathrm{a}\frac{\mathrm{r}^2}{\mathrm{R}^2} + \frac{3}{4}\mathrm{b}\frac{\mathrm{r}}{\mathrm{R}} + \mathrm{c}\right)}{\frac{4}{3}\pi \mathrm{r}^3}$$
$$= \rho_0\left(\frac{3}{5}\mathrm{a}\frac{\mathrm{r}^2}{\mathrm{R}^2} + \frac{3}{4}\mathrm{b}\frac{\mathrm{r}}{\mathrm{R}} + \mathrm{c}\right).$$

Then

$$\omega_0(\mathrm{r}) = \sqrt{\frac{4}{3}\pi\,\mathrm{G}\rho_0(\mathrm{r})} = \sqrt{\frac{4}{3}\pi\,\mathrm{G}\rho_0\left(\frac{3}{5}\mathrm{a}\frac{\mathrm{r}^2}{\mathrm{R}^2} + \frac{3}{4}\mathrm{b}\frac{\mathrm{r}}{\mathrm{R}} + \mathrm{c}\right)}$$

$$= 1.24\cdot 10^{-3}\sqrt{\left(-3.144\frac{\mathrm{r}^2}{\mathrm{R}^2} + 3.849\frac{\mathrm{r}}{\mathrm{R}} + 0.295\right)}. \tag{7.83}$$

At r/R = 0 then $\omega_o(r) = 0.6743{\cdot}10^{-3}\ s^{-1}$; at r/R = 1 then $\omega_o(r) = 1.24{\cdot}10^{-3}\ s^{-1}$; at $\rho_{max} = 8.57\ g/cm^3$ $\omega_o(r) = 1.486{\cdot}10^{-3}\ s^{-1}$, where r/R = 0.49.

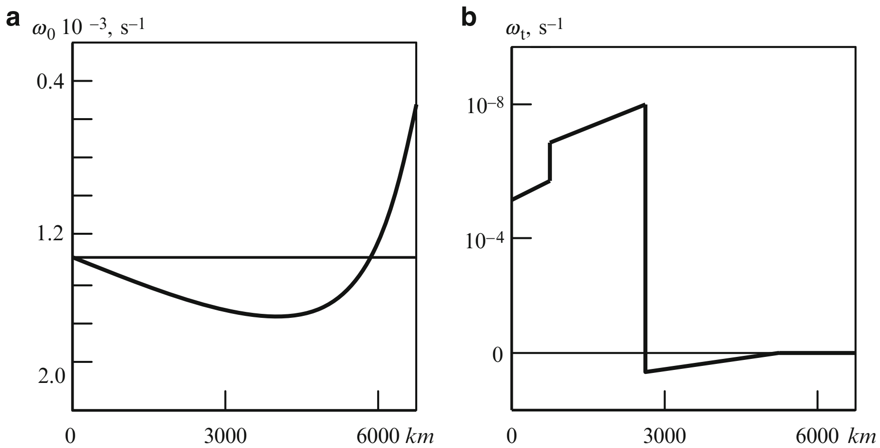

Fig. 7.6 Radial change in virial oscillation frequencies (**a**) and angular velocity of rotation (**b**) according to Eqs. 7.83 and 7.84

Figure 7.6a shows changes in the virial oscillation frequencies of the Earth's shells.

7.10.3 Angular Velocity of Shell Rotation

Angular velocity of the Earth's shell rotations is determined from Eq. 7.68

$$\omega_t(r) = \sqrt{\frac{4}{3}\pi G\rho_t(r)} = \sqrt{\frac{4}{3}\pi G\rho_0(r)\left(\frac{3}{5}a\frac{r^2}{R^2}+\frac{3}{4}b\frac{r}{R}+c\right)k_e(r)}$$

$$= \omega_0(r)\sqrt{\left(\frac{3}{5}\,a\,\frac{r^2}{R^2}+\frac{3}{4}\,b\,\frac{r}{R}+c\right)k_e(r)}$$

$$= \omega_0(r)\sqrt{\left(-3.144\,\frac{r^2}{R^2}+3.8475\,\frac{r}{R}+0.295\right)k_e(r)} \tag{7.84}$$

where $\omega_t(r)$ is the angular velocity of the shell rotation; $\omega_o(r)$ is the shell oscillation frequency which is determined by Eq. 7.83.

The geodynamic parameter $k_e(r)$, which expresses the ratio of the tangential component of the force field and the gravity force acceleration for the upper shell, is approximated as

$$k_e(r) = \frac{\omega_t^2(r)}{\omega_0^2(r)} = \frac{h_c^2}{R^2}$$

where h_c is the distance between the sphere's surface and the density jump; R is the sphere's radius.

At $r/R = 1$ $k_e(r) = 0.003456$; at $r/R = 0$ $k_e(r) = 1$, $\omega_t(0) = \omega_o(0)$, i.e. the virial oscillation frequency corresponds to the gravity pressure of the uniform density masses. In this particular case we are interested in changes of the angular velocity of rotation of the upper (1,000 km) and lower (up to the core border) mantle (2,900 km) shells. Figure 7.6b shows the radial change of the angular velocity of rotation calculated by Eq. 7.84. It is seen that the angular velocity at the lower mantle – outer core is close to zero but changes its direction.

We emphasize once more that Eqs. 7.83 and 7.84 express the third Kepler law which determines radial distribution of both the virial oscillation frequencies and the angular velocities of rotation. Numerical values of these parameters are determined by the radial density distribution law. It also determines the density jumps which mark the effect of the shell's isostatic equilibrium.

7.11 Perturbation Effects Studied on Basis of Dynamic Equilibrium

The most noteworthy effects of dynamics of the Earth are the interrelated phenomena of the precession and nutation of the axis of rotation, tidal effects of the oceans, and atmosphere, the axial obliquity and declination of the plumb-line and the gravity change at each point of the planet's outer force field. The present-day ideas about the nature of these phenomena were formed on the basis of the Earth's hydrostatic equilibrium and since old times were considered as effects of perturbation from the Sun, the Moon and other planets. All the above phenomena represent periodic processes and many observational and analytical works were done for their understanding and description. The present-day studies of these processes are still continuing to be specified and corrected. This is because such topical problems as correct time, ocean dynamics, short and long-term weather and climate changes and other environmental changes are important for every-day human life.

Now, after it was found that the conditions of the hydrostatic equilibrium are not acceptable for study of the Earth's dynamics, we reconsider the nature of the phenomena by applying the concept of the planet's dynamical equilibrium and developing a novel approach to solving the problem.

7.11.1 The Nature of Perturbation for Orbital Motion of a Body

To begin, let us consider physical meaning of the gravitational perturbation for interacting volumetric (but not point) body masses. To the contrary of hydrostatics, where the measure of perturbation in the precession-nutation and the tidal phenomena is the perturbing force, in the dynamic approach that measure of perturbation is power pressure. In Chap. 2 we concluded that the mass points and the vector forces as a physical and mathematical instrument in the problem solution of dynamics of the Earth in its own force field are not applicable. This is because the outer vector central force field of the interacting volumetric masses incorrectly expresses dynamical effects of their interaction. As a result, the kinetic effect of interaction of the mass particles, namely, the kinetic energy of their oscillation, is lost. And also the geometric center of a body is accepted as the gravity center and center of the inertia (reaction). In dynamics it leads to wrong results and conclusions. In this connection we found that in dynamics of the Earth as a self-gravitating body the effect of gravitational interaction of mass particles should be considered as the power pressure. In addition, in this case we are free in our choice of a reference system. Our conclusion does not contradict to Newton's physical ideas which are presented in Book I of his *Principia* where he says: “I approach to state a theory about the motion of bodies tending to each other with centripetal forces, although to express that physically it should be called more correct as pressure. But we are dealing now with mathematics and in order to be understandable for mathematicians let us leave aside physical discussion and apply the force as its usual name”.

Accepting the power pressure as an effect of gravitational interaction, we come to understanding that, in the considered problem of the mutual perturbations between the Earth, the Moon, and the Sun, the interaction results not between the body centers or shells along straight lines, but between the outer force fields of the bodies and between their inner force fields of the shells. Satellite observations show that the outer force field, induced by the Earth's mass, has 4π-outward direction of propagation and acquires a wave nature. We consider this outer wave force field as a physical media by which the bodies transmit their energy. Thus, the Earth and other planets are held and move on the orbits by the power of the outer wave field of the Sun. This power, in terms of its normal and equal to its tangential components, remains valid since the planets separation (see above Sects. 7.5, 7.6 and Chap. 8). In order to demonstrate validity of the above conception let us calculate the mean values of velocity of the Earth and the Moon orbital motion from the frequencies of oscillation of the respective outer wave fields of their parents.

In accordance with Eq. 7.67, the frequency ω_s of oscillation of the self-gravitating Sun's outer force field at the mean distance $R_e$of the Earth's orbit is

$$\omega_s = \sqrt{\frac{GM_s}{R_{es}^3}} = \sqrt{\frac{6.67 \cdot 10^{-8} \cdot 1.99 \cdot 10^{33}}{\left(1.496 \cdot 10^{13}\right)^3}} = 1.9931 \cdot 10^{-7}\, s^{-1},$$

where M_s is the Sun's mass; R_{es} is the mean distance between the Earth and the Sun.

In accordance with wave mechanics, the mean value of the Earth's orbital velocity is

$$v_e = \omega_s R_{es} = 1.9931 \cdot 10^{-7} \cdot 1.496 \cdot 10^{13} = 2.98 \cdot 10^6\, cm/s = 29.9\, km/s$$

If we extend the outer force fields of the Sun and the Earth up to equality of their reduced densities, then in accordance with the same Eq. 7.67 the border between the two interacting fields can be found. By calculation the mean (nodal) value of the Earth's field border is found to extend up to $2.128 \cdot 10^9$ m and the Sun's border – to $1.478 \cdot 10^{11}$ m.

Applying the same procedure, the orbital frequency oscillation ω_e of the Earth's outer force field at the Moon distance R_{me} has value

$$\omega_e = \sqrt{\frac{GM_e}{R_{me}^3}} = \sqrt{\frac{6.67 \cdot 10^{-8} \cdot 5.976 \cdot 10^{27}}{\left(3.844 \cdot 10^{10}\right)^3}} = 2.64907 \cdot 10^{-6}\,\mathrm{s}^{-1}, \qquad (7.85)$$

where M_e is the Earth's mass; R_{me} is the mean distance between the Moon and the Earth.

Then the Moon's orbital velocity is

$$\bar{v}_m = \omega_e R_m = 2.64907 \cdot 10^{-6} \cdot 3.844 \cdot 10^{10} = 1.0183\ \mathrm{km/s}. \qquad (7.86)$$

The Moon's border of the outer force field in the nodal plane extends to the Earth up to $0.72 \cdot 10^8$ m and the Earth's border – to $3.724 \cdot 10^8$ m.

The obtained values of velocities, as well as the values for the pericenters and apocenters, are exactly the same as known from observation. It means that the observed ecliptic inclination relative to the equatorial plane of the Sun and inclination of the Moon's orbit relative to the ecliptic reflect asymmetric distribution of the solar and the planet's masses. It also means that the observed inclination of the Moon's orbit plane and the ecliptic are governed by asymmetric distribution of the Earth's and the Sun's force fields. The force fields of the Earth and the Moon, together with the bodies themselves being local "secondary" inclusions in the powerful force fields of the Sun and the Earth, are obliged to adjust their positions in order to be in dynamic equilibrium. The observed parameters of the orbits and their inclination relative to the plane diameters of the Sun, the Earth and the Moon give a general view of the asymmetric distribution of the body's masses. In particular, the northern hemisphere of the Earth is more massive than the southern one. In the perihelion the northern hemisphere is turned to the less massive hemisphere of the Sun. So that, the polar oblateness of each body controls the location of its pericenter and apocenter, and the equatorial oblateness of each body responds to location of its nodes. Thus, the body motion in the outer force field of its parent occurs under strict conditions of dynamic equilibrium which is also the main condition of its separation. It follows from the condition of dynamic equilibrium that the orbital motion of the Earth and the Moon reflects asymmetry in mass density distribution of the Sun, the Earth, and the Moon and asymmetry in the potential of the outer wave field distribution. Only the structure of the Sun's outer wave field controls the Earth's trajectory at the orbital motion and the Earth's force field manages the orbital motion of the Moon, but not vice versa or somehow else.

7.11.2 *Change of the Outer Force Field and the Nature of Precession and Nutation*

At the right time of motion of the bodies with the outer wave fields, their mutual perturbations are transferred not directly from each body to the other one or from their shells, but through the outer fields by means of the corresponding active and reactive wave pressure of the interacting fields. There is an important dynamic effect of all the perturbations. This is the continuous change in the outer wave field of each body which proceeds from its non-uniform radial distribution of the mass density. As it was earlier shown, the non-uniform radial distribution of mass density initiates the differential rotation of the body shells. And, in accordance with Eqs. 7.67 and 7.68 expressing the third Kepler's law, the reduced body shells' perturbing effects are transferred to the other body by means of the outer wave field. So that the Sun, for instance, through its outer wave field, continuously transfers to the Earth all the perturbations resulting during rotation of the interacting masses of the shells. The Earth, in the framework of the energy conservation law, demonstrates all the perturbations by changes in its orbit turns around the Sun (see below Fig. 7.7).

Earlier it was shown that in the case of non-uniform distribution of mass density the body's potential and kinetic energies have radial and tangential components which induce oscillation and rotation of the shells. It was defined by Eq. 7.82 that the observed daily rotation of the Earth concerns only the upper shell with thickness of ~375 km and reaches the near-by Mohorovičić-Guttenberg discontinuity. By the

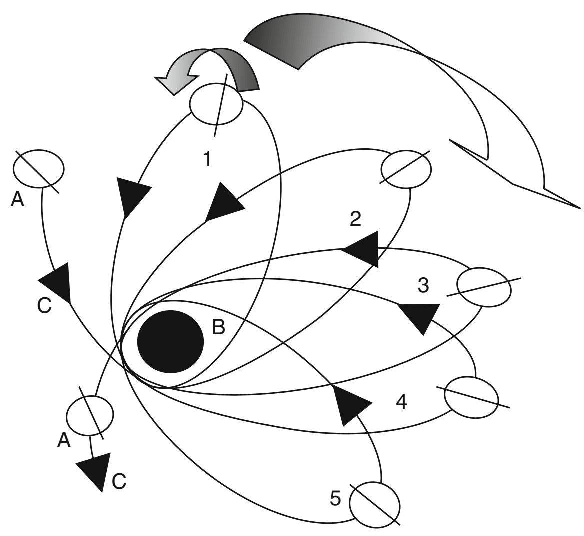

Fig. 7.7 Real picture of motion of a body A in the force field of a body B. Digits identify succession of turns of the body A moving around body B along the open orbit C

same reasoning it is not difficult to find the thickness of the upper shells for the Sun and the Moon correspondingly equal to:

$$h_s = \frac{v_s}{\omega_{0s}(R_s)} \approx \frac{2}{6.28 \cdot 10^{-4}} \approx 1600 \text{ km}, \tag{7.87}$$

$$h_m = \frac{v_m}{\omega_{0m}(R_m)} \approx \frac{4.56 \cdot 10^{-3}}{9.66 \cdot 10^{-4}} \approx 4.72 \text{ km}. \tag{7.88}$$

We do not know real values and angular velocities for the inner shells of the three bodies. These velocities have a direct interrelation with the observed changes in parameters of the orbital motion of the Earth and the Moon including the retrograde motion of the orbital nodes and the apsidal line. In this connection let us try to understand first of all the nature of precession and nutation of the bodies from the point of view of the dynamic approach.

It was noted above that, in accordance with the hydrostatic approach, precession of the equinoxes of the Earth is an effect of the net torque of the Moon and the Sun on the equatorial "bulge" aroused from gravitational attraction. The torque aspires to diminish inclination of the equatorial belt with surplus mass relative to the ecliptic and induce the retrograde motion of the nodal line. In addition, because the ratio of distance between the interacted bodies is changed, then the relationship between the forces is also changed. In this connection the precession is accompanied by nutation (wobbling) motion of the axes of rotation.

Analysis of orbits of the artificial satellite motion around the Earth shows that, in spite of absence of the equatorial "bulge" of mass, the apparatus demonstrates the precession effect. Its orbital plane has a clockwise rotation with retrograde motion of the nodal line. But a new explanation of the phenomenon is given. It appears that the retrograde motion of the nodal line associates with the Earth equatorial and polar oblateness. The amplitude of the nodal line shift depends on the satellite orbit inclination to the Earth's equatorial plane. In the case of the poles' orbital plane the nodal line shift is completely absent. This is because the pole motion excludes both the polar and the equatorial oblatenesses of the Earth. The direction of motion of the apsidal line depends on the satellite's orbit inclination and is determined by the Lentz law.

It is also known that for the other free-of-satellite planets the retrograde motion of the nodal line is also a characteristic phenomenon called the "secular perihelion shift". It was found from observation of Mercury, Venus, Earth and Mars that their secular perihelion shifts are decreased from ~ $40''$ through ~ $8.5''$, ~ $5''$ to ~ $1.5''$ accordingly [Chebotarev 1974].

All these facts imply that the explanation given for the satellites' precession depending on their orbital inclination to the ecliptic is correct. But the nature of this unique phenomenon, characteristic for all celestial bodies, are inconsistent with the hydrostatic approach and should be reconsidered, taking also into account the satellite observations.

7.11.3 Observed Picture of a Body Precession

The precession of the Earth, the Moon and the artificial satellites in the form of motion of an orbital plane toward the backward direction of the body's motion should be considered as a virtual explanation of the phenomenon. In fact, the orbit's plane is a geometric shape traced by the body. And there is no reason to consider its movement without the body itself. There is no difficulty to present the real body motion in space in two opposite directions synchronously. In particular, the actual picture of the Earth, the Moon and the satellite motion in counterclockwise direction and retrograde movement of the nodal line is shown in Fig. 7.7.

Here the satellite is moving in the counterclockwise direction along the unlocked elliptic orbit 1 in the continuously changing (perturbed by oblatenesses) planet's force field. Because of the counterclockwise rotation of the Earth's mass, the satellite in perigee started to move on the orbit 2 and makes a shift in retrograde direction in the ascending and descending nodes. At the same time the eccentricity of the orbit 2 changes by a proper value. Analogously the body passes on orbit 3, 4, 5 and so on. The theory of dynamic equilibrium of the Earth explains the physics of the observed phenomenon as follows.

7.11.4 The Nature of Precession and Nutation

The dynamic equilibrium theory assumes that the Earth is a self-gravitating body, the interacting mass particles of which induce the inner and outer force fields. Separation of the planet's asymmetric shells results by the inner force field and depends on the law of the radial mass density distribution. The normal component of the body's power pressure provides oscillation, and the tangential component induces rotation of the shells having a different angular velocity. At the same time the mantle shells A and the outer shell of the core B may have the same (Fig. 7.8a) or opposite direction (Fig. 7.8b) of rotation depending on the radial mass density distribution.

The seismic data show that the inner core C has a uniform density distribution. Because of this, it does not rotate and its potential energy is realized in the form of oscillation of the interacting particles. The potential E of the outer force field is controlled by integral effect of the interacted masses of all the shells and presented by the reduced shell D having continuously changing power.

The energy of the Earth's outer force field is changed from the body surface in accordance with the 1/r law and at every r is continuously varied because of differences in the angular velocity of rotation of the shell's masses. This force field controls the direction and angular velocity of orbital motion of a satellite. Taking into account the non-uniform and asymmetric distribution of the masses of rotating shells, the change in the trajectory of the body motion is accompanied by a corresponding change in eccentricity of the orbit both at each and subsequent turns.

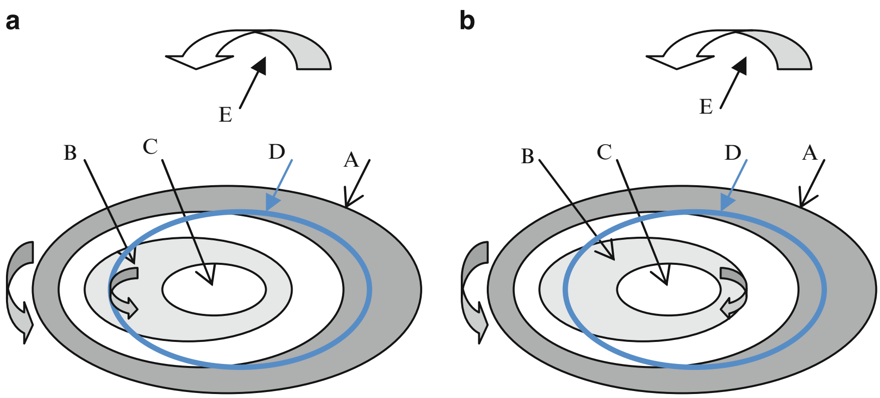

Fig. 7.8 Sketch of rotation of the Earth's shells by action of the inner force field: *A* is the mantle shells; *B* is the outer core; *C* is the inner core; *E* is the outer force field; *D* is the reduced shell of the inner force field of the planet

Its maximum value is reached when the non-uniformities of the rotating masses coincide and the minimal value appears at the opposite position.

It is worth noting that the effect of retrograde motion of the nodal line of the Earth, the Moon and artificial satellites appears to be a common phenomenon because the induced by the Sun and the Earth outer force fields are changing with a finite velocity. The conclusion follows from here that the Sun has the same effects in its shell structure and motion. It is obvious that the other planets with their satellites have the same character of structure and motion.

If one takes into account the effect of a planet's orbital plane inclination to the equatorial plane of the Sun, then the above changes are found to follow the law of 1/r This observable fact proves our conclusion that the changes in the outer force field of a body are controlled by rotation of its reduced inner force shell (see the force shell D on Fig. 7.8). It explains why Mercury has maximal value of the "secular perihelion shift" between the other planets.

Thus, the Earth's orbital motion and retrograde movement of its nodal line are controlled by the Sun's dynamics of the masses through the outer force field. The Earth plays the same role for the Moon and the artificial satellites. As to the nutation motion, then its nature is related to the same peculiarities in the structure and motion of the bodies but the effects of their perturbations are fixed by the axis wobbling.

7.11.5 The Nature of Possible Clockwise Rotation of the Outer Core of the Earth

The question arises why the outer planet's core may have a clockwise rotation. It was shown in Section 2.6 that the law of radial density distribution determines the direction of a body's shell rotation (Fig. 2.2).

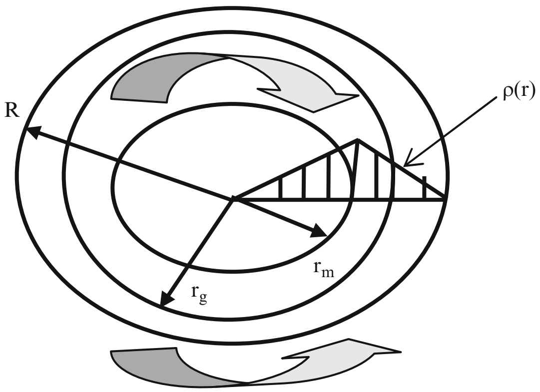

Fig. 7.9 Dependence of the parabolic law of radial density distribution on the shell rotation for the Earth. Here r_m and r_g are the reduced radiuses of inertia and gravitation

It was found that in the case of uniform mass density distribution all energy of the mass interaction is realized in the form of oscillation of the interacting particles (Fig. 2.2a). If the density increases from the body's surface to the center, then there are oscillations and counterclockwise rotation of shells (Fig. 2.2b). Increase of mass density from center to surface leads to oscillation and clockwise rotation with different angular velocities of the body shells (Fig. 2.2c). Finally, the parabolic law of radial density distribution (Fig. 7.9), where the density increases from the surface and then it decreases, leads to oscillation and reverse directions of rotation. Namely, the upper shells have a counterclockwise and the central shells – clockwise rotation. The case demonstrated on Fig. 7.9, obviously, is characteristic for a self-gravitating body.

Note that direction of the body rotation depends on radial density distribution and corresponds with the Lenz right-hand or right-screw rule, well known in electrodynamics. Taking into account the observed effect of the retrograde motion of the satellite nodal line, the gravitational induction of the inner and outer force fields of the Earth has a common nature with electromagnetic induction noted earlier. Just Fig. 7.9 may explain the nature of the retrograde motion of the nodal line of a satellite orbit related to the finite velocity in the potential changes of the outer Earth's force field induced by the interacted mass particles. The continuous and opposite-directed movement of the asymmetric mass density distribution of the mantle and the outer core (Fig. 7.8) seems to be the physical cause of precession, nutation and variation of the inner and outer force fields observed by satellites. This idea is proved by the satellite data about the retrograde motion of the nodal line depending on inclination of its orbital plane with respect to the planet's equatorial plane

It is worth recalling, from the literature, that the idea of dynamical effects of the probably liquid core of the Earth has been discussed among geophysicists for a long time. (Melchior 1972).

7.11.6 The Nature of the Force Field Potential Change

It follows from the above discussion that in the frame of the considered dynamical approach, the variation of potential of the inner and outer force field relates to the non-uniform distribution of mass density of a self-gravitating body. Rotation of the outer and inner shells together with the induction finite velocity delay leads to the observed effects of precession, nutation and variation of their own force fields.

7.11.7 The Nature of the Earth's Orbit Plane Obliquity

Celestial mechanics does not discuss the problem of obliquity of the planet's and satellite's orbit planes and accepts it as an observable fact. The theory of dynamic equilibrium explains this phenomenon and the nature of apocenters and pericenters by asymmetric distribution of masses and by effect of rotation of asymmetric shells of self-gravitating bodies (see Fig. 7.9). In fact, if the mass of the Sun's shells has an asymmetric distribution, then the potential of the outer force field has the same asymmetry. This asymmetry determines inclination of the Earth's orbit plane relative to the plane of the Sun's rotation. Each point of the orbit reflects a condition of dynamical equilibrium of the interacting outer force fields of the planet and the Sun. The position of the Earth's aphelion and perihelion reflects the position of the reduced maximal and minimal concentration of the Sun's mass density in the shells. Because the Sun's asymmetric shells have different angular velocities of rotation, then amplitude of the nodal line will decrease with increasing distance between the mass anomalies and vise versa. The effect of variation of the nodal line is proved by observation. So, the present-day angle of ecliptic inclination to the plane of rotation of the Sun equal to ~7°15 expresses the relation between maximum and minimum concentrations of the reduced mass density of the Sun's shells. An analogous effect is shown by inclination of the Moon's orbital plane to the plane of rotation of the Earth.

7.11.8 The Nature of Chandler's Effect of the Earth Pole Wobbling

As it was noticed, changes in the planet's inner force field are observed in the form of nutation or wobbling of the axis of rotation. The axis itself reflects the dynamics of the upper planet's shell, the thickness of which, by our estimate, is about 375 km. The Moon is rotating about the Sun in the force field of the Earth which is perturbed by its natural satellite. Its maximum yearly perturbation should be the Chandler effect. The Moon's yearly cycle seems to be the ratio of the Earth's to the Moon's month (in days). Then this cycle is $365(30.5/27) \approx 410$ days.

7.11.9 Change in Climate as an Effect of Rotation of the Earth's Shells

The above analysis of dynamical effects of the Earth's shells is based first of all on the data of satellite's orbit changes and measurements of the planet's force field. Unfortunately, a specific feature of an artificial satellite orbital motion is its artificial velocity which is ~16 times higher than the angular velocity of the upper Earth's shell. In this connection all its parameters of satellite motion are unnatural. So, we cannot directly divide the natural component of its nodal retrograde shift in order to get the total picture of perturbations which propagate the Earth's inner shells. This is an experimental problem.

But there are also long term astronomical observations of the Earth's dynamics relative to the far stars, the results of which correspond to the presented ones. In addition, periodicity in rotation of asymmetric inner shells of the Sun can be fixed by climatic changes on the Earth over a long period of time. Such changes were being studied, for instance, by data of the oxygen isotopic composition in mollusk shells over a number of years. Figure 7.10 demonstrates the results of Emiliani (1978) who studied the core obtained during deep sea drilling in the Caribbean basin.

The author obtained the picture of climate change in the Pleistocene era over 730,000 years. It is seen that the periods of climate change vary from 50 to 120,000 years. It means that the pure period of rotation of the asymmetric mass shells of the Sun is absent and the orbital trajectory has not been locked into place during the studied time.

7.11.10 The Nature of Obliquity of the Earth's Equatorial Plane to the Ecliptic

It is obvious that the obliquity of the planet's equatorial plane is related to the polar and equatorial oblateness of the Earth's masses. It follows from Eq. 7.68 that the obliquity, in turn, is determined by the tangential component of the inner force pressure generated by the non-uniform radial mass density distribution.

