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Nuclear Physics of Stars



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Preface

Nuclear processes generate the energy that makes stars shine. The same nuclear processes in stars are responsible for the synthesis of the elements. When stars eject part of their matter through various means, they enrich the interstellar medium with the nuclear ashes and thereby provide the building blocks for the birth of new stars, of planets, and of life itself. The theory of this building of elements is called nucleosynthesis and it is remarkably successful in describing the nuclear processes in stars that are located so far away from us in space and time. It is equally remarkable how the theory predicts these processes based on the quantum mechanical properties of atomic nuclei. Nucleosynthesis, nuclear energy generation in stars, and other topics at the intersection of nuclear physics and astrophysics make up the science of nuclear astrophysics. Like most fields of physics, it involves both theoretical and experimental activities. The purpose of this book is to explain these concepts with special emphasis on nuclear processes and their interplay in stars.

Work on the manuscript for this book started when I was invited to teach a two-week long graduate-level course on "Nuclear Physics of Stars" at the Universitat Politècnica de Catalunya in Barcelona, Spain, in June 2003. During the preparations for the course it became quite obvious that it would be useful to have an up-to-date textbook available. The encouragement I received from many colleagues and students to write such a book was instrumental for my decision to begin work on a manuscript.

After a decade of teaching at the University of North Carolina at Chapel Hill I learned from my students to take no "well-established" fact for granted. They wanted to see derivations of equations when I attempted to state "the obvious." They insisted on more fundamental explanations when I just tried to "wave my hands." The style of the present book is certainly influenced by my teaching experience. Indeed, most equations are derived in the text and special emphasis has been placed on the art work. My main intention is to explain complicated concepts in the simplest and most intuitive manner. In some instances, more elegant formulations of concepts have been presented in the literature. For the manuscript these were considered only if I found

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it impossible to come up with a simpler explanation. Colleagues frequently wanted to know "which review paper" I used in the preparation of a specific section. My strategy was to consult review articles only after I wrote a complete first draft of the section. That way I was forced to comprehend the subject myself from the beginning and to come up with a coherent presentation.

The present book is directed toward advanced undergraduate students, graduate students, and researchers in the fields of nuclear physics and astrophysics. Chapter 1 starts with the basic concepts in nuclear physics and stellar evolution. Chapter 2 develops the theory of nuclear reactions starting from basic quantum mechanical ideas. Nuclear processes in a stellar plasma are discussed in Chapter 3. Chapter 4 contains the most important experimental information needed in order to perform measurements in nuclear astrophysics. Chapter 5 provides a discussion of the theory of stellar nucleosynthesis. The appendices contain sections on basic solutions of the Schrödinger equation, angular momentum selection rules, kinematics, and the theory of angular correlations. At the end of the text, physical constants, mathematical symbols and physical quantities are listed as an aid for the reader. As a prerequisite, the student should have taken an undergraduate course in modern physics with elementary coverage of wave functions. An undergraduate course in quantum mechanics or nuclear physics would also be helpful, but is not required.

The present book goes into considerable depth and, consequently, restrictions in time and space made it unavoidable for me to omit a number of important topics. The instructor who is using this book may wish to supplement the material presented here with information on primordial nucleosynthesis (J. Rich, *Fundamentals of Cosmology*, Berlin: Springer, 2001), cosmic-ray spallation reactions (E. Vangioni-Flam, M. Cassé and J. Audouze, *Phys. Rep.*, Vol. 333, p. 365, 2000), nucleochronology (J. J. Cowan, F.-K. Thielemann and J. W. Truran, *Ann. Rev. Astron. Astrophys.*, Vol. 29, p. 447, 1991), neutrino astrophysics (J. N. Bahcall, *Neutrino Astrophysics*, Cambridge: Cambridge University Press, 1989), *v*-process (Woosley et al., *Astrophys. J.*, Vol. 356, p. 272, 1990), presolar grains (M. Lugaro, *Stardust from Meteorites*, Singapore: World Scientific, 2005) and indirect measurements of astrophysically important nuclear reactions. It is utterly impossible to recommend one, or even a few, references for the last topic which represents a vast field in its own right.

I would certainly not have written this book without the influence of two of my colleagues. I am indebted to Jordi José, who invited me to Barcelona in 2003 and who organized my lectures and my wonderful stay there. I also wish to express my appreciation to Art Champagne, who supported me professionally through all stages during the preparation of the manuscript. A number of people have read through parts of the manuscript and have pro-

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Carrboro, September 2006

Christian Iliadis

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1

1 Aspects of Nuclear Physics and Astrophysics

1.1 History

In 1920 Aston discovered that the mass of the helium atom is slightly less than four times the mass of the hydrogen atom. Immediately afterward, Eddington suggested in his 1920 presidential address to the British Association for the Advancement of Science that Aston's discovery would explain the energy generation of the Sun via the conversion of hydrogen to helium. However, Eddington could not explain the fact that the stellar temperatures inferred from observation were well below those thought necessary to initiate fusion reactions. In 1928 Gamow, and independently Condon and Gourney, calculated the quantum mechanical probability for particles to tunnel through potential barriers and thereby explained the phenomenon of α -particle decay (Gamow 1928, Condon and Gourney 1929). Atkinson and Houtermans used Gamow's results to suggest that quantum mechanical tunneling may explain the energy generation of stars via fusion reactions (Atkinson and Houtermans 1929).

Cockcroft and Walton (1932) initiated the first nuclear reaction using artificially accelerated particles by bombarding and disintegrating lithium nuclei with protons accelerated to several hundred keV energy. Incidentally, the disintegration of lithium into two α -particles is one of the reactions of what would later be called the pp chains. Lauritsen and Crane produced in 1934 a 10-min radioactivity following the bombardment of carbon with protons. It was the first measurement of one of the reactions of what would later be called the CNO cycle.

Atkinson (1936) proposed the fusion of two hydrogen nuclei to deuterium as a source of stellar energy generation. A detailed treatment of this reaction was provided by Bethe and Critchfield who showed that the p + p reaction gives indeed an energy generation of the correct order of magnitude for the Sun (Bethe and Critchfield 1938). The energy production in stars via the CNO cycle was independently discovered by von Weizsäcker (1938) and Bethe (1939). The latter work, in particular, investigated for the first time the rate of energy production and the temperature dependence of the CNO cycle.

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In the following years some of the pioneering ideas of nuclear astrophysics were established. In two papers, Hoyle first presented the theory of nucleosynthesis within the framework of stellar evolution by using the nuclear data available at the time (Hoyle et al. 1946, Hoyle 1954). Nuclear experiments had firmly established that no stable nucleus of mass number 5 or 8 exists in nature. For this reason, it was a mystery how these mass gaps could be bypassed in the synthesis of heavier nuclei from lighter species. Salpeter suggested in 1951 that a small equilibrium concentration of unstable ⁸Be could capture another α -particle to form stable ¹²C and that this "triple- α reaction" could be the main energy source in red giant stars (Salpeter 1952). Hoyle pointed out that the capture probability would be far too small unless an excited state with zero spin and positive parity existed in ¹²C at about 7.7 MeV excitation energy. His remarkable theoretical insight was verified when the level was clearly identified (Dunbar et al. 1953) and its properties determined (Cook et al. 1957), thereby establishing the triple- α reaction as the mechanism to overcome the mass 5 and 8 gaps.

In an influential review, Suess and Urey demonstrated the existence of several double peaks in a greatly improved distribution of observed solar-system abundances (Suess and Urey 1956). It became immediately clear that these abundance peaks were associated with the neutron shell fillings at the magic neutron numbers in the nuclear shell model that Jensen and Goeppert Mayer had developed in 1949. The nucleosynthesis processes for the heavy nuclides beyond iron via neutron captures became later known as the s- and r-process.

Of great importance was the discovery of spectral lines from the element technetium in evolved red giant stars (Merrill 1952). All of the technetium isotopes are unstable and the longest lived isotope has a half-life of $\approx 4.2 \times 10^6$ y. Such half-lives are very short on a cosmological time scale ($\approx 10^{10}$ y) and, consequently, the discovery showed beyond doubt that the technetium must have been produced "recently" within the stars and that the products of nucleosynthesis could indeed reach the stellar surface with the help of mass loss and mixing.

The available knowledge at the time regarding the synthesis of elements was presented in a review article by Burbidge et al. (1957), and independently by Cameron (1957). These papers laid the ground work for the modern theory of nuclear astrophysics. The field has developed since into an exciting discipline with impressive achievements, linking the topics of astronomical observation, nuclear physics experiment, nuclear theory, stellar evolution, and hydrodynamics.

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1.2 Nomenclature

Atomic nuclei consist of protons and neutrons. The symbol *Z* denotes the number of protons and is called *atomic number*. The number of neutrons is denoted by the symbol *N*. The *mass number A* is defined by the integer quantity A = Z + N. It is sometimes also referred to as *nucleon number*. Nuclei with the same number of protons and number of neutrons have the same nuclear properties. They can be represented by the symbol ${}^{A}_{Z}X_{N}$, where X is the element symbol. Any individual nuclear species is called a *nuclide*. Nuclides with the same number of protons, but different number of neutrons (and hence a different mass number *A*) are called *isotopes*. Nuclides of the same mass number, but with different numbers of protons and neutrons are called *isotones*. Nuclides with the same number a different mass number *A* are called *isotones*. Success, isobars, and isotones have different numbers of protons or neutrons and, therefore, their nuclear physics properties are different.

Nuclides can be represented in a two-dimensional diagram, called *chart of the nuclides*. It displays the number of neutrons and protons on the horizontal and vertical axes, respectively. Each square in this diagram represents a different nuclide with unique nuclear physics properties. Figure 1.1 displays a section of the chart of the nuclides, showing the lightest species with $Z \le 15$ and $N \le 20$. The shaded squares represent stable nuclides, while the open squares correspond to unstable nuclides with half-lives in excess of 1 ms. It is obvious that many more unstable than stable nuclides exist in nature. It is also striking that no stable nuclides exist with a mass number of A = 5 or 8. This circumstance has a profound influence on the nucleosynthesis in stars, as will be seen in Chapter 5.

Example 1.1

The nuclide of carbon (Z = 6) with 7 neutrons (N = 7) has a mass number of A = Z + N = 13 and is represented by the symbol ${}_{6}^{13}C_{7}$. Since the element symbol and the number of protons (atomic number) carry the same information, both Z = 6 and N = A - Z = 7 are frequently suppressed in the notation. The carbon species with mass number A = 13 is then unambiguously described by the symbol ${}^{13}C$.

The species ${}^{12}_{6}C_{6}$, ${}^{13}_{6}C_{7}$, and ${}^{14}_{6}C_{8}$ are isotopes of carbon (Z = 6); ${}^{20}_{10}Ne_{10}$, ${}^{20}_{11}Na_{9}$, and ${}^{20}_{12}Mg_{8}$ are isobars of A = 20; ${}^{28}_{14}Si_{14}$, ${}^{29}_{15}P_{14}$, and ${}^{30}_{16}S_{14}$ are isotones of N = 14.



Fig. 1.1 Section of the chart of the nuclides, showing the lightest species with $Z \le 15$ and $N \le 20$. The shaded squares represent stable nuclides, while the open squares correspond to unstable nuclides with half-lives in excess of 1 ms. The only exceptions are the nuclides ⁸Be and ⁹B which have much shorter half-lives. Note that no stable nuclides exist with a mass number of A = 5 or 8.

1.3

Solar System Abundances

It is commonly accepted that the solar system formed from the collapse of a gaseous nebula that had an almost uniform chemical and isotopic abundance distribution. Abundances in the solar system are also similar to those found in many stars, in the interstellar medium of the Sun's neighborhood and in parts of other galaxies. Therefore, it was hoped for a long time that a careful study of solar system abundances would provide a "cosmic" or "universal" abundance distribution, that is, an average abundance distribution which is representative for all luminous matter in the universe. A closer comparison of abundances in the solar system and other parts of the universe shows, however, significant compositional differences. Furthermore, the discovery of presolar grains in primitive meteorites allowed for the first time a very precise chemical and isotopic analysis of interstellar matter. Measurements of isotopic abundances in these presolar grains revealed the existence of very large deviations compared to solar system values. Following common practice in the

recent literature, we will avoid the term "universal" abundances and use instead the expression *solar system abundances* when referring to the abundance distribution in the solar system at the time of its formation. The latter distribution provides an important standard to which reference is frequently made.

There are two major, independent and sometimes complementary, sources of solar system elemental abundances: (i) observations of the solar photosphere, and (ii) analysis of a specific class of meterorites, called CI carbonaceous chondrites. The Sun contains most of the mass in the solar system and is, therefore, representative for the overall composition. On the other hand, planets contain much less mass but they underwent extensive chemical fractionation over the past 4.5 Gy since their formation (Cowley 1995). Among the more than 20,000 recovered meteorites, there are only five known CI carbonaceous chondrites. Although they contain a minuscule amount of matter, they are believed to be among the most primitive objects in the solar system. They show the least evidence for chemical fractionation and remelting after condensation and thus they retained most of the elements (except for a few very volatile species) present in the original matter of the solar nebula. Details on how these abundances are obtained will not be repeated here (see, for example, Arnett 1996, Grevesse and Sauval 1998, Palme and Jones 2003, Lodders 2003). It is sufficient to remark at this point that the abundances derived from the solar photosphere and from primitive meteorites are in remarkable overall agreement (better than \pm 10% for most elements). Solar system *isotopic* abundances are then derived from the *elemental* abundances by using mainly terrestrial isotopic ratios (Rosman and Taylor 1998).

The solar system abundances of the nuclides are shown in Fig. 1.2a versus mass number A. The abundances are normalized to the number of silicon atoms. In cases where two or more stable isobars exist for a specific mass number A, the sum of the individual abundances is shown in the figure. Part b displays the abundances separately for even-A and odd-A nuclides. Almost all the mass is contained in 1 H (71.1%) and 4 He (27.4%). There is an abundance minimum in the A = 5-11 region, corresponding to the elements Li, Be, and B. More than half of the remaining mass (1.5%) is in the form of 12 C and 16 O. The abundances drop slowly with increasing mass number. Another minimum occurs in the A = 41-49 region, around the element Sc. The abundance curve exhibits a maximum in the A = 50-65 region, near the element Fe. The nuclides in this region are referred to as the *iron peak*. Beyond the iron peak, the abundances in general decrease with increasing mass number, although pronounced maxima are clearly visible in the A = 110-150 and A = 180-210regions. Closer inspection of Fig. 1.2b also reveals that even-A nuclides are generally more abundant than odd-A nuclides. Furthermore, the abundance curve for odd-A nuclides is considerably smoother than the one for even-A nuclides.

The outstanding gross features in Fig. 1.2 are the abundance maxima and minima. Specifically, the abundances do not scatter randomly, but instead exhibit a certain regularity and systematics. It is reasonable to assume that the abundances within any group or subgroup of nuclides can be attributed primarily to a specific mechanism of nucleosynthesis. Starting with the work of Suess and Urey (1956), such tables of solar system abundances had an enormous influence on investigations of the origin of the elements and the development of nuclear astrophysics. Not only did it become possible to identify and study various processes of nucleosynthesis that left their distinctive signatures in the abundance distribution, but a connection could also be made to the environments in which these sources of nucleosynthesis operated. All nuclides, with few exceptions, are synthesized in stars. Therefore, the observed solar system abundances offer powerful clues to stellar history and evolution, and by extension, to the chemical evolution of the Galaxy as a whole.

It is fascinating that the structures seen in Fig. 1.2 reflect the nuclear physics properties of various processes occurring in nature. A few very general comments follow below. All of the hydrogen (¹H and ²H) and most of the helium (³He and ⁴He) nuclei originated in the Big Bang (Rich 2001). The most abundant of these, ¹H and ⁴He, are the basic building blocks for the synthesis of heavier and more complex nuclei. A deep abundance minimum occurs in the Li-Be-B region. These nuclides are easily destroyed in fusion reactions with protons (that is, their cross sections are very large). Therefore, their observed solar system abundances must be explained by processes that occur in sites other than stellar interiors. They are thought to be produced via spallation reactions induced by Galactic cosmic rays (Vangioni-Flam, Cassé and Audouze 2000). However, the Big Bang and certain stars did most likely contribute to the production of ⁷Li. All of the heavier nuclides with $A \ge 12$ are produced in stars. The nuclides in the region between ¹²C and ⁴⁰Ca are synthesized via charged-particle nuclear reactions in various stellar burning processes. Reactions between charged particles are subject to the Coulomb repulsion. The larger the charge of the reacting nuclei, the smaller the nuclear reaction probability will become. This circumstance is reflected in the overall decline of the abundance curve from ${}^{12}C$ to ${}^{40}Ca$. The abundance maximum of the iron peak is explained by the fact that these nuclides represent energetically the most stable species (Section 1.5.1). Because of the large Coulomb repulsion, the synthesis of nuclides beyond the iron peak via charged-particle reactions becomes very unlikely. These nuclei are instead produced by the capture of neutrons. The abundances of nuclides in the A > 80 region are on average a factor of 10¹⁰ smaller compared to the hydrogen abundance, as can be seen from Fig. 1.2. The observed narrow and broad peaks in this mass region provide unambiguous evidence for the existence of two distinctive neutron capture processes. All of the above comments are very general and do not explain



Fig. 1.2 Abundances of the nuclides in the solar system at its birth. Number abundances are normalized to the number of silicon atoms (Si = 10^6). Data from Lodders (2003). (a) Sum of all nuclidic abundances at a given value of *A* versus mass number.

The maximum in the A = 50-65 region is referred to as the iron peak. (b) Separate abundance contributions from nuclides with an even or an odd value of A versus mass number. Even-A nuclides are in general more abundant than odd-A nuclides.

any details of the solar system abundance curve. An extensive discussion of the various nucleosynthetic processes will be given in Chapter 5. Information regarding the origin of the solar system nuclides is provided at the end of this book (Section 5.7).

1.4

Astrophysical Aspects

1.4.1

General Considerations

The study of stars is central to astronomy and astrophysics since stars are long-lived objects that are responsible for most of the visible light we observe from normal galaxies. The fusion of light nuclides into heavier species liberates kinetic energy at the expense of mass and serves as the interior source of the energy radiated from the surface. These very same reactions alter the composition of the stellar matter. As already pointed out, all nuclides with masses of $A \ge 12$ are produced in stars. When a star ejects part of its mass into space during certain evolutionary stages, the chemical composition of the interstellar medium will be altered by the thermonuclear debris. The interstellar medium, in turn, plays a key role in providing material out of which new generations of stars form. This cycling of matter between stars and the interstellar medium involves countless stars. By comparing the age of the Galaxy (≈ 14 Gy) with the age of the Sun (≈ 4.5 Gy) we can conclude that the cycling process that gave rise to the solar system abundance distribution operated for almost 10 billion years.

There is unambiguous direct evidence for the nucleosynthesis in stars. First, we already mentioned in Section 1.1 the observation of radioactive technetium in stellar spectra (Merrill 1952). Second, γ -rays from radioactive ²⁶Al were discovered in the interstellar medium by spectrometers onboard satellites (Mahoney et al. 1982, Diehl et al. 1993). The half-life of this nuclide ($\approx 7.17 \times 10^5$ y) is even shorter than that for radioactive technetium, thus demonstrating again that nucleosynthesis is currently active in the Galaxy. Third, neutrinos are predicted to be the byproducts of nuclear processes in stars (Chapter 5). Since they interact very weakly with matter, they escape essentially unimpeded from stellar interiors. Neutrinos from the Sun (Bahcall 1989, Hirata et al. 1990) and from the type II supernova 1987A (Hirata et al. 1987, Bionta et al. 1987) were detected on the Earth, providing another direct test of stellar nucleosynthesis. Fourth, models of supernovae predict the ejection of radioactive ⁵⁶Ni (half-life of 6 days), which then decays to the radioactive daughter nucleus ⁵⁶Co (half-life of 77 days). The subsequent decay of this nuclide (to stable ⁵⁶Fe) is predicted to determine the decline of the light emission from these

stellar explosions. The predictions agree well with the observed light curves of supernovae. Furthermore, photons from the radioactive decay of ⁵⁶Co have been detected directly from supernova 1987A (Matz et al. 1988, Tueller et al. 1990).

The discovery of the existence of two distinct stellar populations by astronomers was also of paramount importance in this respect. The populations are referred to as population I and population II stars. They differ in their age and their content of metals, by which astronomers mean any element other than hydrogen and helium. Population I stars include the Sun and are metal rich. They are young stars, having formed within the past few billion years, and can be found in the disk of the Galaxy. Extreme population I stars represent the youngest, most metal-rich stars and are found in the spiral arms of the Galaxy. Population II stars, on the other hand, are metal poor. They are relatively old and are found in the halo and the bulge of the Galaxy. Extreme population II stars represent the oldest, most metal poor stars and are found in the halo and in globular clusters. Their metal abundance, relative to hydrogen, is smaller by a factor of 100 or more compared to population I stars.

If one assumes that the initial composition of the Galaxy was uniform and if there exists no mechanism capable of concentrating the metals in the disk of the Galaxy, then the Galaxy must have synthesized an overwhelming fraction of its own metals. This argument provides strong support for the theory that nucleosynthesis is a natural process to occur during the evolution of stars. It is then obvious that the metal content of the Galaxy increases with time since the matter out of which stars form is being cycled through an increasing number of stellar generations. Therefore, the differences in metallicity between the two stellar populations suggest that population I stars formed later during the history of the Galaxy when the interstellar medium became much more metal rich.

Nuclear reactions not only explain the bulk solar-system abundance distribution, but are also indispensable for explaining the observed chemical composition of individual stars. Such observations, even for trace elements, are crucial for constraining theoretical models of stars and for better understanding the complicated interplay of stellar hydrodynamics, convection, mixing, mass loss, and rotation. Stellar nucleosynthesis also plays a decisive role for explaining the chemical composition of the interstellar medium and is thus interwined with γ -ray astronomy, the study of primitive meteorites, and the nature of cosmic rays.

1.4.2 Hertzsprung–Russell Diagram

The total amount of radiation emitted per unit time, or the *luminosity*, varies strongly from star to star. The same holds for the effective stellar surface temperature. However, if we plot these two quantities for many individual stars in a diagram, then the result is not a random scatter of points, but most stars fall into several distinct groups. This correlation of stellar luminosity and effective surface temperature represents the single most important relationship of stellar properties. It is referred to as *Hertzsprung–Russell diagram* or colormagnitude diagram. The latter name results from the fact that surface temperature can be expressed in terms of the color of the star, while luminosity is related to the absolute magnitude. An explanation of these relationships can be found in any introductory astronomy textbook. The Hertzsprung–Russell diagram has a profound influence on the theory of stellar evolution and, by extension, on the history of the Galaxy as a whole.

Consider first Fig. 1.3a, showing a Hertzsprung–Russell diagram for a sample of \approx 5000 stars in the solar neighborhood. Each dot corresponds to a single star. The surface temperature increases from right to left in the figure. The vast majority of stars occupy the main sequence (MS), stretching diagonally from the upper left (hot and bright stars) to the lower right (cool and faint stars). The Sun, for example, belongs to the main sequence. In the low and right part (cool and faint stars) of the main sequence one finds the red dwarfs (RD). The subgiant branch (SGB) joins the main sequence and extends in a direction to cooler and brighter stars, where the populated region turns first into the *red clump* (RC), and then into the *red giant branch* (RGB). In a region corresponding to smaller luminosity and higher temperature (lower left), one finds a group of faint and hot stars known as white dwarfs (WD). A well-known example is Sirius B, the companion of Sirius. Some stars are located below the main sequence, but are much brighter than white dwarfs. These are known as subdwarfs (SD). A number of star categories do not appear in the figure. Supergiants (SG) are the brightest stars in the Galaxy and would occupy the upper end of the Hertzsprung-Russell diagram, but are very rare in the solar neighborhood. The cool and faint *brown dwarfs* would appear off scale way down in the lower-right part of the figure, but are too faint to appear in the figure.

A Hertzsprung–Russell diagram for the globular cluster M 3 is shown in Fig. 1.3b. There are about 200 globular clusters in the Galaxy. They are located in a spherical space surrounding the Galactic center, called the halo of the Galaxy. Each cluster consists of 10^4 – 10^6 graviationally bound stars, which are highly concentrated toward the cluster center. An image of the globular cluster M 10 is shown in color Fig. 1 on page 631. Spectroscopic observations revealed that globular clusters are metal poor compared to the Sun, implying that they are rather old and that they formed during the early stages of Galac-

tic evolution. It is commonly accepted that all stars in a typical globular cluster formed around the same time from material of very similar composition. The observation that the stars of a globular cluster occupy distinct regions in the Hertzsprung–Russell diagram must then be explained by differences in the only other major stellar property, that is, their initial mass. As will be shown below, the stellar mass is the most important property influencing the evolution of stars. In fact, the higher the mass, the faster a star will evolve.

Figure 1.3b shows some of the same stellar categories already mentioned in connection with part (a). The densest region is occupied by main-sequence stars. The distinctive kink extending from the main sequence toward cooler and brighter stars is called the *turn-off point* (TO). The subgiant branch stars (SGB) are located on a horizontal part stretching toward the right, which turns upward into the red giant branch (RGB). Three more groups of stars can be clearly distinguished on the left-hand side of the RGB: the asymptotic giant branch (AGB), the red horizontal branch (RHB), and the blue horizontal branch (BHB). As will be seen below, the different groups of stars seen in parts (a) and (b) correspond to different stages of stellar evolution. Globular clusters in particular play an outstanding role in astrophysics since the distinct features in their Hertzsprung–Russell diagrams represent strong constraints for stellar models.

1.4.3

Stellar Evolution of Single Stars

One of the most important goals of the theory of stellar structure and evolution is to understand why certain stars appear only in specific regions of the Hertzsprung–Russell diagram and how they evolve from one region to another. Our aim in this section is to summarize without detailed justification the most important issues related to the nuclear physics of stars. An introduction to stellar evolution can be found in Binney and Merrifield (1998) or Iben (1985). A more comprehensive account is given, for example, in Kippenhahn and Weigert (1990). We will use in this section expressions such as hydrogen burning, helium burning, pp chain, CNO cycle, and so on, to obtain a general idea regarding nuclear processes in stars. These will be explained in depth in Chapter 5.

Theoretical models of stars in hydrostatic equilibrium are constructed in the simplest case by solving a set of four partial differential equations (for radius, luminosity, pressure, and temperature) that describe the structure of a star as a function of the distance from the center and as a function of time. A time sequence of such solutions, or stellar models, represents an *evolutionary track* in the Hertzsprung–Russell diagram. Stellar structure and evolution calculations rely heavily on large scale numerical computer codes. The time changes in the stellar properties are closely related to the energy budget. Energy is generated



Fig. 1.3 Observational Hertzsprung– Russell diagrams, showing visual magnitude versus color index B–V. Each dot corresponds to a star. See the text for an explanation of the labels. (a) Sample of \approx 5000 stars in the solar neighborhood with precisely known distances. The data were acquired by the Hipparcos astrometry satellite. The vast majority of stars occupy the main sequence, stretching diagonally from the hot (blue) and luminous upper left to the cool (red) and faint lower right. The cross hair indicates the position of the Sun. Cer-

tain categories of stars do not appear in the figure, for example, supergiants (SG), which are rare in the solar neighborhood, and brown dwarfs, which are too faint for detection by Hipparcos. (b) Data for the globular cluster M 3. Apparent rather than absolute magnitude is displayed on the vertical axis since the stars have the same distance from the Earth. The RR Lyrae variable stars, located between the red (RHB) and blue (BHB) horizontal branches, are omitted. From Corwin and Carney (2001). by the star via nuclear reactions and gravitational contraction, while energy is continuously lost from the stellar surface via emission of photons and neutrinos. As will become clear in the following discussion, a star spends most of its nuclear burning time fusing hydrogen to helium on the main sequence. Careful observations showed that there is a direct correlation between the mass and the luminosity of a main-sequence star. The greater the total mass of the star, the greater the temperature and pressure in the core, the faster nuclear energy is generated, and the greater the energy output or the luminosity of the star. For example, a 10 M_{\odot} main-sequence star has \approx 3000 times the luminosity of the Sun. Furthermore, the main-sequence lifetime will also depend strongly on the stellar mass because a star burns the nuclear fuel at a rate that is determined by its luminosity. For example, solar-metallicity stars with masses of 1 M_{\odot} , 5 M_{\odot} , and 15 M_{\odot} spend about 10 Gy, 100 My, and 12 My, respectively, on the main sequence. Once a star leaves the main sequence, the evolution speeds up significantly, as will be seen below.

Modern theories have been enormously successful in describing the properties of stars. Nevertheless, many open questions remain unsolved. Stellar evolution is an active research field and it is worthwhile to keep in mind the uncertainties in the model calculations. These reflect our incomplete knowledge of certain processes in stars, including the treatments of energy transport via convection, mass loss, atomic diffusion, turbulent mixing, rotation, and magnetic fields. For binary stars (Section 1.4.4), a host of additional problems is encountered because, first, the model assumption of spherical symmetry must be relaxed and, second, the interaction between the two stars becomes important. We will not discuss these effects in any detail other than to mention that most of them become increasingly important with ongoing stellar evolution. The effects of nuclear physics are deeply interwined with these issues. When we discuss in later chapters the impact of nuclear physics uncertainties on the nuclear energy generation and the nucleosynthesis, it is very important to keep in mind that we are referring only to one piece in a complex puzzle. One of the main goals in nuclear astrophysics is to better understand the inner workings of stars. To this end, a reliable knowledge of nuclear physics is indispensable.

A chart showing the main evolutionary phases for single stars of various initial masses is shown in Fig. 1.4 and will be helpful for the subsequent discussions. The stellar masses are shown on the left-hand side and time increases from left to right.

Premain-sequence stars

When an interstellar gas cloud consisting mainly of hydrogen and helium contracts, gravitational potential energy is transformed into thermal energy and into radiation. The gas is initially in gravitational free fall and most of the lib-

0.013												
0.08	Brown dwarf	D-C										
	Red dwarf	H-C [MS]									He WD	
0.4	Low	н-с рр	H-S ^{1.}		He-C	Н	e-S		3.	-	со	
1.5 -	-mass - star	[MS] CNO	[RGB] U	HeF	H-S [HB,RC]	IAGB]			U	PNN	WD	
2	Inter-	н- с	H-S ^{1.}	He-C H-S		He-S H-S [AGB]			3. D. DNN		CO WD	
4		[MS]	[RGB] U						U			
	mediate	H-C	H-S ^{1.}	He-C		2. H	e-S		3.	-	со	
0	mass	[MS]	[RGB] D U	H-S		U [/	-S \GB]		U	PNN	WD	
9	star	H- C [MS]	H- S ^{1.} [RGB] ^D U	He- C H- S		He-S	C- C 5 He- S [SAGB]	2. D He -S U H -S	3. D U	PNN	ONe WD	
11	Massive	H-C	He-C	C -(C N	e-C	O -C	Si-C	C	;C	BH	
100	star	[MS]	H-S	He 	-s c	-s	Ne -S 	0- S 	SN II/lb/lc		or NS	

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Fig. 1.4 Major evolutionary stages for single stars in different mass ranges. The initial stellar mass is given on the left-hand side. Time increases from left to right. The nuclear fuel in each burning phase is shown in bold. For example, "H-C" refers to hydrogen burning in the core, "He-S" denotes helium burning in a shell, and so on. For lower-mass stars, the meaning of the labels in square brackets is described in the text (see also caption of Fig. 1.5); "DU" de-

notes the different dredge-up events. For massive stars, the three dots indicate that there are additional overlying burning shells (Fig. 1.6); the labels are: "CC" for core collapse, "SN" for supernova, "NS" for neutron star, and "BH" for black hole. Note that the mass ranges are approximate estimates only and depend on the stellar metallicity. For the evolution of stars in the mass range of $M \ge 100 M_{\odot}$, see Woosley, Heger and Weaver (2002), and references therein.

erated energy is not retained but radiated away because the gas is relatively transparent. With increasing density, the opacity increases as well and some of the emitted radiation is retained in the cloud. As a result, the temperature and the pressure begin to rise and the contraction of the central, denser part of the cloud slows down. The increasing temperature causes first a dissociation of hydrogen molecules into atoms, and then an ionization of hydrogen and helium atoms. When a temperature of about 10⁵ K is reached, the gas is essentially ionized. The electrons trap radiation efficiently and, as a result, the pressure and temperature increase and the collapse of the central part of the cloud halts. The premain-sequence star eventually reaches a state of hydrostatic equilibrium, while still accreting matter from the outer parts of the cloud.

The source of energy is gravitational contraction, but the first nuclear reactions start to occur when the central temperature reaches a few million kelvin. Primordial deuterium fuses with hydrogen, a process that is called *deuterium* *burning* (Section 5.1.1), and primordial lithium may be destroyed via interactions with protons (⁷Li + p $\rightarrow \alpha + \alpha$; the notation will be explained in Section 1.5.2). At this stage, energy is transported via convection and most of the star's matter, including surface material, is expected to be processed through the center. Although the nuclear energy release is very small, the reactions change the light element abundances and thus provide valuable information on the central temperatures.

When the temperature reaches several million kelvin, the fusion of hydrogen to helium starts to occur and contributes an increasing fraction to the total energy output. Ultimately, a point will be reached where hydrogen fusion in the core becomes the only source of energy. The star is now in hydrostatic and thermal equilibrium and has reached a location in the Hertzsprung–Russell diagram that is referred to as the *zero age main sequence* (ZAMS). Stars with different initial masses reach the main sequence at different times. For example, the premain-sequence evolution of a $1 M_{\odot}$ star lasts about 75 million years. Different stellar masses populate different locations on the zero age main sequence, which thus represents a line in the Hertzsprung–Russell diagram. Massive stars have higher temperatures, initiate nuclear reactions earlier, and are therefore located on the hotter and brighter part (upper left), while less massive stars will be found on the cooler and fainter part (lower right).

Newly born stars are difficult to observe because they are usually surrounded by a rotating disk of gas and dust. The solar system, for example, presumably formed from such a disk. An example for premain-sequence objects is the *T Tauri stars*. Their lithium abundance is relatively high, indicating that the central temperature has not yet reached large enough values to destroy lithium via nuclear reactions involving protons.

The subsequent fate of stars depends strongly on their initial mass. We will consider the different mass ranges in turn. These main divisions are not sharp but depend somewhat on the chemical composition.

Initial mass of 0.013 $M_{\odot} \lesssim M \lesssim$ 0.08 M_{\odot}

Theory predicts that objects in this mass range never reach the central temperatures required to sustain hydrogen fusion in their cores and are thus unable to generate sufficient nuclear energy to provide pressure support. The search for these very faint and cool stars provides important constraints for stellar evolution theory. Such objects have only been discovered in the mid-1990s and are referred to as *brown dwarfs*. They are predicted to be very abundant in the Galaxy and are, therefore, candidates for the elusive (baryonic) dark matter. Brown dwarfs are fully convective and their energy source in the early stages is provided by gravitational contraction.

Although brown dwarfs are not true stars, they do have enough mass to undergo deuterium burning, a fact that sets them apart from massive planets

like Jupiter. This provides an additional, low-level, source of energy. They also have a relatively high lithium abundance since temperatures remain too low to destroy this element. The outer layers of a brown dwarf can be described by the ideal gas law. The core, however, becomes eventually electron degenerate. As a result, the contraction halts and the brown dwarf slowly cools, at approximately constant radius, by radiating its thermal energy into space. In the Hertzsprung–Russell diagram, a brown dwarf evolves almost vertically downward and straight past the main sequence (Fig. 1.3).

A detailed description of the properties of degenerate matter is given in many modern physics textbooks and is not repeated here. We will summarize a few properties, however, that are also important for our discussion of other stars. Matter becomes degenerate at relatively high densities as a result of the Pauli exclusion principle which states that no more than two spin-1/2 particles (such as electrons) can occupy a given quantum state simultaneously. A degenerate gas strongly resists further compression because electrons cannot move into lower energy levels that are already occupied. Unlike an ideal classical gas, whose pressure is proportional to its temperature, the pressure exerted by a completely degenerate gas does not depend on temperature. Or, in other words, increasing the temperature of a partially degenerate gas has only a small effect on the total pressure. It will be seen later that, when the temperature reaches a sufficiently high value, the degeneracy is lifted, by which we mean that the properties of such a gas revert to those of an ideal classical gas. Furthermore, there exists an upper limit to the pressure provided by a degenerate gas. If gravity exceeds this pressure, the star will collapse despite the presence of the degenerate particles. The maximum value for the mass of a star that can maintain an equilibrium between degeneracy pressure and gravity is called the Chandrasekhar limit. Its precise value depends on the composition. For an electron degenerate gas and matter characterized by two nucleons per electron (for example, ⁴He, ¹²C, or ¹⁶O), the limiting value amounts to $\approx 1.44 M_{\odot}$. Stars that enter a state of electron degeneracy toward the end of their evolution are called white dwarfs. Indeed, white dwarfs with masses in excess of the Chandrasekhar limit are not observed in nature.

Initial mass of 0.08 $M_{\odot} \lesssim M \lesssim$ 0.4 M_{\odot}

Stars in this mass range are sometimes referred to as *red dwarfs* (or M dwarfs). They are the most common type of star in the neighborhood of the Sun. For example, the nearest star to the Sun, Proxima Centauri, is a red dwarf. These stars have sufficient mass to fuse hydrogen to helium (*hydrogen burning*) in their cores via the pp chain. Starting from the zero age main sequence, the red dwarf evolves toward higher luminosity and increasing surface temperature (up and left). All stars that sustain hydrostatic equilibrium by burning hydrogen in their cores are called *main-sequence stars*. Theoretical models indi-

cate that, for example, a $0.1 M_{\odot}$ star of solar metallicity remains on the main sequence for about 6000 Gy. During this time the red dwarf is fully convective, which implies that its entire hydrogen content is available as nuclear fuel. Since the age of the Universe is about 14 Gy, all red dwarfs that we observe must be main-sequence stars. Eventually, they will run out of nuclear fuel, that is, all their hydrogen will be converted to helium. Red dwarfs do not have enough mass to produce the higher temperatures required to fuse helium nuclei. Thus they contract until electron degeneracy sets in. Their volume is constant from then on since the degeneracy pressure resists further compression. They become helium white dwarfs that cool slowly by radiating away their thermal energy.

Initial mass of 0.4 $M_{\odot} \leq M \leq$ 2 M_{\odot}

The evolution of stars in this mass range is considerably more complicated compared to the previous cases. The life of the star starts on the zero age main sequence when hydrogen begins to fuse to helium in the core. In stars with masses below $M \approx 1.5 M_{\odot}$, hydrogen fusion proceeds via the pp chains, while more massive stars burn hydrogen via the CNO cycles. It will be seen later that these different processes affect the stellar structure since they possess very different temperature dependences (Section 5.1). In stars with $M \gtrsim 1.5 M_{\odot}$, the strong temperature dependence of the CNO cycles concentrates the energy production in the center and, as a result, the core transports energy via convection. In stars with $M \lesssim 1.5 M_{\odot}$, the energy generated in the core by the pp chains is transported via radiation.

As an example, we will discuss in the following the evolution of a special star, the Sun (see color Fig. 2 on page 632). The evolutionary track is shown schematically in Fig. 1.5a. The arguments given below follow the numerical results obtained by Sackmann, Boothroyd and Kraemer (1993). The Sun started central hydrogen burning via the pp chains on the zero age main sequence about 4.5 Gy ago. At present the central temperature and density amount to $T \approx 15$ MK and $\rho \approx 150$ g/cm³, respectively, and about one half of the original hydrogen in the core has been consumed so far. The Sun has a very small convective region at the surface, comprising only $\approx 2\%$ of its entire mass. About 4.8 Gy from now, the hydrogen in the core will be exhausted. The Sun will then be located at the bluest and hottest point on the main sequence, called the *turn-off point*. Note that in Fig. 1.5a the track describing nuclear burning on the main sequence in observational Hertzsprung–Russell diagrams represents a band rather than a narrow line.

Hydrogen fusion continues in a thick shell near the core where there is still hydrogen left. The Sun slowly leaves the main sequence at this point. The Sun's center begins to contract in order to generate energy that is no longer

provided by nuclear processes and the contraction causes further heating. As a result, the temperature in the hydrogen burning shell, and the associated nuclear energy generation rate, also increase. Initially, the Sun has not yet developed a fully convective envelope and it is called a subgiant branch star (SGB). Eventually, the envelope becomes fully convective. The extra energy output from the hydrogen burning shell results in a dramatic surface expansion and engulfs the planet Mercury. The Sun becomes a red giant star. While the Sun ascends the red giant branch (RGB), the luminosity increases continuously. Maximum luminosity is achieved on the tip of the red giant branch after about 0.6 Gy from the time when the Sun left the main sequence. During the red giant phase the Sun starts to experience significant mass loss. The contraction of the core during the red giant phase increases the central temperature and density by factors of 10 and 10⁴, respectively, compared to the values at hydrogen ignition. In fact, the core achieves such high densities that the matter becomes electron degenerate. During the RGB phase, the convective envelope deepens significantly until it comprises about 75% of the Sun's mass. This deep convective envelope dredges up the products of hydrogen burning from the outer core. The process is referred to as the "first dredgeup."

When the temperature reaches about $T \approx 0.1$ GK, the helium in the core starts to fuse to carbon and oxygen (*helium burning*). In a normal gas, the extra energy release would cause an expansion. As a result, the temperature would fall and the nuclear energy generation rate would decrease as well. This is the usual manner by which stars adjust to an energy increase in their interior, allowing them to stabilize. However, in a degenerate gas the temperature increase does not affect the pressure. No expansion occurs and, as a result, the temperature increases causing an even higher energy generation rate. As will be seen in Section 5.3, helium burning is highly temperature sensitive. The sequence of events repeats itself, giving rise to a *thermonuclear runaway*. It only terminates after so much energy has been released that the electron degeneracy is lifted. Thus, the ignition of helium in the core results in a violent *core helium flash* (HeF).

It is important to point out that the helium flash does not represent a stellar explosion. The energy during the thermonuclear runaway goes into lifting the electron degeneracy and into the subsequent expansion of the core. The surface luminosity of the star does not increase. In fact, the opposite happens. The surface luminosity declines by two orders of magnitude because the expansion of the core causes the surrounding hydrogen burning shell, which has been supplying all the surface luminosity, to cool and to generate less energy. Eventually, the Sun becomes a *horizontal branch* star, quietly burning helium in the core. The temperatures in the hydrogen shell just above the core are high enough for hydrogen to continue to burn via the CNO cycles. The nuclear energy release in helium fusion is considerably less compared to hydrogen fusion. Therefore, the duration of the core helium burning stage is much shorter than that of the core hydrogen burning stage. The Sun remains on the horizontal branch for about 0.1 Gy, which is typical for all stars in this mass range.

When the helium in the core is exhausted, the core contracts again, heats up, and ignites the helium in a surrounding shell. The Sun now burns nuclear fuel in two shells, helium in a shell surrounding the carbon-oxygen core, and hydrogen in a shell surrounding the helium burning region. The two shells are separated by an intershell region consisting mainly of helium. This stage is referred to as the early asymptotic giant branch phase (E-AGB), because the second ascent of the giant branch merges almost asymptotically with the first giant branch (at least for some stellar masses). While the Sun ascends the asymptotic giant branch, the helium burning shell becomes thermally unstable (Schwarzschild and Härm 1965; see also Section 5.6.1). Energy is not generated at a steady rate, but the hydrogen and helium burning shell alternate as the major contributor to the overall luminosity. For about 90% of the time, the hydrogen burning shell provides the Sun's nuclear energy, while the helium shell is only marginally active. Hydrogen burning adds continuously to the mass of the helium zone, however, so that the temperature and density near this zone rise until energy is generated by helium burning at a rate that is larger than the rate at which it can be carried outward by radiative diffusion. As a result, a thermonuclear runaway occurs. The sudden release of energy pushes out and cools the hydrogen burning shell until it ceases to burn. The helium burning shell is now the only source of nuclear energy. Eventually, the expansion quenches the helium shell flash (or *thermal pulse*) and the Sun contracts again. The hydrogen burning shell reignites and ultimately takes over as the dominant nuclear energy source, until the next thermal pulse occurs about 10⁵ y later. The cycle may repeat many times. This evolutionary stage is called the thermally pulsing asymptotic giant branch (TP-AGB). The total amount of time the Sun spends on the AGB amounts only to about 20 My and is thus very short compared to the main-sequence lifetime. The thermal pulses cause the Sun's radius to vary periodically by a factor of 4, with the peak radius reaching close to the Earth.

The Sun suffers an episode of significant mass loss on the asymptotic giant branch via a strong stellar wind. Thermal pulses are ceasing at this point as the Sun becomes a postasymptotic giant branch star (P-AGB), with only a fraction of its initial mass left and the other part returned to the interstellar medium. As more hydrogen of the envelope is ejected into space, hotter layers are uncovered and the Sun begins to move in the Hertzsprung–Russell diagram toward higher surface temperatures (horizontally to the left). When the surface of the Sun becomes hot enough, the intense ultraviolet radiation ionizes the expanding ejecta, which begin to fluoresce brightly as a *planetary*

nebula (PN). Two examples for planetary nebulae, the Dumbbell Nebula and the Cat's Eye Nebula, are shown in color Figs. 3 and 4 on page 633 and 634, respectively. The residual core is called a *planetary nebula nucleus* (PNN). Eventually, there is no hydrogen envelope left and the hydrogen burning shell extinguishes. The luminosity decreases rapidly causing the evolutionary track to turn downward and slightly to the right. The Sun will then end its existence as a white dwarf with a mass of $\approx 0.5 M_{\odot}$, consisting mainly of carbon and oxygen. It is supported by electron degeneracy pressure and cools slowly by radiating away its thermal energy.

It must be stressed again that in the above discussion the evolution beyond the red giant branch is rather uncertain because of our incomplete knowledge on how to predict convection and mass loss. That these effects will indeed occur has been demonstrated by stellar observations, but a deeper understanding is lacking at present. It is generally accepted that each thermal pulse during the TP-AGB phase provides favorable conditions for another dredge-up episode after the end of flash-burning in the helium shell. The convective envelope reaches deep into the star below the bottom of the hydrogen burning shell and carries the products from hydrogen and helium shell burning, in particular helium and carbon, to the stellar surface. This process is referred to as the "third dredge-up" and increases the carbon abundance in the envelope relative to other elements, for example, oxygen. Stars for which the number ratio of carbon to oxygen exceeds unity are called *carbon stars*. Many of these have been observed and most are believed to correspond to stars in their TP-AGB phase. As will be seen later, AGB stars are also the source of many heavy nuclides with mass numbers beyond A = 60. Stellar models predict that these (s-process) nuclei are also dredged up to the surface where they can be observed in stellar atmospheres. In fact, the first direct evidence that nucleosynthesis takes place in stars and that the products could be mixed to the surface was the observation of radioactive technetium in certain (S-type) carbon stars (Section 1.1). For more information on AGB stars, see Habing and Olofsson (2004).

We are now in a position to understand some other details in the observational Hertzsprung–Russell diagrams shown in Fig. 1.3. The precise location in luminosity and surface temperature of a star on the horizontal branch depends on the chemical composition of the envelope, the size of the helium core at the time of the helium flash, and the mass of the envelope which is influenced by the mass loss during the preceding RGB phase. In a globular cluster, all the stars start out with the same, low-metallicity, composition and their location on the horizontal branch is mainly influenced by mass loss. The more the mass lost from the hydrogen envelope, the hotter the layers in the star are uncovered. Stars with the smallest amount of mass in the hydrogen envelope populate the blue part (BHB), while stars with more hydrogen left in the envelope can be found on the red part (RHB). The horizontal branch intersects the so-called instability strip (which is not related to nuclear burning). Stars located in this narrow and almost vertical band, indicated by the two vertical dashed lines in Fig. 1.5a, are unstable to radial pulsation and are called *RR Lyrae variables*. Their luminosity correlates with both their period (several hours to ≈ 1 day) and their metallicity. Therefore, they are important for determining the distances to globular clusters and for establishing a cosmic distance scale (Binney and Merrifield 1998). Increasing the metallicity has the overall effect of making a star fainter and cooler. Therefore, stars in metal-rich clusters or in the solar neighborhood (Fig. 1.3) accumulate at the red end (right) of the horizontal branch, fairly independent of their envelope mass. This region is called the *red clump* (RC).

The metallicity argument also applies to the subdwarfs (SD). These are in fact main-sequence stars of very low metallicity. They are hotter than solar-metallicity stars at a comparable evolutionary stage and are thus located to the left of the main sequence that is occupied by metal-rich stars.

It should also be clear now why the upper part of the main sequence in Fig. 1.3b is missing. Globular clusters are metal-poor and old, and do not form new stars. The high-mass stars that were originally located on the upper part of the main sequence evolved a long time ago into red giants. Only the slowly evolving low-mass stars are left today on the main sequence. Clearly, with increasing time lower mass stars will eventually become red giants and the main sequence will become shorter. It is interesting that the age of the cluster can be determined from the location of the turn-off point, which is located at the top of the surviving portion of the main sequence. If the distance to the cluster is known by independent means, the luminosity of the stars at the turn-off point can be related to their mass. Stellar evolution models can predict the main-sequence lifetime of stars with a given mass, which must then be nearly equal to the age of the cluster. Such investigations yield ages for the most metal-poor (and presumably oldest) globular clusters of about 12-13 Gy, indicating that these objects formed very early in the history of the Galaxy. This estimate also represents an important lower limit on the age of the Universe (Krauss and Chaboyer 2003).

Initial mass of 2 $M_{\odot} \lesssim M \lesssim$ 11 M_{\odot}

We can divide this mass range into several subranges. Stars with initial masses of $2M_{\odot} \leq M \leq 4M_{\odot}$ evolve obviously faster than less massive stars and their tracks will look quantitatively different from the results shown in Fig. 1.5a. But otherwise they evolve through the same stages as a solar-like star. A major difference, however, arises from the fact that for stars with $M \geq 2M_{\odot}$ the helium core during the RGB phase does not become electron degenerate. Therefore, a helium flash does not occur but instead helium



Fig. 1.5 Schematic evolutionary tracks of (a) the Sun, and (b) massive stars of initial solar composition, in the Hertzsprung-Russell diagram; the luminosity on the vertical axis is given in units of the present solar luminosity. The heavy portions define the locations where major core nuclear burning phases occur. Details of tracks during transitions between major nuclear burning phases are omitted. The meaning of the labels are: main sequence (MS); zero age main sequence (ZAMS); subgiant branch (SGB); red giant branch (RGB); core helium flash (HeF); horizontal branch (HB); early asymptotic giant branch (E-AGB); thermally pulsing asymptotic giant branch

(TP-AGB); post asymptotic giant branch (P-AGB); planetary nebula nucleus (PNN); carbon-oxygen white dwarf (CO-WD). Metal-poor stars in the initial mass range of 0.4 $M_{\odot} \lesssim M \lesssim$ 2 M_{\odot} appear during core helium burning in a region marked by the horizontal dashed line in part (a), depending on the mass loss during the RGB phase. The two dashed diagonal lines indicate the instability strip. In part (b) the core burning phases are labeled by the nuclear fuel: hydrogen (H), helium (He), carbon (C), and so on. The onset of carbon burning is marked by the full circle. Note the vastly different luminosity scale in parts (a) and (b). See the text.

ignites quiescently in the center. Subsequently, these stars make excursions to the left (toward higher temperatures) in the Hertzsprung–Russell diagram and some of them are liable to pass into the instability strip. The observational counterparts of these variable stars are called *classical Cepheids*. They are important for establishing a cosmic distance scale since their observed pulsation period is correlated with their luminosity.

Stars with initial masses of $M \gtrsim 4 M_{\odot}$ experience an additional episode of mixing. Following core helium exhaustion in the core, the structural readjustment to helium shell burning results in a strong expansion, such that the hydrogen burning shell is extinguished as the star begins to ascend the early asymptotic giant branch (E-AGB). At this time the inner edge of the convective envelope penetrates the dormant hydrogen shell, and the products of hydrogen burning are mixed to the surface. This process is referred to as the "second dredge-up." Afterward, the hydrogen shell reignites and the star continues to evolve up the asymptotic giant branch (AGB).

The evolution of stars in the initial mass range of $9 M_{\odot} \leq M \leq 11 M_{\odot}$ is more complicated and less established at present. Models predict a number of important differences compared to the evolution of lower mass stars. We will discuss the evolution of a $10 M_{\odot}$ star with initial solar composition as an example (Ritossa, García-Berro and Iben 1996). The star starts out by burning hydrogen in the core via the CNO cycles for about 10 million years. Following the exhaustion of hydrogen in its center, the star evolves toward the red giant branch where eventually the first dredge-up event occurs. Helium burning starts in the core under nondegenerate conditions and lasts for about 270,000 years. After helium exhaustion, the core contracts and heats up, and the outer layers of the star expand. Thereafter, the hydrogen burning shell extinguishes, while helium continues to burn in a shell surrounding a partially electron degenerate carbon-oxygen core. Eventually, the core becomes sufficiently hot for the fusion of carbon nuclei (carbon burning). When carbon ignites, the star enters the super asymptotic giant branch (SAGB). Carbon burning starts with a thermonuclear runaway (carbon flash) and the energy generation rate from carbon fusion increases greatly. The energy release causes the overlying layers to expand, giving rise to a reduction in the helium shell burning energy generation rate. After a relaxation period, the helium burning shell returns to its prior energy output. Several of these flashes occur over the carbon burning lifetime, which lasts for about 20,000 years. When carbon is exhausted in the center, the electron degenerate core consists mainly of oxygen and neon. After carbon burning extinguishes, the second dredge-up event occurs. Subsequently, the dormant hydrogen shell on top of the helium burning shell is reactivated and a complicated interplay between these two burning shells gives rise to thermal pulses which are driven by helium shell flashes. During this time, the third dredge-up event occurs. Eventually, the hydrogen-rich surface

is removed by a strong stellar wind and the star becomes the central object of a planetary nebula. It ends its existence as a oxygen–neon white dwarf with a mass of $\approx 1.2 M_{\odot}$.

Initial mass of $M\gtrsim$ 11 M_{\odot}

The evolution of stars in this mass range is in many ways fundamentally different compared to our earlier discussion. Schematic evolutionary tracks for $13 M_{\odot}$, $15 M_{\odot}$, $20 M_{\odot}$, and $25 M_{\odot}$ stars are shown in Fig. 1.5b. The case of a $25 M_{\odot}$ star with initial solar composition will be discussed in the following as an example (Chieffi, Limongi and Straniero 1998; Limongi, Straniero and Chieffi 2000; Woosley, Heger and Weaver 2002). The total life of such a massive stars is relatively short and amounts only to ≈ 7 My. The star spends 90% of this time on the main-sequence burning hydrogen to helium via the CNO cycles in the core. When the hydrogen in the center is exhausted, hydrogen burning continues in a shell. The core contracts and heats up until helium is ignited. This new source of nuclear energy heats the overlying hydrogen shell and the outer layers of the star expand greatly. The star becomes a supergiant. These stars show up in the Hertzsprung–Russell diagram at the highest observed luminosities. Examples are Rigel (blue supergiant) and Betelgeuse (red supergiant) in the constellation Orion.

Core helium burning lasts for about 800,000 years and some of the heavy nuclides with masses of A > 60 are synthesized during this stage via neutron captures (s-process; Section 5.6.1). When helium is exhausted in the center, helium burning continues in a shell located beneath the hydrogen burning shell. Eventually, carbon burning starts in the core. These burning stages have already been discussed above.

Stars with initial masses exceeding $\approx 11 M_{\odot}$ are capable of igniting successive burning stages in their cores using the ashes of the previous core burning stage as fuel. Three distinct burning stages follow carbon burning. They are referred to as *neon burning*, oxygen burning, and silicon burning, and will be discussed in detail in Section 5.5. There is a fundamental difference between the initial and the advanced burning stages in the manner by which the nuclear energy generated in the stellar interior is transformed and radiated from the surface. For hydrogen and helium burning, nuclear energy is almost exclusively converted to light. During the advanced burning stages energy is almost entirely radiated as neutrino-antineutrino pairs and the light radiated from the star's surface represents only a very small fraction of the total energy release. Since the neutrino losses increase dramatically during the advanced burning stages and because the nuclear burning lifetime scales inversely with the total luminosity, the evolution of the star rapidly accelerates. For example, silicon burning will last for only about 1 day (Chapter 5). Since the advanced burning stages transpire very quickly, the envelope has insufficient time to re-


Fig. 1.6 Schematic structure of a presupernova star (not to scale). The upper-left side shows the one or two most abundant nuclear species in each region (according to Limongi, Straniero and Chieffi 2000). Nuclear reactions are very temperature dependent. Thus the nuclear burning takes

place in relatively thin shells at the interface between layers of different composition. The nuclear burning shells are labeled on the lower-left side; for example, "H-B" stands for hydrogen burning. This model is sometimes referred to as the "onion shell structure" of a massive star.

act to the structural changes in the stellar interior. Thus, from carbon burning onward, the star will no longer move in the Hertzsprung–Russell diagram, but remains at the position indicated by the solid circle in Fig. 1.5b. Furthermore, since the star spends most of its life burning either hydrogen or helium in the core, these are typically the only phases that we can observe.

The approximate structure of the massive star after the silicon has been exhausted in the core is shown in Fig. 1.6. The star consists now of several layers of different composition that are separated by thin nuclear burning shells. The details of the nucleosynthesis are complicated and will be discussed in Chapter 5. It is sufficient to mention at this point that the heaviest and most stable nuclei (that is, the iron peak nuclei; Section 1.3) are found in the core. In fact, the most abundant nuclide in the core is ⁵⁶Fe. It should also be noted that the luminosity during the red giant phase is so large that the star undergoes a significant mass loss. The effect is more pronounced for stars with $M \ge 30-35 M_{\odot}$ that lose eventually most of their hydrogen envelope. The observational counterparts of such stars are the hot and massive *Wolf–Rayet stars,* which have been observed to lose mass at a rate of $\approx 10^{-5} M_{\odot}$ per year at stellar wind speeds of $\approx 2000 \text{ km/s}$. An image of a Wolf–Rayet star is shown in color Fig. 5 on page 635.

The electron degenerate stellar core has at this point no other sources of nuclear energy to its disposal and grows in mass as the overlying burning shells contribute nuclear ashes. When the mass of the core exceeds the Chandrasekhar limit ($\approx 1.4 M_{\odot}$), the electron degeneracy pressure is unable to counteract gravity, and the core collapses. The core collapse is accelerated by two important effects. First, as the electron density increases, electrons capture onto iron peak nuclei (Section 1.8.4). This removes electrons that were contributing to the pressure. Second, at temperatures of ≈ 5 GK, the thermal radiation becomes sufficiently energetic and intense that the iron peak nuclei are photodisintegrated into lighter and less stable nuclei. This process removes energy that could have provided pressure. At this stage the core of the star is essentially collapsing in free fall. When the density reaches values on the order of the nuclear density ($\approx 10^{14} \text{ g/cm}^3$), the nuclei and free nucleons begin to feel the short-range nuclear force, which is repulsive at very short distances. The inner collapsing core reaches high inward velocities and overshoots the nuclear density. The nuclear potential acts as a stiff spring that stores energy in the compressive phase until it rebounds. The rebounding part of the core encounters infalling matter and thus gives rise to an outward moving shock wave. The very hot and dense inner core has become a protoneutron star with a mass of $\approx 1.5 M_{\odot}$.