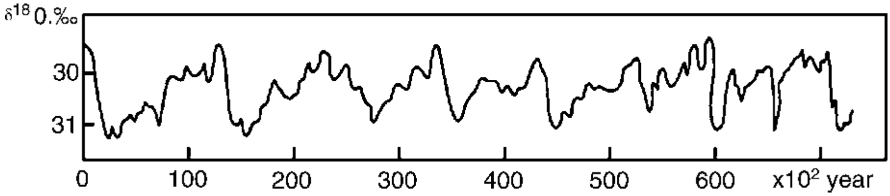

Fig. 7.10 Isotopic composition of oxygen in shells of mollusk *Globigerinoides Sacculifera* within time period 0–730,000 year [Emiliani 1978]

This tangential component of the inner force field induces the inner field of the rotary moments, the energy of which was discussed in Section 6.9 and presented in Table 4. The obliquity value can be obtained from the ratio of the potential energy of the uniform U_o and non-uniform U_t body of the same mass. Accepting this physical idea and the data of Table 4, we write and obtain:

$$\cos\Theta = \frac{U_o}{U_t} = \frac{\alpha_o^2}{\alpha_t{}^2} = \frac{0.6}{0.66} = 0.909, \quad \Theta = 24.5^o, \tag{7.89}$$

where α_o^2 and $\alpha_t^2$are the structural form-factors taken from Table 4.

The error obtained in calculation of obliquity by formula (6.89) equal to about 1^o or $\Delta\alpha_t^2 = 0.006$ – can be explained by the accepted law of the continuous radial distribution of the planet's mass density.

Equation 7.89 expresses the integral effect of the obliquity of the planet's equatorial plane which is observed on the surface of the upper rotating shell. It was shown earlier that the observed obliquity is really an integral dynamical effect of the Earth's mass including the upper part of the Gutenberg shell. But being in a suspended state, relative to the other parts of the body, the upper shell is able to wobble as if on a hinge joint by perturbation from the Sun and the Moon. This effect of the upper shell wobbling gives an impression of the axial wobbling.

By the same cause the obliquity of the ecliptic with respect to the solar equator is determined by the Sun's polar and equatorial oblateness. The trajectory of the Earth's orbital motion at each point is controlled by the outer asymmetric solar force field in accordance with the dynamic equilibrium conditions. And only in the nodes, which are common points for equatorial oblateness of the Sun and the Earth, is the Huygens' effect of the innate initial conditions fixed by the third Kepler's law.

7.11.11 Tidal Interaction of Two Bodies

Let us consider the mechanism and effects of interaction of the outer force pressure of two bodies being in dynamic equilibrium. Come back to the mechanism and conditions of separation of a body mass with respect to its density when a shell with light density is extruded to the surface. Rewrite Eq. 6.71 for acceleration of the gravity force in points A and B of the two body shells (Fig. 7.7b) and their densities ρ_M and ρ_m

$$q_{AB} = 4\pi G r\left(\frac{2}{3}\rho_M - \rho_m\right), \tag{7.90}$$

After the shell with density ρ_m appears on the outer surface of the body, the condition of its separation by Eq. 7.90 will be:

$$\rho_M > 2/3\ \rho_m \tag{7.91}$$

The gravitational pressure will replace the shell up to the radius A + δA, where the condition of its equilibrium reaches $\rho_M = \rho_m$. This condition is kept on the new border line between the body and its upper shell. Taking into account that the shell in any case has a thickness, then, by the Archimedes law, the body will be subject to its hydrostatic pressure. If the separated shell is non-uniform with respect to density, then a component of the tangential force pressure appears in it, and the secondary self-gravitating body (satellite) is formed. The new body will be kept on the orbit by the normal and equal tangential components of the outer force pressure. In this case the reaction of the normal gravitational pressure will be local and non-uniform. If the upper shell is uniform with respect to density, then the reaction of the normal gravitational pressure along the whole surface of the body and the shell remains uniform. In this case the separated shell remains in the form of a uniform ring.

The above schematic description of the physical picture of the separation and creation of a secondary body can be used for construction of a mechanism of the tidal phenomena in the oceans, the atmosphere and the upper solid shell at interaction between the Earth and the Moon. The outer gravitational pressure of the Moon, due to which it maintains itself in equilibrium on the orbit, at the same time renders hydrostatic pressure on the Earth's atmosphere, oceans and upper solid shell through its outer force field. This effect determines the tidal wave in the oceans and takes active part in formation and motion of cyclonic and anti-cyclonic vortexes. In accordance with the Pascal law, the reaction of the Moon's hydrostatic pressure is propagated within the total mass of the ocean water and forms two tidal bulges. Because the upper shell of the Earth is faster-moving relative to motion of the Moon, the front tidal bulge appears ahead of the moving planet. Our perception of the ocean tides as an effect of attraction of the Moon appears to be speculative.

7.12 Dynamics of the Earth's Atmosphere and Ocean

The atmosphere and the oceans (collectively) are both upper shells of the Earth. The first of them occurs in a gaseous and the second one in a liquid phase. These shells exist in the solid Earth's outer force field but the atmosphere exists in dynamical equilibrium, and the oceans are in a weighted transition to a hydrostatic state. The atmosphere as a gaseous shell is totally "dissolved" in the Earth's outer force field in the form of atomic and molecular sub-layers, differentiated with respect to density, and these self-gravitating masses appear in a dynamical equilibrium state. Relatively homogeneous water masses of the oceans have too low a density (approximately 2/3) in comparison with the mineral crust to be an independent, with respect to its dynamics, self-gravitating shell. Therefore, it appears to be suspended in a semi-hydrostatic equilibrium relative to the crust's shell. Its inner gravitational pressure on the shell's surface is equilibrated with the atmospheric and outer gravitational pressure. The surface water is practically found to be in a limiting hydrostatic equilibrium. The small portion of the continuously pumping solar energy varies within 7% in the annual cycle and leads to the dynamical process of water transfer from its liquid to vapor, and vice verse phase.

Dynamics of the main nitrogen and oxygen components with close densities (atomic weights) of the atmospheric masses in a non-perturbed state could be represented by its own virial oscillation and rotation about the solid Earth. But the water vapor which is continuously injected into the gaseous shell, being a component of very large-capacity energy (1 cm^3 of water generates more than 1,000 cm^3 water vapor) appears to be the cause of the stormy dynamical perturbation within the near-surface shell of the atmosphere. The cyclonic activity of the atmospheric vapor has scientific, but mainly practical, interest. The weather and climate change by variation of the solar energy flux through the water vapor dynamics have a negative affect on the biosphere.

In turn, the oceans being in a balanced hydrostatic state, are continuously perturbed both by the inner gravity pressure of the planet related to the density differentiation of the shells and by perturbation from the Sun's and the Moon's outer force fields. All the above dynamical processes seem to be of interest for consideration from the point of view of dynamical equilibrium theory.

In this chapter we search for a solution of Jacobi's virial equation for the non-perturbed atmosphere as the Earth's shell which is affected by both inner and outer perturbations. In order to justify applicability of the virial equation for the study of dynamics of the atmosphere in the framework of a model of a continuous medium, we derive this equation from the Euler equations.

7.12.1 Derivation of the Virial Equation for the Earth's Atmosphere

We consider the problem of global oscillations of the Earth's atmosphere as its shell. We accept that the atmosphere is found to be in dynamical equilibrium both in its own force field and in the outer force field of the Earth. Derivation of the virial equation is done in the framework of the continuous medium model. We accept also that the dynamics of the atmosphere is described by Euler's equations and the medium is composed of an ideal gas.

The Euler equations are written in the form (Landau and Lifshitz 1954)

$$\rho \frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \rho(\mathrm{v}\nabla)\mathrm{v} = -\mathrm{grad}\ \mathrm{p} + \rho \mathrm{F}, \tag{7.92}$$

where ρ is the gas density; p is the gas pressure; $\partial \mathrm{v}/\partial \mathrm{t}$is the rate of velocity change at a fixed point of space; F is the density of mass forces.

In addition, the continuity equation

$$\frac{\partial \mathrm{p}}{\partial \mathrm{t}} + \mathrm{div}\ \rho \mathrm{v} = 0 \tag{7.93}$$

holds for a gas medium.

Multiplying Eq. 7.93 by *v* and summing the product with Eq. 7.1, we obtain

$$\frac{\partial \rho}{\partial \mathrm{t}}\mathrm{v} = -\rho(\mathrm{v}\nabla)\mathrm{v} - \mathrm{vdiv}\ \rho\mathrm{v} - \mathrm{grad\ p} + \rho\mathrm{F}. \tag{7.94}$$

We take the divergence of Eq. 7.94 and note that

$$\frac{\partial}{\partial t}[\mathrm{div}\ \rho\mathrm{v}] = \left[\frac{\partial}{\partial \mathrm{t}}(\mathrm{div}\rho\mathrm{v})\right] = -\frac{\partial^2\rho}{\partial \mathrm{t}^2}.$$

Then Eq. 7.94 can be rewritten in the form

$$\frac{\partial^2\rho}{\partial \mathrm{t}^2} = \mathrm{div}[\rho(\mathrm{v}\nabla)\mathrm{v} + \mathrm{vdiv}(\rho\mathrm{v}) + \mathrm{gradp} - \rho\mathrm{F}] \tag{7.95}$$

Multiplying Eq. 7.95 by $r^2/2$ and integrating the obtained expression over the whole volume of the atmosphere, we have

$$\int\limits_{(\mathrm{V})} \frac{\mathrm{r}^2}{2}\frac{\partial^2\rho}{\partial \mathrm{t}^2}\mathrm{dV} = \int\limits_{(\mathrm{V})} \frac{\mathrm{r}^2}{2}\ \mathrm{div}[\rho(\mathrm{v}\nabla)\mathrm{v} + \mathrm{vdiv}(\rho\mathrm{v}) + \mathrm{grad\ p} - \rho\mathrm{F}]\mathrm{dV}. \tag{7.96}$$

The left-hand side of Eq. 7.96 can be rewritten in the form

$$\int\limits_{(\mathrm{V})} \frac{\mathrm{r}^2}{2}\frac{\partial^2\rho}{\partial \mathrm{t}^2}\mathrm{dV} = \frac{\partial^2}{\partial \mathrm{t}^2}\left(\frac{1}{2}\int\limits_{(\mathrm{V})} \mathrm{r}^2\rho\mathrm{dV}\right) = \ddot{\Phi},$$

where Φ is the Jacobi function of the atmosphere.

Thus, we obtain Jacobi's equation for the Earth's atmosphere derived from the Eulerian equation (7.92) and the continuity equation (7.93) as follows:

$$\ddot{\Phi} = \int\limits_{(\mathrm{V})} \frac{\mathrm{r}^2}{2}\ \mathrm{div}[\rho(\mathrm{v}\nabla)\mathrm{v} + \mathrm{vdiv}(\rho\mathrm{v}) + \mathrm{grad\ p} - \rho\mathrm{F}]\mathrm{dV}. \tag{7.97}$$

We assume that the Earth is a rigid spherical body with mass M and radius R and that the mass of the atmosphere is negligible in comparison with the Earth's mass. We can transform the right-hand side of Eq. 7.97 as follows:

$$\int\limits_{(\mathrm{V})} \frac{\mathrm{r}^2}{2}\mathrm{div}[\rho(\mathrm{v}\nabla)\mathrm{v} + \mathrm{vdiv}(\rho\mathrm{v}) + \mathrm{grad\ p} - \rho\mathrm{F}]\mathrm{dV}$$

$$= 2\mathrm{T} + \mathrm{U} + 3\int\limits_{(\mathrm{V})} \mathrm{pdV} - 4\pi\mathrm{r}^3\mathrm{p}\Big|_{\mathrm{R}}^{\mathrm{R_{max}}},$$

where

$$T = \frac{1}{2} \int\limits_{(V)} \rho v^2 dV$$

is the kinetic energy of the gas of the atmosphere in the Earth's gravitational field;

$$U = GM \int\limits_{(V)} \frac{\rho dV}{r}$$

is the potential energy of the atmosphere in the Earth's field;

$$3 \int\limits_{(V)} p dV$$

is the internal energy of the gas atmosphere;

$$-2\pi r^3 p|_{Re}^{Rmax}$$

is the energy of the outer surface force pressure effecting the atmosphere.

In fact, we can write an expression for the mass forces, taking into account the spherical symmetry of the system considered:
$F = -GM\frac{r}{r^3}$.
Then

$$-\int\limits_{(V)} \frac{r^2}{2} \operatorname{div}(\rho F) dV = \int\limits_{(V)} \frac{r^2}{2} \operatorname{div}\left(\rho GM \frac{r}{r^3}\right) dV = \int\limits_{R}^{Rmax} \frac{r^2}{2} \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{r^2 \rho GM}{r^2}\right) 4\pi^2 dr$$

$$= \frac{4\pi r^2}{2} \rho GM|_R^{Rmax} - 4\pi GM \int\limits_{R}^{Rmax} \rho r dr = U + \frac{4\pi r^2}{2} \rho GM|_R^{Rmax}.$$

Analogously, we have

$$\int\limits_{(V)} \frac{r^2}{2} \operatorname{div}(\operatorname{grad} p) dV = 4\pi \int\limits_{R}^{Rmax} \frac{r^2}{2} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial p}{\Im r}\right) r^2 dr$$

$$= \frac{4\pi r^2}{2} \frac{\partial p}{\partial r}\bigg|_R^{Rmax} - 4\pi r^3 p|_R^{Rmax} + 3 \cdot 4\pi \int\limits_{R}^{Rmax} r^2 p dr$$

$$= 3 \int\limits_{(V)} p dV - 4\pi r^3|_R^{Rmax} + \frac{4\pi r^4}{2} \frac{\partial p}{\partial r}\bigg|_R^{Rmax}.$$

It is easy to show that

$$\int\limits_{(V)} \frac{r^2}{2}\mathrm{div}[\rho(v\nabla)v + v\,\mathrm{div}(\rho v)]dV = \int\limits_{(V)} v^2\rho dV = 2T.$$

Owing to the immobility of the rigid Earth and the atmospheric boundary as well as the condition of continuity for the gas, the equilibrium hydrostatic condition holds along the normal to this border, which can be written in the form

$$\rho\frac{GM}{r^2}|_R = -\frac{\partial p}{\partial r}|_R.$$

If we assume $\rho r^2|_{Rmax} = 0$, $r^4\frac{\partial p}{\partial r}|_{Rmax} = 0$,then

$$\frac{4\pi r^2}{2}\rho GM|_R^{Rmax} + \frac{4\pi r^4}{2}\frac{\partial p}{\partial r}|_R^{Rmax} = 0$$

It follows from this that Jacobi's virial equation (7.97) takes the form

$$\ddot{\Phi} = 2T + U - 4\pi r^3 p|_R + 3\int\limits_{(V)} p dV. \tag{7.98}$$

Finally, using the energy conservation law for a continuum (Sedov 1970) and the conservativity of a system, following from the accepted model (the Earth is a solid body and atmospheric gas is ideal), Eq. 7.98 can be written in the form of the standard Jacobi virial equation

$$\ddot{\Phi} = 2E - U, \tag{7.99}$$

where the total energy of the atmosphere is conserved and equal to

$$E = T + U - 2\pi r^3 p|_R + \frac{3}{2}\int\limits_{(V)} p dV. \tag{7.100}$$

Let us make some important notes concerning the study of the atmosphere by a conventional approach based on use of the virial theorem.

The standard solution of the problem using the same assumption at zero approximation (barometric height formula) expresses the equilibrium condition between the atmospheric gravity and the gas pressure. In our case this condition is satisfied if Jacobi's virial equation (7.99) is averaged with respect to time, i.e. it is reduced to the condition of the hydrostatic equilibrium. But in that case the kinetic energy of the particle interaction is excluded. Assuming that the motion of the gas particles is finite, and after time averaging Eq. 7.99 over a sufficiently large time interval, the validity of the virial theorem is easily shown to be

$$\mathrm{E} = \mathrm{U}/2. \tag{7.101}$$

At known parameters of the atmosphere and the Earth, the potential and total energies of the atmosphere can be estimated as

$$\mathrm{U} = -\frac{\mathrm{GMm}}{\mathrm{R}} \approx -3.2 \cdot 10^{33}\ \mathrm{erg},$$

$$\mathrm{E} \approx -1.6 \cdot 10^{33}\ \mathrm{erg}.$$

7.12.2 Non-perturbed Oscillation of the Atmosphere

Let us now consider the solution of Eq. 7.99 for the spherical model of the atmosphere and find the dependence of its Jacobi function Φ (polar moment of inertia) on time in explicit form.

In the previous chapter it was shown that Eq. 7.99 is resolved both for the uniform medium and for the medium with radial density distribution by some law. In the last case the polar moment of inertia and potential energy are expanded on the uniform and tangential components and instead of Eq. 7.99 two Eqs. 7.61 and 7.62 are written.

Let us consider a solution for the uniform component of the atmosphere whose radial density distribution changes by the barometric equation. The uniform component of Eq. 7.99 has a solution when there is a relationship between the Jacobi function and the potential energy of the system in the form

$$|\mathrm{U}|\sqrt{\Phi} = \mathrm{const.} \tag{7.102}$$

Assume that the Earth has a spherical shape with mass M and radius R and be enveloped by a uniform atmosphere with mass m which has thickness Δ. Then the potential energy of the shell U_a, which is in the gravitational field of the sphere, is

$$\mathrm{U_a} = -\mathrm{GM}\rho_{\mathrm{a}} \int\limits_{\mathrm{R}}^{\mathrm{R}+\Delta} \frac{4\pi \mathrm{r}^2}{\mathrm{r}} \mathrm{dr} = -2\pi \mathrm{GM}\rho_{\mathrm{a}}\left(2\mathrm{R}\Delta + \Delta^2\right),$$

where ρ_a is the mass density of the shell.

The Jacobi function of the atmosphere is

$$\Phi_a = 4\pi\rho_a \frac{1}{2} \int_{R}^{R+\Delta} r^4 = \frac{4\pi\rho_a}{2.5}[5R^4\Delta + 10R^3\Delta^2 + 10R^2\Delta^3 + 5R\Delta^4 + \Delta^5]. \quad (7.103)$$

Expressing the gas density ρ_a through its mass, we can write the relationship (7.102) in the form

$$B = |U|_a\sqrt{\Phi_a} = \frac{GMm}{\sqrt{2}} \frac{3(2+\lambda)}{2(3+3\lambda+\lambda^2)} \sqrt{\frac{3(5+10\lambda+10\lambda^2+5\lambda^3+\lambda^4)}{5(3+3\lambda+\lambda^4)}}, \quad (7.104)$$

where $\lambda = \Delta/R$.

Note that Eq. 7.104 depends only on the ratio of thickness of the shell to the radius of a central body and varies over limited ranges, while λ varies from 0 to ∞. At $\lambda = 0$

$$|U_a|\sqrt{\Phi_a} \to \frac{GMm^{3/2}}{R},$$

and at $\lambda \to \infty$

$$|U_a|\sqrt{\Phi_a} \to \frac{GMm^{3/2}}{R}\left(\frac{27}{20}\right)^{1/2}.$$

On the other hand, the Jacobi function (7.102) expressed through the mass of the shell, is written in the form

$$\Phi_a = \frac{3m}{10} R^2 \frac{3(5+10\lambda+10\lambda^2+5\lambda^3+\lambda^4)}{5(3+3\lambda+\lambda^4)}. \quad (7.105)$$

It follows from (7.14) that the Jacobi function of the atmosphere does not depend only on the value λ but also on the radius of the body R. Moreover, the value Φ_a varies over unlimited ranges when λ runs from 0 to ∞. The same can be said about the potential energy of the atmosphere:

$$U_a = \frac{GMm}{R} \frac{3}{2} \frac{2+\lambda}{2(3+3\lambda+\lambda^2)}$$

Accepting the mass of the atmosphere $m = 10^{21}$ kg, we can find the value B as

$$|U_a|\sqrt{\Phi_a} = \frac{GMm}{R}\left(\frac{mR^2}{2}\right)^{1/2} = 1.0374 \cdot 10^{53}\ [g^{3/2}\,cm^3\,s^{-2}]. \tag{7.106}$$

Taking (7.106) into account, Eq. 7.99 can be written in the form of an equation of virial oscillations of the atmosphere (subscript a at Φ is farther drop):

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}, \tag{7.107}$$

where $A = -2E$; $E = -1.6 \cdot 10^{33}$ erg.

As shown in Chap. 4, at A = const and B = const, Eq. 7.107 has two first integrals, as follows:

$$C = -2A\Phi + 4B\sqrt{\Phi} - \dot{\Phi}^2, \tag{7.108}$$

$$-\arccos\frac{(A/B)\sqrt{\Phi}-1}{\sqrt{1-AC/2B^2}} - \sqrt{1-\frac{AC}{2B^2}}\sqrt{1-\left[\frac{(A/B\sqrt{\Phi}-1)}{\sqrt{1-AC/2B^2}}\right]^2} = \frac{(2A)^{3/2}}{4B}(t-t_0), \tag{7.109}$$

where C and t_o are integration constants. The constant C has a dimension of square angular momentum.

The integrals (7.108) and (7.109) are solutions of the Eq. 7.107. Introducing new variables

$$E'' = -\arccos\frac{(A/B\sqrt{\Phi}-1)}{\sqrt{1-AC/B^2}}, \quad e'' = \sqrt{1-\frac{AC}{2B^2}},$$

and rewriting (7.109) in the form of Kepler's equation

$$E'' - e''\sin E'' = \frac{(2A)^{3/2}}{4B}(t-t_0) = M'',$$

expressions can be obtained for Φ(t) and Φ(t) from the first integrals (7.108) and (7.109) in explicit form, using Lagrange's series (Duboshin 1975):

$$\Phi(t) = \frac{B^2}{A^2}\left[1 + 2e''\cos M'' - e''^2\left(1-\frac{3}{2}\cos^2 M''\right) - \frac{5}{2}e''^3(1-\cos^2 M'')\cos M'' + \ldots\right] \tag{7.110}$$

$$\Phi(t) = \sqrt{\frac{2}{A}}\varepsilon B\left[\sin M + \frac{1}{2}\varepsilon \sin 2M + \frac{\varepsilon^2}{2}\sin M(2\cos^2 M - \sin^2 M + \ldots\right],$$

where $M'' = (2A)^{3/2}(t - t''_0)/n(t - t''_0)$; n'' is the frequency of the atmospheres own virial oscillations; t_o is the time moment when Φ acquires its maximal value.

At $0 < C < 2B^2/A$ the Jacobi function changes in time in accordance with Eq. 7.110. At $C = 2B^2/A$ the Jacobi function is equal to $\Phi = B^2/A^2$, which corresponds to the hydrostatic equilibrium of the system or to the virial theorem.

The solution represents the non-linear periodic pulsation of the Jacobi function of the atmosphere as a whole with period T_v. Using the numerical values of constants $B = 1.03 \cdot 10^{53}$ and $A = –U = 3.2 . 10^{33}$ erg, the period of unperturbed virial oscillations of the Earth atmosphere is equal to

$$T_v = \frac{8\pi B}{(2A)^{3/2}} = 2\pi\sqrt{\frac{R^3}{Gm}} = 5060.7\,\mathrm{s} = 1.4057\,\mathrm{h}. \qquad (7.111)$$

Note that the expression (7.111) for the period of virial oscillations T_v includes three fundamental constants: the body M, radius R and gravity constant G. The simplest combination of these three constants, which gives the dimension of time, coincides with (7.111). In this case the nature of the virial oscillations of the atmosphere can be explained by the change in time of the gravitational potential of the solid Earth. The period (7.111) coincides with the period of revolution of a satellite along a circular orbit with first cosmic velocity $v = 7.9$ km/s, radius of the Earth and with a mathematical pendulum the length of whose filament is equal to the radius of the Earth. This is because the parameters considered are defined by the same constants g, m, and R.

7.12.3 Perturbed Oscillations

We now consider a general approach to solving the problem of perturbed virial oscillations of the atmosphere, taking as an example the perturbations caused by the variation throughout the year of the solar energy flux owing to the ellipticity of the Earth's orbit. We assume that all the dissipative processes that occur during the interaction between the atmosphere and the hydrosphere and in the atmosphere itself are compensated by solar energy. We note, however, that the value of the flux is evidently dependent on time. Assuming also that the eccentricity of the Earth's orbit and the mean total energy of the atmosphere are known and remain unchanged in time, the total energy of the Earth's atmosphere is proportional to the power of the solar energy flux L(x) which reaches the atmosphere at a given point of the orbit.

Then

$$L(t) = L_0 \frac{a^2}{r^2}, \tag{7.112}$$

where Lo is the mean energy flux reaching the Earth's atmosphere; r is the radius of the Earth's orbit; a is the semi-major axis of the Earth's orbit.

Using the property of the elliptical motion, we obtain

$$L(t) = L_0 \left(1 - 2e' \cos E' + e'^2 \cos^2 E'\right)^{-1}, \tag{7.113}$$

where e' is the eccentricity of the Earth's orbit; E' is the eccentric anomaly that characterizes the location of the Earth on the orbit and is linked with time by the Keplerian equation

$$E' - e' \sin E' = n'(t - t_0) = M'_0, \tag{7.114}$$

where $n = 2\pi/\tau$ is the cyclic frequency of the Earth's revolution round the Sun; τ is the period of revolution and is equal to 1 year; t_o is the moment of time required by the Earth to pass through the orbit's perihelion; M' is the mean of the anomaly.

If the eccentricity $e' \leq \bar{e} = 0.6627\ldots$, which is the Laplacian limit, then, using the Lagrangian series, we can obtain expressions for E' and $\varphi(E')$ in the form of an absolute convergent infinite series expanded by entire positive powers of e'.

Note that in order to obtain the expression for $(1–e' \cos E')$ we can write the equality which follows from the Keplerian equation:

$$\frac{1}{(1 - e' \cos E')^2} = \left(\frac{dE'}{dM'}\right)^2.$$

Expanding the eccentric anomaly E' in the Lagrangian series by the power of eccentricity e' [Duboshin 1975], we obtain

$$E' = \sum_{k=0}^{\infty} \frac{e'^k}{k!} \frac{d^{k-1}\left(\sin^k M'\right)}{dM'^{k-1}}. \tag{7.115}$$

This can be rewritten in a more convenient form:

$$E' = \sum_{k=0}^{\infty} e'^k E_k(M'),$$

where

$$E_k(M') = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^{k-1}\left(\sin^k M'\right)}{dM'^{k-1}}$$

As the absolute convergent series can be differentiated term by term, we obtain

$$\frac{a}{r} = \frac{dE'}{dM'} = \sum_{k=0}^{\infty} e'^k \bar{R}_k(M'), \tag{7.116}$$

where

$$\bar{R}_k(M') = \frac{dE_k(M')}{dM'}.$$

Multiplying the series (7.116) by itself, we obtain

$$\frac{a^2}{r^2} = \sum_{k=0}^{\infty} e'^k \bar{R}_k^{(2)}(M'), \tag{7.117}$$

where

$$\bar{R}_k^{(2)}(M') = \sum_{k=0}^{k} \bar{R}_s(M')\bar{R}_{k-s}(M')$$

Then the expression (7.112) for the solar energy flux reaching the atmosphere can be rewritten:

$$L(t) = L_0 \sum_{k=0}^{\infty} e'^k \bar{R}_k^{(2)}(M'). \tag{7.118}$$

In agreement with (7.118), the change of the total energy of the Earth's atmosphere is proportional to the change of the solar energy flux $L(t) - L_o$. Then the expression for the total energy of the atmosphere is

$$E(t) = E + k[L(t) - L_0],$$

where k is a proportionality factor.

Thus, in our problem of virial oscillations of the atmosphere perturbed by the solar energy flux varying during the motion of the Earth along the orbit, the equation can be written as

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} + X(e', M'), \tag{7.119}$$

where $X(e', M')$ is the perturbation function, which has the form

$$X(e', M') = 2k[L(t) - L_0]\left[\sum_{k=0}^{\infty} e'^k R_K^{(2)}(M') - 1\right]. \tag{7.120}$$

To estimate the geophysical effect of the variation of the solar energy within a time period of 1 year, we introduce the perturbation function (7.120) into Eq. 7.119 of perturbed oscillations, to an accuracy of squared eccentricity, i.e.,

$$X(M') = kL_0[2e' \cos M' + \frac{5}{2}e'^2 \cos 2M], \tag{7.121}$$

where $kL_o \approx 3 \cdot 10^{31}$ erg.

In this case, the expressions for the Jacobi function of the atmosphere and its first derivative to an accuracy of *e* have the form

$$\Phi(M) = \Phi_0(1 - 2e\cos M), \tag{7.122}$$

$$\dot{\Phi}(M) = 2\Phi_0 n \sin M, \tag{7.123}$$

Then, differentiating the expression for the eccentricity of the perturbed oscillations $e = (1-AC/2B^2)^{1/2}$ with respect to time and using Lagrange's method of varying arbitrary constants, we obtain

$$\frac{de}{dt} = \frac{A\dot{\Phi}(M)X(M')}{2B^2 e} = \frac{A\Phi_0 kL_0 ne'}{B^2}\Big[\sin(M + M') + \sin(M - M') + \frac{5}{2}e'\sin(M + 2M') + \frac{5}{2}e'\sin(M - 2M')\Big]. \tag{7.124}$$

Integrating (7.124) with respect to time, we obtain the law of variation of the virial oscillation eccentricity as a first approximation of the perturbation theory:

$$e(t) = \bar{e} - \frac{A\Phi_0 ne'}{B^2}\Big[\frac{\cos(M + M')}{n + n'} + \frac{\cos(M - M')}{n - n'} + \frac{5}{2}e'\frac{\cos(M + 2M')}{n + 2n'} + \frac{5}{2}e'\frac{\cos(M - 2M')}{n - 2n'}\Big] \tag{7.125}$$

Finally, putting the expression for the eccentricity of the perturbed oscillations into (7.122), we obtain

$$\Phi(t) = \Phi_0 + \frac{1}{2}\alpha_1[\cos(2M + M') + \cos M'] + \frac{1}{2}\alpha_2[\cos(2M - M') + \cos M'] + \frac{1}{2}\alpha_3[\cos(2M + 2M') + \cos 2M'] + \frac{1}{2}\alpha_4[\cos(2M - 2M') + \cos(2M'], \tag{7.126}$$

where

$$\alpha_1 = \frac{2A\Phi_0^2 kL_0 ne'}{B^2(n+n')}, \qquad \alpha_2 = \frac{2A\Phi_0^2 kL_0 ne'^2}{B^2(n-n')},$$

$$\alpha_3 = \frac{5A\Phi_0^2 kL_0 ne'^2}{B^2(n+2n')}, \qquad \alpha_4 = \frac{5A\Phi_0^2 kL_0 ne'^2}{B^2(n-2n')},$$

$$M = n(t-t_0), \qquad M' = n'(t-t'_0)$$

$$n = 1.2 \cdot 10^{-3}\,s^{-1}, \qquad n = 1.2 \cdot 10^{-7}\,s^{-1}$$

It follows from (7.125) that a contribution of the long-periodic part of the first-order perturbations to the variation of the Jacobi function of the Earth's atmosphere is

$$\Phi_a(t) = \text{const} + 1.44 \cdot 10^{-3}\Phi_0 \cos M' + 1.44 \cdot 10^{-3} e'\Phi_0 \cos 2M', \tag{7.127}$$

where $\Phi_o = 1.06 \cdot 10^{39}$ g · cm^2; $\Phi_a/I_a = ¾$.

Assuming in first approximation that the rotations of the Earth and the atmosphere are synchronous, and using the law of conservation of angular momentum for the Earth-atmosphere system as $(I_\oplus + I_a)\omega_\oplus = \text{const}$, we obtain

$$\frac{\Delta I_a}{I_\oplus} = -\frac{\Delta\omega}{\omega_\oplus} = \frac{\Delta T}{T_0}, \tag{7.128}$$

where ω_o is the angular velocity of the Earth's daily rotation; $T = 8.64 \cdot 10^4$s; $I_\oplus = 8.04 \cdot 10^{44}$ g cm^2.

It is easy to show that the Earth's rate of rotation has annual variations with daily amplitude of variation of about 2 ms duration and can be approximated in our estimate by the sum of two harmonics with a period of 1 year and half a year respectively. This estimate is in good agreement with the observed data of the seasonal variation of the Earth's angular velocity.

7.12.4 Resonance Oscillation

We now consider the solution for identification of the resonance frequencies of the perturbed oscillations of the atmosphere due to the change of solar energy flux during the Earth's motion along an elliptic orbit.

Let us assume that the Earth's atmosphere satisfies all the conditions needed for writing the Eq. 7.107 of unperturbed virial oscillations. We can solve this equation because its two first integrals of motion (7.108) and (7.109) are known. We accept from the solution of Eq. 7.107 that the Jacobi function Φ changes in time with the period $\tau'' = 5{,}060.7\ \mathrm{s}^{-1}$ and the frequency $n'' = 2\pi/\tau'' = 0.00124\ \mathrm{s}^{-1}$. Assuming also that the perturbation of the Earth's atmosphere is affected only by the change of power of the solar radiation flux during the year owing to the ellipticity of the Earth's orbit, then the equation of the perturbed virial oscillations should have the form of (7.28) and the perturbation function $X(e', M')$, represented by (7.120), should be a periodic function of time with period $\tau' = 31{,}556{,}929.9747$ s and frequency $n' = 2\pi/\tau' = 1.9910638\ \mathrm{s}^{-1}$.

We use the Picard method to obtain the solution of Eq. 7.119, which in this case is written as

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} + 2kL_0\left[\sum_{k=0}^{\infty} e'^2 R_k^{(2)}(M') - 1\right], \tag{7.129}$$

where $A = -2E_o$; $B = |U_a|\sqrt{\Phi_a}$; $e' = 0.014$ is the eccentricity of the Earth's orbit; $M' = n'(t - t_o) = E' - e' \sin E'$ is the mean anomaly determined from the Kepler equation; E' is the eccentric anomaly.

Using Lagrange's method of variation of the arbitrary constants C and t, which determine the solution of Eq. 7.129 in the form

$$\Phi = \Phi(C, t''_0, t), \tag{7.130}$$

$$\dot{\Phi} = \dot{\Phi}(C, t''_0, t), \tag{7.131}$$

we can write the system of differential equations (5.19) and (5.20) (see Chap. 5) determining the change of C и t_o'' in time:

$$\frac{dC}{dt} = -2\dot{\Phi}X(M'), \tag{7.132}$$

$$\frac{dt''_0}{dt} = -2\dot{\Phi}X(M')\Psi(\Phi, C), \tag{7.133}$$

where the function $\Psi(\Phi, C)$ was defined earlier, and $X(M')$ is a periodic function of the argument M' with the period 2π.

Substituting the expressions for Φ and $\dot{\Phi}$ from (7.130) and (7.131) into (7.132) и (7.133), we obtain the system of two differential equations:

$$\frac{dC}{dt} = F_1(M', M'', C, t''_0), \tag{7.134}$$

$$\frac{dt''_0}{dt} = F_2(M', M'', C, t''_0), \tag{7.135}$$

where F_1 and F_2 are periodic functions of the arguments M' and M'' with the period 2π.

Owing to the periodicity of the right-hand side of the system of Eqs. 7.134 and 7.44, they can be expanded into a double Fourier series. In this case the system (7.134) and (7.135) can be written:

$$\frac{dC}{dt} = \left\{ A_{00} + \sum_{k',k''=-\infty}^{\infty} \left[A_{k',k''} \cos(k'M' + k''M'') + B_{k',k''} \sin(k'M' + k''M'') \right] \right\}, \tag{7.136}$$

$$\frac{dC}{dt} = \left\{ a_{00} + \sum_{k',k''=-\infty}^{\infty} \left[a_{k',k''} \cos(k'M' + k''M'') + b_{k',k''} \sin(k'M' + k''M'') \right] \right\}. \tag{7.137}$$

Here the coefficients A_{oo} , $A_{k'k''}$, $B_{k',k''}$, a_{oo}, $a_{k'k''}$, $b_{k',k''}$ do not depend on M' and M'', but are functions of the unknown quantities C and t_o''.

Using the Picard procedure, we determine $C^{(1)}$ and $t_o''^{(1)}$ in the first approximation by substituting the constant values $C^{(o)}$ and $t_o''^{(o)}$ into the expressions for F_1 and F_2. The values $C^{(o)}$ and $t_o''^{(o)}$ could be found through the initial conditions of $\Phi_0$and $\dot{\Phi}_0$using corresponding formulas (7.108) and (7.109) which describe the unperturbed virial oscillations.

After integration of the system (7.136) and (7.137) with respect to time, we have

$$\begin{aligned} C^{(1)} = C^{(0)} + A_{00}^{(0)}(t - t_0) + \sum_{k',k''\to-\infty}^{\infty} \frac{1}{k'n' + k''n''} \Big\{ A_{k',k''}^{(0)} [\cos(k'M' + k''M'') \\ - \cos(k'M'_0 + k''M''_0)] - B_{k',k''}^{(0)} [\sin(k'M' + k''M'') \\ - \sin\left(k'M'_0 + k''M''_0)\right] \Big\}, \end{aligned} \tag{7.138}$$

$$\begin{aligned} t''^{(1)}_0 = t''^{(0)}_0 + a_{00}^{(0)}(t - t_0) + \sum_{k',k''\to-\infty}^{\infty} \frac{1}{k'n' + k''n''} \Big\{ a_{k',k''}^{(0)} [\cos(k'M' + k''M'') \\ - \cos(k'M'_0 + k''M''_0)] - b_{k',k''}^{(0)} [\sin(k'M' + k''M'') \\ - \sin\left(k'M'_0 + k''M''_0)\right] \Big\}, \end{aligned} \tag{7.139}$$

where $C^{(1)}$ and $t_o''^{(1)}$ are the arbitrary constants which determine solution of the Eq. 7.28; $A^{(0)}_{k',k''}$, $B^{(0)}_{k',k''}$, $a^{(0)}_{00}$, $a^{(0)}_{k',k''}$, $b^{(0)}_{k',k''}$ are corresponding coefficients of the system (7.136) and (7.137) after replacing C и t_o'' by $C^{(o)}$ and $t_o''^{(o)}$.

Thus we have obtained the analytical structure of the solutions (7.138) and (7.139) known in general perturbation theory which have three classes of terms: constant, periodic and secular. Of the periodic terms, the most important are the resonance terms, i.e. those quantities $n_s = k'n' + k''n''$ which are substantially less then both n” and n’. These terms give a series of long periodic inequalities (their number is infinite) and they allow prediction of the development of the natural processes within relatively long intervals of time.

Let us calculate, as an example, such lower resonance frequencies which have climatic significance:

$$n_{2:12471} = (2.12415 - 12471 \cdot 1.9910638)10^{-7} = 0.556 \cdot 10^{-7}s^{-1},$$

$$n_{3:18706} = (3.12415 - 18706 \cdot 19910638)10^{-7} = 0.161 \cdot 10^{-7}s^{-1},$$

$$n_{11:68589} = (11.12415 - 68589 \cdot 1.9910638)10^{-7} = 0.07 \cdot 10^{-7}s^{-1}$$

and so on, and corresponding to those frequencies the periods are:

$$\tau_{2:12471} = 3.6 \text{ years}; \quad \tau_{3:18706} = 12.36 \text{ years}, \quad \tau_{11:68598} \approx 28 \text{ years}.$$

It should, however be kept in mind that the first approximation obtained in the framework of perturbation theory is in good agreement with observations within short (not cosmogenic) intervals of time.

Note that one can find analogously the resonance frequencies and the periods of virial oscillations which occur owing to other perturbations, such as, for example, diurnal perturbations, because of the rotation of the Earth around its axis and its latitudinal perturbations, etc. As is well known, the mean solar day has period $\tau = 8.64 \cdot 10^4$ s and frequency $n_s = 2\pi/\tau = 7.27 \cdot 10^{-5}\,s^{-1}$. Then calculating the resonance frequency $n_s = k'n' \vec{+} k''n'' << n, n''$, we obtain:

$$n_{1:17} = (1 \cdot 124.17 - 17 \cdot 7.27)10^{-5} = 0.58 \cdot 10^{-5}\,s^{-1},$$

$$n_{2:24} = (2 \cdot 124.17 - 24 \cdot 7.27)10^{-5} = 1.16 \cdot 10^{-5}\,s^{-1},$$

and so on, and corresponding periods:

$$\tau_{1:17} = 12.5 \text{ d}; \quad \tau_{2:24} = 6.24 \text{ d andsoon.}$$

For the monthly Moon perturbations, when $\tau = 2.352672 \cdot 10^6$ s and $n_s = 2.67 \cdot 10^{-6}\,s^{-1}$, we have

$$n_{1:465} = (1 \cdot 1241.7 - 2.67 \cdot 465)10^{-6} = 0.15 \cdot 10^{-6}\,s^{-1},$$

$$n_{2:930} = (2 \cdot 1241.7 - 2.67 \cdot 930)10^{-6} = 0.3 \cdot 10^{-6}\,s^{-1}$$

and so on, and corresponding periods:

$$\tau_{1:465} = 1.3 \text{ years}; \quad \tau_{2:930} = 0.66 \text{ years.}$$

7.12.5 Observation of the Virial Eigenoscillations of the Earth's Atmosphere

It was predicted in Sect. 7.12.2, by means of the solution of Jacobi's virial equation, that the eigenoscillations of the Earth's atmosphere with a period of $T_v = 1^h.4$ and frequency $\omega = 1.24 \cdot 10^{-3}\,s^{-1}$ exists. This solution describes the periodic change of the Jacobi function of the Earth's atmosphere in time and can be expressed in the form of a trigonometric Fourier series expanded by entire multiple values of argument M related to t as

$$M = \frac{2\pi}{T_v}(t - t'_0),$$

where t'_o is the moment of time defining the phase of the virial oscillations.

The first four terms of this series are

$$\Phi(t) = \Phi_0\left[1 + \frac{3}{2}e^2 + \left(-2e + \frac{e^3}{4}\right)\cos M - \frac{e^2}{2}\cos 2M - \frac{e^3}{3}\cos 3M + \ldots\right], \quad (7.140)$$

where Φ_o is the mean value of the Jacobi function, determined by the virial theorem; e is the parameter of the virial oscillations of the atmosphere which characterizes the amplitude of the Jacobi function change; $e \ll 1$.

It has been shown theoretically that the period of chance of the Jacobi function of the atmosphere depends on the value of its total energy, and in the case of non-perturbed atmosphere is equal to $1^h.4$. In the framework of the model considered with spherically symmetric atmosphere, the change of Jacobi function takes place owing to the change of the radial mass density distribution of the atmosphere having the same period. Direct experimental test of this statement is difficult because the value of the Jacobi function cannot be measured directly to prove the expression (7.140). But, as it was shown in Sections 2.2, permanent changes in the Earth's Jacobi function (polar moment of inertia) is fixed by artificial satellites and earthquake measurements. Moreover, the process of virial oscillations is accompanied by synchronous changes of pressure, temperature, air moisture, magnetic field intensity and other measurable geophysical parameters at the Earth's surface. In addition, from the condition of a global scale of the virial oscillations, it follows that all the geophysical parameters are pulsating with the same period and are coherent both within the considerable interval of time and in space over all the Earth's surface as well as vertically. The expression for the virial pulsations of the atmospheric pressure p(t) and temperature T(t) can be expanded into a Fourier series (7.140) as follows:

$$p(t) = p_0\left[1 + \frac{1}{2}e^2 + \left(2e + \frac{3}{4}e^3\right)\cos M + \frac{5}{2}e^2\cos 2M + \ldots\right], \quad (7.141)$$

$$T(t) = T_0\left[1 + \frac{1}{2}e^2 + \left(2e + \frac{3}{4}e^3\right)\cos M + \frac{5}{2}e^2\cos 2M + \ldots\right], \quad (7.142)$$

where p_o and T_o are the mean values of the atmospheric pressure and temperature averaged over all the Earth's surface and through the mass of the atmosphere respectively.

We shall use for this expansion the analysis and interpretation of the experimental data. As is well known, regular observation of the atmospheric pressure and temperature at various points of the Earth prevents the discovery of a rigorous periodicity in changes of atmospheric parameters, especially for short periods. There are a number of reasons for this, including their variability which defines the dynamics of air masses at any point of observation. The parameters recognized and studied until now are seasonal, diurnal and semidiurnal periodicity, in addition to variation of the atmospheric parameters connected with the motion of the planet along the elliptical orbit around the Sun, with the obliquity of the axis of rotation to the ecliptic, and with perturbations caused by the Moon.

In our case, in order to prove that the predicted oscillations of various geophysical parameters with period 1h.4 exist, we used the applicable spectral analysis of experimental data.

Oscillation of the temperature. We now describe the results of spectral analysis of temperature data recorded by our colleagues in the Central Hydrometeorological Observatory at the Ostankino TV Tower, Moscow. Let us consider two sets X_1 and Y_1 representing regular records of the air temperature variation of 34 h duration, each of which were obtained simultaneously in July 1971 at heights of 503 and 83 m (Fig. 7.11). The discreteness interval of the numerical record of the temperature was 120 s, the interval of the gauge was ~60 s, and the sensitivity was около 0.1°C. The sets X_1 and Y_1 contain 1,024 discrete values of recorded temperature starting at 10 h 34 min on 17 July through 21 h 34min on 18 July. Spectral analysis of the data received was carried out by computer using the method of quick Fourier transformation with the program developed by A.B. Leybo and V.Yu. Semenov.

Figure 7.11a shows the recorded power spectra S_{xx} and S_{yy} of the sets X_1 and Y_1. With the help of this initial information, we calculated the function of the mutual spectral density $S_{xx} = S'_{xy} + iS''_{xy}$. Then the function of the mutual coherence was found to be

$$Co^2 = \frac{|S_{xy}|^2}{|S_{xx}||S_{yy}|} \tag{7.143}$$

and the function of the phase difference was defined as

$$\Delta\varphi = \mathrm{atctg}\frac{S''_{xy}}{S'_{xy}}. \tag{7.144}$$

They are both plotted in Figs. 7.11b, c.

It is known that the range of the confidence intervals for estimating the phase difference $\Delta\varphi$ tends to zero as Co^2 runs to unity. Hence the higher the value of the function of the mutual coherence, the higher the stability of the phase difference of the harmonics of the two processes X and Y. Therefore, the relationship between the harmonics is probable for those frequencies where the value of Co^2 is close to

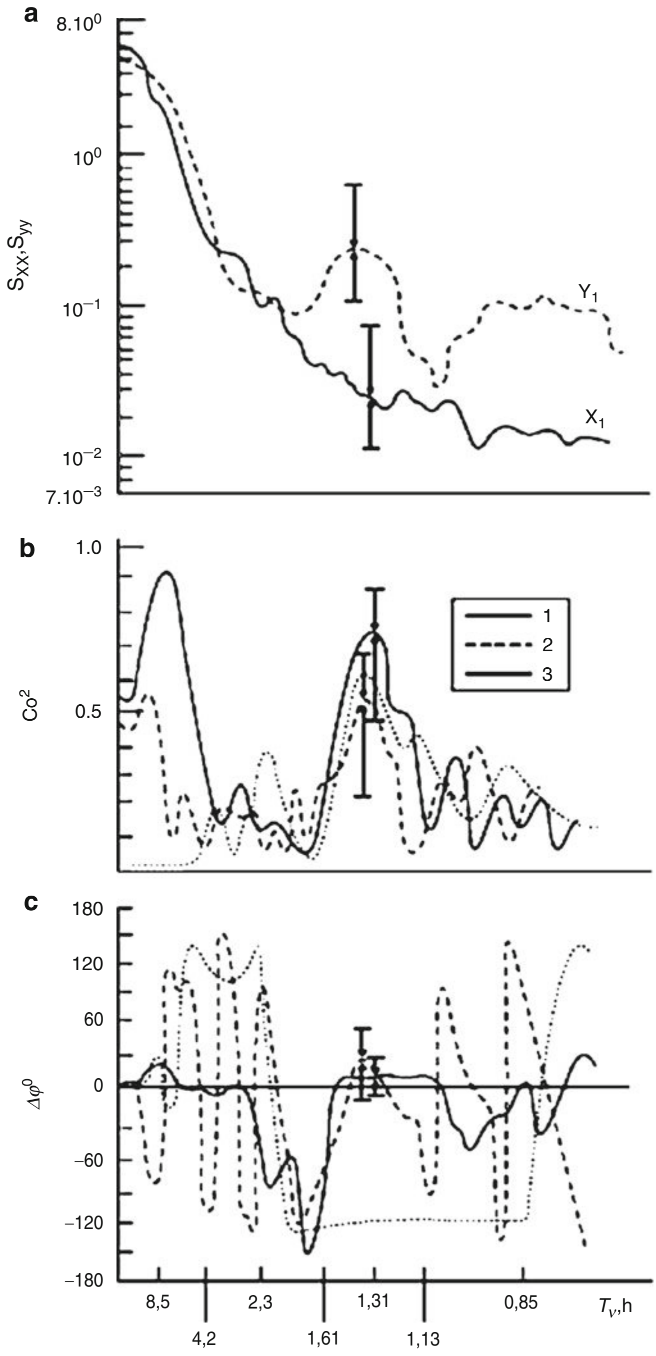

Fig. 7.11 Power spectra of sets of temperature variation at heights of 503 and 83 m (**a**); functions of mutual coherence of the sets X_1 and Y_1 (*1*), X_2 and Y_2 (*2*), X_3 and Y_3 (*3*) (**b**); functions of phase difference of the sets X_1 and Y_1 (*1*), X_2 and Y_2 (*2*), X_3 and Y (*3*) (**c**). (*Arrows* show the range of 95% of the confidence intervals)

unity. At the same values of Co^2 the meaning of the phase difference is lost because of the wide range of the confidence interval.

We can see from Fig. 7.11b that in the vicinity of the period $T_v = 1^h.4$, predicted by theory, the function of the mutual coherence of the sets X_1 and Y_1 is significant, i.e. has high probability of not being equal to zero, and sometimes of being even greater then 0.7. Note also that the phase difference function in the neighborhood of the period of oscillations T_v is equal to zero, which indicates that the harmonic constituents of the temperature variations with the period T_v within the considered interval of time are coherent at heights of 503 and 83 m.

Twice as much extended time for sets X_1 and Y_1 does not change the general character of the discovered regularity. Figures 7.11b, c demonstrate values of the mutual coherence function Co^2 and the phase difference $\Delta\varphi$ of two sets X_2 and Y_2 each containing 2,048 experimental points and recorded synchronously with the same discreteness interval. Recording was continued for 3 days starting at 10 h 34 min on 17 July through 6 h 54 min on 20 July 1971. However, reduction of time for the sets leads to an increase in values of the mutual coherence function on the low frequencies, but the values of the frequencies corresponding to the period T_v do not increase. This proves the theoretically predicted conclusion concerning the existence of a coherent harmonic with the period T_v.

Figure 7.11b plots the mutual coherence function for two sets of temperature variations recorded at 17 h 2 min in each case. These sets were recorded at a height of 503 m with the same discreteness interval as discussed above, but in different years. Set X_3 covers a time interval from 10 h 34 min on 17 July until 3 h 36 min on 18 July 1971, and Y_3 was recorded from 9 h 14 min 30 July to 2 h 16 min on 31 July 1971. We can see from the plot that in this case the mutual coherence function acquires a value equal to 0.6 at $T = 1$ h.31, which proves the theory of the steady state virial oscillations within sufficiently long intervals of time.

Let us estimate the amplitude of temperature virial oscillations in the neat surface layer of the atmosphere. For this purpose we recall that the value of the power spectrum of a process at a given value of the frequency is proportional to the square of the amplitude value of the harmonics with the same frequency in Fourier analysis of the process. We plot the power spectrum of a sinusoid of some known amplitude, for example, equal to 1°C with period $1^h.4$ and a given discreteness $\tau \approx 120$ s. The value of the power spectrum of the sinusoid for the frequency related to period $T_v = 1^h.4$ was found equal to 3.4 grad^2 per hertz, and the value of the power spectrum of temperature micro-fluctuation at heights of 83–503 m in July 1971, according to the Observatory recording, was 0.3 grad^2 per hertz. It is easy to obtain the estimated value of the amplitude of the temperature virial oscillations, which is 0.3°C.

Oscillation of the pressure. We now consider our results of the spectrum analysis of the atmospheric pressure records made by means of a microbarograph designed by V.N. Bobrov in the Institute of Earth Magnetism and Radio Wave Propagation, Russian Academy of Sciences. Analysing the records, we took the pressure ordinates correct to 1 mm at amplitude of pressure oscillation within several cm (at microbarograph sensitivity equal to 0.02 mb/mm). The spectral analysis was done using the methodology described above.

Figure 7.12a shows the power spectrum of two sets X_4 and Y_4 containing 1,024 of the pressure ordinates taken with discreteness interval 76.5 s covering the time

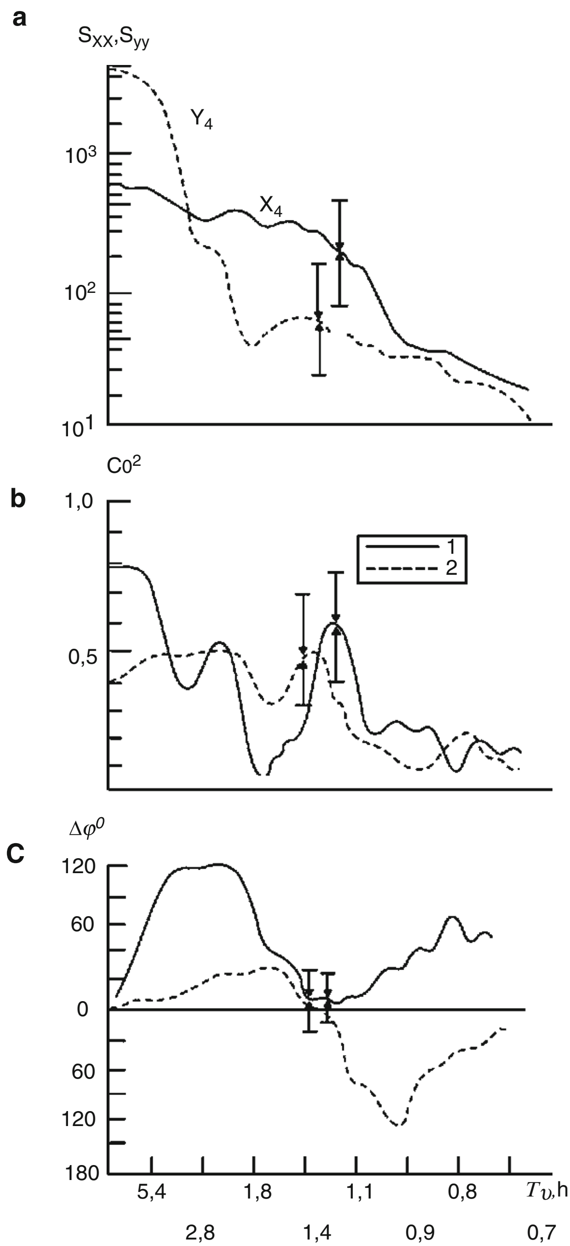

Fig. 7.12 Power spectra of the sets X_4 and Y_4 of atmospheric pressure micro-fluctuations northeast of the coastal area of the Caspian Sea (**a**); functions of mutual coherence of the sets X_4 and Y_4 (*1*) and X_5 and Y_5 (*2*) (**b**); functions of the phase difference of the sets X_4 and Y_4 (*1*) and X_5 and Y_5 (*2*) (**c**). (*Arrows* show the range of 95% of the confidence intervals)

interval from 18 h on 31 August through 15 h 30 min of 1 September 1977 and from 15 h 00 min on 1 September through 12 h 30 min on 2 September 1977.

The experimental data of micro-fluctuations of the atmospheric pressure were obtained at a near surface layer of the northeast coastal area of the Caspian Sea by our colleagues from the Institute of Earth Magnetism and Radio Wave Propagation. Using the function of mutual spectral density S_{xx}, we calculated the functions of mutual coherence Co^2 and phase difference $\Delta\varphi$, shown in Fig. 7.12b, c.

The spectral densities were analyzed by averaging the values of seven experimental points (the number of degrees of freedom equal to 14). The graph of spectral densities was plotted in relative units and on logarithmic scale (in order to have the same range of confidence intervals for any value of the spectral density).

Figure 7.12b shows that in the vicinity of the frequency value corresponding to the period $T_v = 1^h.4$, the coherence coefficient is significant and is equal to $Co^2 = 0.6$. It is also important to note that, for the same frequency, the phase difference is stable and close to zero (see Fig 7.12c). These facts prove the theory of the existence of coherent oscillations of the Earth's atmosphere with period close to T_v.

Figures 7.12b, c also show the curves of mutual coherence and phase difference for the other two sets X_5 and Y_5 plotted on a base of 512 experimental points of atmospheric pressure recorded within time intervals starting at 16 h 56 min on 28 August and at 2 h 45 min on 29 August and at 18 h 00 min 31 August through 3 h 49 min on 1 September 1977, respectively. The mutual coherence function has a value equal to 0.5 in the vicinity of T_V, and the function of the phase difference has a value close to zero. This also proves the conclusion discussed above.

We can also show that virial pulsation of the Earth's atmosphere with period $T_v = 1^h.4$ is observed in both mid and low latitudes. For this purpose we studied records of micro-fluctuations of the atmospheric pressure obtained by the same researchers during their expedition to Cuba.

Let us analyze two sets of experimental data X_6 and Y_6 taken from their records of micro-fluctuations of the atmospheric pressure with discreteness interval equal to 180 s. The process X_6 covers a time interval starting at 19 h 00 min on 7 May through 20 h 36 min on 8 May 1976 and the process Y_6 covers an interval from 21 h 54 min on 7 May through 23 h 30 min on 9 May 1976. Figure 7.13a shows the power spectra of the processes X_6 and Y_6. The functions of mutual coherence and phase difference which, in the vicinity of the period T_v, have values $Co^2 = 0.56$ and $\Delta\varphi = 0°$ respectively, are shown in Fig. 7.13b, c. The same figures also show functions of mutual coherence and phase difference for two other sets, X_7 and Y_7, representing 256 experimental points each and analyzed with the same discreteness intervals. They cover time intervals starting at 21 h 37 min on 25 April through 10 h 23 min on 26 April and on 20 h 00 min on 28 April on 9 h 14 min on 29 April 1976. The value of function $Co^2 = 0.6$ and $\Delta\varphi = 0°$ relative to frequencies with period $T_v = 1^h.4$.

Our estimate of the amplitude of virial oscillations of the atmospheric pressure made by using the above procedure gives the value of 0.1 mbar.