While the shock wave moves outward through the outer core region, it loses energy by photodisintegrating the iron peak nuclei. Furthermore, energy is removed from the shock wave by the emission of neutrinos. It takes about 1 s after core collapse, and about 10 ms after the core has bounced, for the shock wave to reach the outer edge of the core. At this time the shock wave has lost all of its kinetic energy and it stalls. How exactly the shock is revived and how it will ultimately propagate through the stellar layers beyond the iron core and disrupt the star in a core collapse supernova explosion is still unknown. The stalled shock wave is thought to be revived by the neutrinos and antineutrinos that emerge from the hot and dense proto-neutron star, a fraction of which is absorbed by protons and neutrons behind the shock (Bethe and Wilson 1985).

Once the shock wave is revived by the neutrino energy deposition, it propagates outward beyond the iron core and compresses and heats each of the overlying shells of the star. Some of the shells experience, after hydrostatic burning prior to core collapse, another episode of nucleosynthesis which proceeds on timescales of a few seconds and is called *explosive nuclear burning*. The silicon (²⁸Si) and oxygen (¹⁶O) in the first layers that the shock wave encounters (Fig. 1.6) are quickly converted to iron peak nuclei at high temperatures (≈ 5 GK). It will be shown in Section 5.5.5 that under such conditions the most abundant product nuclide originating from these layers is ⁵⁶₂₈Ni₂₈. By the time the shock wave reaches the other layers of the star, the temperatures achieved are much smaller and hence these are ejected into space with less nuclear processing. Nevertheless, some species are predominantly made in these layers, among them the ²⁶Al observed in the interstellar medium (Section 1.7.5). The deepest regions that are ejected are characterized by a large abundance of free neutrons. These possibly give rise to the nucleosynthesis of many heavy nuclei in the A > 60 mass range via neutron capture (r-process; Section 5.6.2).

The above scenario for the core collapse of a massive star is responsible for supernovae of types II and Ib/Ic. It must be stressed that the explosion mechanism is far from understood at present. It is also not clear if a neutron star or, after fallback of material onto the core, a black hole is left behind as the central remnant. These issues are the subject of active current research. In many respects, however, current models of core collapse supernovae agree with observation. In particular, observations of supernova 1987A, which exploded in the Large Magellanic Cloud in 1987, were of outstanding importance in this respect (see color Fig. 6 on page 636). Since it was located so close to us, the event could be studied in much greater detail than any other supernova. For example, a burst of neutrinos had long been predicted by theory and was indeed detected in this event (Section 1.4.1). Furthermore, current models correctly predict the amount of the ejected radioactive ⁵⁶Ni which, after decay first to ⁵⁶Co and then to stable ⁵⁶Fe, gives rise to the tail in the light curves of core collapse supernovae. A famous type II supernova remnant, the Crab Nebula, is shown in color Fig. 7 on page 637.

The association of massive stars with supernovae of type II and type Ib/Ic was made some time ago. The different supernova types are classified observationally according to their spectra. Spectra of type II supernovae contain hydrogen lines, while those of type I supernovae do not. Type I supernovae whose spectra show absorption caused by the presence of silicon are referred to as type Ia supernovae; otherwise they are classified as type Ib or Ic supernovae (the latter distinction is based on a helium line feature in the spectrum). Type II supernovae tend to occur in the arms of spiral galaxies, but not in early-type galaxies. Type Ib or Ic supernovae also seem to occur in spiral arms. On the other hand, type Ia supernovae show no such preference. Since the spiral arms contain many massive (and thus young) stars and early-type galaxies do not contain such objects, the observations suggest that massive stars are the progenitors of type II and type Ib/Ic supernovae, but not of type Ia supernovae. Type Ib/Ic supernovae are thought to result from the core collapse of Wolf-Rayet stars that lost their hydrogen envelope to a strong stellar wind or to a companion star before the explosion. The supernova rate in our Galaxy amounts to about two events per century. Most of them are predicted to be type II and type Ib/Ic supernovae, while the contribution from type Ia supernovae amounts only to $\approx 15\%$. The latter objects will be discussed below. For more information on these issues, and the related topic of the evolution of stars with $M \ge 100 M_{\odot}$, see Woosley, Heger and Weaver (2002).

1.4.4 Binary Stars

Perhaps as many as one half of all stars are members of binary star systems according to recent statistics. If the stars are members of a close binary system, then they will significantly influence each other's evolution. In a close binary system, the separation may range from a few times the radii of the stars to a situation where both stars share a common envelope (contact binaries). Consider the binary star system shown in Fig. 1.7. Each star is surrounded by a hypothetical surface marking its gravitational domain. This surface is referred to as the *Roche lobe* and its intersection with the equatorial plane is shown as a dashed figure-eight curve. The location where the two Roche lobes touch (that is, where the effects of gravity and rotation cancel each other) is called the *inner Lagrangian point*. When one of the stars evolves off the main sequence and becomes a red giant, it may fill its Roche lobe. Material is then free to flow from that star through the inner Lagrangian point onto its companion. Many different kind of stars may be members of close binary systems and the transfer of mass from one star to another gives rise to very interesting phenomena (Iben 1991). In the following we will focus on binary systems that contain a compact object, either a white dwarf or a neutron star.

Type la supernovae

Type Ia supernovae are among the most energetic stellar explosions in the Universe. They sometimes even outshine their host galaxies. An image of the type Ia supernova 1994D is shown in color Fig. 8 on page 638. Their light curves—which are powered by the decay of radioactive ⁵⁶Ni—and spectra are in general homogeneous. However, there are important differences. For example, the spread in peak luminosity among type Ia supernovae amounts to a factor of \approx 15. It turns out that the peak luminosity is correlated with the rate of brightness decline (Phillips 1993). Since this correlation can be used to compensate for the peak luminosity spread, type Ia supernovae are important candidates for establishing a cosmological distance scale (see below). There are other important differences, such as a spread in the expansion velocity at the photospheres even for similarly bright events, that support the conclusion that type Ia supernovae represent a class of a certain diversity (Leibundgut 2000).

A detailed understanding of type Ia supernovae is still lacking. Many different models have been proposed to explain these events. It is also not clear if a single model can account for all observations. We will focus here on one of the most popular models that may describe at least the majority of type Ia supernovae.

The favored scenario involves a carbon–oxygen white dwarf in a close binary star system that accretes matter via Roche lobe overflow from a companion main-sequence or red giant star. The rate of mass accretion must be relatively large ($\approx 10^{-7} M_{\odot}$ per year) in order to avoid mass loss through a nova-like event (see below). When the white dwarf grows to a critical mass near the Chandrasekhar limit, carbon ignites under degenerate conditions and a thermonuclear runaway occurs (Section 1.4.3). The energy release from the nuclear burning ($\approx 10^{44}$ J) is so large that it disrupts the white dwarf at high velocity within a time scale of seconds. A significant fraction of the initial carbon and oxygen is consumed and, in general, neither a neutron star nor a black hole is left behind in the explosion. For SN 1572 (Tycho's supernova; see color Fig. 9 on page 639) the likely companion has been identified as a solar-like star, supporting the above scenario (Ruiz-Lapuente et al. 2004).

The nucleosynthesis depends on the temperatures and densities achieved in different layers of matter. In the hottest and densest regions, the explosion converts most of the matter to radioactive ⁵⁶Ni (via nuclear statistical equilibrium at low neutron excess; see Section 5.5.5). The decay of this nuclide, and the subsequent decay of the daughter nucleus ⁵⁶Co, then gives rise to the observable emission of type Ia supernovae. In other words, the amount of ⁵⁶Ni synthesized determines the absolute brightness of the event. The outer regions that attain smaller temperatures and densities may undergo explosive silicon or oxygen burning and give rise to the production of intermediate-mass nuclei. Elements from oxygen to calcium are indeed observed in the spectral evolution during the peak phase of type Ia supernovae.

An important unresolved issue is related to the propagation of the thermonuclear burning front. Two burning modes can be distinguished. One possibility is a detonation in which the nuclear flame propagates as a supersonic front. In this case, the flame compresses the material and increases the temperature to the point of ignition. The energy release from the ignited material behind the flame supports its propagation. Another possibility is a deflagration in which the nuclear burning proceeds subsonically. Here, the energy release from the burning material heats the next layer and ignites it. These two modes are not exclusive and a transition from one mode to another may occur during the explosion. Related to this issue is the question of where precisely (near or off center) and at how many locations the ignition occurs.

Type Ia supernovae are fascinating objects in their own right, but a deeper understanding of the explosion is also important for cosmology. Their light curves are relatively homogeneous, that is, their intrinsic brightness is known to within some range. By measuring their apparent luminosity it becomes hence possible to estimate their distance. Furthermore, since type Ia supernovae are so bright they can be observed across billions of light years. For these reasons, type Ia supernovae are used as "standard candles" for establishing cosmological distances. By recording both their apparent luminosity and their redshifts, observations of very distant type Ia supernovae provide a

measure for the expansion history of the Universe. It is found that the expansion is accelerating, driven by the elusive dark energy (Riess et al. 1998, Perlmutter et al. 1999). The profound cosmological implications provide strong motivation for improving models of type Ia supernovae. For more information on the stellar models, see Höflich (2006), and references therein.

Classical novae

Classical novae are stellar explosions that occur in close binary systems. In this case, hydrogen-rich matter is transferred via Roche lobe overflow from a low-mass main-sequence star to the surface of a compact white dwarf. The transferred matter does not fall directly onto the surface but is accumulated in an accretion disk surrounding the white dwarf. Typical accretion rates amount to $\approx 10^{-10}$ – $10^{-9} M_{\odot}$ per year. A fraction of this matter spirals inward and accumulates on the white dwarf surface, where it is heated and compressed by the strong surface gravity. At some point, the bottom layer becomes electron degenerate. Hydrogen starts to fuse to helium (via the pp chains) during the accretion phase and the temperature increases gradually. The electron degeneracy prevents an expansion of the envelope and eventually a thermonuclear runaway occurs near the base of the accreted layers. At this stage the nuclear burning is dominated by explosive hydrogen burning via the (hot) CNO cycles. Both the compressional heating and the energy release from the nuclear burning heat the accreted material until an explosion occurs.

The classical nova rate in the Galaxy is about ≈ 35 per year and thus they occur much more frequently than supernovae (Section 1.4.3). Contrary to type Ia supernovae, which disrupt the white dwarf, all classical novae are expected to recur with periods of $\approx 10^4$ – 10^5 years. The luminosity increase during the outburst amounts to a factor of $\approx 10^4$. A classical nova typically ejects $\approx 10^{-5}$ – $10^{-4} M_{\odot}$ of material, with mean ejection velocities of $\approx 10^3$ km/s. Note that there are other types of novae, such as dwarf novae or nova-like variables. However, these are not related to thermonuclear burning.

Optical, infrared, and ultraviolet spectra of classical novae reveal the presence of many elements in the expanding nova shells that are strongly overabundant compared to solar system values. For example, the observed overabundances of carbon and oxygen in all classical novae demonstrate that at some time during the evolution of the outburst the accreted material must have been mixed to a certain degree with matter from the white dwarf. This dredge-up of material, in fact, gives rise to a more energetic explosion (by increasing the number of CNO catalyst nuclei; Section 5.2). The observation of an overabundance of neon in some classical novae showed that these outbursts do not involve a carbon-oxygen white dwarf, but a more massive white dwarf of oxygen-neon composition. The latter objects result from the evolution of intermediate mass stars with initial masses of $9 M_{\odot} \leq M \leq 11 M_{\odot}$

(Fig. 1.4). The presence of large amounts of matter from the white dwarf core in the ejecta may imply that the white dwarf in a classical nova system is losing mass as a result of subsequent outbursts. Thus these objects are unlikely to become progenitors of type Ia supernovae. Other observed overabundances, for example, of nitrogen, silicon, or sulfur, are the result of nuclear processing during the explosive burning of hydrogen. An image of Nova Cygni 1992 is shown in color Fig. 10 on page 640.

Stellar model calculations indicate that the peak phase of explosive nuclear burning in classical novae lasts typically for several hundred seconds. The characteristics of the outburst depend on the white dwarf mass and luminosity, the mass accretion rate, and the chemical composition for both the accreted and the white dwarf material. For example, it has been demonstrated that the lower the mass accretion rate, the larger the amount of accreted mass before the thermonuclear runaway is initiated. A more massive accreted layer, in turn, gives rise to a higher pressure in the bottom layers and hence a more violent explosion. On the other hand, if a too large accretion rate is assumed, no thermonuclear runaway is initiated. Simulations also indicate that classical nova outbursts on the surface of the heavier oxygen–neon white dwarfs achieve higher peak temperatures than those exploding on carbon–oxygen cores. For more information on classical novae, see José, Hernanz and Iliadis (2006) and Starrfield, Hix and Iliadis (2006).

Type I X-ray bursts

A number of close binary star systems involve a neutron star as a compact object. A neutron star has a mass of $\approx 1.4 M_{\odot}$, a radius of about 10–15 km, and a density on the order of 10^{14} g/cm³. These binary star systems belong to a class of objects that are called X-ray binaries. The accretion of matter from the companion on the surface of the neutron star gives rise to a large gravitational energy release. As a result, the temperatures near the neutron star surface are high ($\approx 10^7$ K) and the persistent thermal emission occurs at X-ray energies.

In high-mass X-ray binaries, the companion is a massive ($\geq 5 M_{\odot}$) population I star, while the neutron star has a strong magnetic field. The matter is accreted at relatively high rates and is funneled along the magnetic field lines onto the magnetic poles. This creates a hot spot of X-ray emission and, if the rotational axis of the neutron star is inclined with respect to the magnetic axis, this gives rise to an X-ray pulsar. Typical rotation periods range from 0.1 s to a fraction of an hour. The rotational periods for some X-ray pulsars have been observed to decrease, indicating that the neutron stars spin up as a result of accretion of matter.

In low-mass X-ray binaries, the companion is a low mass ($\leq 1.5 M_{\odot}$) population II star and matter is transferred to a weakly magnetized neutron star via Roche lobe overflow. Many of these systems produce, apart from the per-

sistent X-ray emission, bursts in the X-ray intensity (Lewin, van Paradijs and Taam 1993). In a rare variety, called type II X-ray bursts, the bursts occur in rapid succession and are separated by a few minutes. The profile of each burst rises and falls abruptly. They are most likely associated with a sudden increase in the mass transfer rate caused by instabilities in the accretion disk.

The large majority of bursts belong to the class of type I X-ray bursts. In this case, the X-ray luminosity typically increases by an order of magnitude. They are believed to be of thermonuclear origin, unlike the X-ray binary varieties discussed above. When hydrogen- and helium-rich matter from the low-mass companion is first accreted in a disk and then falls onto the surface of the neutron star, the temperatures and densities are high enough to fuse hydrogen continuously to helium via the (hot) CNO cycles. The accreted or synthesized helium, however, is not fusing yet but sinks deeper into the neutron star atmosphere. Eventually the helium is ignited via the triple- α reaction under electron degenerate conditions and a thermonuclear runaway occurs. The helium flash triggers the explosive burning of the outer region consisting of a mixture of hydrogen and helium. This is just one possible scenario. In other models the ignition occurs in pure helium or in mixed hydrogen-helium accreted material. The details of the nucleosynthesis depend on the temperatures and densities achieved in the various burning layers. Calculations show that in the innermost and hottest layers elements up to-and perhaps beyond-the iron peak are synthesized. After the termination of a burst, a new shell of matter is accreted and the cycle repeats.

The above model explains the basic features of type I X-ray bursts. A burst lasts typically for less than 1 min and repeats after several hours to days. The luminosity profile shows a rapid rise within \approx 1–10 s, caused by the sudden nuclear energy release, and a slower decline on the order of \approx 5–100 s, reflecting the cooling of the neutron star surface. Some bursts show millisecond oscillations of the X-ray flux. These have been suggested to arise from a surface wave in the nuclear burning layer or perhaps from anisotropies in the nuclear burning caused by a spreading hot spot on the surface of a rapidly spinning neutron star.

Stellar models of type I X-ray bursts are sensitive to a number of parameters and assumptions, such as the mass accretion rate, rotation, the number of ignition points, the propagation of the burning front across the neutron star surface, and the composition of the accreted matter.

It is unlikely for any significant amount of accreted and processed matter to escape the large gravitational potential of the neutron star. Therefore, type I X-ray bursts are probably not important contributors to the chemical evolution of the Galaxy. They are important, however, for probing the properties of neutron stars, such as the mass, radius, and the composition. For more information, see Schatz and Rehm (2006) and references therein.



Fig. 1.7 Binary star system. Each star is surrounded by a hypothetical surface, called the Roche lobe, that marks its gravitational domain. The intersection of the equatorial plane with the Roche lobes is shown as a dashed curve. The location where the two Roche lobes touch is called the inner Lagrangian point. See the text.

1.5

Masses, Binding Energies, Nuclear Reactions, and Related Topics

1.5.1

Nuclear Mass and Binding Energy

The most fundamental property of the atomic nucleus is its mass. Early mass measurements showed that the total nuclear mass, m_{nuc} , is less than the sum of masses of the constituent nucleons. We may write

$$m_{\rm nuc} = Zm_{\rm p} + Nm_{\rm n} - \Delta m \tag{1.1}$$

According to the Einstein relationship between mass and energy, the *mass de*fect Δm is equivalent to an energy of $\Delta E = \Delta m \cdot c^2$. The quantity ΔE is referred to as *nuclear binding energy*. It is defined as the energy released in assembling a given nucleus from its constituent nucleons, or equivalently, the energy required to separate a given nucleus into its constituent nucleons. We may express the binding energy as

$$B(Z, N) = (Zm_{\rm p} + Nm_{\rm n} - m_{\rm nuc}) c^2$$
(1.2)

A plot of measured binding energies per nucleon, B(Z, N)/A, of the most stable isotope for each mass number *A* is shown in Fig. 1.8. Most of these

nuclides (which are stable in the laboratory) have binding energies between 7 and 9 MeV per nucleon. Nuclides with mass numbers in the range of A =50–65 have the largest binding energies per nucleon. They are the iron peak species which we already encountered in Section 1.3. It appears that nature favors the abundances of the most tightly bound and most stable nuclides, as will be explained in detail in later chapters. The most tightly bound nuclides of all are ⁶²Ni, ⁵⁸Fe, and ⁵⁶Fe with binding energies per nucleon of B(Z, N)/A= 8794.549 ± 0.010 keV, 8792.221 ± 0.012 keV, and 8790.323 ± 0.012 keV, respectively (Audi, Wapstra and Thibault 2003). Lighter or heavier nuclei are less tightly bound. It follows that nuclear processes liberate energy as long as the binding energy per nucleon of the final product(s) exceeds the binding energy per nucleon of the initial constituents. Consequently, nuclear energy can be liberated by the fusion of nuclei lighter than iron, or by the fission of nuclei heavier than iron. For example, if a star consists initially of pure hydrogen (¹H), an energy of \approx 7 MeV per nucleon can be liberated by fusing hydrogen to helium (⁴He), or more than 8 MeV per nucleon is liberated by fusing hydrogen to ⁵⁶Fe.

Example 1.2

The binding energies per nucleon of deuterium (²H or d) and helium (⁴He or α) are given by B(d)/A = 1.112 MeV and $B(\alpha)/A = 7.074$ MeV. Calculate the energy released when two deuterium nuclei are combined to form one ⁴He nucleus.

First, we calculate the binding energies of deuterium and ⁴He:

$$B(d) = \frac{B(d)}{A}A = (1.112 \text{ MeV}) \cdot 2 = 2.224 \text{ MeV}$$
$$B(\alpha) = \frac{B(\alpha)}{A}A = (7.074 \text{ MeV}) \cdot 4 = 28.296 \text{ MeV}$$

By combining two deuterium nuclei to one ⁴He nucleus, the total energy release amounts to

 $(28.296 \,\mathrm{MeV}) - (2.224 \,\mathrm{MeV}) - (2.224 \,\mathrm{MeV}) = 23.85 \,\mathrm{MeV}$

corresponding to a value of 5.96 MeV per nucleon.

1.5.2

Energetics of Nuclear Reactions

A nuclear interaction may be written symbolically as

$$0 + 1 \rightarrow 2 + 3$$
 or $0(1,2)3$ (1.3)



Fig. 1.8 (a) Experimental binding energies per nucleon, B(Z, N)/A, of the most stable nuclide for each mass number A. (b) Expanded section showing the region of the iron peak. The nuclides with the largest binding energies per nucleon are ⁶²Ni, ⁵⁸Fe, and ⁵⁶Fe. Data from Audi, Wapstra and Thibault (2003).

where 0 and 1 denote two colliding nuclei before the interaction, while 2 and 3 denote the interaction products. Most nuclear interactions of astrophysical interest involve just two species before and after the interaction. If species 0 and 1 are identical to species 2 and 3, then the interaction is called *elastic* or *inelastic* scattering. Otherwise, the above notation refers to a *nuclear reaction*. Photons may also be involved in the interaction. If species 2 is a photon, then

the interaction is called *radiative capture reaction*. If species 1 is a photon, then we are considering a *photodisintegration reaction*. All of these interactions will be discussed in later chapters.

Figure 1.9 shows schematically the energetics of nuclear reactions and can be used to illustrate a number of relationships that will be employed frequently in the following chapters. The vertical direction represents energy. Consider first part (a), showing a reaction $0 + 1 \rightarrow 2 + 3$, where all species involved in the interaction are particles with rest mass. The rest masses of 0 and 1 (before the reaction) and of 2 and 3 (after the reaction) are indicated by horizontal solid lines. The total relativistic energy in a nuclear reaction must be conserved. Thus, one may write

$$m_0c^2 + m_1c^2 + E_0 + E_1 = m_2c^2 + m_3c^2 + E_2 + E_3 \quad \text{or}$$

$$Q_{01\to23} \equiv m_0c^2 + m_1c^2 - m_2c^2 - m_3c^2 = E_2 + E_3 - E_0 - E_1 \quad (1.4)$$

where E_i are kinetic energies and m_i are rest masses. The difference in masses before and after the reaction, or the difference in kinetic energies after and before the reaction, is equal to the energy release and is referred to as the *reaction Q-value*. If *Q* is positive, the reaction releases energy and is called *exothermic*. Otherwise the reaction consumes energy and is called *endothermic*. Apart from a few exceptions, the most important nuclear reactions in stars are exothermic (Q > 0). Note that Eq. (1.4) is applicable in any reference frame. The difference between center-of-mass and laboratory reference frame is discussed in Appendix C. The quantities E_{01} and E_{23} in Fig. 1.9a represent the total kinetic energies in the center-of-mass system before and after the reaction, respectively. It is apparent that the center-of-mass kinetic energies and the *Q*-value are related by

$$E_{23} = E_{01} + Q_{01 \to 23} \tag{1.5}$$

Part (b) shows a radiative capture reaction $0 + 1 \rightarrow \gamma + 3$. In this case we find accordingly

$$m_0 c^2 + m_1 c^2 + E_0 + E_1 = m_3 c^2 + E_3 + E_\gamma \quad \text{or} Q_{01 \to \gamma 3} \equiv m_0 c^2 + m_1 c^2 - m_3 c^2 = E_3 + E_\gamma - E_0 - E_1$$
(1.6)

Center-of-mass kinetic energies and the Q-value are now related by

$$E_{\gamma 3} = E_{01} + Q_{01 \to \gamma 3} \tag{1.7}$$

where $E_{\gamma3}$ denotes the sum of the energy of the emitted photon (E_{γ}) and the center-of-mass kinetic energy of the recoil nucleus 3. The latter contribution is usually very small so that one can frequently set $E_{\gamma3} \approx E_{\gamma}$ (see Appendix C).

The reaction *Q*-value for a radiative capture reaction is equal to the energy released when nuclei 0 and 1 combine to form a composite nucleus 3. If one

would add this very same amount of energy to nucleus 3, then it becomes energetically possible for nucleus 3 to separate into the fragments 0 and 1. Thus, the *particle separation energy* of nucleus 3 is equal to the Q-value of the reaction $0 + 1 \rightarrow \gamma + 3$, that is, $S_{3\rightarrow 01} = Q_{01\rightarrow\gamma3}$. Separation energies will be used frequently in the following chapters. Their values depend on the nuclear properties of species 0, 1, and 3. For example, suppose we start out with a stable nucleus in Fig. 1.1 and remove one neutron at a time. As a result, we move in the chart of the nuclides to the left toward increasingly proton-rich nuclei. The farther we move away from the group of stable nuclei, the larger the proton-neutron imbalance becomes, and the less energy is required to remove a proton from a given nucleus. In other words, the proton separation energy S_p decreases. After a certain number of neutrons have been removed, a nuclide is eventually reached for which S_p becomes negative. Such nuclides are called proton unstable since they decay via the emission of a proton. The line in the chart of the nuclides with $S_p = 0$ (on the proton-rich side) is referred to as proton dripline. Similarly, if we remove from a given stable nucleus protons instead of neutrons, then we would move in the chart of the nuclides downward. The neutron-proton imbalance increases while the neutron separation energy S_n decreases with each removal of a proton. The line with $S_n = 0$ (on the neutron-rich side) defines now the *neutron dripline*. Particle driplines play an important role in certain stellar explosions (Chapter 5).

1.5.3

Atomic Mass and Mass Excess

Direct measurements of nuclear masses are complicated by the presence of the atomic electrons. Atomic masses, on the other hand, can be measured with very high precision. For this reason, experimental mass evaluations tabulate atomic rather than nuclear masses. These quantities are related by

$$m_{\text{atom}}(A, Z) = m_{\text{nuc}}(A, Z) + Zm_{\text{e}} - B_{\text{e}}(Z)$$
 (1.8)

where m_e and B_e denote the electron mass and the electron binding energy in the atom, respectively. Nuclear reactions conserve the total charge. Therefore, one may replace nuclear by atomic masses since the same number of electron rest masses is added on both sides of a reaction equation. An error is introduced by this approximation because of the difference in the electron binding energies in the atom. However, this contribution is very small compared to the nuclear mass differences and can usually be neglected. In the following we will be using atomic rather than nuclear masses, unless noted otherwise.

Frequently, a quantity called *atomic mass excess* (in units of energy) is introduced, which is defined by

$$M.E. \equiv (m_{atom} - Am_u)c^2 \tag{1.9}$$





Fig. 1.9 Energy level diagrams to illustrate the energetics of nuclear reactions. The vertical direction represents an energy scale. Part (a) corresponds to a situation where all species participating in the reaction are particles with rest mass. In part (b) one of the species is a photon. See the text.

where the integer *A* is the mass number. The quantity m_u denotes the (unified) atomic mass unit, u, which is defined as one-twelfth of the mass of the neutral ¹²C atom. Numerically, one finds $m_u c^2 = 931.494$ MeV. The *Q*-value for a reaction $0 + 1 \rightarrow 2 + 3$ can be expressed in terms of the mass excess as

$$Q = m_0 c^2 + m_1 c^2 - m_2 c^2 - m_3 c^2$$

= $(m_0 c^2 + m_1 c^2 - m_2 c^2 - m_3 c^2) + (A_2 m_u c^2 + A_3 m_u c^2 - A_0 m_u c^2 - A_1 m_u c^2)$
= $(M.E.)_0 + (M.E.)_1 - (M.E.)_2 - (M.E.)_3$ (1.10)

Clearly, using atomic masses or atomic mass excesses gives precisely the same result when calculating reaction Q-values. If positrons are involved in a reaction, then the Q-value obtained by using atomic masses (or atomic mass excesses) includes the annihilation energy $2m_ec^2 = 1022$ keV of the positron with another electron from the environment, as will be shown below. In numerical expressions, we will frequently be using the quantity

$$M_i = \frac{m_i}{m_u} \tag{1.11}$$

which is called *relative atomic mass* of species *i* and is given in atomic mass units, u. The relative atomic mass for a given species is numerically close to its (integer) mass number, but for accurate work the former quantity should be used. An evaluation of atomic masses is presented in Audi, Wapstra and Thibault (2003). Mass measurement techniques and various theoretical mass models are reviewed in Lunney, Pearson and Thibault (2003).

Experimental values for atomic mass excesses, binding energies, and relative atomic masses for the light nuclides are listed in Table 1.1. Note that $(M.E.)_{12_C} \equiv 0$ by definition. The following example illustrates their use for calculating *Q*-values.

Example 1.3

Calculate the *Q*-values for the reactions (i) ${}^{17}\text{O} + p \rightarrow \alpha + {}^{14}\text{N}$ and (ii) $p + p \rightarrow e^+ + \nu + d$ by using the information listed in Table 1.1. (The symbols e^+ and ν denote a positron and a neutrino, respectively).

(i) For the ${}^{17}O(p,\alpha){}^{14}N$ reaction we find from Eq. (1.10)

$$Q = (M.E.)_{17O} + (M.E.)_{1H} - (M.E.)_{14N} - (M.E.)_{4He}$$

= [(-808.81) + (7288.97) - (2863.42) - (2424.92)] keV = 1191.83 keV

Exactly the same result is obtained if atomic masses are used. (ii) For the $p(p,e^+\nu)d$ reaction one obtains

$$Q = (m_{1H} + m_{1H} - m_{2H})c^2 = (M.E.)_{1H} + (M.E.)_{1H} - (M.E.)_{2H}$$

= 2 × (7288.97 keV) - (13135.72 keV) = 1442.22 keV

This value includes the annihilation energy $2m_ec^2 = 1022$ keV of the positron with another electron from the environment, as can be seen from

$$Q = [m_{1_{\rm H}} + m_{1_{\rm H}} - m_{2_{\rm H}}]c^2 = [(m_{\rm p} + m_{\rm e}) + (m_{\rm p} + m_{\rm e}) - (m_{\rm d} + m_{\rm e})]c^2$$

= $[m_{\rm p} + m_{\rm p} - m_{\rm d} + m_{\rm e}]c^2 = [(m_{\rm p} + m_{\rm p} - m_{\rm d} - m_{\rm e}) + 2m_{\rm e}]c^2$

The symbols ¹H, ²H and p, d in the above expression denote atomic and nuclear masses, respectively.

1.5.4

Number Abundance, Mass Fraction, and Mole Fraction

The number density of nuclei *i* in a stellar plasma, N_i , is equal to the total number of species *i* per unit volume. Avogadro's number N_A is defined as the number of atoms of species *i* which makes M_i grams, that is, $N_A = M_i/m_i = M_i/m_i$

Tab. 1.1 Experimental values of the atomic mass excess (M.E.), binding energy per nucleon (B/A), and relative atomic mass (M) for some light nuclides in the $A \leq 20$ mass region. Errors are not listed. From Audi, Wapstra and Thibault (2003).

Α	Elt.	M.E. (keV)	B/A (keV)	<i>M</i> (u)
1	n	8071.3171	0.0	1.0086649157
	Н	7288.97050	0.0	1.00782503207
2	Н	13135.7216	1112.283	2.0141017778
3	Н	14949.8060	2827.266	3.0160492777
	He	14931.2148	2572.681	3.0160293191
4	He	2424.91565	7073.915	4.00260325415
6	Li	14086.793	5332.345	6.015122795
7	Li	14908.14	5606.291	7.01600455
	Be	15770.03	5371.400	7.01692983
8	Li	20946.84	5159.582	8.02248736
	Be	4941.67	7062.435	8.00530510
	В	22921.5	4717.16	8.0246072
9	Li	24954.3	5037.84	9.0267895
	Be	11347.6	6462.76	9.0121822
10	Be	12606.7	6497.71	10.0135338
	В	12050.7	6475.07	10.0129370
11	Be	20174.	5952.8	11.021658
	В	8667.9	6927.71	11.0093054
	С	10650.3	6676.37	11.0114336
12	В	13368.9	6631.26	12.0143521
	С	0.0	7680.144	12.0000000
13	В	16562.2	6496.40	13.0177802
	С	3125.0113	7469.849	13.0033548378
	Ν	5345.48	7238.863	13.00573861
14	С	3019.893	7520.319	14.003241989
	Ν	2863.4170	7475.614	14.0030740048
	0	8007.36	7052.308	14.00859625
15	С	9873.1	7100.17	15.0105993
	Ν	101.4380	7699.459	15.0001088982
	0	2855.6	7463.69	15.0030656
16	Ν	5683.7	7373.81	16.0061017
	0	-4737.00141	7976.206	15.99491461956
17	Ν	7871.	7286.2	17.008450
	0	-808.81	7750.731	16.99913170
	F	1951.70	7542.328	17.00209524
18	Ν	13114.	7038.5	18.014079
	0	-781.5	7767.03	17.9991610
	F	873.7	7631.605	18.0009380
19	0	3334.9	7566.39	19.003580
	F	-1487.39	7779.015	18.99840322
	Ne	1751.44	7567.375	19.0018802
20	F	-17.40	7720.131	19.99998132
	Ne	-7041.9313	8032.240	19.9924401754
	Na	6848.	7298.6	20.007351

 $6.022 \times 10^{23} \text{ mol}^{-1}$. The *mass density* is then given by $\rho = N_i m_i = N_i M_i / N_A$ if only species *i* is present, or by $\rho = (1/N_A) \sum_i N_i M_i$ for a mixture of species. We write

$$\frac{\sum_{i} N_{i}M_{i}}{\rho N_{A}} = \frac{N_{1}M_{1}}{\rho N_{A}} + \frac{N_{2}M_{2}}{\rho N_{A}} + \frac{N_{3}M_{3}}{\rho N_{A}} + \cdots$$

$$= X_{1} + X_{2} + X_{3} + \cdots = \sum_{i} X_{i} = 1$$
(1.12)

where the quantity

$$X_i \equiv \frac{N_i M_i}{\rho N_A} \tag{1.13}$$

represents the fraction of the mass that is bound in species *i* and, therefore, is called the *mass fraction*. A related quantity is the *mole fraction*, defined by

$$Y_i \equiv \frac{X_i}{M_i} = \frac{N_i}{\rho N_A} \tag{1.14}$$

In a stellar plasma, the number density N_i will change if nuclear transmutations take place. But it will also change as a result of variations in the mass density caused by compression or expansion of the stellar gas. In situations where the mass density of the stellar plasma varies, it is of advantage to express abundances in terms of the quantity Y_i instead of N_i . In a simple expansion of matter without nuclear reactions or mixing, the former quantity remains constant, whereas the latter quantity is proportional to the mass density ρ .

Strictly speaking, the mass density ρ is not a conserved quantity even if no compression or expansion of the stellar gas occurs. The reason is that nuclear transmutations transform a fraction of the nuclear mass into energy or leptons (for example, electrons or positrons) and vice versa. To avoid this difficulty, the density is sometimes defined as $\rho_A = (1/N_A) \sum_i N_i A_i$ in terms of the number of nucleons (that is, the mass number A_i) instead of the relative atomic mass M_i , since the number of nucleons is always conserved in a nuclear transmutation. The mass fraction of Eq. (1.13) should in principle be replaced by the *nucleon fraction* $X_i = N_i A_i / (\rho_A N_A)$. However, the difference between mass density and nucleon density, or between mass fraction and nucleon fraction, is very small and the distinction is usually not important numerically. In order to avoid confusion, we will be using in this book mass densities and mass fractions. For more information on abundances see, for example, Arnett (1996).

Example 1.4

The mass fractions of ¹H and ⁴He at the time of the Sun's birth are equal to 0.71 and 0.27, respectively. Calculate the ratio of the corresponding number densities.

From Eq. (1.13) and Table 1.1 we find

$$\frac{N({}^{1}\mathrm{H})}{N({}^{4}\mathrm{He})} = \frac{\frac{\rho N_{A}X({}^{1}\mathrm{H})}{M({}^{1}\mathrm{H})}}{\frac{\rho N_{A}X({}^{4}\mathrm{He})}{M({}^{4}\mathrm{He})}} = \frac{M({}^{4}\mathrm{He})}{M({}^{1}\mathrm{H})}\frac{X({}^{1}\mathrm{H})}{X({}^{4}\mathrm{He})} = \frac{(4.0026\,\mathrm{u})}{(1.0078\,\mathrm{u})}\frac{0.71}{0.27} = 10.4$$

1.5.5 Decay Constant, Mean Lifetime, and Half-Life

The time evolution of the number density N (or of the absolute number of nuclei \mathcal{N}) of an unstable nuclide is given by the differential equation

$$\left(\frac{dN}{dt}\right) = -\lambda N \tag{1.15}$$

The quantity λ represents the probability of decay per nucleus per time. Since it is constant for a given nuclide under specific conditions (constant temperature and density), it is referred to as *decay constant*. Integration of the above expression immediately yields the radioactive decay law for the number density of undecayed nuclei remaining after a time *t*,

$$N = N_0 e^{-\lambda t} \tag{1.16}$$

where N_0 is the initial number density at t = 0. The time it takes for the number density *N* to fall to one-half of the initial abundance, $N_0/2 = N_0 e^{-\lambda T_{1/2}}$, is called the *half-life* $T_{1/2}$, with

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.69315}{\lambda}$$
(1.17)

The time it takes for N to fall to 1/e = 0.36788 of the initial abundance, $N_0/e =$ $N_0 e^{-\lambda \tau}$, is called the *mean lifetime* τ , with

$$\tau = \frac{1}{\lambda} = 1.4427 \, T_{1/2} \tag{1.18}$$

If a given nuclide can undergo different competing decays (for example, γ and β -decay, or different γ -ray transitions), then the total decay probability in

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Eqs. (1.15)–(1.18) is given by the sum of the decay probabilities for the individual processes. Hence

$$\lambda = \sum_{i} \lambda_{i}$$
 or $\frac{1}{\tau} = \sum_{i} \frac{1}{\tau_{i}}$ (1.19)

where the quantities λ_i and τ_i are called *partial* decay constant and *partial* halflife, respectively. The product of the absolute number of nuclei, \mathcal{N} , and the decay constant determines the number of decays per unit time and is referred to as the *activity*, $A \equiv \lambda \mathcal{N} = -d\mathcal{N}/dt$. Common units of the activity are the curie (1 Ci = 3.7×10^{10} decays per second) and the becquerel (1 Bq = 1 decay per second). It must be emphasized that Eqs. (1.15)–(1.19) apply to any nuclear decay, such as β -decay, α -particle decay, γ -ray decay of excited levels, and the destruction of nuclei via nuclear reactions in a stellar plasma, as will be shown later.

1.6

Nuclear Shell Model

A detailed treatment of the nuclear shell model is beyond the scope of this book. Basic discussions are presented in many introductory nuclear physics texts (for example, Krane 1988). For a more detailed account, the reader is referred to DeShalit and Talmi (1963) or Brussaard and Glaudemans (1977). In the following we will summarize some of the important assumptions and predictions of the model. Our aim is to better understand how nuclear properties, such as binding energies, spins, and parities, can be explained from the underlying configurations of the nucleons. These considerations are also important because a number of nuclear structure properties that are mentioned in this text, for example, reduced γ -ray transition strengths, weak interaction matrix elements, and spectroscopic factors, are frequently computed by using the shell model.

The *atomic* shell model has been enormously successful in describing the properties of atoms. In the case of an atom, the heavy nucleus represents a center for the Coulomb field in which the light electrons move independently in first-order approximation. The spherical Coulomb potential is given by $V_C = Ze^2/r$, with *Z* the atomic number, *e* the electron charge, and *r* the distance between nucleus and electron. Solving the Schrödinger equation for this system yields the electron orbits, or shells, that are characterized by various quantum numbers. In general, several of these (sub-)shells are almost degenerate in energy and together they form major shells. The rules for building up the atomic electron configuration follow immediately from the Pauli exclusion principle, stating that no more than two spin-1/2 particles can occupy a given quantum state simultaneously. The shells are then filled up with elec-

trons in order of increasing energy. We thus obtain an inert core of filled shells and some number of valence electrons that primarily determine the atomic properties. Atoms with all states of the major shells occupied exhibit a high stability against removal or addition of an electron. These are the inert gases.

The application of a similar model to the atomic nucleus encounters a number of obvious complications. First, the nuclear interaction is very different from the Coulomb interaction and, moreover, the nature of the nucleonnucleon interaction is not precisely known. Second, there are two kinds of elementary particles present in the nucleus (protons and neutrons) as opposed to the atomic case (electrons). Third, there is no heavy center of force for the nucleons. Despite these complications, the nuclear shell model has been highly successful in describing many properties of nuclei. Its basic assumption is that the interaction of each nucleon with all the other protons and neutrons in the nucleus is well approximated by some average potential V(r). A single nucleon moves independently in this potential and can be described by a single-particle state of discrete energy and constant angular momentum.

The independent motion of the nucleons can be understood qualitatively in the following manner. According to the Pauli exclusion principle, no more than two protons or neutrons can exist in a given quantum state. The singleparticle levels are filled with nucleons up to some level, depending on how many nucleons are present. Consider now a single nucleon, occupying some intermediate single-particle level, moving through the nucleus. It is well known that the nuclear force has a short range and, therefore, we expect that the actual nuclear potential will strongly fluctuate. The nucleon may collide with other protons or neutrons, but it cannot gain or lose energy easily since the neighboring levels are already occupied and thus cannot accept an additional nucleon. Of course, it may gain a large amount of energy and hence move to a higher lying, unoccupied single-particle level. But such collisions with a significant energy transfer are less likely to occur. Consequently, the motion of the nucleon will often be fairly smooth.

1.6.1

Closed Shells and Magic Numbers

We will start from the assumption that the interaction between one nucleon and all the other nucleons in the nucleus can be approximated by a suitable single-particle potential. In the simplest case, it consists of a central potential (for example, a harmonic oscillator potential or a Woods–Saxon potential) and a strong spin–orbit coupling term. The solutions of the Schrödinger equation for such a potential are bound single-particle states that are characterized by the values of the radial quantum number n, orbital angular momentum quantum number ℓ , and total angular momentum quantum number j (the latter is obtained from the coupling $\vec{j} = \vec{\ell} + \vec{s}$ where *s* denotes the intrinsic spin equal to 1/2 for protons or neutrons). In particular, the energies of the single-particle states depend explicitly on the values of *n*, ℓ , and *j*. The single-particle states are energetically clustered in groups and thus reveal a shell structure. Each state of given *j* can be occupied by a maximum number of (2j + 1) identical nucleons, corresponding to the number of magnetic substates ($m_j = -j, -j + 1, \ldots, j - 1, j$), and thus represents a *subshell*. Several different subshells lying close in energy can be grouped together and form a *major shell*. Furthermore, each single-particle state possesses a definite parity (Appendix A), given by $\pi = (-1)^{\ell}$. The shells are filled up according to the Pauli exclusion principle.

The single-particle levels for either protons or neutrons are shown in Fig. 1.10 where the horizontal direction represents an energy scale. The lefthand side (a) displays the single-particle energies of a harmonic oscillator potential as a function of the oscillator quantum number $N = 2(n-1) + \ell$, corresponding to the total number of oscillator quanta excited. Part (b) shows the single-particle energies of a Woods-Saxon potential. This potential is more realistic but mathematically less tractable. It is defined by $V(r) = V_0 [1 + e^{(r-R_0)/a}]^{-1}$, where V_0 , R_0 , and a denote the potential depth, the potential radius, and the diffuseness, respectively. Note that in part (a) each single-particle state of given N consists in general of states with different values of ℓ . These have the same energies and are thus called *degenerate*. The degeneracy does not occur for the more realistic Woods-Saxon potential, that is, states with different values of ℓ possess different energies. It is customary to use the spectroscopic notation s, p, d, f, g, ... for states with orbital angular momenta of $\ell = 0, 1, 2, 3, 4, \dots$, respectively. If an additional spin–orbit term is added to the potential, then each state of given ℓ value (except $\ell = 0$) can have a total angular momentum of either $j = \ell + 1/2$ or $j = \ell - 1/2$ (Appendix B). Since ℓ is an integer, *j* must be of odd half-integer value. Part (c) shows how the spin–orbit term splits each state with $\ell > 0$ into two levels. The number of identical particles (protons or neutrons) that can occupy a state of given *j* amounts to (2j + 1) and is presented in part (d). Part (e) displays the singleparticle states in spectroscopic notation as $n\ell_i$. Note that the quantum number *n* corresponds to the order in which the various states of given ℓ and *j* appear in energy. Thus, $1s_{1/2}$ is the first $\ell = 0$, j = 1/2 state, $2s_{1/2}$ is the second, and so on. The parities of the single-particle levels are shown in part (f), and part (g) indicates the subtotal of the number of identical nucleons that can fill all the states up to a given level.

It is important to point out that the spin–orbit coupling term is so strong that it changes the energies of the single-particle states significantly. For example, consider the N = 3 and 4 oscillator shells. The g-state ($\ell = 4$) in part (b) splits into two levels, $1g_{7/2}$ and $1g_{9/2}$. Since the spin–orbit coupling is strong the

 $1g_{9/2}$ state is depressed and appears to be close in energy to the $2p_{1/2}$, $1f_{5/2}$, and $2p_{3/2}$ states that originate from the N = 3 oscillator shell. There is now an energy gap at a subtotal nucleon (or occupation) number of 50 and, consequently, this group of states forms a major shell. Similar arguments apply to other groups of levels. It can be seen from Fig. 1.10 that gaps (or major shell closures) in the single-particle energy spectrum appear at occupation numbers of 2, 8, 20, 28, 50, 82, and 126. These are referred to as *magic numbers*.

It should be obvious that nuclei with filled major shells of protons or neutrons exhibit an energetically favorable configuration, resulting in an extra stability compared to neighboring nuclei with only partly filled shells. The magic numbers manifest themselves in many observed nuclear properties, such as masses, particle separation energies, nuclear charge radii, electric quadrupole moments, and so on. As an example, Fig. 1.11 shows the difference of the measured ground-state atomic mass excess from its mean value that is calculated by using a smooth semiempirical mass formula. At the locations of magic neutron numbers, the atomic mass excess is smaller, resulting in a smaller atomic mass and a larger binding energy according to Eqs. (1.2) and (1.9). Another example will be given later in connection with neutron capture cross sections (Fig. 5.61). Such observations provide unambiguous evidence for the shell structure of nuclei. As will become apparent in Section 5.6, the synthesis of the heavy elements is strongly influenced by the magic neutron numbers of N = 50, 82, and 126. It has to be emphasized again that the magic numbers as they are observed in nature can only be reproduced if a strong spin-orbit coupling term is introduced into the independent-particle potential.

1.6.2

Nuclear Structure and Nucleon Configuration

The shell model not only predicts the properties of closed shell nuclei, but also the properties of nuclei with partly filled shells. The nuclear properties follow directly from the configuration of the nucleons: (i) the binding energy or mass of the nucleus is determined by the single-particle energies (due to the independent motion of the nucleons in an average potential) and by the mutual interaction of the valence nucleons (that is, those located outside a closed major shell); (ii) the total angular momentum of the nucleus (or the *nuclear spin*) is obtained by coupling the angular momenta of the independent single-particle states according to the quantum mechanical rules for vector addition (Appendix B); and (iii) the total parity of the nucleus is determined by the product of the parities for all nucleons.

Consider first a nucleus with completely filled subshells. In each subshell *j* all magnetic substates m_j are occupied and thus the sum of j_z over all nucleons in the subshell must be zero. In other words, the nucleons in a completely



Fig. 1.10 Approximate sequence of singleparticle states for identical nucleons (protons or neutrons). See the text. The magic numbers (given in boxes on the right-hand side) appear at the energy gaps and correspond to the cumulative number of nucleons up to that state. The level pattern represents qualitative features only. This holds specifically for states with $N \ge 4$ where the level order differs for protons (which are subject to the Coulomb interaction) and neutrons. Reprinted with permission from P. J. Brussaard and P. W. M. Glaudemans, Shell-Model Applications in Nuclear Spectroscopy (Amsterdam: North-Holland, 1977). Copyright by P. J. Brussaard.

filled subshell must couple to an angular momentum of zero. Furthermore, since (2j + 1) is an even number, the total parity of the nucleons amounts to $\pi = +1$. Indeed, the observed spin and parity of nuclei with closed subshells (or closed major shells) amount to $J^{\pi} = 0^+$ (for example, ${}_{2}^{4}\text{He}_{2}$, ${}_{6}^{12}\text{C}_{6}$, ${}_{6}^{14}\text{C}_{8}$, ${}_{8}^{14}\text{O}_{6}$, ${}_{8}^{28}\text{Si}_{14}$, ${}_{16}^{32}\text{S}_{16}$, or ${}_{20}^{40}\text{Ca}_{20}$). A closed-shell nucleus can only be excited by promoting at least one nucleon to a higher lying, unoccupied, subshell. This is consistent with the observation that the first excited states of such nu-



Fig. 1.11 Difference between experimental ground-state atomic mass excess (Audi et al. 2003) and the mass excess predicted by the spherical macroscopic part of the finite-range droplet (FRDM) mass formula (Möller et al. 1995) versus neutron number.

clei are usually found at relatively high excitation energies. Nuclei with partly filled shells may have excited states that result from a recoupling of the angular momenta only. This explains why in such cases the observed excitation energies are significantly smaller.

By considering Fig. 1.10 we can easily explain the quantum numbers for the ground states of nuclei when a single nucleon is located outside a closed subshell. In this case, the ground-state spin and parity is given by the lowest single-particle state available to the valence nucleon. For example, we find $J^{\pi} = 1/2^{-}$ for ${}_{6}^{13}C_{7}$, $J^{\pi} = 5/2^{+}$ for ${}_{8}^{17}O_{9}$, $J^{\pi} = 1/2^{+}$ for ${}_{14}^{29}Si_{15}$, or $J^{\pi} = 3/2^{+}$ for ${}_{16}^{33}S_{17}$. A single valence nucleon outside a closed subshell behaves in this respect the same as a single "hole" in an otherwise filled subshell. The groundstate spin and parity, for example, of ${}_{14}^{27}Si_{13}$ amounts to $J^{\pi} = 5/2^{+}$ because it has a single neutron hole in the $1d_{5/2}$ shell.

The situation is not as obvious when the subshells are only partly filled. We observe experimentally that the ground states of all doubly even nuclei possess a spin and parity of $J^{\pi} = 0^+$. For example, this applies to ${}^{26}_{12}\text{Mg}_{14}$ although neither the protons nor the neutrons completely fill the subshells. We can explain this observation by assuming that it is energetically favorable for pairs of protons or neutrons to couple to a total spin and parity of $J^{\pi}_{\text{pair}} = 0^+$. This *pairing effect* also influences the proton and neutron separation energies of neighboring nuclei, as will be seen in Section 5.6. The shell model can then be used to predict the ground-state spins and parities for odd-*A* nuclei. For example, consider ${}^{23}_{10}\text{Ne}_{13}$. All the protons couple pairwise to quantum numbers

of 0^+ , as do 12 of the neutrons. The lowest available level for the odd neutron is the $1d_{5/2}$ state (Fig. 1.10) and thus the ground-state spin of ²³Ne amounts to $J^{\pi} = 5/2^+$. These simplistic considerations reproduce many of the observed ground-state spins, but fail in some cases. According to the above arguments one would expect a ground-state spin and parity of $J^{\pi} = 5/2^+$ for ²³₁₁Na₁₂, but instead we observe $J^{\pi} = 3/2^+$. The discrepancy is explained by the fact that the interplay of many nucleons in an unfilled shell is rather complicated so that an even number of protons or neutrons does not always couple to a total angular momentum of J = 0 for the ground state. This is especially true for excited nuclear levels.

In all but the simplest situations, the nucleon configuration must be taken into account explicitly. Further complications arise since a given nuclear level may be described by a mixed configuration, that is, by different nucleon configurations that couple to the same value of J^{π} . In such cases, large-scale shell model calculations must be performed with numerical computer codes. The shell model has been enormously successful in explaining the structure of nuclei. It is frequently used in nuclear astrophysics in order to calculate nuclear quantities that have not yet been measured in the laboratory. Reduced γ -ray transition strengths (Section 1.7.2) or weak interaction transition strengths (Section 1.8.3), for instance, depend on nuclear matrix elements that connect an initial (decaying) state with a final state. The matrix elements can be calculated numerically with the shell model in a straightforward manner once an appropriate form for the transition operator (for the electromagnetic or weak interaction) is assumed. Another important quantity in nuclear astrophysics is the spectroscopic factor. It will be explained in Section 2.5.7 how this property can be used for estimating an unknown cross section of a nuclear reaction $A + a \rightarrow B$. The spectroscopic factor is defined in terms of the overlap integral between the final state wave function of *B* and the initial state wave function of A + a. It does not depend on a transition operator, but only on a wave function overlap, and thus can be calculated rather reliably for many nuclei.

1.7

Nuclear Excited States and Electromagnetic Transitions

1.7.1

Energy, Angular Momentum, and Parity

Every nucleus exhibits excited states. They can be populated by many different means, for example, nuclear reactions, β -decays, thermal excitations (see below), inelastic electron or particle scattering, and Coulomb excitation. Each nuclear level is characterized by its excitation energy E_x , which is defined as the binding energy difference between the level in question and the

ground state of the nucleus. For the ground state we have, as per definition, $E_x = 0$. In the laboratory, each excited level of energy E_i can make a transition to a lower lying state of energy E_f via three different processes that are all induced by the electromagnetic interaction: (i) γ -ray emission, (ii) internal conversion, and (iii) internal pair formation. *Internal conversion* refers to a process where an excited nucleus de-excites by transferring its energy directly, that is, in a single step, to an orbital electron. *Internal pair formation* denotes the de-excitation of a nucleus by creating an electron-positron pair, in which case the de-excitation energy must exceed twice the value of the electron rest energy ($2m_ec^2$). Although the three processes can in principle compete with each other, the emission of a γ -ray is by far the most important one for nuclear astrophysics and will be discussed in the following.

In a γ -ray transition between two nuclear levels, the energy of the emitted photon is given by

$$E_{\gamma} = E_i - E_f - \Delta E_{\rm rec} \tag{1.20}$$

where the origin of the *recoil shift* ΔE_{rec} is described in Appendix C.1. We are mainly concerned here with γ -ray energies in the range of 100 keV to 15 MeV. For such energies the recoil shift is very small and can usually be neglected. Hence we may use in most cases $E_{\gamma} \approx E_i - E_f$. This assumes that the excited nucleus decays from rest. If the decaying level is populated via a nuclear reaction, then another correction (the Doppler shift) must also be taken into account (Appendix C.1). In any case, the emitted γ -rays will exhibit discrete energies. If E_f corresponds to the ground state, then no further emission of γ -rays is possible. Otherwise, de-excitation of the nucleus by emission of one or more photons before reaching the ground state is likely to occur.

The emitted (or absorbed) electromagnetic radiation can be classified according to the angular momentum $L\hbar$ which is carried by each photon, and according to its parity (Appendix B). The angular momentum carried away by the photon determines the multipolarity of the radiation. A value of L for the angular momentum corresponds to 2^{L} -pole radiation with its characteristic angular distribution for the emitted intensity. For example, L = 1 and L = 2 correspond to dipole (2¹) and quadrupole (2²) radiation, respectively. Two identical radiation patterns for a given value of L may correspond to different waves, "electric" 2^L-pole radiation and "magnetic" 2^L-pole radiation, which differ through their parity. For example, E2 and M1 correspond to electric quadrupole radiation and magnetic dipole radiation, respectively. In a γ -ray transition between two nuclear levels the total angular momentum and parity of the system (nucleus plus electromagnetic field) are conserved. The conservation laws give rise to certain selection rules that must be fulfilled for an emission (or absorption) of radiation of given character to occur. The quantum mechanical rules are explained in Appendix B.

1.7.2 **Transition Probabilities**

A detailed discussion of the quantum theory for the interaction of nuclei with electromagnetic radiation is beyond the scope of this book. We will instead summarize the most important steps in the derivation of the transition probability. For more information, see Blatt and Weisskopf (1952).

The decay constant (that is, the probability per unit time) for the emission of electromagnetic radiation of a given character (for example, E1 or M2) in a transition connecting two given nuclear levels can be calculated using perturbation theory. The result is (Blatt and Weisskopf 1952)

$$\lambda(\overline{\omega}L) = \frac{8\pi}{L[(2L+1)!!]^2} \frac{1}{\hbar} \left(\frac{E_{\gamma}}{\hbar c}\right)^{2L+1} B(\overline{\omega}L)$$
(1.21)

with E_{γ} and L the energy and multipolarity of the radiation, respectively; $\overline{\omega}$ denotes either electric (E) or magnetic (M) radiation and the double factorial is defined as $(2L+1)!! \equiv 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2L+1)$. The quantity $B(\overline{\omega}L)$ is called the reduced transition probability. It contains the wave functions of the initial and final nuclear states, and the multipole operator, that is, the operator responsible for changing the initial to the final state while simultaneously creating a photon of proper energy, multipolarity, and character. Reduced transition probabilities can be calculated by using nuclear structure models, for example, the shell model (Section 1.6). In the simplest case one may assume that the nucleus consists of an inert core plus a single nucleon, that the γ -ray transition is caused by this nucleon changing from one shell-model state to another, and that the radial wave functions of the initial and final states are constant over the nuclear interior and vanish outside the nucleus. With these assumptions one obtains the *Weisskopf estimates* for the γ -ray transition probabilities, which are given below for the lowest-and as will be seen, most importantmultipolarities:

 $\lambda_{W}(E1)\hbar = 6.8 \times 10^{-2} A^{2/3} E_{\gamma}^{3}, \qquad \lambda_{W}(M1)\hbar = 2.1 \times 10^{-2} E_{\gamma}^{3}$ (1.22)

$$\lambda_{\rm W}({\rm E2})\hbar = 4.9 \times 10^{-8} A^{4/3} E_{\gamma}^5, \qquad \lambda_{\rm W}({\rm M2})\hbar = 1.5 \times 10^{-8} A^{2/3} E_{\gamma}^5$$
 (1.23)

$$\lambda_W(\text{E3})\hbar = 2.3 \times 10^{-14} A^2 E_{\gamma}^7, \qquad \lambda_W(\text{M3})\hbar = 6.8 \times 10^{-15} A^{4/3} E_{\gamma}^7 \qquad (1.24)$$

In these numerical expressions, A denotes the mass number of the decaying nucleus, the photon energy E_{γ} is in units of MeV, and the Weisskopf estimates are in units of eV. (It will be shown later that the product $\lambda\hbar$ is equal to a γ -ray partial width).

The Weisskopf estimates for the γ -ray decay probability are shown in Fig. 1.12 versus γ -ray energy for emitted radiations of different multipolarity and character. It is apparent that the quantity λ_W rises strongly with increasing γ -ray energy. We will be using in later chapters the relation

 $\Gamma = \lambda \hbar \sim E_{\gamma}^{2L+1}$, as predicted by the Weisskopf estimates, when describing the energy dependence of γ -ray partial widths. Also, the decay probability depends strongly on the multipolarity *L* and the character $\overline{\omega}$ of the radiation. Furthermore, according to the selection rules (Appendix B), electric and magnetic radiations of the same multipolarity cannot be emitted together in a transition between two given nuclear levels. For a transition connecting two levels of opposite parities, we find from Fig. 1.12 the inequalities

$$\lambda_W(E1) \gg \lambda_W(M2) \gg \lambda_W(E3) \gg \cdots$$
 (1.25)

In this case, the lowest multipole permitted by the selection rules usually dominates. In particular, if E1 radiation is allowed it will dominate the transition strength in the vast majority of astrophysical applications. For a transition connecting two levels of the same parity, one obtains

$$\lambda_W(M1) \gg \lambda_W(E2) \gg \lambda_W(M3) \gg \cdots$$
 (1.26)

However, experimentally measured γ -ray transition strengths do not support the conclusion that M1 transitions are always faster than E2 transitions if both radiations are allowed by the selection rules. In fact, the actual decay strengths may deviate strongly from the Weisskopf estimates since the latter are obtained by using rather crude assumptions. It turns out that for many transitions the observed decay constants are several orders of magnitude smaller than the theoretically predicted value of λ_W , indicating a poor overlap in the wave functions of the initial and final nuclear levels. On the other hand, for E2 transitions it is found that the observed decay probability frequently exceeds the Weisskopf estimate by large factors. This indicates that more than one nucleon must be taking part in the transition and that the excitation energy of the decaying level is stored in the collective in-phase motion of several nucleons.