Figure 7.13b shows the function of mutual coherence of two sets of atmospheric pressure X_8 and Y_8 taken with discreteness interval $\Delta t = 6$ min and recorded in various years and at different points of the globe: (a) in Cuba within a time interval

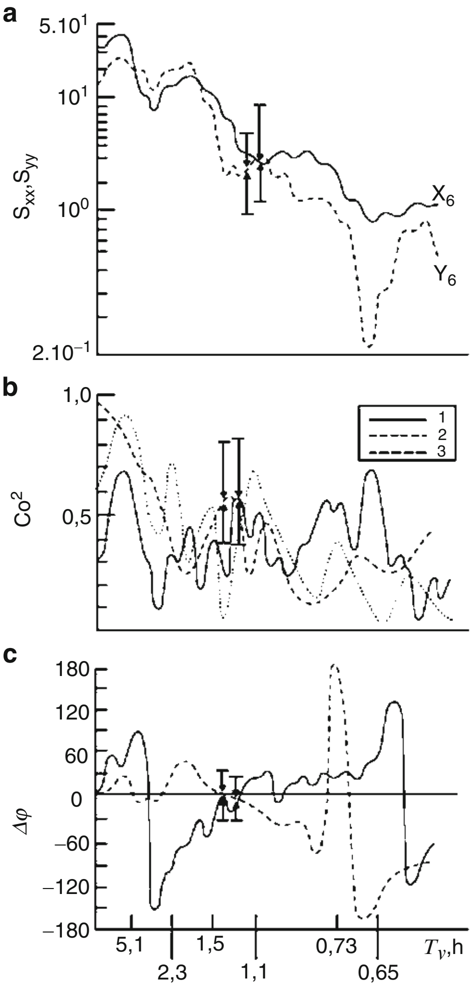

Fig. 7.13 Power spectra of the sets X_6 and Y_6 of atmospheric pressure variations in Cuba (**a**); functions of mutual coherence of the sets X_6 and Y_6 (*1*) and X_7 and Y_7 (*2*) and X_8 and Y_8 (*3*) (**b**); functions of the phase difference of the sets X_6 and Y_6 (*1*) and X_7 and Y_7 (*2*) (**c**). (*Arrows* show the range of 95% of the confidence intervals)

starting at 19 h 00 min on 7 May through 2 h 30 min 8 May 1976; (b) on northeastern shore of the Caspian Sea at 19 h 00 min on 4 September through 20 h 30 min on 5 September 1977. We also observed here that the function Co^2 takes a value of about 0.65 in the vicinity of the period $T_v = 1^h.4$. This also proves our hypothesis.

Analogous results were obtained on analysis of the experimental data of micro-fluctuation of the atmospheric pressure recorded at the Black Sea. The analyses were made with discreteness interval equal to 120 s.

We have shown the existence of harmonics with period close to $T_v = 1^h.4$, coherent within long time intervals for micro-fluctuations of atmospheric pressure and temperatures in the near surface layer of the atmosphere, obtained at various points of the globe and in different seasons of the year. The resulting data from analyses of experimental records of micro-fluctuations of pressure and temperature as well as the geomagnetic field of the Earth are given in Table 7.2.

7.12.6 Dynamics of the Oceans

It follows from Eq. 7.71 that the world ocean is found to be in a suspended state because the density value of its water is by far less (2/3) than the mean density of the solid Earth. Accepting this criterion, we may assume that in the earlier stage of creation of the planet its hydrosphere remained in the gaseous phase. Only after irradiation of the corresponding part of the potential energy has the water vapor condensed into the liquid phase (see Chap. 9). Taking into account the observed geographic distribution of the Atlantic and Pacific oceanic basins, we also may assume that their floors were formed as a consequence of the planet's equatorial oblateness during formation. Applying the same argument and geography of location, one may conclude that the Indian oceanic floor was formed on the same base at the Earth's polar flattening and due to the asymmetric deformation of the southern hemisphere (see Chap. 2).

The oceans have their own potential energy value which is equal to $U \approx 2 \cdot 10^{32}$ erg. This value is by four degrees less of their oscillating and rotating kinetic energy. It means that the oceans stay in hydrostatic equilibrium in the outer force field of the solid Earth which provides rotational motion of this shell

This suspended stay of equilibrium of the oceans determines their dynamics. According to Eq. 7.71, angular velocity of the rotating oceanic water, because of low density, is less than that of the solid Earth. Therefore, the oceanic water in its rotary motion has lower angular velocity with respect to the solid Earth. Encountering the continents on their way, the ocean waters form the latitudinal currents along both American and Asian continents. The observed multiply disordered regional and local linear and eddy currents are the consequences of different perturbations, which dominate over the regular currents due to the above energy effects.

The regular currents, which move along the east shores of the continents, play an important role as heating system in formation of weather and climate processes in the middle and high latitudes.

7.12.7 The Nature of the Weather and Climate Changes

The atmosphere and oceans comprise a common natural system which controls the weather and climate on the planet. Both shells, being uniform in density and staying in semi-hydrostatic and semi-dynamic equilibrium in the outer force field of the solid Earth, are affected by virial oscillations of the planet and rotate with a

Table 7.2 Experimental data after reciprocal spectral analysis

Place of observation and parameters	Data and hours of observation	Frequency points (h)	Discreteness (Δt, s)	Time (h)	$Co^2/\ \Delta\varphi^\circ$
Caspian Sea, pressure	X:31.08 (18-00)–01.09.1977 (05-05) Y:28.08 (17-00)–29.08.1977 (04-05)	512/11	78.5	1.39	0.5/13
Caspian Sea, pressure	X:31.08 (18-00)–01.09.1977 (15-38) Y:01.09 (15-38)–02.09.1977 (12-41)	1024/22	78.5	1.36	0.56/6
Caspian Sea, pressure	X:28.08 (16-45)–29.08.1977 (04-26) Y:09.09 (22-00)–10.08.1978 (09-30)	512/11	78.5	1.39	0.47/0
Black Sea, pressure	X:13.08 (22-30)–14.08.1979 (11-00) Y:14.08 (11-00)–14.08.1979 (23-30)	512/12,5	87.7	1.39	0.76/−1.4
Moscow, temper. (503 m)	X:24.01 (08-14)–25.01.1973 (01-16) Y:25.01 (02-30)–25.01.1973 (19-32)	512/17	120	1.42	0.60/0
Moscow, temper. (50, 83 m)	X:17.07 (10-34)–18.07.1971 (03-36) Y:17.07 (10-34)–18.07.1971 (03-36)	512/17	120	1.42	0.67/−30
Moscow, temper. (503, 83 m)	X:17.07 (10-34)–18.07.1971 (20-36) Y:17.07 (10-34)–18.07.1971 (03-36)	1024/17	120	1.26	0.72/4
Moscow, temper. (503, 83 m)	X:17.07 (10-34)–20.07.1971 (06-54)	2048/68	120	1.28	0.55/10

(continued)

Table 7.2 (continued)

Place of observation and parameters	Data and hours of observation	Frequency points (h)	Discreteness (Δt, s)	Time (h)	$Co^2/\Delta\varphi°$
	Y:17.07 (10-34)–20.07.1971 (06-54)				
Moscow, temper. (503, 83 m)	X:17.07 (10-34)–18.07.1971 (03-36)	512/17	120	1.31	0.62/0
	Y:30.07 (09-14)–31.07.1977 (02-16)				
Cuba, pressure	X:25.04 (21-37)–26.04.1976 (10-23)	256/13	180	1.42	0.57/10
	Y:28.04 (20-00)–29.04.1976 (09-14)				
Cuba, pressure	X:07.05 (19-00)–08.05.1976 (20-36)	512/26	180	1.28	0.59/1.7
	Y:08.05 (21-54)–09.00.1976 (23-30)				
Cuba, pressure	X:16.11 (12-00)–18.11.1976 (15-12)	1024/51	180	1.42	0.50/−17
	Y:07.05 (19-00)–09.05.1976 (22-15)				
Cuba, magnetic field	X:16.02 (12-00)–18.02.1976 (15-12)	1024/51	180	1.65	0.88/120
Cuba, pressure	Y:16.02 (12-00)–18.02.1976 (15-12)			1.38	0.50/160
Cuba, pressure	X:07.05 (19-00)–08.05.1976 (20-30)	256/26	360	1.1	0.65/−40
Caspian Sea, pressure	Y:14.08 (11-00)–14.08.1979 (23-30)				

corresponding angular velocity. However, irregular seasonal pumping of solar energy, because of ellipticity of the Earth's orbit and the precession and nutation perturbation of the upper solid shell, leads to continuous redistribution of solar energy in the latitude and longitude directions. Those perturbations cause permanent change in balance of the evaporated water from the ocean surface and lead to changes in baric topography and in trajectories of the cyclonic vortexes which carry clouds of moisture. From point of view of the considered theory, those are the perturbations that appear to be the cause of weather and climate change.

The principles of perturbation and resonance oscillation of the atmosphere presented in this chapter could be used as a basis for development of analytical solution of the problem.

7.13 Lyapunov Stability of Motion in Jacobi Dynamics

It is important to note that in the dynamic approach to the solution of problem of the system's dynamics, the integral characteristics of a system (Jacobi function and total energy), presents in Jacobi's equation, are immanent to their own integrals. Estimating the Lyapunov stability of motion of a system, they play the role of Lyapunov functions.

Studying Lyapunov stability of the virial oscillations of celestial bodies, we use the Duboshin criterion, which is applicable when permanent perturbations are present. For conservative systems, the potential energy of the system plays the role of such a perturbation. Thus, the nature of the virial oscillations can be understood as an effect of non-linear resonance between the kinetic and the potential energies.

7.13.1 Lyapunov Stability of Motion of a System Described in Terms of Co-ordinates and Integral Characteristics

Let us recall the definition of stability of motion according to Lyapunov (Duboshin 1952). If for any $\varepsilon > 0$ there exist $\delta > 0$ such that for any perturbations of the initial data x_{oj}, satisfying the condition $| x_{oj}| < \delta$, the inequalities $|x_{oj}| < \varepsilon$ $(j = 1, 2, \ldots, n)$ holds, then the perturbed motion is stable; otherwise it is unstable.

Studying the stability of motion according to Lyapunov, we shell use his theorem (Duboshin 1952) which follows:

If for differential equations of a perturbed motion the function V of fixed sign (positive defined or negative defined) can be found (Lyapunov function), whose derivative by vitrue of these equations is the function of the constant sign opposite to the sign of V or identically equal to zero, then the unperturbed motion is stable.

The theory of Lyapunov stability considers variations in the initial data. Thus the equations for the perturbed and unperturbed motions ate the same, and one can therefore write Lyapunov's equation for the variation of perturbations (sequential perturbations) in the form

$$\dot{x}_j = X_j\left(x_1, \ldots, x_j \ldots, x_n, t\right) \quad j = 1, 2, \ldots, n \tag{7.145}$$

where

$$X_j\left(x_1, \ldots, x_j \ldots, x_n, t\right) = P_j\left(f_i + x_1, \ldots, f_j + x_j, , \ldots, f_n + x_n\right) - P_j\left(f_i, \ldots, f_j, , \ldots, f_n t\right)$$

is equal to the difference between the two particular solutions of one and the same equation of motion of the system which was obtained by subtracting the two equations. The right-hand sides x_j (. . .) of Eq. 7.145 are the functions equal to zero at the stationary point $x_{j=}0$ (j = 1, 2, . . ., n), i.e. in the absence of initial data perturbations. Thus, according to Lyapunov, the problem of stability can be reduced to the problem of stability of the zero (trivial) solution of the perturbation equations, having on the right-hand side at the origin of the new co-ordinates (x_j) the function x_j (. . .) identically equal to zero.

Investigating the Lyapunov stability of the system whose equations of motion are written in co-ordinates, the difficulty is to find the Lyapunov function V. Progress in this field has been achieved owing to the development of Chetaev's method (Chetaev 1962), which we shall employ.

The method is as follows. If the first m integrals $F_1(x_1,\ldots, x_n) = h_1,\ldots,$ $F_m(x_1,\ldots, x_n) = h_m$ of the perturbed motion are known, then the Lyapunov function can be sought in the form

$$V = \lambda_1[F_1 - F_1(0)] + \ldots + \lambda_m[F_m - F_m(0)] + \mu_1\left[F_1^2 - F_1^2(0)\right] + \ldots + \left[F_m^2 - F_m^2(0)\right]. \tag{7.146}$$

In writing the equations for the system's dynamics in terms of integral characteristics it appears that the latter (Jacobi function and total energy) are immanent to their own integrals. Let us illustrate this statement by well-known examples.

Example 1. Stability of stationary motion in the two-body problem.

Let r_o be the radius of circular motion of mass m in the two-body problem; ω is its angular velocity. Then according to Kepler's law we write

$$\omega^2 r_0^3 = Gm = \mu = \text{const.} \tag{7.147}$$

If r, ψ, θ are co-ordinates of the elliptic motion, then the integrals of their perturbed motion are as follows:

$$T + U = \frac{m}{2}\left[(\dot{r})^2 + r^2(\theta)^2 + r^2\cos^2\theta\left(\dot{\psi}\right)^2\right] - \mu\frac{m}{r} = \frac{m}{2}h_1, \tag{7.148}$$

$$\frac{\partial T}{\partial \dot{\psi}} = mr^2\cos\theta\dot{\psi} = mh_2. \tag{7.149}$$

Introducing the perturbations x_j,

$$r = r_o + x_1; \quad \dot{r} = 0 + x_2,$$

$$\theta = x_3; \quad \dot{\theta} = 0 = x_4,$$

$$\dot{\psi} = \omega + x_5$$

Then integrals (7.148) and (7.149) of the perturbed motion can be rewritten in the form

$$F_1 = x_2^2 + (r_0 + x_1)^2 x_4^2 + (r_0 + x_1)^2 \cos^2 x_3 (\omega + x_5)^2 - \frac{2\mu}{r_0 + x_1} = h_1, \quad (7.150)$$

$$F_2 = (r_0 + x_1)^2 \cos_2 x_3 (\omega + x_5) = h_2. \quad (7.151)$$

Let us write the Lyapunov function according to Chetaev's method as follows:

$$V = F_1 - F_1(0) + \lambda[F_2 - F_2(0)] + k\left[F_2^2 - F_2^2(0)\right]$$

The function V is of fixed sign when $2\omega + \lambda + 2kr_0^2\omega = 0$. It is a positive definite function when the condition $k > 3r_0^2$ is satisfied.

All the conditions needed for application of the stability theorem mentioned earlier are satisfied at $k, > 3r_0^2$ and the motion in the accepted co-ordinates is therefore stable.

To estimate the stability of Keplerian motion in terms of integral (volumetric) characteristics, only one integral of motion is needed, namely the angular momentum M_o:

$$M_0 \propto \Phi\omega. \quad (7.152)$$

It should be pointed out that these integral characteristics (Φ and ω) form a canonical pair since their product is proportional to the action of the system

$$h \propto \Phi\omega \quad (7.153)$$

and the Jacobi function Φ is the integral characteristic of motion. It is easy to verify that Keplerian motion is unstable at $M_o = 0$, i.e. when linear motion takes place.

Example 2. Stability of a pendulum at vertical oscillation of its fixed point of suspension (Kapitza 1951).

In contrast to the case considered above, this example is related to the problems of stability of the system at permanent perturbations (e.g. due to dissipation of feedback of energy). Here the equations of unperturbed motion (without dissipation of energy) and perturbed motion (when dissipation of energy takes place) are different. The procedure of subtraction of equations does not lead to the equation with the right-hand side identically equal to zero in the stationary point (for trivial solution).

Such systems can be studied with the help of an approach developed by Duboshin (1952). Duboshin's criterion of stability of motion at permanent perturbations is as follows (1978): Unperturbed motion is stable et permanent perturbations $R_s(t/x_\sigma)$ if for any given arbitrary positive $\varepsilon \leq A$ there exist such positive numbers $\lambda \leq \varepsilon$ and r depending on ε that at any initial data x_s^o satisfying the conditions $\left|X_s^o\right| \leqslant \lambda$ and at any functions F_s satisfying at $|x_s| \leq \varepsilon$ at $t > t_o$ the inequalities

$$|R_s(t/x_\sigma)| < r \tag{7.154}$$

all the solutions $x_s(t)$ of the system of differential equations for $t > t_o$ satisfy the conditions $|x_s(t)| < \varepsilon$. In other cases the motion is called unstable at permanent perturbations, whatever their amplitude r is.

We shell employ Duboshin's criterion while studying the stability of a pendulum's oscillations and the stability of the virial oscillations.

The equation of motion of a pendulum near the stable (in the absence of oscillations of the fixed point) lowest equilibrium state is written

$$\ddot{x} + \left(\frac{g}{l} + d\frac{\omega^2}{l}\cos\omega t\right)x = 0, \tag{7.155}$$

where x is the angle measured from the equilibrium state; ℓ is the length of the pendulum; d is the amplitude of vertical oscillations of the fixed point; and ω is the frequency of oscillations of the fixed point.

For a pendulum oscillating near the lowest stable equilibrium state, the loss of stability (the parametric resonance) appears at the frequency of oscillations

$$\omega = \frac{2}{n}\sqrt{\frac{g}{\ell}} \tag{7.156}$$

where n is a natural number.

For oscillations of a pendulum near the unstable upper state, the condition of stability of the motion is expressed as

$$\omega > \frac{\sqrt{2g\ell}}{d}. \tag{7.157}$$

Thus variation of the system's parameters affects the stability of the motion. It should be noted that Eq. 7.155 itself (even if the dependence of amplitude d on time is taken into account) is written in linearized form relative to x, although the amplitudes of swing can be large and even rotation can be considered. Therefore, the more correct form of the equation will be, without linearization:

$$\ddot{x} + \left(\frac{g}{\ell} + d\frac{\omega^2}{\ell}\cos\omega t\right)\sin x = 0. \tag{7.158}$$

In this connection, let us consider the case when d = 0 and the effect of potential energy of the free rotation of the system is taken into account.

$$\ddot{x} = 0, \quad \dot{x} = \Omega = \text{const},$$

where Ω is the angular velocity.

The energy of rotation here is equal to the kinetic energy

$$T = \frac{1}{2} m\ell\Omega^2$$

The phase of revolution of a pendulum is written as $x = \Omega t + x_o$, where x_o is the initial phase .The potential energy of the system is $U = (mg/\ell)\sin x$, where U is proportional to sinx.

In this case Eq. 7.158 of the perturbed motion of a pendulum can be considered as a non-linear resonance of the potential and kinetic energies.

Unperturbed motion (rotation) of the system has angular velocity depending on its energy $T = 1/2m\ell^2\Omega$ and therefore

$$\Omega = \sqrt{\frac{2T}{m\ell^2}}$$

i.e.

$$\Omega \propto T^{1/2} \tag{7.159}$$

Thus, the unperturbed motion is non-linear although harmonic. Non-linearity is developed here in non-isochronous oscillations.

The perturbation function, which is the potential energy, oscillates with the same frequency as the perturbed system itself but has a phase shift. The curves indicating the variation of the phase of revolution (unperturbed motion) and the potential energy (perturbation function) are plotted in Fig. 7.14.

In this figure τ denotes the period of revolution of the pendulum. Curve 1 represents the phase of revolution, which is a function of time: $x = \Omega t$. Curve 2 is parallel to curve 1 but is shifted in the phase by 2π. Curve 3 represents the phase of the potential energy, which is the following function of time:

$$U = \frac{mg}{\ell}\sin x\alpha\left(x - \frac{x^3}{3} + \ldots\right)$$

In a linear approximation we have $U \propto x$ and thus the phase $x = \Omega t$ coincides with the phase of revolution. It follows from Fig. 7.14 that if the value of the coefficient mg grows, then curve 3 intersects curve 2. The differences of phase of revolution and perturbation reaches the value 2π and the character of the motion changes. Therefore, if the potential energy is considered as a permanently acting perturbation, the existence of the bifurcation point of the system with respect to permanent mg is evident.

Let us write all the forgoing in analytical form. The equation of motion is

$$\ddot{x} + \frac{g}{\ell}\cos x = 0. \tag{7.160}$$

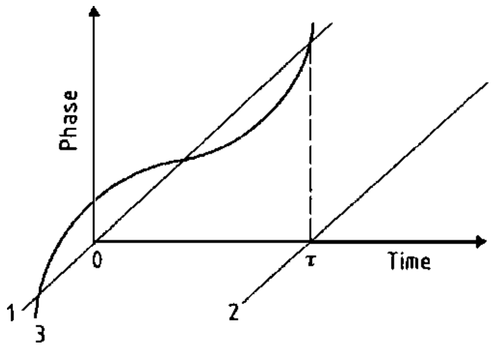

Fig. 7.14 Variation of the phase of revolution (unperturbed motion) and the potential energy (perturbation function) for a pendulum of vertical oscillations of its fixed point of suspension

Multiplying (7.160) by 2 $\dot{x}$ and integrating over t, one obtains

$$(\dot{x})^2 + \frac{2g}{\ell}\sin x + b = 0 \tag{7.161}$$

or

$$(\dot{x})^2 = -b - + \frac{2g}{\ell}\sin x. \tag{7.162}$$

The left-hand side of the last equation obtains non-negative values. Therefore there exist two qualitatively different types of solution depending on the value of parameter b. If b has negative values ($-b > 0$) and $|b|>2g/\ell$, then any value of x is possible ($|\sin x|\leqslant\ell$and thus . If $|b|<2g/\ell$, then only those values of x are possible at which the inequality $|\sin x|\leqslant|b|/(2g/\ell)<1$ holds.

Thus at $g/\ell = |b|/2$ we obtain bifurcation of the system by the parameter g/ℓ; at $g/\ell\leqslant|b|/2$ we obtain rotation of the system; and at $g/\ell>|b|/2$ we obtain oscillations of the system. It can be seen that the kinetic energy T of the system oscillations is periodically overpumping from the system into potential energy (perturbation energy V) and back, but at any time the total energy remains, $E = T + U = \text{const}$. The amplitude of swing of a system is the difference between the phase of unperturbed motion (rotation) and that of perturbation (the potential energy), which is responsible for the existence of the non-linear resonance (note that unperturbed motion is non-linear and non-isochronous, and the difference between the phase of rotation and perturbation is less than 2π.

There is no contradiction here or for estimates of stability of motion according to Lyapunov, since in Lyapunov's theory sufficiently small initial perturbations can be taken into consideration. Here we are studying the whole range of perturbation magnitudes. At the same time, it is obvious how Duboshin's criterion (7.154) works. If the stability of rotation of the system is studied at permanently acting perturbations whose role is played by the potential energy, then the value of perturbation is bound from above by a constant value until the bifurcation point $g/\ell = |b|/2$ and the rotation of he system occurs.

Now the general case of stability of motion can be easily considered, when $d \neq 0$ in Eqs. 7.155–7.158. The term (d cosωt)sinx = v in Eq. 7.158 can be considered as an external perturbation of the free (rotating) system and thus the system becomes unstable when it passes the bifurcation point (the parameter d has the same dimension as the term mg, i.e. it is proportional to energy). Therefore, the energy of the new perturbation should be greater than or equal to the energy of previous perturbations. The energy of the initial perturbations is equal to $mg\ell$ and that of the new perturbation to $mv^2/2$, where v is the velocity of movement of the fixed point of a pendulum equal to dω. The energy of the new perturbation is

$$\frac{mv^2}{2} = \frac{md^2\omega^2}{2}$$

The following inequalities should hold:

$$\frac{mv^2}{2} > mg\ell$$

$$\frac{d^2\omega^2}{2} > g\ell \quad \text{or} \quad d > \sqrt{\frac{2g\ell}{\omega}}$$

Thus, we have shown that by using Duboshin's criterion in studying the stability of motion of a pendulum with vertical oscillations of its fixed point of suspension in the integral characteristics, the result can be obtained directly. The analogous result (7.157), following from Mathieu's theory when studying the dynamics of the system in co-ordinates, requires more complicated solutions.

Lyapunov's theory provides the possibility of studying the stability of motion relative to different generalized co-ordinates and associated with their momenta. But the choice of suitable variables for the particular system cannot be made if the physics of the studied phenomenon is ignored.

Keplerian motion of a particle in an elliptical orbit can serve as an example of such a case, being unstable if determined relative to the radius-vector of a system. But this motion will be stable if it is defined relative to the variable $z = r - p/\ell + e\cos\psi$, where p and e are the eccentricity parameters of an ellipse, and r and ψ are its polar co-ordinates. The number of generalized co-ordinates should be equal to the number of degrees of freedom of a system (where co-ordinates are linearly independent). If the integral characteristics are used for the description of motion of a system, the number of characteristics must be complemented up to the number of degrees of freedom, at the expense of unimportant variables.

In the case of the two-body problem (taking the law of conservation of momentum into account) we should have the two canonical pairs. They are action-angle (usually mentioned) and Jacobi function-frequency (there is no example of application of the last pair in other works).

Duboshin's criterion allows the stability of a system to be studied when its potential energy is considered (partially or as a whole) as perturbation. Such an approach to the use of potential energy is employed in quantum mechanics, where all the potential energy is considered as perturbation (Landau and Lifshitz 1963). In this connection the virial oscillations of celestial bodies from the physical viewpoint should be considered as an effect of non-linear resonance between the kinetic and potential energies.

Comparing the approaches to estimations of the system's stability in a system of co-ordinates and in integral characteristics, let us point out the following. The method, which uses a full co-ordinate-momenta description of the system's dynamics, is too detailed, and classification of the studied system with respect to stability is therefore impossible. There are an infinite number of cases of systems based on it which can be chosen.

Using the integral approach, the following classification is possible. For example, in Keplerian motion, relative to the ratio of the integrals (energy and angular momentum), all possible types of orbit can be subdivided into four classes: two are of stable motion (elliptical and hyperbolic) and two are unstable (circular and parabolic).

A system's motion described by co-ordinates and momenta can be classified as unstable only in a formal way that can be accounted for by the choice of co-ordinate system. In fact, since we do not know the solution of the problem beforehand, the wrong co-ordinate system might be chosen.

In order to study the Lyapunov stability of a system, the Lyapunov function must be known. However, there is no standard rule prescribing how to find it; it is, rather, an art.

Considerable progress has been made in this field in the development of Chetaev's method. In this connection let us point out two facts:

(a) Chetaev's method for obtaining the Lyapunov function uses the combination of integrals of motion of the system. It is obvious that this way of choosing the Lyapunov function has considerable advantages.
(b) The integrals (and not their combination) are not positively defined functions of co-ordinates and momenta, which is why they are not used in such an analysis. They cannot be co used since arbitrary (and not their own) co-ordinate-momenta ate strange to these integrals. In practice, for many soluble problems, combination of a number of integrals of motion of a system corrects this deficiency. However, this can be done only for systems with small numbers of degrees of freedom which resemble a more general description of a system.

In describing a system in terms of integral characteristics, the characteristics themselves are immanent to their own integrals of a system and therefore they play the role of the Lyapunov function. In this case it is not even necessary to carry out formal mathematical analysis to study the stability of a system.

7.13.2 Stability of Virial Oscillations According to Lyapunov

Let us first estimate the stability of motion which is described by the classical virial theorem, i.e. when

$$\ddot{\Phi}_{\text{virial}} = 0. \tag{7.163}$$

The equation of the perturbed oscillations is in this case Jacobi's equation with the same value of the constant $\alpha^2\beta$:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}$$

The perturbation function in this case is

$$\eta = \Phi - \Phi_{\text{virial}}. \tag{7.164}$$

This function satisfies the equation

$$\eta = \ddot{\Phi} - \ddot{\Phi}_{\text{virial}} = -A + \frac{B}{\sqrt{\eta + \Phi_{\text{virial}}}} = x(\eta), \tag{7.165}$$

where $x(\eta)|_{r=0} = 0$

Here the Jacobi function plays the role of the co-ordinate, and, since $x(0) = 0$, we obtain the classical problem of stability of the trivial solution according to Lyapunov. Using the form of $x(\eta)$ and taking into account that, when the condition of the virial theorem holds at any moment in time, Φ_{virial} is a constant and A and B are such that $- A + (B/\Phi) = 0$. It follows that $x(\eta)$ is not a function of the defined sign. Therefore, the stability of the virial state does not follow automatically; it follows only on the condition that, in a virial state, the function Φ_{virial} assumes minimal values among all the possible configurations of the system. But the last condition is even more speculative then our hypothesis of constancy of the $\alpha^2\beta$ product.

Thus, the general, and the main, case remains the one when Φ does not remains constant. The case described by Φ_{virial} can be considered as a smooth state relative to the main state, i.e. it is related to the condition when averaging over the period of oscillations has been carried out.

Now let us consider the stability of virial oscillations of a conservative system according to Lyapunov described by

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} \tag{7.166}$$

when the product of the form factors $\alpha^2\beta$ is not a constant, i.e. variation of the parameter B is allowed.

When there are no variations of B, Eq. 7.166 is written

$$\ddot{\Phi}_1 = -A + \frac{B_1}{\sqrt{\Phi_1}} \tag{7.167}$$

and when variations of the parameter B occur, the equation is written

$$\ddot{\Phi}_2 = -A + \frac{B_2}{\sqrt{\Phi_2}}, \tag{7.168}$$

where A is a constant in the absence of energy dissipation.

The perturbation function η is written

$$\eta = \Phi_2 - \Phi_1. \tag{7.169}$$

Then the perturbation equation is written

$$\ddot{\eta} = \frac{B_2}{\sqrt{\Phi_1 - \eta}} - \frac{B_1}{\sqrt{\Phi_1}} = x(\eta). \tag{7.170}$$

If η is a continuous function, we have

$$x(\eta)|_{r=0} = 0 \tag{7.171}$$

If η is a discontinuous function, then

$$x(\eta)|_{r=0} \neq 0 \tag{7.172}$$

If η is continuous function < i.e. variation of $\alpha^2\beta$ is continuous, we obtain the classical problem of Lyapunov stability of the trivial solution of the perturbed equation, and all the comments are ordinary, i.e. the motion is stable (the necessary and sufficient conditions hold).

At discontinuous variations of $\alpha^2\beta$ (isolated singular points), the stability problem can be studied using Duboshin's criterion. As was shown in Chap. 5, the equation for dissipative system can be reduced to a system of linear equations with constant coefficients. When the ratio of frequencies for each of these equations is a rational number, the solution can be given by a closed algebraic curve (i.e. a set of isolated points). If the ratio of frequencies is an irrational number, the curve corresponding to the solution fills the whole set of possible values of the Jacobi equation. Here the last possibility corresponds to continuous variation of the function η.

It is interesting to point out that the well-known three-body problem theorem by Poincaré studies this case. There, instead of the mass of one body, in the course of the proof of the theorem, the continuity is employed.

But as Wintner (1941) has pointed out, the result obtained by Poincaré corresponds to the case when the masses of the bodies are not fixed. This note made by Wintner is rather general and can be related to the product of the form factors $\alpha^2\beta$ which should not be considered as a continuous function but rather as a set of isolated points.

On the whole, study of the stability of virial oscillations gives us evidence that the proof of the constancy of the product $|U|\Phi$ for existing celestial bodies is a sufficient but not a necessary condition. The necessary condition for the proof of the above relationship follows from the direct derivation of the equation of virial oscillations of celestial bodies from Einstein's equation and also by showing that Schwartzschild's solution in the framework of the general relativity theory is the solution of Jacobi's equation when $\Phi = 0$.

7.13.3 Stability of Virial Oscillations of Celestial Bodies as Dissipative Systems

Let us now estimate the stability of oscillations of celectial bodies emitting electromagnetic energy. This is the case for systems existing in nature.

For a conservative system, Jacobi's equation is written

$$\ddot{\Phi}_c = 2E_c - U \tag{7.173}$$

where the subscript c indicates that the corresponding characteristics of the system relate to a conservative system, and the subscript d, below, denotes a dissipative system.

Let us study the stability using Eq. 7.173, assuming dissipation of energy, which can be written

$$\ddot{\Phi}_d = 2E_d - U$$

A variation of the solution of Eq. 7.173 is written $\eta = \Phi_d - \Phi_c$.

Then

$$\eta = 2(E_d - E_c) = 2E_\gamma = At,$$

where t is the time elapsed from the beginning of evolution.

According to Duboshin's criterion, the system (7.173) which emits energy will be stable at any finite time interval, ranging from $t = t_o$ to $t = \tau$, where τ is the bifurcation time of the system (see Chap. 8).

Chapter 8
Creation and Evolution of the Solar System Bodies

The study of creation and evolution of celestial bodies and their systems is one of the main subjects in astrophysics, cosmogony and cosmology. It is known from observation that the matter of the Solar System bodies is very identical with respect to its substantial and chemical content and from this point of view all the bodies are of common origin. But the attempts to find general mechanism of creation of the bodies encounter irresistible contradiction. The point is that the planets having only ~0.015% of the system's mass possess 98% of the orbital angular momentum. At the same time ~99.85% of the Sun's masses produce no sooner than 2% of the moment of momentum which, in both cases, is accepted to be a conservative parameter. Also, the specific (for unit of the mass) orbital angular momentum is increased with the distance from the Sun. These results follow from the problem solution based on the hydrostatic approach.

We study the problem in the framework of the Jacobi dynamics and with the assumption that all evolutionary processes are developed as a consequence of the energy loss through radiation. And also we consider virial oscillations of celestial bodies to be a mechanism of generation and release of the energy emitted.

Our study appeared to be successful, since, applying dynamic approach, we resolve the above discrepancy. It was found that the orbital planetary moment of momentum and the angular momentum of the axial rotating body are kinetic effects which originate from different components of the potential energy. It was shown in Sect. 7.2 of Chap. 7, that axial rotation is a dynamical effect which is provided by tangential component of the body's potential energy. Rotating energy comprises very small portion of the total potential energy. It equals to $\sim 3 \cdot 10-^{2}$ for the Earth and $\sim 10^{-4}$ for the Sun from their total energy . Rotating energy is the function of the radial density distribution law and appears as an effect of interaction of the uniform and non-uniform mass particles with respect to their density. For the uniform body its rotary energy is equal to zero. This is physics of the body's angular momentum.

Quite different origin has orbital moment of momentum. Interaction (collision and scattering) of the body's mass particles is accompanied by continuous redistribution of the mass density. According to the Roche's tidal dynamics (see Sect. 7.5 of Chap. 7), redistribution of the density leads to the shell separation. As it follows

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from Eq. 7.71 of Chap. 7, when the body's upper shell density reaches value more than 2/3 of the mean density of all others shells, then the upper shell becomes weightless. From physical point of view it means that the own force field of the upper shell reaches dynamical equilibrium with respect to the parent's outer force field. If the density of such a shell around the body has non-uniform distribution, then its tangential component of the potential energy converts it into secondary body with mean density according to Eq. 7.70. In the case of its uniform density, the shell is preserved as a parent's ring. In general case the shell is decomposed on parts of different masses. The orbital motion of a body now is resulted from the parental body by its outer force field induced by the normal component of the body's potential energy. In this case the orbital velocity of the newly created body should be equal to the first cosmic velocity of the parental body on its surface and the direction of motion is determined by the Lenz law. Just this energy (and corresponding velocity) is the function of the orbital moment of momentum of the secondary body.

So, it is clear from the above that the induced outer force field, which is formed by the normal component of the solar potential energy, operates the orbital motion of a secondary body. Doing so, the secondary body conserves the value of the parental energy at the time of its creation. To the contrary, the body's own shell rotation is provided by the tangential component of the potential energy. From here, the energy and the orbital moment of momentum are conservative parameters. The body's total energy is also conservative parameter, but the tangential component of energy and the corresponding angular momentum are not conservative values (see minus sign in Eq. 7.50 of Chap. 7).

8.1 The Third Kepler's Law as a Kinematics Basis for the Problem Solution of Creation of the Solar System Bodies

Using the above physics, we discovered a very interesting phenomenon, which opens the way for solving the problem. It appears that the mean orbital velocities and periods of revolution of all the planets and asteroids are equal to the first cosmic velocity and corresponding period of the contracting Protosun, having its radius equal to the semi-major axes of each planet's orbit. The same was happened with the planets' satellites. The subsequent expansion of the space has not broken the above regularity.

Tables 8.1 and 8.2 demonstrate the observable and calculated values of the orbital periods of revolution of the planets, asteroids (small planets) and satellites obtained by applying first cosmic velocities of the Protosun and the protoplanets which prove the above said.

Table 8.1 Observable orbital periods of revolution of the planets around the Sun and calculated periods of oscillation of its corresponding outer shell

Planets	Orbital radius, $R \times 10^{11}$ (m)	Observable period of revolution T (year)	Calculating period of oscillation T_1 (year)
Mercury	0.579	0.24	0.2408
Venus	1.082	0.62	0.6153
Earth	1.496	1.0	1.00
Mars	2.28	1.88	1.8823
Vesta	3.53	3.63	3.7594
Juno	3.997	4.37	4.3733
Ceres	4.13	4.6	4.598
Themis	4.68	5.539	5.5397
Jupiter	7.784	11.86	11.8781
Saturn	14,271	29.48	29.4802
Uranus	28.708	84.01	84.1951
Neptune	44.969	164.8	164.9185
Pluto	59.466	248.09	250.8882

Table 8.2 Observable orbital periods of revolution of the satellites around the planets and calculated periods of oscillation of their corresponding outer shells

Planets	Satellites	Orbital radius $R \times 10^6$, m	Observable period of revolution T, year	Calculated period of revolution T_1, year
Earth	Moon	384.4	27.32	27.4103
Mars	Phobos	9.4	0.319	0.3208
	Deimos	23.5	1.262	1.2604
Jupiter	V	181	0.498	0.4973
	Io	422	1.769	1.7706
	Europa	671	3.551	3.5508
	Ganymede	1,070	7.155	7.154
	Callisto	1,880	16.69	16.6709
	XIII	11,100	240.92	239.0960
	VII	11,750	259.14	259.5899
	XII	21,000	620.77	660.7744
	1X	23,700	758.90	745.1833
Satutn	Janus	151.5	0.7	0.6956
	Mimas	185.6	0.94	0.9431
	Enceladus	238.1	1.37	1.3704
	Tethys	294.7	1.89	1.8869
	Dione	377.4	2.74	2.7366
	Titan	1,212.9	15.95	15.7548
	Iapetus	3,560.8	79.33	79.2494
	Phoebe	12,944	548.2	549.2722
Uranus	Cordelia	49.751	0.3350	0.3348
	Cupid	74.8	0.618	0.6172
	Miranda	129.39	1.4135	1.4043
	Ariel	191.02	2.5204	2.5189
	Umbriel	266.3	4.1442	4.1463
	Titania	435.91	8.7058	8.6840
	Oberon	583.52	13.4632	13.4503

(continued)

Table 8.2 (continued)

Planets	Satellites	Orbital radius $R \times 10^6$, m	Observable period of revolution T, year	Calculated period of revolution T_1, year
Neptune	Triton	354.8	5.877	5.8523
	Nereid	5513.4	360.14	359.8227
Pluto	Charon	19.571	6.387	9.5065
	Nix	48.675	24.856	37.2873
	Hydra	64.780	38.206	54.2482

The first cosmic velocity v_1 of the Protosun's and protoplanetary bodies and the period of oscillation of the corresponding outer shell T_1 of the created bodies were calculated by the formulae from which, in fact, the third Kepler's law follows:

$$\nu_1 = \omega R = R\sqrt{\frac{Gm}{R^3}} = \sqrt{\frac{Gm}{R}}, \quad T1 = \frac{2\pi}{\omega} = \frac{2\pi R}{\nu_1}, \quad \frac{(2\pi)^2}{T_1^2} = \frac{Gm}{R^3},$$

where m is the body's mass; G is the gravity constant; R is the semi-major axis; $\omega = \nu_1/R$ is the frequency of virial oscillation of the outer shell, which appears to be equal to the angular velocity of the orbital motion. Note, the frequency of virial oscillation of the outer weighty shell does not equal to its angular velocity because the frequency is the parameter of the force field.

For example, when the Protosun's radius R extended up to the present day Earth's orbit ($m = 1.99 \cdot 10^{30}$ kg, $R = 1.496 \cdot 10^{11}$ m), then its first cosmic velocity was equal to

$$\nu_1 = \omega R = \sqrt{\frac{Gm_s}{R}} = \sqrt{\frac{6,67 \cdot 10^{11} \cdot 1,99 \cdot 10^{30}}{1,496 \cdot 10^{11}}} = 29786.786\,\text{m/s}$$
$$= 29.786786\ \text{km/s}.$$

This value corresponds to the observed mean orbital velocity of the Earth.

The period of oscillation of the interacted mass particles of the Protosun's outer shell ($R = 1.496 \cdot 10^{11}$ m, $v_1 = 29786.786$ m/s) was equal to

$$T_1 = \frac{2\pi R}{\nu_1} = \frac{6,28 \cdot 1.496 \cdot 10^{11}}{29786.786} = 3.1540428 \cdot 10^7 \text{s} = 1\ \text{year},$$

which is equal to the observed period of the planet's orbital revolution.

When the Protoearth's radius R extended up to the present day Moon's orbit ($m_e = 5.976 \cdot 10^{24}$ kg, $R = 3.844 \cdot 10^8$ m), then its first cosmic velocity was equal to

$$\nu_1 = \sqrt{\frac{Gm_e}{R}} = \sqrt{\frac{6,67 \cdot 10^{11} \cdot 5.976 \cdot 10^{24}}{3.844 \cdot 10^8}} = 1018.3018\ \text{m} = 1.0183918\ \text{km/s},$$

which is the present day Moon's mean orbital velocity.

The period of oscillation of the interacted mass particles of the Protoearth's outer shell (R = 3.844 10^8 m, v_1 = 1018.3018 m/s) was equal to

$$T_1 = \frac{2\pi R}{\nu_1} = \frac{2 \cdot 3.14 \cdot 3,844 \cdot 10^8}{1018.3018} = 23.706449 \cdot 10^5 s = 27.438019 \text{ days},$$

which corresponds to the present day Moon's period of orbital revolution.

The obtained results mean that all the planets and satellites were launched by first cosmic velocity of the self-gravitating Protosun and protoplanets after their outer shells acquired weightlessness. As it was said above, the process of evolutionary loss of energy by emission led to redistribution and differentiation of the body's mass density: increase it in the inner shells and decrease in the outer one by the light components dilution. In general, due to this process and contraction in the form of separation of the matter, the shell separation of a body with respect to density was developed up to the state of weightlessness and self-gravitation of the outer shell's matter and creation of the body.

The discovered regularity of the Solar system's planets and satellites creation seems to be valid for the process of separation of the Protosun itself and the other ptotostars from the protogalaxy Milky Way. If we accept for the Galaxy's known astrometric data (mass $m_g = 2.5 \cdot 10^{41}$ kg, distance of the Sun from the Galaxy center $R_s = 2.5 \cdot 10^{20}$ m), then it is not difficult to calculate that the first cosmic velocity of the proto-Galaxy, which size was limited by the Sun's semi-major orbital axes, is equal to 230 km/s, and the orbital period of revolution is $220 \cdot 10^6$ year. The values are close to those found by observation, namely: mean orbital velocity of the Sun is called as (230–250) km/s, and the orbital period of revolution $T_s = (220–250) \cdot 10^6$ year.

The observed today picture of the Milky Way, consisting of a bar-shaped core surrounded by a disc of gaseous matter and stars, which create two major and four or more smaller logarithmic spiral arms, prove the generally common mechanism of creation of the galactic system. This picture demonstrates that the huge of mass and size of the Protogalaxy rotating body was subjected during evolution by the polar and the equatorial oblateness. Due to redistribution of mass density and after reaching a state of their weightlessness the stars were separated from the outer shell in different surface regions. It appears that the main parts of the mass were separated in two regions (in the pericenter and the apocenter) and less in all others. But because the first cosmic velocity is the function of radius of the body, the regions of separation in the space formed the logarithmic spirals of the moving stars in accordance with the third Kepler's law. The same logarithmic spirals were formed also by the planets and the satellites in the Solar System.

The following initial values of density ρ_i and radius R_i of the Protosun and protoplanets can be obtained on the basis of their dynamic equilibrium state.

The Protosolar cloud has separated from the Protogalaxy body when its outer shell in the equatorial domain has reached the value of the first cosmic velocity. In fact, the gaseous cloud should represent chemically non-homogeneous rotating body. As it follows from Roche's dynamics (see Chap. 7, Eq. 7.71), the mean density of the gaseous Protogalaxy outer shell should be $\rho_s = 2/3\rho_g$. The condition $\rho_s = 2/3\rho_g$ is the starting point of separation and creation of the Protosun from the outer Protogalaxy shell. Accepting the above described mechanism of formation of the secondary body, we can find the mean density of the Protogalaxy at the Protosun separation as

$$\rho_g = \frac{m_g}{\frac{4}{3}\pi R^3} = \frac{2.5 \cdot 10^{41}}{\frac{4}{3} \cdot 3.14 \cdot \left(2.5 \cdot 10^{20}\right)^3} = 1.67 \cdot 10^{-21}\mathrm{kg/m^3} = 1.67 \cdot 10^{-24}\mathrm{g/cm^3}.$$

The mean density of the separated proto-Galaxy shell is

$$\rho_s = 2/3\rho_g = 2/3 \cdot 1.67 \cdot 10^{-24} = 1.11 \cdot 10^{-24}\mathrm{g/cm^3}.$$

In accordance with Eq. 6.70, the mean density and radius of the initially created Protosun body should be

$$\rho_s = 2\rho_g = 2 \cdot 1.67 \cdot 10^{-24} = 3.34 \cdot 10^{-24}\mathrm{g/cm^3};$$

$$R_s = \sqrt[3]{\frac{2 \cdot 10^{33}}{\frac{4}{3}3,34 \cdot 10^{-24}}} = 7.5 \cdot 10^{18}\mathrm{cm} = 7.5 \cdot 10^{16}\mathrm{m}$$

The mean density and the radius of the initially created Protojupiter, Protoearth and Protomoon are as follows:

the ProtoJupiter : $\rho_j = 2 \cdot 10^{-9}\mathrm{g/cm^3}, \quad R_j = 6.2 \cdot 10^{13}\mathrm{cm} = 6.2 \cdot 10^{11}\mathrm{m};$

the ProtoEarth : $\rho_e = 2.85 \cdot 10^{-7}\mathrm{g/cm^3}, \quad R_e = 1.9 \cdot 10^{11}\mathrm{cm} = 1.19 \cdot 10^{9}\mathrm{m};$

the Protomoon : $\rho_m = 5 \cdot 10^{-4}\mathrm{g/cm^3}, \quad R_m = 1.1 \cdot 10^{9}\mathrm{cm} = 1.1 \cdot 10^{7}\mathrm{m};$

Analogous unified process was repeated for all the planets and their satellites.

Creation of the other small bodies like comets, meteors and meteorites are also found their explanation within the considered mechanism and physics. In fact, the only condition for separation of outer body's shell is its weightlessness (its corresponding mean density relative to the body's mean density), but not a limit of some amount of mass. In this connection any volume and amount of mass could probability be separated at any time. For example, we found by calculation that the short-periodic Encke's Comet (1970 I, T = 3.302 year) has semi-major orbital axis $R \approx 1.5 \cdot 10^{11}$ m and has separated from the Proto-Sun after small planet Vesta and before the Mars. The short-periodic Halley's Comet (1910 II, T = 76.1 year) has semi-major orbital axis $R = 2.7 \cdot 10^{12}$ m and has separated from the Proto-Sun after Saturn and before Jupiter. The long-periodic Ikeya-Seki's Comet (1965 III, T = 874 year) has semi-major orbital axis $R = 1.35 \cdot 10^{14}$ m and has separated from the Proto-Sun before the Pluto. Like the asteroid belt between the Jupiter and Mars, the comet belts should definitely exist between the orbits of all the Jupiter group planets. As to the meteors and meteorites, they all should be separated from the planets by the same way. From point of view of dynamical equilibrium of their orbital motion the orbits of all the small bodies (comets, meteors and meteorites) should have large eccentricities and dip angles of inclination to the equator of their central bodies. This is because of probable oblateness of the proto-Sun body, where its polar regions should have higher values of the first cosmic velocity. Those small bodies and meteorites, which have not reached or have later on lost dynamical equilibrium, fell down on the planet's or satellite's surface.

As it was shown in the Table 8.1, the small planets of the asteroid belt separated from the Protosun by the same mechanism. From point of view of the orbital motion and first cosmic velocities there are no any features of their separation from a broken planet.

The above consideration takes off an old misunderstanding about the difference in the orbital planet's and the Sun's moment of momentum. The secondary body conserves the creation energy and orbital moment of momentum in accordance with the third Kepler's law. As to direction of body's axial rotation and orbital revolution, then these parameters enter by the inner and outer force field, like in electrodynamics, in accordance with the Lenz law. As to the specific (for unit of the mass) orbital moment momentum of the planets and satellites which increases along with distance from the central body, the explanation of this gives the increasing radius from the central body

The revealed physics and kinematics of creation and separation of the Solar System bodies prove the Huygens' law of motion on semi-cubic parabola of his watch pendulum, which synchronously follows the Earth motion. This curve was called evolute. And the curve perpendicular to the series of tangents to the evolute is called evolvent. The relationship between the evolute and evolvent represents the relationship between function and its derivative or between function and its integral. These relations exist not locally like in mathematical analysis but in integral form and geometrically visible. While plotting a series of the evolvents with fixed lengths of the pendulum a peculiarity of the same type is appeared in each point of the initial curve of evolvent. The peculiarity is the semi-cubic parabola of type $x^2 = y^3$ or $x = y^{3/2}$. This is just universal law of a body motion in the nature, which is the consequence of the simple fact. The degree 3/2 is the ration between the body's mass volume, which generates the energy, and the body's surface, which emit this energy. And also, in any task of motion we always have some initial conditions, which the moving object is inherited. In the case of the Huygens' oscillating pendulum the suspension filament starts unrolling in a fixed point. In the case of a celestial body, creation of satellite starts in a fixed point of its parental body where the initial conditions are transferred by the third Kepler's law.

This is because just in a fixed point of dynamical equilibrium between the generated volumetric energy (cubic degree of radius) and the irradiated from outer surface energy (square degree of radius) is broke.

First cosmic velocity was practically applied by man only in twentieth century. The nature seems to use it perpetually as the main instrument of the Universe evolution. Our Universe seems to be a pulsating system and its basic infinitesimall particle is ~10^{-36} g in weight (see calculation later on in Sect. 8.3) is responsible for the system's equilibrium. Because of the matter evolution and energy conservation law is the process, continuing infinitely long time. Some details related to the evolution of a gaseous sphere we discuss below.

8.2 Evolution by Radiation and Gravitational Contraction of a Gaseous Sphere

We consider here several problems in the gravitational evolution of a gaseous sphere based on the generalized virial theorem and the relationship between the potential energy and the moment of inertia of the sphere in the form

$$-U\sqrt{I} = \alpha^2 \frac{Gm^2}{R}\sqrt{m(\beta R)^2} = \alpha^2\beta G m^{5/2}, \tag{8.1}$$

where U is the potential energy of the sphere; I is the polar moment of inertia; G is the gravitational constant; m is the body mass; R is the sphere radius; and α^2 and β^2 are dimensionless structural parameters depending on the radial mass density distribution of the spherical body.

From (8.1), and taking into account Eqs. 2.43, 2.45, 7.54 and 7.55, we have the following relationships between the structural form factors:

$$\alpha^2 = \frac{r_g^2}{R^2} \quad \text{and} \quad \beta^2 = \frac{r_m^2}{R^2}, \tag{8.2}$$

$$\alpha^2\beta = a = \text{const}, \tag{8.3}$$

where $\alpha^2 = \alpha_o^2 + \alpha_t^2 + \alpha_\gamma^2$; $\beta^2 = \beta_0^2 - \beta_t^2$; $\alpha_0^2 = \beta_0^2 = 0.6$; $2\alpha_t^2 = \beta_t^2$; $\alpha_o^2\beta_0 = a_0 =$ const; $\alpha_t^2\beta_t = a_t =$ const; r_g and r_m are the reduced radius of gravity and radius of inertia; α_o, β_o, α_t, α_γ, β_t, are form factors of the normal, tangential and dissipative components of the energy for non-uniform mass density distribution of a system.

In Chaps. 6 and 7 we found that the constancy of the form factors product (8.3) is independent of the body mass, radius and radial mass density distribution for spherical and elliptic bodies. Eq. 8.3 is therefore a key expression in our further consideration.

8.2.1 Equilibrium Boundary Conditions for a Gravitating Gaseous Sphere

It is well known that polytropic models require the boundary mass density of a gravitating body to be rigorously equal to zero. Hence this condition gives us no opportunity to consider any physical processes during evolution.

If Eq. 8.1 for the spherical and elliptical gravitating system is valid, it allows us to consider convenient boundary conditions which can be used in the study of evolutionary problems.

In deriving the physical boundary conditions for a self-gravitating and rotating gaseous sphere, we consider its rotation as an effect of the tangential component of energy generated by the interacted non-uniform particles. As it was shown in Chap. 7, the ellipticity of the body is formed not as a result of its rotation but because of its self-gravitation. The key relationship (8.3) used here as the basis of our consideration prevents any possible errors. When we have to introduce the moment of inertia, the rotating sphere boundary at the equator will be defined by Kepler's law.

The fact that gaseous sphere boundary equilibrium conditions differ from those of the interior explains the difference between a free molecular boundary particle movement and an internal chaotic one. It is a consequence of the discrete matter structure dominant at the boundary (Jeans 1919).

Let us now consider the thermodynamic boundary conditions. Surely, we can define the boundary temperature only in the case of its real existence which, in turn, depends on the existence of the thermodynamic equilibrium between matter and radiation. Otherwise, it cannot be considered as black body radiation, and the Stefan-Boltzmann equation is inapplicable.

Thermodynamic equilibrium at the boundary can be reached only when the energy and momentum carried away by the radiation flow is greater than that carried away by the flow of particles from the sphere surface per unit time. Such a surface cannot increase further without disturbing the thermodynamic equilibrium.

We shall consider the evolutionary process of the gaseous sphere to be a successive series of hydrodynamic states in equilibrium. We shell also assume that the radiation energy loss causes the sphere to contract during the time periods between the equilibrium states.

Taking these ideas into account, we can express the hydrodynamic equilibrium at the boundary either by an expression representing particle flow 'locking' by the gravitational force, or, equivalently, by an equation showing the absence of particle dissipation from the boundary surface, which can be written in the form

$$\frac{Gm\mu}{R^2} = \frac{\mu\bar{v}^2}{R}, \tag{8.4}$$

where μ is the mass of the particle, and $\bar{v}$ is the velocity of the particle heat movement at the sphere boundary of the pole (more precisely it is the velocity of a particle running from the gravitational field).

For gravitational contraction between any two equilibrium states, Eq. 8.4 must be written as

$$\frac{Gm\mu}{R^2} > \frac{\mu\bar{v}^2}{R}. \tag{8.5}$$

Let us prove that the expression (8.4) for the gaseous spherical body boundary satisfies the virial relations.

First we consider one particle at the sphere boundary surface with mass μ and moving in the volumetric central field of the body with mass m and radius R. Then it is easy to see that

$$\left(\frac{\mu\ddot{R}^2}{2}\right) = \mu\left(\overline{\ddot{R}R} + \left(\dot{\bar{R}}\right)^2\right). \tag{8.6}$$

The kinetic energy K_p of the particle is

$$\mu\left(\dot{\bar{R}}\right)^2 = \frac{2\mu v^2}{2} = 2K_p. \tag{8.7}$$

From Newton's law we have

$$\ddot{\bar{R}} = -\frac{Gm}{R^3}R. \tag{8.8}$$

The potential energy U_p of the particle in the gravitational field of the body is

$$\mu\overline{\ddot{R}R} = -\frac{Gm\mu}{R^3}\left(\overline{RR}\right) = -\frac{Gm\mu}{R} = U_p. \tag{8.9}$$

Therefore

$$\frac{d^2}{dt^2}\left(\frac{\mu R^2}{2}\right) = U_p + 2K_p. \tag{8.10}$$

Summing over all particles at the boundary layer and neglecting their interaction energy, we obtain

$$\frac{d^2}{dt^2}\left(\frac{m_s R^2}{2}\right) = U_s + 2K_s, \tag{8.11}$$

where m_s is the mass of the boundary spherical layer.

Or finally

$$\frac{3}{4}\ddot{I}_s = U_s + 2K_s,$$

$$\ddot{\Phi}_s = U_s + 2K_s \tag{8.12}$$

which represents Jacobi's virial equation for a spherical gaseous layer.

The exchange of particles between the gaseous body and its boundary layer takes place at the same radius R and lasts for a short time while the total mass of the layer remains constant. So Eq. 8.12 is rigorous.

The solution of Eq. 8.12 will be exactly the same as that obtained in Chap. 4 for a gravitating sphere, except that corresponding parameters of the sphere must be replaced by those of the boundary layer.

If one time-averages over time intervals which are longer than the period of boundary-layer oscillations, then the left-hand side of Eq. 8.12 tends to zero (i.e. the layer enters into the outer force field) and a quasi-equilibrium boundary state is obtained determined by the generalized classical virial relation between the potential and kinetic energies:

$$\dot{\Phi}_s = U_s + 2K_s. \tag{8.13}$$

Thus, we have proved that Eq. 8.4 written for the gaseous sphere boundary is a virial relation. We shell use this expression further in solving the problem of contraction velocity for gravitating gaseous sphere.

8.2.2 *Velocity of Gravitational Contraction of a Gaseous Sphere*

In considering the evolution of a gaseous sphere, one does not usually take into account its rotation because the total kinetic energy exceeds the rotational energy. Other authors who accepted the rotation of the gaseous sphere could not manage with the angular momentum accepted as conservative value during contraction (Zeldovich and Novikov 1967; Spitzer 1968; Alfvén and Arrhenius 1970).

It was shown in Chaps. 2 and 7 that the main part of kinetic energy of a celestial body is represented by oscillatory energy of the interacting elementary particles. The rotational part is much smaller of oscillatory energy and appears to be an indication of degree of the body matter non-homogeneity. Slow rotating bodies like the Sun, Mercury, Venus, and Moon have more homogeneous density distribution. Their part of rotational energy from the total kinetic one is ~$1/10^4$. For the other planets of the Solar System this figure is ~1/300. It follows from (7.50) of Chap. 7 that the value of oscillatory energy for a body as a whole is a conservative parameter. The value of rotary energy is a changeable parameter.