The Weisskopf estimates are very useful since they provide a standard against which to compare observed transition strengths. The latter are frequently quoted in *Weisskopf units*, defined as

$$M_W^2(\overline{\omega}L) \equiv \frac{\lambda(\overline{\omega}L)}{\lambda_W(\overline{\omega}L)} = \frac{\Gamma(\overline{\omega}L)}{\Gamma_W(\overline{\omega}L)} \quad \text{or} \quad \lambda(\overline{\omega}L) = M_W^2(\overline{\omega}L) \text{ W.u. (1.27)}$$

This definition removes the strong energy dependence of the decay probability. Several thousand observed γ -ray transitions were analyzed in this manner and their transition strengths in Weisskopf units have been presented separately according to the multipolarity and character of the radiation (Endt 1993 and references therein). The resulting distributions of transition strengths extend from some small value of $M_W^2(\overline{\omega}L)$, which is strongly influenced by the sensitivity of the detection apparatus, to the largest observed transition probability. The latter value defines for each combination of $\overline{\omega}L$ a *recommended upper*



Fig. 1.12 Weisskopf estimate of the γ -ray decay probability for pure electric (E) and magnetic (M) multipole radiations emitted in transitions between two nuclear levels of energy difference E_{γ} . The γ -ray partial width Γ_W is equal to the product $\lambda_W \hbar$. The curves are calculated for A = 20 and a nuclear radius of $R = 1.20A^{1/3}$ fm = 3.3 fm.

limit (RUL). For the mass region A = 5-44 the following values have been reported (Endt 1993)

RUL(E1) = 0.5 W.u.,	RUL(M1) = 10 W.u.
RUL(E2) = 100 W.u.,	RUL(M2) = 5 W.u.
RUL(E3) = 50 W.u.,	RUL(M3) = 10 W.u.

These values are important for estimating the maximum expected γ -ray decay probability for an unobserved transition (Problem 1.5). It is tempting to estimate *average* decay strengths based on the centroids of the observed transition strength distributions (see Fig. 2 in Endt 1993). However, one has to be very careful since the "averages" (as well as the "lower limits") depend on the γ -ray detection limit and thus may be expected to decrease with an improvement in the sensitivity of the detection equipment.

1.7.3

Branching Ratio and Mixing Ratio

So far we discussed γ -ray transitions of specific multipolarity L and character $\overline{\omega}$. In practice, however, a given initial state may decay to a number of different final states. Furthermore, each transition connecting two given states may proceed via a mixture of radiations according to the selection rules. These complications can be described by introducing two new quantities, the branching ratio and the mixing ratio. In the following we will express these quantities in terms of the γ -ray decay probability in units of energy, $\Gamma = \lambda \hbar$,

which is also referred to as the γ -ray partial width. Consider Fig. 1.13 showing the γ -ray decay of an initial excited level *i*. The *total* γ -ray width of the initial state can be expressed in terms of *partial* γ -ray widths that each correspond to a transition to a specific final state *j* as

$$\Gamma_{\text{tot}} = \sum_{j} \Gamma_{j} \tag{1.28}$$

Assuming that the initial state decays only by γ -ray emission, the γ -ray branching ratio is defined by

$$B_j \equiv \frac{\Gamma_j}{\Gamma_{\rm tot}} \times 100\% \tag{1.29}$$

and is usually given in percent. Each γ -ray branch may result from radiations of different multipolarities L and characters $\overline{\omega}$. Although the selection rules may allow for three or more possibilities (for example, a $2^+ \rightarrow 1^+$ transition may proceed via M1, E2, or M3 radiations), in most practical cases not more than the lowest two values of $\overline{\omega}L$ need to be taken into account. If we assume that only radiations with $\overline{\omega}'L$ and $\overline{\omega}L + 1$ contribute to the transition (M1 and E2 in the above example), the partial γ -ray width is given by

$$\Gamma_{i}(\overline{\omega}L+1;\overline{\omega}'L) = \Gamma_{i}(\overline{\omega}L+1) + \Gamma_{i}(\overline{\omega}'L)$$
(1.30)

The γ -ray multipolarity mixing ratio is defined as

$$\delta_j^2 \equiv \frac{\Gamma_j(\overline{\omega}L+1)}{\Gamma_j(\overline{\omega}'L)} \tag{1.31}$$

By combining Eqs. (1.28)–(1.31) we may express the individual widths in terms of the total width as

$$\Gamma_j(\overline{\omega}'L) = \frac{1}{1+\delta_j^2} \frac{B_j}{100} \Gamma_{\text{tot}}$$
(1.32)

$$\Gamma_j(\overline{\omega}L+1) = \frac{\delta_j^2}{1+\delta_i^2} \frac{B_j}{100} \Gamma_{\text{tot}}$$
(1.33)

A highly excited nuclear state with many different decay probabilities to lower lying levels will preferably decay via those transitions that correspond to the largest decay strengths, that is, via emission of radiations with the smallest multipoles. If a given level is located, say, above at least 20 lower lying states, then the observed γ -ray decays from this level are in almost all instances either of dipole (E1 or M1, depending on the parity of the initial and final level) or E2 character. This empirical finding is called the "dipole or E2 rule" (Endt 1990) and is useful for estimating unknown spin and parities of nuclear levels.



Fig. 1.13 Energy level diagram showing the γ -ray decay of an initial state *i* to the ground state (0) and to two excited states (1, 2). The branching ratio B_j represents the relative intensity of a particular decay branch as a percentage of the total intensity and δ_j denotes the multipolarity mixing ratio.

1.7.4

Gamma-Ray Transitions in a Stellar Plasma

In a hot plasma, excited states in a given nucleus are thermally populated, for example, through absorption of photons (photoexcitation), Coulomb excitation by surrounding ions, inelastic particle scattering, and other means. The time scale for excitation and de-excitation (for example, via emission and absorption of photons) in a hot stellar plasma is usually—with the important exception of isomeric states (see below)—much shorter than stellar hydrody-namical time scales, even under explosive conditions (Fowler, Caughlan and Zimmerman 1975). These excited levels will participate in nuclear reactions and β -decays, as will be explained later, and thus their population must in general be taken into account. For a given nuclide in a nondegenerate plasma at thermodynamic equilibrium, the ratio of the number density of nuclei in excited state μ , denoted by N_{μ} , and the total number density of nuclei, N, is given by a Boltzmann distribution (Ward and Fowler 1980)

$$P_{\mu} = \frac{N_{\mu}}{N} = \frac{g_{\mu}e^{-E_{\mu}/kT}}{\sum_{\mu}g_{\mu}e^{-E_{\mu}/kT}} = \frac{g_{\mu}e^{-E_{\mu}/kT}}{G}$$
(1.34)

with $g_{\mu} \equiv (2J_{\mu} + 1)$, J_{μ} and E_{μ} the statistical weight, spin and excitation energy, respectively, of state μ ; the quantity k denotes the Boltzmann constant and T is the plasma temperature. The sum over μ in the denominator includes the ground state and is referred to as the *partition function* G. Note that Eq. (1.34) follows directly from statistical thermodynamics and encompasses all the different processes for the excitation and de-excitation of levels (that

is, not only the emission and absorption of photons). Clearly, the thermal population of excited nuclear levels becomes more important with increasing temperature and lower excitation energy. These properties of Eq. (1.34) are explored in Problem 1.6.

1.7.5

Isomeric States and the Case of ²⁶AI

In most cases the nuclear levels decaying by γ -ray emission have very high transition probabilities, corresponding to half-lives that are generally less than 10^{-9} s. However, quite a few cases have been observed where the half-lives are longer by many orders of magnitude, sometimes amounting to seconds, minutes or even days. Such long-lived excited nuclear levels are referred to as *isomeric states* (or isomers, or metastable states) and the corresponding γ -ray decays are called isomeric transitions. We will denote these levels with the superscript "*m*" (^AX^m).

The two aspects that are mainly responsible for the long half-lives of isomeric states are (i) a large difference for the spins of the isomeric and the final nuclear level, and (ii) a relatively small energy difference between the two levels. The first aspect implies a large γ -ray multipolarity (for example, M4 or E5). The second aspect implies a small γ -ray energy. According to Eq. (1.21), both of these effects have the tendency to reduce the decay probability substantially).

We will illustrate some of the complexities that arise from the presence of an isomer by discussing the important case of ²⁶Al. An energy level diagram is displayed in Fig. 1.14. Focus first only on the left-hand part, showing the ground state ($E_x = 0$, $J^{\pi} = 5^+$) and three excited states ($E_x = 228 \text{ keV}$, $J^{\pi} = 0^+$; $E_x = 417 \text{ keV}$, $J^{\pi} = 3^+$; and $E_x = 1058 \text{ keV}$, $J^{\pi} = 1^+$) in ²⁶Al. According to the selection rules, the direct γ -ray de-excitation of the first excited state at $E_x = 228 \text{ keV}$ would require the emission of M5 radiation. The γ -ray decay probability for such a high multipolarity is very small and thus the first excited state is an isomer (²⁶Al^m). It decays via a β -transition (which is much more likely to occur than the M5 γ -ray transition) to the ground state of ²⁶Mg with a half-life of $T_{1/2}(^{26}Al^m) = 6.34 \text{ s}$. The ²⁶Al ground state is also β -unstable and decays with a half-life of $T_{1/2}(^{26}Al^g) = 7.17 \times 10^5$ y mainly to the first excited state at $E_x = 1809 \text{ keV}$ in ²⁶Mg. This level, in turn, de-excites quickly via γ -ray emission of E2 character.

Interestingly, photons with an energy of 1809 keV originating from the interstellar medium have been detected first by the HEAO-3 spacecraft (Mahoney et al. 1982), and subsequently by other instruments. The E_x = 1809 keV level in ²⁶Mg decays so quickly (within a fraction of a second) that, if it is populated via nuclear reactions in the interiors of stars, the emitted 1809 keV photons would immediately be absorbed by the surrounding matter and would never be able to escape from the stellar production site. However, suppose instead that ²⁶Al^g is synthesized via nuclear reactions in the stellar interior. The long half-life of the ground state provides ample opportunity for this species to be expelled from a star into the interstellar medium, where it then decays so that the emitted photons can reach the Earth. Note that only the decay of the ground state, but not the decay of the isomer, in ²⁶Al gives rise to the emission of 1809 keV γ -rays.

An all-sky map of the 1809 keV γ -ray line, obtained by the COMPTEL telescope aboard the Compton Gamma Ray Observatory (CGRO), is shown in color Fig. 11 on page 641. The discovery of ²⁶Al^g in the interstellar medium is of paramount importance, as already pointed out (Section 1.4.1). It clearly demonstrates that nucleosynthesis is currently active since the ²⁶Al^g half-life is short compared to the time scale of Galactic chemical evolution ($\approx 10^{10}$ y). From the observed γ -ray intensity it is estimated that the production rate of 26 Al^g in the Galaxy amounts to $\approx 2M_{\odot}$ per 10⁶ y. The origin of the Galactic ²⁶Al^g is still controversial at present. However, the observational evidence favors massive stars as a source. For example, the all-sky map of the 1809 keV γ -ray line shows that ²⁶Al^g is confined along the Galactic disk and that the measured intensity is quite clumpy and asymmetric. Furthermore, the measurement of the Doppler shift of the 1809 keV line demonstrated clearly that the ²⁶Al^g co-rotates with the Galaxy and hence supports a Galaxy-wide origin for this species (Diehl et al. 2006). Recent stellar model calculations for massive stars suggest that ²⁶Al^g is mainly produced in type II supernovae during explosive carbon and neon burning (Section 1.4.3). A smaller fraction is possibly synthesized in Wolf-Rayet stars during core hydrogen burning and in the subsequent type Ib/Ic supernova explosion. For more information, see Limongi and Chieffi (2006).

We noted above that in a hot stellar plasma most nuclear levels quickly achieve thermal equilibrium since the time scales for excitation and deexcitation are very short. However, this is not necessarily the case for isomeric states. For example, the γ -ray transition probabilities for the de-excitation of the ²⁶Al isomer at $E_x = 228$ keV and for its population from the ground state via absorption of radiation depend on the same reduced transition strength. Since the emission or absorption of M5 radiation is unlikely, the ground and isomeric states in ²⁶Al cannot achieve thermal equilibrium directly (that is, Eq. (1.34) is not generally valid in this case). Thermal equilibrium may nevertheless be achieved indirectly via transitions involving higher lying levels in ²⁶Al.

Consider again Fig. 1.14. In this case, the ground state and the isomer can communicate via the $E_x = 417$ keV state (0 $\leftrightarrow 417 \leftrightarrow 228$) or via the $E_x = 1058$ keV state (0 $\leftrightarrow 417 \leftrightarrow 1058 \leftrightarrow 228$). Higher lying ²⁶Al states also play

a role as the temperature is increased, but have been omitted in the figure for clarity. The thermal equilibration of ²⁶Al can be calculated by solving numerically a set of linear differential equations that describe all possible γ -ray and β -decay transitions. For some of these (indicated by thick arrows) the experimental transition strengths are known, while for others (thin arrows) the transition strengths have to be calculated by using the shell model (Section 1.6). The procedure is described in detail in Coc, Porquet and Nowacki (1999) and Runkle, Champagne and Engel (2001) and is not repeated here. The resulting effective lifetime of ²⁶Al versus temperature is displayed in Fig. 1.15. The solid line is obtained numerically by taking explicitly the equilibration of the ground and isomeric states via thermal excitations involving higher lying levels into account. The dashed curve is calculated analytically by assuming that the ground and isomeric states are in thermal equilibrium (Example 1.5). Below T = 0.1 GK, the effective lifetime is given by the laboratory lifetime of 26 Al^g ($\tau = 1.4427 T_{1/2} = 3.3 \times 10^{13}$ s). Above T = 0.4 GK, the ground and isomeric states are in thermal equilibrium. At intermediate temperatures, T =0.1–0.4 GK, the equilibration of ²⁶Al via higher lying levels results in an effective lifetime that differs significantly from the thermal equilibrium value.

We focussed here on the case of 26 Al. Other important examples of isomers in nuclear astrophysics are 176 Lu^{*m*} (Zhao and Käppeler 1991) and 180 Ta^{*m*} (Wisshak et al. 2001). For a distinction between the kind of isomer discussed above (also called *spin-isomer*) and other types of isomers (*shape-* and *K-isomers*), see Walker and Dracoulis (1999).

1.8

Weak Interaction

The strong nuclear force and the electromagnetic force govern the nuclear reactions that are of outstanding importance for the energy generation and the nucleosynthesis in stars. However, weak interactions also play an important role in stars for several reasons. First, when a radioactive nuclide is produced during the nuclear burning, its decay via weak-interaction processes will compete with its destruction via nuclear reactions, as will become apparent in Chapter 5. Second, weak interactions determine the *neutron excess parameter* during the nucleosynthesis, which is defined as

$$\eta \equiv \sum_{i} (N_i - Z_i) Y_i = \sum_{i} \frac{(N_i - Z_i)}{M_i} X_i \quad \text{with} \quad -1 \le \eta \le 1 \quad (1.35)$$

where N_i , Z_i , M_i , Y_i , and X_i denote the number of neutrons and protons, the relative atomic mass (in atomic mass units), the mole fraction, and the mass fraction, respectively. The sum runs over all nuclides *i* in the plasma. Note that $\eta = 0$ if only N = Z nuclei (⁴He, ¹²C, ¹⁶O, and so on) are present. The



Fig. 1.14 Energy level schemes of ²⁶Al and ²⁶Mg, showing the lowest lying states in each nuclide. The vertical arrows represent γ -ray decays, while the diagonal arrows indicate β -decay transitions. Only the transitions indicated by the thick arrows have been observed experimentally. The transitions shown as thin arrows play an important role in the equilibration of the ground state and the isomer at $E_{\chi} = 228$ keV in ²⁶Al.

Note that the direct γ -ray de-excitation of the isomer is strongly inhibited by the selection rules. The presence of ²⁶Al^g in the interstellar medium is inferred from the observed intensity of the 1809 keV γ -ray, originating from the de-excitation of the first excited state in ²⁶Mg. A small β -decay branch of the ²⁶Al ground state to the E_x = 2938 keV $(J^{\pi} = 2^+)$ level in ²⁶Mg is omitted in the figure for clarity. See the text.

quantity η represents physically the number of *excess* neutrons per nucleon in the plasma and can only change as a result of weak interactions. The neutron excess must be monitored carefully in stellar model computations, since it is important for the nucleosynthesis during the late burning stages in massive stars and during explosive burning (Section 5.5). Furthermore, we already mentioned that electron capture is very important for the dynamic behavior of the core collapse in massive stars before a type II supernova explosion because it reduces the number of electrons available for pressure support (Section 1.4.3). Third, neutrinos emitted in weak interactions affect the energy budget of stars and thus influence models of stellar evolution and explosion.

We will focus here on the process of nuclear β -decay, which involves the proton, neutron, electron, positron, neutrino, and antineutrino, and will summarize some concepts that are important in the present context. For more information on weak interaction processes in stars see, for example, Langanke and Martínez-Pinedo (2000).



Fig. 1.15 Effective lifetime of ²⁶Al as a function of temperature. The solid line is adopted from Coc, Porquet and Nowacki (1999) and Runkle, Champagne and Engel (2001). It was obtained numerically by taking explicitly the equilibration of the ground and isomeric states via thermal excitations involving higher lying levels into account. At each temperature, the calculation was

started with a given amount of pure ²⁶Al^g. The value of $\tau_{\rm eff}(^{26}{\rm Al})$ is then defined by the time necessary for the total (ground plus isomeric state) ²⁶Al abundance to decline by 1/e. The dashed curve is calculated analytically by assuming that the ground and isomeric states are in thermal equilibrium (Example 1.5).

1.8.1

Weak Interaction Processes

Consider first the free neutron. It decays into a proton under the influence of the weak interaction via

$$n \to p + e^- + \overline{\nu} \tag{1.36}$$

where e^- and $\overline{\nu}$ denote an electron and antineutrino, respectively. The halflife of the free neutron amounts to $T_{1/2} = 10.2$ min. This decay is slower by many orders of magnitude compared to typical nuclear reaction time scales or electromagnetic decay probabilities and demonstrates that the interaction causing β -decay is indeed very weak. The most common weak interaction processes in nuclear β -decay are listed below:

$^{A}_{Z}X_{N} \rightarrow ^{A}_{Z+1}X'_{N-1} + \mathrm{e}^{-} + \overline{\nu}$	β^- -decay (electron emission)	(1.37)
${}^A_Z X_N \rightarrow {}^A_{Z-1} X'_{N+1} + \mathrm{e}^+ + \nu$	β^+ -decay (positron emission)	(1.38)
${}^{A}_{Z}X_{N} + \mathrm{e}^{-} \rightarrow {}^{A}_{Z-1}X_{N+1}' + \nu$	electron capture	(1.39)

$${}^{A}_{Z}X_{N} + \nu \rightarrow {}^{A}_{Z+1}X'_{N-1} + e^{-} \qquad \text{neutrino capture} \qquad (1.40)$$

$${}^{A}_{Z}X_{N} + \overline{\nu} \rightarrow {}^{A}_{Z-1}X'_{N+1} + e^{+}$$

$$(1.41)$$
Here e^+ , ν , and $\overline{\nu}$ denote a positron, neutrino, and antineutrino, respectively. In each of these interactions, the decaying nuclide changes its chemical identity, but the mass number *A* remains the same. The light particles e^- , e^+ , ν , and $\overline{\nu}$ are leptons, that is, they do not interact via the strong nuclear force.

The first three decays represent the most common weak interaction processes of radioactive nuclei in the laboratory. Consider as an example the β -decay of ${}^{64}_{29}Cu_{35}$. It may proceed via ${}^{64}_{29}Cu_{35} \rightarrow {}^{64}_{30}Zn_{34} + e^- + \overline{\nu} \ (\beta^-\text{-decay}), {}^{64}_{29}Cu_{35} \rightarrow {}^{64}_{28}\text{Ni}_{36} + e^+ + \nu \ (\beta^+\text{-decay}), \text{ or } {}^{64}_{29}\text{Cu}_{35} + e^- \rightarrow {}^{64}_{28}\text{Ni}_{35} + \nu \ (\text{electron capture}).$ When the electron is captured from the atomic K-shell, the process is called K capture. Neutrino capture is observed, for example, in the reaction ${}^{37}_{17}\text{Cl}_{20} + \nu \rightarrow {}^{37}_{18}\text{Ar}_{19} + e^-$, which has been used for the detection of solar neutrinos (Davis, Harmer and Hoffman 1968). Antineutrinos produced by nuclear power plants have been observed via the process $p + \overline{\nu} \rightarrow e^+ + n$ (Reines and Cowan 1959).

Positron emission and electron capture populate the same daughter nuclide. In later chapters, both of these decays will sometimes be considered together, while at other times it will be important to distinguish between these processes. We will be using the following abbreviated notation. The β -decay of ⁶⁴Cu to ⁶⁴Ni, irrespective of the specific process, will be denoted by ⁶⁴Cu($\beta^+\nu$)⁶⁴Ni. When we would like to make specific reference to the positron emission or electron capture, we write ⁶⁴Cu(e⁺ ν)⁶⁴Ni or ⁶⁴Cu(e⁻ ν)⁶⁴Ni, respectively. The β^- -decay of ⁶⁴Cu to ⁶⁴Zn will be denoted by ⁶⁴Cu($\beta^-\nu$)⁶⁴Zn, irrespective of the fact that an antineutrino rather than a neutrino is emitted in this decay.

1.8.2 Energetics

The total energy release in nuclear β -decay can be expressed by the difference of the *atomic* masses before and after the interaction. We find (Problem 1.7)

$$Q_{\beta^{-}} = \left[m({}^{A}_{Z}X_{N}) - m({}^{A}_{Z+1}X'_{N-1}) \right] c^{2} \qquad \beta^{-} \text{-decay}$$
(1.42)

$$Q_{e^+} = \left[m(^A_Z X_N) - m(^A_{Z-1} X'_{N+1}) - 2m_e \right] c^2 \qquad \text{positron emission} \qquad (1.43)$$

$$Q_{EC} = \left[m \begin{pmatrix} A \\ Z \end{pmatrix} N - m \begin{pmatrix} A \\ Z-1 \end{pmatrix} \right] c^2 - E_b \qquad \text{electron capture} \qquad (1.44)$$

where m_e and E_b denote the electron mass and the atomic binding energy of the captured electron, respectively. The released energy is almost entirely transferred to the emitted leptons. For example, in β^- -decay we have $Q_{\beta^-} = K_e + E_{\nu}$, where K_e and E_{ν} denote the kinetic electron energy and the total neutrino energy, respectively. Since there are three particles after the interaction, the electron and neutrino energy distributions must be continuous, ranging from zero to Q_{β^-} for each lepton. In electron capture, only one lepton

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is emitted and thus the neutrino is monoenergetic, with $Q_{EC} = E_{\nu}$. Furthermore, this decay mode is accompanied by X-ray emission since the vacancy in the atomic shell caused by the captured electron is quickly filled by other atomic electrons. Electron capture competes in general with positron emission since both decay modes populate the same daughter nucleus. However, if the difference in *atomic* masses amounts to $[m(_ZA_N) - m(_{Z-1}A_{N+1})]c^2 < 2m_ec^2 = 1022$ keV, then only electron capture is energetically allowed.

It must be emphasized that for positron emission in a stellar plasma, the energy release calculated from the mass difference of parent and daughter nucleus alone, $Q'_{e^+} = [m(^A_Z X_N) - m(^A_{Z-1} X'_{N+1})]c^2$, includes the annihilation energy $2m_ec^2 = 1022$ keV of the positron with another electron from the environment, as can be seen by comparison with Eq. (1.43). Therefore, the quantity Q'_{e^+} rather than Q_{e^+} is of primary interest when calculating the energy release of positron emission in a stellar plasma. Of course, Q'_{e^+} must be properly corrected for neutrino losses (see below).

We considered so far only β -decay transitions involving nuclear ground states. If a transition proceeds to an excited state in the daughter nucleus, then we have to replace Q_i by $Q_i^{gs} - E_x$ in Eqs. (1.42)–(1.44), where Q_i^{gs} and E_x denote the ground-state energy release and the excitation energy, respectively. Sometimes a β -decay populates levels in the daughter nucleus that are unstable by emission of light particles (protons, neutrons, or α -particles). These transitions give rise to β -delayed particle decays. They compete with transitions to bound states in the daughter nucleus. Therefore, both of these processes have to be distinguished carefully when modeling the nucleosynthesis in certain scenarios. For example, consider the β -decay of ²⁹S which proceeds with about equal probability to bound states in ²⁹P and to excited ²⁹P levels that are unbound by proton emission. In the first case, ²⁹S decays to the final nucleus ²⁹P via ²⁹S $\rightarrow e^+ + \nu + ^{29}P$, while in the second case ²⁹S decays to the final nucleus ²⁸Si via ²⁹S $\rightarrow e^+ + \nu + ^{29}P^*$ and ²⁹P* $\rightarrow ^{28}Si + p$. These processes can be distinguished by using the notations ²⁹S($e^+\nu$)²⁹P and ²⁹S($e^+\nu$ p)²⁸Si.

The neutrinos released in nuclear β -decay interact so weakly with matter that they are lost from the star unless the density is very large ($\rho \ge 10^{11} \text{ g/cm}^3$). Consequently, the average neutrino energy must usually be subtracted from the total nuclear energy liberated when considering the energy budget of a star. An approximate expression for the average neutrino energy loss in β^- -decay or positron emission is given by (Fowler, Caughlan and Zimmerman 1967)

$$\overline{E}_{\nu}^{\beta} \approx \frac{m_{\rm e}c^2}{2} w \left(1 - \frac{1}{w^2}\right) \left(1 - \frac{1}{4w} - \frac{1}{9w^2}\right) \tag{1.45}$$

where $w = (Q_{\beta} + m_e c^2)/m_e c^2$. The energy release of the β -decay, Q_{β} , is given by Eqs. (1.42) and (1.43), and may need to be corrected for the excitation en-

ergy if the transition proceeds to an excited state in the daughter nucleus. As already noted above, the neutrinos emitted in electron capture are monoenergetic.

Neutrino emission is also important for the transport of energy from the stellar interior to the surface, from which the energy can be radiated. During the early evolutionary stages of stars, internal energy is mainly transported by mechanisms such as radiative diffusion or convection. As a result, the rate of energy outflow is related to the temperature gradient of the star. At high temperature ($T > 10^9$ K), however, a relatively large number of photons have energies in excess of the threshold for pair production, $\gamma \rightarrow e^+ + e^-$ (Section 4.2.2). The positron and electron, in turn, may either annihilate via $e^+ + e^- \rightarrow 2\gamma$ or via $e^+ + e^- \rightarrow \nu + \overline{\nu}$. These neutrinos emerge directly from their point of origin and will escape from the star. In fact, during the late evolutionary stages of massive stars, this (non-nuclear) production of neutrino–antineutrino pairs represents the dominant energy loss mechanism. The energy outflow is in this case directly determined by the neutrino production rate. Neutrino energy losses rise strongly with temperature and have a profound influence on the stellar evolution of massive stars (Section 1.4.3 and Chapter 5).

1.8.3

Beta-Decay Probabilities

A detailed discussion of the theory of weak interactions in nuclei is beyond the scope of the present book. A modern account can be found, for example, in Holstein (1989). Here we will focus on the elementary Fermi theory of β -decay which explains satisfactorily lifetimes and the shapes of electron (or positron) energy distributions. Fermi's theory of β -decay is discussed in most introductory nuclear physics texts (see, for example, Krane 1988). We will initially assume that the β -decay occurs under laboratory conditions. Beta-decays in stellar plasmas will be addressed afterward. The rate of nuclear β -decay can be calculated from Fermi's golden rule of time-dependent, first-order perturbation theory (Messiah 1999). In order to illustrate the most important results, we will first discuss β^- -decay, although the derived expressions are equally valid for positron emission. The case of electron capture is subsequently discussed.

Electron or positron emission

The probability N(p) dp per unit time that an electron (or positron) with linear momentum between p and p + dp is emitted can be written as

$$d\lambda = N(p) dp = \frac{2\pi}{\hbar} \left| \int \Psi_f^* H \Psi_i \, dV \right|^2 \frac{dn}{dE_0} = \frac{2\pi}{\hbar} \left| H_{fi} \right|^2 \frac{dn}{dE_0} \tag{1.46}$$

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where Ψ_i and Ψ_f are the total wave functions before and after the decay, respectively, *H* is the Hamiltonian associated with the weak interaction, and *dV* is a volume element. The factor dn/dE_0 denotes the number of final states per unit energy. A given transition is obviously more likely to proceed if there is a large number of accessible final states. The experimental evidence shows that the shapes of many measured electron (or positron) energy distributions are dominated by the factor dn/dE_0 . The integral H_{fi} (or matrix element), which depends only very weakly on energy, determines the overall magnitude of the decay probability. It can be expressed in terms of the separate wave functions of the final nuclear state (ψ_f) and of the leptons (ϕ_e , ϕ_v) after the decay as

$$H_{fi} = g \int \left[\psi_f^* \phi_e^* \phi_\nu^* \right] \Omega \Psi_i \, dV \tag{1.47}$$

where the constant g determines the strength of the interaction. For electron (or positron) decay, the total wave function before the transition is equal to the wave function of the parent nucleus, $\Psi_i = \psi_i$. The operator Ω describes the transition from nuclear level ψ_i to level ψ_f . The emitted neutrino (or antineutrino) can be treated as a free particle because it interacts only weakly. The emitted electron (or positron) can also be treated as a free particle because it has a relatively high velocity and is little affected by the nuclear Coulomb field. Thus we may approximate the lepton wave functions by plane waves, normalized within the nuclear volume V, and expand the exponentials according to

$$\phi_{\mathsf{e}}(\vec{r}) = \frac{1}{\sqrt{V}} e^{-i\vec{p}\cdot\vec{r}/\hbar} \approx \frac{1}{\sqrt{V}} \left(1 + \frac{i\vec{p}\cdot\vec{r}}{\hbar} + \cdots \right)$$
(1.48)

$$\phi_{\nu}(\vec{r}) = \frac{1}{\sqrt{V}} e^{-i\vec{q}\cdot\vec{r}/\hbar} \approx \frac{1}{\sqrt{V}} \left(1 + \frac{i\vec{q}\cdot\vec{r}}{\hbar} + \cdots \right)$$
(1.49)

where \vec{p} and \vec{q} are the linear momenta of the electron (or positron) and the neutrino (or antineutrino), respectively. Consider, for example, the emission of an electron in β^- -decay with a typical kinetic energy of 1 MeV. The relativistic electron momentum amounts in this case to p = 1.4 MeV/*c*. For a nuclear radius of $r \approx 5$ fm we find then a value of $pr/\hbar = 0.035$. Hence, the second term in the expansion of Eq. (1.48) is usually very small and, therefore, the electron wave function is approximately constant over the nuclear volume. Similar arguments apply to the neutrino wave function. In the simplest case, one may then retain just the first, leading, term in Eqs. (1.48) and (1.49). It follows

$$|H_{fi}|^2 = \frac{1}{V^2} \left| g \int \psi_f^* \Omega \psi_i \, dV \right|^2 = \frac{1}{V^2} g^2 M^2 \tag{1.50}$$

The nuclear matrix element *M* describes the transition probability between the initial and final nuclear levels. A proper relativistic treatment of β -decay

shows that there are in fact two different matrix elements with different strengths that may contribute to the overall transition probability. Thus, we have to replace Eq. (1.50) by

$$|H_{fi}|^2 = \frac{1}{V^2} \left(G_V^2 M_F^2 + G_A^2 M_{\rm GT}^2 \right)$$
(1.51)

where G_V and G_A are the vector and axial-vector coupling constants, and M_F and M_{GT} are referred to as Fermi and Gamow–Teller matrix element, respectively. It can be shown that no interference term between vector and axial-vector interaction occurs. The two matrix elements depend on the structure of the initial and final nuclear states and can be calculated by using the shell model (Section 1.6).

The above nonrelativistic treatment of the nucleons and the assumption of constant lepton wave functions over the nuclear volume results in nuclear matrix elements that are independent of the lepton energies and define the so-called *allowed* β -*decay transitions*. In some decays, however, it turns out that angular momentum and parity selection rules prevent allowed transitions. In such cases, the next terms in the plane wave approximations of Eqs. (1.48) and (1.49) have to be taken into account and the nuclear matrix element is no longer independent of energy. These transitions are termed *forbidden* since they are much less likely to occur than allowed decays. The degree by which a transition is forbidden depends on how many terms in the plane wave approximation need to be taken into account until a nonvanishing nuclear matrix element is obtained. The second term gives rise to *first-forbidden* transitions, the third to *second-forbidden*, and so on. We will consider in the following only allowed β -decay transitions.

The density of final states, dn/dE_0 , in Eq. (1.46) determines for allowed transitions the shape of the electron (or positron) energy distribution. It is given by (Problem 1.10)

$$\frac{dn}{dE_0} = \frac{dn_e dn_\nu}{dE_0} = \frac{(4\pi)^2 V^2}{h^6} p^2 \, dp \, q^2 \, dq \frac{1}{dE_0} \tag{1.52}$$

The final state (or total decay) energy is $E_0 = Q = K_e + E_\nu$, where Q is the energy release for the transition under consideration (see Eqs. (1.42) and (1.43); if the decay proceeds to an excited state, Q must account for the excitation energy). Since the neutrino mass is very small, we may use $m_\nu c^2 \approx 0$, so that $q = E_\nu/c = (E_0 - K_e)/c$ and $dq/dE_0 = 1/c$. A correction must be applied to Eq. (1.52) that takes into account the Coulomb interaction between the daughter nucleus and the emitted electron or positron. The electron in β^- -decay feels an attractive Coulomb force, while the positron plane wave in

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Eq. (1.48) has to be replaced by a distorted wave. The correction factor is referred to as *Fermi function*, F(Z', p), and depends on the electron or positron momentum and the charge of the daughter nucleus. The function F(Z', p) can be calculated numerically and is tabulated in Gove and Martin (1971).

It follows from Eqs. (1.46), (1.51), and (1.52) that

$$d\lambda = N(p) dp = \frac{1}{2\pi^3 \hbar^7 c^3} \left(G_V^2 M_F^2 + G_A^2 M_{\rm GT}^2 \right) F(Z', p) p^2 (E_0 - K_{\rm e})^2 dp \quad (1.53)$$

This distribution vanishes for p = 0 and at the endpoint where the maximum electron or positron kinetic energy is equal to the total decay energy, $K_e^{\max} = E_0 = Q$. Hence, a measurement of the momentum or energy distribution in a given decay yields a value for the total energy release in β -decay. Total relativistic energy, kinetic energy, and linear momentum of the electron or positron are related by

$$E_{\rm e} = K_{\rm e} + m_{\rm e}c^2 = \sqrt{(m_{\rm e}c^2)^2 + (pc)^2}$$
(1.54)

The total decay constant is then given by the integral

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{\left(G_V^2 M_F^2 + G_A^2 M_{GT}^2\right)}{2\pi^3 \hbar^7 c^3} \int_0^{p_{\max}} F(Z', p) p^2 (E_0 - K_e)^2 dp$$

= $\frac{m_e^5 c^4}{2\pi^3 \hbar^7} \left(G_V^2 M_F^2 + G_A^2 M_{GT}^2\right) f(Z', E_e^{\max})$ (1.55)

The dimensionless quantity

$$f(Z', E_{\rm e}^{\rm max}) = \frac{1}{m_{\rm e}^5 c^7} \int_0^{p_{\rm max}} F(Z', p) p^2 (E_{\rm e}^{\rm max} - E_{\rm e})^2 \, dp \tag{1.56}$$

is referred to as the *Fermi integral* and depends only on the charge Z' of the daughter nucleus and on the maximum total energy of the electron, E_e^{max} . Numerical values of $f(Z', E_e^{\text{max}})$ have also been tabulated. For the derivation of Eq. (1.55) we used the relationships $p_{\text{max}}c = \sqrt{(E_e^{\text{max}})^2 - (m_ec^2)^2}$ and $E_0 - K_e = K_e^{\text{max}} - K_e = E_e^{\text{max}} - E_e$ that are obtained from Eq. (1.54).

We can rewrite Eq. (1.55) as

$$f(Z', E_{\rm e}^{\rm max})T_{1/2} = \frac{2\pi^3\hbar^7}{m_{\rm e}^5c^4} \frac{\ln 2}{\left(G_V^2 M_F^2 + G_A^2 M_{\rm GT}^2\right)}$$
(1.57)

The quantity $f(Z', E_e^{\max})T_{1/2}$ is called the *ft-value* and is experimentally obtained from measurements of the half-life and the maximum energy of the emitted electrons or positrons. The ft-value is a standard measure for the strength of a particular β -decay transition and yields information about the nuclear matrix elements and the coupling constants.

Electron capture

The decay constant for allowed electron capture can be obtained in a similar manner. Recall that in this case the energy spectrum of the emitted neutrino is not continuous, but monoenergetic with $Q_{EC} = E_0 = E_{\nu}$. Instead of Eq. (1.46) we write

$$\lambda = \frac{2\pi}{\hbar} \left| \int \Psi_f^* H \Psi_i \, dV \right|^2 \frac{dn}{dE_0} = \frac{2\pi}{\hbar} |H_{fi}|^2 \frac{dn_\nu}{dE_0} \tag{1.58}$$

The density of final states in this case is given by (Problem 1.10)

$$\frac{dn_{\nu}}{dE_0} = \frac{Vq^2}{2\pi^2\hbar^3}\frac{dq}{dE_0} = \frac{VE_{\nu}^2}{2\pi^2\hbar^3c^3}$$
(1.59)

where we used $E_{\nu} = qc$. The total wave functions before and after the decay are now given by $\Psi_i = \psi_i \phi_e$ and $\Psi_f = \psi_f \phi_\nu$ (the subscripts have the same meaning as before). Usually an electron from the atomic K shell is captured because these have the largest probability of being near the nucleus. But the electron is now in a bound state and cannot be described by a free-particle plane wave. One can approximate ϕ_e by the electron wave function ϕ_K of the K orbit at the location of the nucleus,

$$\phi_{e}(\vec{r}) = \phi_{K}(\vec{r}) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_{0}}\right)^{3/2} e^{-Zr/a_{0}}$$
$$\approx \phi_{K}(0) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_{0}}\right)^{3/2} = \frac{1}{\sqrt{\pi}} \left(\frac{Zm_{e}e^{2}}{\hbar^{2}}\right)^{3/2}$$
(1.60)

with *Z* the atomic number of the parent nucleus. The constant a_0 denotes the Bohr radius, $a_0 = \hbar^2 / (m_e e^2) = 0.0529$ nm. For the neutrino wave function ϕ_v we use again only the first (constant) term in the plane wave approximation.

From Eqs. (1.49), (1.58)–(1.60) one finds for the decay constant of allowed electron capture

$$\lambda_K = 2 \frac{Z^3 m_e^3 e^6}{\pi^2 \hbar^{10} c^3} \left(G_V^2 M_F^2 + G_A^2 M_{\rm GT}^2 \right) E_\nu^2 \tag{1.61}$$

where the matrix elements are defined as before in terms of initial and final state nuclear wave functions. Note that these are identical to the matrix elements that occur in Eq. (1.51) for positron emission since they connect the very same nuclear states. The additional factor of 2 in Eq. (1.61) arises because either of the two electrons in the K shell can be captured. The transition probability for the weaker L-capture can be calculated in a similar manner. It is obvious that the electron capture probability increases strongly with the charge Z of the parent nucleus. This is the reason for the fact that electron capture is

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greatly favored over positron emission in heavy nuclei. The above expression must be corrected for relativistic effects and the influence of the shielding of the nuclear Coulomb field by the outer electrons. Such corrections have been calculated numerically and are tabulated, for example, in Gove and Martin (1971).

Fermi and Gamow–Teller transitions

We already commented on the classification of β -decays into allowed and forbidden transitions. In the first case, the leptons do not remove any orbital angular momentum. In the latter case, the radiations are inhibited because angular momentum conservation requires the leptons to carry off orbital angular momentum or because the parities of the initial and final nuclear states are mismatched. The allowed radiations are further subdivided into *Fermi transitions* and *Gamow–Teller transitions*. They can only occur (that is, the corresponding matrix elements M_F or M_{GT} are nonzero only) if certain selection rules are satisfied for the nuclear spins (J_i , J_f) and parities (π_i , π_f) of the initial and final nuclear states connected by the transition:

$$\Delta J \equiv |J_i - J_f| = 0, \qquad \pi_i = \pi_f \qquad \text{for Fermi transitions} \qquad (1.62)$$

$$\Delta J \equiv |J_i - J_f| = 0 \text{ or } 1, \qquad \pi_i = \pi_f \qquad \text{for Gamow-Teller transitions} \qquad (but \text{ not } J_i = 0 \to J_f = 0) \qquad (1.63)$$

It follows that one can study these cases separately since decays with $0 \rightarrow 0$ $(\Delta J = 0)$ and $\pi_i = \pi_f$ represent pure Fermi transitions ($M_{\text{GT}} = 0$), while decays with $\Delta J = 1$ and $\pi_i = \pi_f$ are pure Gamow–Teller transitions ($M_F = 0$). Examples for pure Fermi and Gamow–Teller transitions are ¹⁴O \rightarrow ¹⁴N + e⁺ + ν ($J_i = 0^+ \rightarrow J_f = 0^+$) and ⁶He \rightarrow ⁶Li + e⁻ + $\overline{\nu}$ ($J_i = 0^+ \rightarrow J_f = 1^+$). The decay of the free neutron in Eq. (1.36), on the other hand, represents a mixed transition. From studies of such decays, the values of the coupling constants G_V and G_A can be deduced (see, for example, Wilkinson 1994).

In the laboratory, where the parent nucleus is usually in its ground state, β -decay transitions proceed to all energetically accessible states in the daughter nucleus. The total decay constant is given by the sum of transition probabilities for all of these β -decay branches. Such laboratory β -decay constants or half-lives are independent of temperature and density. Experimental values of $T_{1/2}$ are tabulated in Audi et al. (2003) and this reference will be used as a source of terrestrial half-lives throughout this book, unless mentioned otherwise.

1.8.4 Beta-Decays in a Stellar Plasma

Consider now the weak interaction processes that take place when β -decays occur in a stellar plasma at elevated temperature *T* and density ρ . In a hot plasma, excited states in the parent nucleus are thermally populated and these excited levels may also undergo β -decay transitions to the ground state or to excited states in the daughter nucleus. The total β -decay rate in a stellar plasma, λ_{β}^* , is given by the weighted sum of the individual transition rates, λ_{ij} , according to

$$\lambda_{\beta}^{*} = \sum_{i} P_{i} \sum_{j} \lambda_{ij} \tag{1.64}$$

The sum on *i* and *j* is over parent and daughter states, respectively. The population probabilities, P_i , of excited states in a nondegenerate plasma at thermodynamic equilibrium are given by Eq. (1.34). Since the quantity P_i is temperature dependent, it follows immediately that λ_{β}^* will also depend on temperature. In fact, if the decay constants for excited state β -decays are larger than the one for ground-state β -decay, then the total decay constant λ_{β}^* may become strongly temperature dependent. Clearly, even if the ground state of the parent nucleus is stable in the laboratory, it may nevertheless undergo β -decay in a hot stellar plasma. Similar considerations apply to the β -decay of the daughter nucleus. In the laboratory, it cannot decay back to the parent nucleus because the transition is energetically forbidden. In a hot plasma, however, β -decay transitions may occur from thermally populated excited states in the daughter nucleus to the ground state or to excited states in the parent nucleus. The situation is schematically shown in Fig. 1.16. In practice, one finds



Fig. 1.16 Beta-decays (a) in the laboratory, and (b) in a hot stellar plasma. The vertical direction corresponds to an energy scale. For reasons of clarity, only two levels are shown in the parent nucleus X and the daughter nucleus X'. The ground and first excited state are labeled by 0 and 1,

respectively. In the laboratory, the β -decay proceeds from the ground state of nucleus X to levels in nucleus X', while far more β -decay transitions are energetically accessible in a stellar plasma owing to the thermal excitation of levels (dashed vertical arrows). See the text.

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that most of the transition probability for β^- -decay or positron emission in a hot stellar plasma arises from the first few levels in a given parent nucleus. Note that the β^- -decay rate becomes also density dependent at sufficiently large values of ρ when the electron gas is degenerate. In fact, the decay rate decreases with increasing density since the number of final states available for the emitted electron to occupy is reduced (Langanke and Martínez-Pinedo 2000).

Example 1.5

In the laboratory, β^+ -decays of the nuclide ²⁶Al have been observed both from the ground state ($J^{\pi} = 5^+$) and from the first excited (isomeric) state ($J^{\pi} = 0^+$) located at an excitation energy of $E_x = 228$ keV (Fig. 1.14). The ground state decays via positron emission to excited levels in the daughter nucleus ²⁶Mg (we will neglect a small electron capture branch) with a half-life of $T_{1/2}^{gs} =$ 7.17×10^5 y, while the first excited state decays to the ²⁶Mg ground state with a half-life of $T_{1/2}^m = 6.345$ s. Above a temperature of T = 0.4 GK, both of these ²⁶Al levels are in thermal equilibrium (Fig. 1.15). Calculate the *stellar* half-life of ²⁶Al when the plasma temperature amounts to T = 2 GK.

According to Eq. (1.64), the decay constant of 26 Al in the stellar plasma is given by

$$\lambda_{\beta}^{*} = P_{\rm gs}\lambda_{\rm gs} + P_{m}\lambda_{m} = P_{\rm gs}\frac{\ln 2}{T_{1/2}^{\rm gs}} + P_{m}\frac{\ln 2}{T_{1/2}^{m}}$$

where the subscripts "gs" and "m" denote the ground state and the first excited state, respectively. The thermal population probability P_i (that is, the fraction of ²⁶Al nuclei residing in either the ground or first excited state) can be calculated from Eq. (1.34) (a numerical expression for the quantity kT is given in Section 3.1.1). Thus

$$\begin{split} \lambda_{\beta}^{*} &= \frac{\ln 2}{g_{gs}e^{-E_{gs}/kT} + g_{m}e^{-E_{m}/kT}} \left[\frac{g_{gs}e^{-E_{gs}/kT}}{T_{1/2}^{gs}} + \frac{g_{m}e^{-E_{m}/kT}}{T_{1/2}^{m}} \right] \\ &= \frac{\ln 2}{(2 \cdot 5 + 1) + (2 \cdot 0 + 1)e^{-0.228/kT}} \left[\frac{(2 \cdot 5 + 1)}{T_{1/2}^{gs}} + \frac{(2 \cdot 0 + 1)e^{-0.228/kT}}{T_{1/2}^{m}} \right] \\ &= \frac{\ln 2}{11 + e^{-0.228/0.0862T_{9}}} \left[\frac{11}{T_{1/2}^{gs}} + \frac{e^{-0.228/0.0862T_{9}}}{T_{1/2}^{m}} \right] \\ &\approx \frac{\ln 2}{11} \left[\frac{e^{-0.228/0.0862T_{9}}}{6.345 \, \text{s}} \right] = 9.93 \times 10^{-3} e^{-2.646/T_{9}} \quad (\text{s}^{-1}) \end{split}$$

Hence we find at $T = 2 \text{ GK} (T_9 = 2)$

 $\lambda_{\beta}^{*} = 9.93 \times 10^{-3} e^{-2.646/2.0} \,\mathrm{s}^{-1} = 0.0026 \,\mathrm{s}^{-1}$

and the stellar half-life of ²⁶Al amounts to $T_{1/2}^* = \ln 2/\lambda_{\beta}^* = 270$ s. The result is only valid for densities of $\rho \leq 10^6$ g/cm³, since at higher densities electron capture needs to be taken into account (see below). The results from the above method for calculating the stellar half-life of ²⁶Al are shown as the dashed line in Fig. 1.15. The values are only correct for temperatures in the range of T =0.4–5 GK. At lower temperatures, the ground and isomeric states are not in thermal equilibrium (Section 1.7.5), while at higher temperatures the thermal populations of other excited states in ²⁶Al have to be taken into account.

We will now discuss the interesting case of electron capture. It will be shown later (Section 3.1.1) that the average thermal energies at the temperatures typical for the interior of main-sequence stars and red giants amount to \approx 1 keV and a few tens of keV, respectively. For most atoms, however, the ionization energies are smaller than these values. Therefore, most nuclei in these environments possess few, if any, bound electrons. The decay constant for bound electron capture, given by Eq. (1.61), may thus be very small or even zero. In the hot interiors of stars, however, there is an appreciable density of free electrons. Hence, β -decays can proceed by capture of (free) electrons from the continuum. The probability of continuum electron capture is proportional to the free electron density at the location of the nucleus and is inversely proportional to the average electron velocity which depends on the plasma temperature. Consequently, the rate of continuum electron capture depends on the local electron temperature and the density. At lower stellar temperatures, a given parent nucleus may not be completely ionized. In that case, both bound and continuum electron capture contribute to the total decay constant.

At low densities, the kinetic energies of the free electrons are usually small. At very high densities, however, the (Fermi) energy of the degenerate electrons may become sufficiently large to cause nuclei to undergo continuum capture of energetic electrons, even if they are stable under laboratory conditions. Of course, electron capture transitions involving thermally excited nuclear levels must also be taken into account according to Eq. (1.64).

Moreover, at high temperature (T > 1 GK) quite a large number of photons have energies in excess of the threshold energy for pair production (Section 4.2.2). Although a positron annihilates quickly in the stellar plasma with an electron, the pair production rate becomes eventually so large at high temperatures that the positron density is a significant fraction of the electron density. Thus, capture of continuum positrons by nuclei must be considered in addition to continuum electron capture.

The decay constant for continuum electron capture can be obtained easily for a given nuclide if its laboratory decay constant for bound electron capture is known. The ratio of stellar to laboratory decay constant is approximately equal to the ratio of the electron densities at the nucleus for the stellar and lab-

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oratory environments, that is, the ratio of probabilities for finding an electron at the nucleus where it can be captured. An order of magnitude estimate for the ratio of electron capture probabilities is given by

$$\frac{\lambda_{\text{star}}}{\lambda_{\text{lab}}} \approx \frac{n_{\text{e}^-} \langle F(Z, p) \rangle}{2N_A |\phi_{\text{e}}(0)|^2} \tag{1.65}$$

where $n_{e^-}/N_A = \rho(1 - \eta)/2$ is the electron density (Fowler, Caughlan and Zimmerman 1967), η is the neutron excess parameter given by Eq. (1.35), and $|\phi_e(0)|^2$ is given by Eq. (1.60). The Fermi function F(Z, p) accounts for the fact that the wave function of the captured electron is distorted by the nuclear Coulomb field. Since the electron velocities in the plasma are given by a distribution, the Fermi function must be averaged over the electron velocities. It can be seen from Eq. (1.65) that the ratio $\lambda_{\text{star}}/\lambda_{\text{lab}}$ depends on the density and composition (through n_{e^-}), and on the temperature (through $\langle F(Z, p) \rangle$). Note that the above expression is independent of nuclear matrix elements. For more information, including a discussion of induced continuum electron capture (that is, when a nuclide is stable in the laboratory), see Bahcall (1964).

It is obvious from the above considerations that many different transitions contribute to the stellar decay rate of a given nucleus. In the laboratory, the decay proceeds from the ground state of parent nucleus X to energetically accessible states in the daughter nucleus X'. In a stellar plasma, the labels "parent" and "daughter" can alternatively apply to both nuclei. For example, in the laboratory ⁵⁶Mn decays to the stable nuclide ⁵⁶Fe via ⁵⁶Mn($\beta^- \nu$)⁵⁶Fe. At high temperatures and densities, however, ⁵⁶Fe decays via continuum electron capture, ⁵⁶Fe(e⁻, ν)⁵⁶Mn, and via positron emission through thermally populated ⁵⁶Fe states, ⁵⁶Fe(e⁺ ν)⁵⁶Mn.

The estimation of stellar β -decay rates essentially reduces to the calculation of (i) nuclear matrix elements by using some model of nuclear structure (for example, the shell model; Section 1.6), and (ii) the appropriate Fermi functions and integrals for all energetically accessible transitions from the parent to the daughter nucleus. The calculations can be constrained and tested by experimental measurements of half-lives and Gamow–Teller strength distributions. Stellar weak interaction rates and the associated neutrino energy losses for a range of temperatures and densities are tabulated in Fuller, Fowler and Newman (1982) (for the proton, neutron, and nuclides with A = 21-60), Oda et al. (1994) (for A = 17-39), and Langanke and Martínez-Pinedo (2001) (for A = 45-65). Figure 1.17 shows as an example the temperature dependence of the stellar decay constants for the electron capture (solid line) and positron emission (dashed line) of ⁵⁶Co. Note that ⁵⁶Co decays in the laboratory to ⁵⁶Fe by bound state electron capture with a half-life of $T_{1/2} = 77.2$ d.

Finally, we will briefly discuss a neutrino energy loss mechanism that becomes important at very high temperatures and densities. It is referred to as the *Urca process* (Gamow and Schoenberg 1940) and consists of alternate elec-

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Fig. 1.17 Stellar decay constants versus temperature for the electron capture (solid line) and positron emission (dashed line) of ⁵⁶Co. The electron capture decay constant is calculated for the condition $\rho(1 - \eta)/2 = 10^7$ g/cm³ and increases with rising density. In the laboratory, ⁵⁶Co decays to ⁵⁶Fe by bound electron capture with a decay constant of $\lambda_{lab} = 1.0 \times 10^{-7}$ s⁻¹ ($T_{1/2} = \ln 2/\lambda_{lab} = 77.2$ d). Data from Langanke and Martínez-Pinedo (2000).

tron captures and β^- -decays involving the same pair of parent and daughter nuclei

$${}^{A}_{Z}X_{N}(e^{-},\nu){}^{A}_{Z-1}X'_{N+1}(\beta^{-}\nu){}^{A}_{Z}X_{N}\dots$$
(1.66)

The net result of two subsequent decays gives ${}_Z^A X_N + e^- \rightarrow {}_Z^A X_N + e^- + \nu + \overline{\nu}$. A neutrino–antineutrino pair is produced with no change in the composition, but energy in the form of neutrinos is lost from the star. It is obvious from energy arguments that both the electron capture and the β^- -decay cannot occur spontaneously. The first step may be induced by continuum electron capture of energetic electrons when the density is high, while the second step may proceed from thermally populated excited states when the temperature is high. In the end, thermal energy is lost every time a pair of interactions goes to completion. The mechanism represents an efficient cooling process that will not only depend on temperature and density, but also on the composition of the stellar plasma. The Urca process is thought to be vital for understanding the explosion mechanism in some models of type Ia supernovae (Section 1.4.4).

Problems

1.1 Determine the number of protons, *Z*, and the number of neutrons, *N*, for the nuclides ¹⁸F, ⁵⁶Ni, ⁸²Rb, ¹²⁰In, ¹⁵⁰Gd, and ²³⁵U.

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1.2 How much energy is released in the following reactions: (i) ${}^{3}\text{He}(d,p){}^{4}\text{He}$; (ii) ${}^{17}\text{O}(p,\gamma){}^{18}\text{F}$; (iii) ${}^{12}\text{C}(\alpha,\gamma){}^{16}\text{O}$; and (iv) ${}^{13}\text{C}(\alpha,n){}^{16}\text{O}$? Assume that the reactions involve nuclei in their ground states only. Use the results presented in Table 1.1.

1.3 Consider the chain of radioactive decays, $1 \rightarrow 2 \rightarrow 3$, where 1, 2, and 3 denote a parent, daughter, and final nuclide respectively. Assume that initially only the parent nuclei are present, that is, $N_1(t = 0) = N_0$, $N_2(t = 0) = 0$, $N_3(t = 0) = 0$, and that species 3 is stable. (i) Set up the differential equation describing the abundance change of species 2 and find the time evolution of the daughter abundance, $N_2(t)$. (ii) Find the time evolution of the final nuclide abundance, $N_3(t)$. (iii) Examine the abundances N_1 , N_2 , and N_3 at small values of *t*. Keep only linear terms in the expansion of the exponential function and interpret the results.

1.4 With the aid of Fig. 1.10, predict the spins and parities of ¹⁹O, ³¹P, and ³⁷Cl for both the ground state and the first excited state. Compare your answer with the observed values. These can be found in Endt (1990) and Tilley et al. (1995).

1.5 Suppose that an excited state with spin and parity of 2^+ in a nucleus of mass A = 20 decays via emission of a γ -ray with a branching ratio of 100% to a lower lying level with spin and parity of 0^+ . Assume that the γ -ray energy amounts to $E_{\gamma} = E_i - E_f = 6$ MeV. Estimate the maximum expected γ -ray transition probability $\Gamma = \lambda \hbar$.

1.6 Consider a nucleus in a plasma at thermal equilibrium. Calculate the population probabilities of the ground state ($E_0 = 0$) and of the first three excited states ($E_1 = 0.1$ MeV, $E_2 = 0.5$ MeV, $E_3 = 1.0$ MeV). Perform the computations for two temperatures, $T = 1.0 \times 10^9$ K and 3.0×10^9 K, and assume for simplicity that all states have the same spin value.

1.7 Derive the relationships of Eqs. (1.42)–(1.44) from the differences in *nuclear* masses before and after the decay.

1.8 How much energy is released in the following β -decays: (i) ⁷Be(e⁻, ν)⁷Li; (ii) ¹⁴C($\beta^{-}\nu$)¹⁴N; and (iii) ¹⁸F(e⁺ ν)¹⁸O? Assume that the decays involve nuclei in their ground states only. Use the results presented in Table 1.1.

1.9 Calculate the average neutrino losses in the decays ${}^{13}N(e^+\nu){}^{13}C$ and ${}^{15}O(e^+\nu){}^{15}N$. Assume that the positron emissions involve the ground states of the parent and daughter nuclei only. Use the results presented in Table 1.1.

1.10 Derive Eq. (1.52) for the density of final states. Recall that the final state contains both an electron and a neutrino. You have to count the states in the six-dimensional *phase space* that is defined by three space and three linear momentum coordinates. The unit volume in phase space is h^3 .