The solution of the virial equation obtained earlier enables us to propose the following mechanism for gravitational contraction of a gaseous sphere. During each period of the sphere's oscillation, a certain amount of energy is lost through radiation. Hence, the contraction amplitude is larger than the expansion amplitude. The difference between the two amplitudes is the value of the gaseous sphere contraction averaged over one period of oscillation. Taking into account the adiabatic invariant relation (Landau and Lifshitz 1973a), we shell consider the problem of the gravitational contraction of a gaseous sphere using the virial relations and the key relationships (8.1) and (8.3). Note that we consider here the process of evolution without loss of body equilibrium.

Since we consider the evolution process of a gaseous sphere as a successive moment from one equilibrium state to another, it is natural that the minimum time interval for averaging varying parameters should be a little larger than that required for establishing the hydrodynamic equilibrium. So it is not difficult to control the variations of parameters during evolution which are not in contradiction with the equilibrium. (Later, we shell consider these restrictions to be nonexistent).

It is convenient for our purpose to write the generalized virial theorem in the form

$$-U = -2(E - E_\gamma) \equiv 2(E_\gamma - E), \tag{8.14}$$

where $E = U + K$ is the total energy of the gaseous sphere which is a constant over time; E_γ is the electromagnetic energy radiated up to the considered moment of time; K is kinetic energy which includes the energy of rotation and oscillation of the interacted mass particles; E and U are negative parameters.

The time derivative of E_γ is the gaseous sphere luminosity L which is a function of the sphere radius R and the boundary surface temperature T_o:

$$\frac{d}{dt}\left(E_\gamma\right) = L = 4\pi\sigma R^2 T_0^4, \tag{8.15}$$

where σ is the Stefan-Boltzmann constant.

From Eq. 8.14 it follows that

$$\frac{d}{dt}\left(E_\gamma\right) \equiv \frac{d}{dt}\left(E_\gamma - E\right) + \frac{1}{2}\frac{d}{dt}(-U).$$

The potential energy is in turn a function of the radius R:

$$-U = \alpha^2 \frac{Gm^2}{R}.$$

The time derivative of $(-U)$ is

$$\frac{d}{dt}(-U) = v_c \frac{d}{dR}(-U),$$

where $v_c = dR/dt$ is the gaseous sphere contraction velocity. To find this velocity we write

$$\frac{1}{2} v_c \frac{d}{dR}\left(\alpha^2 \frac{Gm^2}{R}\right) = \frac{dE_\gamma}{dt} = L$$

and finally, with the help of Eq. 8.3, we obtain

$$v_c = \frac{8\pi\sigma}{Gm^2} \frac{R^2 T_0^4}{(d/dR)(\alpha^2/R)}. \tag{8.16}$$

From Eq. 8.16 it is easy to see that v_c contains two unknown functions: $\alpha^2 = \alpha^2(R)$ and $T_o = T_o(R)$.

As was found in Chap. 2, the structural form factor α^2, as well as β^2, is the function of radial mass density distribution of the sphere. In Chap. 7 we considered this function presented by (8.2) and (8.3). It was found that the contraction velocity of the gaseous sphere depends on the mass density redistribution which determines kinetic energy of the body and its shells. So, the function $\beta = \beta(R)$ can be found from the condition of kinetic energy conservation of the body's upper shell after its separation.

It follows from (8.3) that during the gravitational contraction of the gaseous sphere its radius $R \to R_1$ and $\beta^2 \to 1$ (where R_1 is the orbital radius of separation). If $R \to R_1$ then velocity of rotation $v \to v_1$ (v_1 is the first cosmic velocity).

The kinetic energy of the body's upper shell before K_b and after K_a shell is written as

$$K_b = I\omega^2 = \beta^2 m\omega^2 R^2, \tag{8.17}$$

$$K_a = mv_1^2 = m_s\omega^2 R_1^2, \tag{8.18}$$

where I is the polar moment of inertia of the body; ω is the frequency of the radial oscillations; m and m_s are the body and its upper shell mass; $R - R_1$ is the thickness of the upper shell or the contraction value.

From Eqs. 8.17 and 8.18 we can write

$$\beta^2 = \frac{m_s\omega^2 R_1^2}{m\omega^2 R^2} = \kappa \frac{R_1^2}{R^2},$$

$$\beta = \sqrt{\kappa}\frac{R_1}{R},$$

$$\alpha^2 = \frac{a}{\beta} = \frac{a}{\sqrt{\kappa}} \frac{R}{R_1}, \tag{8.19}$$

where κ is the ratio of the Protosun's mass to the mass of a separated body.

Thus, we obtained an expression for α as a function of R, which is valid when the kinetic energy of the upper body's shell conserves in the orbital motion of the separated creature.

Let us now try to obtain the relationship between the gaseous sphere boundary temperature T_o and the radius R. We introduced the virial equilibrium boundary conditions by Eq. 8.4. This equilibrium was defined as particle flow 'locking' by the gravitational force, or, equivalently, by an equation showing the absence of particle dissipation from the boundary surface. Let us now rewrite it:

$$\frac{Gm\mu}{R^2} = \frac{\mu\bar{v}^2}{R}. \tag{8.20}$$

The heat velocity $\bar{v}^2$ depends on the boundary temperature T_o as

$$\mu\bar{v}^2 = 3kT_0, \tag{8.21}$$

where k is the Boltzmann constant.

Therefore we can rewrite the condition for particle flow 'locking' (8.20) with the help of Eq. 8.21 as

$$\frac{Gm\mu}{3k} = T_0R. \tag{8.22}$$

From the law of equal energy distribution over the degrees of freedom for the case of a gas particle mixture in equilibrium, it follows that

$$\mu_1\bar{v}_1^2 = \mu_2\bar{v}_2. \tag{8.23}$$

It is easy to see from (8.22) that the equilibrium radius of a gaseous sphere depends on the chemical composition of the gas. This conclusion follows from Eqs. 8.22 and Eq. 7.71 of Chap. 7, where the mechanical equilibrium condition of a body's upper shell is considered. Those results explains the effect of the particle flow 'locking' on the pole by the gravitational force which is based on the concept of mass and radiation equilibrium. Care must therefore be taken when the gas mixture is analyzed, i.e. if there are a small number of particles with light masses, the mixture will dissipate easily and the particles flow 'locking' will take place in the case of the heavier particles of the gas mixture.

When the quantities of the various mass particles are approximately equal, the particle flow 'locking' condition can be found only by a numerical solution. The gaseous sphere radius can be determined only after the equilibrium equation is solved, and to solve it we must consider all the given types and concentrations of particles in the flow. Formally, we can apply the effective particle mass μ which depends on a value averaged over all the particle masses. The problem can also be solved by numerical methods for a gas mixture consisting of many particles, and especially when the processes of ionization and recombination and chemical reactions occur.

Another interesting phenomenon, which we shell discuss, arises from the fact that electromagnetic forces are much stronger then gravitational forces. When some electrons escape the gravitating body it becomes positive by charges that create huge forces which tend to stop the process of electron dissipation. That is why it is necessary to use the proton mass μ_p when the gaseous cloud consists of neutral hydrogen partly ionized at the gaseous sphere boundary surface (the position of the boundary shell is specified by the radius R). The flow of electrons will be 'locked' by the extra forces appearing as a result of their primary dissipation. In addition, this uncompensated positive charge should have a drift at the boundary surface and small flow of cold plasma should be observed.

In the course of contraction of the gaseous sphere and the increase of its average temperature, the process of gas ionization should also increase. When the flow of electrons is large enough, and the limiting equilibrium between the gravitational forces and the charged protons is achieved, the protons should also start to run off the body's gravitational field. In this case, the increasing electron flux has to be 'locked' by electrostatic forces. The boundary equilibrium change from the proton 'locking' to electron 'locking' should start at this moment.

Thus, we come to the conclusion that at least two phases of gaseous sphere evolutions should exist: that of the proton and that of the electron, with a transitional domain between them which can be calculated by numerical methods in each specific case.

Figure 8.1 illustrates all that we have said. The process of gravitational contraction of the gaseous sphere is represented by the curve ABCD. Within the AB range, the body equilibrium is kept by the gravitational field 'locking' of the proton flow (the proton phase). Within the same range of sphere contraction, the radius R decreases while the temperature T_o increases. Point B is the critical one; here the transformation of equilibrium boundary conditions from proton 'locking' to electron 'locking' begins. The process spreads up to point C. While the sphere radius decreases in the range BC, the boundary temperature remains constant.

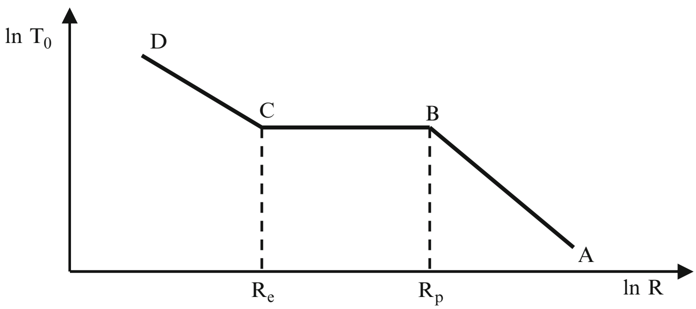

Fig. 8.1 Proton AB and electron CD equilibrium phases of the boundary shell of contracting gaseous sphere

In the electron equilibrium phase in the range CD, we can see that during the contraction process the boundary temperature increases again.

Let us check the derived expression (8.22) and the conclusion concerning the existence of two boundary equilibrium phases on the observed Sun data.

First, we calculate the numerical value of T_oR in the CGS system with the help of Eq. 8.22. Assuming numerical values for proton and electron masses, we obtain

$$T_pR_p = A_p = \frac{Gm\mu_p}{3k} = \frac{6.67 \cdot 10^{-8} \cdot 2 \cdot 10^{33} \cdot 1.67 \cdot 10^{-24}}{3 \cdot 1.38 \cdot 10^{-16}} = 5 \cdot 10^{17} \text{cm} \cdot \text{K},$$

$$T_eR_e = A_e = \frac{Gm\mu_e}{3k} = \frac{6.67 \cdot 10^{-8} \cdot 2 \cdot 10^{33} \cdot 9.1 \cdot 10^{-28}}{3 \cdot 1.38 \cdot 10^{-16}} = 2.73 \cdot 10^{14} \text{cm} \cdot \text{K}.$$

For the contemporary Sun we know that $R = 7 \cdot 10^{10}$ cm and $T_o = 5{,}000$ K so that $T_oR = 3.5 \cdot 10^{14}$ cm · K.

As at $T_o = 5{,}000$ K, where gas ionization must be fairly complete, we have a very good coincidence of the calculated and the observed data for the products T_oR and T_eR_e. For this temperature the proton radius of the Sun R_p is equal to 10^{14} T_oR cm, which corresponds to the orbit radius of Jupiter.

Thus, we have found α, β and T_o as functions of the radius R. We can now obtain the gaseous sphere contraction velocity. We rewrite Eq. 8.16:

$$\nu_c = \frac{8\pi\sigma}{Gm^2} \frac{(RT_0)^4}{R^2(d/dR)(\alpha^2 R)} \tag{8.24}$$

and, using (8.19), we can evaluate the denominator as

$$R^2 \left| \frac{d}{dR} \left(\frac{\alpha^2}{R} \right) \right| = R^2 \left| \frac{d}{dR} \left[\frac{a}{\sqrt{\kappa}} \frac{1}{R} \frac{R}{R_1} \right] \right| = \frac{a}{2\sqrt{\kappa}} \frac{R}{R_1}.$$

Finally, we write contraction velocity v_c as

$$\nu_c = \frac{16\pi\sigma}{Gm^2} \frac{A^4}{a} \sqrt{\kappa} \frac{R_1}{R}, \tag{8.25}$$

where $A = A_e = R_eT_e$ and $A = A_p = R_pT_p$ are for the electron and the proton phases of the gaseous sphere evolution, respectively.

Let us use Eq. 8.25 to obtain the contraction velocity and the time of contraction of the Protosun during the proton and the electron phase of the gaseous sphere evolution using its corresponding parameters.

If we take for the proton phase of the Protosun, after its separation from the Protoalaxy, $A_p = 5 \cdot 10^{17}$ cm · K, initial radius $R = 7.5 \cdot 10^{18}$ cm, final radius of the proton phase evolution (at the asteroid belt, after separation of the Protojupiter),

$R_1 = 4.2 \cdot 10^{13}$ cm, a = 0.46, and $\sigma = 5.76 \cdot 10^{-5}$ erg · cm^{-2} · s · (K)^4 as initial, we obtain

$$\overline{\nu}_{csp} = \frac{16 \cdot 3.14 \cdot 5.76 \cdot 10^{-5} \left(5 \cdot 10^{17}\right)^4}{6.67 \cdot 10^{-8} \left(2 \cdot 10^{33}\right)^2 \cdot 0.46} \cdot \sqrt{\frac{2 \cdot 10^{33}}{2.65 \cdot 10^{30}}} \cdot \frac{4.2 \cdot 10^{13}}{7.5 \cdot 10^{18}}$$
$$= 2.23 \cdot 10^5 \text{cm} \cdot \text{s}^{-1},$$

$$t_{sp} = \frac{7.5 \cdot 10^{18}}{2.23 \cdot 10^5} = 3.36 \cdot 10^{13} \text{s} = 1.06 \cdot 10^6 \text{years.}$$

We can find now the contraction velocity and the time of contraction of the Protosun during the electron phase of the gaseous sphere evolution. We take now for the electron phase $A_e = 2.73 \cdot 10^{14}$ cm·K, initial radius of the Protossun, after separation of the Protojupiter, $R = 4.2 \cdot 10^{13}$ m, final radius of the electron phase let be present day value $R_1 = 7 \cdot 10^{10}$ m, a = 0.46, and $\sigma = 5.76 \cdot 10^{-5}$ erg · cm^{-2} · s · (K)^4. Then we obtain

$$\nu_{cse} = \frac{16 \cdot 3.14 \cdot 5.76 \cdot 10^{-5} \left(2.73 \cdot 10^{14}\right)^4}{6.67 \cdot 10^{-8} \left(2 \cdot 10^{33}\right)^2 \cdot 0.46} \cdot \sqrt{\frac{2 \cdot 10^{33}}{1.18 \cdot 10^{28}}} \cdot \frac{7 \cdot 10^{10}}{4.2 \cdot 10^{13}}$$
$$= 8.96 \cdot 10^{-5} \text{cm} \cdot \text{s}^{-1},$$

$$t_{se} = \frac{4.2 \cdot 10^{13}}{8.96 \cdot 10^{-5}} = 4.7 \cdot 10^{17} \text{s} = 14.9 \cdot 10^9 \text{years.}$$

The found values show that the contemporary Solar System has formed during the proton phase (the Jupiter group of planets) within one million years and during the electron phase (the Earth group of planets) within next 15 billion years. Here we have not taken into account the effects of chemistry of the gaseous sphere on the equilibrium boundary conditions of the evolutionary process. But the obtained figures of evolution time show that our calculations give good approximation to the reality.

8.2.3 The Luminosity–Mass Relationship

To obtain the luminosity-mass we again consider the gaseous sphere evolution plot given in Fig. 8.1. It follows from (8.15) that in proton (AB) and electron (CD) evolutionary phases, the gaseous sphere luminosity is proportional to $1/R^2$. The boundary surface temperature T_o remains practically constant during the transition period (BC) when the equilibrium transformation from the proton to the electron phase takes place. But the gaseous sphere luminosity will decrease sharply. One can see that the luminosity decrease here is proportional to

$$L \propto \frac{\mu_p^2}{\mu_e^2}, \tag{8.26}$$

i.e. it is proportional to the ratio of the proton and electron mass squared as the gaseous sphere surface decreases proportionally to R^2. Thus, while going from point B to point C of the plot, the luminosity of the contracting body decreases by six orders of magnitude. We can suppose that the observed variations of variable star brightness are related to their virial energy pulsations, when stars are at the stage of evolution being considered.

As shown in the previous section, the most continuous period of proton or electron phase evolution is on the right end of the plot intercept (AB) and (CD). For these principal evolution time intervals we can write

$$L = 4\pi\sigma R^2 T_o^4 = \frac{(RT_o)^4}{R^2} \propto m^4. \tag{8.27}$$

This expression, derived from our theoretical considerations, is in good agreement with the well-known luminosity-mass relation which follows from observations. That is why Eq. 8.22 can be considered as an additional relation between the luminosity, the radius and the boundary surface temperature.

Let us take one more example. In Campbell's work (1962), 13 elliptical galaxies from the Virgo Cluster are considered and an analysis of the radius-mass relation for the observed data is given. To interpret these data, Jeans' relation (Jeans 1919) is used:

$$Gm\mu = \frac{3}{2}kT_oR \quad \text{or} \quad \frac{m}{R} = \frac{3}{2}\frac{3kT_o}{2G\mu}, \tag{8.28}$$

where μ is the proton mass.

On the plot presented in this work reflecting the mass-radius dependence, all the points are found to lie on a straight line with slope corresponding to $T_o \approx 1.5 \cdot 10^7$ K. Campbell concludes from this that the Jeans condition of self-gravitational instability is valid.

We note that Jeans' formula was derived on the assumption of low gas temperature and that all the kinetic energy of the gas is used for particle heat movement. The radiation energy was not taken into account.

Because of absence of direct temperature measurements, the theoretically found high temperature values at very steep line slopes need other explanations. We mast stress that in the observational data presented, the distance to the objects (in relative units) have been found with high degree of precision so that the experimentally derived constancy of the line slope should be trusted.

We interpret Campbell's data on the basis of our expression (8.22) where we consider the mass-radius relation to be dependent on electron temperature. That is why, contrary to Jeans, we write

$$\frac{m}{R_e} = \frac{3kT_e}{G\mu_e}.$$

Now the value of the boundary surface temperature of Campbell's galaxies is $T_o \approx 4{,}000$ K. This value corresponds to the usual boundary temperatures of celestial bodies whose evolution goes according to the electron phase of the equilibrium.

Hence, the experimental data presented by Campbell in his paper confirms once more the validity of Eq. 8.22 and the assumption of the existence of two evolutionary phases for celestial bodies.

In connection with the interpretation of Campbel's data, it is possible to use Eq. 8.22 to obtain the limiting temperature which should be reached by a gaseous sphere in its evolution. We write (8.22) as

$$\frac{Gm}{c^2}\frac{1}{R} = \frac{3kT_e}{\mu c^2} \quad \text{or} \quad \frac{R_g}{R} = \frac{3kT_o}{\mu c^2}. \tag{8.29}$$

Hence, during the evolution of a gaseous sphere through the electron phase of equilibrium, when $R \to R_g < T_o \to \mu_e c^2/3\,k$ or, equally,

$$3kT_0 \to \mu_e c^2 \approx 0.5\ \text{MeV},$$

$$T \approx 5 \cdot 10^9\ \text{K}.$$

This means that the temperature of the bodies approaches the electron temperature.

8.2.4 *Bifurcation of a Dissipative System*

In Chap. 5 we considered the dynamics of a dissipative system assuming that its evolution is a consequence of the loss of energy due to its radiation. Let us consider the problem in some details.

Jacobi's virial equation for a system was written as

$$\ddot{\Phi} = -A_0[1 + q(t)] + \frac{B}{\sqrt{\Phi}}, \tag{8.30}$$

where the function $A_o[1 + q(t)] = E - E_\gamma$ increases monotonically, reflecting the change of the total energy of a system as a function of time, and E_γ is the energy radiated up to time $t[E_\gamma > 0]$.

The solution of Eq. 8.30 was found to be

$$\begin{aligned} &-\arccos W + \arccos W_0 - \sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}} = \sqrt{1 - W^2} \\ &+\sqrt{1 - \frac{A_0 C}{2B^2}}\sqrt{1 - W_0^2} = \pm\frac{[2A_9(1 + q(t))]^{3/2}}{4B}(t - t_0). \end{aligned} \tag{8.31}$$

Equations of the discriminant curves which bound oscillations of the moment of inertia (Jacobi function) (see Fig. 5.1) are

$$\sqrt{I_{1,2}} = \frac{2B}{A_0[1 + q(t)]}\left\{1 \pm \sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}}\right\}. \tag{8.32}$$

From analysis of the solution of Eq. 8.30 it follows that the dissipative system during its evolution must inevitable reach the state when its stability breaks; that moment (see Fig. 5.1) can be defined by the point O_b which is the physical bifurcation point. The position of the point can be defined by Eq. 8.32 as

$$\frac{2B^2}{A_0[1 + q(t_b)]} = C, \tag{8.33}$$

Where $q(t_b)$ is parameter of the bifurcation point which can be found from condition (8.33)

$$q(t_b) = \frac{2B^2}{A_0 C} - 1. \tag{8.34}$$

The moment of inertia (Jacobi function) of the system corresponding to the bifurcation point, where the discriminant lines coincide, is

$$I_b = \frac{B}{A_0\left(1 + \frac{2B^2}{A_0 C} - 1\right)} = \frac{C^2}{4B^2}. \tag{8.35}$$

To find the moment of time of t_b where the system reaches its bifurcation point, one must know the law of energy radiation of the body q(t) or $E_\gamma(t)$, entering Eq. 8.30.

We give below our model solution for $E_\gamma(t)$.

The solution for the energy $E_\gamma(t)$ radiation up to t is based on the assumed existence of the proton and the electron phases of evolution for celestial bodies proposed in this chapter. On this basis, we have found a relationship between the body luminosity L and its radius R. During ‘smooth’ intervals of the body evolution, when $E_\gamma(t)$ is a continuous and monotonic function of time, the following relation holds:

$$\frac{Gm\mu_p}{3k} = RT_0, \tag{8.36}$$

where μ_p is the mass of the particle (proton or electron) which provides the boundary heat equilibrium of the body; k is the Boltzmann constant; and T_o is the gaseous sphere boundary temperature.

Let us write down the expression for the body luminosity L in relation to the time derivative of E_γ:

$$\frac{dE_\gamma}{dt} = L = 4\pi\sigma R^2 T_0^4, \tag{8.37}$$

where σ is the Stefan-Boltzmann constant.

Now we shell find an explicit expression for $E_\gamma(t)$ with the initial condition

$$E_\gamma(t_0)|_{t_0=0} = 0.$$

Equation (8.37) between the limits 0 and t can be integrated with the help of (8.36):

$$E_\gamma(t) = \int_0^t 4\pi\sigma R^2 T_0^4 dt = \int_0^t \frac{4\pi\sigma R^4 T_0^4}{R^2} dt = \int_0^t \frac{4\pi\sigma \left(Gm\mu_p\right)^4}{(3k)^4} \frac{1}{R^2} dt = \int_0^t \frac{K}{R^2} dt, \tag{8.38}$$

Where $K = 4\pi\sigma\left(Gm\mu_p\right)^4(3k)^4$.

Now let us make use of the expression (8.25) for the velocity of the gravitational contraction of the gaseous sphere v_c , which we had found earlier in this chapter:

$$v_c = \frac{dR}{dt} = \frac{32}{3} \frac{\pi\sigma}{Gm^2} \left(\frac{Gm\mu_p}{3k}\right)^4 \frac{\sqrt{\kappa}}{a} \sqrt[4]{\frac{R_1}{R}}, \tag{8.39}$$

Integrating this equation,

$$\int_0^R R^{1/4} dR = -\frac{32}{3} \frac{\pi\sigma}{Gm^2} \left(\frac{Gm\mu_p}{3k}\right)^4 \frac{1}{a} \sqrt[4]{\kappa^2 R_1} \int_0^t dt,$$

we obtain

$$\frac{4}{5} R^{5/4} - \frac{4}{5} R_0^{5/4} = -\frac{32}{3} \frac{\pi\sigma}{Gm^2} \left(\frac{Gm\mu_p}{3k}\right)^4 \frac{1}{a} \left(\sqrt[4]{\kappa^2 R_1}\right) t. \tag{8.40}$$

Then

$$R = \left(-Dt + R_0^{5/4}\right)^{4/5},$$

where

$$D = \frac{40\pi\sigma}{3Gm^2}\left(\frac{Gm\mu_p}{3k}\right)^4 \frac{1}{a}\sqrt[4]{\kappa^2 R_1}.$$

Finally, substituting the found expression for (8.40) into (8.38), we have

$$\begin{aligned} E_\gamma(t) &= \int_0^t \frac{Kdt}{\left(-Dt + R_0^{5/4}\right)^{8/5}} = \frac{5K}{3D}\left[\left(R_0^{5/4} - Dt\right)^{-3/5} - R_0^{3/4}\right] \\ &= \frac{5}{3}\frac{K}{D}\left[\frac{1}{\left(R_0^{5/4} - Dt\right)} - \frac{1}{R_0^{3/4}}\right] \end{aligned} \tag{8.41}$$

Thus we have obtained an expression in explicit form which we can use to calculate the energy loss by radiation during the time intervals of 'smooth' evolution of celestial bodies and hence find the parameters of the bifurcation point of a dissipative system.

8.3 Cosmo–Chemical Effects

From the analysis of the solution of Eq. 8.30 for a dissipative system, we found that, during the period of energy dissipation, the primary celestial body reaches a bifurcation point, characterized by separation of its outer shell which angular frequency coincides with frequency of virial oscillations. According to our theory of bifurcational creation of secondary bodies (in Alfvén's definition), some portion of the mass of the rotating primordial cloud reaches equilibrium relative to the tidal forces of the whole cloud at the bifurcation point, and moves further in a Kepler's orbit. As a result, during the subsequent dissipation of energy, the primary body continues its contraction by means of redistribution of the mass density without a separated secondary body. This secondary body conserves the corresponding angular moment $M_1 = m\nu_1 R_1 = m\nu_1^2/\omega$ which in fact is the kinetic energy divided by frequency of the interacted mass particles. In accordance with (8.3), the value of this tangential component of the kinetic energy equal to doubled potential energy ($2\beta_t = \alpha_t$) at the moment of a secondary body separation.

It is commonly known that when both the gravitational and electromagnetic interactions are taken into account, the condition to attain an equilibrium state by some portion of the mass (secondary body) can be written in the form suggested by Chandrasekhar and Fermi (1953):

$$\int\limits_{(V)} \left[\rho\bar{v}^2 + 3p + \frac{H^2 + E^2}{8\pi} - \frac{(\nabla U)^2}{8\pi G}\right] dV = 0, \tag{8.42}$$

where ρ is the density of the substance of the secondary body; *v* the mean velocity; p the internal pressure; H and E are the components of the electromagnetic field; G the gravitational constant; V the volume of the system; and ∇U the gradient of the gravitational field.

Since the bifurcational point of a system is characterized by the zero amplitude of the virial oscillations, the kinetic terms in Eq. 8.42 are small compared to the mass terms. In this case, Eq. 8.42 can be rewritten as

$$\int\limits_{(V)} \left[3p - \frac{(\nabla U)^2}{8\pi G}\right] dV \approx 0,$$

or

$$\int\limits_{(V)} 3p dV \approx 0.1 \frac{Gm}{R}, \tag{8.43}$$

where the coefficient 0.1 represents the electromagnetic component in expansion of the potential energy (8.43) found by astronomical observation of the equilibrium nebulae (Ferronsky et al. 1996).

The left-hand side of (8.43) is proportional to the energy of the Coulomb interactions of the charged particles (electrons, protons, ionized atoms and molecules). The right-hand side of this expression is proportional to the energy of the gravitational interaction of the particles.

Thus, assuming the separated secondary body to have mass m, radius R and the average mass of its constituent particles to be μ, expression (8.43) can be rewritten in the form of an equality of the energies of the gravitational and Coulomb interactions

$$\frac{m}{\mu} \frac{e^2}{R\sqrt[3]{\mu/m}} = 0.1 \frac{Gm^2}{R}, \tag{8.44}$$

where $e = 4.8 \cdot 10^{-10}$ e.s.u. is the electron charge.

Expression (8.44) is the equivalent of

$$m\mu \propto \frac{e^3}{G^{3/2}}. \tag{8.45}$$

Table 8.3 Critical and averaged masses of the constituent particle for the Planets

Planets	m_c (g)	μ_a (g)	μ_a (aum)
Mercury	$0.33 \cdot 10^{27}$	$0.78 \cdot 10^{-21}$	469
Earth	$5.97 \cdot 10^{27}$	$0.18 \cdot 10^{-21}$	114
Jupiter	$2 \cdot 10^{30}$	$0.00 \cdot 10^{-23}$	6.02
Saturn	$0.57 \cdot 10^{30}$	$1.87 \cdot 10^{-23}$	11.3
Uranus	$0.087 \cdot 10^{30}$	$4.79 \cdot 10^{-23}$	28.8

The last expression relates the critical mass m_c of the separated secondary body to the averaged mass μ_a of its constituent particles (electron, proton, molecules), responsible for the hydrodynamic equilibrium of the body, as

$$m_c\mu_a^2 \propto \left(\frac{e^2}{G^{3/2}}\right) = \text{const} = 2 \cdot 10^{-16} g^3. \tag{8.46}$$

To illustrate this relationship, we determined the average values for the masses of the individual particles constituting the planets, stars and galaxes.

Planets: Table 8.3 shows critical masses of the constituent particles for the planets of the Solar System.

Thus, assuming that the bifurcation theory describes the formation of the Solar System correctly, the particles determining the hydrodynamic gas pressure in the case of the considered planet at the moment of their separation from the proto-solar cloud could have been composed of such elements as H, He, O, Si, Mn, Fe in atomic or molecular form. The average masses of the particles obtained can be used as a criterion in the development of cosmo-chemical models of planets with a complicated chemical composition at the moment of their separation from the proto-solar cloud and also for the construction of their chemical evolution models.

Stars: From (8.46) can be found the boundary values for all stellar critical masses, corresponding to the masses of the proton and the electron – particles which can be responsible for the hydrodynamic pressure inside the stellar cloud at the moment of separation at the bifurcation point of the proto-galactic cloud.

For the mass of the proton $\mu_p = 1.6 \cdot 10^{-24} g, \quad m_c = 10^{32} g.$
For the mass of the electron $\mu_e = 0.9 \cdot 10^{-27} g, \quad m_c = 2 \cdot 10^{38} g.$
In the case of $\mu_a = \sqrt{\mu_p \mu_a} = 0,4 \cdot 10^{-25} g, \quad m_c = 10^{35} g.$

Therefore, considering a typical stellar mass to be ~ 10^{33} g, we obtain in the framework of the bifurcation theory of formation of celestial bodies that the hydrodynamic equilibrium of the gas at the moment of separation of the protostellar cloud is supported both by electron and proton.

Galaxes: The presence of the factor $(e^2/G)^{3/2}$ in the right-hand side of (8.46) allows us to carry out the following transformations:

$$m_c\mu_a^2 = \left(\frac{e^2}{\hbar c}\right)^{3/2} \left(\frac{\hbar c}{G}\right) = \left(\frac{1}{137}\right)^{3/2} m_p^3, \tag{8.47}$$

where ħ is Planck's constant; c the velocity of light; and m_p the Planck's mass.

Thus, in the right-hand side of (8.47) there are two fundamental constants: the Planck mass m_p ($2.2 \cdot 10^{-5}$ g) and the fine-structure constant $\alpha = 1/137$. The presence of the constant α in the right-hand side of (8.47), being the universal constant of the weak and electromagnetic interactions, shows that this relation is applicable not only to electromagnetic but also to weak interactions. Then, putting the experimentally found values for the neutrino mass $\mu_\nu = 10^{-30}$ g (Shirkov 1980) into (8.44), we obtain

$$m_c = \frac{2 \cdot 10^{-16}}{\left(10^{-30}\right)^2} = 2 \cdot 10^{44} g. \tag{8.48}$$

This mass, following from (8.48), is a typical mass of galaxies. Therefore, in the framework of the bifurcation theory of formation of celestial bodies, the hydrodynamic equilibrium (8.41) of the substances of galaxies at the moment of their formation can be provided by the pressure of neutrinos.

Universe: In the framework of the virial oscillation theory, the evolution of the Universe can be described by a pulsating model (for $c =$ constant) of the system of material elementary particles. Such a system indefinitely long time exists. The mass of the particle responsible for hydrodynamic equilibrium of the Universe at the moment of its maximal compression (singularity stage) can be obtained from the same expression (8.46). Assuming $m_c \approx 10^{56}$ g we obtain

$$\mu_a \approx 10^{36} g. \tag{8.49}$$

In the bifurcation theory the maximal average mass of particles in cosmic space can be determined from the condition $\mu_a = m_c$. Then,

$$\mu_{max} = 6 \cdot 10^{-6} g.$$

This value is close to the Planck mass.

8.4 Direct Derivation of the Equation of Virial Oscillation from Einstein's Equation

Weinberg (1972) reduced Einstein's equation for homogeneous isotropic space, with the help of the Robertson-Walker metric, to the following scalar form:

$$3\ddot{R} = -4G(\rho + 3p)R, \tag{8.50}$$

$$\ddot{R}R + 2\left(\dot{R}\right)^2 + 2k = 4\pi G(\rho - p)R^2, \tag{8.51}$$

where R is the radius of the Universe; p radiation pressure (mass defect); and ρ the density of matter without mass defect.

Multiplying Eq. 8.50 by R/3 and summing it with (8.51), we obtain:

$$\left(\ddot{R}^2\right) + 2k = 8\pi G R^2 \left(\frac{1}{3}\rho - p\right). \tag{8.52}$$

When $\rho < <p$ and $\rho R^3 = \text{const}$ (dust cloud), and taking into account that for curved space (Landau and Lifshitz 1973b)

$$\rho R^3 = \frac{m}{2\pi^2}, \tag{8.53}$$

where m is the total mass of the particles constituting the cloud, expression (8.53) is transformed into

$$\left(\ddot{R}^2\right) + 2k = \frac{8\pi}{3} G \frac{m}{2\pi^2} \frac{1}{R}. \tag{8.54}$$

Since from the Jacobi function we have $\Phi = mR^2/2$, Eq. 8.54 can be rewritten as

$$\ddot{\Phi} + km = \frac{2}{3\pi} Gm^2 \sqrt{\frac{m}{2}} \frac{1}{\sqrt{\Phi}} \tag{8.55}$$

or

$$\ddot{\Phi} + km = \frac{\sqrt{2}}{3\pi} \sqrt{G^2 m^5} \frac{1}{\sqrt{\Phi}}. \tag{8.56}$$

Finally, the equation of virial oscillations can be easily obtained in the known form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}, \tag{8.57}$$

where $A = km = E$ is the total energy, and B is a constant equal to $Gm^{5/2}$ multiplied by a factor which depends on the realization of the mass defect and on the period of the $\alpha^2\beta$ form factors (equal to $1/\sqrt{2}$).

When $p = \rho/3$, the equation of virial oscillations for radiation can be obtained from Eq. 8.52:

$$\ddot{\Phi} = -A.$$

Equations 8.50 and 8.51 are valid for all natural systems which exhibit a central symmetry of mass distribution. For celestial bodies Eq. 8.52 is written as

$$\rho R^3 = \frac{3m}{4\pi}.$$

Then, from (8.52) it follows that

$$(\ddot{R})^2 + 2k = \frac{8\pi}{3} GR^2 \left(1 - \frac{3p}{\rho}\right) = \frac{2Gm}{R}\left(1 - \frac{3p}{\rho}\right).$$

Now, Eq. 8.57 becomes

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}\left(1 - \frac{3p}{\rho}\right). \quad (8.58)$$

As Weinberg (1972) pointed out, the inequality $0 < 3p \leq \rho$ holds for celestial bodies, and in the most general case we can write

$$з = (\gamma - 1)(\rho - n\mu)$$

where n is the density of particles and μ is the mass of a particle.

Therefore, $(\rho - n\mu)$ is the mass defect and γ is the politropic index, which for stable system ranges from 0 to 5/3 for non-relativistic objects, and $\gamma \geq 4/3$ for ultra-relaticvistic objects. For $\gamma > 5/3$, the body expands indefinitely, and at $\gamma \leq 4/3$ collapse of the body occurs.

For actually existing celestial bodies, where the absence of heat equilibrium is taken into account (in the case of a discrete system), pressure is defined as (Weinberg 1972).

$$p = \frac{1}{3}[\rho + f(\rho, n)],$$

where $f(\rho, n) = T^o_\alpha$ is a function of the energy density ρ and the density n (number of particles per unit volume). This function is equal to zero in the ultra-relativistic limit and in the non-relativistic limit it is equal to $[-n\mu + (\rho - n\mu)] = -2n\mu + \rho$.

In both limiting cases, pressure p is

$$p = \frac{1}{3}\rho \quad \text{and} \quad p = \frac{2}{3}(\rho - n\mu).$$

Hence, in Eq. 8.58 the undetermined factor in B is equal to zero and $[(2\mu n/\rho) - 1)$ or $(1 - (2\Delta/\rho)]$, where $\Delta = \mu n - \rho$ is the mass defect.

Finally, taking into account the mass defect in Eq. 8.58 shows that the constant $B = B_o D$, where B_o is of Newtonean nature $(aGm^{5/2})$ and D, a relativistic correction, is smaller than 1.

Now let us estimate this correction D in the case of the white dwarf and the neutron star models according to Weinberg.

The equation determining the density of particles of particles when Fermi-Dirac statistics hold can be written as

$$n = \frac{k_F^3}{3\pi\hbar^2},$$

where n is the number of particles in the volume; k_F the radius of the Fermi sphere; and ħ is Planck's constant.

The density of matter of a star is written as

$$\rho = n\mu_p n_p,$$

where μ_p is the mass of a proton and n_p the average number of protons in a nuclei.

The critical density of matter in a star is

$$\rho_{cr} = \frac{\mu_p n_p {\mu_e}^3}{3\pi\hbar^3},$$

where μ_e is the electron mass.

Introducing the new variables $Z_1 = \rho/\rho_{cr}$ and $Z_2 = \rho/\rho_{cr}$, the equation of state for white dwarfs can be rewritten as follows:

$$Z_1 = \frac{3\mu_e}{\mu_p} F_1(Z_1),$$

$$Z_2 = \frac{3\mu_e}{\mu_p} F_2(Z_2),$$

where F_1 and F_2 are some transcendental functions.

For neutron stars the critical density is

$$\rho_{cr} = \frac{\mu_p^4}{3\pi\hbar^3}$$

and the equations of state are written

$$Z_1 = 3F_1(Z_1),$$

$$Z_2 = 3F_2(Z_2).$$

Solving the equations of state for the two limiting cases when $\rho << \rho_{cr}$ (i.e. when the polytropic indexes are 5/3 and 4/3 respectively), we obtain for white dwarfs, respectively,

$$\rho_e = \frac{3}{2}p \quad \text{and} \quad \rho_e = 3p.$$

For neutron stars, in the limiting cases ($\rho << \rho_{cr}$) and ($\rho >> \rho_{cr}$), we have the same form of relations:

$$\rho = \frac{3}{2}p \quad \text{and} \quad \rho = 3p,$$

where ρ is the total density of matter.

In the ultra-relativistic limit, the relativistic correction will have very large values ($D = 0$), which means that the total collapse of the star (Oppenheimer-Volkoff limit) is leading to the formation of a black hole.

Note that besides the gravitational interaction, there are only two types of known interactions: the unified electroweak and the strong. Thus it follows that there cannot be any other types of collapse, since the collapse of white dwarfs corresponds to the first type of interaction and the formation of neutron stars to the second.

Thus, we have obtained the equation of virial oscillations (8.58) directly in the most general case and without having to assume the constancy of the form factor product $\alpha^2\beta$. Since the same equation follows from Jacobi's equation with the use of the hypothesis, we conclude that the relation $\alpha^2\beta = \text{const}$ was proven.

We should also note that modern astrophysical studies of the oscillation of celestial bodies in the non-relativistic approximation are based on the supposition that these movements have a homologous structure (Misner et al. 1973; Weinberg 1972; Frank-Kamenetsky 1959; Zeldovich and Novikov 1967). It can easily be verified that the supposition of homology is a sufficient condition to prove the constancy of the form factor product $\alpha^2\beta$ which is the main point in the derivation of the equation of virial oscillations from Jacobi's equation.

The mathematical formulation of the homologous motion of matter in the course of oscillation of a celestial body is written as follows:

$$r(t) = t(0) \cdot f(t),$$

where r(t) is the radius of a given layer-shell of the body and f(t) is an arbitrary function of time.

Let us introduce the Lagrange co-ordinates, where m is the mass inside the sphere of radius r, and dm is the mass of shell of radius r and thickness dr. According to the property of Lagrange co-ordinates, they are independent of time.

Then, the Jacobi function and the potential energy are written as:

$$\Phi = \frac{1}{2}\int_0^m r^2 dm,$$

$$|U| = G\int_0^m \frac{m dm}{r}.$$

Using the assumption that the motion is homologous, these expressions can be rewritten:

$$\Phi = \frac{1}{2} f^2(t) \int_0^m r^2(0) dm,$$

$$U = \frac{G}{f(t)} G \int_0^m \frac{m dm}{r(0)}.$$

Integrals on the right-hand side of these expressions do not depend on time and are therefore constants. Thus, the product $U^2\Phi$ does not depend on time and is also a constant.

Note that in the works of the authors mentioned above, the formula for the pulsation frequency of celestial bodies has been obtained assuming small amplitudes and the validity of the harmonic law of pulsations. Our approach allows the same frequency of pulsations to be obtained without the above restricting assumptions. Moreover, by comparing the two expressions which give equivalent results, it is possible to obtain the politropic index which enters into the astrophysical formula for the frequency of pulsations.

Chapter 9
The Nature of Electromagnetic Field of a Celestial Body and Mechanism of Its Energy Generation

The hydro-magnetic dynamo, action of which is provided by the planet's liquid metal core or the solar gas plasma, is the most popular idea for explanation of a body electromagnetic field generation. Its essence is in the motion of the conducting liquid core where self-excitation of the electric and magnetic poloidal (meridional) and toroidal (parallel) fields are happened. During rotation of the inner planet's shells with different angular velocities, in the case of asymmetric thermal convection of the shell mass, the intensity of the fields is increased. This condition, for example, for the Earth is achieved because the rotation and magnetic axes are not coincided and the thermal convection supposedly takes place. But physically justified theory of the observed planet's and solar phenomenon of electromagnetic field is absent. There is no explanation of mechanism of generation of the energy of this field except of general physical principle of the mass and charge interaction. Also the ideas or hypotheses about source of refilling of the planets energy which is spent for the gravitational and thermal irradiation are absent. The only source of the solar and star irradiated energy is accepted to be the interior nuclear fusion.

In order to find solution of the problem, in this Chapter we discuss a novel idea based on the innate capacity of body's energy for performing motion. As it was shown in Chaps. 2 and 3, the energy is the measure of the motion and interaction of particles of any kind of body's matter. The various forms of energy are interconvertible and its sum for a system remains constant. The above unique properties of the energy, with its oscillating mode of the motion in our dynamics, make it possible to consider the nature of the electromagnetic and gravitational effects of celestial bodies as interconnected events.

It was shown in Chap. 7 that the body's gravitational (potential) energy results in the body's matter volumetric pulsations, having oscillating regime, frequencies of which depend on the mass density. In our consideration the planets and stars are accepted as self-gravitating bodies. Their dynamics is based on the own internal force field and the potential and kinetic energies are controlled by the energy of oscillation of the polar moment of inertia, i.e. by interaction of the body's elementary particles.

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Applying the dynamical approach and the results obtained, we show below that the nature of creation of the electromagnetic field and mechanism of its energy generation appears to be the effect of the volumetric gravitational oscillation of the body's masses. This effect is also characteristic for any celestial body.

9.1 Electromagnetic Component of the Interacted Masses

It was shown in Sect. 7.2 that the electromagnetic energy is a component of the expanded analytical expression of the potential energy. The expansion was done by means of the auxiliary function of the density variation relative to its mean value. The expression of the body's potential gravitational energy in the expanded form (7.48) was found as

$$U = \alpha^2 \frac{GM^2}{R} = \left[\frac{3}{5} + 3\int_0^1 \psi x dx + \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 dx\right]\frac{GM^2}{R}, \tag{9.1}$$

where U is the potential energy of the gravitational interaction; α^2 is the form-factor of the force function; G is gravity constant; M is the body mass; R is its radius; $\Psi(s)$ is the auxiliary function of radial density distribution relative to its mean value.

We have considered and applied the two first right-hand side terms of Eq. 9.1. The third term in dimensionless form represents an additive part of the potential energy of the interaction of the non-uniformities between themselves, which was written as

$$\frac{9}{2}\lambda = \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 dx \equiv \frac{9}{2}\int_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2 dx. \tag{9.2}$$

where $\lambda = \int_0^1 \left(\frac{\psi}{x}\right)^2 dx \geq 0$.

The non-uniformities are determined as the difference between the given density of a spherical layer and the mean density of the body within the radius of the considered layer. For interpretation of the third term we apply the analogy of electrodynamics (Ferronsky et al. 1996). Each particle there generates an external field, which determines its energy. The energies of some other interacted particles and their own charges are determined by this field. As far as the potential of the field is expressed by means of the Poisson equation through the density of charge in the same point, then the total energy can be presented in additive form through the application of the squared field potential. If the body mass is considered as a moving system, then the Maxwell radiation field applies.

In our solution the dimensionless third term of the field energy is written as

$$\frac{9}{2}\lambda = \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 dx \equiv \frac{9}{2}\int_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2 dx \equiv \frac{9}{2}\int_0^1 E^2 dV, \tag{9.3}$$

where $E = \Psi/x^2$ is dimensionless form of the electromagnetic field potential which is a part of the gravitational potential; Ψ plays role of the charge; $dV = x^2dx$ is the volume element in dimensionless form.

In order to determine the numerical value of λ, the calculations for a sphere with different laws of radial density distribution including the politropic model were done (Ferronsky et al. 1996). These models were used in our earlier numerical calculations of the form factors α^2 and β^2. The results, presented in Tables 9.1 and 9.2, show that for the density distributions which have physical meaning (Dirac's envelopes, Gaussian and exponential distributions) and also for the politrops with index 1.5, the parameter λ has the same constant value. We interpret this fact for a steady-state dynamical system as evidence of the existence of equilibrium radiation between a celestial body and the external flow. The numerical value of the parameter λ is equal to 0.022. There is also an observational confirmation of this conclusion. Spitzer (1968) demonstrates observational results of nebulae of different mass and size in Table 3.2 of his book, which we reproduce here in Table 9.3.

One can see that for masses of solar order and up to huge size the value of m/R^2 remains constant. This fact proves the statement of the physical meaning of the expression (9.2) which is the equilibrium radiation of a celestial body.

Table 9.1 Numerical values of the parameter λ for a sphere with different laws of density distribution

Law of radial density distribution	α^2	β	$Q = \alpha^2\beta$	9λ
$\rho = \rho_o$	0.6	0.77	0.46	0
$\rho = \rho_o\,(1 - r/R)$	0.74	0.63	0.47	0.086
$\rho = \rho_o\,[1-(r/R)^2]$	0.71	0.65	0.47	0.060
$\rho = \rho_o \exp(-k\, r/R)$	0.16 k	3.45/k	0.53	0.19
$\rho = \rho_o \exp[-(k\, r/R)^2$	$\sqrt{k/2\,\pi}$	$1.8\sqrt{1/k}$	0.49	0.19
$\rho = \rho_o\, \delta(1 - r/R)$	0.5	1.0	0.5	0.20

Table 9.2 Numerical values of the parameter λ for politropic models of a sphere

Index of politrope	α^2	β	$Q = \alpha^2\beta$	9λ
0	0.6	0.77	0.46	0
1	0.75	0.62	0.465	0.08
1.5	0.87	0.55	0.475	0.24
2	1.0	0.48	0.482	0.43
3	1.5	0.34	0.5	1.31
3.5	2.0	0.26	0.52	2.26

Table 9.3 Observational parameters of equilibrium nebulae

Parameters	Visible dark nebulae Small globula	Large globula	Intermediate cloud	Large cloud
m/m_{Sun}	>0.1	3	$8 \cdot 10^2$	$1.8 \cdot 10^4$
R (pc)	0.03	0.25	100	20
п (п/cm^3)	$>4 \cdot 10^4$	$1.6 \cdot 10^3$	100	20
$m/\pi R^2$ (g/cm^2)	$>10^{-2}$	$3 \cdot 10^{-3}$	$3 \cdot 10^{-3}$	$3 \cdot 10^{-3}$

Thus, the virial approach to the problem solution of the Earth global dynamics gives a novel idea about the nature of the planet's electromagnetic field. The energy of this field appears to be the component of the potential energy of the interacted masses. The question arises about the mechanism of the body's energy generation, which provides radiation in wide range of the wave spectrum from radio through thermal and optical to x and γ rays.

9.2 Potential Energy of the Coulomb Interaction of Mass Particles

In Sect. 6.4.4 of Chap. 6 with the help of model solution, we showed that for the Coulomb interactions of the charged particles, constituting a celestial body, the relationship between the potential energy of a self-gravitating system and its Jacobi function holds.

Considering a one-component, ionized, quasi-neutral and gravitating gaseous cloud with a spherical symmetrical mass distribution, we found that the form-factor entering the expression for the potential energy of the Coulomb interaction acquires the same physical meaning what it has in the expression for the potential energy of the gravitational interaction of the masses. It represents the effective shell to which the charges in the sphere are reduced.

The considered task about the potential energy of the Coulomb interactions of the charged particles proves the legitimacy of solution of the virial equation of dynamical equilibrium for study electromagnetic effects of a celestial body.

9.3 Emission of Electromagnetic Energy by a Celestial Body as an Electric Dipole

In Chap. 5 we considered the solution of the virial equation of dynamical equilibrium for dissipative systems written in the form

$$\ddot{\Phi} = -A_0[1 + q(t)] + \frac{B}{\sqrt{\Phi}}. \tag{9.4}$$

Here the function of the energy emission [1−q(t)] was accepted on the basis of the Stefan-Boltzmann law without explanation of the nature of the radiation process. Now, after the analysis of the relationship between the potential energy and the polar moment of inertia, considered in the previous section, and taking into account the observed relationship by artificial satellites, we try to obtain the same relation for the celestial body as an oscillating electric dipole (Ferronsky et al. 1987).

Equation 9.4 for a celestial body as a dissipative system can be rewritten as

$$\ddot{\Phi} = -A_0 + \frac{B}{\sqrt{\Phi}} + X(t - t_0),$$

where $X(t - t_o)$ is the perturbation function sought, expressing the electromagnetic energy radiation of the body as

$$X(t - t_0) = E_\gamma(t - t_0).$$

The electromagnetic field formed by the body is described by Maxwell's equations, which can be derived from Einstein's equations written for the energy-momentum tensor of electromagnetic energy. In this case only the general property of the curvature tensor in the form of Bianchi's contracted identity is used. We recall briefly this derivation sketch (Misner et al. 1975).

Let us write Einstein's equation in geometric form:

$$G = 8\pi T, \tag{9.5}$$

where G is an Einstein tensor and T is an energy-momentum tensor.

In the absence of mass, the energy-momentum tensor of the electromagnetic field can be written in arbitrary co-ordinates in the

$$4\pi T^{\mu\nu} = F^{\mu\alpha}F^{\nu\beta}g_{\alpha\beta} - \frac{1}{4}g^{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}, \tag{9.6}$$

where $g_{\alpha\beta}$ is the metric tensor in co-ordinates, and $F^{\mu\nu}$ the tensor of the electromagnetic field.

From Bianchi's identity

$$\nabla G \equiv 0, \tag{9.7}$$

where ∇ is a covariant 4–delta, follows the equation expressing the energy-momentum conservation law:

$$\nabla T \equiv 0. \tag{9.8}$$

In the component form, the equation is

$$F^{\mu\alpha}{}_{;\sigma}g_{\alpha\tau}F^{\sigma\tau} + F^{\mu\alpha}{}_{;\tau}g_{\alpha\sigma}F^{\tau\alpha} = g^{\mu\nu}(F_{\nu\tau;\sigma} + F_{\sigma\nu;\tau})F^{\sigma\tau}. \tag{9.9}$$

After a series of simple transformations, we finally have

$$F^{\beta v}{}_{;v} = 0. \tag{9.10}$$

here and above, the symbol ' ; ' defines covariant differentiation.

To obtain the total power of the electromagnetic energy emitted by the body, Maxwell's equations should be solved, taking into account the motion of the charges constituting the body. In the general case, the expressions for the scalar and vector potentials are

$$4\pi\phi = \int\limits_{(V)} \frac{[\rho]dV}{R}, \tag{9.11}$$

$$4\pi\overline{A} = \int\limits_{(V)} \frac{[j]dV}{R}, \tag{9.12}$$

where ρ and j are densities of charge and current; [j] denotes the retarding effect (i.e. the value of function j at the time moment t – R/c); R is the distance between the point of integration and that of observation, and c the velocity of light.

In this case, however, it seems more convenient to use the Hertz vector Z of the retarded dipole p(t – R/c) (Tamm 1976). The Hertz vector is defined as

$$4\pi Z = \frac{1}{R}\rho\left(t - \frac{R}{c}\right). \tag{9.13}$$

Electromagnetic field potentials of the Hertz dipole can be determined from the expressions

$$\phi = -\mathrm{div} Z, \tag{9.14}$$

$$\overline{A} = \frac{1}{c}\frac{dZ}{dt}. \tag{9.15}$$

Moreover, the Hertz vector satisfies the equation

$$\Box Z \equiv \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)Z = 0, \tag{9.16}$$

where □ is the d'Alembertian operator.

The intensities of the electric and magnetic fields E and H are expressed in terms $\overline{Z}$ by means of the equations

$$\overline{H} = \mathrm{rot}\dot{\overline{Z}}, \tag{9.17}$$

$$\overline{E} = \text{grad div}\overline{Z} - \frac{1}{c}\ddot{\overline{Z}}. \tag{9.18}$$

The radiation of the system can be described with the help of the Hertz vector of the dipole p = qr, where q is the charge and r the distance of the vector from the charge (+q) to (–q).

From the sense of the retardation of the dipole p(t – R/c) we can write the following relations:

$$\frac{d\bar{p}}{\partial R} = -\frac{1}{c}\dot{\bar{p}}; \quad \frac{d^2\bar{p}}{dR^2} = \frac{1}{c}\ddot{\bar{p}}.$$

Then the components of the fields E and H of the dipole are as follows:

$$H_\varphi = \frac{\sin\theta}{c^2R}\ddot{\bar{p}}\left(t - \frac{R}{c}\right), \tag{9.19}$$

$$E_\theta = \frac{\sin\theta}{c^2R}\ddot{\bar{p}}\left(t - \frac{R}{c}\right), \tag{9.20}$$

where θ is the angle between p and $\bar{R}$; $H_\varphi \perp E_\theta$ and $\perp R$; the other components of E and H in the wave zone are tending to zero quicker than 1/R in the limit $R \to \infty$.

The flax of energy (per unit area) is equal to

$$S = \frac{c}{4\pi}E_\theta H_\varphi = \frac{1}{4\pi c^2}\frac{\sin\theta}{R^2}\left(\ddot{\bar{p}}\right)^2. \tag{9.21}$$

The total energy radiated per unit time is given by

$$\oint\!\!\oint S d\sigma = \frac{2}{3c^3}\left(\ddot{\bar{p}}\right)^2 \tag{9.22}$$

Thus, transforming the dissipative system to an electric dipole by means of the Hertz vector, we have reduced the task of a celestial body model construction to the determination of the dipole charges +Q and –Q through the effective parameters of the body.

This problem can be solved by equating expression (9.22) for the total radiation of a celestial body as an oscillating electric dipole. In addition, the relation for the black body radiation expressing through effective parameters was presented below in Sect. 8.2.3.

The expression (9.22) for the total rate of the electromagnetic radiation *J* of the electric dipole can be written in the form (Landau and Lifshitz 1973b)

$$J = \frac{2}{3}\frac{Q^2}{c^3}\left(\ddot{\bar{r}}\right), \tag{9.23}$$

where Q is the absolute value of each of the dipole charges, and r is the vector distance between the polar charges of the dipole. Its length in our case is equal to the effective radius of the body.

In our elliptic motion model of the two equal masses the vector r satisfies the equation

$$\ddot{\bar{r}} = -Gm\frac{\bar{r}}{r^3}. \qquad (9.24)$$

Thus, the total rate of the electromagnetic radiation of the dipole is

$$J = \frac{2}{3}\frac{Q^2}{c^3}\frac{(Gm)^2}{r^4}. \qquad (9.25)$$

In order to obtain the average flux of electromagnetic energy radiation, the value of the factor $1/r^4$ should be calculated averaged during the time period of one oscillation. Using the angular momentum conservation law, we can replace the time-averaging by angular averaging, taking into consideration that

$$dt = \frac{mr^2}{2M}d\varphi, \qquad (9.26)$$

where M is angular momentum, and φ is the polar angle.

The equation of the elliptical motion is

$$\frac{1}{r} = \frac{1}{a(1-\varepsilon^2)}(1+\varepsilon\cos\varphi), \qquad (9.27)$$

where *a* is the semi-major axis, and ε is the eccentricity of the elliptical orbit.

The value of $1/r^4$ can be found by integration. In our case of small eccentricities, we neglect the value of ε^2 and write

$$\overline{\left(\frac{1}{r^4}\right)} = \frac{1}{a^4}. \qquad (9.28)$$

Finally we obtain

$$\bar{J} = \frac{2}{3}\frac{Q^2}{c^3}\frac{Gm^2}{a^4}. \qquad (9.29)$$

Earlier it was shown (Ferronsky et al. 1987) that

$$\bar{J} = 4\pi\sigma\frac{1}{a^2}A_c^4, \qquad (9.30)$$

where σ is the Stefan-Boltzmann constant; $A_e = Gm\mu_e/3k$ is the electron branch constant; μ_e is the electron mass; and k is the Boltzmann constant.

Equating relations (9.29) and (9.30), we find the expression for the effective charge Q as follows:

$$Q = \sqrt{6\pi\sigma}\frac{A_e^2}{cr_g}, \tag{9.31}$$

where $r_g = Gm/c^2$ is the gravitation radius of the body.