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2 Nuclear Reactions

2.1 Cross Sections

The cross section σ is a quantitative measure for the probability that an interaction will occur. In the following we define several quantities which are displayed in Fig. 2.1. Suppose that a beam of \mathcal{N}_b particles per unit time t, covering an area A, is incident on a target. The number of *nonoverlapping* target nuclei within the beam is \mathcal{N}_t . We assume that the total number of interactions that occur per unit time, \mathcal{N}_R/t , is equal to the total number of emitted (nonidentical) interaction products per unit time, \mathcal{N}_e/t . If the interaction products are scattered incident particles, then we are referring to elastic scattering. If the interaction products have an identity different from the incident particles, then we are referring to a reaction. The number of interaction products emitted at an angle θ with respect to the beam direction into the solid angle $d\Omega$ is $\mathcal{N}_e^{d\Omega}$. The area perpendicular to the direction θ covered by a radiation detector is given by $dF = r^2 d\Omega$. The cross section is defined by

$$\sigma \equiv \frac{(\text{number of interactions per time})}{(\text{number of incident particles per area per time})(\text{number of target nuclei within the beam})} = \frac{(\mathcal{N}_R/t)}{[\mathcal{N}_h/(tA)] \mathcal{N}_t}$$
(2.1)

We will use this general definition to describe reaction probabilities in astrophysical plasmas and in laboratory measurements of nuclear reactions. In the latter case, two situations are frequently encountered: (i) if the beam area, A, is larger than the target area, A_t , then

$$\frac{\mathcal{N}_R}{t} = \frac{\mathcal{N}_b}{tA} \,\mathcal{N}_t \sigma \tag{2.2}$$

and the number of reactions per unit time is expressed in terms of the incident *particle flux*, $N_b/(tA)$, the number of target nuclei, N_t , and the cross section; (ii) if the target area, A_t , is larger than the beam area, A, then

$$\frac{\mathcal{N}_R}{t} = \frac{\mathcal{N}_b}{t} \frac{\mathcal{N}_t}{A} \sigma \tag{2.3}$$

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and the number of reactions per unit time is expressed in terms of the incident *particle current*, N_b/t , the total number of target nuclei within the beam per area covered by the beam, N_t/A , and the cross section. Of course, for a homogeneous target, N_t/A is equal to the *total* number of target nuclei divided by the *total* target area A_t . The latter quantity is easier to determine in practice. We can also express the total cross section, σ , and the differential cross section, $d\sigma/d\Omega$, in terms of the number of emitted interaction products

$$\frac{\mathcal{N}_e}{t} = \sigma \frac{(\mathcal{N}_b/t)}{A} \mathcal{N}_t \tag{2.4}$$

$$\frac{\mathcal{N}_e^{d\Omega}}{t} = \left(\frac{d\sigma}{d\Omega}\right) \frac{(\mathcal{N}_b/t)}{A} \mathcal{N}_t \, d\Omega \tag{2.5}$$

If we define $N_{et} \equiv N_e / N_t$, that is, the number of emitted interaction products per target nucleus, then we obtain

$$\sigma = \frac{(\mathcal{N}_{et}/t)}{(\mathcal{N}_b/t)(1/A)} \quad \text{and} \quad \left(\frac{d\sigma}{d\Omega}\right) = \frac{(\mathcal{N}_{et}^{d\Omega}/t)}{(\mathcal{N}_b/t)(1/A)}\frac{1}{d\Omega}$$
(2.6)

With the definition of a flux or current density *j* as the number of particles per time per area, we can write for the beam and emitted interaction products

$$j_b = \frac{(\mathcal{N}_b/t)}{A} \tag{2.7}$$

$$j_{et} = \frac{(\mathcal{N}_{et}^{d\Omega}/t)}{dF}$$
(2.8)

For the total and differential cross section one finds

$$\sigma = \frac{(\mathcal{N}_{et}/t)}{j_b} \tag{2.9}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{j_{et}\,dF}{j_b\,d\Omega} = \frac{j_{et}r^2\,d\Omega}{j_b\,d\Omega} = \frac{j_{et}r^2}{j_b} \tag{2.10}$$

These quantities are related by

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) \, d\Omega \tag{2.11}$$

Common units of nuclear reaction and scattering cross sections are

$$1 b \equiv 10^{-24} \text{ cm}^2 = 10^{-28} \text{ m}^2$$

$$1 \text{ fm}^2 = (10^{-15} \text{ m})^2 = 10^{-30} \text{ m}^2 = 10^{-2} \text{ b}$$

In this chapter, all kinematic quantities are given in the center-of-mass system (Appendix C), unless noted otherwise.



Fig. 2.1 Typical nuclear physics counting experiment, showing a beam of \mathcal{N}_b particles per unit time, \mathcal{N}_t nonoverlapping target nuclei within the beam area A, \mathcal{N}_e interaction products and a detector of area dF. The detector is located at an angle of θ with respect to the incident beam direction. The two situations correspond to as follows: (a) the target area is larger than the beam area; and (b) the beam area is larger than the target area.

2.2

Reciprocity Theorem

Consider the reaction $A + a \rightarrow B + b$, where A and a denote the target and projectile, respectively, and B and b are the reaction products. The cross section of this reaction is fundamentally related to that of the reverse reaction, $B + b \rightarrow A + a$, since these processes are invariant under time-reversal, that is, the direction of time does not enter explicitly in the equations describing these processes. At a given total energy, the corresponding cross sections $\sigma_{Aa \rightarrow Bb}$ and $\sigma_{Bb \rightarrow Aa}$ are not equal but are simply related by the phase space available in the exit channel or, equivalently, by the number of final states per unit energy interval in each case. The number of states available for momenta between p and p + dp is proportional to p^2 (Messiah 1999). Hence

$$\sigma_{Aa \to Bb} \sim p_{Bb}^2$$
 and $\sigma_{Bb \to Aa} \sim p_{Aa}^2$ (2.12)

The linear momentum and the de Broglie wavelength are related by $\lambda = h/p$. The wave number *k* of the free particle is defined in terms of the de Broglie wavelength by $\lambda \equiv 2\pi/k$. Hence, we have $p = mv = \hbar k$. It follows (Blatt and Weisskopf 1952) that

$$\frac{k_{Aa}^2 \sigma_{Aa \to Bb}}{(1+\delta_{Aa})} = \frac{k_{Bb}^2 \sigma_{Bb \to Aa}}{(1+\delta_{Bb})}$$
(2.13)

This expression is called *reciprocity theorem* and it holds for differential as well as total cross sections. The factors $(1 + \delta_{ij})$ account for the fact that cross sections between identical particles in the entrance channel are twice those between different particles, other factors being equal.

When particles with spin are involved in the reactions, then the above equation must be modified by multiplying the density of final states by their statistical weights. Since there are $(2j_i + 1)$ states of orientation available for a particle with spin j_i , we can write for unpolarized particles

$$\frac{k_{Aa}^2(2j_A+1)(2j_a+1)\sigma_{Aa\to Bb}}{(1+\delta_{Aa})} = \frac{k_{Bb}^2(2j_B+1)(2j_b+1)\sigma_{Bb\to Aa}}{(1+\delta_{Bb})}$$
(2.14)

$$\frac{\sigma_{Bb\to Aa}}{\sigma_{Aa\to Bb}} = \frac{(2j_A+1)(2j_a+1)}{(2j_B+1)(2j_b+1)} \frac{k_{Aa}^2(1+\delta_{Bb})}{k_{Bb}^2(1+\delta_{Aa})}$$
(2.15)

It follows that the cross section $\sigma_{Bb\to Aa}$ can be easily calculated, independently from any assumptions regarding the reaction mechanism, if the quantity $\sigma_{Aa\to Bb}$ is known experimentally or theoretically. Equation (2.15) is applicable to particles with rest mass as well as to photons. It must be emphasized that the symbols *A*, *a*, *b*, and *B* do not only refer to specific nuclei but, more precisely, to specific states. In other words, the reciprocity theorem connects the same nuclear levels in the forward as in the reverse reaction.

The reciprocity theorem has been tested in a number of experiments. An example is shown in Fig. 2.2. Compared are differential cross sections for the reaction pair ${}^{24}Mg(\alpha,p){}^{27}Al$ (open circles) and ${}^{27}Al(p,\alpha){}^{24}Mg$ (crosses), connecting the ground states of ${}^{24}Mg$ and ${}^{27}Al$. Both reactions were measured at the same center-of-mass total energy and angle. The differential cross sections exhibit a complicated structure, presumably caused by overlapping broad resonances. Despite the complicated structure, it can be seen that the agreement between forward and reverse differential cross section is excellent. Such results support the conclusion that, to this accuracy, nuclear reactions are invariant under time-reversal. See also Blanke et al. (1983).

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Fig. 2.2 Experimental test of the reciprocity theorem for the reaction pair ²⁴Mg(α ,p)²⁷Al (open circles) and ²⁷Al(p, α)²⁴Mg (crosses), connecting the ground states of ²⁴Mg and ²⁷Al. The differential cross sections of both reactions are shown for the same total energy and detection angle in the center-of-

mass system. The cross sections have also been adjusted to compensate for differences in spins. Reprinted with permission from W. von Witsch, A. Richter and P. von Brentano, Phys. Rev., Vol. 169, p. 923 (1968). Copyright (1968) by the American Physical Society.

2.3

Elastic Scattering and Method of Partial Waves

2.3.1

General Aspects

The interactions between nucleons within a nucleus and between nucleons participating in nuclear reactions have to be described using quantum mechanics. The fundamental strong interaction is very complicated and not precisely known. We know from experiments that it is of short range. Furthermore, it exhibits a part which is attractive at distances comparable to the size of a nucleus and another part which is repulsive at very short distances. Because of the complexity of this nucleon-nucleon interaction it is necessary to employ approximations. Instead of calculating all the interactions between all nucleons exactly, one frequently resorts to using effective potentials. These describe the behavior of a nucleon, or a group of nucleons (such as an α -particle), in the effective (average) field of all the other nucleons. Because of the approximate nature of this approach, the resulting effective potentials are usually tailored to specific reactions and energies and thus lack generality. The most widely used approximate potentials are called *central potentials*. They depend only on the magnitude of the radius vector, but not on its direction, that is, $V(\vec{r}) = V(r)$. Since the nuclear potential is of short range, we will consider

here only potentials that for large distances $(r \to \infty)$ approach $V(r) \to 0$ faster than 1/r.

In this section, we will initially focus on the problem of elastic scattering. The formalism is then extended to include nuclear reactions. A general treatment of nuclear scattering involves solutions of the time-dependent Schrödinger equation, that is, the scattering of wave packets. However, the most important physics aspects can be derived by considering the much simpler stationary problem of solving the time-independent Schrödinger equation. No further assumptions about the nuclear potential are made here. We will derive the general formalism which relates the observed scattering cross section to the wave function far away from the scattering center. The cross section will be expressed in terms of so-called phase shifts. In order to determine the latter quantity, knowledge of the wave function in the nuclear region is necessary. These considerations will be discussed in subsequent sections.

The scattering process is schematically shown in Fig. 2.3. Consider a beam of monoenergetic particles incident on a target along the *z*-direction. The value and the uncertainty of the *z*-component of the linear momentum are given by $p_z = \text{const}$ and $\Delta p_z = 0$, respectively. It follows immediately from the Heisenberg uncertainty principle ($\Delta p_z \Delta z \approx \hbar$) that $\Delta z \rightarrow \infty$. Hence, the incoming wave has a large extent in the *z*-direction, that is, the process is nearly stationary. Furthermore, we assume for the *x*- and *y*-components of the linear momentum $p_x = p_y = 0$. This implies, according to $\lambda_i = h/p_i$, that $\lambda_x = \lambda_y \rightarrow \infty$. In other words, the incoming particles are represented by a wave of very large wavelength in the *x*- and *y*-directions, that is, an incident plane wave.

The stationary scattering problem is described by the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r})$$
(2.16)

At the position of the target nucleus we cannot specify the total wave function further without assuming an explicit nuclear potential. However, far away from the scattering center, at the position of our detector, we can express the total wave function as a sum of two stationary waves: an incoming plane wave and an outgoing spherical wave. Therefore, we start for the total wave function at large distances with the ansatz

$$\psi_T(\vec{r}) = N\left[e^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{ikr}}{r}\right], \qquad r \to \infty$$
(2.17)

The term $e^{i\vec{k}\cdot\vec{r}}$ represents a plane wave traveling in the *z*-direction (a free particle). The second term contains a spherical wave (e^{ikr}), a quantity $f(\theta)$ called

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Fig. 2.3 Schematic representation of the scattering process. A plane wave is incident along the *z*-direction on a scattering center (target) which gives rise to an outgoing spherical wave. Note the significant differences in the dimensions of the collimator (\approx mm), target nuclei (\approx fm), and detector distance (\approx cm), which are typical for nuclear physics experiments.

scattering amplitude, and the factor 1/r since the scattered intensity must obey an inverse square law; *N* is an overall normalization factor.

2.3.2

Relationship Between Differential Cross Section and Scattering Amplitude

The particle density (in units of inverse volume) is given by $P = \psi^* \psi$ and the current density (in units of inverse area per time) of beam particles or scattered particles with velocity v can be written as j = vP. For the incoming plane wave we can write

$$j_b = v_b (N e^{-ikz}) (N e^{ikz}) = v_b N^2$$
(2.18)

whereas we obtain for the scattered spherical wave

$$j_s = v_s \left[Nf^*(\theta)e^{-ikr}\frac{1}{r} \right] \left[Nf(\theta)e^{ikr}\frac{1}{r} \right] = v_s N^2 |f(\theta)|^2 \frac{1}{r^2}$$
(2.19)

Substitution of j_b and j_s into Eq. (2.10) yields

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{j_s r^2}{j_b} = |f(\theta)|^2 \tag{2.20}$$

since for elastic scattering we can assume that $v_b = v_s$. The important result here is that the differential cross section is equal to the square of the scattering amplitude.

2.3.3 The Free Particle

It is instructive to consider first the force-free particle. The plane wave $e^{i\vec{k}\cdot\vec{r}}$ represents a free particle of momentum $\vec{p} = \hbar\vec{k}$ and energy $E = \hbar^2 k^2 / (2m)$. The potential is V(r) = 0 and, therefore, we have $f(\theta) = 0$. If we choose the *z* axis along \vec{k} (see Fig. 2.4), the plane wave can be written as

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = e^{ikr(z/r)} = e^{ikz}$$
(2.21)

which is independent of the angle ϕ . Since $\vec{L} = \vec{r} \times \vec{p}$, we only need to consider values of m = 0 for the magnetic quantum number. In this case, the spherical harmonics are given by (see Eq. (A.9))

$$Y_{\ell 0} = \sqrt{\frac{2\ell + 1}{4\pi}} P_{\ell}(\cos \theta)$$
(2.22)

where $P_{\ell}(\cos \theta)$ is a Legendre polynomial. With the substitutions $E = p^2/(2m) = \hbar^2 k^2/(2m)$ and $\rho \equiv kr$, the radial equation for the free particle can be written as (see Eq. (A.23))

$$\frac{d^2 u_\ell}{d\rho^2} + \left[1 - \frac{\ell(\ell+1)}{\rho^2}\right] u_\ell = 0$$
(2.23)

The solutions, $j_{\ell}(kr)$, are called *spherical Bessel functions* (Section A.2) and we can write for the asymptotic values

$$u_{\ell}^{\text{f.p.}} = (kr)j_{\ell}(kr) = \sin\left(kr - \ell\pi/2\right), \qquad r \to \infty$$
(2.24)

The eigenfunctions of the free particle, $j_{\ell}(kr)P_{\ell}(\cos\theta)$, form a complete orthonormal set. Therefore, we expand the plane wave according to

$$e^{ikz} = \sum_{\ell=0}^{\infty} c_{\ell} j_{\ell}(kr) P_{\ell}(\cos\theta)$$
(2.25)

The derivation of the expansion coefficients, which is not repeated here (see, for example, Messiah 1999), yields $c_{\ell} = (2\ell + 1)i^{\ell}$. Thus

$$e^{ikz} = \sum_{\ell=0}^{\infty} (2\ell+1)i^{\ell} j_{\ell}(kr) P_{\ell}(\cos\theta)$$
(2.26)

It can be seen that the plane wave with linear momentum kr has been expanded into a set of *partial waves*, each having an orbital angular momentum of $\hbar \sqrt{\ell(\ell + 1)}$, an amplitude of $(2\ell + 1)$, and a phase factor of i^{ℓ} . For very large distances appropriate for any experimental detector geometry, we find for the free particle

$$\psi_T^{\text{f.p.}} = e^{ikz} = \sum_{\ell=0}^{\infty} (2\ell+1)i^{\ell} \frac{\sin(kr - \ell\pi/2)}{kr} P_{\ell}(\cos\theta), \qquad r \to \infty$$
(2.27)



Fig. 2.4 Linear and angular momenta of the free particle. The vector \vec{p} points along the *z* axis, while the projection of \vec{L} on the *z* axis is zero (m = 0).

Using the relationship $\sin x = (i/2)(e^{-ix} - e^{ix})$ we write

$$\psi_T^{\text{f.p.}} = \frac{1}{2kr} \sum_{\ell=0}^{\infty} (2\ell+1)i^{\ell+1} \left[e^{-i(kr-\ell\pi/2)} - e^{i(kr-\ell\pi/2)} \right] P_\ell(\cos\theta), \qquad r \to \infty$$
(2.28)

For the special case of s-waves ($\ell = 0$), we have $u_0^{\text{f.p.}} = \sin(kr)$ instead of Eq. (2.24) (see also Eq. (A.26)). Consequently, Eqs. (2.27) and (2.28) are not only valid for $r \to \infty$, but apply in this case to all distances.

2.3.4 Turning the Potential On

For a central potential (Section 2.3.1) with $V(r) \neq 0$ and $f(\theta) \neq 0$, only the solution to the radial equation will change. Instead of $u_{\ell}^{\text{f.p.}}$ we have to write u_{ℓ} . The two functions $u_{\ell}^{\text{f.p.}}$ and u_{ℓ} essentially differ only for small r where $V(r) \neq 0$. For large distances r we have V(r) = 0 and both functions must satisfy the same radial equation. We write

$$u_{\ell} = \sin(kr - \ell\pi/2 + \delta_{\ell}), \qquad r \to \infty$$
(2.29)

This solution can differ at most from the radial wave function of the free particle (Eq. (2.24)) by a phase shift δ_{ℓ} , which arises from the different *r* dependence in the region where $V(r) \neq 0$. Note that for s-waves ($\ell = 0$), Eq. (2.29) applies again to all distances outside the potential.

Similar to the case of the free particle (see Eq. (2.25)), we can expand the total wave function into partial waves

$$e^{ikz} + f(\theta)\frac{e^{ikr}}{r} = \sum_{\ell=0}^{\infty} b_{\ell} \frac{u_{\ell}(kr)}{kr} P_{\ell}(\cos\theta)$$
(2.30)

The expansion coefficients are given by $b_{\ell} = (2\ell + 1)i^{\ell}e^{i\delta_{\ell}}$ (Problem 2.1). Thus

$$\psi_T = e^{ikz} + f(\theta) \frac{e^{ikr}}{r}$$

= $\sum_{\ell=0}^{\infty} (2\ell+1) i^{\ell} e^{i\delta_{\ell}} \frac{\sin(kr - \ell\pi/2 + \delta_{\ell})}{kr} P_{\ell}(\cos\theta), \quad r \to \infty$ (2.31)

Using the relation $\sin x = (i/2)(e^{-ix} - e^{ix})$ we write

$$\psi_T = \frac{1}{2kr} \sum_{\ell=0}^{\infty} (2\ell+1)i^{\ell+1} \left[e^{-i(kr-\ell\pi/2)} - e^{2i\delta_\ell} e^{i(kr-\ell\pi/2)} \right] P_\ell(\cos\theta), \qquad r \to \infty$$
(2.32)

Comparison to the total wave function of the free particle (Eq. (2.28)) clearly shows that the potential modifies at large distances each outgoing spherical wave by a factor of $e^{2i\delta_{\ell}}$ and thereby shifts each outgoing spherical wave by a phase δ_{ℓ} .

2.3.5 Scattering Amplitude and Elastic Scattering Cross Section

We solve first for the scattering amplitude $f(\theta)$ by writing

$$f(\theta)\frac{e^{ikr}}{r} = \psi_T - \psi_T^{\text{f.p.}} = \frac{1}{2kr} \sum_{\ell=0}^{\infty} (2\ell+1)i^{\ell+1} \left[e^{i(kr-\ell\pi/2)} \left(1 - e^{2i\delta_\ell}\right) \right] P_\ell(\cos\theta)$$
(2.33)

Using $e^{i\pi\ell/2} = \cos(\pi\ell/2) + i\sin(\pi\ell/2) = i^{\ell}$ and the identity $e^{i\delta}\sin\delta \equiv (i/2)(1 - e^{2i\delta})$ yields

$$f(\theta) = \frac{i}{2k} \sum_{\ell=0}^{\infty} (2\ell+1) \left(1 - e^{2i\delta_{\ell}}\right) P_{\ell}(\cos\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_{\ell}} \sin\delta_{\ell} P_{\ell}(\cos\theta)$$
(2.34)

It is again apparent that the effect of the scattering potential is to shift the phase of each outgoing partial wave.

The differential elastic scattering cross section can be written as

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm el} = f^*(\theta)f(\theta) = \frac{1}{4k^2} \left|\sum_{\ell=0}^{\infty} (2\ell+1)\left(1-e^{2i\delta_\ell}\right)P_\ell(\cos\theta)\right|^2$$
$$= \frac{1}{k^2} \left|\sum_{\ell=0}^{\infty} (2\ell+1)\sin\delta_\ell P_\ell(\cos\theta)\right|^2 \tag{2.35}$$

The interference terms involving different functions $P_{\ell}(\cos \theta)$ generally give rise to nonisotropic angular distributions. Using the orthogonality relation for Legendre polynomials,

$$\int_{d\Omega} P_{\ell}(\cos\theta) P_{\ell'}(\cos\theta) d\Omega = \frac{4\pi}{2\ell+1} \delta_{\ell\ell'}$$
(2.36)

where $\delta_{\ell\ell'}$ denotes a Kronecker symbol, we obtain for the total elastic scattering cross section

$$\sigma_{\rm el} = \int \left(\frac{d\sigma}{d\Omega}\right)_{\rm el} d\Omega = \sum_{\ell=0}^{\infty} \sigma_{\rm el,\ell}$$
(2.37)

$$\sigma_{\rm el,\ell} = \frac{\pi}{k^2} (2\ell+1) \left| 1 - e^{2i\delta_\ell} \right|^2 = \frac{4\pi}{k^2} (2\ell+1) \sin^2 \delta_\ell \tag{2.38}$$

For the special case of s-waves ($\ell = 0$) we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm el,0} = \frac{1}{k^2}\sin^2\delta_0 \tag{2.39}$$

$$\sigma_{\rm el,0} = \frac{4\pi}{k^2} \sin^2 \delta_0 \tag{2.40}$$

and the angular distribution becomes isotropic (that is, independent of θ). It follows that the cross section is entirely determined by the phase shifts δ_{ℓ} . It is also apparent that $\delta_{\ell} \to 0$ as $V(r) \to 0$ for all ℓ .

So far we assumed that at least one particle participating in the interaction is uncharged. If both nuclei are charged, then we have to replace the phase δ_{ℓ} for the short-range nuclear potential by $\delta_{\ell} + \sigma_{\ell}$, where σ_{ℓ} is the phase shift due to the long-range Coulomb potential. The Coulomb phase shift can be calculated analytically (see Eq. (D.13)). We write

$$1 - e^{2i(\delta_{\ell} + \sigma_{\ell})} = \left(1 - e^{2i\sigma_{\ell}}\right) + e^{2i\sigma_{\ell}}\left(1 - e^{2i\delta_{\ell}}\right)$$
(2.41)

and the scattering amplitude can be expressed as

$$f(\theta) = \frac{i}{2k} \sum_{\ell=0}^{\infty} (2\ell+1) \left[1 - e^{2i(\delta_{\ell} + \sigma_{\ell})} \right] P_{\ell}(\cos \theta)$$

$$= \frac{i}{2k} \sum_{\ell=0}^{\infty} (2\ell+1) \left(1 - e^{2i\sigma_{\ell}} \right) P_{\ell}(\cos \theta)$$

$$+ \frac{i}{2k} \sum_{\ell=0}^{\infty} (2\ell+1) e^{2i\sigma_{\ell}} \left(1 - e^{2i\delta_{\ell}} \right) P_{\ell}(\cos \theta)$$
(2.42)

The first term describes the scattering from a pure Coulomb field (Rutherford scattering). The second term contains the phase shifts δ_{ℓ} and σ_{ℓ} . It is obvious that the cross section will exhibit interference terms corresponding to the scattering from both the nuclear and the Coulomb potential.

2.3.6 Reaction Cross Section

We can now consider the possibility that a nuclear reaction occurs, that is, any process which is different from elastic scattering (for example, particle capture or inelastic scattering). A specific set of conditions (momentum, quantum numbers, and so on) for the outgoing particle is called a *channel*. A more precise definition of this concept will be given in later sections. Elastic scattering, inelastic scattering to a final excited state x, inelastic scattering to a different excited final state y, and so on, all correspond to different channels.

Suppose first that elastic scattering is the only possible process. In that case as many particles enter as exit from an imaginary sphere surrounding the target nucleus (Fig. 2.5a). As a result, the integral over the current density j_T , corresponding to the total wave function ψ_T for elastic scattering, is zero

$$\int_{d\Omega} j_T \, d\Omega = 0 \tag{2.43}$$

Suppose now that nonelastic processes occur as well. In that case a fraction of the incoming particles will either change kinetic energies, for example, in inelastic scattering (n,n'), or change identity, for example, in particle capture (n, γ). A number of incoming particles will disappear and, consequently, there will be a net current of particles into the sphere (Fig. 2.5b). The rate of disappearance from the elastic channel corresponds to the reaction cross section. Formally, we can write

$$\sigma_{\rm re} = \frac{r^2}{j_b} \int_{d\Omega} j_T \, d\Omega \tag{2.44}$$

Recall that the wave function ψ_T , corresponding to the current density j_T , represents the wave function for elastic scattering only. In the following, an expression is derived which relates the reaction cross section to the phase shifts. We start from the quantum mechanical expression for the current density (Messiah 1999),

$$j = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial r} - \frac{\partial \psi}{\partial r}^* \psi \right)$$
(2.45)

From this expression we find for the incoming plane wave e^{ikz}

$$j_b = \frac{\hbar}{2mi} \left[e^{-ikz} (e^{ikz}ik) - e^{-ikz} (-ik)e^{ikz} \right] = \frac{\hbar k}{m}$$
(2.46)

Substitution of the total elastic scattering wave function ψ_T (see Eq. (2.32)) into

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Fig. 2.5 Representation of the current density if (a) scattering is the only possible process, and (b) both elastic and nonelastic processes occur. In part (a) the same number of particles enter and exit from an imaginary sphere surrounding the target nucleus and the integral over the total current density is zero. In part (b) a number of incoming particles disappear because of reactions and thus there is a net current of particles into the sphere.

Eq. (2.45) yields, after some algebra,

$$j_{T} = \frac{\hbar}{4mkr^{2}} \left\{ \left| \sum_{\ell=0}^{\infty} (2\ell+1)i^{\ell+1}e^{i\ell\pi/2}P_{\ell}(\cos\theta) \right|^{2} - \left| \sum_{\ell=0}^{\infty} (2\ell+1)i^{\ell+1}e^{2i\delta_{\ell}}e^{-i\ell\pi/2}P_{\ell}(\cos\theta) \right|^{2} \right\}$$
(2.47)

With the orthogonality relation for Legendre polynomials (see Eq. (2.36)), one finds

$$\sigma_{\rm re} = \sum_{\ell=0}^{\infty} \sigma_{\rm re,\ell} \tag{2.48}$$

$$\sigma_{\mathrm{re},\ell} = \frac{\pi}{k^2} (2\ell+1) \left(1 - \left| e^{2i\delta_\ell} \right|^2 \right) \tag{2.49}$$

We require $|e^{2i\delta_\ell}|^2 \leq 1$ since otherwise $\sigma_{\rm re}$ becomes negative. In general, the phase shift δ_ℓ will be a complex number, that is, $\delta_\ell = \delta_{\ell_1} + i\delta_{\ell_2}$. For the special case that δ_ℓ is real, one finds $|e^{2i\delta_\ell}|^2 = 1$. In other words, reactions cannot occur and elastic scattering is the only possible process. The allowed range of values for $\sigma_{{\rm re},\ell}$ and $\sigma_{{\rm el},\ell}$ is represented by the shaded region in Fig. 2.6. Recall that the expression for the elastic scattering cross section (see Eq. (2.38)) holds only for uncharged particles. The maximum elastic scattering cross section occurs at $e^{2i\delta_\ell} = -1$, yielding

$$\sigma_{\text{el},\ell}^{\max} = \frac{4\pi}{k^2} (2\ell + 1) \quad \text{and} \quad \sigma_{\text{re},\ell} = 0$$
 (2.50)

The maximum reaction cross section is obtained for $e^{2i\delta_{\ell}} = 0$, leading to

$$\sigma_{\rm re,\ell}^{\rm max} = \sigma_{\rm el,\ell} = \frac{\pi}{k^2} (2\ell + 1) \tag{2.51}$$



Fig. 2.6 Upper and lower limit of elastic scattering cross section for a given reaction cross section. Values inside the shaded region are allowed, while those outside the shaded region are impossible. The quantity $e^{2i\delta_\ell}$ is real for all points located on the solid curve.

It follows that elastic scattering may occur without any reactions taking place, but reactions can never occur without elastic scattering being present. When the reaction cross section is at maximum, its value is equal to the elastic scattering cross section.

Traditionally, the theory of scattering has been applied in order to study the nature of the nuclear potential. Usually, the differential cross section $d\sigma/d\Omega$ is given by experiment and it is desired to find the corresponding potential V(r). The experimental phase shifts δ_{ℓ} are obtained by fitting the cross section formula to experimental angular distribution data, provided that a satisfactory fit is achieved by means of a small number of terms in the partial wave expansion. This procedure is repeated for several values of the incident energy. One then attempts to find a potential V(r), which reproduces the observed phase shifts, by solving the Schrödinger equation numerically for each value of ℓ .

2.4 Scattering by Simple Potentials

The cross section is determined by the phase shifts. The latter can be obtained from the wave function in the nuclear region that is generated by an explicit nuclear potential. In this section, we will consider the case of s-wave ($\ell = 0$) scattering of neutral and spinless particles. Two simple potentials will be discussed explicitly: (i) an attractive square-well potential, and (ii) an attractive square-well plus square-barrier potential. Although very simple, these models contain qualitatively most of the physics that will be encountered later in the discussion of far more complex situations. We will specifically calculate the phase shifts δ_0 and the intensity of the wave function in the region of the potential by solving the radial Schrödinger equation. It will be seen how the properties of the potential determine the phase shift and the wave function intensity.

2.4.1 Square-Well Potential

The potential is displayed in Fig. 2.7. For $\ell = 0$ the radial equation becomes (Appendix A.1)

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2} [E - V(r)]u = 0$$
(2.52)

For a constant potential, V(r) = const, we obtain with $\hat{k}^2 = (2m/\hbar^2)(E - V)$ the radial equation

$$\frac{d^2u}{dr^2} + \hat{k}^2 u = 0 \tag{2.53}$$

The general solution in terms of complex exponentials is given by

$$u = \alpha e^{i\hat{k}r} + \beta e^{-i\hat{k}r} \tag{2.54}$$

We will consider the two regions $r < R_0$ and $r > R_0$ separately. For $r < R_0$ we have E - V > 0 and, therefore,

$$u_{\rm in} = A'e^{iKr} + B'e^{-iKr}, \qquad K^2 = \frac{2m}{\hbar^2}(E+V_0)$$

= $A'[\cos(Kr) + i\sin(Kr)] + B'[\cos(Kr) - i\sin(Kr)]$ (2.55)

At the boundary we require $u_{in}(0) = 0$, otherwise the radial wave function u(r)/r will diverge at r = 0. It follows immediately that $u_{in}(0) = A' + B' = 0$ and the cosine terms in Eq. (2.55) disappear. Hence

$$u_{\rm in} = A' i \sin(Kr) - A' [-i \sin(Kr)] = 2iA' \sin(Kr) = A \sin(Kr)$$
(2.56)



Fig. 2.7 Three-dimensional square-well potential of radius R_0 and potential depth V_0 . The horizontal line indicates the total particle energy *E*. For the calculation of the transmission coefficient, it is necessary to consider a one-dimensional potential step that extends from $-\infty$ to $+\infty$. See the text.

where we used the definition $A \equiv 2iA'$. In the region $r > R_0$, one finds again E - V > 0, and the general solution is given by

$$u_{\text{out}} = C'e^{ikr} + D'e^{-ikr}, \qquad k^2 = \frac{2m}{\hbar^2}E$$

= $C'[\cos(kr) + i\sin(kr)] + D'[\cos(kr) - i\sin(kr)]$
= $i[C' - D']\sin(kr) + [C' + D']\cos(kr) = C''\sin(kr) + D''\cos(kr)$
(2.57)

It is convenient to rewrite this expression. The sum of sin *x* and $\cos x$ gives again a sine function which is shifted along the *x*-axis. Using $C'' = C \cos \delta_0$ and $D'' = C \sin \delta_0$ we can formally write

$$u_{\rm out} = C \left[\sin(kr) \cos \delta_0 + \cos(kr) \sin \delta_0 \right]$$
(2.58)

With $sin(x \pm y) = sin x cos y \pm cos x sin y$ one finds

$$u_{\rm out} = C\sin(kr + \delta_0) \tag{2.59}$$

The solutions u_{in} (see Eq. (2.56)) and u_{out} (see Eq. (2.59)) will be used below.

Transmission probability

We are interested in the transmission probability from the outer to the inner region. It is convenient to start from the wave function solutions in terms of complex exponentials (see Eqs. (2.55) and (2.57)). It should be pointed out that for real potentials, the transmission probability is only defined for the one-dimensional case (see, for example, Messiah 1999). Instead of considering the

three-dimensional potential shown in Fig. 2.7, we will assume that the particles are incident from the right-hand side, that a one-dimensional potential steps down at a distance of $x = R_0$ by an amount of V_0 , and that the potential step extends to $-\infty$. We obtain for the one-dimensional radial wave functions

$$u_{\rm in} = A' e^{iKx} + B' e^{-iKx} \tag{2.60}$$

$$u_{\text{out}} = C'e^{ikx} + D'e^{-ikx} \tag{2.61}$$

Although we do not have to consider the time-dependent Schrödinger equation here, it is instructive to investigate the full time-dependent solution, which is obtained by multiplying the complex exponentials by the factor $e^{-i\omega t}$, with $\omega = E/\hbar$. It can easily be seen, for example, that the second term of u_{in} corresponds to a plane wave that propagates into the negative x direction. The first and second terms of u_{out} correspond to plane waves reflected from the boundary at R_0 and moving toward R_0 , respectively. We are interested in the scattering process. The particle density of incident projectiles, for example, is given by $|D'e^{-ikx}|^2 = |D'|^2$. The current density (or flux) of incident particles is given by the product of particle density and velocity in the outer region, $j_{inc} = v_{out}|D'|^2$ (Section 2.3.2). Similarly, one finds for the transmitted or reflected particle flux $j_{trans} = v_{in}|B'|^2$ or $j_{refl} = v_{out}|C'|^2$, respectively. It follows for the probability that an individual particle will be transmitted from the outer to the inner region

$$\hat{T} = \frac{j_{\text{trans}}}{j_{\text{inc}}} = \frac{v_{\text{in}}|B'|^2}{v_{\text{out}}|D'|^2} = \frac{K|B'|^2}{k|D'|^2}$$
(2.62)

The quantity \hat{T} is called the *transmission coefficient*.

The continuity condition requires that the wave functions and their derivatives are continuous at the boundary $x = R_0$,

$$(u_{\rm in})_{R_0} = (u_{\rm out})_{R_0} \tag{2.63}$$

$$\left(\frac{du_{\rm in}}{dx}\right)_{R_0} = \left(\frac{du_{\rm out}}{dx}\right)_{R_0} \tag{2.64}$$

We obtain

$$A'e^{iKR_0} + B'e^{-iKR_0} = C'e^{ikR_0} + D'e^{-ikR_0}$$
(2.65)

$$\frac{K}{k} \left(A' e^{iKR_0} - B' e^{-iKR_0} \right) = \left(C' e^{ikR_0} - D' e^{-ikR_0} \right)$$
(2.66)

Setting A' = 0, since there is no plane wave approaching the boundary R_0 from the left-hand side, and eliminating C' yields

$$\frac{K}{k}\left(-B'e^{-iKR_{0}}\right) = B'e^{-iKR_{0}} - 2D'e^{-ikR_{0}} \quad \text{or} \quad \frac{B'}{D'} = 2\frac{e^{-iKR_{0}}}{e^{-iKR_{0}}}\frac{k}{K+k} \quad (2.67)$$

For the transmission coefficient we find with Eqs. (2.62) and (2.67)

$$\hat{T} = \frac{K}{k} \frac{|B'|^2}{|D'|^2} = 4 \frac{kK}{(K+k)^2} = 4 \frac{\frac{2m}{\hbar^2} \sqrt{(E+V_0)E}}{\left[\sqrt{\frac{2m}{\hbar^2}(E+V_0)} + \sqrt{\frac{2m}{\hbar^2}E}\right]^2}$$
(2.68)

We will use this result later in connection with the continuum theory of nuclear reactions (Section 2.6).

Phase shift and resonance phenomenon

The quantity \hat{T} describes the transmission probability from the right- to the left-hand side in Fig. 2.7. We have considered so far only the amplitude ratio of two waves: one approaching the boundary R_0 from the right, the other one receding from R_0 to the left. We will now consider the full radial wave function solution for the three-dimensional case. We start from Eqs. (2.56) and (2.59),

$$u_{\rm in} = A\sin(Kr) \tag{2.69}$$

$$u_{\rm out} = C\sin(kr + \delta_0) \tag{2.70}$$

From the continuity condition (see Eqs. (2.63) and (2.64)) one finds

$$A\sin(KR_0) = C\sin(kR_0 + \delta_0) \tag{2.71}$$

$$AK\cos(KR_0) = Ck\cos(kR_0 + \delta_0) \tag{2.72}$$

First, we divide both equations to solve for the phase shift δ_0 . The result is

$$\frac{1}{K}\tan(KR_0) = \frac{1}{k}\tan(kR_0 + \delta_0)$$
(2.73)

$$\delta_0 = -kR_0 + \arctan\left[\frac{k}{K}\tan(KR_0)\right] \tag{2.74}$$

This expression can be rewritten in terms of the total energy as

$$\delta_0 = -\frac{\sqrt{2mE}}{\hbar}R_0 + \arctan\left[\sqrt{\frac{E}{E+V_0}}\tan\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}R_0\right)\right]$$
(2.75)

It can be seen that the phase shift is determined by the properties of the potential (R_0 , V_0) and the properties of the particle (E,m). For $V_0 \rightarrow 0$ one finds $\delta_0 \rightarrow 0$, as already pointed out above. The cross section can be calculated simply from the phase shift (see Eq. (2.40)). Second, one can solve for $|A|^2/|C|^2$, that is, the ratio of wave function intensities in the interior ($r < R_0$) and exterior regions ($r > R_0$). By squaring and adding Eqs. (2.71) and (2.72) we obtain

$$\frac{|A|^2}{|C|^2} = \frac{k^2}{k^2 + [K^2 - k^2]\cos^2(KR_0)} = \frac{E}{E + V_0 \cos^2\left(\frac{\sqrt{2m(E+V_0)}}{\hbar}R_0\right)}$$
(2.76)

where the identity $\sin^2(kr + \delta) + \cos^2(kr + \delta) = 1$ has been used.

Plots of $|A|^2/|C|^2$ and δ_0 versus *E* for the scattering of a neutron by a squarewell potential are shown in Fig. 2.8. A potential depth of $V_0 = 100$ MeV and a potential radius of $R_0 = 3$ fm are assumed. The quantity $|A|^2/|C|^2$ measures the relative intensity of the wave function in the interior region $r < R_0$. It is apparent that $|A|^2/|C|^2$ oscillates between extreme values. This remarkable behavior is referred to as *resonance phenomenon*. At certain discrete energies E_i (resonance energies) the probability for finding the particle inside the boundary $r < R_0$ is at maximum. It can also be seen that each resonance shifts the phase δ_0 by some amount. The resonances occur at energies at which $\cos^2(KR_0) = 0$ in Eq. (2.76), that is, $KR_0 = (n + 1/2)\pi$. Hence

$$K = \frac{\left(n + \frac{1}{2}\right)\pi}{R_0} = \frac{2\pi}{\lambda_{\rm in}}$$
(2.77)

/

$$\lambda_{\rm in} = \frac{2R_0}{\left(n + \frac{1}{2}\right)} = \frac{R_0}{\left(\frac{n}{2} + \frac{1}{4}\right)} \tag{2.78}$$

with λ_{in} the wavelength in the interior region. Since $(n/2 + 1/4) = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \dots$ it follows that resonances occur when precisely (n/2 + 1/4) wavelengths fit into the interior region. At those wavelengths, the derivative of the interior wave function (a sine function; see Eq. (2.69)) at the radius R_0 is zero. As can be seen in Fig. 2.9, *n* also corresponds to the number of wave function nodes in the region $r < R_0$. For the resonance energies we obtain from Eq. (2.77)

$$E_n = \frac{\hbar^2}{2m} \frac{\pi^2}{R_0^2} \left(n + \frac{1}{2} \right)^2 - V_0 \tag{2.79}$$

In the above example of neutron scattering by a square-well potential of depth $V_0 = 100$ MeV and radius $R_0 = 3$ fm, one has $(\hbar \pi)^2 / (2mR_0^2) = 22.648$ MeV. We obtain

$$E_2 = 41.5 \,\mathrm{MeV}, \qquad E_3 = 177.4 \,\mathrm{MeV}, \qquad E_4 = 358.6 \,\mathrm{MeV}, \dots$$
(2.80)

No physical solution exists for n = 0 or 1, that is, for the potential depth chosen it is not possible to match the interior and exterior wave functions by fitting either 1/4 or 3/4 wavelengths into the region $r < R_0$. In other words, there are no solutions with either no node or only one node in the interior region.

The results obtained from the above formalism are illustrated qualitatively in Fig. 2.10 showing radial wave functions for different depths of an attractive square-well potential. The bombarding energy is low (that is, the wavelength is large compared to R_0) and held constant. In part (a) the potential depth is zero (free particle) and the wave function is given by a sine function. In part (b), the potential depth increases and, therefore, the wavelength in the



Fig. 2.8 (a) Ratio of wave function intensities in the interior ($r < R_0$) and exterior ($r > R_0$) region, $|A|^2/|C|^2$, and (b) phase shift δ_0 versus total energy *E* for the scattering of neutrons $(2m/\hbar^2 = 0.0484 \text{ MeV}^{-1} \text{ fm}^{-2})$ by a square-well potential (Fig. 2.7). For the potential depth and the radius, values of $V_0 = 100 \text{ MeV}$ and $R_0 = 3 \text{ fm}$, respectively, are assumed. The curves show the resonance phenomenon.

interior decreases according to

$$\frac{\lambda_{\rm in}}{2\pi} = \frac{1}{K} = \frac{1}{\sqrt{(2m/\hbar^2)(E+V_0)}}, \qquad \lambda_{\rm in} = \frac{h}{\sqrt{2m(E+V_0)}}$$
(2.81)

The values and derivatives of the inside and outside wave functions can only be matched by shifting the outside solution inward. This is the physical meaning of a phase shift. If the potential depth is increased further, the wavelength in the interior becomes smaller and the exterior wave must shift inward, until exactly 1/4 wavelength fit into the interior region. When this happens, the derivative of the wave function at R_0 becomes zero corresponding to a maximum amplitude inside the potential region. The system is in resonance as shown in part (c). A further increase in the potential depth results in: a decreasing amplitude in the interior (part d); a minimum interior amplitude because of poor



Fig. 2.9 Two simplest solutions for the radial wave function inside the square-well potential. Both solutions give rise to a resonance since the derivative of the wave function at the potential radius R_0 is zero. The solutions are characterized by the number of wave function nodes *n* in the interior region ($r < R_0$). They are shown here for illustrative purposes. Note that neither of these functions represent physical solutions for the conditions adopted in Fig. 2.8.

wave function matching conditions (part e); and the appearance of the first node in the interior region (part f).

A plot of $|A|^2/|C|^2$ versus potential depth V_0 is shown in Fig. 2.11. A total energy of E = 1 MeV and a potential radius of $R_0 = 3$ fm are assumed. Solving Eq. (2.79) for the potential depth V_0 yields

$$V_{0,n} = \frac{\hbar^2}{2m} \frac{\pi^2}{R_0^2} \left(n + \frac{1}{2} \right)^2 - E$$
(2.82)

Thus, we expect resonances to occur at $V_{0,0} = 4.7$ MeV, $V_{0,1} = 49.9$ MeV, $V_{0,2} = 140.5$ MeV, $V_{0,3} = 276.4$ MeV, $V_{0,4} = 457.6$ MeV, and so on (with n = 0, 1, 2, 3, 4 nodes in the interior region, respectively), in agreement with the results displayed in Fig. 2.11.

2.4.2

Square-Barrier Potential

In the following we will again consider the simple case of s-wave ($\ell = 0$) scattering. In addition to an attractive square well, the potential displays a repulsive square barrier. This is a simple model for a nuclear reaction if a barrier is present. For example, the Coulomb potential provides a barrier in reactions involving charged particles. By solving the Schrödinger equation explicitly, we will find the probability for transmission through the potential barrier and the intensity of the wave in the interior region. The potential is displayed in Fig. 2.12. We will consider the three regions I, II, III separately. In each region, the potential is constant and, assuming $\ell = 0$, we again obtain



Fig. 2.10 Square-well potential (left) and corresponding radial wave function solutions (right) for different potential depths. For a given depth of the potential, the values and derivatives of the inside and outside wave functions must be matched by shifting the outside solution. The phase shift is a measure for this displacement. In part (c) the derivative of the wave function at R_0 is zero and the system is in resonance.

with $\hat{k}^2 = (2m/\hbar^2)(E - V)$ the radial equation (Appendix A.1)

$$\frac{d^2u}{dr^2} + \hat{k}^2 u = 0 \tag{2.83}$$

For region I, we have E - V > 0 and, therefore,

$$u_{\rm I} = Ae^{iKr} + Be^{-iKr}, \qquad K^2 = \frac{2m}{\hbar^2}(E + V_0)$$

= $A'\sin(Kr)$ (2.84)

The solution is the same as the one obtained in the study of the square-well potential (see Eq. (2.56)). In region II, we have E - V < 0 and $k_{\rm II}$ becomes
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Fig. 2.11 Plot of $|A|^2/|C|^2$ versus potential depth V_0 for the scattering of neutrons by a square-well potential. For the total energy and the potential radius, values of E = 1 MeV and $R_0 = 3$ fm are assumed. The maxima show resonances corresponding to n = 0, 1, 2, 3, and 4 radial wave function nodes in the interior region.

imaginary. The solution can be written in terms of real exponentials as

$$u_{\rm II} = Ce^{ik_{\rm II}r} + De^{-ik_{\rm II}r}, \qquad k_{\rm II}^2 = \frac{2m}{\hbar^2}(E - V_1) = i^2 \frac{2m}{\hbar^2}(V_1 - E) \equiv i^2 \kappa^2$$

= $Ce^{-\kappa r} + De^{\kappa r}$ (2.85)

In region III, we have again E - V > 0, and the general solution is given by

$$u_{\text{III}} = Fe^{ikr} + Ge^{-ikr}, \qquad k^2 = \frac{2m}{\hbar^2}E$$
$$= F'\sin(kr + \delta_0) \tag{2.86}$$

The solution is the same as the one obtained in the study of the square-well potential (see Eq. (2.59)).

Transmission through the barrier

First, we are interested in the transmission probability through the potential barrier. It is convenient to start from the wave function solutions in terms of complex exponentials (see Eqs. (2.84)–(2.86)). We must again perform the calculation for the one-dimensional case. Instead of considering the three-dimensional potential shown in Fig. 2.12, we will assume that the particles are incident from the right-hand side, that they encounter at a distance of $x = R_1$ a one-dimensional step barrier of height V_1 , that at a distance of $x = R_0$ the potential steps down to $-V_0$, and that this potential continues to $-\infty$. We



Fig. 2.12 Three-dimensional square-well potential of radius R_0 and potential depth V_0 and a repulsive square-barrier potential of thickness $R_1 - R_0$ and height V_1 . The total particle energy (horizontal line) is smaller than the barrier height, $E < V_1$. For the calculation of the transmission coefficient, it is necessary to consider a one-dimensional potential that extends from $-\infty$ to $+\infty$. See the text.

obtain for the one-dimensional radial wave functions

$$u_{\rm I} = Ae^{iKx} + Be^{-iKx} \tag{2.87}$$

$$u_{\rm II} = Ce^{-\kappa x} + De^{\kappa x} \tag{2.88}$$

$$u_{\rm III} = F e^{ikx} + G e^{-ikx} \tag{2.89}$$

The second term of $u_{\rm I}$ corresponds to a plane wave that propagates into the negative *x* direction, whereas the first and second terms of $u_{\rm III}$ correspond to plane waves reflected from the barrier and moving toward the barrier, respectively. The transmission coefficient is then given by $\hat{T} = j_{\rm trans}/j_{\rm inc} = (K|B|^2)/(k|G|^2)$ (see Eq. (2.62)).

The continuity condition (see Eqs. (2.63) and (2.64)) requires that the wave functions and their derivatives are continuous at the boundaries $x = R_0$ and $x = R_1$

$$(u_{\rm I})_{R_0} = (u_{\rm II})_{R_0} \qquad (u_{\rm II})_{R_1} = (u_{\rm III})_{R_1} \qquad (2.90)$$

$$\left(\frac{du_{\rm I}}{dx}\right)_{R_0} = \left(\frac{du_{\rm II}}{dx}\right)_{R_0} \qquad \qquad \left(\frac{du_{\rm II}}{dx}\right)_{R_1} = \left(\frac{du_{\rm III}}{dx}\right)_{R_1} \tag{2.91}$$

We obtain specifically

$$Ae^{iKR_0} + Be^{-iKR_0} = Ce^{-\kappa R_0} + De^{\kappa R_0}$$
(2.92)

$$i\frac{\kappa}{\kappa}\left(Ae^{iKR_0} - Be^{-iKR_0}\right) = -Ce^{-\kappa R_0} + De^{\kappa R_0}$$
(2.93)

$$Ce^{-\kappa R_1} + De^{\kappa R_1} = Fe^{ikR_1} + Ge^{-ikR_1}$$
(2.94)

$$-Ce^{-\kappa R_{1}} + De^{\kappa R_{1}} = i\frac{k}{\kappa} \left(Fe^{ikR_{1}} - Ge^{-ikR_{1}} \right)$$
(2.95)

Adding and subtracting pairs of equations yields

$$A\left(1+i\frac{K}{\kappa}\right)e^{iKR_{0}}+B\left(1-i\frac{K}{\kappa}\right)e^{-iKR_{0}}=2De^{\kappa R_{0}}$$
(2.96)

$$A\left(1-i\frac{K}{\kappa}\right)e^{iKR_{0}}+B\left(1+i\frac{K}{\kappa}\right)e^{-iKR_{0}}=2Ce^{-\kappa R_{0}}$$
(2.97)

$$2De^{\kappa R_1} = F\left(1 + i\frac{k}{\kappa}\right)e^{ikR_1} + G\left(1 - i\frac{k}{\kappa}\right)e^{-ikR_1}$$
(2.98)

$$2Ce^{-\kappa R_1} = F\left(1 - i\frac{k}{\kappa}\right)e^{ikR_1} + G\left(1 + i\frac{k}{\kappa}\right)e^{-ikR_1}$$
(2.99)

Elimination of the coefficients *C* and *D*, and using the definitions $\alpha \equiv 1 + iK/\kappa$ and $\beta \equiv 1 + ik/\kappa$ gives

$$A\alpha e^{iKR_0} + B\alpha^* e^{-iKR_0} = e^{-\kappa(R_1 - R_0)} \left(F\beta e^{ikR_1} + G\beta^* e^{-ikR_1} \right)$$
(2.100)

$$A\alpha^{*}e^{iKR_{0}} + B\alpha e^{-iKR_{0}} = e^{\kappa(R_{1}-R_{0})} \left(F\beta^{*}e^{ikR_{1}} + G\beta e^{-ikR_{1}}\right)$$
(2.101)

Of interest is the transmission coefficient \hat{T} of a wave incident from the righthand side on the potential barrier. Since there is no wave approaching the barrier from the left-hand side, we set A = 0. We can also eliminate F and obtain

$$B\left[\alpha^{*}\beta^{*}e^{\kappa\Delta} - \alpha\beta e^{-\kappa\Delta}\right] = G\left[(\beta^{*})^{2} - \beta^{2}\right]e^{-i(kR_{1} - KR_{0})} = -2i\frac{k}{\kappa}Ge^{-i(kR_{1} - KR_{0})}$$
(2.102)

where we used $\Delta \equiv R_1 - R_0$. The transmission coefficient is then given by

$$\hat{T} = \frac{K}{k} \frac{|B|^2}{|G|^2} = \frac{4Kk/\kappa^2}{|\alpha^*\beta^*e^{\kappa\Delta} - \alpha\beta e^{-\kappa\Delta}|^2}$$
(2.103)

Using the relation $\sinh^2 z = (1/4)(e^{2z} + e^{-2z}) - 1/2$ yields after some algebra

$$\hat{T} = \frac{Kk}{[K+k]^2 + [\kappa^2 + K^2 + k^2 + K^2 k^2 / \kappa^2] \sinh^2(\kappa\Delta)}$$
(2.104)



Fig. 2.13 Transmission coefficient \hat{T} versus energy E for the scattering of neutrons by the square-barrier potential shown in Fig. 2.12. The potential properties are (a) $V_0 = 100$ MeV, $V_1 = 10$ MeV, $R_0 = 3$ fm, R_1 = 8 fm, and (b) V_0 = 50 MeV, V_1 = 10 MeV, R_0 = 3 fm, R_1 = 8 fm. The drastic drop of the transmission coefficient at small energies is apparent.

In terms of energies one finds explicitly

$$\frac{1}{\hat{T}} = \frac{1}{\sqrt{E(E+V_0)}} \left\{ \left[2E + V_0 + 2\sqrt{E(E+V_0)} \right] + \left[E + V_0 + V_1 + \frac{E(E+V_0)}{V_1 - E} \right] \sinh^2 \left[\sqrt{(2m/\hbar^2)(V_1 - E)} \Delta \right] \right\}$$
(2.105)

This result is remarkable since it shows that a particle approaching the potential barrier from the right-hand side can reach the left-hand side even if its total energy is less than the barrier height. This is referred to as the *tunnel effect* and is of central importance for charged-particle reactions in stars, as will be shown in Chapter 3.

Plots of \hat{T} versus *E* for the scattering of neutrons are shown in Fig. 2.13. The values used are (a) $V_0 = 100$ MeV, $V_1 = 10$ MeV, $R_0 = 3$ fm, $R_1 = 8$ fm, and (b) $V_0 = 50$ MeV, $V_1 = 10$ MeV, $R_0 = 3$ fm, $R_1 = 8$ fm. It can be seen that the transmission coefficient drops rapidly with decreasing energy E. It is also apparent from the absolute magnitude of \hat{T} that the intensity of the wave receding from the barrier to the left-hand side is much smaller compared to the intensity of the wave approaching the barrier from the right-hand side.

Frequently, the case of a low bombarding energy or a thick barrier is of interest,

$$\kappa \Delta = \frac{\sqrt{2m(V_1 - E)}}{\hbar} (R_1 - R_0) \gg 1$$
(2.106)

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In this case we can approximate the denominator in Eq. (2.103) by

$$\left|\alpha^{*}\beta^{*}e^{\kappa\Delta} - \alpha\beta e^{-\kappa\Delta}\right|^{2} \approx \left|\alpha^{*}\beta^{*}e^{\kappa\Delta}\right|^{2}$$
(2.107)

After some algebra we obtain

$$\hat{T} \approx 4 \frac{\sqrt{E(E+V_0)}(V_1 - E)}{V_1(V_0 + V_1)} e^{-2\kappa(R_1 - R_0)}$$
(2.108)

The energy dependence of the transmission coefficient is entirely dominated by the exponential factor. For physically reasonable values of E, V_0 , and V_1 , the coefficient in front of the exponential is of the order of unity. Hence we find

$$\hat{T} \approx e^{-(2/\hbar)} \sqrt{2m(V_1 - E)(R_1 - R_0)}$$
(2.109)

This important result, which strictly applies to the s-wave ($\ell = 0$) scattering of neutral particles, will be used later in connection with the transmission through the Coulomb barrier (Section 2.4.3).

Resonances

In the previous section, we derived the transmission probability for a onedimensional square-barrier potential. The full radial wave function solution for the three-dimensional case will now be considered. It is interesting to have a closer look at the situation. For region I we expect again a resonance phenomenon due to good wave function matching conditions. For region III we expect again a phase shift in order to match the solutions smoothly at $r = R_1$. The barrier in region II provides an extra complication. Here, the wave function u_{II} is given by real exponentials and, depending on the relative magnitude of the coefficients *C* and *D*, may represent a decreasing, an increasing, or a more complicated function of the radius *r*.

As we did in the study of the square-well potential, we will calculate the energy dependence of the phase shift δ_0 and of $|A'|^2$, the intensity of the wave in the interior region $r < R_0$. We start from the wave function solutions (see Eqs. (2.84)–(2.86))

$$u_{\rm I} = A' \sin(Kr) \tag{2.110}$$

$$u_{\rm II} = Ce^{-\kappa r} + De^{\kappa r} \tag{2.111}$$

$$u_{\rm III} = F' \sin(kr + \delta_0) \tag{2.112}$$

and apply again the continuity condition (see Eqs. (2.63) and (2.64)) to the boundaries $r = R_0$ and $r = R_1$. It follows that

$$A'\sin(KR_0) = Ce^{-\kappa R_0} + De^{\kappa R_0}$$
(2.113)

$$A'\frac{K}{\kappa}\cos(KR_0) = -Ce^{-\kappa R_0} + De^{\kappa R_0}$$
(2.114)

$$Ce^{-\kappa R_1} + De^{\kappa R_1} = F' \sin(kR_1 + \delta_0)$$
(2.115)

$$-Ce^{-\kappa R_1} + De^{\kappa R_1} = F'\frac{k}{\kappa}\cos(kR_1 + \delta_0)$$
(2.116)

Solving for δ_0 by eliminating A', F', C, and D yields

$$\delta_{0} = -kR_{1} + \arctan\left[\frac{k}{\kappa}\frac{\sin(KR_{0})\left(e^{-\kappa\Delta} + e^{\kappa\Delta}\right) + \frac{K}{\kappa}\cos(KR_{0})\left(e^{\kappa\Delta} - e^{-\kappa\Delta}\right)}{\sin\left(KR_{0}\right)\left(e^{\kappa\Delta} - e^{-\kappa\Delta}\right) + \frac{K}{\kappa}\cos(KR_{0})\left(e^{-\kappa\Delta} + e^{\kappa\Delta}\right)}\right]$$
(2.117)

For $k \to 0$ (or $E \to 0$) we obtain $\delta_0 \to 0$.

The wave intensity in the interior region, $|A'|^2$, is found by eliminating the constants *C*, *D*, and the phase shift δ_0 . Furthermore, we use the expressions $e^{2x} + e^{-2x} = 4 \sinh^2 x + 2$ and $e^x - e^{-x} = 2 \sinh x$. The tedious algebra is not given here explicitly. The result is

$$\frac{|F'|^2}{|A'|^2} = \sin^2(KR_0) + \left(\frac{K}{k}\right)^2 \cos^2(KR_0) + \sin^2(KR_0) \sinh^2(\kappa\Delta) \left[1 + \left(\frac{\kappa}{k}\right)^2\right] + \cos^2(KR_0) \sinh^2(\kappa\Delta) \left[\left(\frac{K}{\kappa}\right)^2 + \left(\frac{K}{k}\right)^2\right] + \sin(KR_0) \cos(KR_0) \sinh(2\kappa\Delta) \left[\left(\frac{K}{\kappa}\right) + \left(\frac{K}{\kappa}\right) \left(\frac{\kappa}{k}\right)^2\right]$$
(2.118)

The energy dependence of the quantities $|A'|^2/|F'|^2$, $|A'|^2/(|F'|^2\hat{T})$ and δ_0 for neutron scattering by a square-barrier potential is shown in Fig. 2.14, where the transmission coefficient \hat{T} is obtained from the approximation of Eq. (2.109). We assume values of $V_0 = 100$ MeV, $V_1 = 10$ MeV, $R_0 = 3$ fm, $R_1 = 8$ fm (dashed lines) and $V_0 = 50$ MeV, $V_1 = 10$ MeV, $R_0 = 3$ fm, $R_1 = 8$ fm (solid lines). The figure reflects both the effects of the barrier transmission and the resonance phenomenon. For a potential depth of $V_0 = 100$ MeV, no resonance occurs over the energy range shown and the plot looks almost identical to the corresponding part in Fig. 2.13. Consequently, the quantity $|A'|^2/(|F'|^2\hat{T})$ is almost constant with energy. Furthermore, the phase shift varies smoothly with energy. For a potential depth of $V_0 = 50$ MeV, on the other hand, the

interior wave function solution has a large amplitude due to good matching conditions. The resulting resonance is clearly seen in part (a). The shape of the resonance is distorted by the barrier transmission coefficient. In part (b), the effects of the barrier transmission are removed and, consequently, the shape of the resonance becomes symmetric. It is also evident that the resonance shifts the phase by a significant amount. This method of removing the transmission coefficient from the wave function intensity or the cross section is of crucial importance in nuclear astrophysics, as will be seen in Chapter 3.

A plot of $|A'|^2/|F'|^2$ versus potential depth V_0 in the region $r < R_0$ is shown in Fig. 2.15. The graph is obtained for the potential parameters $V_1 = 10$ MeV, $R_0 = 3$ fm, $R_1 = 8$ fm, and E = 5 MeV. Several resonances are apparent which become broader with increasing value of V_0 . By changing V_0 we change the wavelength in the interior region (see Eq. (2.81)). As was the case for the simple square-well potential (Section 2.4.1), the resonances result from favorable wave function matching conditions at the boundaries. The first resonance corresponds to a wave function with no node in the region $r < R_0$. The second resonance corresponds to one node, the third resonance to two nodes, and so on. Comparison to Fig. 2.11 shows that the resonances are much narrower because of the repulsive square-barrier potential.

As a final example, Fig. 2.16 shows schematically the radial wave functions for three cases. In part (a) the potential depth is zero. The amplitude of the wave function in the interior is very small and reflects primarily the transmission through the barrier. In part (b), the amplitude in the interior is at maximum due to favorable matching conditions. The system is in resonance with no wave function node in the interior. Part (c) displays the wave function for the second resonance, showing one node in the interior region.

2.4.3

Transmission Through the Coulomb Barrier

The low-energy s-wave transmission coefficient for a square-barrier potential (see Eq. (2.109)) can be easily generalized since a potential barrier of arbitrary shape may be divided into thin slices of width dr. The total s-wave transmission coefficient is then given by the product of the transmission coefficients for each slice,

$$\hat{T} = \hat{T}_1 \cdot \hat{T}_2 \cdot \ldots \cdot \hat{T}_n \approx \exp\left[-\frac{2}{\hbar} \sum_i \sqrt{2m(V_i - E)} (R_{i+1} - R_i)\right]$$
$$\xrightarrow[n \text{ large}]{} \exp\left[-\frac{2}{\hbar} \int_{R_0}^{R_c} \sqrt{2m[V(r) - E]} dr\right]$$
(2.119)





Fig. 2.14 Energy dependence of the quantities (a) $|A'|^2/|F'|^2$, (b) $|A'|^2/(|F'|^2\hat{T})$ and (c) δ_0 for neutron scattering by a squarebarrier potential (Fig. 2.12). The properties of the potential are $V_1 = 10$ MeV, $R_0 = 3$ fm and $R_1 = 8$ fm. The dashed and solid lines are obtained for potential depths of V_0 =

100 MeV and V_0 = 50 MeV, respectively. The curves represent the effects of both the barrier transmission and the resonance phenomenon. In part (b), the effects of the barrier transmission are removed and the shape of the resonance becomes symmetric (solid line).

For the important case of the Coulomb potential, displayed in Fig. 2.17, we obtain

$$\hat{T} \approx \exp\left(-\frac{2}{\hbar}\sqrt{2m}\int_{R_0}^{R_c}\sqrt{\frac{Z_0Z_1e^2}{r}-E}\,dr\right)$$
(2.120)

with Z_0 and Z_1 the charge of the projectile and target, respectively. The quantity R_0 is the radius of the square-well potential and defines the height

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Fig. 2.15 Plot of $|A'|^2/|F'|^2$ versus potential depth V_0 in the interior region for the scattering of neutrons by a square-barrier potential. The curve is calculated for the parameters $V_1 = 10$ MeV, $R_0 = 3$ fm, $R_1 = 8$ fm and E = 5 MeV. The maxima correspond to resonances which result from favorable wave function matching conditions at the boundaries.

of the Coulomb barrier, $V_C = Z_0 Z_1 e^2 / R_0$. Numerically, one finds $V_C = 1.44Z_0 Z_1 / R_0$ (MeV), with R_0 in units of femtometers. The quantity R_c is the distance at which the incoming particle would be reflected classically. It is referred to as *classical turning point* and is defined by $E = Z_0 Z_1 e^2 / R_c$ or $E/V_C = R_0 / R_c$. The integral in Eq. (2.120) can be solved analytically. Rewriting the above expression by using the definition of the classical turning point yields

$$\hat{T} \approx \exp\left(-\frac{2}{\hbar}\sqrt{2mZ_0Z_1e^2}\int_{R_0}^{R_c}\sqrt{\frac{1}{r}-\frac{1}{R_c}}\,dr\right)$$
(2.121)

Substitution of $z \equiv r/R_c$ gives

$$\hat{T} \approx \exp\left(-\frac{2}{\hbar}\sqrt{2mZ_0Z_1e^2} \int_{R_0/R_c}^1 \sqrt{\frac{1}{zR_c} - \frac{1}{R_c}} R_c \, dz\right) \\ = \exp\left(-\frac{2}{\hbar}\sqrt{\frac{2m}{E}} Z_0 Z_1 e^2 \int_{R_0/R_c}^1 \sqrt{\frac{1}{z} - 1} \, dz\right)$$
(2.122)

The result is

$$\hat{T} \approx \exp\left(-\frac{2}{\hbar}\sqrt{\frac{2m}{E}}Z_0 Z_1 e^2 \left[\arccos\sqrt{\frac{E}{V_C}} - \sqrt{\frac{E}{V_C}\left(1 - \frac{E}{V_C}\right)}\right]\right)$$
(2.123)



Fig. 2.16 Schematic representation of radial wave functions (thin solid lines) for the scattering of neutral particles by a square-barrier potential (thick solid line) of different potential depths V_0 . The wave functions are appropriate for (a) no resonance, (b) the first resonance with no wave function node in the interior region ($r < R_0$), and (c) the second resonance with one node in the interior.

For low energies compared to the Coulomb barrier height, $E/V_C \ll 1$, we use the expansion $\arccos \sqrt{x} - \sqrt{x(1-x)} \approx \pi/2 - 2\sqrt{x} + x^{3/2}/3$ and obtain

$$\hat{T} \approx \exp\left(-\frac{2}{\hbar}\sqrt{\frac{2m}{E}}Z_0 Z_1 e^2 \left[\frac{\pi}{2} - 2\sqrt{\frac{E}{V_C}} + \frac{1}{3}\left(\frac{E}{V_C}\right)^{3/2}\right]\right) \\ = \exp\left(-\frac{2\pi}{\hbar}\sqrt{\frac{m}{2E}}Z_0 Z_1 e^2 \left[1 + \frac{2}{3\pi}\left(\frac{E}{V_C}\right)^{3/2}\right] + \frac{4}{\hbar}\sqrt{2mZ_0 Z_1 e^2 R_0}\right) (2.124)$$

The first term in the exponential is larger than the third term by a factor of $(\pi/4)\sqrt{V_C/E}$ and therefore dominates the transmission coefficient. The third term vanishes in the limit $R_0 \rightarrow 0$ and represents a correction due to a finite radius to which the projectile must penetrate. The larger the radius R_0 , the smaller the penetration distance (Fig. 2.17) and the larger the transmission coefficient will become. The second term represents a correction factor to the first term when the energy becomes a significant fraction of the Coulomb



Fig. 2.17 Attractive nuclear square-well potential ($r < R_0$) plus repulsive Coulomb potential ($r > R_0$), shown as thick solid line. The transmission coefficient can be calculated analytically by dividing the Coulomb barrier of height $V_C = Z_1 Z_2 e^2 / R_0$ into infinitesimally thin square-barrier potentials. The radius R_c at which the total particle energy *E* (thin solid line) is equal to the

Coulomb potential, $E = Z_1 Z_2 e^2 / R_c$, is referred to as the classical turning point. The thick dashed line indicates a small negative (attractive) potential U_s that results from the polarization of the electron–ion plasma (electron screening), giving rise to a modification of both the Coulomb potential and the energetics of the reaction (Section 3.2.6).

barrier height. The leading term of the s-wave Coulomb barrier transmission coefficient for small energies compared to the Coulomb barrier height,

$$\hat{T} \approx \exp\left(-\frac{2\pi}{\hbar}\sqrt{\frac{m}{2E}}Z_0Z_1e^2\right) \equiv e^{-2\pi\eta}$$
(2.125)

is called the *Gamow factor* and will play an important role in the discussion of thermonuclear reaction rates for charged particles (Section 3.2.1). The quantity η is the Sommerfeld parameter. Numerically, we find

$$2\pi\eta = 0.989534Z_0Z_1\sqrt{\frac{1}{E}\frac{M_0M_1}{M_0+M_1}}$$
(2.126)

where the energy *E* is in MeV and the relative atomic masses M_i are in units of u.

1082 Nuclear Reactions2.5Theory of Resonances

2.5.1 General Aspects

Up to now we have discussed wave function intensities, phase shifts, and transmission probabilities for simple nuclear potentials. In the following, the resulting cross sections will be considered. Initially, we will restrict ourselves again to the case of s-wave scattering of neutral particles, that is, the complications of the Coulomb and centripetal barriers are disregarded. The total elastic scattering and reaction cross sections are then given by Eqs. (2.40) and (2.49),

$$\sigma_{\rm el,0} = \frac{\pi}{k^2} \left| 1 - e^{2i\delta_0} \right|^2 = \frac{4\pi}{k^2} \sin^2 \delta_0 \tag{2.127}$$

$$\sigma_{\rm re,0} = \frac{\pi}{k^2} \left(1 - \left| e^{2i\delta_0} \right|^2 \right) \tag{2.128}$$

The cross sections are entirely determined by the phase shift δ_0 .

It is interesting to plot the total elastic scattering cross sections for the potential models considered in Sections 2.4.1 and 2.4.2. They are shown in Fig. 2.18 for (a) a square-well potential with $R_0 = 3$ fm, $V_0 = 100$ MeV; (b) a squarebarrier potential with $R_0 = 3$ fm, $R_1 = 8$ fm, $V_0 = 100$ MeV, $V_1 = 10$ MeV; and (c) a square-barrier potential with $R_0 = 3$ fm, $R_1 = 8$ fm, $V_0 = 50$ MeV, $V_1 =$ 10 MeV. We expect resonances in parts (a) and (c), as is apparent from Figs. 2.8 and 2.14. However, a resonance is clearly observed only in Fig. 2.18c. And even in this case the resulting shape of the total elastic scattering cross section looks complicated.

Up to now we considered single-particle potentials. The spacing of resonances, referred to as single-particle resonances, calculated by these models amounts to many MeV. However, experiments performed since the 1930s frequently showed closely spaced resonances (sometimes a few keV or less apart) of very narrow widths. For example, Fig. 2.19 shows the experimental elastic scattering cross section of neutrons on ¹⁶O. In contrast to our theoretical results obtained so far, a very complicated structure consisting of several resonances with different widths is clearly observed. The solid line in Fig. 2.19 represents a calculation using a single-particle potential. It reproduces only one of the many resonances shown. Although some observed resonances can indeed be described by single-particle potentials, in the vast majority of cases the single-particle picture is clearly not appropriate for the explanation of the observed rapid cross section variations.

At this point we suspect that the interactions of many nucleons inside the nucleus are complicated and cannot be expressed in terms of a single radial wave function generated by a single-particle potential. In the following, we



Fig. 2.18 Elastic scattering cross sections for s-wave neutrons versus energy for the simple potentials discussed in Sections 2.4.1 and 2.4.2: (a) square-well potential with $R_0 = 3$ fm, $V_0 = 100$ MeV; (b) square-barrier potential with $R_0 = 3$ fm, $R_1 = 8$ fm, $V_0 = 100$ MeV, $V_1 = 10$ MeV; (c) square-barrier potential with $R_0 = 3$ fm, $R_1 = 8$ fm, $V_0 = 50$ MeV, $V_1 = 10$ MeV. A resonance is clearly observed only in part (c).

will develop a different model in order to describe a nuclear resonance in general terms without using an explicit assumption for the nuclear potential. In fact, a specific assumption regarding the many-particle nuclear potential may even be undesirable since at this point neither the nuclear forces between the nucleons nor their motion in the nuclear interior are precisely known.

2.5.2 Logarithmic Derivative, Phase Shift, and Cross Section

Since we will not make reference to a specific nuclear potential, this model will not be able to predict the absolute magnitude of cross sections. In fact, most



Fig. 2.19 Experimental total cross section for the elastic scattering ¹⁶O(n,n)¹⁶O (data points) and calculated elastic scattering cross section using a (Woods–Saxon) single-particle potential (solid line). The potential reproduces only one of the observed resonances but cannot account for the entire observed structure. Reprinted with permission from G. D. Westin and J. L. Adams, Phys. Rev. C, Vol. 4, p. 363 (1972). Copyright (1972) by the American Physical Society.

resonance theories reformulate the cross sections in terms of known quantities of the nuclear exterior (penetration and shift factors) and unknown quantities of the nuclear interior (reduced widths). Our goal is to predict relative cross sections "near" a resonance. The only assumptions we make regarding the nuclear potential is the existence of a relatively well-defined spherical nuclear surface at r = R and that the projectile and target have no nuclear interaction outside of this radius.

At this boundary the interior and exterior wave functions and their derivatives have to be matched,

$$u_{\ell}^{\text{in}}(R) = u_{\ell}^{\text{out}}(R) \quad \text{and} \quad \left(\frac{du_{\ell}^{\text{in}}(r)}{dr}\right)_{r=R} = \left(\frac{du_{\ell}^{\text{out}}(r)}{dr}\right)_{r=R} \quad (2.129)$$

By dividing both expressions and by introducing a dimensionless quantity, called the *logarithmic derivative at the boundary*,

$$f_{\ell} \equiv R \left(\frac{1}{u_{\ell}(r)} \frac{du_{\ell}(r)}{dr} \right)_{r=R} = R \left(\frac{d\ln u_{\ell}(r)}{dr} \right)_{r=R}$$
(2.130)

we can rewrite the conditions of Eq. (2.129) as

$$f_{\ell}(u_{\ell}^{\rm in}) = f_{\ell}(u_{\ell}^{\rm out})$$
 (2.131)

In other words, the calculation of f_{ℓ} with u_{ℓ}^{in} and u_{ℓ}^{out} must yield the same value. Obviously, the quantity f is related to the slope of the wave function at the radius r = R.

We start from the expression for the total wave function in the exterior region r > R (see Eq. (2.32)). For s-waves ($\ell = 0$) it reduces to

$$\psi_{T,\text{out}} = Ae^{ikr} + Be^{-ikr}, \qquad k^2 = \frac{2mE}{\hbar^2} = -\frac{i}{2kr}e^{2i\delta_0}e^{ikr} + \frac{i}{2kr}e^{-ikr} = \frac{i}{2kr}\left(e^{-ikr} - e^{2i\delta_0}e^{ikr}\right) = \frac{1}{2kr}e^{i\delta_0}\left[e^{-i(kr+\delta_0)} - e^{i(kr+\delta_0)}\right] = \frac{1}{kr}e^{i\delta_0}\sin(kr+\delta_0)$$
(2.132)

where the first expression ($Ae^{ikr} + Be^{-ikr}$) generally holds in the force-free region (Eqs. (2.57) and (2.86); see also Appendix A.1). Recall the meaning of the above equation. The outgoing spherical wave e^{ikr} is multiplied by a factor $e^{2i\delta_0}$ which effectively shifts the wave by an amount of δ_0 .

As already implied in Sections 2.3.3 and 2.3.4, for the special case of s-waves, Eq. (2.32) (and thus Eq. (2.132)) is not only valid at large distances but applies to all distances r > R. Furthermore, the spherical harmonic for $\ell = 0$ is constant (see Eq. (A.9)) and, therefore, the total wave function is given by the radial wave function

$$\psi_{T,\text{out}} = \frac{i}{2kr} \left(e^{-ikr} - e^{2i\delta_0} e^{ikr} \right) = \frac{u_{\text{out}}(r)}{r}$$
(2.133)

The cross section is determined by the phase shift δ_0 . We will first find a relationship between δ_0 and f_0 and will then express the cross section in terms of f_0 . From Eqs. (2.130) and (2.133) one obtains

$$\frac{f_0}{R} = \left(\frac{1}{u_{\text{out}}(r)}\frac{du_{\text{out}}(r)}{dr}\right)_{r=R} = \frac{-ike^{-ikR} - ike^{2i\delta_0}e^{ikR}}{e^{-ikR} - e^{2i\delta_0}e^{ikR}}$$
(2.134)

Solving for $e^{2i\delta_0}$ gives

$$e^{2i\delta_0} = \frac{f_0 + ikR}{f_0 - ikR} e^{-2ikR}$$
(2.135)

For the elastic scattering cross section (see Eq. (2.127)) we find

$$\sigma_{\text{el},0} = \frac{\pi}{k^2} \left| 1 - \frac{f_0 + ikR}{f_0 - ikR} e^{-2ikR} \right|^2 = \frac{\pi}{k^2} \left| e^{2ikR} - \frac{f_0 + ikR}{f_0 - ikR} \right|^2$$
$$= \frac{\pi}{k^2} \left| -\frac{2ikR}{f_0 - ikR} + e^{2ikR} - 1 \right|^2 = \frac{\pi}{k^2} \left| A_{\text{res}} + A_{\text{pot}} \right|^2$$
(2.136)

with

$$A_{\rm res} = -\frac{2ikR}{f_0 - ikR}$$
 and $A_{\rm pot} = e^{2ikR} - 1$ (2.137)

It can be seen that A_{res} has a maximum if $f_0 = 0$, consistent with our identification of resonances as a slope of zero for the radial wave function at the boundary r = R.

Similarly, using $f_0 = \text{Re } f_0 + i \text{ Im } f_0 = g + ih$, one obtains for the reaction cross section (see Eq. (2.128))

$$\sigma_{\rm re,0} = \frac{\pi}{k^2} \left(1 - \left| \frac{f_0 + ikR}{f_0 - ikR} e^{-2ikR} \right|^2 \right) = \frac{\pi}{k^2} \left[1 - \left(\frac{g + ih + ikR}{g + ih - ikR} \right) \left(\frac{g - ih - ikR}{g - ih + ikR} \right) \right] = \frac{\pi}{k^2} \frac{-4hkR}{g^2 + h^2 - 2hkR + k^2R^2} = \frac{\pi}{k^2} \frac{-4kR \operatorname{Im} f_0}{(\operatorname{Re} f_0)^2 + (\operatorname{Im} f_0 - kR)^2}$$
(2.138)

Only A_{res} depends on the interior region r < R through f_0 . Consequently, only this term can give rise to resonances and A_{res} is called *resonance scattering amplitude*. The term A_{pot} can be interpreted as follows. Suppose that $f_0 \rightarrow \infty$. In that case $A_{\text{res}} = 0$. From the definition of f_0 (see Eq. (2.130)) this implies u(R) = 0 and, therefore, an infinitely high potential for r < R (that is, the sphere of radius R is impenetrable). Thus, the quantity A_{pot} is called *hard-sphere potential scattering amplitude*. Note also that Im $f_0 \leq 0$; otherwise $\sigma_{\text{re},0}$ becomes negative. For the special case that f_0 is real (Im $f_0 = 0$), the reaction cross section disappears, $\sigma_{\text{re},0} = 0$. Therefore, f_0 must be complex for reactions to occur.

It is also interesting to consider the elastic scattering phase shift δ_0 . From Eq. (2.135) one finds

$$2i\delta_0 = \ln(f_0 + ikR) - \ln(f_0 - ikR) - 2ikR$$
(2.139)

Assuming Im $f_0 = 0$ (or $\sigma_{re,0} = 0$) we find by using the identity $\ln(a + ib) = (1/2) \ln(a^2 + b^2) + i \arctan(b/a)$

$$\delta_{0} = \frac{1}{2i} \left[\frac{1}{2} \ln \left(f_{0}^{2} + k^{2} R^{2} \right) + i \arctan \left(\frac{kR}{f_{0}} \right) \right]$$
$$- \frac{1}{2i} \left[\frac{1}{2} \ln \left(f_{0}^{2} + k^{2} R^{2} \right) + i \arctan \left(\frac{-kR}{f_{0}} \right) \right] - kR$$
$$= \arctan \left(\frac{kR}{f_{0}} \right) - kR = \beta_{0} + \varphi_{0}$$
(2.140)

with

$$\beta_0 = \arctan\left(\frac{kR}{f_0}\right) \quad \text{and} \quad \varphi_0 = -kR \quad (2.141)$$

The phase shift δ_0 is expressed as a sum of two terms. The first term, β_0 , depends on the scattering potential through f_0 and can give rise to resonances.

The second term, φ_0 , is independent of the scattering potential. It corresponds to the phase shift for hardsphere scattering, since $\delta_0 = \varphi_0$ for $f_0 \rightarrow \infty$ (or u(R) = 0).

2.5.3 Breit–Wigner Formulas

The logarithmic derivative at the boundary, f_0 , has to be known in order to calculate $\sigma_{el,0}$ and $\sigma_{re,0}$. For the derivation of f_0 we need to make some assumptions regarding the wave function in the nuclear interior (r < R). Remember that the general solution with constant wave number K in the interior (see Eqs. (2.55) and (2.84))

$$u_{\rm in} = Ae^{iKr} + Be^{-iKr} \tag{2.142}$$

only applies for the simple assumption of a constant potential V(r) = const (Section 2.4). Clearly, the actual nuclear potential will be rather complicated since for r < R the wave function of the incident particle will depend on the variables of all the other nucleons involved. Nevertheless, we will approximate the interior wave function, in the closest vicinity of the nuclear boundary only, by the above expression.

The complex amplitudes *A* and *B* depend on the properties of the nuclear system. We have to allow for a phase difference ζ between incoming (e^{-iKr}) and outgoing (e^{iKr}) spherical waves. Furthermore, one has to account for the possibility that the particle is absorbed in the nuclear interior due to reaction processes, that is, the amplitude of the outgoing wave e^{iKr} will generally be smaller than the amplitude of the incoming wave e^{-iKr} . We start with the ansatz

$$A = Be^{2i\zeta}e^{-2q} \tag{2.143}$$

where both ζ and q are real numbers. We also require $q \ge 0$ since no more particles can return than entered the nucleus originally. From Eqs. (2.142) and (2.143) one finds

$$u_{\rm in} = Be^{2i\zeta}e^{-2q}e^{iKr} + Be^{-iKr} = \frac{B}{2}\left[e^{-i(Kr+\zeta+iq)} + e^{i(Kr+\zeta+iq)}\right]2e^{(i\zeta-q)}$$

= $2Be^{(i\zeta-q)}\cos(Kr+\zeta+iq)$ (2.144)

The logarithmic derivative of the radial wave function must be continuous at r = R. Substitution of Eq. (2.144) into Eq. (2.130) yields

$$f_0 = R \left(\frac{1}{u_{\rm in}(r)} \frac{du_{\rm in}(r)}{dr}\right)_{r=R} = R \frac{-2Be^{(i\zeta-q)}K\sin(KR+\zeta+iq)}{2Be^{(i\zeta-q)}\cos(KR+\zeta+iq)}$$
$$= -KR\tan(KR+\zeta+iq)$$
(2.145)

Clearly, f_0 depends on the energy-dependent quantities K, ζ , and q. If one knew these variables, then one could calculate resonance energies and cross sections directly from the properties of the nuclear interior. Unfortunately, this is not the case. Thus, our strategy will be to express the cross sections σ_{el} and σ_{re} near a single resonance in terms of measurable quantities.

We will assume that the argument of the tangent, $KR + \zeta + iq$, is a smooth function of energy *E*. Furthermore, if q = 0 then f_0 becomes real, and the reaction cross section disappears. Recall that a resonance corresponds to a large wave function amplitude in the nuclear interior, implying a slope of zero for the radial wave function at r = R (Fig. 2.10). One can define *formal resonance energies* E_{λ} by the condition

$$f_0(E_\lambda, q) = -KR \tan(KR + \zeta + iq) = 0$$
(2.146)

Of course, there is a whole set of such energies. Let us consider any one of them and study the behavior of f_0 near E_{λ} .

In the following it is assumed that the absorption in the nuclear interior is weak compared to the elastic scattering process, that is $|q| \ll 1$. Expansion of $f_0(E, q)$ near E_λ and q = 0 into a Taylor series in both q and E gives

$$f_0 \approx f_0(E_\lambda, q) + (E - E_\lambda) \left(\frac{\partial f_0}{\partial E}\right)_{E_\lambda, q=0} + q \left(\frac{\partial f_0}{\partial q}\right)_{E_\lambda, q=0}$$
(2.147)

For the last term, one finds with Eq. (2.145)

$$q\left(\frac{\partial f_0}{\partial q}\right)_{E_{\lambda},q=0} = -qKR \left[\frac{\partial}{\partial q}\tan(KR + \zeta + iq)\right]_{E_{\lambda},q=0} = -iqKR$$
(2.148)

since at the resonance energy E_{λ} one has $\tan x = 0$ (see Eq. (2.146)) and thus $d(\tan x)/dx = \cos^{-2} x = 1$. It follows

$$f_0 \approx (E - E_\lambda) \left(\frac{\partial f_0}{\partial E}\right)_{E_{\lambda}, q = 0} - iqKR = \operatorname{Re} f_0 + i\operatorname{Im} f_0$$
(2.149)

Recall that $(\partial f_0 / \partial E)_{E_{\lambda},q=0}$ is a real quantity since q = 0 implies a vanishing reaction cross section. Substitution of Eq. (2.149) into Eqs. (2.137) and (2.138) gives for the resonance scattering amplitude and the reaction cross section

$$A_{\rm res} = \frac{-\frac{2ikR}{(\partial f_0/\partial E)_{E_{\lambda},q=0}}}{(E - E_{\lambda}) - \frac{i(kR + qKR)}{(\partial f_0/\partial E)_{E_{\lambda},q=0}}}$$
(2.150)

$$\sigma_{\rm re,0} = \frac{\pi}{k^2} \frac{\frac{(2KK)(2qKK)}{(\partial f_0 / \partial E)^2_{E_{\lambda},q=0}}}{(E - E_{\lambda})^2 + \frac{(qKR + kR)^2}{(\partial f_0 / \partial E)^2_{E_{\lambda},q=0}}}$$
(2.151)

We introduce the following definitions (the subscripts e and r stand for "elastic" and "reaction," respectively):

$$\Gamma_{\lambda e} \equiv -\frac{2kR}{(\partial f_0 / \partial E)_{E_{\lambda},q=0}} \qquad (\text{particle width}) \qquad (2.152)$$

$$\Gamma_{\lambda \mathbf{r}} \equiv -\frac{2qKR}{(\partial f_0/\partial E)_{E_{\lambda},q=0}} \qquad (\text{reaction width}) \qquad (2.153)$$

$$\Gamma_{\lambda} \equiv \Gamma_{\lambda e} + \Gamma_{\lambda r} \qquad (\text{total width}) \qquad (2.154)$$

Only the quantity $\Gamma_{\lambda r}$ depends on the parameter *q* describing absorption in the nuclear interior. The new quantities $\Gamma_{\lambda e}$, $\Gamma_{\lambda r}$, and Γ_{λ} have units of energy since f_0 , kR, and KR are dimensionless. All widths refer to the resonance λ of interest. Also, f_0 depends on the channel through which the reaction is initiated. Rewriting Eqs. (2.150) and (2.151) in terms of the newly defined quantities yields, after some algebra, for the elastic scattering and reaction cross sections (see Eqs. (2.136) and (2.138))

$$\sigma_{\rm el,0} = \frac{\pi}{k^2} \left| \frac{i\Gamma_{\lambda \rm e}}{(E - E_{\lambda}) + i\Gamma_{\lambda}/2} + e^{2ikR} - 1 \right|^2$$
$$= \frac{\pi}{k^2} \left[2 - 2\cos(2kR) + \frac{\Gamma_{\lambda \rm e}^2 - \Gamma_{\lambda \rm e}\Gamma_{\lambda} + \Gamma_{\lambda \rm e}\Gamma_{\lambda}\cos(2kR) + 2\Gamma_{\lambda \rm e}(E - E_{\lambda})\sin(2kR)}{(E - E_{\lambda})^2 + \Gamma_{\lambda}^2/4} \right] \quad (2.155)$$

$$\sigma_{\rm re,0} = \frac{\pi}{k^2} \frac{\Gamma_{\lambda \rm e} \Gamma_{\lambda \rm r}}{(E - E_\lambda)^2 + \Gamma_\lambda^2 / 4}$$
(2.156)

The last two expressions are referred to as *Breit–Wigner formulas* for s-wave neutrons.

Plots of $\sigma_{el,0}$ and $\sigma_{re,0}$ for incident neutrons versus energy *E* near a resonance are shown in Fig. 2.20. We use the values R = 3 fm and $E_{\lambda} = 1$ MeV and assume energy-independent partial widths of $\Gamma_{\lambda} = 10.1$ keV and $\Gamma_{\lambda e} = 10$ keV. Several interesting aspects can be noticed. First, the full width at half maximum of the $\sigma_{re,0}$ curve (FWHM = 10.1 keV) corresponds precisely to the value of Γ_{λ} . Therefore, we identify this parameter with the *total resonance width*. The quantities $\Gamma_{\lambda e}$ and $\Gamma_{\lambda r}$ correspond then to *partial widths* for the scattering and reaction channel, respectively. Second, far away from the resonance $(|E - E_{\lambda}| \gg \Gamma_{\lambda})$ only the hardsphere potential scattering amplitude A_{pot} will contribute to the cross section. We obtain from Eq. (2.155) a value of $\sigma_{el,0} \approx$ $(2\pi/k^2)[1 - \cos(2kR)] \approx 100$ fm², shown as dashed line in Fig. 2.20. Third, the numerator in the expression for $\sigma_{el,0}$ (see Eq. (2.155)) contains an interference term $2\Gamma_{\lambda e}(E - E_{\lambda}) \sin(2kR)$ which will change sign below and above the resonance. Clearly, the structure seen in Fig. 2.20a is caused by destructive

 $(E < E_{\lambda})$ and constructive $(E > E_{\lambda})$ interference of A_{res} and A_{pot} . This interference also causes the full width at half maximum in the elastic scattering cross section at the resonance to be different from the value of Γ_{λ} . Fourth, remember that a resonance corresponds to a value of zero for the logarithmic derivative f_0 at the nuclear boundary. The implication is that in the scattering process the particle enters the nucleus with significant probability only near resonance. Off resonance the particle is almost entirely reflected at the boundary, and the wave function inside is weak. The resonance scattering is thus ascribed to the inside of the nucleus and the potential scattering to its surface.

The present results help us understand the complicated structures observed in the scattering cross sections for an explicit nuclear potential, as shown in Fig. 2.18. The structures are caused, in part, by interference effects between the re-emission of the incident particle by the nucleus and the scattering near the nuclear surface. An additional complication is introduced by the fact that single-particle potential models predict several resonances that may interfere with each other.

We consider now the elastic scattering phase shift near a resonance, $\delta_0 = \beta_0 + \varphi_0 \approx \beta_0$. For Im $f_0 = 0$ (q = 0) we obtain from Eqs. (2.140), (2.149), and (2.152)

$$\beta_0 \approx \arctan\left[\frac{kR}{(E - E_\lambda)(\partial f_0 / \partial E)_{E_\lambda, q = 0}}\right] = \arctan\left[\frac{\Gamma_{\lambda e}}{2(E_\lambda - E)}\right]$$
(2.157)

At the resonance energy, $E = E_{\lambda}$, the argument of the arctan function becomes infinite and thus $\beta_0 = \pi/2$. Furthermore, since $d(\arctan x)/dx = (1 + x^2)^{-1}$, one finds for the energy derivative of the resonance elastic scattering phase shift at $E = E_{\lambda}$

$$\left(\frac{d\beta_0}{dE}\right)_{E_{\lambda}} = \frac{1}{2} \left[\frac{(d\Gamma_{\lambda e}/dE)(E_{\lambda} - E) + \Gamma_{\lambda e}}{(E_{\lambda} - E)^2 + (\Gamma_{\lambda e}/2)^2} \right]_{E_{\lambda}} = \frac{2}{(\Gamma_{\lambda e})_{E_{\lambda}}}$$
(2.158)

Hence, the resonance phase shift at $E = E_{\lambda}$ amounts to $\pi/2$ while its energy derivative determines the particle width. For the special case that $\Gamma_{\lambda e}$ is nearly constant with energy (for example, for a narrow resonance), we find from Eq. (2.157) that $\beta_0 = \pi/4$ at $E = E_{\lambda} - \Gamma_{\lambda e}/2$ and $\beta_0 = 3\pi/4$ at $E = E_{\lambda} + \Gamma_{\lambda e}/2$. Thus the particle width $\Gamma_{\lambda e}$ becomes equal to the energy interval over which β_0 increases from $\pi/4$ to $3\pi/4$. The above techniques involving the resonance phase shift are frequently employed for the calculation of particle partial widths.

Let us now consider the particle width $\Gamma_{\lambda e}$ in more detail. We define (see Eq. (2.152))

$$\Gamma_{\lambda e} = -\frac{2kR}{(\partial f_0/\partial E)_{E_{\lambda},q=0}} \equiv 2P_0\gamma_{\lambda e}^2 \quad \text{with} \quad \gamma_{\lambda e}^2 \equiv -\left(\frac{\partial f_0}{\partial E}\right)_{E_{\lambda},q=0}^{-1}$$
(2.159)



Fig. 2.20 Plots of (a) $\sigma_{el,0}$ and (b) $\sigma_{re,0}$ versus energy *E* for incident neutrons near a resonance at $E_{\lambda} = 1$ MeV. A value of R = 3 fm is chosen for the radius. The widths are assumed to be energy independent with values of $\Gamma_{\lambda} = 10.1$ keV and $\Gamma_{\lambda e} = 10$ keV. See discussion in the text.

where the particle width has been split into two factors. The first factor, $P_0 = kR$, depends on the channel energy through the factor kR and on the conditions outside the nucleus. The second factor, $\gamma_{\lambda e}^2$, is called the *reduced width* and it incorporates all the unknown properties of the nuclear interior. The quantity $\gamma_{\lambda e}^2$ is characteristic of the resonance and the channel under consideration, and is independent of the channel energy *E*. The energy dependence of the partial widths $\Gamma_{\lambda e}$ and $\Gamma_{\lambda r}$ has to be taken into account when calculating the cross section for broad resonances, as will be explained in later sections.

2.5.4

Extension to Charged Particles and Arbitrary Values of Orbital Angular Momentum

The one-level Breit–Wigner formulas (see Eqs. (2.155) and (2.156)) have been obtained near a formal resonance energy E_{λ} assuming (i) neutrons as incident

particles, (ii) an orbital angular momentum of $\ell = 0$, and (iii) interactions of spinless particles. The basic structure of the cross section expressions derived here is also applicable to the much more general case. Although the general expressions are more complicated in appearance compared to the results for s-wave neutrons, no new physical ideas are involved. The properties of the nuclear interior enter into the cross sections only through the logarithmic derivative f_{ℓ} of the wave function $u_{\ell}(r)$ at the nuclear boundary r = R.

In the following, the formulas will be generalized to arbitrary values of ℓ and to interacting charged particles. We will not derive the results in detail here (see, for example, Blatt and Weisskopf 1952) but will only sketch some results of the derivation. Of special interest to us is the modified reaction cross section $\sigma_{re,\ell}$.

The radial wave function solutions of the Schrödinger equation outside the nuclear surface are no longer given by incoming and outgoing spherical waves (e^{-ikr} and e^{ikr}), as in the case of $\ell = 0$ neutrons (see Eq. (2.132)), but are given in terms of the functions F_{ℓ} and G_{ℓ} . For neutrons, these represent spherical Bessel and Neumann functions, $F_{\ell} = (kr)j_{\ell}(kr)$ and $G_{\ell} = (kr)n_{\ell}(kr)$, respectively, while for charged particles they correspond to the regular and irregular Coulomb wave functions (Appendix A.3). The radial wave function outside the nuclear boundary is given in terms of F_{ℓ} and G_{ℓ} by

$$u_{\ell}(r) = Au_{\ell}^{+}(r) + Bu_{\ell}^{-}(r), \qquad r > R$$

= $Ae^{-i\sigma_{\ell}}[G_{\ell}(r) + iF_{\ell}(r)] + Be^{i\sigma_{\ell}}[G_{\ell}(r) - iF_{\ell}(r)]$ (2.160)

where u_{ℓ}^{-} and u_{ℓ}^{+} correspond, for large distances, to incoming and outgoing spherical waves, respectively. The quantity σ_{ℓ} denotes the Coulomb phase shift and determines the purely Rutherford (electrostatic) scattering. For $\ell = 0$ neutrons, the above expression reduces to our previous result (Eq. (2.132); see Problem 2.4).

It is of advantage to introduce two real quantities, called *shift factor* S_{ℓ} and *penetration factor* P_{ℓ} , which are completely determined by the conditions outside the nucleus. We obtain with Eqs. (2.160) and (A.18)

$$R\left(\frac{1}{u_{\ell}^{+}(r)}\frac{du_{\ell}^{+}(r)}{dr}\right)_{r=R}$$

$$=R\left[\frac{G_{\ell}(dG_{\ell}/dr) + F_{\ell}(dF_{\ell}/dr) + iG_{\ell}(dF_{\ell}/dr) - iF_{\ell}(dG_{\ell}/dr)}{F_{\ell}^{2} + G_{\ell}^{2}}\right]_{r=R}$$

$$\equiv S_{\ell} + iP_{\ell}$$
(2.161)

$$S_{\ell} = R \left[\frac{F_{\ell}(dF_{\ell}/dr) + G_{\ell}(dG_{\ell}/dr)}{F_{\ell}^{2} + G_{\ell}^{2}} \right]_{r=R} \text{ and}$$

$$P_{\ell} = R \left(\frac{k}{F_{\ell}^{2} + G_{\ell}^{2}} \right)_{r=R}$$

$$(2.162)$$

The new quantities depend on the wave number k, the channel radius R, the orbital angular momentum ℓ , and on the charge parameter η (see Eq. (A.32)). For $\ell = 0$ neutrons, $F_{\ell} = (kr)j_0(kr) = \sin(kr)$ and $G_{\ell} = (kr)n_0(kr) = \cos(kr)$ (Appendix A.2), and we obtain from Eq. (2.161) $P_0 = kR$ and $S_0 = 0$. In other words, the shift factor vanishes if there is no barrier. With the quantities P_{ℓ} and S_{ℓ} , the reaction cross section can be derived in a similar way as presented in the previous section. The calculation is not repeated here (see Blatt and Weisskopf 1952). The result is the Breit–Wigner formula

$$\sigma_{\mathrm{re},\ell} = (2\ell+1)\frac{\pi}{k^2} \frac{\Gamma_{\lambda \mathrm{e}} \Gamma_{\lambda \mathrm{r}}}{(E-E_r)^2 + \Gamma_{\lambda}^2/4}$$
(2.163)

with

$$\Gamma_{\lambda e} \equiv -\frac{2P_{\ell}(E)}{(\partial f_{\ell}/\partial E)_{E_{\lambda},q=0}} = 2P_{\ell}(E)\gamma_{\lambda e}^{2} \qquad (\text{particle width}) \qquad (2.164)$$

$$\Gamma_{\lambda \mathbf{r}} \equiv -\frac{2qKR}{(\partial f_{\ell}/\partial E)_{E_{\lambda},q=0}}$$
 (reaction width) (2.165)

$$\Gamma_{\lambda} = \Gamma_{\lambda e} + \Gamma_{\lambda r}$$
 (total width) (2.166)

$$E_r \equiv E_{\lambda} + \frac{S_{\ell}(E)}{(\partial f_{\ell}/\partial E)_{E_{\lambda},q=0}} = E_{\lambda} - S_{\ell}(E)\gamma_{\lambda e}^2 \qquad \text{(observed resonance)}$$

The similarity between Eq. (2.163) and the result obtained earlier for s-wave neutrons (see Eq. (2.156)) is apparent. The meaning of P_{ℓ} and S_{ℓ} becomes clear now. The penetration factor appears in the particle width expression since an incident particle must penetrate to the nuclear surface for a reaction to occur. The shift factor appears in the level shift expression and it causes the observed resonance energy E_r to be different from the formal resonance energy (or level energy) E_{λ} . Both the resonance energy shift and the particle width also depend on the properties of the nuclear interior through the reduced width $\gamma_{\lambda e}^2$.

The penetration factor is closely related to the transmission coefficient. Both quantities describe the same physical concept, but are defined in slightly different ways. The former quantity is independent of the nuclear interior while the latter is defined in terms of the ratio of current densities in the interior and exterior regions (see Eqs. (2.62) and (2.103)). However, the energy dependences of both quantities should be very similar. The penetration factor can

where

be calculated analytically for neutrons. The expressions are not repeated here (see Blatt and Weisskopf 1952). It is sufficient to mention that for small neutron energies the neutron partial widths behave as $\Gamma_{\ell}(E) \sim P_{\ell}(E) \sim (kR)^{2\ell+1} \sim E^{\ell+1/2}$. For charged particles, on the other hand, the calculation of penetration factors is much more involved. Various analytical approximations exist for estimating $P_{\ell}(E)$ (see, for example, Clayton 1983). The reader should be aware, however, that some of these approximations are not always accurate and that it is more reliable to compute penetration factors directly from numerical values of the Coulomb wave functions (Appendix A.3). The energy dependence of the s-wave penetration factor at low energies compared to the Coulomb barrier height ($E \ll V_C$) is given by Eqs. (2.124) and (2.125). For higher orbital angular momenta, the charged-particle penetration factors behave at low energies as $P_{\ell}(E) \sim \exp[-a/\sqrt{E} - b\ell(\ell + 1)]$ (Problem 2.2), where the first exponential term represents the Gamow factor.

Numerical values for the factors P_{ℓ} and S_{ℓ} are displayed in Fig. 2.21 for ¹²C + p and ¹²C + n in order to illustrate some important points. The curves are obtained by using a radius of $R = 1.25(12^{1/3} + 1) = 4.1$ fm. The different energy dependences of P_{ℓ} for protons and neutrons is striking. The penetration factors for both protons and neutrons drop for decreasing energy, but the former values drop significantly faster since the Coulomb barrier has to be penetrated in addition to the centripetal barrier (for $\ell > 0$). The energy dependence of P_{ℓ} is similar for protons of all ℓ values, while for neutrons the energy dependence varies for different ℓ values. At higher energies ($E \approx 3$ MeV) we obtain $P_{\ell} \approx 1$ for protons and neutrons. Note that the curve for the s-wave ($\ell = 0$) penetration factor of 12 C + n is simply given by $P_0 = kR$ (see earlier). The shift factors vary far less with energy compared to the penetration factors. In fact, for both neutrons and protons, S_{ℓ} is almost constant below an energy of a few hundred keV. Of course, one finds $S_0 = 0$ for neutrons as already noted above.

The straight lines for $P_{\ell}(E)$ at low neutron energies in the log–log plot of Fig. 2.21 are a consequence of the energy dependence $P_{\ell}(E) \sim E^{\ell+1/2}$. The slopes of the curves are equal to $\ell + 1/2$ since $\log P_{\ell}(E) \sim \log E^{\ell+1/2} = (\ell + 1/2) \log E$. A similar procedure can be applied to charged particles. A graphic illustration of the energy dependence of the penetration factors for ${}^{12}\text{C} + \text{p}$ is given in Fig. 2.22. Since we have $\log P_{\ell}(E) \sim -a/\sqrt{E} - b\ell(\ell + 1)$, straight lines are obtained at low energies when $\log P_{\ell}(E)$ is plotted versus $-1/\sqrt{E}$. The slopes are similar at low energies where they are determined by the tunneling probability through the Coulomb barrier, while the intercepts depend on the value of ℓ . The straight lines shown in Fig. 2.22 represent a useful tool when checking or interpolating values of $P_{\ell}(E)$ that are obtained numerically from computer codes.



Fig. 2.21 Penetration (top) and shift factors (bottom) for ${}^{12}C + p$ (left) and ${}^{12}C + n$ (right). In each panel, the curves show the results for orbital angular momenta of $\ell = 0, 1, 2, \text{ and } 3$. All curves are calculated for a radius of $R = 1.25(12^{1/3} + 1) = 4.1$ fm. The much stronger energy dependence of the penetration factor for protons compared to neutrons is apparent.



Fig. 2.22 Penetration factor versus $-1/\sqrt{E}$ for the ¹²C + p reaction. At low energies compared to the Coulomb barrier height ($E \ll V_C$), straight lines are obtained for each value of ℓ .

Up to now we have not specified the reaction channel. Suppose that there are only two channels open for the resonance λ of interest, channel α and channel β . According to Eq. (2.163), the reaction cross sections near resonance in channels α and β are given by

$$\sigma_{\alpha,\mathrm{re},\ell} = (2\ell+1)\frac{\pi}{k_{\alpha}^{2}} \frac{\Gamma_{\lambda\alpha}\Gamma_{\lambda\mathrm{r}\alpha}}{(E_{\alpha}-E_{r\alpha})^{2}+(\Gamma_{\lambda\alpha}+\Gamma_{\lambda\mathrm{r}\alpha})^{2}/4} = \sigma_{(\alpha,\beta)}$$
(2.168)
$$\sigma_{\beta,\mathrm{re},\ell} = (2\ell+1)\frac{\pi}{k_{\beta}^{2}} \frac{\Gamma_{\lambda\beta}\Gamma_{\lambda\mathrm{r}\beta}}{[(E_{\alpha}+Q)-(E_{r\alpha}+Q)]^{2}+(\Gamma_{\lambda\beta}+\Gamma_{\lambda\mathrm{r}\beta})^{2}/4} = \sigma_{(\beta,\alpha)}$$
(2.169)

It follows directly from the reciprocity theorem, $k_{\alpha}^2 \sigma_{(\alpha,\beta)} = k_{\beta}^2 \sigma_{(\beta,\alpha)}$ (see Eq. (2.13)), that the reaction width of the (α,β) reaction is equal to the entrance channel width of the (β,α) reaction, and vice versa.

2.5.5 *R*-Matrix Theory

It is important to summarize the assumptions we made in the derivation of the reaction cross section formula (see Eq. (2.163)): (i) the spins of the interacting nuclei are zero, (ii) the nucleus has a sharp radius, and (iii) a specific resonance corresponds to a logarithmic derivative of zero at the nuclear boundary. In the formal theory of resonance reactions (*R*-matrix theory) all of these assumptions are relaxed. We will not derive the formalism in any detail (see, for example, Breit 1959, or Lane and Thomas 1958) but will instead present some of the main results. We are specifically interested in the application of the general theory to the case of a single and isolated resonance. As will be seen, the main physical ideas of the formalism developed so far will not change in the formal theory.

Consider again Eq. (2.149), but in order to describe the simplest possible case we will assume that elastic scattering is the only allowed process (q = 0). In that case

$$f_0 = (E - E_\lambda) \left(\frac{\partial f_0}{\partial E}\right)_{E_\lambda, q=0}$$
(2.170)

By using the definitions of the logarithmic derivative f_0 (see Eq. (2.130)) and of the reduced width $\gamma_{\lambda e}^2$ (see Eq. (2.159)) we find near a particular level energy E_{λ}

$$\frac{1}{f_0} = \frac{1}{R} \left(\frac{u_{\rm in}(r)}{du_{\rm in}(r)/dr} \right)_{r=R} = \frac{(\partial f_0/\partial E)_{E_\lambda,q=0}^{-1}}{E - E_\lambda} = \frac{\gamma_{\lambda e}^2}{E_\lambda - E} \equiv \Re$$
(2.171)

The quantity \Re is called *R*-function. When the energy *E* is not close to E_{λ} , the *R*-function is obtained by summing over all resonances λ . In general,

elastic scattering will not be the only possible process, but other channels are present as well. In order to take these into account, the *R*-function becomes the *R*-matrix,

$$\Re_{c'c} = \sum_{\lambda} \frac{\gamma_{\lambda c'} \gamma_{\lambda c}}{E_{\lambda} - E}$$
(2.172)

Physically, the *R*-matrix relates the value of the wave function in the internal region to its derivative at each channel entrance. The above equation gives the energy dependence of the *R*-matrix explicitly in terms of the energy-independent parameters $\gamma_{\lambda c}$ and E_{λ} . The poles of the *R*-matrix, that is, the energies E_{λ} , are real and hence each of the elements $\Re_{c'c}$ represents a real number. Furthermore, the energies E_{λ} are independent of the channels *c* and *c'*. In other words, the poles of every matrix element $\Re_{c'c}$ occur at the same energies E_{λ} .

We need to be more precise when defining a *reaction channel c*. The quantity *c* denotes a set of quantum numbers { $\alpha(I_1I_2)s\ell$, *JM*} with

$\alpha(I_1I_2)$	a specific pair of nuclei 1 and 2, with spins of I_1 and I_2 ,
	in a specific state of excitation (thus an excited state of
	1 or 2 would correspond to a different α)
$\vec{s} = \vec{I}_1 + \vec{I}_2$	channel spin, with $ I_1 - I_2 \le s \le I_1 + I_2$
ℓ	orbital angular momentum
<i>Ī,</i> М	total spin and its component, with $ec{J}=ec{s}+ec{\ell}$

For the entrance channel consisting of a projectile and a target nucleus, we set $\vec{I}_1 = \vec{j}_p$ and $\vec{I}_2 = \vec{j}_t$. Conservation of the total angular momentum restricts the possible *J* values of the resonance which can be populated in the reaction to (Appendix B)

$$\vec{J} = \vec{\ell} + \vec{j}_p + \vec{j}_t \tag{2.173}$$

Each of these spins has (2I + 1) orientations in space, which are determined by the magnetic quantum number $m_I = 0, 1, ..., \pm I$. Thus there are $(2\ell + 1)(2j_p + 1)(2j_t + 1)$ different sets of spin orientations, corresponding to different quantum states of the system. For an unpolarized beam and target, each such state has the same probability, that is $[(2\ell + 1)(2j_p + 1)(2j_t + 1)]^{-1}$. Therefore, the cross section has to be multiplied by the relative probability that the unpolarized projectiles and target nuclei will be found to have a total spin of *J*, which is given by

$$g(J) = \frac{2J+1}{(2j_p+1)(2j_t+1)(2\ell+1)}$$
(2.174)

From the *R*-matrix, the cross sections and phase shifts can be derived for any number of resonances and channels (Lane and Thomas 1958). In the following, we will only focus on a particularly simple but useful case, that is, the reaction cross section near an isolated resonance λ of spin *J*. The one-level, many channel approximation of *R*-matrix theory (or generalized one-level Breit–Wigner formula) for the cross section of a reaction (α , α'), involving charged or neutral particles with projectile and target spins of j_p and j_t , is given by

$$\sigma_{\rm re}(\alpha,\alpha') = \frac{\pi}{k^2} \frac{2J+1}{(2j_p+1)(2j_t+1)} \frac{\left(\sum_{\ell s} \Gamma_{\lambda c}\right) \left(\sum_{\ell' s'} \Gamma_{\lambda c'}\right)}{(E-E_{\lambda}-\Delta_{\lambda})^2 + \Gamma_{\lambda}^2/4}$$
(2.175)

with

$$\Gamma_{\lambda c}(E) = 2P_{c}(E)\gamma_{\lambda c}^{2} \qquad (particle width) \qquad (2.176)$$

$$\Gamma_{\lambda}(E) = \sum \Gamma_{\lambda c''}(E) \qquad (total width) \qquad (2.177)$$

$$\Delta_{\lambda}(E) = \sum_{c''}^{c''} \Delta_{\lambda c''}(E) \qquad (\text{total level shift}) \qquad (2.178)$$
$$\Delta_{\lambda c}(E) = -[S_c(E) - B_c]\gamma_{\lambda c}^2 \qquad (\text{partial level shift}) \qquad (2.179)$$
$$\beta(E) = \arctan \frac{\Gamma_{\lambda}(E)}{\Gamma_{\lambda}(E)} \qquad (\text{resonance elastic scattering})$$

$$p(E) = \arctan \frac{1}{2[E_{\lambda} - E + \Delta_{\lambda}(E)]}$$
 (resonance elastic scattering
phase shift) (2.180)

The parameter B_c will be described later. The penetration and shift factors refer to the nuclear radius. In principle, one can chose an arbitrary radius beyond the range of the nuclear force so that the external wave functions reflect the solutions of the wave equation containing only the Coulomb interaction. However, it is also desirable to chose *R* as small as possible so that the characteristic quantities of the resonance theory contain primarily information concerning the nuclear interaction. Commonly, the interaction radius *R* is the smallest separation distance of the nuclear pair at which the nuclear potential is negligible. This radius is customarily chosen in *R*-matrix theory as $R = r_0(A_t^{1/3} + A_p^{1/3})$, with a radius parameter in the range of $r_0 = 1.0-1.5$ fm.

The above expression for the reaction cross section (see Eq. (2.175)) contains certain complications with respect to practical applications. This comes about because, in general, the energy dependence of the penetration and shift factor has to be taken into account. The quantity P_c is strongly energy dependent, but the energy dependence of S_c is weak (Fig. 2.21). The usual approximation procedure, called the *Thomas approximation* (Thomas 1951), is to expand the level shift linearly with respect to energy. We call the energy at which the cross section $\sigma_{re}(\alpha, \alpha')$ has a maximum the *observed resonance energy* E_r . It is defined by the requirement

$$E_r - E_\lambda - \Delta_\lambda(E_r) = 0 \tag{2.181}$$

The boundary condition parameter B_c in Eq. (2.179), defined as the real and arbitrary value of the logarithmic derivative of the radial wave function in channel *c* at the radius *R*, determines the eigenvalues E_{λ} (in previous sections, we used implicitly the zero derivative condition, $B_c = 0$). It is customarily chosen as $B_c = S_c(E_r)$ so that the level shift Δ at the observed resonance energy E_r becomes zero,

$$\Delta_{\lambda c}(E_r) = -[S_c(E_r) - S_c(E_r)]\gamma_{\lambda c}^2 = 0 \quad \text{and} \quad E_r = E_\lambda$$
(2.182)

With the expansion

$$\Delta_{\lambda}(E) \approx \Delta_{\lambda}(E_r) + (E - E_r) \left(\frac{d\Delta_{\lambda}}{dE}\right)_{E_r}$$
(2.183)

we obtain by using Eq. (2.181)

$$E_{\lambda} + \Delta_{\lambda} - E \approx E_r - E + (E - E_r) \left(\frac{d\Delta_{\lambda}}{dE}\right)_{E_r} = (E_r - E) \left[1 - \left(\frac{d\Delta_{\lambda}}{dE}\right)_{E_r}\right]$$
(2.184)

Substitution into Eq. (2.175) yields

$$\sigma_{\rm re}(\alpha,\alpha') = \frac{\pi}{k^2} \frac{2J+1}{(2j_p+1)(2j_t+1)} \frac{\left(\sum_{\ell s} \Gamma_{\lambda c}\right) \left(\sum_{\ell' s'} \Gamma_{\lambda c'}\right)}{(E_r - E)^2 [1 - (d\Delta_{\lambda}/dE)_{E_r}]^2 + \Gamma_{\lambda}^2/4}$$

Dividing the numerator and denominator by $[1 - (d\Delta_{\lambda}/dE)_{E_r}]^2$ gives

$$\sigma_{\rm re}(\alpha,\alpha') = \frac{\pi}{k^2} \frac{2J+1}{(2j_p+1)(2j_t+1)} \frac{\left(\sum_{\ell_s} \Gamma^o_{\lambda c}\right) \left(\sum_{\ell' s'} \Gamma^o_{\lambda c'}\right)}{(E_r - E)^2 + (\Gamma^o_{\lambda})^2/4}$$
(2.185)

where the "observed" widths $\Gamma_{\lambda i}^{o}$ are given in terms of the previously defined "formal" widths $\Gamma_{\lambda i}$ (see Eq. (2.176)) by

$$\Gamma^{o}_{\lambda c} \equiv \frac{\Gamma_{\lambda c}}{1 - (d\Delta_{\lambda}/dE)_{E_{r}}} = \frac{\Gamma_{\lambda c}}{1 + \left(\sum_{c''} \gamma^{2}_{\lambda c''} \frac{dS_{c''}}{dE}\right)_{E_{r}}}$$
(2.186)

The main advantage of using Eq. (2.185) compared to Eq. (2.175) is that the complication of an energy-dependent shift factor in the denominator is absent.

Since the former expression has a simpler (Lorentzian) structure, it is used in the vast majority of applications. However, we had to introduce a new quantity. The reader must be careful when applying Eq. (2.185) in the analysis of experimental data. It has to be understood that the partial widths thus obtained represent "observed" widths. As can be seen from Eq. (2.186), the difference between "observed" and "formal" partial width may be substantial for levels with a large reduced width. We can also introduce an "observed" reduced width by writing

$$\Gamma^{o}_{\lambda c} = \frac{2P_{c}(E)\gamma^{2}_{\lambda c}}{1 + \left(\sum_{c''}\gamma^{2}_{\lambda c''}\frac{dS_{c''}}{dE}\right)_{E_{r}}} = 2P_{c}(E)(\gamma^{o}_{\lambda c})^{2}$$
(2.187)

As a general guide, partial widths have to be interpreted as "observed" quantities whenever a Lorentzian structure is assumed for the cross section (for example, in reaction rate calculations, mean lifetime measurements, or thick target yields).

Finally, we express the resonance phase shift and its energy derivative in terms of the "observed" total width. We obtain from Eqs. (2.180) and (2.184) immediately

$$\beta = \arctan \frac{\Gamma_{\lambda} / [1 - (d\Delta_{\lambda} / dE)_{E_r}]}{2(E_r - E)} = \arctan \frac{\Gamma_{\lambda}^o}{2(E_r - E)}$$
(2.188)

and, similar to Eq. (2.158),

$$\left(\frac{d\beta}{dE}\right)_{E_r} = \frac{2}{(\Gamma^o_\lambda)_{E_r}} \tag{2.189}$$

This expression is frequently used in calculations of "observed" particle partial widths (see Section 2.5.7).

2.5.6

Experimental Tests of the One-Level Breit–Wigner Formula

The total cross section for neutrons incident on a target consisting of a natural isotopic mixture of cadmium is shown in Fig. 2.23. The data are fitted by a one-level Breit–Wigner formula, superimposed on a 1/v background (Section 2.6). It is obvious that the agreement with the data is extremely accurate. The Breit–Wigner formula describes reliably the shape of resonances if their widths are small compared to their energy separation.