We have thus demonstrated that it is possible to construct a simple model of the radiation emitted by a celestial body, using only the effective radius and the charge of the body. Moreover, it was shown a practical method of determining the effective charge using the body temperature from observed data.

The logical question is raised what is mechanism of the energy generation of the bodies which they emitted in the wide range of oscillating frequencies spectrum. Let us consider this important question at least in first approximation.

9.4 Quantum Effects of Generated Electromagnetic Energy

The problem of the energy generation technology for human practical use has been solved far ago. In the beginning it was understood how to transfer the wind and fair energy into the energy of translational and rotary motion. Later on people have learned about production of the electric and atomic energy. Technology of the thermo-nuclear fusion energy generation is the next step. It is assumed that the Sun replenishes its emitted energy by the thermo-nuclear fusion of hydrogen, helium and carbon. The Earth thermal energy loss is considered to be filled up by convection of the masses and thermal conductivity. But the source of energy for convection of the masses is not known.

The obtained solution of the problem of volumetric pulsations for a self-gravitating body based on their dynamical equilibrium creates real physical basis to formulate and solve the problem. In fact, if a body performs gravitational pulsations (mechanical motions of masses) with strict parameters of contraction and expansion of any as much as desired small volume of the mass, then such a body, like a quantum generator, should generate electromagnetic energy by means of its transformation from mechanical form through the forced energy levels transitions and their inversion on both the atomic and nuclear levels. In short, the considered process represents transfer of mechanical energy of the mass pulsation to the energy of electromagnetic field (Fig. 9.1).

An interpretation of the process can be presented as follows. While pulsating and acting in regime of the quantum generator, the body should generate and emit coherent electromagnetic radiation. Its intensity and wave spectrum should depend on the body mass, its radial density distribution and chemical (atomic) content.

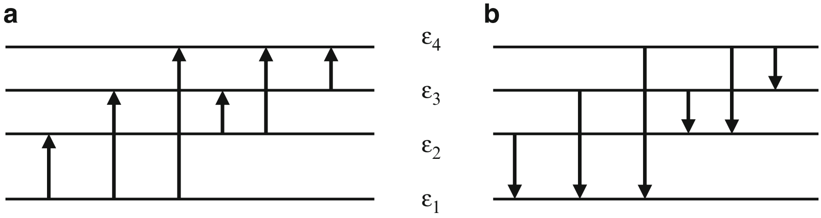

Fig. 9.1 Quantum transition of energy levels at contraction phase of the body mass (**a**) and inversion at the phase of its expansion (**b**); ε_1, ε_2 , ε_3 , ε_4 are levels of energy

As it was shown in Sect. 7.4, the body with uniform density and atomic content provides pulsations of uniform frequency within the entire volume. In this case, the energy generated during the contraction phase will be completely absorbed at the expansion phase. The radiation appeared at the body's boundary surface must be in equilibrium with the outer flux of radiation. The phenomenon like this seems to be characteristic for the equilibrated galaxy nebulae and for the Earth water vapor in anti-cyclonic atmosphere.

The pulsation frequencies of the shell-structured bodies are different but steady for each shell density. In the case of density increase to the body center, the radiation generating at the contraction phase will be partially absorbed by an overlying stratum at the expansion phase. The other part of radiation will be summed up and transferred to the body surface. That radiation forms an outer electromagnetic field and is equilibrated by interaction with the outer radiation flax. The rest of the non-equilibrated and more energetic in the spectrum of radiation moves to the space. The coherent radiation which reaches the boundary surface has a pertinent potential and wave spectrum depending on mass and atomic content of the interacted shells in accordance with Moseley law. The Earth emits infrared thermal radiation in an optical short wave range of spectrum. The Sun and other stars cover the spectrum of electromagnetic radiation from radio- through optical, x and gamma ray of wave ranges. The observed spectra of star radiation show that total mass of a body takes part in generation and formation of surface radiation. According to the accepted parabolic low of density distribution of the Earth it has maximum density value near the lower mantel boundary. The value of the outer core density has jump-like fall and the inner core density seems to be uniform up to the body center. The discussed mechanism of the energy generation is justified by the observed seismic data of density distribution. It is assumed, that the excess of generated electromagnetic energy from the outer core comes to the inner core and keeps there the pressure of dynamical equilibrium at the body pulsation during the entire time of the evolution. The parabolic distribution of density seems to be characteristic for most of the celestial bodies.

In connection with the discussed problem it is worth to consider the equilibrium conditions between radiation and matter on the body boundary surface.

9.5 The Nature of the Star Emitted Radiation Spectrum

We assume that the Novas and Supernovas after explosion and collapse pass into neutron stars, white dwarfs, quasars, black holes and other exotic creatures which emit electromagnetic radiation in different ranges of the wave spectrum. The discussed in the book effects based on dynamical equilibrium evolution of self-gravitating celestial bodies allow the exotic stars to be interpreted from a new position. We consider the observed explosions of stars as a natural logical step of evolution related to their mass differentiation with respect to the density. The process is completed by separation of the upper 'light' shell. At the same time the wave parameters of the generated energy of the star after shell separation are changed because of changes in density and atomic contents. As a result, the frequency intensity and spectrum of the coherent electromagnetic radiation on the boundary surface are changing. For example, instead of radiation in optical range the coherent emission in *x* or gamma ray range takes place. But the body's dynamical equilibrium should remain during all the time of evolution. The loss of the upper body shell leads to decrease of the angular velocity and increase of the oscillation frequency. The idea of the star gravitational collapse seems to be an effect of the hydrostatic equilibrium theory.

As to the high temperature on the body surface, the order of which from Rayleigh -Jeans' equation is 10^7 K and more, then in our interpretation as applying Eq. 8.33 for evolution of a star of solar mass at the electron phase (Fig. 8.1), the limiting temperature $T_0 \to \mu_e c^2/3k$ or (Ferronsky et al. 1996)

$$3kT_0 \to \mu_e c^2 \approx 0.5 \text{ MeV},$$

$$T \approx 5 \cdot 10^9 \text{ K}.$$

This means that on the body surface the gas approaches to the electron temperature because the velocity of its oscillating motion runs to *c*.

The energy is a quantitative measure of interaction and motion of all the forms of the matter. In accordance with the low of conservation the energy does not disappear and does not appear itself. It only passes from one form to another. For a self-gravitating body the energy of mechanical oscillations, induced by the gravitational interactions, passes to electromagnetic energy of the radiation emission and vise versa. The process results by the induced quantum transition of the energy levels and their inversion. Here transition of the gravitational energy into electromagnetic and vice versa results in the self-oscillating regime. In the outer space of the body's border the emitted radiation energy forms the equilibrium electromagnetic field. The non-equilibrium part of the energy in corresponding wave range of the spectrum is irradiated to the outer space. The irreversible loss of the emitted energy is compensated by means of the binding energy (mass defect) at the fission and fusion of molecules, atoms and nuclei. The body works in the regime of a quantum generator. Those are conclusions followed from the theory based on the body dynamical equilibrium.

9.6 Temperature of the Relict Radiation

The conclusion about the constancy of the equilibrium radiation parameter λ in (9.2) enables to obtain the numerical value of its temperature. For this we equate the expression of the black radiation energy and the volumetric density of that part of the gravitational energy which corresponds to the radiation energy in the Maxwell theory, that is

$$\sigma T^4 = \lambda \frac{Gm^2}{R} \cdot \frac{1}{4/3\pi R^3}, \tag{9.32}$$

where σ is the Stefan-Boltzmann's constant and T is absolute temperature.

From here

$$T = \sqrt[4]{\frac{3}{4\pi}\frac{\lambda G}{\sigma}} \cdot \sqrt{\frac{m}{R^2}}. \tag{9.33}$$

In accordance with Spitzer (1968), the value of $m/\pi R^2$ for globulae of solar to galactic mass is equal to $3 \cdot 10^{-3}$ g/cm^2 ($m/R^2 = 10^{-2}$) Our data shows that $\lambda = 0.022$. Then from Eq. 9.33 the value of $T \approx 4$ K. This is the observed relict radiation..

Let us now consider expression (9.32) from the point of view of the solar system. The expression can be written in the form

$$(RT)^4 = \frac{\lambda G m^2}{4/3\pi\sigma}.$$

From here with the solar mass

$$RT = \sqrt{m} \cdot \sqrt[4]{\frac{\lambda G}{4/3\pi\sigma}} = 5 \cdot 10^{17} \text{cm} \cdot {}^{o}\text{K}.$$

This numerical value of RT was obtained in our work (Ferronsky et al. 1979a; 1987) as the value of the proton branch of the Sun evolution. This theoretical solution has been proved by observational data of Spitzer. The electron branch of the evolution was also found there. Its RT value is 2,000 times less than that of the proton branch in accordance with the ratio of their masses. Hence, the radius of the Protosun cloud is ~10^{18} cm for the proton branch of the evolution and ~ 10^{13} cm for the electron branch. This is possible explanation for the division of the solar system planets in two groups.

The planetary masses are by 10^3–10^7 times less than the solar mass and their radii are 10^{-2}–10^{-4} solar radius. This estimation follows from the recalculation of the proton and electron branches of evolution. Therefore, the ratio m/R^2 for the planets is only 10–100 times more than that of a protostar obtained by Spitzer. The radiation of the planets and satellites should be in equilibrium with the other flux which is the internal solar flax. In this connection the equilibrium temperature of the planets and satellites should be 3–10 times more than the star's temperature of 10–100°K. The existing direct measurements give evidence of two peaks in the relict radiation spectrum (4 K and 20 K). It means that our physics and numerical estimates are reasonable.

In conclusion we wish to stress that the discovered by the artificial satellites relationship between the gravitational field (potential energy) and the polar moment of inertia of the Earth leads to understanding the nature an mechanism of the planet's energy generation as the force function of all the dynamical processes release in the form of oscillation and rotation of the matter. Through the energy nature we understand the unity of forms of the gravitational and electromagnetic interactions which, in fact, are the two sides of the same natural effect.

Chapter 10
Conclusions

It might appear surprising that the integral approach to description of dynamics of natural systems, which has a number of obvious advantages, has been developed far less well than the differential hydrostatic one. However, if we consider the development of the apparatus of mathematical physics from this viewpoint, the picture changes completely.

In fact, as soon as the concept of the field was formulated – even through at first this concept was a purely mathematical one (e.g. of the electrostatic and magnetic fields) – Gauss' theorem relating to the flux of a field vector through a closed surface was put forward. This integral characteristic of a field enclosed within a surface is an invariant of the field. In the case of electrostatics it is charge which give rise to the field.

The concept of vector flux through a closed surface has been generalized and developed. For example, such a generalization is Stokes' theorem relating to the circulation of a vector around a closed circuit, which can be used to identify vortex sources in vector fields. These theorems, which by their very nature are distinctly integral ones, have served as the basis for the whole mathematical theory of continuum mechanics, the electromagnetic theory of Maxwell and Poisson's theory of Newtonian gravitation.

Thus, the development of the mathematical apparatus of physics has taken the course of the integral approach to the description of natural phenomena. The concepts of divergence and the rotor introduced in this connection have served as instruments for finding the sources and sinks of a field and its vortices.

However, the idea of the continuity of a field, which gave rise to these concepts itself placed a limit on them, because the size of the region in which the charge was enclosed by a surface had to tend to zero. The Gaussian surface integral was thus replaced by divergence as a differential operation.

Circulation was similarly replaced by the rotor as a differential operation. It is these operations which are used in the Maxwellian field theory. This is because of the erroneous idea that the electric charges giving rise to the field are themselves continuous quantities distributed over the volume and also over the surface of dielectrics and conductors. The theorems of Gauss and Stokes are therefore limited to volumes shrinking to nil, and the theory became a purely differential one. This situation was later improved by Lorentz, who introduced into the field discrete

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charge points of finite magnitude scattered in empty space. According to his theory, Maxwell's equations remain applicable in the empty space between the small regions enclosing point singularities. On the closed surfaces surrounding these regions containing field singularities, the solutions to the field equations satisfy integral conditions. The flax of the field vector through these surfaces is equal to the sum of discrete charges enclosed by the total surface,

With the solution averaged over space, Lorentz' theory led to Maxwell's theory, which was in fact his objective. This is how the integral approach to the description of natural phenomena came into being.

The same approach was used by Einstein in the interpretation of his general theory of relativity and for deriving the equations of motion of matter in accordance with Newton's theory from his own equations.

It is, of course, well known that Einstein constructed his general theory of relativity as a relativistic theory of gravitation. For this, he first wrote Newton's equations in the form of field equations using Poisson's equation, and then gave the latter a relativistic, generalized character.

Einstein went further and abandoned inertial counting system, which had been accorded a position of privilege. Thus, the invariance was no longer assumed to be Lorenzian but universal in relation to any improper continuous transformation. Here, use was also made of Lorentz' idea, which we have mentioned earlier, of the discrete nature of the distribution of matter. Matter is concentrated in point singularities of a field, and between them there is empty space, for which Einstein's field equations hold true. The equations are not satisfied at singular points, which must be surrounded by closed surfaces. For the latter, the integral relations of Gauss in tern hold true, i.e. the flux of the field through these surfaces is equal to the charges found inside them. It should be emphasized once again that the actual fields inside these regions need not satisfy the conditions of the Einstein's equations.

Einstein's theory is, therefore, by its very nature and because of the basis on which it is constructed, an integral one. This fact is not usually realized, which is why we draw attention to it. It is by this condition, which in mathematical terms amounts to the requirement that the divergence of the original tensor should become exactly nil, that the nature of Einstein's tensor is uniquely defined. Such a tensor is one, the divergence of which is twice the contracted Bianchi identity for the Riemann curvature tensor.

If all the singularities of a field are surrounded by small spheres, in the space between them the field will everywhere be regular and its equations can be expanded in descending series in terms of the reciprocals of the velocities of light. Equating the coefficients in terms of the same powers, we obtain a series of equations. Every such system contains new quantities not found in the previous systems and is easily solved.

The motion of singularities (i.e. of particles) is determined by virtue of the fact that the left-hand sides of the systems of equations being solved satisfy four identities. The right-hand side of these equations must therefore also satisfy these identities or, with the singularities taken into account, the integral conditions. In the absence of singularities these conditions are automatically satisfied and provide nothing new. But if there are present, they determine the equations of motion. Einstein followed all calculations through and obtained Newton's equations. This method can also be used when gravitational and magnetic fields exist simultaneously, and the result of the calculation is positive. In this way, Einstein showed that even the classical interaction of mass points is caused by the non-linearity of the field equations. This fact is usually emphasized, but the role of integral conditions tends not to be mentioned.

Einstein's equations therefore contain Newton's equations and thus also their solutions and combinations.

Jacobi's virial equation is derived from Newton's equations and consequently, must itself be contained in Einstein's equations. However, it is not immediately apparent whether Newton's or Jacobi's equation is the more fundamental. Newton's equations were obtained by Einstein from his second-order equations by approximation. Jacobi's equation was obtained from Einstein's by the method of oscillation moments, also in second-order but by an exact method. This makes Jacobi's equation the more fundamental one; moreover, unlike Newton's equation, it remains integral and dynamical in nature.

As we have mentioned, the way in which the whole problem is formulated gives Jacobi's moment equation an exact, closed from which in fact solves the problem itself. In the case of the universe the problem is also one of its non-steady-state nature. A clever solution to this problem was found earlier by Friedmann. His solution is a solution to Jacobi's equation or to the smoothed Einstein equation. This is an analogue of Maxwell's equation in the form of a smoothed Lorentzian equation for charge points.

For the empty space between point singularities an anisotropic solution to Einstein's equation has been found (also by indirect means). This solution is Kasner's metric. Analysis of this metric shows that the empty space being considered pulsates. It is compressed on two axes, expands on one, and vice versa. Since this solution has been obtained for the case of space without matter, i.e. without its interaction, so that the law of interaction is without significance, the oscillatory nature of processes in nature is universal. The solution, however, is a formal one and its physical significance needs to be elucidated.

In fact, in Newton's well known law of gravitation for two masses it is assumed that these are mass points. Otherwise, the inverse-square law ceases to apply to their interaction. This in turn contravenes the law of remote screening mentioned in Chap. 1, which makes it impossible for approximately isolated (conservative) systems to exist.

The law of gravitation thus permits the existence of infinitely small radii of curvature and thereby of an infinitely large curvature of space time, i.e. of singularities. There are other examples of motion towards or away from a singularity, such as the formation of stars and planets, the expansion of the universe etc. Newton's law of gravitation therefore non-explicitly reflects the conditions for the existence of singularities, and the generalization of his theory by Einstein retains and, on the basis of the principle of equivalence, clearly demonstrates these singularities.

Singularities are therefore an empirical fact. So what are they?

In accordance with Einstein's theory, curvature is produced by mass. Consequently, empty-space time is not abstract emptiness but a physical vacuum with its own structure and also an analogue of mass, which in fact reflects Kasner's solution to Einstein's equation. This view is now widely held. In most models a vacuum is considered to be a quantum-mechanical system of virtual particles and to behave in a way similar to an elastic medium. Belinsky et al. (1970) studied the behavior of Einstein's equation for non-empty space-time but near a singularity. They showed that with increasing proximity to (distance from) the singularity a moment is approached at which the vacuum curvature exceeds the curvature from matter and the solution to Einstein's equation again becomes Kasner's solution.

Its solution, however, is a case of uniform – although anisotropic – space-time. Belinsky, Lifshitz and Khalatnikov also examined the case of inhomogeneous space-time and came to conclusion that the nature of the solution was the same but that the Kasner parameters were dependent on the co-ordinates and time.

In the case of further evolution of the Kasner solution with expansion of space away from the singularity, the original anisotropic space in gradually converted into isotropic space, i.e. into the Friedmann model, which is a solution to the second-order virial equation.

The oscillatory law of the dynamics of natural processes is thus a universal law of nature. It should, however, be noted that into all the approaches mentioned above the concept of finite time and of a beginning of time counting has been introduced. In some models there is also the concept of the end of the world. Only in one of them (in which the average density of matter for the space being considered is strictly determined) do the periodically alternating processes of expansion and contraction infinitely. It is this mode which is determined by the solution to Jacobi's virial equation.

A special feature of the Kasner solution for the general anisotropic case of space-time is the appearance in it of dependence of metric coefficients of time in accordance with the $| t |^{2/3}$ law, where t is a time interval. This law was found for the most general case in which there is no external symmetry, i.e. no symmetry which is not associated only with the internal arrangement of singularities.

The sources of the important relation $| t |^{2/3}$ go back to Kepler, who found experimentally the law in accordance with which the squares of the periods of rotation of bodies of the solar system are the cubes of the semi-axes of the ellipses in which they undergo motion.

It was pointed out in Chap. 6 that in Newton's theory about the attraction of mass points such a law is also found to be asymptotic for the case in which n bodies collide simultaneously. It was also shown there that within this asymptotic limit the simultaneous collision of n bodies leads to a homologous configuration. And for it in turn the condition of the applicability of Jacobi's general virial equation with two functions holds true. Thus, using a solution of the Kasner type, the applicability of Jacobi's virial equation within the asymptotic limit of simultaneous collision between n bodies which was found earlier for Newton's theory is extended to the case of the solution of Einstein's general equation. This indicates the universal nature of Jacobi's virial equation in dynamics.

Let us note further important aspect of the solutions under consideration, which relates to the change of Kasner epochs. Their number is infinitely independent of whether the world has a beginning and end. This occurs as a result of a decrease in the duration of an individual epoch as a singularity is approached.

Let us now consider yet another aspect of the fundamental nature of Jacobi's virial equation. As we have already pointed out, Newton's law of gravitation permits the existence of a curvature in space-time, which is derived from Einstein's theory. However, there is one fundamental difference between the two theories. According to Newton, the gravitational interaction is a long-range one, corresponding to an infinite velocity of propagation of the interaction. Einstein assumes a short-range interaction. It is propagated at finite velocity (at the velocity of light). Consequently, Newton's theory is formulated in terms of Euclidian geometry. Nevertheless, with both theories space-time is distorted.

Newton's theory is constructed on the basis of a simple empirical law of Kepler's and does not make use of another empirical law, namely the principle of equivalence derived from the experiments of Eötvös.

So what common ground is there between the theories?

The fact is that Newton's theory is constructed as Newtonean mechanics plus his own law of gravitation. In Newtonean mechanics there are three axioms, but the type of interaction is not determined; this is done experimentally. In generalized Newton's theory, it is the mechanics that should have been generalized and not the type of interaction.

With Einstein the type of interaction is replaced by the principle of equivalence. The mechanics, on the other hand, is generalized in accordance with the principle of the invariance of equations. Long-range interaction is thus not involved here, and the type of interaction makes no difference.

Jacobi's virial equation, which was obtained from Newton's equations, also does not so much generalize the type of interaction law, in the way that this was done in his (Jacobi's) conclusions, as take into account the mass defect (potential energy). It is therefore linked with the principle of equivalence. The mass defect, in turn, is determined by a system that has already been formed and, consequently, does not depend on the type of interaction during the process of formation (long-range or short-range).

As was thought by Wintner, Jacobi's virial equation therefore reflects the type of interaction law only integrally over the whole period of time in which the mass defect is formed. Also, if there is no delay, as in the case of Newton's long-range law of gravitation, it will be simultaneously a specific and instantaneous type of interaction, as pointed out by Wintner.

If a delay does take place, for example in accordance with Einstein's short-range interaction law, instead of a specific, instantaneous type of interaction, the equation will include an expression which has been strongly averaged over time, and the dependence on the type of interaction will cease to be of significance. It will be replaced by an assertion about the dependence on instantaneous mass or on the mass defect which has built up over a long time.

This is the answer to the question posed. At the same time, the strength of the Jacobi equation is evident. Since in the general theory of relativity the usual problems in the framework of a short time interval – and even the classical two-body problem – are not solved, the enormous practical significance of solving Jacobi's virial equation becomes obvious. The fact that there are oscillations even in empty space-time indicates the exclusively fundamental nature of this equation.

Moreover, it has now become obvious that Jacobi's virial equation, which was obtained from Newton's equations, is a particular case of more general virial equation derived from Einstein's equations. This equation will thus be studied from the most general global points of view, namely that of empty space-time, which will not be called a vacuum, and that of models of an evolving universe. It should be noted here that the models that have so far been developed from Jacobi's equation of an open, a closed and a pulsating universe have been obtained automatically as its natural solutions as a function of the source data – the quantities of total moment and mass defect. In this case, all possible types of solution are encompassed, and the question of the completeness of the set of possible models of the universe is thereby solved.

Let us now consider an example which demonstrates the use of the integral approach for constructing a complete closed theory based on Hooke's law. The theory concerned is the theory of elasticity.

In this theory, for any volume of a continuum, only quantities and parameters which ate integral from the point of view of an external observer are considered, namely deformation, stress and modulus of elasticity. The elements of the volume interact through their surface. A quantitative measure of their interaction is provided by strains, and a quantitative measure of the results of interaction by relative changes in the external dimensions of elements, in other words, their deformation. The internal structure of the material is demonstrated quantitatively by means of integral parameters, namely the mass density, the modulus of elasticity and the Poisson coefficient.

The interaction between the element of interest of a body and the external world takes place through external surface and volume forces. The external surface forces act only on the surfaces of the whole body and not on that of any of its elements. External volume forces amount to the application of surface forces to the surfaces of any element, and thereby to tensions. Here, external surface forces do not come into the equilibrium equations but into the boundary conditions of the problem and are thus excluded as forces.

It is important to stress this point. It was mentioned earlier by Hertz, who set himself the problem of constructing a system of mechanics without forces. The fact that he was relatively unsuccessful is because in his days Minkowski's idea about the unity of space-time was as yet unknown. The link between static and dynamics was not as clear as it would be after Minkowski.

Exposition of the theory of elasticity usually begins with the formulation of Hooke's law in the form of the relation between deformation and force in the context of the tensions or compression of a uniform beam. The elongation or compression of the beam is in linear proportion to the force applied to its ends.

This relation is then established more precisely by taking into account the dimension of the area of the beam's cross-section, the concept of tension as a ratio of force to the cross-section and the elongation of the beam as a function of the initial length. The deformation is thus the ratio of relative elongation to the total length. The intrinsic properties of the material of the beam are taken into account by introducing the concept of Young's modulus, which has the dimensionality of the volume density of energy. Only then is it established that, with longitudinal tension and compression, the beam is simultaneously undergoing transverse compression and tension. The relative deformations about the three axes will be in proportion to one another. The link between the relative deformations is determined by the Poisson coefficient, which is also a parameter characterizing the material of the beam. Usually, this aspect is not stressed. For our problem, however, it is of particular interest.

First, the beam has dimensions (length, height, width, surface area), which are determined only from the position of an external observer; secondly, all forces are also fixed from the beginning; thirdly, the internal arrangement of the beam is fixed by integral parameters (the modulus of elasticity and the Poisson coefficient); fourthly, when external forces are applied, the beam reacts as a unified whole and simultaneously in three dimensions. Since here the relative deformations are in proportion to one another with a constant Poisson coefficient, the principle of the superimposition of deformations is applicable. The whole theory is therefore found to be strictly linear.

Finally, it should be noted that two states of the beam are always considered – the initial and the final states before and after the application of the forces. One of them is generally the equilibrium state. If these two states of one and the same system occur at different times, displacement deformations are replaced by velocity deformations. In this case the approach followed takes the form of the theory of viscous or liquid media of gases. As far as Hooke's law itself is concerned, for its purposes deformations and stresses are characterized by tensors of second order and the set of coefficients linking them in a linear fashion is an elasticity tensor of fourth order.

For a fluid, Hooke's law is written in the form of Pascal's law. This way of writing it expresses the condition of equilibrium of the medium, where the stresses on the main axes are equal to the pressure of the fluid. Another condition of equilibrium for a fluid is the law of the conservation of matter.

If in the context of Hooke's law we move to the point of view of Minkowskian unified space-time, and effect a Lorentzian transformation from a stationary system of co-ordinates to a moving one, the equilibrium conditions in accordance with Pascal's law or in the form of any other Hookean tension law can be expressed as Euler equations and as an equation of the continuity of the medium. Here it is important to note that, when deriving the Euler equations of motion, it is not obligatory to use Newton's second law of mechanics and that a Hookean system equilibrium equation can be used.

Nor are any dynamic laws used to justify the Minkowski approach, which is based directly on experimental values and is considered to be valid.

It should be noted that, in the context of Hooke's law, a rigorous solution can be found to Jacobi's virial equation for conservative system. In this case Hooke's law determines the constancy of the product of potential energy and of the Jacobi function; this constancy is written in the form$|U|\sqrt{\Phi} = aGm^{5/2}$.

In this relation the coefficient $a = \alpha^2\beta$ (which stands for the product of form factors included in the expressions for the potential energy and the Jacobi function) acts as a modulus of the dynamic elasticity of the system. It remains a constant and reflects the constancy of the law of mass density distribution of the system within the limits of its elastic deformations with virial oscillations. The deformation of the system is characterized by its integral parameter the Jacobi function – and the stresses are determined by the term $Gm^{5/2}/U$. As a result, the virial pulses of the system will be strictly periodic, and the deformations will be found to be elastic and therefore reversible.

On this basis it was shown in Chap. 7 that the parameters of the virial oscillations of the Earth, which are detected, can be used as if the Earth were an elastic body for determining its potential energy. This option remains open for when natural systems are being examined in the framework of other models of continuous and discrete media.

We have mentioned a number of aspects of the universality of Jacobi dynamics in the examination of natural systems. We shell now consider the prospects for solving a number of practical problems in the context of the dynamic approach.

One of these problems is the dynamics of the Solar System, of its evolution and its origin. In Chap. 8 we made first step in this direction and obtained the basic common solution on creation of celestial bodies and their systems. It appears that any rank of new celestial body (from galaxy to meteorite and even to molecule and atom) is born by self-gravitating parent as consequence of loss of its energy by radiation. It means that the stage of self-gravitation and separation mast be changed by the stage of gravitation and joining of the matter. Thus, the present day stage of expansion of the Universe, after total separation of the matter, should come to the stage of its contraction and gathering. Generally saying, our Universe is a closed pulsating and perpetual system. New more detailed solutions in this direction are desirable.

In the context of the dynamic approach a new problem of dynamics of the self-gravitating Earth and its interaction with the Sun and the Moon were considered in Chap. 7. The found normal and tangential components of the potential and kinetic energies of a self-gravitating body made it possible to understand mechanism of separation of the body's shells, their oscillation and rotation by the inner force field. It was understood that the induced outer force field, which has all the properties of the electromagnetic field, acquires the property to conserve the irradiated energy and potential in the orbital motion of its secondary body. But because of limited velocity of propagation of the changing potential the orbital trajectory is found to be open. This fact is proved both by the artificial satellites and the observed precession of all the planets and the moons. The found important effect makes it possible to interpret inner structure of the Sun, the Earth, the Moon and other celestial bodies.

And also it rises the problem of improving of the Kepler's approximation of the Earth's and other body's orbits which are found to be too rough.

In our opinion, the dynamics of the microcosm is a very interesting application of Jacobi dynamics. This book takes only the first step in this direction. It is shown that Jacobi dynamics is also applicable for the solution of this type of problem. An attractive idea is to use the dynamic approach for studying the physics of molecules, atoms and nuclei as dissipative systems, which might lead to discovery of many interesting effects.

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