The resonance reaction theory developed so far does not only apply to unbound states, but to bound states as well. In the latter case, the Breit–Wigner formula allows for the calculation of the cross section wing of a *subthreshold resonance* (Example 2.1). The Breit–Wigner formula has important applications



Fig. 2.23 Total cross section for neutrons incident on a target consisting of a natural isotopic mixture of cadmium. The data are fitted by a one-level Breit–Wigner formula, superimposed on a 1/v background. The deduced resonance parameters are $E_{\lambda} = 0.176 \text{ eV}, \Gamma = 0.115 \text{ eV}, \text{ and } \sigma_{\text{max}} = 7.2 \times 10^{-21} \text{ cm}^2$. The Breit–Wigner formula

reproduces the shape of resonances accurately if their widths are small compared to their energy separation. Reprinted with permission from H. H. Goldsmith, H. W. Ibser and B. T. Feld, Rev. Mod. Phys., Vol. 19, p. 259 (1947). Copyright (1947) by the American Physical Society.

in nuclear astrophysics, especially in cases where the cross section of interest cannot be measured directly and has to be estimated theoretically. For example, consider the following situation which is frequently encountered in practice. Data have been obtained in some higher lying bombarding energy range. The energy range of interest for stellar fusion, however, is located outside the range for which data have been measured. By fitting the existing data to a Breit–Wigner formula, one obtains the resonance energy and widths as phenomenological parameters which are then used to extrapolate the cross section to the energy region of interest.

Frequently, the widths of astrophysically important resonances are rather small (less than a few eV) and it is experimentally no longer feasible to measure the cross section directly at specific energies near the resonance. What is directly measured in such cases is the integral under the resonance cross section curve. The Breit–Wigner formula provides an accurate equation for integrating the resonance cross section, resulting in convenient analytical expressions for narrow-resonance reaction rates (Section 3.2.4) and thick-target yields (Section 4.8.1).

The total cross section for several overlapping resonances of different spins and parities can be described by an incoherent sum of one-level Breit–Wigner formulas. If two resonances have the same J^{π} value, however, they may interfere and the resulting expressions become more complicated. Also, the differential cross sections of two broad resonances may interfere even if their J^{π} values are different.

It must be emphasized again that the resonance theory described here is not capable of predicting resonance energies and widths. These quantities are treated as phenomenological parameters. Absolute cross sections can only be obtained either by fitting resonance data, or if the resonance energies and partial (or reduced) widths are independently known from other sources (Section 2.5.7). In the following numerical example, the one-level Breit–Wigner formula will be applied to a subthreshold resonance.

Example 2.1

The *Q*-value of the ²⁰Ne(p,γ)²¹Na reaction amounts to Q = 2431.3 keV. The ²¹Na level at $E_x = 2425$ keV ($J^{\pi} = 1/2^+$) is located just below the proton threshold and corresponds to a subthreshold s-wave ($\ell = 0$) resonance at a center-of-mass energy of $E_r = -6.4$ keV (Fig. 2.24a). The (formal) reduced proton width for this level can be obtained from (d,n) proton transfer reaction measurements (Terakawa et al. 1993). The value is $\gamma_{p,\ell=0}^2 = 1.41 \times 10^6$ eV. The $E_x = 2425$ keV level decays to the ground state with a probability (branching ratio) of 1 (100%) via emission of M1/E2 radiation (Appendix B). The value of the (formal) γ -ray partial width at E_r , obtained from the measured mean lifetime of the state (Anttila, Keinonen and Bister 1977), amounts to $\Gamma_{\gamma}(E_r) = 0.30$ eV. Calculate the contribution of this level to the astrophysical *S*-factor (the *S*-factor is defined in Section 3.2.1) versus bombarding energy below 2 MeV.

In this case only two channels are open. The level can decay via emission of either a proton or a γ -ray. We may write the Breit–Wigner formula (see Eq. (2.175)) as

$$\sigma_{20\text{Ne}+p}(\mathbf{p},\gamma) = \frac{\pi}{k^2} \frac{2J+1}{(2j_p+1)(2j_t+1)} \frac{\Gamma_{\mathbf{p},\ell=0}\Gamma_{\gamma,\text{M1/E2}}}{(E-E_\lambda-\Delta_\lambda)^2 + (\Gamma_{\mathbf{p},\ell=0}+\Gamma_{\gamma,\text{M1/E2}})^2/4}$$

The cross section has a maximum at the observed resonance energy $E_r = E_{\lambda} + \Delta_{\lambda}(E_r) = E_{\lambda}$ (see Eqs. (2.181) and (2.182)) since we chose the boundary condition as $\Delta_{\lambda}(E_r) = 0$. Therefore, we set $E_{\lambda} = -6.4$ keV. We find the energy-dependent proton width from the expression $\Gamma_{p,\ell=0}(E) = 2P_{\ell=0}(E)\gamma_{p,\ell=0}^2$ (see Eq. (2.176)). The energy dependence of the γ -ray partial width is given by $\Gamma_{\gamma,L} \sim E_{\gamma}^{2L+1}$ (see Eq. (1.21)), with E_{γ} the γ -ray energy and L the γ -ray multipolarity. The M1/E2 multipolarity mixing ratio (see Eq. (1.31)) for this level is not known. It is sufficient to assume here that the transition to the ground state ($E_f = 0$) proceeds via pure M1 emission. Thus

$$\frac{\Gamma_{\gamma,\mathrm{M1}}(E)}{\Gamma_{\gamma,\mathrm{M1}}(E_r)} = \left[\frac{E_{\gamma}(E)}{E_{\gamma}(E_r)}\right]^{2L+1} = \left[\frac{E+Q-E_f}{E_r+Q-E_f}\right]^{2L+1} = \left[\frac{E+Q}{E_r+Q}\right]^3$$

The influence of the γ -ray channel on the level shift can be neglected. From Eqs. (2.178) and (2.179) one finds

$$\Delta_{\lambda}(E) \approx \Delta_{p,\ell=0}(E) = -[S_{\ell=0}(E) - S_{\ell=0}(E_r)]\gamma_{p,\ell=0}^2$$

We obtain from the definition of the astrophysical S-factor (see Eq. (3.70))

$$S_{20\text{Ne}+p}(\mathbf{p},\gamma) = E e^{2\pi\eta} \sigma_{20\text{Ne}+p}(p,\gamma)$$

=
$$\frac{Ee^{2\pi\eta} \frac{\pi}{k^2} \frac{2J+1}{(2j_p+1)(2j_t+1)} 2P_{\ell=0}(E)\gamma_{p,\ell=0}^2 \Gamma_{\gamma,\text{M1}}(E_r) \left(\frac{E+Q}{E_r+Q}\right)^3}{\left\{E - E_r + [S_{\ell=0}(E) - S_{\ell=0}(E_r)]\gamma_{p,\ell=0}^2\right\}^2 + [\Gamma(E)]^2/4}$$

with $\Gamma(E) = \Gamma_{p,\ell=0}(E) + \Gamma_{\gamma,M1}(E)$. Numerically we find (with *E* in MeV and M_i in u)

$$2\pi\eta = 0.989534 Z_p Z_t \sqrt{\frac{1}{E} \frac{M_t M_p}{M_t + M_p}}$$

$$E\frac{\pi}{k^2} = 6.56618216 \times 10^{-1} \frac{M_t + M_p}{M_t M_p} \qquad (\text{MeV b})$$

$$\frac{2J+1}{(2j_p+1)(2j_t+1)} = \frac{2 \cdot \frac{1}{2} + 1}{\left(2 \cdot \frac{1}{2} + 1\right)(2 \cdot 0 + 1)} = 1$$

$$S_{\ell=0}(E_r) = -1.537$$

The penetration and shift factors are directly computed from the Coulomb wave functions (see Eq. (2.162)). The resulting calculated *S*-factor for the ground-state transition in the 20 Ne(p, γ)²¹Na reaction is shown as a solid line in Fig. 2.24b. The data points display the experimental *S*-factor. These results represent one of the very few examples in nuclear physics where a tail of a subthreshold resonance is observed without interference from unbound states or direct radiative capture.

It must be emphasized that the solid line does not represent a fit to the data. It is calculated by using the Breit–Wigner formula with parameters (resonance energy, proton and γ -ray partial widths) that are obtained from independent experiments (that is, *not* from capture measurements). It should also be noted that the total width at the resonance energy amounts to $\Gamma(E_r) = \Gamma_{\gamma}(E_r) = 0.3 \text{ eV}$. In other words, the *S*-factor is extrapolated over 1500 keV/0.3 eV = 5 × 10⁶ resonance widths. The agreement between experiment and calculation is remarkable and provides strong support for the applicability of the Breit–Wigner formula to isolated resonances.



Fig. 2.24 (a) Level scheme of ²¹Na showing a subthreshold s-wave ($\ell = 0$) resonance in ²⁰Ne + p, corresponding to a level at $E_x =$ 2425 keV ($J^{\pi} = 1/2^+$) which is located just below the proton threshold. (b) Astrophysical *S*-factor of the ²⁰Ne(p, γ)²¹Na reaction versus center-of-mass proton energy for the γ -ray transition to the ground state of ²¹Na. The data points display the measured *S*-

factor (from Rolfs and Rodney 1975), while the solid line shows the result of the calculation explained in the text. Note that the solid line is not a fit to the data. The agreement between data and calculation is remarkable since the Breit–Wigner formula had to be extrapolated over more than 10⁶ resonance widths.

2.5.7

Partial and Reduced Widths

We have seen how the resonance cross section can be expressed in terms of resonance energies and reduced widths. For some reactions, however, no cross section data are available. In such cases it becomes important to estimate the cross section theoretically. The Breit–Wigner formula can only be used for this purpose if the resonance energies and reduced widths are known from independent sources (see Example 2.1). Resonances that are generated by simple explicit potentials are discussed in Section 2.4. Such *single-particle resonances* are generally broad at higher bombarding energies and their energy separation is large. In contrast to these, many measured resonances are very narrow and their spacing is small (Fig. 2.19). These resonances could not be explained by single-particle potentials and it was therefore necessary to develop a theory of resonances without reference to a specific nuclear potential (Section 2.5.1). The reduced widths depend on, as yet unknown, properties of the nuclear interior and are treated as phenomenological parameters.

According to Bohr (1936), the observed resonances correspond to virtual states in the nucleus. These virtual states are not single-particle levels, but are the result of the interactions of many nucleons. This many-nucleon picture is also referred to as *compound nucleus description*. The close spacing of the observed resonances is then explained by the fact that there are many different ways by which a large number of nucleons can be excited. The observed resonances are then caused by the rapid variation of the total nuclear wave function of the target-plus-projectile system with energy. In the following we will develop this picture quantitatively. Our goal will be to relate the reduced widths to nuclear properties which can be estimated by using models of nuclear structure.

Consider the total wave function of the target-plus-projectile system, Ψ , with $H\Psi = E\Psi$. The total Hamiltonian *H*, although unknown, may be written as

$$H = H_{\xi}^{t} + E_{K}^{p}(r) + \sum_{i=1}^{A} V_{i}(\xi_{i}, x)$$

= $\left[H_{\xi}^{t} + E_{K}^{p}(r) + \overline{V}(r)\right] + \left[-\overline{V}(r) + \sum_{i=1}^{A} V_{i}(\xi_{i}, x)\right] = H_{0} + H'$ (2.190)

with H_{ξ}^{t} the Hamiltonian of the target nucleus consisting of A nucleons, $E_{K}^{p}(r)$ the kinetic energy of the projectile, $V_{i}(\xi_{i}, x)$ the interactions between each target nucleon with the projectile, and $\overline{V}(r)$ an average potential of the projectile in the field of the target nucleus. The quantity H_{0} is the single-particle Hamiltonian, and H' describes the *residual interaction* (that is, the deviation from an average potential). Without the residual interaction, the potential $\overline{V}(r)$ would give rise to single-particle resonances, corresponding to single-particle levels in the target-plus-projectile system. However, the quantity H' causes the single-particle levels to split into a large number of distinct levels. Each of these states corresponds to a complicated mixture of configurations and is described by a complicated sum of wave functions. Consequently, the logarithmic derivative of the radial wave function at the nuclear boundary, that is, the reduced width γ_{Ar}^{2} will in general be different for each virtual state.





Fig. 2.25 Level scheme and cross section versus energy. A singleparticle Hamiltonian H_0 generates the single-particle levels p_1 , p_2 , p_3 , p_4 . The actual Hamiltonian H gives rise to a splitting of each singleparticle level into many actual states. The latter show up as a fine structure in the cross section versus energy curve. See the text.

Such levels which are described by the configurations of many nucleons are referred to as *compound-nucleus levels*.

The situation is shown schematically in Fig. 2.25. The single-particle Hamiltonian H_0 gives rise to the single-particle levels p_1 , p_2 , p_3 , p_4 . The levels p_1 and p_2 are bound states. The actual Hamiltonian H causes a splitting of each single-particle level into many actual states. These can be observed as resonances in the cross section (thin solid line). However, the single-particle character does not get entirely lost. If the measured cross section curve is averaged over the observed fine structure, then the single-particle resonances are approximately recovered (thick solid line). We may also say that, in this picture, each reduced width $\gamma_{\lambda c}^2$ of an actual level belongs to a definite single-particle level p_i . The entire set of reduced widths can then be split up into groups, each group corresponding to a definite value of p_i .

In the following, we will consider the simple case of only one open nucleon channel (elastic only) for compound nucleus formation or decay. It is useful at this point to express the particle width in a different way. We have seen that Γ_{λ} corresponds to the total width of a resonance λ , or the total width of a virtual level in the compound nucleus. A finite level width, in turn, implies a finite mean lifetime τ of the level, since $\Gamma_{\lambda}\tau_{\lambda} \approx \hbar$. Therefore, we can identify $1/\tau_{\lambda} \approx \Gamma_{\lambda}/\hbar$ with the decay (or formation) probability of the level per unit time. A partial width $\Gamma_{\lambda c}$ corresponds then to the decay (or formation) probability of


Fig. 2.26 Level scheme of compound nucleus *C* showing a single unbound (or virtual) state λ which may decay by emission of particles (a) or photons (γ). The full width at half maximum (FWHM) of the resonance in the cross section versus energy curve corresponds to the total width Γ_{λ} which is equal to the sum of all partial widths. The latter quantities are a measure for the decay (or formation) probability of level λ through a particular channel *c*.

level λ through a particular channel *c*. The decay of a compound nucleus state into two channels is shown schematically in Fig. 2.26.

The partial width $\Gamma_{\lambda c}$ will now be determined from the flux of the particles through the only open channel *c*. The probability per unit time, $\Gamma_{\lambda c}/\hbar$, for the emission of a particle is given by the number of particles per second leaving the channel. This number can be calculated by integrating the current (see Eq. (2.45)) through a sphere of radius *R* over the full solid angle,

$$\frac{\Gamma_{\lambda c}}{\hbar} = \int_{d\Omega} R^2 j \, d\Omega = \int_{d\Omega} R^2 \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial r} - \frac{\partial \psi^*}{\partial r}^* \psi \right)_{r=R} \, d\Omega \tag{2.191}$$

With $\psi = Y(\theta, \phi)R_c(r) = Y(\theta, \phi)u_c(r)/r$ (Appendix A) we find

$$\frac{\Gamma_{\lambda c}}{\hbar} = \frac{\hbar}{2mi} \int_{d\Omega} R^2 \left[\frac{u_c^*}{r} \frac{d}{dr} \left(\frac{u_c}{r} \right) - \frac{u_c}{r} \frac{d}{dr} \left(\frac{u_c^*}{r} \right) \right]_{r=R} |Y|^2 d\Omega$$
$$= \frac{\hbar}{2mi} \int_{d\Omega} R^2 \frac{1}{R^2} \left(u_c^* \frac{du_c}{dr} - u_c \frac{du_c^*}{dr} \right)_{r=R} |Y|^2 d\Omega$$
(2.192)

The radial wave function u_c of the compound state can be expanded in terms of single-particle radial eigenfunctions u_{pc} , which form a complete set of orthonormal functions. The eigenfunctions u_{pc} describe a single nucleon moving in a single-particle potential. We may write

$$u_c(R) = \sum_p A_{\lambda pc} u_{pc}(R)$$
(2.193)

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The above discussion of compound levels implies that, at a given energy, one particular single-particle state p contributes mainly to the width of level λ . This means that for a given level λ one of the terms in the sum of Eq. (2.193) is much larger than the others. Hence

$$u_c(R) \approx A_{\lambda pc} u_{pc}(R) \tag{2.194}$$

By using the normalization of the spherical harmonic Y and the definition of the logarithmic derivative, $f_{pc}(E) = R(u_{pc}^{-1} du_{pc}/dr)_{r=R}$ (see Eq. (2.130)), we obtain from Eqs. (2.192) and (2.194)

$$\Gamma_{\lambda c} = \frac{\hbar^2}{2mi} A_{\lambda pc}^2 \left(u_{pc}^* \frac{du_{pc}}{dr} - u_{pc} \frac{du_{pc}^*}{dr} \right)_{r=R}$$

= $\frac{\hbar^2}{2miR} A_{\lambda pc}^2 \left(u_{pc}^* u_{pc} f_{pc} - u_{pc} u_{pc}^* f_{pc}^* \right)_{r=R} = \frac{\hbar^2 |u_{pc}(R)|^2}{2miR} A_{\lambda pc}^2 (f_{pc} - f_{pc}^*)$
(2.195)

Since we describe a decaying compound state, the radial wave function for r > R is given by $u_{pc}(r) = Au_{pc}^+(r)$, that is, we have B = 0 for a purely outgoing wave (see Eq. (2.160)). This condition is equivalent to $f_{pc}(E) = S_c + iP_c$ (see Eq. (2.161)). It follows that

$$\Gamma_{\lambda c} = \frac{\hbar^2 |u_{pc}(R)|^2}{2miR} A_{\lambda pc}^2 [(S_c + iP_c) - (S_c - iP_c)] = 2\frac{\hbar^2}{mR^2} P_c A_{\lambda pc}^2 \frac{R}{2} |u_{pc}(R)|^2$$
(2.196)

This can be expressed as

$$\Gamma_{\lambda c} = 2 \frac{\hbar^2}{mR^2} P_c C^2 S \theta_{pc}^2$$
(2.197)

with

$$C^2 S = A_{\lambda pc}^2$$
 (spectroscopic factor) (2.198)

$$\theta_{pc}^2 = \frac{\kappa}{2} |u_{pc}(R)|^2$$
 (dimensionless single-particle reduced width) (2.199)

Comparison to Eq. (2.176) shows that the reduced width $\gamma_{\lambda c}^2$ has been reformulated in terms of a constant (\hbar^2/mR^2) and the quantities θ_{pc}^2 and C^2S .

Strictly speaking, the quantities S and C^2 denote a spectroscopic factor (Section 1.6.2) and the square of an isospin Clebsch–Gordan coefficient, respectively. The former quantity is frequently calculated using the nuclear shell model (Section 1.6), while the latter depends on the nuclear reaction (see, for example, Brussaard and Glaudemans 1977). In the present context of partial widths, only the product C^2S is of interest. The spectroscopic factor depends

on the many-nucleon structure of level λ and is a measure for the relative probability that an actual compound state λ can be described by the singleparticle state p. The structure of Eq. (2.197) emphasizes that the partial width for nucleon emission from a compound level can be thought of as a product of three factors: (i) the probability that the nucleons will arrange themselves in a configuration corresponding to the final state, C^2S , (ii) the probability that the single nucleon will appear at the boundary, $|u_{pc}(R)|^2$, and (iii) the probability that the single nucleon will penetrate the Coulomb and angular momentum barriers, P_c . By introducing a single-particle partial width

$$\Gamma_{\lambda pc} = 2 \frac{\hbar^2}{mR^2} P_c \theta_{pc}^2 \tag{2.200}$$

we may also express Eq. (2.197) as

$$\Gamma_{\lambda c} = C^2 S \, \Gamma_{\lambda pc} \tag{2.201}$$

In other words, the spectroscopic factor can be written as the ratio of the two quantities $\Gamma_{\lambda c}$ and $\Gamma_{\lambda pc}$. Since both of these partial widths are strongly energy dependent through the penetration factor P_c , they have to be calculated at the *same* incident energy *E*.

It is apparent that there are two different methods of estimating partial widths for nucleon channels once the spectroscopic factor C^2S has been obtained by independent means. If Eq. (2.197) is used, then the penetration factor P_c and the dimensionless single-particle reduced width θ_{pc}^2 must be computed. On the other hand, if Eq. (2.201) is employed, then the single-particle partial width $\Gamma_{\lambda pc}$ has to be calculated. This can be achieved, for example, by solving the Schrödinger equation numerically for the elastic scattering of nucleons by an appropriate single-particle potential (Schiffer 1963, Iliadis 1997). The single-particle partial width is then directly obtained from the slope of the resonance phase shift at the resonance energy (see Eq. (2.189)). The former method is computationally more convenient if values of θ_{pc}^2 are already available.

Numerical values of the dimensionless single-particle reduced width θ_{pc}^2 for protons are reported in Iliadis (1997) (Fig. 2.27) and Barker (1998). The results were obtained by calculating u_{pc} for a Woods–Saxon single-particle potential (Section 1.6.1). The value of θ_{pc}^2 depends on the interaction radius R, the orbital angular momentum ℓ , and the number of nodes of the radial wave function in the nuclear interior. The numerical values shown in Fig. 2.27 have been obtained with $R = 1.25(A_p^{1/3} + A_t^{1/3})$ fm. Obviously, for estimates of $\Gamma_{\lambda c}$ the quantities θ_{pc}^2 and P_c have to be computed at the same radius R. The θ_{pc}^2 values from Iliadis (1997) represent "observed" quantities, while the results from Barker (1998) represent "formal" quantities. Unfortunately, the dimensionless single-particle reduced width θ_{pc}^2 is frequently set equal to unity in the litera-

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Fig. 2.27 "Observed" dimensionless single-particle reduced width θ_{pc}^2 for ¹²C + p, ²²Na + p, ³¹P + p, and ⁴⁰Ca + p versus center-of-mass proton energy. In each panel, the curves correspond to different orbital angular momenta ($\ell = 0, 1, 2, \text{ and } 3$) and have been computed for a radius of $R = 1.25(A_p^{1/3} + A_t^{1/3})$ fm. Reprinted from C. Iliadis, Nucl. Phys. A, Vol. 618, p. 166 (1997). Copyright (1997), with permission from Elsevier.

ture. In this case, a significant error is introduced in the estimation of partial widths.

Frequently, a resonance cannot be observed directly. This happens, for example, if the cross section is too small or if the target is radioactive. In such cases, the formalism discussed above can be used to estimate the absolute reaction cross section. Once the spectroscopic factor is either calculated using the nuclear shell model (Section 1.6), or measured using transfer reactions, the particle partial width can be estimated in a straightforward way. The reaction cross section is then obtained by applying the Breit–Wigner formula.

It is interesting to investigate the reliability of Eq. (2.197) for the calculation of proton partial widths. Figure 2.28 shows a comparison of measured and estimated proton partial widths for compound levels in ²⁵Al, ²⁷Al and ³¹P. The ratio of partial widths, $\Gamma_{exp}/\Gamma_{\lambda c}$, is shown in part (a) versus the value of C^2S , and in part (b) versus the observed resonance energy E_r . The values of $\Gamma_{\lambda c}$ are estimated from Eq. (2.197) by using proton spectroscopic factors measured in (³He,d) transfer reactions and by computing θ_{pc}^2 and P_c numerically. The experimental proton widths Γ_{exp} were directly measured in resonance elastic



Fig. 2.28 Ratio of measured and estimated proton partial widths, $\Gamma_{exp}/\Gamma_{\lambda c}$, for levels in ²⁵Al, ²⁷Al, and ³¹P (a) versus the value of C^2S , and (b) versus the observed resonance energy E_r . The experimental proton widths Γ_{exp} were directly measured in resonance elastic proton scattering or proton capture reactions. The calculated values $\Gamma_{\lambda c}$ are obtained from Eq. (2.197) by using spectroscopic factors measured in (³He,d) transfer studies.

proton scattering or proton capture reactions. The error bars of the displayed ratios consider only the uncertainties of the experimental proton widths. It can be seen that experimental and estimated proton partial widths agree on average within \approx 50%. In fact, we expect that the parametrizations of $\Gamma_{\lambda c}$ (see Eqs. (2.197) and (2.201)) are more accurate than this since we have entirely neglected the errors in the measured transfer spectroscopic factors. Further systematic studies are needed.

Example 2.2

An important resonance in the ¹⁷F(p, γ)¹⁸Ne reaction occurs at a center-ofmass energy of $E_r = 600$ keV ($J^{\pi} = 3^+$). The spectroscopic factors for this resonance are known from independent measurements (that is, neutronstripping on the mirror target nucleus ¹⁷O). Their values are $(C^2S)_{\ell=0} = 1.01$ and $(C^2S)_{\ell=2} \approx 0$. Estimate the "observed" proton partial width for this resonance. 138 2 Nuclear Reactions

We write with Eq. (2.197)

$$\begin{split} \Gamma^{o}_{\mathbf{p},\ell=0} &= 2 \frac{\hbar^2}{mR^2} \, P_{\ell=0}(C^2 S)_{\ell=0}(\theta^{o}_{\mathbf{p},\ell=0})^2 \\ &= 2(2.22 \times 10^6 \, \text{eV})(8.10 \times 10^{-3})(1.01)(0.45) = 16.3 \, \text{keV} \\ \Gamma^{o}_{\mathbf{p},\ell=2} &= 2 \frac{\hbar^2}{mR^2} \, P_{\ell=2}(C^2 S)_{\ell=2}(\theta^{o}_{\mathbf{p},\ell=2})^2 \\ &= 2(2.22 \times 10^6 \, \text{eV})(7.90 \times 10^{-5})(\approx 0)(0.45) \approx 0 \end{split}$$

The values of $(\theta^o_{p,\ell=0})^2$ and $(\theta^o_{p,\ell=2})^2$ are obtained by interpolating the results for ${}^{12}C + p$ and ${}^{22}Na + p$ shown in Fig. 2.27. The estimated "observed" proton partial width is

$$\Gamma_p^o = \Gamma_{p,\ell=0}^o + \Gamma_{p,\ell=2}^o = 16.3 \, \text{keV}$$

The calculated result is in excellent agreement with the experimental value of $\Gamma_p = (18 \pm 2)$ keV that was directly measured in ${}^{17}F(p,p){}^{17}F$ elastic scattering studies (Bardayan et al. 2000).

Example 2.3

The "observed" proton partial width for the s-wave ($\ell = 0$) resonance at a center-of-mass energy of $E_r = 214 \text{ keV} (J^{\pi} = 1/2^+) \text{ in } {}^{24}\text{Mg}(p,\gamma)^{25}\text{Al}$ was directly measured. The result is $\Gamma^o_{p,\ell=0} = (1.40 \pm 0.12) \times 10^{-2} \text{ eV}$ (Powell et al. 1999). Estimate the proton spectroscopic factor for the corresponding compound state.

First, we calculate the "observed" single-particle proton width from Eq. (2.200),

$$\Gamma^{o}_{\lambda pc} = 2 \frac{\hbar^2}{mR^2} P_c(\theta^o_{pc})^2 = 2(1.84 \times 10^6 \,\mathrm{eV})(4.56 \times 10^{-8})(0.59)$$
$$= 9.90 \times 10^{-2} \,\mathrm{eV}$$

We obtain from Eq. (2.201)

$$(C^2 S)_{\ell=0} = \frac{\Gamma^o_{p,\ell=0}}{\Gamma^o_{\lambda\nu c}} = \frac{(1.40 \pm 0.12) \times 10^{-2} \,\mathrm{eV}}{9.90 \times 10^{-2} \,\mathrm{eV}} = 0.14 \pm 0.01$$

The result is in excellent agreement with the value of $(C^2S)_{\ell=0} = 0.14$ measured independently in the proton transfer reaction ${}^{24}Mg({}^{3}He,d){}^{25}Al$ (Peterson and Ristinen 1975).

2.6 Continuum Theory

It is interesting to discuss the extreme case where a projectile approaching the target in a particular channel α is very unlikely to reappear in the entrance channel once it has penetrated into the nuclear interior. The condition is fulfilled, for example, if the number of open channels is very large. This is typically the case when the energy of the incident particle is much higher than the first few excitation energies of the target nucleus (say, E > 3 MeV for target masses of A > 50). The condition may also be fulfilled at low energies if the incident particle initiates a reaction with a large positive *Q*-value (say, Q > 2 MeV). In these cases we expect that, once the incident particle is inside the nucleus, it exchanges its energy rapidly with the other nucleons and the probability that it leaves by the same channel α is very small.

For simplicity, s-wave neutrons are considered again as incident particles. For the interior wave function we find from Eq. (2.142)

$$u_{\rm in} \sim e^{-iKr} \tag{2.202}$$

It has the form of an ingoing wave only since it does not return. This is only a rough approximation since it is impossible to represent the motion of the incident particle inside the nucleus as a function of *r* only. However, it represents the main features of the dependence of the wave function on *r*. The logarithmic derivative of the radial wave function must be continuous at r = R. Hence (see Eq. (2.130))

$$f_0 = R \left(\frac{1}{u_{\rm in}(r)} \frac{du_{\rm in}(r)}{dr}\right)_{r=R} = R \frac{\left[\frac{d}{dr}(Be^{-iKr})\right]_{r=R}}{Be^{-iKr}} = -iKR$$
(2.203)

Substitution into Eq. (2.138) yields immediately for the reaction cross section (since Re $f_0 = 0$ and Im $f_0 = -KR$)

$$\sigma_{\rm re,0} = \frac{\pi}{k^2} \left(1 - \left| e^{2i\delta_0} \right|^2 \right) = \frac{\pi}{k^2} \frac{4kK}{(K+k)^2}$$
(2.204)

The wave number inside the nucleus, *K*, is the only information regarding the interior which enters into this expression. Comparison to Eqs. (2.51) and (2.68) shows that the reaction cross section for s-wave neutrons can be interpreted as the product of the maximum cross section, π/k^2 , and the s-wave transmission coefficient, \hat{T}_0 ,

$$\sigma_{\mathrm{re},0} = \sigma_{\mathrm{re},0}^{\mathrm{max}} \hat{T}_0 \tag{2.205}$$

where

$$\hat{T}_0 = 1 - \left| e^{2i\delta_0} \right|^2 \tag{2.206}$$

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Since we assumed that the projectile is not re-emitted by the compound nucleus into the entrance channel α , the reaction cross section σ_{re} here is identical with the cross section $\sigma_{\alpha C}$ for the formation of the compound nucleus through channel α . Also, disregarding the possibility that the incident particle can return via the entrance channel means that Eq. (2.204) cannot give rise to resonances. For this reason, the above method for determining the cross section is referred to as *continuum theory*.

From Eq. (2.204) we can also estimate the s-wave reaction cross section for neutrons at low incident energies *E*. For $k \ll K$, the wave number *K* in the interior does not change much with variations in *k* (see Fig. 2.7) and one finds

$$\sigma_{\rm re,0} = \frac{\pi}{k^2} \frac{4kK}{(K+k)^2} \approx \frac{4\pi}{Kk} \sim \frac{1}{k} \sim \frac{1}{v} \sim \frac{1}{\sqrt{E}}$$
(2.207)

where $p = \hbar k$ and v is the velocity of the incident neutron. The result is independent of the reaction mechanism and is referred to as $1/v \, law$ for reactions induced by s-wave neutrons. Reaction cross sections for ${}^{3}\text{He}(n,p){}^{3}\text{H}$, ${}^{6}\text{Li}(n,\alpha){}^{3}\text{H}$ and ${}^{10}\text{B}(n,\alpha){}^{7}\text{Li}$ are displayed in Fig. 4.15a. Below a neutron energy of $\approx 1 \text{ keV}$, the cross sections follow the $1/v \, \text{law}$.

Equations (2.204)–(2.206) are obtained under the assumption of s-wave neutrons as incident particles. They can be easily generalized for any projectile and orbital angular momentum (Blatt and Weisskopf 1952). The cross section for the formation of the compound nucleus through channel α is then given by

$$\sigma_{\alpha C} = \frac{\pi}{k^2} \sum_{\ell} (2\ell+1) \hat{T}_{\ell}(\alpha) \tag{2.208}$$

where

$$\hat{T}_{\ell}(\alpha) = 1 - \left| e^{2i\delta_{\alpha\ell}} \right|^2 \tag{2.209}$$

is the transmission coefficient of channel α for orbital angular momentum ℓ and $\delta_{\alpha\ell}$ is the corresponding phase shift in channel α for elastic scattering by an appropriate potential. The potential must be complex for reactions to occur; otherwise the phase shift will be real and the transmission coefficient vanishes. This is consistent with our earlier discussion in Section 2.3.6. Transmission coefficients are usually calculated numerically from so-called *optical model potentials* which represent the average nuclear potential. For more information on optical model potentials, see Satchler (1990).

2.7 Hauser–Feshbach Theory

In Section 2.5 we considered the case where a reaction proceeds through an isolated narrow resonance. We will now discuss the other extreme situation. With increasing excitation energy in the compound nucleus, the resonances become broader and are located closer together. There is a continuous transition from sharp, isolated levels to the so-called continuum where levels overlap so much that little structure remains in the cross section. In other words, the cross section varies smoothly with energy. The reaction cross section, averaged over any resonance structure, is derived in the following.

The total angular momentum *J* and parity π of the compound nucleus will be conserved in a reaction (α , α'). The average cross section is then given by a sum of contributions from separate *J* and π ,

$$\langle \sigma_{\rm re}(\alpha, \alpha') \rangle = \sum_{J\pi} \langle \sigma_{\rm re}(\alpha, \alpha') \rangle^{J\pi}$$
 (2.210)

Recall that α denotes a pair of particles (including their state of excitation) in a particular channel (Section 2.5.5). Unprimed and primed quantities refer to the incoming and outgoing channel of the reaction, respectively. Next, we factor each term $\langle \sigma_{\rm re}(\alpha, \alpha') \rangle^{J\pi}$ into a cross section for compound nucleus formation through channel α and a branching ratio for decay into channel α' ,

$$\langle \sigma_{\rm re}(\alpha,\alpha') \rangle^{J\pi} = \sigma_{\alpha C}^{J\pi} \frac{G_{\alpha'}^{J\pi}}{\sum\limits_{\alpha''} G_{\alpha''}^{J\pi}}$$
(2.211)

The quantities $G_{\alpha}^{J\pi}$ represent probabilities for the decay into a specific outgoing channel, where the sum over α'' in the denominator is over all channels to which the compound nucleus can decay ($\sum_{\alpha''} G_{\alpha''}^{J\pi} = 1$). The factorization of the cross section in Eq. (2.211) reflects the independence of formation and decay of the compound nucleus while still fulfilling the requirement of total angular momentum and parity conservation. Substitution of the reciprocity theorem (see Eq. (2.14))

$$(2I_1+1)(2I_2+1)k_{\alpha}^2 \langle \sigma_{\rm re}(\alpha,\alpha') \rangle^{J\pi} = (2I_1'+1)(2I_2'+1)k_{\alpha'}^2 \langle \sigma_{\rm re}(\alpha',\alpha) \rangle^{J\pi}$$
(2.212)

into Eq. (2.211) gives

$$\frac{G_{\alpha'}^{J\pi}}{G_{\alpha}^{J\pi}} = \frac{(2I_1'+1)(2I_2'+1)k_{\alpha'}^2\sigma_{\alpha'C}^{J\pi}}{(2I_1+1)(2I_2+1)k_{\alpha}^2\sigma_{\alpha'C}^{J\pi}}$$
(2.213)

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where I_1 and I_2 are the spins of the particles in channel α . Summation over all channels α'' yields (since $\sum_{\alpha''} G_{\alpha''}^{J\pi} = 1$)

$$G_{\alpha'}^{J\pi} = \frac{(2I_1'+1)(2I_2'+1)k_{\alpha'}^2\sigma_{\alpha'C}^{J\pi}}{\sum\limits_{\alpha''}(2I_1''+1)(2I_2''+1)k_{\alpha''}^2\sigma_{\alpha''C}^{J\pi}}$$
(2.214)

For the formation of the compound nucleus one can use Eq. (2.208),

$$\sigma_{\alpha C} = \sum_{J\pi} \sigma_{\alpha C}^{J\pi} = \frac{\pi}{k_{\alpha}^2} \sum_{\ell} (2\ell + 1) \hat{T}_{\ell}(\alpha)$$
(2.215)

Since the cross section is averaged over many overlapping resonances, we expect that the transmission coefficient does not depend on *J*. Therefore

$$\sigma_{\alpha C} = \frac{\pi}{k_{\alpha}^2} \sum_{\ell} (2\ell+1) \sum_{J=|\ell-s|}^{\ell+s} \sum_{s=|I_1-I_2|}^{I_1+I_2} \frac{2J+1}{(2I_1+1)(2I_2+1)(2\ell+1)} \hat{T}_{\ell}(\alpha) \quad (2.216)$$

The quantities *I*, *s*, and ℓ have the same meanings as in Section 2.5.5 and refer to a specific channel α . The factor in front of the transmission coefficient takes the number of different spin orientations into account (Section 2.5.5). Rearranging the order of summation yields

$$\sigma_{\alpha C} = \frac{\pi}{k_{\alpha}^2} \sum_{J\pi} \frac{2J+1}{(2I_1+1)(2I_2+1)} \sum_{s=|I_1-I_2|}^{I_1+I_2} \sum_{\ell=|J-s|}^{J+s} \hat{T}_{\ell}(\alpha)$$
(2.217)

Comparison of Eqs. (2.215) and (2.217) then gives

$$\sigma_{\alpha C}^{J\pi} = \frac{\pi}{k_{\alpha}^2} \frac{2J+1}{(2I_1+1)(2I_2+1)} \sum_{s\ell} \hat{T}_{\ell}(\alpha)$$
(2.218)

Combining Eqs. (2.210), (2.211), (2.214), and (2.218) results in

$$\langle \sigma_{\rm re}(\alpha, \alpha') \rangle = \sum_{J\pi} (2I_1' + 1)(2I_2' + 1)k_{\alpha'}^2 \frac{\sigma_{\alpha C}^{J\pi} \sigma_{\alpha' C}^{J\pi}}{\sum_{\alpha''} (2I_1'' + 1)(2I_2'' + 1)k_{\alpha''}^2 \sigma_{\alpha'' C}^{J\pi}}$$

$$= \frac{\pi}{k_{\alpha}^2} \sum_{J\pi} \frac{2J + 1}{(2I_1 + 1)(2I_2 + 1)} \frac{\left[\sum_{s\ell} \hat{T}_{\ell}(\alpha)\right] \left[\sum_{s'\ell''} \hat{T}_{\ell''}(\alpha')\right]}{\sum_{\alpha''} \sum_{s''\ell''} \hat{T}_{\ell''}(\alpha'')}$$

$$(2.219)$$

This is the *Hauser–Feshbach formula* for energy-averaged cross sections (Hauser and Feshbach 1952, Vogt 1968). The quantity α refers to the incoming channel of the reaction and thus I_1 and I_2 are the spins of the target and projectile, respectively. The sum over α'' is again over all channels that are energetically

accessible for the decay of the compound nucleus at the total energy in the entrance channel. The sums over J^{π} , ℓ , and *s* run over all values allowed by the selection rules for angular momentum coupling (Appendix B): π is positive or negative; J = 0, 1, 2, ... for *A* even, or $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$ for *A* odd; *s* takes on all integer values between $|I_1 - I_2|$ and $I_1 + I_2$; ℓ takes on all even values between |J - s| and J + s if the pair α has the same parity as π and all odd values otherwise. We assume here that the transmission coefficients are independent of the channel spin (that is, the potential has no spin–orbit term) and, therefore, the sum over *s* in Eq. (2.219) becomes a simple multiplicative factor. See Problem 2.6.

The transmission coefficients $\hat{T}_{\ell}(\alpha)$ are determined by complex phase shifts $\delta_{\alpha\ell}$ (see Eq. (2.209)) that are usually calculated numerically from optical model potentials (Section 2.6). Recall that these represent the average nuclear potential only. Consequently, the transmission coefficients describe the formation probability of single-particle levels. In other words, the reaction cross section calculated from Eq. (2.219) cannot account for the fine structure shown in Fig. 2.25, but corresponds to the average cross section shown as the thick solid line.

The Hauser–Feshbach theory is also applicable if a channel involves the emission or absorption of γ -rays (Cowan, Thielemann and Truran 1991). A correction must be applied to Eq. (2.219) because the processes of compound nucleus formation and decay are not completely independent of each other as can be shown by a more involved derivation of the Hauser–Feshbach formula using the resonance theory (Vogt 1968). This *width-fluctuation correction* enhances the cross section for weak reaction channels at the cost of stronger ones and is most important near thresholds, where additional channels become energetically accessible, and for reactions with few open channels.

Recall that α also specifies the state of excitation of a pair of particles in a particular channel. In practical applications one is mostly interested in cross sections obtained by summing or averaging over specific sets of excited states. For example, what is usually measured in the laboratory is the quantity $\langle \sigma_{re}(\alpha, \alpha') \rangle$, with α representing the ground states of target and projectile, summed over excited states in the outgoing channel α' . Or, if the reaction takes place in a hot stellar plasma, $\langle \sigma_{re}(\alpha, \alpha') \rangle$ must be averaged over excited states in the entrance channel α (Section 3.1.5). In such cases, Eq. (2.219) is still valid if each of the transmission coefficients in the numerator is replaced by sums of transmission coefficients over the excited states in question. In exceptional cases, all the final states for compound nucleus decay and their quantum numbers are experimentally known. The Hauser–Feshbach formula can then be applied with essentially no adjustable parameters. In most cases of practical interest, however, the compound nucleus may decay to levels beyond the highest excited state for which energy, spin, and parity are explicitly

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known. The transmission coefficients in the numerator and denominator of Eq. (2.219) must then be modified to include terms that integrate a nuclear level density over the energy region beyond the known levels. This requires the development of expressions for the density of states as a function of excitation energy, spin, and parity. The evaluation of the overall cross section is then reduced to the problem of determining the required transmission coefficients and nuclear level densities. For a detailed discussion of these quantities in connection with the Hauser–Feshbach model see, for example, Rauscher et al. (1997) or Arnould and Goriely (2003).

Figure 2.29 shows as an example the cross section for the ⁶⁴Ni(p, γ)⁶⁵Cu reaction at bombarding energies between 1 and 4 MeV. The ⁶⁴Ni(p,n)⁶⁴Cu reaction has a *Q*-value of ≈ -2.5 MeV, which means that at a bombarding energy close to 2.5 MeV the neutron channel opens and the (p,n) reaction will start to compete with the (p,γ) reaction. Since the total incoming flux must be constant, the opening of a new reaction channel corresponds to a reduction of flux into all other reaction channels. As a result, the cross section of the (p,γ) reaction drops substantially at the neutron threshold, giving rise to a so-called *competition cusp*. The dashed curve in Fig. 2.29 was obtained by using Eq. (2.219) and is in qualitative agreement with the measurements. The theoretical description of the data is significantly improved if width fluctuation corrections are taken into account (solid line). A discussion of the Hauser–Feshbach model in the context of thermonuclear reaction rates is given in Section 3.2.7.



Fig. 2.29 Cross section versus bombarding energy for the ⁶⁴Ni(p, γ)⁶⁵Cu reaction. Beyond an energy of \approx 2.5 MeV the endothermic ⁶⁴Ni(p,n)⁶⁴Cu reaction is energetically allowed. The sharp drop in the cross section at the neutron threshold reflects the decrease of the flux in all other decay channels of the compound nucleus ⁶⁵Cu.

The curves show the results of Hauser– Feshbach statistical model calculations with (solid line) and without (dashed line) width fluctuation corrections. Reprinted from F. M. Mann et al., Phys. Lett. B, Vol. 58, p. 420 (1975). Copyright (1975), with permission from Elsevier.

Problems

2.1 Show by substituting Eq. (2.27) into Eq. (2.30) that the expansion coefficients are given by $b_{\ell} = (2\ell + 1)i^{\ell}e^{i\delta_{\ell}}$. It is helpful for the derivation to write the sine functions as complex exponentials and to group separately the terms with e^{ikr} and e^{-ikr} .

2.2 The s-wave ($\ell = 0$) transmission coefficient at low energies compared to the Coulomb barrier height is given by Eq. (2.124). Derive the transmission coefficient at low energies for the Coulomb and centripetal potentials by substituting $V(r) = Z_0 Z_1 e^2 / r + \ell(\ell + 1)\hbar^2 / (2mr^2)$ into Eq. (2.119). The simplest procedure is to expand the square root in the integrand before integration.

2.3 Suppose that a hypothetical resonance occurs in the A(p, γ)B reaction. The "observed" proton and γ -ray partial widths amount to $\Gamma_p^o = 50$ meV and $\Gamma_{\gamma}^o = 50$ meV, respectively. Assume that no other reaction channels are open. Use the one-level Breit–Wigner formula to calculate the ratio of reaction cross sections at E_r and $E_r + \Gamma^o$ (Γ^o denotes the total resonance width). Disregard the small energy dependence of the wave number k and of the partial widths.

2.4 Show explicitly that the general solution of Eq. (2.160) reduces for $\ell = 0$ neutrons to $u_0(r) = Ae^{ikr} + Be^{-ikr}$ (see Eq. (2.132)).

2.5 The $E_r^{cm} = 518$ keV ($J^{\pi} = 1^-$) s-wave resonance (Fig. 3.11) in the ${}^{13}C(p,\gamma){}^{14}N$ reaction (Q = 7550 keV) has an "observed" proton and γ -ray partial width of $\Gamma_p^o = 37$ keV and $\Gamma_{\gamma}^o = 9.4$ eV, respectively, at the resonance energy. Both values are given here in the center-of-mass system. They are derived from the results reported in King et al. (1994). The latter value corresponds to the γ -ray partial width of the *E*1 transition to the ${}^{14}N$ ground state ($J^{\pi} = 1^+$). By using the energy dependences of the partial widths, find for this particular resonance the center-of-mass energy at which $\Gamma_p^o \approx \Gamma_{\gamma}^o$. Approximate the s-wave penetration factor by the Gamow factor and disregard the small energy dependence of the dimensionless single-particle reduced width.

2.6 Consider the ²³Na(p, α)²⁰Ne reaction, leading to the ²⁰Ne ground state ($J^{\pi} = 0^+$), at a center-of-mass proton energy of $E_p \approx 0.4$ MeV [$J^{\pi}(^{23}\text{Na}) = 3/2^+$, $J_p^{\pi} = 1/2^+$]. The proton separation energy of ²⁴Mg (or the *Q*-value for the ²³Na(p, γ)²⁴Mg reaction) is $S_p = 11.693$ MeV (Audi, Wapstra and Thibault 2003). Hence, the compound nucleus ²⁴Mg has an excitation energy near 11.7 MeV + 0.4 MeV ≈ 12 MeV. At this energy, ²⁴Mg can decay by proton emission to the ²³Na ground state and by α -particle emission to the ²⁰Ne ground state or the ²⁰Ne first-excited state ($E_x = 1.63$ MeV, $J^{\pi} = 2^+$). Determine the energy-averaged cross section by writing down all terms of Eq. (2.219) up to and including J = 2.

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3 Thermonuclear Reactions

3.1

Cross Sections and Reaction Rates

The *Q*-value represents the energy released in a particular nuclear reaction. Of importance in a stellar environment, however, is the total nuclear energy liberated in a stellar plasma per unit volume. The latter aspect depends on two additional factors, the nuclear cross section and the velocity distribution of the particles in the plasma. The nuclear cross section is a measure for the probability per pair of interacting nuclei 0 and 1 that a nuclear reaction will occur. The total cross section (in units of area) is defined by Eq. (2.1). In general, nuclear cross sections depend on the relative velocity of the target-plus-projectile system, that is, $\sigma = \sigma(v)$.

Using Eq. (2.1), we may write the rate of a nuclear reaction (number of reactions per time *t* and unit volume *V*) as

$$\frac{\mathcal{N}_R}{V \cdot t} = (\sigma \mathcal{N}_t) \left(\frac{\mathcal{N}_b}{V \cdot A \cdot t} \right) = \sigma \frac{\mathcal{N}_t}{V} \frac{\mathcal{N}_b}{A \cdot t} = \sigma \frac{\mathcal{N}_t}{V} v \frac{\mathcal{N}_b}{V}$$
(3.1)

with the current density (number of particles per time and per area) given by $j_b = N_b/(At) = vN_b/V$.

3.1.1

Particle-Induced Reactions

Consider a reaction involving four species, $0 + 1 \rightarrow 2 + 3$, where both the projectile (0) and the target (1) are represented by particles with rest mass (that is, neither 0 nor 1 represents a photon). With the definition $r_{01} \equiv N_R / (Vt)$, we obtain for the reaction rate

$$r_{01} = N_0 N_1 v \sigma(v) \tag{3.2}$$

where $N_0 \equiv N_t / V$ and $N_1 \equiv N_b / V$ are the number densities of the interacting particles (in units of particles per volume). In a stellar plasma at thermodynamic equilibrium, the relative velocity of the interacting nuclei 0 and 1 is not constant, but there exists a distribution of relative velocities, described by the probability function P(v). In this case P(v) dv is the probability that the relative velocity of the interacting nuclei is in the range between v and v + dv,

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with

$$\int_0^\infty P(v) \, dv = 1 \tag{3.3}$$

We may generalize the reaction rate for a distribution of relative velocities by writing

$$r_{01} = N_0 N_1 \int_0^\infty v P(v) \sigma(v) \, dv \equiv N_0 N_1 \langle \sigma v \rangle_{01} \tag{3.4}$$

where $\langle \sigma v \rangle_{01}$ is the reaction rate per particle pair and $N_0 N_1$ is the total number density of pairs of nonidentical nuclei 0 and 1. For identical particles, the total number density of pairs is given by

$$\frac{N_0(N_0-1)}{2} \xrightarrow[N_0]{\text{arge}} \frac{N_0^2}{2}$$
(3.5)

and we obtain for the reaction rate the general expression

$$r_{01} = \frac{N_0 N_1 \langle \sigma v \rangle_{01}}{(1 + \delta_{01})} \tag{3.6}$$

where δ_{01} is the Kronecker symbol. The number of reactions per unit volume and time is given by the product of the number of particle pairs and the reaction rate per particle pair. The latter quantity contains the nuclear physics information. In practice, it is the quantity $N_A \langle \sigma v \rangle_{01}$ (where N_A denotes the Avogadro constant) in units of cm³mol⁻¹s⁻¹ rather than $\langle \sigma v \rangle_{01}$ which is tabulated and presented in the literature. For the case of three-particle reactions or decays, see Fowler, Caughlan and Zimmerman (1967). In a stellar plasma, the kinetic energy available to nuclei is that of their thermal motion. Therefore, the reactions initiated by this motion are called *thermonuclear reactions*. With few exceptions, nuclei in a stellar plasma move nonrelativistically and are nondegenerate (see, for example, Wolf 1965). Thus in most cases the velocities of nuclei can be described by a Maxwell-Boltzmann distribution. The probability for the occurrence of a nuclear reaction depends on the *relative* velocities between the interacting nuclei. If the velocity distributions of the interacting nuclei at thermodynamic equilibrium are separately described by Maxwell-Boltzmann distributions, then it follows that the relative velocities between the two species of nuclei will also be Maxwellian (Clayton 1983).

We may write for the Maxwell-Boltzmann distribution

$$P(v) dv = \left(\frac{m_{01}}{2\pi kT}\right)^{3/2} e^{-m_{01}v^2/(2kT)} 4\pi v^2 dv$$
(3.7)

which gives the probability that the relative velocity has a value between v and v + dv. The Boltzmann constant is given by $k = 8.6173 \times 10^{-5} \text{ eV/K}$,

T is the temperature, and m_{01} is the reduced mass $m_{01} = m_0 m_1 / (m_0 + m_1)$ (Appendix C.2). With $E = m_{01}v^2/2$ and $dE/dv = m_{01}v$ we may write the velocity distribution as an energy distribution,

$$P(v) dv = P(E) dE = \left(\frac{m_{01}}{2\pi kT}\right)^{3/2} e^{-E/kT} 4\pi \frac{2E}{m_{01}} \frac{dE}{m_{01}} \sqrt{\frac{m_{01}}{2E}}$$
$$= \frac{2}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} \sqrt{E} e^{-E/kT} dE$$
(3.8)

The *velocity* distribution has a maximum at $v_T = \sqrt{2kT/m_{01}}$, corresponding to an energy of E = kT. The *energy* distribution has a maximum at E = kT/2. For the reaction rate per particle pair we obtain

$$\langle \sigma v \rangle_{01} = \int_0^\infty v P(v) \sigma(v) \, dv = \int_0^\infty v \sigma(E) P(E) \, dE$$

= $\left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty E \, \sigma(E) \, e^{-E/kT} \, dE$ (3.9)

Numerically we obtain for the reaction rate at a given temperature T

$$N_A \langle \sigma v \rangle_{01} = \frac{3.7318 \times 10^{10}}{T_9^{3/2}} \sqrt{\frac{M_0 + M_1}{M_0 M_1}} \int_0^\infty E \,\sigma(E) \, e^{-11.605 \, E/T_9} \, dE$$
(cm³ mol⁻¹ s⁻¹) (3.10)

where the center-of-mass energy *E* is in units of MeV, the temperature T_9 in GK ($T_9 \equiv T/10^9$ K), the relative atomic masses M_i in u and the cross section σ in b (1 b $\equiv 10^{-24}$ cm²). Clearly, the reaction rate depends critically on the cross section σ which differs for each nuclear reaction.

Figure 3.1a shows the factor $(kT)^{-3/2}Ee^{-E/kT}$, which contains mainly the Maxwell–Boltzmann distribution, versus energy *E* for three different scenarios: (i) the Sun's core (T = 15 MK), (ii) a nova (T = 300 MK), and (iii) a supernova (T = 5 GK). Each displayed curve increases linearly at small energies, reaches a maximum at E = kT, and then decreases exponentially and approaches zero for large values of *E*. The term kT is numerically given by $kT = 86.173 T_9$ (keV) = 0.086173 T_9 (MeV), and is displayed in Fig. 3.2. The maxima of the curves in Fig. 3.1a occur at $E_{max} = kT = 1.3$ keV, 26 keV, and 431 keV.

For neutron-induced reactions, such as (n,γ) or (n,α) , the reaction rate is frequently expressed in terms of a Maxwellian-averaged cross section,

$$N_A \langle \sigma \rangle_T \equiv \frac{N_A \langle \sigma v \rangle}{v_T} = \frac{1}{v_T} N_A \int_0^\infty v P(v) \sigma_n(v) \, dv$$
$$= \frac{4}{\sqrt{\pi}} \frac{N_A}{v_T^2} \int_0^\infty v \sigma_n(v) \left(\frac{v}{v_T}\right)^2 e^{-(v/v_T)^2} \, dv$$
(3.11)



Fig. 3.1 (a) The factor $(kT)^{-3/2}E e^{-E/kT}$ that occurs in the rate expression for reactions induced by particles with rest mass (see Eq. (3.10)) at three different temperatures, T = 0.015 GK, 0.3 GK and 5 GK; these conditions are encountered in the Sun, in classical novae and in type II su-

pernovae, respectively; (b) The factor $E_{\gamma}^2/(e^{E_{\gamma}/kT}-1)$ that occurs in the expression of the decay constant for photodisintegration reactions (see Eq. (3.18)) at three different temperatures, T = 1 GK, 2.5 GK, and 5 GK.



Fig. 3.2 The energy maximum of the Maxwell–Boltzmann velocity distribution as a function of temperature.

with $v_T = \sqrt{2kT/m_{01}}$ being the thermal velocity (that is, the maximum of the *velocity* distribution). The quantity $\langle \sigma \rangle_T$ rather than $\langle \sigma v \rangle$ is frequently presented in the literature. The usefulness of the above expression will become apparent in later sections.

3.1.2 Photon-Induced Reactions

If species 2 is a photon, then the process $\gamma + 3 \rightarrow 0 + 1$ is called a *photodis-integration reaction*. The current density may be written as $j_b = N_b/(At) = cN_b/V$, with *c* being the speed of light. With the definitions $r_{\gamma 3} \equiv N_R/(Vt)$, $N_3 \equiv N_t/V$, $N_\gamma \equiv N_b/V$ we obtain from Eq. (3.1)

$$r_{\gamma 3} = N_3 N_\gamma c \sigma(E_\gamma) \tag{3.12}$$

The cross section depends on the γ -ray energy. Furthermore, in a stellar plasma at thermodynamic equilibrium, the number density of photons is not constant, but depends on the stellar temperature and on the γ -ray energy. We may generalize the reaction rate by writing

$$r_{\gamma 3} = N_3 \int_0^\infty c N_\gamma(E_\gamma) \sigma(E_\gamma) \, dE_\gamma \tag{3.13}$$

For the decay constant (probability of decay per nucleus per second) we find

$$\lambda_{\gamma}(3) = \frac{r_{\gamma 3}}{N_3} = \int_0^\infty c N_{\gamma}(E_{\gamma}) \sigma(E_{\gamma}) \, dE_{\gamma}$$
(3.14)

The energy density of electromagnetic waves with frequencies between v and v + dv at temperature *T* is given by the Planck radiation law

$$u(\nu) \, d\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \, d\nu \tag{3.15}$$

With the substitution $E_{\gamma} = h\nu$ we find for the energy density

$$u(E_{\gamma}) dE_{\gamma} = \frac{8\pi}{(hc)^3} \frac{E_{\gamma}^3}{e^{E_{\gamma}/kT} - 1} dE_{\gamma}$$
(3.16)

The number of photons with energies between E_{γ} and $E_{\gamma} + dE_{\gamma}$ per unit volume at a temperature *T* is then

$$N_{\gamma}(E_{\gamma}) dE_{\gamma} = \frac{u(E_{\gamma})}{E_{\gamma}} dE_{\gamma} = \frac{8\pi}{(hc)^3} \frac{E_{\gamma}^2}{e^{E_{\gamma}/kT} - 1} dE_{\gamma}$$
(3.17)

With Eq. (3.14) we obtain for the photodisintegration decay constant at a given temperature

$$\lambda_{\gamma}(3) = \frac{8\pi}{h^3 c^2} \int_0^\infty \frac{E_{\gamma}^2}{e^{E_{\gamma}/kT} - 1} \sigma(E_{\gamma}) \, dE_{\gamma} \tag{3.18}$$

Since most photodisintegration reactions are endothermic ($Q_{\gamma 3 \rightarrow 01} < 0$) the lower integration limit is actually given by the threshold energy, $E_t = Q_{01 \rightarrow \gamma 3}$, of the reaction. Note that $\lambda_{\gamma}(3)$ does not depend on the stellar density.



Fig. 3.3 Energy level diagram comparing photodisintegration reactions of different threshold energies, E_t . The right-hand side shows schematically the factor $E_{\gamma}^2/(e^{E_{\gamma}/kT}-1)$ (see Fig. 3.1b). Only photons with energies above the threshold ($E_t = Q_{01 \rightarrow \gamma3}$ for photon γ ; $E'_t = Q_{45 \rightarrow \gamma3}$ for photon γ') can initiate a photodisintegration reaction and contribute to the decay constant given by Eq. (3.18).

Figure 3.1b shows the factor $E_{\gamma}^2/(e^{E_{\gamma}/kT}-1)$ versus γ -ray energy for three different scenarios: (i) T = 1 GK (kT = 86 keV), (ii) T = 2.5 GK (kT =215 keV), and (iii) T = 5 GK (kT = 431 keV). The maxima of the curves occur at $E_{\gamma,max} \approx 1.6 kT = 140$ keV, 349 keV and 700 keV. The number of photons is not conserved, but is determined by the conditions of thermal equilibrium. For many important photodisintegration reactions, the threshold energies are much larger than the location of the maxima of the factor $E_{\gamma}^2/(e^{E_{\gamma}/kT}-1)$, that is, $E_t \gg E_{\gamma,max}$. Figure 3.3 compares the situation for two photodisintegration reactions of different threshold energies and with $E_t \gg E_{\gamma,max}$. Clearly, the integral $\lambda_{\gamma}(3) \sim \int_{E_t}^{\infty} E_{\gamma}^2 (e^{E_{\gamma}}/kT-1)^{-1} \sigma(E_{\gamma}) dE_{\gamma}$ will be smaller for the reaction with the larger threshold energy if both reactions have similar photodisintegration cross sections.

3.1.3

Abundance Evolution

Consider first a reaction between two nuclei 0 and 1, and disregard other processes. The reaction rate for 0 + 1 is related to the mean lifetime τ of the nuclear species in the stellar plasma. The rate of change of the abundance (in terms of the number density) of nucleus 0 due to reactions with nucleus 1 can be expressed as

$$\left(\frac{dN_0}{dt}\right)_1 = -\lambda_1(0)N_0 = -\frac{N_0}{\tau_1(0)}$$
(3.19)

where $\lambda \equiv 1/\tau$ is the decay constant. By using the reaction rate (see Eq. (3.6)), we may also write

$$\left(\frac{dN_0}{dt}\right)_1 = -(1+\delta_{01})r_{01} = -(1+\delta_{01})\frac{N_0N_1\langle\sigma v\rangle_{01}}{(1+\delta_{01})} = -N_0N_1\langle\sigma v\rangle_{01} \quad (3.20)$$

The Kronecker symbol appears since for identical nuclei each reaction destroys two particles. From Eqs. (3.19), (3.20), and (1.13) we obtain the relations

$$r_{01} = \frac{\lambda_1(0)N_0}{(1+\delta_{01})} = \frac{1}{(1+\delta_{01})} \frac{N_0}{\tau_1(0)}$$
(3.21)

$$\tau_1(0) = \frac{N_0}{(1+\delta_{01})r_{01}} = \frac{1}{N_1 \langle \sigma v \rangle_{01}} = \left(\rho \frac{X_1}{M_1} N_A \langle \sigma v \rangle_{01}\right)^{-1}$$
(3.22)

$$\lambda_1(0) = \frac{1}{\tau_1(0)} = N_1 \langle \sigma v \rangle_{01} = \rho \frac{X_1}{M_1} N_A \langle \sigma v \rangle_{01}$$
(3.23)

The decay constant of a nucleus for destruction via a particle-induced reaction depends explicitly on the stellar density and, as will be seen later, implicitly on stellar temperature through the reaction rate. If species 0 can be destroyed by several different reactions, its total lifetime is given by

$$\frac{1}{\tau(0)} = \sum_{i} \frac{1}{\tau_i(0)}$$
(3.24)

The above expressions are very useful and will be applied frequently in the discussion of nuclear burning stages (Chapter 5). The following example shows their use in determining the preferred process (reaction or β -decay) by which a particular nucleus is destroyed in a stellar plasma.

Example 3.1

In a stellar plasma, the nucleus ²⁵Al may be destroyed by the capture reaction ²⁵Al(p, γ)²⁶Si or by β^+ -decay ($T_{1/2} = 7.18$ s). Neglecting other processes, determine the dominant destruction process at a stellar temperature of T = 0.3 GK assuming a reaction rate of $N_A \langle \sigma v \rangle = 1.8 \times 10^{-3}$ cm³ mol⁻¹ s⁻¹. Assume a stellar density of $\rho = 10^4$ g/cm³ and a hydrogen mass fraction of $X_{\rm H} = 0.7$.

Using Eqs. (1.18) and (3.22) we obtain for the mean lifetime of both processes

$$\beta^{+} \text{-decay: } \tau_{\beta^{+}}(^{25}\text{Al}) = \frac{T_{1/2}}{\ln 2} = \frac{7.18 \text{ s}}{0.693} = 10.36 \text{ s}$$

p capture: $\tau_{p}(^{25}\text{Al}) = \left(\rho \frac{X_{\text{H}}}{M_{\text{H}}} N_{A} \langle \sigma v \rangle \right)^{-1}$
$$= \left[(10^{4} \text{ g/cm}^{3}) \cdot \frac{0.7}{1.0078 \text{ u}} \cdot (1.8 \times 10^{-3} \text{ cm}^{3} \text{ s}^{-1} \text{ mol}^{-1}) \right]^{-1} = 0.08 \text{ s}$$

Thus the proton capture reaction is the dominant destruction mechanism of 25 Al under these conditions.

Consider now the influence of several nuclear processes (reactions, photodisintegrations, β -decays) together on the abundance evolution of a particular nucleus in a stellar plasma. As a specific example we will choose again ²⁵Al (Fig. 3.4). It may be produced by a number of processes that are represented by solid lines, including ²⁴Mg(p, γ)²⁵Al, ²²Mg(α ,p)²⁵Al, ²⁵Si(β + ν)²⁵Al, ²⁶Si(γ ,p)²⁵Al, and so on. On the other hand, it is destroyed by the processes shown as dotted lines, such as ²⁵Al(p, γ)²⁶Si, ²⁵Al(α ,p)²⁸Si, ²⁵Al(β + ν)²⁵Mg, ²⁵Al(γ ,p)²⁴Mg, and so forth. The time evolution of the ²⁵Al abundance is described by the expression

$$\frac{d(N_{25}_{Al})}{dt} = N_{H}N_{24}_{Mg}\langle\sigma v\rangle_{24}_{Mg(p,\gamma)} + N_{4}_{He}N_{22}_{Mg}\langle\sigma v\rangle_{22}_{Mg(\alpha,p)}
+ N_{25}_{Si}\lambda_{25}_{Si(\beta^{+}\nu)} + N_{26}_{Si}\lambda_{26}_{Si(\gamma,p)} + \cdots
- N_{H}N_{25}_{Al}\langle\sigma v\rangle_{25}_{Al(p,\gamma)} - N_{4}_{He}N_{25}_{Al}\langle\sigma v\rangle_{25}_{Al(\alpha,p)}
- N_{25}_{Al}\lambda_{25}_{Al(\beta^{+}\nu)} - N_{25}_{Al}\lambda_{25}_{Al(\gamma,p)} - \cdots$$
(3.25)

In general, if the only sources of abundance change are nuclear processes (that is, no expansion or mixing of matter), then the abundance evolution of nucleus *i* is given by the differential equation

$$\frac{dN_i}{dt} = \left[\sum_{j,k} N_j N_k \langle \sigma v \rangle_{jk \to i} + \sum_l \lambda_{\beta,l \to i} N_l + \sum_m \lambda_{\gamma,m \to i} N_m\right] \\ - \left[\sum_n N_n N_i \langle \sigma v \rangle_{ni} + \sum_o \lambda_{\beta,i \to o} N_i + \sum_p \lambda_{\gamma,i \to p} N_i\right]$$
(3.26)

The terms in the first and second parentheses represent all processes producing and destroying nucleus *i*, respectively. In the first parenthesis, the three terms stand for: the sum over all reactions producing nucleus *i* via reactions between *j* and *k*; the sum over all β -decays of nuclei *l* leading to *i*; and the sum over all photodisintegrations of nuclei *m* leading to *i*. Similar arguments apply to the terms in the second parenthesis. If a reaction between nonidentical particles ($j \neq k$) creates two nuclei *i* (for example, ⁷Li + p $\rightarrow \alpha + \alpha$), then $N_j N_k \langle \sigma v \rangle_{jk \to i}$ has to be replaced by $2N_j N_k \langle \sigma v \rangle_{jk \to i}$. If a reaction between identical particles (j = k) produces only one particle *i* (for example, p + p \rightarrow d), then $N_j N_k \langle \sigma v \rangle_{jk \to i}$ must be replaced by $N_j^2 \langle \sigma v \rangle_{jj \to i}/2$. The above expression holds without modification for all other reactions involving identical particles. For the inclusion of three-particle reactions see, for example, Chieffi, Limongi and Straniero (1998). Note that it is of advantage to express Eq. (3.26) in terms



Fig. 3.4 Relevant part of the chart of the nuclides showing processes that create (solid arrows) or destroy (dashed arrows) the species ²⁵Al.

of mole fractions Y (Section 1.5.4) instead of number densities N if the mass density changes during the nucleosynthesis. In most discussions of nuclear burning stages and processes in Chapter 5 we will make the assumption of constant density ρ and thus the use of either N or Y is appropriate.

In any realistic situation, we have to consider the evolution of not just one nuclide, but of several (sometimes many) species simultaneously. For each nuclide we can set up an expression of the form given by Eq. (3.26). Such a system of coupled, nonlinear ordinary differential equations is called a *nuclear reaction network*. In the simplest cases, we will solve the reaction network analytically. In more complex situations, however, the system of equations must be solved numerically. We do not concern ourselves with the numerical techniques of solving a reaction network. These are described in detail by Arnett (1996), Timmes (1999), or Hix and Meyer (2006).

Sometimes the solutions of nuclear reaction networks reveal certain fundamental properties which simplify the interpretation of the results. The most important of these properties are called *steady state* and *equilibrium*. A steadystate solution exists if for some part of the reaction network the time derivatives of all abundances, dN_i/dt , are zero or nearly zero. This implies that in Eq. (3.26) the sum of all destruction terms is balanced by the sum of all creation terms. An equilibrium solution is more restrictive and applies to a situation where the abundances of a pair of nuclei (or of a group of nuclei) are locally balanced because of (almost) equally strong forward and reverse reactions (see Section 3.1.4). We will make frequent use of both concepts in discussions of nucleosynthesis.

3.1.4

Forward and Reverse Reactions

It was shown in Section 2.2 that the cross sections of a forward and a reverse reaction are fundamentally related by the reciprocity theorem. Here, we will derive a number of expressions for the corresponding reaction rates. For a reaction involving only particles with rest mass, $0 + 1 \rightarrow 2 + 3$, we obtain

from Eq. (2.15) with $p^2 = 2mE$

$$\frac{\sigma_{23\to01}}{\sigma_{01\to23}} = \frac{(2j_0+1)(2j_1+1)}{(2j_2+1)(2j_3+1)} \frac{m_{01}E_{01}}{m_{23}E_{23}} \frac{(1+\delta_{23})}{(1+\delta_{01})}$$
(3.27)

For a reaction involving photons, $0 + 1 \rightarrow \gamma + 3$, we obtain with $p^2 = E_{\gamma}^2/c^2$

$$\frac{\sigma_{\gamma3\to01}}{\sigma_{01\to\gamma3}} = \frac{(2j_0+1)(2j_1+1)}{2(2j_3+1)} \frac{2m_{01}c^2E_{01}}{E_{\gamma}^2} \frac{1}{(1+\delta_{01})}$$
(3.28)

where $(2j_{\gamma} + 1) = 2$ since the photon has only two polarization directions (Messiah 1999).

If the forward reaction $0+1\to 2+3$ and corresponding reverse reaction $2+3\to 0+1$ involve only particles with rest mass, then we find for the reaction rates

$$N_A \langle \sigma v \rangle_{01 \to 23} = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E_{01} \sigma_{01 \to 23} e^{-E_{01}/kT} dE_{01}$$
(3.29)

$$N_A \langle \sigma v \rangle_{23 \to 01} = \left(\frac{8}{\pi m_{23}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E_{23} \sigma_{23 \to 01} e^{-E_{23}/kT} dE_{23}$$
(3.30)

The kinetic energies are related by $E_{23} = E_{01} + Q_{01 \rightarrow 23}$ (see Eq. (1.5)). It follows (see also Fowler, Caughlan and Zimmerman 1967)

$$\frac{N_A \langle \sigma v \rangle_{23 \to 01}}{N_A \langle \sigma v \rangle_{01 \to 23}} = \left(\frac{m_{01}}{m_{23}}\right)^{1/2} \frac{\int_0^\infty E_{23} \sigma_{23 \to 01} e^{-E_{23}/kT} dE_{23}}{\int_0^\infty E_{01} \sigma_{01 \to 23} e^{-E_{01}/kT} dE_{01}} \\
= \frac{(2j_0 + 1)(2j_1 + 1)(1 + \delta_{23})}{(2j_2 + 1)(2j_3 + 1)(1 + \delta_{01})} \left(\frac{m_{01}}{m_{23}}\right)^{3/2} e^{-Q_{01 \to 23}/kT} \quad (3.31)$$

Obviously, $N_A \langle \sigma v \rangle_{01 \to 23}$ and $N_A \langle \sigma v \rangle_{23 \to 01}$ refer to the same stellar temperature *T*.

To find the relationship between forward and reverse reactions if species 2 is a photon, we start from Eqs. (3.9) and (3.18),

$$\lambda_{\gamma}(3) = \frac{8\pi}{h^3 c^2} \int_0^\infty \frac{E_{\gamma}^2}{e^{E_{\gamma}/kT} - 1} \,\sigma_{\gamma 3 \to 01} \,dE_{\gamma} \tag{3.32}$$

$$N_A \langle \sigma v \rangle_{01 \to \gamma 3} = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E_{01} \sigma_{01 \to \gamma 3} e^{-E_{01}/kT} dE_{01}$$
(3.33)

From Eq. (3.28) we find

$$\frac{\lambda_{\gamma}(3)}{N_{A}\langle\sigma v\rangle_{01\to\gamma3}} = \frac{\frac{8\pi}{h^{3}c^{2}}\int_{0}^{\infty}\frac{E_{\gamma}^{2}}{e^{E_{\gamma}/kT}-1}\frac{(2j_{0}+1)(2j_{1}+1)}{(2j_{3}+1)(1+\delta_{01})}\frac{m_{01}c^{2}E_{01}}{E_{\gamma}^{2}}\sigma_{01\to\gamma3}\,dE_{\gamma}}{\left(\frac{8}{\pi m_{01}}\right)^{1/2}\frac{N_{A}}{(kT)^{3/2}}\int_{0}^{\infty}E_{01}\,\sigma_{01\to\gamma3}\,e^{-E_{01}/kT}\,dE_{01}}$$
(3.34)



Fig. 3.5 Comparison of the exact expression $E_{\gamma}^2/(e^{E_{\gamma}/kT}-1)$ (solid line) and the approximation $E_{\gamma}^2/e^{E_{\gamma}/kT}$ (dashed line) at a stellar temperature of T = 5 GK. For a sufficiently large threshold energy (in this case, for $E_t > 1.5$ MeV), the difference between both expressions is negligible.

The energies are related by $E_{01} + Q_{01 \rightarrow \gamma 3} = E_{\gamma}$, as shown in Fig. 3.3. Note that most capture reactions have positive Q-values (that is, Q < 0 for the corresponding reverse photodisintegration reactions), otherwise nucleus 3 would be unstable by particle emission. Furthermore, many capture reactions have large Q-values, on the order of several MeV. In this case, the integration over γ -ray energy E_{γ} will not start at 0 but at a threshold energy of $E_t = Q_{01 \rightarrow \gamma 3}$ as explained above (Fig. 3.3). Since this implies $E_{\gamma} \gg kT$, we may use the approximation $e^{E_{\gamma}/kT} - 1 \approx e^{E_{\gamma}/kT}$. Figure 3.5 shows the factor $E_{\gamma}^2/(e^{E_{\gamma}/kT} - 1)$ (solid line) and the approximation $E_{\gamma}^2/e^{E_{\gamma}/kT}$ (dashed line) versus γ -ray energy for a stellar temperature of T = 5 GK. Clearly, if the threshold energy $E_t = Q_{01 \rightarrow \gamma 3}$ is sufficiently large, then the difference between the two expressions is negligible. The approximation also holds for smaller Q-values below 1 MeV if charged particles are involved in the process. The photodisintegration cross section is then suppressed at low energies (due to the tunnel effect) where the deviation between the solid and dashed lines in Fig. 3.5 is largest. However, the approximation may not be valid for (n, γ) reactions with small Q-values.



Fig. 3.6 (a) Reaction rate ratio of forward and reverse reactions versus temperature. The curves correspond to different values of $Q_{01\rightarrow23}$. (b) Ratio of decay constants for photodisintegration reaction and corresponding capture reaction versus temperature for different values of $Q_{01\rightarrow\gamma3}$.

With the approximation $e^{E_{\gamma}/kT} - 1 \approx e^{E_{\gamma}/kT}$ we obtain from Eq. (3.34) (see also Fowler, Caughlan and Zimmerman 1967)

$$\frac{\lambda_{\gamma}(3)}{N_{A}\langle\sigma v\rangle_{01\to\gamma3}} = \frac{\frac{8\pi}{h^{3}c^{2}}(kT)^{3/2}m_{01}c^{2}}{\left(\frac{8}{\pi m_{01}}\right)^{1/2}N_{A}}\frac{(2j_{0}+1)(2j_{1}+1)}{(2j_{3}+1)(1+\delta_{01})} \\ \times \frac{\int_{0}^{\infty}E_{01}e^{-(E_{01}+Q_{01\to\gamma3})/kT}\sigma_{01\to\gamma3}dE_{\gamma}}{\int_{0}^{\infty}E_{01}e^{-E_{01}/kT}\sigma_{01\to\gamma3}dE_{01}} \\ = \left(\frac{2\pi}{h^{2}}\right)^{3/2}\frac{(m_{01}kT)^{3/2}}{N_{A}}\frac{(2j_{0}+1)(2j_{1}+1)}{(2j_{3}+1)(1+\delta_{01})}e^{-Q_{01\to\gamma3}/kT} \quad (3.35)$$

Figure 3.6a shows the ratio of reaction rates, $N_A \langle \sigma v \rangle_{23 \to 01} / N_A \langle \sigma v \rangle_{01 \to 23} \approx e^{-Q_{01 \to 23}/kT}$, for reactions involving particles with rest mass, where the factor containing the spins and reduced masses in Eq. (3.31) is set equal to unity. The different curves correspond to different values of $Q_{01 \to 23}$. For a positive *Q*-value, the ratio $N_A \langle \sigma v \rangle_{23 \to 01} / N_A \langle \sigma v \rangle_{01 \to 23}$ is always less than unity. It is apparent that the reverse reaction becomes important at sufficiently large temperatures and at small *Q*-values. Figure 3.6b shows the ratio of decay constants, $\lambda_{\gamma}(3)/\lambda_1(0) = \lambda_{\gamma}(3)/[\rho(X_1/M_1)N_A \langle \sigma v \rangle_{01 \to \gamma3}]$, for reactions involving photons, where the factor containing the spins in Eq. (3.35) and the term X_1/M_1 are set equal to unity. For the density an arbitrary value of $\rho = 10^3$ g/cm³ has been chosen. For all curves shown, the value of $Q_{01 \to \gamma3}$ (that is, for the capture reaction) is positive. It can be seen that the ratio of photodisintegration and capture reaction decay constants can exceed unity and may become very large, depending on the values of temperature and reaction *Q*-value.



Fig. 3.7 Sections of the nuclidic chart indicating interactions between nuclei. Arrows that point vertically up show (p,γ) reactions; those pointing vertically down represent (γ,p) photodisintegrations and those pointing diagonally down to the right correspond to β^+ -decays. (a) The reaction chain ¹¹B+p \leftrightarrow ¹²C and ¹²C+p \leftrightarrow ¹³N at elevated temperatures. (b) A situation depicting an equilibrium between the forward reaction $A+a \rightarrow \gamma+B$ and the reverse reaction $B+\gamma \rightarrow a+A$; see Section 3.1.6.

The strong Q-value dependence of the ratio $\lambda_{\gamma}(3)/\lambda_1(0)$ in Fig. 3.6b has an important consequence. Capture reactions involving target nuclei with an even number of protons and neutrons usually have small Q-values, while the Q-values are larger for capture reactions involving an odd number of protons or neutrons. In other words, a relatively large amount of energy is released if an energetically favorable even-even structure can be achieved as a result of the capture process. As an example, consider the reaction chain ^{11}B + p \leftrightarrow ^{12}C and $^{12}C + p \leftrightarrow ^{13}N$ shown in Fig. 3.7a. The corresponding *Q*-values are $Q_{^{11}B+p}$ = 16 MeV and $Q_{^{12}C+p}$ = 2 MeV. Clearly, at elevated temperatures the photodisintegration of ¹²C will be a relatively slow process while the photodisintegration of ¹³N will be much faster. As a consequence, the abundance of ¹²C will be enhanced over that of the neighboring (and less stable) nuclei ¹¹B and ¹³N. The net effect of photodisintegration processes in stellar plasmas at elevated temperatures is to convert nuclei to more stable species. These considerations will be especially important for the advanced burning stages of massive stars (see Section 5.5).

3.1.5

Reaction Rates at Elevated Temperatures

Until now we considered only reactions involving nuclei in their ground states. However, at elevated stellar temperatures the nuclei will be thermally excited, for example, through photoexcitation, inelastic particle scattering,

and other means. These excited states may also participate in nuclear reactions. We already mentioned in Section 1.7.4 that for a nondegenerate plasma in thermodynamic equilibrium, the ratio of the number density $N_{i\mu}$ of nuclei *i* in excited state μ and the total number density N_i of nuclei *i* is given by a Boltzmann distribution

$$P_{i\mu} = \frac{N_{i\mu}}{N_i} = \frac{g_{i\mu}e^{-E_{i\mu}/kT}}{\sum_{\mu}g_{i\mu}e^{-E_{i\mu}/kT}} = \frac{g_{i\mu}e^{-E_{i\mu}/kT}}{G_i}$$
(3.36)

where the symbols have the same meaning as in Eq. (1.34).

Of primary astrophysical importance is the reaction rate involving thermally excited nuclei, $N_A \langle \sigma v \rangle^*$, rather than the rate involving nuclei in the ground state, $N_A \langle \sigma v \rangle$. For the reaction $0 + 1 \rightarrow 2 + 3$ the rate including thermally excited states is obtained by summing over all transitions to relevant excited states in nuclei 2 and 3, and by appropriately averaging over combinations of excited states in nuclei 0 and 1. The number densities N_i entering the reaction rate expression, $r_{01} = N_0 N_1 \langle \sigma v \rangle^*_{01 \rightarrow 23}$, refer to the total number of nuclei *i* per unit volume. For the sake of simplicity, we will disregard in the following excited states of the light particles 1 and 2 in the entrance or exit channel (which is a valid assumption for the proton, neutron, and α -particle). We write

$$N_A \langle \sigma v \rangle_{01 \to 23}^* = \sum_{\mu} P_{0\mu} \sum_{\nu} N_A \langle \sigma v \rangle_{01 \to 23}^{\mu \to \nu}$$

$$= \frac{\sum_{\mu} g_{0\mu} e^{-E_{0\mu}/kT} \sum_{\nu} N_A \langle \sigma v \rangle_{01 \to 23}^{\mu \to \nu}}{\sum_{\mu} g_{0\mu} e^{-E_{0\mu}/kT}}$$
(3.37)

where μ and ν are labels for states in the target nucleus 0 and the residual nucleus 3, respectively. Note that laboratory experiments usually provide information for the calculation of the quantity

$$N_A \langle \sigma v \rangle_{01 \to 23} = \sum_{\nu} N_A \langle \sigma v \rangle_{01 \to 23}^{\text{g.s.} \to \nu}$$
(3.38)

that is, the rate involving all transitions from the ground state of the target nucleus 0 to the ground state and to excited states of the residual nucleus 3. Usually, cross sections involving excited target nuclei cannot be measured directly in the laboratory and have to be calculated by using theoretical models. From Eqs. (3.37) and (3.38) we find

$$N_{A}\langle \sigma v \rangle_{01 \to 23}^{*} = \frac{N_{A} \langle \sigma v \rangle_{01 \to 23}^{*}}{N_{A} \langle \sigma v \rangle_{01 \to 23}} N_{A} \langle \sigma v \rangle_{01 \to 23} = R_{tt} N_{A} \langle \sigma v \rangle_{01 \to 23}$$
$$= \frac{\sum_{\mu} g_{0\mu} e^{-E_{0\mu}/kT} \frac{\sum_{\nu} N_{A} \langle \sigma v \rangle_{01 \to 23}^{\mu \to \nu}}{\sum_{\nu} N_{A} \langle \sigma v \rangle_{01 \to 23}^{\xi \to \to \nu}} N_{A} \langle \sigma v \rangle_{01 \to 23}$$
(3.39)

Numerical values for the *stellar enhancement factor* R_{tt} , estimated by using the Hauser–Feshbach statistical model (Section 2.7), are given in Angulo et al. (1999) and Rauscher and Thielemann (2000). The above expressions are also valid if general thermodynamic equilibrium has not been attained, as long as the excited states are in equilibrium with the ground state (Fowler, Caughlan and Zimmerman 1967, 1975). Of primary astrophysical interest are the abundances of all levels which will decay to the ground state after the final cooling of the stellar event. Therefore, the sums over μ and ν include all bound states up to an energy at which the levels become unbound and decay primarily via particle emission. Similar statements apply to the reverse reaction $2 + 3 \rightarrow 0 + 1$. An explicit expression for $N_A \langle \sigma v \rangle_{01 \rightarrow 23}^*$ that is applicable to the special case of narrow resonances will be derived in Section 3.2.4.

The relationships between forward and reverse reaction rates derived in Section 3.1.4 assume that all interacting nuclei are in their ground states. They also need to be modified in order to take thermally excited states into account. As a simple example, consider the situation shown in Fig. 3.8. A laboratory measurement of the cross section for the reaction $0 + 1 \rightarrow 2 + 3$ at a constant bombarding energy *E* considers only target nuclei 0 in their ground state and sums over all transitions to the ground state or to excited states in the final nucleus 3. In a stellar plasma, on the other hand, both the target and the residual nucleus may be thermally excited, and all possible transitions between excited states μ and ν in nuclei 0 and 3, respectively, have to be taken into account.

Suppose that there are a number of excited states in nucleus 0 and in nucleus 3 which are all in thermal equilibrium with their respective ground states. Furthermore, assume that the light particles 1 and 2 have no excited states. The *stellar* rates for forward and reverse reactions are then obtained by appropriately averaging over initial states and summing over final states. The expression for the forward stellar rate is given by Eq. (3.37), while for the reverse stellar rate one has

$$N_{A} \langle \sigma v \rangle_{23 \to 01}^{*} = \sum_{\nu} P_{3\nu} \sum_{\mu} N_{A} \langle \sigma v \rangle_{23 \to 01}^{\nu \to \mu}$$

$$= \frac{\sum_{\nu} g_{3\nu} e^{-E_{3\nu}/kT} \sum_{\mu} N_{A} \langle \sigma v \rangle_{23 \to 01}^{\nu \to \mu}}{\sum_{\nu} g_{3\nu} e^{-E_{3\nu}/kT}}$$
(3.40)

Into this expression we substitute our earlier result (see Eq. (3.31))

$$\frac{N_A \langle \sigma v \rangle_{23 \to 01}^{\nu \to \mu}}{N_A \langle \sigma v \rangle_{01 \to 23}^{\mu \to \nu}} = \frac{g_{0\mu} g_1 (1 + \delta_{23})}{g_{3\nu} g_2 (1 + \delta_{01})} \left(\frac{m_{01}}{m_{23}}\right)^{3/2} e^{-Q_{01 \to 23}^{\mu \to \nu}/kT}$$
(3.41)

We also make the nonrelativistic approximations $m_{01} = m_{01}^{\mu}$ and $m_{23} = m_{23}^{\nu}$. For the ground states, we have $e^{-E_0/kT} = e^{-E_3/kT} = 1$. By using $Q_{01\to 23} =$



Fig. 3.8 Reactions between the pairs of nuclei 0 + 1 and 2 + 3. (a) In the laboratory the nuclei 0 and 1 are usually in their ground states and transitions may occur to excited levels of nuclei 2 and 3. (b) In the stellar plasma, excited levels participate in the reaction in both the entrance and the exit channel. Only a single excited state is shown in each channel for reasons of clarity.

 $Q_{01\to23}^{\mu\to\nu} + E_{3\nu} - E_{0\mu}$ (Fig. 3.8b) one obtains from Eqs. (3.37), (3.40), and (3.41)

$$\frac{N_A \langle \sigma v \rangle_{23 \to 01}^*}{N_A \langle \sigma v \rangle_{01 \to 23}^*} = \frac{(1 + \delta_{23})}{(1 + \delta_{01})} \left(\frac{m_{01}}{m_{23}}\right)^{3/2} \frac{g_0 g_1 G_0^{\text{norm}}}{g_2 g_3 G_3^{\text{norm}}} e^{-Q_{01 \to 23}/kT}$$
(3.42)

The quantity $Q_{01\rightarrow23}$ denotes the *Q*-value connecting the ground states and G_i^{norm} is the *normalized partition function*

$$G_i^{\text{norm}} = \frac{G_i}{g_i} = \frac{\sum_{\mu} g_{i\mu} e^{-E_{i\mu}/kT}}{g_i}$$
(3.43)

with $g_{i\mu}$ and $E_{i\mu}$ the statistical weight and excitation energy of state μ in nucleus *i*, respectively; g_i is the statistical weight of the ground state of nucleus *i*. Numerical values of G_i^{norm} versus temperature are tabulated in Angulo et al. (1999) and Rauscher and Thielemann (2000).

Equation (3.42) holds for any number of excited states in the target and residual nucleus. It is also independent of the reaction mechanism (for example, nonresonant versus resonant process, number and properties of narrow resonances, and so on). We can easily generalize this result by allowing for excitations in the nuclei 1 and 2. Numerically, we find from Eq. (3.42) for reactions involving only particles with rest mass

$$\frac{N_A \langle \sigma v \rangle_{23 \to 01}^*}{N_A \langle \sigma v \rangle_{01 \to 23}^*} = \frac{(2j_0 + 1)(2j_1 + 1)(1 + \delta_{23})}{(2j_2 + 1)(2j_3 + 1)(1 + \delta_{01})} \left(\frac{G_0^{\text{norm}} G_1^{\text{norm}}}{G_2^{\text{norm}} G_3^{\text{norm}}}\right) \left(\frac{M_0 M_1}{M_2 M_3}\right)^{3/2} e^{-11.605 \, Q/T_9}$$
(3.44)

and from Eq. (3.35) for reactions involving photons

$$\frac{\lambda_{\gamma}^{*}(3 \to 01)}{N_{A} \langle \sigma v \rangle_{01 \to \gamma 3}^{*}} = 9.8685 \times 10^{9} T_{9}^{3/2} \frac{(2j_{0}+1)(2j_{1}+1)}{(2j_{3}+1)(1+\delta_{01})} \left(\frac{G_{0}^{\text{norm}} G_{1}^{\text{norm}}}{G_{3}^{\text{norm}}}\right) \left(\frac{M_{0}M_{1}}{M_{3}}\right)^{3/2} e^{-11.605 \, Q/T_{9}}$$

$$(3.45)$$

with j_i and M_i being the ground-state spins and masses (in u) of the nuclei, Q the ground-state Q-value of the forward reaction $0 + 1 \rightarrow 2 + 3$ or $0 + 1 \rightarrow \gamma + 3$ (in MeV), and $T_9 \equiv T/10^9$ K. In the following sections we will mostly suppress the asterisk, with the understanding that rates or decay constants must take into account the effects of thermally excited states if appropriate.

Example 3.2

Evaluations of experimental thermonuclear reaction rates (Angulo et al. 1999, Iliadis et al. 2001) list laboratory reaction rates and have to be modified for use in stellar model calculations. Consider as an example the ${}^{32}S(p,\gamma){}^{33}Cl$ reaction at a stellar temperature of T = 10 GK ($T_9 = 10$). For the laboratory reaction rate (assuming that the ${}^{32}S$ target nuclei are in the ground state) a value of $N_A \langle \sigma v \rangle_{{}^{32}S+p} = 1.05 \times 10^4$ cm³ mol⁻¹ s⁻¹ is reported in Iliadis et al. (2001). Calculate the stellar rate for the forward reaction and the stellar decay constant for the reverse reaction.

The stellar reaction rate (considering thermally excited ³²S nuclei) is given by

$$N_A \langle \sigma v \rangle_{^{32}\text{S+p}}^* = R_{tt} N_A \langle \sigma v \rangle_{^{32}\text{S+p}}$$

= 0.83 \cdot 1.05 \times 10⁴ cm³ mol⁻¹ s⁻¹ = 8.72 \times 10³ cm³ mol⁻¹ s⁻¹

with a value of R_{tt} = 0.83 adopted from Rauscher and Thielemann (2000).

The corresponding stellar decay constant for the photodisintegration of ³³Cl is obtained from the spins $j_{32_{\rm S}} = 0$, $j_{\rm p} = 1/2$, $j_{33_{\rm Cl}} = 3/2$, the value $Q_{32_{\rm S+p}} = 2.2765$ MeV, and from the normalized partition functions $G_{32_{\rm S}}^{\rm norm} = 1.6$, $G_{\rm p}^{\rm norm} = 1$, $G_{33_{\rm Cl}}^{\rm norm} = 1.9$ (Rauscher and Thielemann 2000)

$$\begin{split} \lambda_{\gamma}^{*}(^{33}\text{Cl} \to ^{32}\text{S} + \text{p}) &= \frac{\lambda_{\gamma}^{*}(^{33}\text{Cl} \to ^{32}\text{S} + \text{p})}{N_{A}\langle \sigma v \rangle_{^{32}\text{S} + \text{p}}^{*}} N_{A}\langle \sigma v \rangle_{^{32}\text{S} + \text{p}}^{*} \\ &= 9.8685 \times 10^{9} \cdot 10^{3/2} \frac{1 \cdot 2}{4 \cdot 1} \left(\frac{1.6 \cdot 1}{1.9}\right) \left(\frac{32.0 \cdot 1.0}{33.0}\right)^{3/2} \\ &\times e^{-11.605 \cdot 2.2765/10} \cdot 8.72 \times 10^{3} \,\text{cm}^{3} \,\text{mol}^{-1} \,\text{s}^{-1} \\ &= 7.79 \times 10^{13} \,\text{s}^{-1} \end{split}$$

3.1.6 Reaction Rate Equilibria

Consider a forward and reverse reaction involving four particles with rest mass, $0 + 1 \rightarrow 2 + 3$ and $2 + 3 \rightarrow 0 + 1$. The overall reaction rate for $0 + 1 \leftrightarrow 2 + 3$ is then given by

$$r = r_{01 \to 23} - r_{23 \to 01} = \frac{N_0 N_1 \langle \sigma v \rangle_{01 \to 23}}{(1 + \delta_{01})} - \frac{N_2 N_3 \langle \sigma v \rangle_{23 \to 01}}{(1 + \delta_{23})}$$
(3.46)

For equilibrium conditions (r = 0) we find from Eqs. (3.44) and (3.46) for the ratio of nuclidic abundances

$$\frac{N_2 N_3}{N_0 N_1} = \frac{(1+\delta_{23})}{(1+\delta_{01})} \frac{\langle \sigma v \rangle_{01 \to 23}}{\langle \sigma v \rangle_{23 \to 01}}
= \frac{(2j_2+1)(2j_3+1)}{(2j_0+1)(2j_1+1)} \frac{G_2^{\text{norm}} G_3^{\text{norm}}}{G_0^{\text{norm}} G_1^{\text{norm}}} \left(\frac{m_{23}}{m_{01}}\right)^{3/2} e^{Q_{01 \to 23}/kT}$$
(3.47)

Similarly, for reactions involving photons we find for the overall reaction rate $0+1\leftrightarrow\gamma+3$

$$r = r_{01 \to \gamma 3} - r_{\gamma 3 \to 01} = \frac{N_0 N_1 \langle \sigma v \rangle_{01 \to \gamma 3}}{(1 + \delta_{01})} - \lambda_\gamma(3) N_3$$
(3.48)

and for equilibrium conditions (r = 0) we obtain from Eqs. (3.23), (3.45), and (3.48) the expression

$$\frac{N_3}{N_0 N_1} = \frac{1}{(1+\delta_{01})} \frac{\langle \sigma v \rangle_{01 \to \gamma 3}}{\lambda_{\gamma}(3)} = \frac{1}{(1+\delta_{01})} \frac{1}{N_1} \frac{\lambda_1(0)}{\lambda_{\gamma}(3)} \\
= \left(\frac{h^2}{2\pi}\right)^{3/2} \frac{1}{(m_{01}kT)^{3/2}} \frac{(2j_3+1)}{(2j_0+1)(2j_1+1)} \frac{G_3^{\text{norm}}}{G_0^{\text{norm}} G_1^{\text{norm}}} e^{Q_{01 \to \gamma 3}/kT} \tag{3.49}$$

The last expression is referred to as the Saha statistical equation.

The equilibrium condition can also be expressed in terms of abundance evolutions. Suppose that 0 and 3 denote heavy nuclei and 1 and 2 represent light particles (protons, neutrons or α -particles). The *partial* rates of change of isotopic abundances N_0 and N_3 that are caused by the processes $0 + 1 \rightarrow 2 + 3$ and $2 + 3 \rightarrow 0 + 1$, respectively, are given by (see Eq. (3.20))

$$\left(\frac{dN_0}{dt}\right)_{01\to23} = -r_{01\to23} \tag{3.50}$$

$$\left(\frac{dN_3}{dt}\right)_{23\to01} = -r_{23\to01} \tag{3.51}$$

We may visualize these processes by flows of material from species 0 to 3 and vice versa. Therefore, the partial rates of change of abundances, $(dN_0/dt)_{01\rightarrow 23}$ and $(dN_3/dt)_{23\rightarrow 01}$, are referred to as abundance flows. The net abundance flow f between two species 0 and 3 is given by the difference between forward and reverse abundance flow

$$f_{03} \equiv \left| \left(\frac{dN_0}{dt} \right)_{01 \to 23} - \left(\frac{dN_3}{dt} \right)_{23 \to 01} \right| = |r_{01 \to 23} - r_{23 \to 01}| \\ = |N_0 N_1 \langle \sigma v \rangle_{01 \to 23} - N_2 N_3 \langle \sigma v \rangle_{23 \to 01}|$$
(3.52)

The equilibrium condition can be expressed by either of the following relations:

$$\left(\frac{dN_0}{dt}\right)_{01\to23} \approx \left(\frac{dN_3}{dt}\right)_{23\to01} \gg f_{03} \approx 0 \tag{3.53}$$

$$\phi_{03} \equiv \frac{|r_{01 \to 23} - r_{23 \to 01}|}{\max(r_{01 \to 23}, r_{23 \to 01})} \approx 0 \tag{3.54}$$

In this case the net abundance flow is much smaller in absolute magnitude than either the forward flow or the reverse flow. Contrary to the steady-state assumption (Section 3.1.3), the equilibrium condition does not imply constant abundances N_0 or N_3 . Those may indeed change if nuclei 0 and 3 are linked to other species by nuclear processes. The equilibrium condition refers to the (near) equality of forward and reverse abundance flows between a pair of nuclei. When a group of several pairs of nuclei comes into equilibrium, for example, via the processes $(p,\gamma) \leftrightarrow (\gamma,p)$, $(n,\gamma) \leftrightarrow (\gamma,n)$ and $(\alpha,\gamma) \leftrightarrow (\gamma,\alpha)$, the resulting solution of the reaction network is called a *quasiequilibrium*. For more information on equilibria, see Arnett (1996).

In the following we will discuss reactions involving photons in more detail. During the complex interplay involving several different nuclear reactions and β -decays it happens frequently that a particular reaction converting nucleus A by particle capture to nucleus B $(A + a \rightarrow B)$ exhibits a small Qvalue. If the stellar plasma can attain sufficiently high temperatures, then the photodisintegration of nucleus *B* has to be taken into account and may significantly alter the nucleosynthesis.

Consider Fig. 3.7b, showing a number of different nuclei involved in proton captures, photodisintegrations, and β^+ -decays. Suppose now that the Q-value for the capture reaction $A(a, \gamma)B$ is relatively small (less than a few hundred keV) and that the stellar temperature is high. An equilibrium between the abundances of nuclei A and B is established for two necessary conditions,

$$\lambda_{A \to B} > \lambda_{A \to A'}$$

$$\lambda_{B \to A} > \lambda_{B \to C} + \lambda_{B \to B'}$$
(3.55)
(3.56)

$$\lambda_{B \to A} > \lambda_{B \to C} + \lambda_{B \to B'} \tag{3.56}$$

If the first condition is not fulfilled, then nucleus *B* is bypassed altogether. If the second condition is not fulfilled, then there is no process that creates nucleus *A* after it has been destroyed. We will also assume that the photodisintegration of *C* is negligible (that is, $\lambda_{C \to C'} > \lambda_{C \to B}$), so that *C* does not come into equilibrium with *A* and *B*. It is now of interest to determine which path the nucleosynthesis will follow, either bypassing nucleus *B* via β^+ -decay $(A \to A')$, or via the competing reaction sequence through nucleus *B* to either *C* or *B'* $[A \to B \to (C \text{ or } B')]$. The latter process, $A \to B \to C$, is referred to as *sequential two-particle capture*. (For a distinction between sequential and direct two-particle capture, see Grigorenko and Zhukov 2005.)

Suppose that an equilibrium between the abundances of nuclei *A* and *B* has been established. The reaction rate for conversion of nucleus *A* to either *C* (via capture of particle *a*) or *B*' (via β^+ -decay) is then given by the expressions (see Eq. (3.21))

$$r_{A \to (C \text{ or } B')} = N_B^e \lambda_{B \to C} + N_B^e \lambda_{B \to B'}$$
(3.57)

$$r_{A\to(C \text{ or } B')} = N_A^e \lambda_{A\to B\to C} + N_A^e \lambda_{A\to B\to B'} = N_A^e \lambda_{A\to B\to(C \text{ or } B')}$$
(3.58)

where N_A^e and N_B^e denote the equilibrium abundances of *A* and *B*, respectively. From Eqs. (3.57) and (3.58) we obtain

$$\lambda_{A \to B \to (C \text{ or } B')} = \frac{N_B^e}{N_A^e} (\lambda_{B \to C} + \lambda_{B \to B'})$$
(3.59)

For the equilibrium abundance ratio N_B^e/N_A^e we use the Saha equation (see Eq. (3.49))

$$\frac{N_B}{N_A N_a} = \frac{\langle \sigma v \rangle_{A \to B}}{\lambda_{B \to A}} = \frac{1}{N_a} \frac{\lambda_{A \to B}}{\lambda_{B \to A}}$$
$$= \left(\frac{h^2}{2\pi}\right)^{3/2} \frac{1}{\left(m_{Aa} kT\right)^{3/2}} \frac{(2j_B + 1)}{(2j_A + 1)(2j_a + 1)} \frac{G_B^{\text{norm}}}{G_A^{\text{norm}} G_a^{\text{norm}}} e^{Q_{A \to B}/kT}$$
(3.60)

Thus

$$\lambda_{A \to B \to (C \text{ or } B')} = \frac{\lambda_A \to B}{\lambda_{B \to A}} \left(\lambda_{B \to C} + \lambda_{B \to B'} \right)$$
$$= N_a \left(\frac{h^2}{2\pi} \right)^{3/2} \frac{1}{(m_{Aa}kT)^{3/2}} \frac{(2j_B + 1)}{(2j_A + 1)(2j_a + 1)}$$
$$\times \frac{G_B^{\text{norm}}}{G_A^{\text{norm}} G_a^{\text{norm}}} e^{Q_{A \to B}/kT} \left(\lambda_{B \to C} + \lambda_{B \to B'} \right)$$
(3.61)

Numerically, we find

$$\lambda_{A\to B\to(C \text{ or } B')} = 1.0133 \times 10^{-10} \rho \frac{X_a}{M_a} \left(\frac{M_B}{M_A M_a}\right)^{3/2} \frac{g_B}{g_A g_a} \left(\frac{G_B^{\text{norm}}}{G_A^{\text{norm}} G_a^{\text{norm}}}\right) \times T_9^{-3/2} e^{11.605 Q_{A\to B}/T_9} \left(\lambda_{B\to C} + \lambda_{B\to B'}\right)$$
(3.62)

where the normalized partition functions account for the influence of thermally excited levels and the *Q*-value is in MeV. Note that the quantity $\lambda_{A \to B \to (C \text{ or } B')}$ introduced in Eq. (3.58) has a slightly different meaning from the usual decay constant of Eq. (3.19). The latter describes the decay probability of a particular nucleus per time, while the former represents the probability for the conversion of nucleus *A* along a specified path ($A \to B \to (C \text{ or } B')$), in this case). This distinction becomes important for identical particles. If the process $A \to B \to (C \text{ or } B')$ destroys two (or three) identical particles *A*, then the right-hand sides of Eqs. (3.61) and (3.62) must be multiplied by a factor of 2 (or 3) in order to calculate the decay constant of nucleus *A*.

Obviously, the path $A \rightarrow B \rightarrow (C \text{ or } B')$ becomes more important with increasing values of $\lambda_{A \to B}$, $\lambda_{B \to C}$, or $\lambda_{B \to B'}$, and decreasing values of $\lambda_{B \to A}$. It must be emphasized that the ratio $\lambda_{A \to B} / \lambda_{B \to A}$ is independent of the cross section, and depends mainly on the value of $Q_{A \to B}$. Also, note that we replaced the equilibrium abundance N_B^e/N_A^e by the ratio $\lambda_{A\to B}/\lambda_{B\to A}$ which in turn is determined by the reciprocity theorem (Section 3.1.4). Thus we made no assumptions regarding the specific processes occurring between nuclei A and B. Those include, for example, particle capture and photodisintegration, or particle inelastic scattering and particle decay. Consequently, the above expression is also valid for negative values of $Q_{A \rightarrow B}$, that is, if nucleus *B* decays by (direct) particle emission. The factor $\rho e^{Q_{A \to B}/T}$ implies that $\lambda_{A \to B \to (C \text{ or } B')}$ becomes smaller for decreasing Q-value or increasing temperature, but becomes larger for increasing density. It will be shown in Chapter 5 how the interplay of temperature, density, Q-values, half-lives and reaction rates influences sensitively the most likely nucleosynthesis path. The decay constant for the case that three nuclei A, B and C come into equilibrium will be addressed in Problem 3.1. See also Schatz et al. (1998).

Example 3.3

The following situation occurs in hydrogen burning environments at high temperatures (thermonuclear explosions). Consider the specific case shown in Fig. 3.9a. The reaction ${}^{21}Mg + p \rightarrow \gamma + {}^{22}Al$ has a small estimated *Q*-value of $Q_{21Mg+p} = 163$ keV. At T = 0.6 GK, $\rho = 10^4$ g/cm³ and $X_H/M_H = 0.7$ the following decay constants are obtained from tabulated reaction rates

and β -decay half-lives:

$$\lambda_{A \to B} = \lambda_{^{21}Mg \to ^{22}Al} = 1.1 \times 10^3 \text{ s}^{-1}$$
$$\lambda_{A \to A'} = \lambda_{^{21}Mg \to ^{21}Na} = 5.6 \text{ s}^{-1}$$
$$\lambda_{B \to A} = \lambda_{^{22}Al \to ^{21}Mg} = 3.4 \times 10^7 \text{ s}^{-1}$$
$$\lambda_{B \to C} = \lambda_{^{22}Al \to ^{23}Si} = 3.1 \times 10^4 \text{ s}^{-1}$$
$$\lambda_{B \to B'} = \lambda_{^{22}Al \to ^{22}Mg} = 26.2 \text{ s}^{-1}$$

The above decay constants fulfill the conditions of Eqs. (3.55) and (3.56). Thus, an equilibrium between the abundances of ²¹Mg and ²²Al is quickly established. We would like to determine if the nucleosynthesis proceeds via ²¹Mg β^+ -decay or via sequential two-proton capture to ²³Si. From Eq. (3.61) we obtain

$$\begin{split} \lambda_{21}{}_{Mg \to 22}{}_{Al \to (2^{3}\text{Si or } 22}{}_{Mg}) &= \frac{\lambda_{21}{}_{Mg \to 2^{2}\text{Al}}}{\lambda_{22}{}_{Al \to 2^{1}\text{Mg}}} \left(\lambda_{22}{}_{Al \to 2^{3}\text{Si}} + \lambda_{22}{}_{Al \to 22}{}_{Mg}\right) \\ &= \frac{1.1 \times 10^{3} \,\text{s}^{-1}}{3.4 \times 10^{7} \,\text{s}^{-1}} \left(3.1 \times 10^{4} \,\text{s}^{-1} + 26.2 \,\text{s}^{-1}\right) = 1.0 \,\text{s}^{-1} \end{split}$$

which has to be compared to

$$\lambda_{21}_{Mg \to 21}_{Na} = 5.6 \, \text{s}^{-1}$$

Hence, the nucleosynthesis path via ${}^{21}Mg(\beta^+\nu){}^{21}Na$ is favored by a factor of 5.6/1.0 = 5.6.

Example 3.4

One of the most important reactions involving α -particles is the *triple-\alpha reaction* (3 α). It proceeds in two steps: (i) $\alpha + \alpha \rightarrow {}^{8}Be$, and (ii) ${}^{8}Be + \alpha \rightarrow {}^{12}C$. The *Q*value for the first step is $Q_{\alpha+\alpha\rightarrow^8Be} = -92.1$ keV and, therefore, ⁸Be is particle unstable (that is, it decays by breaking up into two α -particles). This breakup is much faster compared to the fusion of two α -particles into ⁸Be and, consequently, an equilibrium is established between the abundances of ⁴He and ⁸Be. The second step involves the capture of another α -particle on the small equilibrium abundance of ⁸Be, as shown in Fig. 3.9b. Estimate the decay constant, $\lambda_{\alpha+\alpha+\alpha\rightarrow^{12}C}$, for the 3α reaction at a temperature of T = 0.3 GK and density of $\rho = 10^5$ g/cm³, assuming a mass fraction of $X_{\alpha} = 1$ and $N_A \langle \sigma v \rangle_{\alpha + {}^8Be \rightarrow {}^{12}C} =$ 1.17×10^{-2} cm³ mol⁻¹ s⁻¹ (Caughlan and Fowler 1988).

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Fig. 3.9 Sections of the nuclidic chart depicting reaction rate equilibria involving (a) $(p,\gamma) \leftrightarrow (\gamma,p)$ reactions and (b) the 3α reaction. In each case, equilibrium has been achieved between the two nuclei shown as gray squares.

From Eqs. (3.61) and (3.62) one finds

$$\lambda_{\alpha+\alpha+\alpha\to^{12}C} = 3N_{\alpha} \left(\frac{h^2}{2\pi}\right)^{3/2} \frac{1}{(m_{\alpha\alpha}kT)^{3/2}} \frac{g_{^8Be}}{g_{\alpha}g_{\alpha}} e^{Q_{\alpha+\alpha\to^8Be}/kT} \lambda_{^8Be+\alpha\to^{12}C}$$
$$= 1.0133 \times 10^{-10} \rho \frac{X_{\alpha}}{M_{\alpha}} 3 \left(\frac{M_{^8Be}}{M_{\alpha}M_{\alpha}}\right)^{3/2} \frac{g_{^8Be}}{g_{\alpha}g_{\alpha}} T_9^{-3/2}$$
$$\times e^{11.605 Q_{\alpha+\alpha\to^8Be}/T_9} \lambda_{^8Be+\alpha\to^{12}C}$$

Since three identical particles are destroyed by the 3α reaction, we have $3r_{\alpha\alpha\alpha} = N_{\alpha}\lambda_{\alpha\alpha\alpha}$ and a factor of 3 has been included in the above expression. At this temperature we adopt for all normalized partition functions a value of $G_i^{\text{norm}} = 1$ (Rauscher and Thielemann 2000). The spins of the α -particle and of ⁸Be are $j_i = 0$, thus $g_i = 1$.

With the substitution $\lambda_{^{8}Be+\alpha\rightarrow^{12}C} = \rho(X_{\alpha}/M_{\alpha})N_{A}\langle\sigma v\rangle_{^{8}Be+\alpha\rightarrow^{12}C}$ (see Eq. (3.23)) we find

$$\lambda_{\alpha+\alpha+\alpha\to^{12}C} = 1.0133 \times 10^{-10} (10^5)^2 \left(\frac{1}{4.0}\right)^2 3 \left(\frac{8.0}{4.0 \cdot 4.0}\right)^{3/2} (0.3)^{-3/2} \times e^{-11.605 \cdot 0.0921/0.3} \cdot 1.17 \times 10^{-2} = 1.35 \times 10^{-4} \,\mathrm{s}^{-1}$$

At very low (T < 100 MK) and very high (T > 2 GK) stellar temperatures, the decay constant for the 3α reaction cannot be calculated with the above expression and the formalism becomes more involved (Nomoto, Thielemann and Miyaji 1985; Angulo et al. 1999).
3.1.7 Nuclear Energy Generation

Suppose that the forward reaction $0 + 1 \rightarrow 2 + 3$ is exothermic. The nuclear energy released per reaction is given by the *Q*-value. The energy production per unit time and unit mass is then given by

$$\varepsilon_{01\to23} = \frac{Q_{01\to23}r_{01\to23}}{\rho} = \frac{Q_{01\to23}}{\rho} \frac{N_0 N_1 \langle \sigma v \rangle_{01\to23}}{(1+\delta_{01})}$$
(3.63)

Similarly, for the endothermic reverse reaction we obtain

$$\varepsilon_{23\to01} = -\frac{Q_{01\to23}}{\rho} \frac{N_2 N_3 \langle \sigma v \rangle_{23\to01}}{(1+\delta_{23})}$$
(3.64)

$$\varepsilon_{\gamma 3 \to 01} = -\frac{Q_{01 \to 23}}{\rho} N_3 \lambda_\gamma(3) \tag{3.65}$$

At higher temperatures the reverse reaction has to be taken into account and the overall energy generation for the process $0 + 1 \leftrightarrow 2 + 3$ is $\varepsilon_{01 \rightarrow 23} + \varepsilon_{23 \rightarrow 01}$ for reactions involving particles with rest mass and $\varepsilon_{01 \rightarrow \gamma3} + \varepsilon_{\gamma3 \rightarrow 01}$ if species 2 is a photon.

If reactions produce electrons, positrons, or γ -rays, then their energy is retained in the stellar plasma. Neutrinos, on the other hand, interact so weakly with the medium that they escape from the site of thermonuclear burning, with the important exceptions of the big bang and supernova core collapse. Since the neutrino energy is usually not deposited in the star, it has to be subtracted from the *Q*-value when calculating the nuclear energy generation.

The energy generation rate can also be expressed by using Eqs. (3.20), (3.21), and (3.63) as

$$\varepsilon_{01\to23} = \frac{Q_{01\to23}}{\rho} \frac{N_0 \lambda_1(0)}{(1+\delta_{01})} = -\frac{Q_{01\to23}}{\rho(1+\delta_{01})} \left(\frac{dN_0}{dt}\right)_1$$
(3.66)

The total (time-integrated) released energy is obtained from

$$\int \varepsilon_{01\to23} dt = -\int_{N_{0,\text{initial}}}^{N_{0,\text{final}}} \frac{Q_{01\to23}}{\rho(1+\delta_{01})} (dN_0)_1 = \frac{Q_{01\to23}}{\rho(1+\delta_{01})} (\Delta N_0)_1$$
(3.67)

with $(\Delta N_0)_1 = N_{0,\text{initial}} - N_{0,\text{final}}$ the change in the abundance of nucleus 0 because of reactions with nucleus 1. Numerically we find from Eq. (1.13)

$$\int \varepsilon_{01\to23} dt = \frac{N_A Q_{01\to23}}{M_0 (1+\delta_{01})} (\Delta X_0)_1 \qquad (\text{MeV/g})$$
(3.68)

where $Q_{01\rightarrow 23}$ and M_0 are in units of MeV and u, respectively.

3.2 Nonresonant and Resonant Thermonuclear Reaction Rates

In previous sections we defined thermonuclear reaction rates, derived expressions for reaction rate ratios for forward and reverse reactions, and discussed reaction rate equilibria. In none of the expressions derived so far have we made specific reference to the nuclear reaction cross section $\sigma(E)$. However, this quantity is essential for calculating the reaction rate. In the following we will discuss how to derive the thermonuclear reaction rate for particle- and photon-induced reactions.

The reaction rate for a particle-induced reaction is given by (see Eq. (3.10))

$$N_A \langle \sigma v \rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E \,\sigma(E) \, e^{-E/kT} \, dE \tag{3.69}$$

Once the cross section $\sigma(E)$ has either been measured or estimated theoretically, the quantity $N_A \langle \sigma v \rangle$ can always be found by solving the above integral numerically. In fact, if the cross section has a complicated energy dependence there is usually no alternative to this procedure. On the other hand, if the energy dependence of the cross section is relatively simple, then the reaction rate can be calculated analytically. In this section we will discuss such analytical expressions for several reasons. First, an analytical rather than numerical description provides additional insight into stellar fusion reactions. Second, in certain situations (for example, for narrow resonances) the cross-section curve is not known explicitly, and hence the rate cannot be integrated numerically. Third, an analytical description also allows for improved estimates in cases where the reaction rate has to be extrapolated to the region of interest.

Two extreme cases will be discussed in detail, which apply to a large number of nuclear reactions. The first case refers to cross sections that vary smoothly with energy (*nonresonant cross sections*). The second case applies to cross sections which vary strongly in the vicinity of a particular energy (*resonant cross sections*).

3.2.1

Nonresonant Reaction Rates for Charged-Particle-Induced Reactions

The measured cross section for the ${}^{16}O(p,\gamma){}^{17}F$ reaction is shown in Fig. 3.10a. The cross section varies smoothly at higher energies, but drops at low energies by several orders of magnitude due to the decreasing transmission probability through the Coulomb barrier. The reaction rates may be obtained either by numerical integration or by using analytical expressions that will be derived in this section. At this point we introduce the *astrophysical S-factor*, *S*(*E*), defined

$$\sigma(E) \equiv \frac{1}{E} e^{-2\pi\eta} S(E)$$
(3.70)

This definition removes both the 1/E dependence of nuclear cross sections (see Eq. (2.49)) and the s-wave Coulomb barrier transmission probability (see Eq. (2.125)). Recall that the Gamow factor $e^{-2\pi\eta}$ is only an approximation for the s-wave transmission probability at energies well below the height of the Coulomb barrier. However, even if a particular fusion reaction proceeds via p- or d-partial waves, the removal of the strongly energy-dependent swave transmission probability from the cross section will result in an S-factor with a greatly reduced energy dependence. This is demonstrated in Fig. 3.10b, showing the S-factor for the ${}^{16}O(p,\gamma){}^{17}F$ reaction. Clearly, the S-factor varies far less with energy compared to the cross section. For reasons that will become clear later in this section, the S-factor is also a useful concept in the case of broad resonances. As an example, Fig. 3.11a shows the cross section for the ${}^{13}C(p,\gamma){}^{14}N$ reaction, while the corresponding S-factor is displayed in Fig. 3.11b. The much reduced energy dependence of the S-factor compared to the cross section is again evident. The above arguments are analogous to those we made in Section 2.4.2 in connection with the simple square-barrier potential and the removal of the transmission probability from the wave intensity in the nuclear interior (Fig. 2.14).

With the definition of the *S*-factor, we write for the nonresonant reaction rate (see Eqs. (2.125) and (3.69))

$$N_A \langle \sigma v \rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty e^{-2\pi\eta} S(E) e^{-E/kT} dE$$

= $\left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty \exp\left(-\frac{2\pi}{\hbar} \sqrt{\frac{m_{01}}{2E}} Z_0 Z_1 e^2\right) S(E) e^{-E/kT} dE$
(3.71)

with Z_i the charges of target and projectile. First, suppose that the astrophysical *S*-factor is constant, $S(E) = S_0$. We find

$$N_A \langle \sigma v \rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} S_0 \int_0^\infty e^{-2\pi\eta} e^{-E/kT} dE$$
(3.72)

with

$$kT = 0.086173 T_9 \quad (MeV)$$

$$2\pi\eta = 0.989534 Z_0 Z_1 \sqrt{\frac{M_0 M_1}{M_0 + M_1} \frac{1}{E}}$$

by



Fig. 3.10 (a) Experimental cross section and (b) astrophysical *S*-factor of the ¹⁶O(p, γ)¹⁷F reaction. Note the strongly varying cross section on a logarithmic scale in part (a) and the smooth behavior of the *S*-factor on a linear scale in part (b). Data from Angulo et al. (1999).

where the relative atomic masses M_i and the energy E are in units of u and MeV, respectively. The integrand has an interesting energy dependence. The factor $e^{-E/kT}$, originating from the Maxwell–Boltzmann distribution, approaches zero for large energies, whereas the term $e^{-1/\sqrt{E}}$, reflecting the Gamow factor, approaches zero for small energies. Clearly, the major contribution to the integral will come from energies where the product of both factors is near its maximum.

Figure 3.12a illustrates the situation for the reaction ${}^{12}C(\alpha,\gamma){}^{16}O$ at T = 0.2 GK. The dashed and the dashed-dotted lines show the factor $e^{-E/kT}$ and $e^{-2\pi\eta}$, respectively. The solid line shows the integrand $e^{-E/kT}e^{-2\pi\eta}$. Note the logarithmic scale, indicating the small magnitude of the integrand compared to the Gamow and Maxwell–Boltzmann factors. The solid line in Fig. 3.12b shows the integrand on a linear scale, displaying a relatively sharp peak.





Fig. 3.11 (a) Experimental cross section and (b) astrophysical *S*-factor of the ¹³C(p, γ)¹⁴N reaction. Note the much-reduced energy dependence in part (b). The low-energy *S*-factor tail of the broad resonance at $E \approx 0.5$ MeV can also be described by the nonresonant reaction rate formalism, as shown in this section. A narrow resonance at E = 0.45 MeV has been omitted from the figure. Data from Angulo et al. (1999).

Part (b) also indicates the maximum of the Maxwell–Boltzmann distribution (arrow), which occurs at kT = 17 keV. However, the integrand peaks at an energy of $E_0 = 315$ keV, which is much larger than kT, indicating that most of the reactions occur in the high-energy tail of the Maxwell–Boltzmann distribution. It appears that the Gamow factor effectively shifts the integrand to higher energies and, therefore, the integrand is commonly referred to as the *Gamow peak*. The Gamow peak represents the relatively narrow energy range over which most nuclear reactions occur in a stellar plasma.



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Fig. 3.12 (a) Maxwell–Boltzmann factor $(e^{-E/kT}; \text{ dashed line})$ and Gamow factor $(e^{-2\pi\eta}; \text{ dashed-dotted line})$ versus energy for the ${}^{12}\text{C}(\alpha,\gamma){}^{16}\text{O}$ reaction at a temperature of T = 0.2 GK. The product $e^{-E/kT}e^{-2\pi\eta}$, referred to as the Gamow peak, is shown as solid line. (b) The same Gamow factor

shown on a linear scale (solid line). The maximum occurs at $E_0 = 0.32$ MeV while the maximum of the Maxwell–Boltzmann distribution is located at kT = 0.017 MeV (arrow). The dotted line shows the Gaussian approximation of the Gamow peak.

The location E_0 of the maximum of the Gamow peak can be found from the first derivative of the integrand in Eq. (3.72) with respect to E,

$$\frac{d}{dE} \left(-\frac{2\pi}{\hbar} \sqrt{\frac{m_{01}}{2E}} Z_0 Z_1 e^2 - \frac{E}{kT} \right)_{E=E_0} = \frac{\pi}{\hbar} Z_0 Z_1 e^2 \sqrt{\frac{m_{01}}{2}} \frac{1}{E_0^{3/2}} - \frac{1}{kT} = 0 \quad (3.73)$$

Thus

$$E_{0} = \left[\left(\frac{\pi}{\hbar}\right)^{2} \left(Z_{0}Z_{1} e^{2}\right)^{2} \left(\frac{m_{01}}{2}\right) (kT)^{2} \right]^{1/3}$$

= 0.1220 $\left(Z_{0}^{2}Z_{1}^{2} \frac{M_{0}M_{1}}{M_{0} + M_{1}} T_{9}^{2}\right)^{1/3}$ (MeV) (3.74)

where in the numerical expression M_i are the relative atomic masses of projectile and target in units of u.

The energy E_0 is the most effective energy for nonresonant thermonuclear reactions. Figure 3.13 shows the Gamow peak energy E_0 versus temperature for a number of proton- and α -particle-induced reactions. It is obvious that the Gamow peak energy increases with increasing target-projectile charge. The open circles indicate the height V_C of the Coulomb barrier. It is important to point out that, except for the highest temperatures near T = 10 GK, we find $E_0 \ll V_C$ and thus the interacting charged nuclei must always tunnel through the Coulomb barrier.

Figure 3.14 shows the Gamow peak at a temperature of T = 30 MK for three reactions: (i) p + p, (ii) ${}^{12}C + p$, and (iii) ${}^{12}C + \alpha$. It demonstrates a crucial aspect of thermonuclear burning in stars. Not only does the Gamow peak shift to higher energies for increasing target and projectile charges, but the area under the curves decreases rapidly as well. Suppose, for example, that a mixture of different nuclei is present in the stellar plasma at a particular time. Then those reactions with the smallest Coulomb barrier account frequently for most of the nuclear energy generation and will be consumed most rapidly, while reactions with larger Coulomb barriers usually do not contribute significantly to the energy production.

The Gamow peak may be approximated by a Gaussian function having a maximum of the same size and of the same curvature at $E = E_0$. From Eq. (3.74) we write

$$\exp\left(-\frac{2\pi}{\hbar}\sqrt{\frac{m_{01}}{2E}}Z_0Z_1e^2 - \frac{E}{kT}\right) = \exp\left(-\frac{2E_0^{3/2}}{\sqrt{E}kT} - \frac{E}{kT}\right)$$
$$\approx \exp\left(-\frac{3E_0}{kT}\right)\exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right]$$
(3.75)

where the 1/e width Δ of the Gaussian is obtained from the requirement that the second derivatives match at E_0 . Thus

$$\frac{d^2}{dE^2} \left(\frac{2E_0^{3/2}}{\sqrt{E}kT} + \frac{E}{kT} \right)_{E=E_0} = \frac{3}{2} \frac{1}{E_0 kT}$$
(3.76)

$$\frac{d^2}{dE^2} \left(\frac{E - E_0}{\Delta/2}\right)_{E=E_0}^2 = \frac{2}{(\Delta/2)^2}$$
(3.77)

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Fig. 3.13 Location of the Gamow peak maximum versus temperature for a number of proton-induced (upper part) and α -particle-induced (lower part) reactions. The open circles on the right-hand side show the Coulomb barrier height, $V_C = 1.44 Z_0 Z_1 / R_0$, with V_C and R_0 in units of MeV and fm, respectively.

Setting the right-hand sides of the last two expressions equal and solving for Δ gives

$$\Delta = \frac{4}{\sqrt{3}}\sqrt{E_0kT} = 0.2368 \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} T_9^5\right)^{1/6}$$
(MeV) (3.78)

Since usually $kT \ll E_0$, it is apparent that the width Δ of the Gamow peak is smaller than E_0 . Figure 3.15 shows the Gamow peak width Δ versus temperature for a number of proton- and α -particle-induced reactions. It can be seen that the Gamow peak width increases with increasing Coulomb barrier. Thermonuclear reactions occur mainly over an energy window from $E_0 - \Delta/2$





Fig. 3.14 The Gamow peaks for the p + p, ${}^{12}C$ + p, and ${}^{12}C$ + α reactions at a temperature of *T* = 0.03 GK.

to $E_0 + \Delta/2$, except in the case of narrow resonances (see later). For increasing charges of target or projectile, this window shifts to higher energies and becomes broader.

The nonresonant thermonuclear reaction rates can be calculated by replacing the Gamow peak with a Gaussian. From Eqs. (3.72) and (3.75) one finds

$$N_A \langle \sigma v \rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} S_0 \int_0^\infty e^{-2\pi\eta} e^{-E/kT} dE$$

$$\approx \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} S_0 e^{-3E_0/kT} \int_0^\infty \exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right] dE$$
(3.79)

The lower integration limit can be extended to $-\infty$ without introducing a significant error. The value of the integral over the Gaussian is then $\sqrt{\pi}\Delta/2$. For a constant *S*-factor we obtain

$$N_A \langle \sigma v \rangle = N_A \sqrt{\frac{2}{m_{01}}} \frac{\Delta}{(kT)^{3/2}} S_0 e^{-3E_0/kT}$$
(3.80)

Alternatively, one finds with the substitution $\tau \equiv 3E_0/(kT)$ and Eqs. (3.74) and (3.78)

$$N_A \langle \sigma v \rangle = N_A \sqrt{\frac{2}{m_{01}}} \frac{\Delta}{(kT)^{3/2}} S_0 e^{-\tau} \tau^2 \frac{(kT)^2}{9E_0^2}$$

= $\left(\frac{4}{3}\right)^{3/2} \frac{\hbar}{\pi} \frac{N_A}{m_{01} Z_0 Z_1 e^2} S_0 \tau^2 e^{-\tau}$ (3.81)

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Fig. 3.15 The width of the Gamow peak versus temperature for a number of proton-induced (upper part) and α -particle-induced (lower part) reactions.

One of the most striking features of thermonuclear reaction rates is their temperature dependence. The temperature dependence of $N_A \langle \sigma v \rangle$ (and of the energy production rate ε) near some energy $T = T_0$ can be derived by introducing a power law

$$N_A \langle \sigma v \rangle_T = N_A \langle \sigma v \rangle_{T_0} (T/T_0)^n \tag{3.82}$$

where

$$\ln N_A \langle \sigma v \rangle_T = \ln N_A \langle \sigma v \rangle_{T_0} + n(\ln T - \ln T_0)$$
(3.83)

$$\frac{\partial \ln N_A \langle \sigma v \rangle_T}{\partial \ln T} = n \tag{3.84}$$

With $\tau = 3E_0/(kT) = cT^{2/3}/T = cT^{-1/3}$ and $N_A \langle \sigma v \rangle_T = c'T^{-2/3}e^{-\tau}$, we may also write

$$\ln N_A \langle \sigma v \rangle_T = \ln c' - \frac{2}{3} \ln T - \tau \tag{3.85}$$

$$n = \frac{\partial \ln N_A \langle \sigma v \rangle_T}{\partial \ln T} = -\frac{2}{3} - \frac{\partial \tau}{\partial \ln T} = -\frac{2}{3} - \tau \frac{\partial \ln(cT^{-1/3})}{\partial \ln T} = -\frac{2}{3} + \frac{\tau}{3}$$
(3.86)

Hence

$$N_A \langle \sigma v \rangle_T = N_A \langle \sigma v \rangle_{T_0} (T/T_0)^{(\tau-2)/3}$$
(3.87)

The parameter $\tau = 3E_0/(kT)$ is numerically given by

$$\tau = 4.2487 \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} \frac{1}{T_9} \right)^{1/3}$$
(3.88)

Values of τ are shown in Fig. 3.16 versus temperature for a number of reactions. For example, at T = 15 MK one obtains $\tau = 13.6$ for the p + p reaction, yielding for the exponent of *T* a value of $n \approx 3.9$. On the other hand, at T = 200 MK we obtain $\tau = 54.88$ for the ¹²C + α reaction, resulting in $n \approx 17.6$. Clearly, the striking temperature dependence of thermonuclear reaction rates has an important impact on stellar models. Small temperature fluctuations, which are likely to occur during the course of stellar evolution, will cause dramatic changes in energy production. Therefore, either an effective mechanism must exist in order to stabilize the star, or in circumstances where this is not possible, a thermonuclear explosion is likely to occur.

Two corrections to the nonresonant reaction rate formalism derived so far will now be considered. The first correction is necessary since we have replaced the asymmetric Gamow peak by a symmetric Gaussian (see Eq. (3.75)) where the area under the latter function is given by $e^{-\tau}\sqrt{\pi}\Delta/2$ (see Eq. (3.80)). Figure 3.12b compares the two functions for the reaction ${}^{12}C(\alpha,\gamma){}^{16}O$ at T = 0.2 GK. The solid line shows the Gamow peak while the dotted curve displays the Gaussian approximation. The reaction rate must be multiplied by a correction factor that represents the ratio of the areas under these two curves,

$$F(\tau) = \frac{\int_0^\infty \exp\left(-\frac{2\pi}{\hbar}\sqrt{\frac{m_{01}}{2E}}Z_0Z_1e^2 - \frac{E}{kT}\right) dE}{e^{-\tau}\sqrt{\pi}\Delta/2}$$
$$= \frac{2}{\sqrt{\pi}\sqrt{E_0kT}}\frac{\sqrt{3}}{4}e^{\tau}\int_0^\infty \exp\left(-\frac{2E_0^{3/2}}{kT}\frac{1}{\sqrt{E_0\epsilon}} - \frac{E_0\epsilon}{kT}\right)E_0 d\epsilon$$
$$= \sqrt{\frac{\tau}{\pi}}\frac{e^{\tau}}{2}\int_0^\infty \exp\left[-\frac{\tau}{3}\left(\epsilon + \frac{2}{\sqrt{\epsilon}}\right)\right]d\epsilon$$
(3.89)

where we have introduced the dimensionless variable $\epsilon \equiv E/E_0$. It can be seen that the correction factor *F* is a function of τ only. It is also clear from

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Fig. 3.16 Numerical values of the parameter τ versus temperature for a number of proton- and α -particle-induced reactions. Note that τ is dimensionless.

Fig. 3.16 that τ is usually a relatively large number. Therefore, it is of advantage to expand *F* in terms of a small parameter that varies inversely with τ . The result is (see Problem 3.2)

$$F(\tau) \approx 1 + \frac{5}{12\tau} \tag{3.90}$$

Figure 3.17 shows values of $F(\tau)$ versus temperature for a number of reactions. It can be seen that the correction factor is usually small (less than a few percent) at low temperatures. Its magnitude increases with rising temperature and lowering of the Coulomb barrier.

A second correction is necessary since for many nonresonant reactions the *S*-factor is not constant, but varies with energy. In most cases, it is sufficient to expand the experimental or theoretical *S*-factor into a Taylor series around E = 0,

$$S(E) \approx S(0) + S'(0)E + \frac{1}{2}S''(0)E^2$$
(3.91)

where the primes indicate derivatives with respect to *E*. Substitution of this expansion into Eq. (3.71) yields a sum of integrals, where each integral can be expanded into powers of $1/\tau$. As a result of this procedure, which is not given explicitly here, one has to replace in Eq. (3.81) the constant *S*₀ by an effective *S*-factor. The result is (Fowler, Caughlan and Zimmerman 1967)

$$N_A \langle \sigma v \rangle = \left(\frac{4}{3}\right)^{3/2} \frac{\hbar}{\pi} \frac{N_A}{m_{01} Z_0 Z_1 e^2} S_{\text{eff}} \tau^2 e^{-\tau}$$
(3.92)





Fig. 3.17 Correction factor $F(\tau)$ versus temperature for a number of reactions.

$$S_{\text{eff}}(E_0) = S(0) \left[1 + \frac{5}{12\tau} + \frac{S'(0)}{S(0)} \left(E_0 + \frac{35}{36} kT \right) + \frac{1}{2} \frac{S''(0)}{S(0)} \left(E_0^2 + \frac{89}{36} E_0 kT \right) \right]$$
(3.93)

The first terms in the square bracket correspond to the factor $F(\tau)$ due to the asymmetry of the Gamow peak, while the other terms arise from corrections due to the *S*-factor variation with energy. Numerically, one finds (Lang 1974)

$$N_A \langle \sigma v \rangle = \frac{C_1}{T_9^{2/3}} e^{-C_2/T_9^{1/3}} \left(1 + C_3 T_9^{1/3} + C_4 T_9^{2/3} + C_5 T_9 + C_6 T_9^{4/3} + C_7 T_9^{5/3} \right) \quad (\text{cm}^3 \,\text{mol}^{-1} \,\text{s}^{-1})$$
(3.94)

$$C_1 = 7.8324 \times 10^9 \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} \right)^{1/6} S(0) \sqrt{\frac{M_0 + M_1}{M_0 M_1}}$$

$$C_2 = 4.2475 \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} \right)^{1/3}$$

$$C_3 = 9.810 \times 10^{-2} \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} \right)^{-1/3}$$

$$C_4 = 0.1220 \frac{S'(0)}{S(0)} \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} \right)^{1/3}$$

$$C_5 = 8.377 \times 10^{-2} \frac{S'(0)}{S(0)}$$

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$$C_6 = 7.442 \times 10^{-3} \frac{S''(0)}{S(0)} \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} \right)^{2/3}$$
$$C_7 = 1.299 \times 10^{-2} \frac{S''(0)}{S(0)} \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} \right)^{1/3}$$

where M_i is the relative atomic mass in u, and the quantities S(0), S'(0), and S''(0) are in units of MeV·b, b, and b/MeV, respectively.

Figure 3.18 shows schematically three situations that are frequently encountered in practice. The data shown in part (a) display a very slowly varying S-factor. In this case, a linear Taylor expansion (solid line) seems appropriate for describing the data. If the Gamow peak is located, say, around E =0.7 MeV, then the nonresonant reaction rates may be calculated from the fit coefficients S(0) and S'(0) to high accuracy. Depending on the hydrodynamical conditions at the astrophysical environment, however, the Gamow peak may be located at energies which are not directly accessible with present experimental techniques (say, below E = 0.3 MeV in part (a) of the figure). In this case, the Taylor expansion may be used to extrapolate the S-factor to the energy range of the Gamow peak at low energies. This procedure represents the simplest way to estimate the reaction rates from cross section data obtained at higher energies if no data are available in the Gamow peak region. Although frequently applied in practice, one has to be careful with this method, and a more reliable S-factor extrapolation based on theoretical nuclear models (Chapter 2) is desirable in this case.

A different situation is shown in part (b). Here, the data below E = 0.65 MeV may be best described by a quadratic Taylor expansion and the reaction rates are then evaluated using Eq. (3.94). However, at higher energies the Taylor expansion will diverge (in this case positively) and does no longer describe the data. Thus, the calculated reaction rates become inaccurate at temperatures at which a substantial fraction of the Gamow peak is located beyond E = 0.65 MeV. For this reason, the nonresonant reaction rate expression (see Eq. (3.94)) is sometimes multiplied by a cutoff factor (Fowler, Caughlan and Zimmerman 1975)

$$f_{\rm cutoff} \approx e^{-(T_9/T_{9,\rm cutoff})^2} \tag{3.95}$$

where $T_{9,\text{cutoff}}$ corresponds to the temperature at which a substantial fraction of the Gamow peak lies in an energy region at which the *S*-factor parametrization starts to deviate from the data (the vertical dotted line in Fig. 3.18b). Beyond this temperature, the reaction rates have to be evaluated by different means.

Consider now the situation shown in Fig. 3.18c. In this case, the data display a resonance at E = 0.8 MeV corresponding to an *S*-factor which varies strongly with energy. However, it can be seen that below $E \approx 0.55$ MeV the wing of the



Fig. 3.18 Schematic representation of the *S*-factor versus energy for (a) a very slowly varying *S*-factor curve; (b) an energy-dependent *S*-factor; and (c) a broad resonance. See discussion in the text.

resonance varies rather slowly. Therefore, if one is mainly interested in stellar temperatures at which the Gamow peak is located below E = 0.55 MeV, then one may apply the nonresonant reaction rate formalism to the low-energy tail of the broad resonance (see Eq. (3.94)). As was the case before, the calculated reaction rate has to be cut off at higher temperatures corresponding to energies at which the *S*-factor expansion deviates from the data (dotted line in Fig. 3.18c).

The *S*-factor for nonresonant reactions sometimes shows a strong energy dependence so that a Taylor series expansion is no longer applicable. Although analytical descriptions are reported in the literature for such cases (see, for example, Fowler, Caughlan and Zimmerman 1975), it is usually more reliable to integrate the reaction rates numerically (see Eq. (3.69)).

Example 3.5

The measured *S*-factor for the reaction ${}^{12}C(p,\gamma){}^{13}N$ below E = 0.5 MeV is shown in Fig. 3.19. A broad resonance appears at $E \approx 0.4$ MeV. The *S*factor below E = 0.23 MeV varies smoothly with energy and has been expanded around E = 0 into a quadratic Taylor series, with coefficients of S(0) $= 1.34 \times 10^{-3}$ MeV·b, $S'(0) = 2.6 \times 10^{-3}$ b, and $S''(0) = 8.3 \times 10^{-2}$ b/MeV (Adelberger et al. 1998). (i) For a temperature of T = 0.03 GK, determine the location and width of the Gamow peak, and the temperature sensitivity of the reaction rates. (ii) Determine the maximum temperature at which the reaction rates can be calculated reliably with the *S*-factor parametrization given above.

From Eqs. (3.74), (3.78), (3.87), and (3.88) we find

$$E_0 = 0.1220 \left(1^2 6^2 \frac{1.0 \cdot 12.0}{1.0 + 12.0} 0.03^2 \right)^{1/3} \text{ MeV} = 0.038 \text{ MeV}$$
$$\Delta = 0.2368 \left(1^2 6^2 \frac{1.0 \cdot 12.0}{1.0 + 12.0} 0.03^5 \right)^{1/6} \text{ MeV} = 0.023 \text{ MeV}$$
$$\tau = 4.2487 \left(1^2 6^2 \frac{1.0 \cdot 12.0}{1.0 + 12.0} \frac{1}{0.03} \right)^{1/3} = 44.0$$

and thus $N_A \langle \sigma v \rangle_T \sim (T/T_0)^{(44.0-2)/3} = (T/T_0)^{14.0}$

The quadratic *S*-factor expansion describes the data reliably only below E = 0.23 MeV. We ask for the temperature range at which an insignificant fraction of the Gamow peak lies beyond E = 0.23 MeV, that is, $E_0(T) + \Delta(T) = 0.23$ MeV. From Figs. 3.13 and 3.15 it can be seen that this condition is fulfilled only at $T \le 0.2$ GK. Therefore, we expect that the reaction rates calculated with the given *S*-factor parameterization are reliable below this temperature. The situation is illustrated in Fig. 3.19.

3.2.2

Nonresonant Reaction Rates for Neutron-Induced Reactions

Neutrons that are produced in a star quickly thermalize and their velocities are given by a Maxwell–Boltzmann distribution. For a smoothly varying neutron cross section, the reactions are most likely to occur near the maximum of the Maxwell–Boltzmann distribution, that is, at thermal energies of $E_T = kT$ or thermal velocities of $v_T = \sqrt{2kT/m_{01}}$ (see Section 3.1.1 and Fig. 3.1a). For s-wave neutrons ($\ell = 0$) of low velocity, the reaction cross section is inversely





Fig. 3.19 Experimental *S*-factor versus energy for the ${}^{12}C(p,\gamma){}^{13}N$ reaction. The solid line represents an *S*-factor expansion that describes the data below *E* = 0.23 MeV. The dashed line shows the Gamow peak for a temperature of *T* = 0.2 GK. Data are from Angulo et al. (1999).

proportional to the neutron velocity (see Eq. (2.207)),

$$\sigma \sim \frac{1}{v} \sim \frac{1}{\sqrt{E}} \tag{3.96}$$

Strictly speaking, if charged particles are released in neutron-induced processes, the cross section is modified by the transmission probability of the emitted particle through the Coulomb and centripetal barriers. However, since many neutron-induced reactions are exothermic with Q-values in excess of several MeV, the transmission coefficient of the charged particle is approximately constant. Under these circumstances the 1/v law applies not only to (n,γ) reactions but also to reactions such as (n,p) or (n,α) (Fig. 4.15a). Furthermore, the 1/v law is also valid in certain cases where resonant contributions give rise to a smoothly varying reaction cross section for s-wave neutrons. As an example, suppose that a neutron-induced reaction proceeds through the low-energy wing of a broad resonance. Setting $E \ll E_r$ in the Breit–Wigner formula (see Eq. (2.185)) and using the low-energy dependence of the neutron partial width (assuming that the partial width for the exit channel is approximately constant) yields $\sigma_{\ell=0} \sim (1/v^2)\Gamma_{\ell=0} \sim (1/v^2)v \sim 1/v$. Neutron capture by heavy nuclei with large $Q_{n\gamma}$ values is another important example. In this case, the reaction proceeds through many broad and overlapping resonances. These resonances are difficult to resolve experimentally so that a measurement yields an average cross section that varies smoothly with energy. The cross section is then given by $\sigma_{\ell=0} = \sigma_{\ell=0}^{\max} \cdot \hat{T} \sim 1/v$ (see Eq. (2.207)), where \hat{T} is the s-wave transmission coefficient.

For $\sigma \sim 1/v$, or $S \equiv \sigma v$ = const, we obtain (see Eqs. (3.3) and (3.4))

$$N_A \langle \sigma v \rangle = N_A \int_0^\infty v P(v) \sigma(v) \, dv = N_A \sigma v = N_A S = \text{const}$$
(3.97)

The reaction rate is independent of temperature and, in principle, could be determined from σ measured at any velocity v. In practice, however, the nonresonant neutron cross section does not always follow the simple 1/v law for any of the following reasons: (i) the s-wave neutron energies are no longer small, (ii) a new reaction channel becomes energetically accessible, or (iii) higher partial waves may contribute to the neutron cross section.

In the latter case, the velocity or energy dependence can be found from the expression $\sigma_{\ell} \sim (1/v^2)\Gamma_{\ell}$. At low energies we can use $\Gamma_{\ell}(E) \sim (vR)^{2\ell+1} \sim E^{\ell+1/2}$ (Section 2.5.4) and obtain $\sigma_{\ell} \sim v^{-1}$, v, v^3 (or $\sigma_{\ell} \sim E^{-1/2}$, $E^{1/2}$, $E^{3/2}$) for $\ell = 0, 1, 2$, respectively. Again, the above dependences on v (or E) do not apply in exceptional situations where the neutron binding energy (or $Q_{n\gamma}$) becomes comparable to the neutron kinetic energy since in such cases the influence of the exit channel must also be taken into account. With the above energy dependences of the different partial waves, the reaction rate is

$$N_A \langle \sigma v \rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E\sigma(E) \, e^{-E/kT} \, dE \sim \int_0^\infty E^{\ell+1/2} e^{-E/kT} \, dE$$
(3.98)

The integrand, $E^{\ell+1/2}e^{-E/kT}$, represents the stellar energy window in which most of the nonresonant neutron-induced reactions take place. It is plotted in Fig. 3.20 for different ℓ -values (solid lines) and is compared to the Maxwell– Boltzmann factor, $E e^{-E/kT}$ (dashed line). All curves are plotted for kT =30 keV and are normalized to the same maximum value. It can be seen that the centripetal barrier shifts the window of effective stellar energies. The maximum of the integrand occurs at $E_{\text{max}} = (\ell + 1/2)kT$. The influence of the centripetal barrier on nonresonant neutron-induced reaction rates is far smaller compared to the influence of the Coulomb barrier on nonresonant charged-particle reactions. As an approximate rule it can be assumed that the Maxwell–Boltzmann distribution provides a reliable estimate for the effective energy window in the case of nonresonant neutron-induced reactions.

If the product $S \equiv \sigma v$ is not constant but varies with velocity, it may be expanded into a Taylor series around E = 0 in terms of v or \sqrt{E} ,

$$\sigma v = S(\sqrt{E}) \approx S(0) + \dot{S}(0)\sqrt{E} + \frac{1}{2}\ddot{S}(0)E$$
(3.99)

where the dots indicate derivatives with respect to $\sqrt{E} \sim v$ and S(0), $\dot{S}(0)$, $\ddot{S}(0)$ are empirical constants. The energy dependence of the cross section is





Fig. 3.20 The factor $E^{\ell+1/2}e^{-E/kT}$ versus neutron energy, representing the stellar energy window in which most of the nonresonant neutron-induced reactions take place, for different values of the orbital angular momentum ℓ . The dashed curve shows the Maxwell–Boltzmann factor $Ee^{-E/kT}$ for comparison. All curves are calculated for kT = 30 keV.

then given by

$$\sigma(E) \approx \sqrt{\frac{m_{01}}{2E}} \left(S(0) + \dot{S}(0)\sqrt{E} + \frac{1}{2}\ddot{S}(0)E \right)$$
(3.100)

Substitution into Eq. (3.10) yields for the reaction rate (Problem 3.3)

$$N_A \langle \sigma v \rangle = N_A \left(S(0) + \frac{2}{\sqrt{\pi}} \dot{S}(0) \sqrt{kT} + \frac{3}{4} \ddot{S}(0) kT \right)$$
(3.101)

Numerically we find

$$N_A \langle \sigma v \rangle = 6.022 \times 10^{23} S(0) \left(1 + 0.3312 \frac{\dot{S}(0)}{S(0)} \sqrt{T_9} + 0.06463 \frac{\ddot{S}(0)}{S(0)} T_9 \right) \qquad (\text{cm}^3 \,\text{mol}^{-1} \,\text{s}^{-1})$$
(3.102)

with $\dot{S}(0)/S(0)$ and $\ddot{S}(0)/S(0)$ in units of MeV^{-1/2} and MeV⁻¹, respectively.

For many neutron-induced reactions, especially neutron captures, the reaction rate is expressed in terms of the Maxwellian-averaged cross section (Section 3.1.1),

$$N_A \langle \sigma v \rangle = N_A \langle \sigma \rangle_T v_T = N_A \frac{4}{v_T \sqrt{\pi}} \int_0^\infty v \sigma(v) \left(\frac{v}{v_T}\right)^2 e^{-(v/v_T)^2} dv \qquad (3.103)$$

For s-wave neutrons at low energy, $\sigma v = v_T \sigma(v_T) = v_T \sigma_T = \text{const}$, it follows

$$N_A \langle \sigma \rangle_T v_T = N_A \frac{4}{v_T \sqrt{\pi}} v_T \sigma(v_T) \int_0^\infty \left(\frac{v}{v_T}\right)^2 e^{-(v/v_T)^2} dv = N_A v_T \sigma(v_T)$$
(3.104)

and the Maxwellian-averaged cross section, $\langle \sigma \rangle_T$, is equal to the cross section measured at thermal velocity, σ_T . For different velocity dependences, for example, $\sigma = \text{const}$ or $\sigma \sim 1/v^2$, direct substitution into the above equation yields $\langle \sigma \rangle_T = 1.13\sigma_T$, while for p-wave capture ($\sigma \sim v$) one obtains $\langle \sigma \rangle_T =$ $1.5\sigma_T$. Thus, for a smoothly changing cross section, a measurement of σ at a single velocity, v_T , provides a reaction rate that is not too far off from its true magnitude. However, in order to obtain precise values for the reaction rate, the cross section is measured in practice over a range of neutron energies in the effective stellar window that is given by the Maxwell–Boltzmann distribution. For more details, see Beer, Voss and Winters (1992).

3.2.3

Nonresonant Reaction Rates for Photon-Induced Reactions

The majority of astrophysically important photodisintegration reactions, $\gamma + 3 \rightarrow 0 + 1$, have not been measured directly. Their reaction rates are most conveniently derived from the corresponding reverse particle-induced reaction rate by applying the reciprocity theorem (Section 3.1.4). Nevertheless, a number of photodisintegration reactions have been measured directly and it is interesting to investigate some general properties of their decay constants. From Eqs. (3.18) and (3.28) we find

$$\lambda_{\gamma}(3) = \frac{8\pi m_{01}}{h^3} \frac{(2j_0+1)(2j_1+1)}{(2j_3+1)} \int_0^\infty \frac{E_{\gamma} - Q_{01 \to \gamma 3}}{e^{E_{\gamma}/kT} - 1} \sigma_{01 \to \gamma 3} \, dE_{\gamma} \tag{3.105}$$

with $E_{01} = E_{\gamma} - Q_{01 \rightarrow \gamma 3}$. Recall that the above expression applies only to the forward and reverse reaction for a specific pair of initial and final states (Section 3.1.4). For simplicity we will assume that the photodisintegration proceeds between the ground states of nuclei 3 and 0, while nucleus 1 denotes a light particle (p, n or α). In this case, $Q_{01 \rightarrow \gamma 3}$ is the ground-state *Q*-value of the forward reaction. With the approximation $e^{E_{\gamma}/kT} - 1 \approx e^{E_{\gamma}/kT}$ (Section 3.1.4) one obtains

$$\lambda_{\gamma}(3) = \frac{8\pi m_{01}}{h^3} \frac{(2j_0+1)(2j_1+1)}{(2j_3+1)} \int_0^\infty (E_{\gamma} - Q_{01\to\gamma3}) e^{-E_{\gamma}/kT} \sigma_{01\to\gamma3} dE_{\gamma}$$
(3.106)

We must distinguish between the emission of charged particles, (γ, p) or (γ, α) , and the emission of neutrons, (γ, n) . For nonresonant charged-particle emission, the cross section is given by Eq. (3.70). For a nearly constant *S*-factor the decay constant is

$$\lambda_{\gamma}(3) \sim \int_{0}^{\infty} (E_{\gamma} - Q_{01 \to \gamma 3}) e^{-E_{\gamma}/kT} \frac{e^{-2\pi\eta}}{E_{01}} S(E_{01}) dE_{\gamma}$$
$$\sim S(E_{0}) e^{-Q_{01 \to \gamma 3}/kT} \int_{0}^{\infty} e^{-2\pi\eta} e^{-E_{01}/kT} dE_{01}$$
(3.107)

The integrand is equal to the Gamow peak (Section 3.2.1) for the forward reaction. Clearly, the concept of a Gamow peak is also useful for photodisintegration reactions involving the emission of charged particles. Since the Gamow peak is located at E_0 and has a 1/e width of Δ (see Eqs. (3.74) and (3.78)), we expect that for the photodisintegration reaction the γ -ray energy range of effective stellar burning is centered at

$$E_{\gamma}^{\rm eff} = E_0 + Q_{01 \to \gamma 3} \tag{3.108}$$

and has a width of Δ . For rising temperature, E_0 will increase and thus E_{γ}^{eff} will shift to a larger value. It is also apparent from Eq. (3.107) that, compared to the rate of the forward capture reaction (see Eq. (3.72)), the decay constant $\lambda_{\gamma}(3)$ has an additional temperature dependence through the term $e^{-Q/kT}$.

The situation is very different if a neutron is emitted in a nonresonant photodisintegration reaction. For small neutron energies, we found that the energy dependence of the (n,γ) cross section is given by $\sigma_{\ell} \sim E^{\ell-1/2}$ (Section 3.2.2). This cross section behavior was derived under the assumption of relatively small neutron energies compared to the neutron binding energy. Most (n,γ) reactions have relatively large *Q*-values and, therefore, we can substitute this expression into Eq. (3.106). Thus

$$\lambda_{\gamma}(3) \sim \int_{0}^{\infty} (E_{\gamma} - Q_{01 \to \gamma 3}) e^{-E_{\gamma}/kT} E_{01}^{\ell - 1/2} dE_{\gamma} \sim \int_{0}^{\infty} e^{-E_{\gamma}/kT} (E_{\gamma} - Q_{01 \to \gamma 3})^{\ell + 1/2} dE_{\gamma}$$
(3.109)

It was already mentioned that for neutron-capture reactions the energy window of effective stellar burning is located at $E_n^{\text{eff}} = (\ell + 1/2)kT$ (Fig. 3.20). Hence we expect that the effective energy window for the reverse (γ ,n) reaction is located at $E_{\gamma}^{\text{eff}} = (\ell + 1/2)kT + Q_{n\gamma}$ (Problem 3.4). As an example, Fig. 3.21 shows the integrand in Eq. (3.109) for the ¹⁴⁸Gd(γ ,n)¹⁴⁷Gd reaction. The two curves correspond to temperatures of T = 2 and 3 GK and are calculated assuming emission of s-wave neutrons ($\ell = 0$). The *Q*-value for the ¹⁴⁷Gd(n, γ)¹⁴⁸Gd reaction amounts to $Q_{n\gamma} = 8.984$ MeV. Thus, the photodisintegration reaction can only proceed for γ -ray energies in excess of the threshold value $E_{\gamma} = Q_{n\gamma}$. Between T = 2 and 3 GK, the maximum of the integrand



Fig. 3.21 Integrand in Eq. (3.109) versus γ -ray energy at two temperatures (T = 2 and 3 GK) for the photodisintegration reaction ¹⁴⁸Gd(γ ,n)¹⁴⁷Gd. The ground-state Q-value for the (forward) capture reaction amounts to $Q_{n\gamma} = 8.984$ MeV (Audi, Wap-

stra and Thibault 2003). This value is equal to the neutron separation energy of ¹⁴⁸Gd. Both curves are plotted for the emission of s-wave neutrons ($\ell = 0$). The integrand represents the γ -ray energy window of effective stellar burning.

shifts by only \approx 43 keV, a value which is barely noticeable in the figure. Therefore, the effective energy window for (γ ,n) reactions of astrophysical interest is located closely to the reaction threshold, independent of the temperature. This behavior is in stark contrast compared to the much larger energy shift of the Gamow peak in (γ ,p) or (γ , α) reactions. Also, note that the magnitude of the integrand increases by more than a factor of 10⁷ between *T* = 2 and 3 GK, emphasizing the dramatic temperature dependence of decay constants for (γ ,n) reactions.

3.2.4

Narrow-Resonance Reaction Rates

In the previous sections, reaction rates for smoothly varying *S*-factors were discussed. In this section we will discuss the other extreme case, that is, strongly varying *S*-factors caused by resonances. We will consider here resonances which are isolated and narrow. The first condition implies that the level density in the compound nucleus is relatively small so that the resonances do not overlap significantly in amplitude. Several different definitions are used in the literature for a *narrow resonance*. Here, a resonance is called narrow if the corresponding partial widths are approximately constant over the total resonance width (or, Γ less than a few keV).

An isolated resonance is conveniently described by the one-level Breit– Wigner formula (see Eq. (2.185))

$$\sigma_{\rm BW}(E) = \frac{\lambda^2}{4\pi} \frac{(2J+1)(1+\delta_{01})}{(2j_0+1)(2j_1+1)} \frac{\Gamma_a \Gamma_b}{(E_r - E)^2 + \Gamma^2/4}$$
(3.110)

where j_i are the spins of target and projectile, J and E_r are the spin and energy of the resonance, Γ_i are the resonance partial widths of entrance and exit channel, and Γ is the total resonance width. Each partial width has to be summed over all possible values of orbital angular momenta and channel spins. The wave number is substituted by the de Broglie wavelength $\lambda = 2\pi/k = 2\pi\hbar/\sqrt{2m_{01}E}$ in order to avoid confusion with the symbol for the Boltzmann constant. The factor $(1 + \delta_{01})$ is included since the cross section for identical particles in the entrance channel increases by a factor of 2. Note that in the above expression the widths are expressed in terms of "observed" quantities (that is, the Thomas approximation is used; see Section 2.5.5) since it will simplify the calculations substantially. For most narrow resonances, this approximation introduces a negligible error.

The reaction rates for a single narrow resonance can be calculated using Eqs. (3.10) and (3.110),

$$N_A \langle \sigma v \rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E \sigma_{\rm BW}(E) e^{-E/kT} dE$$
$$= N_A \frac{\sqrt{2\pi}\hbar^2}{(m_{01}kT)^{3/2}} \omega \int_0^\infty \frac{\Gamma_a \Gamma_b}{(E_r - E)^2 + \Gamma^2/4} e^{-E/kT} dE$$
(3.111)

where $\omega \equiv (2J + 1)(1 + \delta_{01})/[(2j_0 + 1)(2j_1 + 1)]$. For a sufficiently narrow resonance, the Maxwell–Boltzmann factor $e^{-E/kT}$ and the partial widths Γ_i are approximately constant over the total width of the resonance. They may be replaced by their value at E_r and the integral can be calculated analytically. Thus

$$N_A \langle \sigma v \rangle = N_A \frac{\sqrt{2\pi\hbar^2}}{(m_{01}kT)^{3/2}} e^{-E_r/kT} \omega \frac{\Gamma_a \Gamma_b}{\Gamma} 2 \int_0^\infty \frac{\Gamma/2}{(E_r - E)^2 + \Gamma^2/4} dE$$
$$= N_A \frac{\sqrt{2\pi\hbar^2}}{(m_{01}kT)^{3/2}} e^{-E_r/kT} \omega \frac{\Gamma_a \Gamma_b}{\Gamma} 2\pi$$
$$= N_A \left(\frac{2\pi}{m_{01}kT}\right)^{3/2} \hbar^2 e^{-E_r/kT} \omega \gamma$$
(3.112)

where we used the definition $\omega \gamma \equiv \omega \Gamma_a \Gamma_b / \Gamma$. The quantity $\omega \gamma$ is proportional to the area under the resonance cross section, or equivalently, to the product of maximum cross section, $\sigma_{BW}(E = E_r) = (\lambda_r^2 / \pi) \omega \Gamma_a \Gamma_b / \Gamma^2$, and the total width Γ of the resonance,

$$\Gamma \cdot \sigma_{\rm BW}(E = E_r) = \Gamma \cdot \frac{\lambda_r^2}{\pi} \omega \frac{\Gamma_a \Gamma_b}{\Gamma^2} = \frac{\lambda_r^2}{\pi} \omega \gamma$$
(3.113)

Therefore, $\omega\gamma$ is referred to as the *resonance strength*. It is clear from the above expressions that the reaction rates for narrow resonances depend only on the energy and the strength of the resonance, but not on the exact shape of the cross section curve. This is a fortunate circumstance since, as will be seen later, for most narrow resonances the partial and total widths are experimentally not known.

If several narrow and isolated resonances contribute to the cross section, then their contributions to the reaction rate add incoherently. Numerically, one finds

$$N_A \langle \sigma v \rangle = \frac{1.5399 \times 10^{11}}{\left(\frac{M_0 M_1}{M_0 + M_1} T_9\right)^{3/2}} \sum_i (\omega \gamma)_i e^{-11.605 E_i / T_9} \qquad (\text{cm}^3 \text{mol}^{-1} \text{s}^{-1}) \quad (3.114)$$

where *i* labels different resonances, $(\omega \gamma)_i$ and E_i are in units of MeV, and M_i are the relative atomic masses in u.

The temperature dependence of the reaction rate for a single narrow resonance can be found by performing a calculation similar to the one that was applied to the nonresonant case. Starting from Eqs. (3.82) and (3.84) we find with $N_A \langle \sigma v \rangle_T = c T^{-3/2} e^{-c' E_r/T}$

$$\ln N_A \langle \sigma v \rangle_T = \ln c - \frac{3}{2} \ln T - c' \frac{E_r}{T}$$
(3.115)

$$n = \frac{\partial \ln N_A \langle \sigma v \rangle_T}{\partial \ln T} = -\frac{3}{2} - c' E_r \frac{\partial (T^{-1})}{\partial \ln T} = \frac{c' E_r}{T} - \frac{3}{2}$$
(3.116)

Hence

$$N_A \langle \sigma v \rangle_T = N_A \langle \sigma v \rangle_{T_0} (T/T_0)^{c'E_r/T - 3/2} = N_A \langle \sigma v \rangle_{T_0} (T/T_0)^{11.605 E_r/T_9 - 3/2}$$
(3.117)

where E_r in the last term is given in units of MeV. Figure 3.22 shows the exponent *n* versus temperature for several values of the resonance energy E_r . Clearly, the temperature sensitivity of narrow-resonance reaction rates increases for decreasing temperatures and increasing resonance energies. Depending on the values of *T* and E_r , narrow-resonance reaction rates may even be more temperature sensitive than nonresonant reaction rates.

In the following we will discuss the influence of the partial widths Γ_a and Γ_b on the reaction rates of a single narrow resonance by using a capture reaction $(0 + 1 \rightarrow \gamma + 3)$ as an example. Suppose further that only two channels are open, the particle channel (Γ_a) and the γ -ray channel (Γ_{γ}). The total width is $\Gamma = \Gamma_a + \Gamma_{\gamma}$. Experimental γ -ray partial widths typically amount to ≈ 1 meV–eV. Most neutron partial widths are in the range of ≈ 10 meV–keV. Neither of these partial widths are extremely sensitive to the value of E_r .



Fig. 3.22 The temperature sensitivity of narrow-resonance reaction rates for various values of the resonance energy.

Charged-particle partial widths, on the other hand, are governed by the transmission probability through the Coulomb barrier and are very sensitive to the resonance location, especially at low energies (Section 2.5.4).

Suppose first that the charged-particle width is smaller than the γ -ray partial width, a situation which is typical for low resonance energies (say, below $E_r \approx 0.5$ MeV). Since $\Gamma_a \ll \Gamma_\gamma$ we obtain from the definition of the resonance strength

$$\omega\gamma = \omega \frac{\Gamma_a \Gamma_\gamma}{\Gamma_a + \Gamma_\gamma} \approx \omega \frac{\Gamma_a \Gamma_\gamma}{\Gamma_\gamma} = \omega \Gamma_a \tag{3.118}$$

Thus the resonance strength depends only on the charged-particle partial width. Depending on the precise value of the resonance energy, and to a lesser extent, on the spectroscopic factor (see Eq. (2.197)), the resonance strength may become very small. Experimental studies of such low-energy resonances in charged-particle reactions represent a serious challenge for the nuclear experimentalist (Chapter 4). For a very narrow resonance, only the small energy region near E_r contributes to the reaction rate. Nevertheless, the concept of a Gamow peak is also useful for narrow resonances if the resonance strength is determined by the charged-particle particle partial width. This can be seen by ex-

pressing the narrow-resonance reaction rates for $\Gamma_a \ll \Gamma_\gamma$ (and $\Gamma \approx \Gamma_\gamma$) as

$$N_A \langle \sigma v \rangle \sim \int_0^\infty E \sigma_{\rm BW}(E) e^{-E/kT} dE \sim \int_0^\infty E \frac{1}{E} \frac{\Gamma_a \Gamma_\gamma}{(E_r - E)^2 + \Gamma_\gamma^2 / 4} e^{-E/kT} dE$$
$$\sim \int_0^\infty \frac{P_\ell(E) \Gamma_\gamma}{(E_r - E)^2 + \Gamma_\gamma^2 / 4} e^{-E/kT} dE$$
$$\sim \int_0^\infty \frac{\Gamma_\gamma}{(E_r - E)^2 + \Gamma_\gamma^2 / 4} e^{-2\pi\eta} e^{-E/kT} dE$$
(3.119)

where the energy *dependence* of the penetration factor $P_{\ell}(E)$ is approximated by the Gamow factor $e^{-2\pi\eta}$. Hence, the integrand can be written as the product of two factors: (i) the Gamow peak $e^{-2\pi\eta}e^{-E/kT}$, and (ii) a resonant *S*-factor curve of Lorentzian shape. Note that the Lorentzian has a FWHM of Γ_{γ} and a maximum height of $4/\Gamma_{\gamma}$. Thus, for a narrow resonance a change in Γ_{γ} has no influence on the area under the Lorentzian curve. It is obvious from Eq. (3.119) that, if a reaction cross section exhibits a number of narrow resonances, then those resonances located in the region of the Gamow peak (at energies between $E_0 - \Delta/2$ and $E_0 + \Delta/2$) will be the major contributors to the total reaction rates. In other words, if there are resonances located in the Gamow peak, then other resonances located either below or above the Gamow peak are of minor importance. The situation is represented in Fig. 3.23a. The dashed line shows the Maxwell–Boltzmann factor $e^{-E/kT}$, calculated for T = 0.4 GK, whereas the dashed-dotted line displays the Gamow factor. The solid lines show the Gamow peak and the narrow resonance S-factors. In this example, the narrow resonances at $E_r = 0.2$, 0.4 and 0.6 MeV will dominate the total reaction rates, while the resonances at $E_r = 0.05$ and 0.8 MeV will be far less important.

Suppose now that the γ -ray partial width is smaller than the particle width, $\Gamma_a \gg \Gamma_{\gamma}$. This situation typically occurs for charged particles at higher resonance energies (say, above $E_r \approx 0.5$ MeV where the particle partial width is frequently $\Gamma_a \gg 1$ eV), or for neutrons (except perhaps at very low energies). In this case we obtain from the definition of the resonance strength

$$\omega\gamma = \omega \frac{\Gamma_a \Gamma_\gamma}{\Gamma_a + \Gamma_\gamma} \approx \omega \frac{\Gamma_a \Gamma_\gamma}{\Gamma_a} = \omega \Gamma_\gamma \tag{3.120}$$

The resonance strength depends only on the γ -ray partial width and will typically be on the order of 1 meV–eV. The precise value is determined by the complicated nuclear configurations involved in the reaction. It is important to realize that "a most important energy window," such as the Gamow peak for charged particles or the Maxwell–Boltzmann distribution for neutrons, does not exist if $\Gamma_a \gg \Gamma_{\gamma}$. Figure 3.23b shows as an example the factor $e^{-E/kT}$ at T = 0.4 GK (dashed line) together with three narrow resonances (solid lines) at





Fig. 3.23 The influence of narrow resonances on reaction rates for the case (a) $\Gamma_{\gamma} \gg \Gamma_{a}$; (b) $\Gamma_{\gamma} \ll \Gamma_{a}$. The Maxwell–Boltzmann and Gamow factors are shown as dashed and dashed-dotted lines, respectively. The solid line in part (a) displays the Gamow peak. The sharp peaks indicate only the position of narrow resonances. Note that in part (a) the displayed resonances have different strengths, while those in part (b) are assumed to have similar strengths.

locations of $E_r = 0.2$, 0.4 and 0.6 MeV. The resonances are assumed to have similar strengths $\omega \gamma \approx \Gamma_{\gamma}$. For each resonance, only the region over the narrow resonance peak will contribute to the reaction rate. The reaction rate contribution of a narrow resonance increases rapidly for decreasing resonance energy, according to the factor $e^{-E/kT}$. Hence, the resonance at $E_r = 0.2$ MeV will clearly dominate the total reaction rate (note the vertical logarithmic scale in Fig. 3.23b). The smaller the resonance energy, the larger the reaction rate contribution will be, as long as $\Gamma_a \gg \Gamma_{\gamma}$. Consequently, it becomes very important to locate all the low-energy resonances.

It is often stated in the literature that in charged-particle reactions all resonances located within the Gamow peak ($E_0 \pm \Delta/2$) may contribute significantly to the total reaction rates. It should be obvious from the above considerations that this assumption represents an oversimplification, since it applies





0.3

Fig. 3.24 The fractional contribution of narrow resonances to the total reaction rate versus resonance energy for (a) 27 Al(p, γ) 28 Si at T =3.5 GK and (b) 24 Mg(α , γ) 28 Si at T = 2.5 GK. The main contribution arises in both cases from resonances with energies of $E_i < E_0$. The Gamow peak is shown as dotted line. The resonance energies and strengths are adopted from Endt (1990) and Iliadis et al. (2001).

only to resonances for which the total width is dominated by the γ -ray partial width ($\Gamma_a \ll \Gamma_{\gamma}$). With increasing energy a point will be reached in any capture reaction where the particle partial width will dominate over the γ ray partial width ($\Gamma_a \gg \Gamma_{\gamma}$), and for these resonances a Gamow peak does not exist. Therefore, one should in general not assume that all resonances located throughout the region $E_0 \pm \Delta/2$ contribute significantly to the total rates. The assumption is especially unjustified at higher stellar temperatures. This is demonstrated in Fig. 3.24, showing the fractional contribution of each resonance to the total rate, $N_A \langle \sigma v \rangle_i / N_A \langle \sigma v \rangle_{\text{total}}$, versus resonance energy for: (i) ²⁷Al(p, γ)²⁸Si at T = 3.5 GK, and (ii) ²⁴Mg(α , γ)²⁸Si at T = 2.5 GK. All known resonances with center-of-mass energies in the range of $E_i = 0.2$ –3.8 MeV in part (a) and $E_i = 1.1$ –4.3 MeV in part (b) are included in the figure. The dotted curves show the corresponding Gamow peaks. Clearly, the main contribution to the total reaction rate derives from resonances that are located *below* the

center of the Gamow peak, that is, $E_i < E_0$. The same applies to other (p,γ) and (α,γ) reactions on targets in the mass range A = 20–40 at temperatures of T = 0.5–10 GK. A survey reveals that a better estimate of an effective energy window for resonant capture reactions at $T \ge 0.5$ GK is the energy region from $\approx 0.3 E_0$ to $\approx E_0$. In the following chapters, we will still use the Gamow peak $(E_0 \pm \Delta/2)$ as the conventional effective energy window, but the reader should keep in mind that this represents a crude estimate, especially at higher temperatures.

It is demonstrated above that narrow resonances in the range of effective stellar energies have a dramatic effect on reaction rates. Therefore, it is important to locate all narrow resonances that could contribute to the total reaction rates. The situation is shown in Fig. 3.25. As a first step, one typically measures the reaction of astrophysical interest, 0 + 1, down to an energy of E_{min} , representing the smallest energy achievable in the laboratory (the dotted line in Fig. 3.25). Charged-particle cross sections below an energy of E_{min} become so small due to Coulomb barrier considerations that present experimental techniques are not sensitive enough for direct measurements. In a second step, therefore, the energy range between E = 0 and E_{min} is investigated by means of indirect measurements. Such studies populate the astrophysically important levels in the compound nucleus *C* by using reactions X+x other



Fig. 3.25 Energy level diagram, showing narrow resonances in the reaction 0 + 1 (left-hand side) and the corresponding levels in the compound nucleus *C*. The locations of two Gamow peaks at different temperatures are displayed as hatched bars. Below an energy of E_{\min} (dotted line) charged-particle measurements are not feasible. In this case one may estimate the reaction rates by measuring nuclear structure properties of levels in nucleus *C* via a reaction X + x.

than the one of direct astrophysical interest (Section 4.1). From the measured nuclear properties (excitation energies, spins, parities, spectroscopic factors, and so on) of the compound levels close to the particle threshold, the resonance energies and strengths of astrophysically important resonances can be estimated.

For the influence of experimental uncertainties of E_r , $\omega\gamma$, and C^2S on the resulting narrow-resonance reaction rates, the reader is referred to Thompson and Iliadis (1999).

Example 3.6

Suppose that four hypothetical narrow s-wave resonances occur at low energies in the ²⁰Ne(p, γ)²¹Na reaction. The resonance energies are $E_r = 10$ keV, 30 keV, 50 keV, and 100 keV. The corresponding resonance strengths are $\omega\gamma = 7.24 \times 10^{-33}$ eV, 3.81×10^{-15} eV, 1.08×10^{-9} eV, and 3.27×10^{-4} eV. Each of these values has been obtained by assuming $\Gamma_p \ll \Gamma_\gamma$ and $C^2S = 1$. Which resonance do you expect to dominate the total reaction rates at T = 0.02 GK and 0.08 GK?

At T = 0.02 GK, the Gamow peak location (see Eqs. (3.74) and (3.78)) is $E_0 \pm \Delta/2 = 40 \pm 10$ keV. Only the resonances at $E_r = 30$ keV and 50 keV are located in the Gamow peak and, therefore, these will dominate the reaction rates. At T = 0.08 GK, we obtain $E_0 \pm \Delta/2 = 100 \pm 30$ keV. Only the resonance at $E_r = 100$ keV is located in the Gamow peak and thus will dominate the total reactions rates. See also Problem 3.5.

We will now consider two issues that are important at elevated temperatures when a capture reaction, for example, (p,γ) , (n,γ) or (α,γ) , proceeds through narrow resonances. The first concerns the influence of excited target states on the reaction rates. From Eq. (3.37) we find for the *stellar* rate of the capture reaction $0 + 1 \rightarrow \gamma + 3$

$$N_A \langle \sigma v \rangle = \sum_{\mu} P_{0\mu} N_A \langle \sigma v \rangle^{\mu} = \frac{\sum_{\mu} g_{0\mu} e^{-E_{0\mu}/kT} N_A \langle \sigma v \rangle^{\mu}}{\sum_{\mu} g_{0\mu} e^{-E_{0\mu}/kT}}$$
(3.121)

where μ sums over the levels in the target nucleus 0 including the ground state, while excited states in the light particle 1 are neglected (a safe assumption for protons, neutrons or α -particles). The subscript "01 \rightarrow 23" is suppressed for clarity and it is assumed that the reaction rate $N_A \langle \sigma v \rangle^{\mu}$ has already been properly summed over transitions to excited final states ν in nucleus 3. All other symbols have exactly the same meaning as in Section 3.1.5. Suppose now that the reaction rate $N_A \langle \sigma v \rangle^{\mu}$ for a specific target state μ is determined by a number of narrow resonances that are labeled by ρ . From Eq. (3.112) we

find

$$N_A \langle \sigma v \rangle^{\mu} = \sum_{\rho} N_A \langle \sigma v \rangle_{\rho}^{\mu} = \sum_{\rho} N_A \left(\frac{2\pi}{m_{01}kT} \right)^{3/2} \hbar^2 e^{-E_{\rho\mu}/kT} (\omega\gamma)_{\rho\mu}$$
$$= \sum_{\rho} N_A \left(\frac{2\pi}{m_{01}kT} \right)^{3/2} \hbar^2 e^{-E_{\rho\mu}/kT} \frac{g_{\rho}}{g_{0\mu}g_1} \frac{\Gamma_{\rho\mu}\Gamma_{\rho\gamma}}{\Gamma_{\rho}}$$
(3.122)

where g_{ρ} , $g_{0\mu}$, and g_1 are the statistical weights of the resonance, of the level in target nucleus 0 and of the light particle 1, respectively, with $\omega_{\rho\mu} \equiv$ $g_{\rho}/(g_{0\mu}g_1)$; $\Gamma_{\rho\mu}$, $\Gamma_{\rho\gamma}$, and Γ_{ρ} are the particle partial width for resonance formation from target level μ , the (decay) γ -ray partial width, and the total width, respectively, of resonance ρ ; $E_{\rho\mu}$ is the energy of resonance ρ for target level μ and $E_{\rho\mu_0} = E_{\rho\mu} + E_{0\mu}$ is the energy of resonance ρ for the target ground state μ_0 . An energy level diagram is shown in Fig. 3.26. From Eqs. (3.36), (3.43), (3.121), and (3.122) we obtain

$$N_A \langle \sigma v \rangle = \frac{\sum_{\mu} g_{0\mu} e^{-E_{0\mu}/kT} \sum_{\rho} N_A \left(\frac{2\pi}{m_{01}kT}\right)^{3/2} \hbar^2 e^{-E_{\rho\mu}/kT} \frac{g_{\rho}}{g_{0\mu}g_1} \frac{\Gamma_{\rho\mu}\Gamma_{\rho\gamma}}{\Gamma_{\rho}}}{\sum_{\mu} g_{0\mu} e^{-E_{0\mu}/kT}} = \frac{1}{G_0^{\text{norm}}} \sum_{\rho} N_A \langle \sigma v \rangle_{\rho}^{\mu_0} \sum_{\mu} \frac{\Gamma_{\rho\mu}}{\Gamma_{\rho\mu_0}}$$
(3.123)

Hence, the total stellar rate is given by a sum over narrow resonance groundstate rates, $N_A \langle \sigma v \rangle_{\rho}^{\mu_0}$, where each resonance term is modified by a factor of $(1 + \Gamma_{\rho\mu_1}/\Gamma_{\rho\mu_0} + \Gamma_{\rho\mu_2}/\Gamma_{\rho\mu_0} + \cdots)$, with $\Gamma_{\rho\mu}/\Gamma_{\rho\mu_0}$ denoting the ratio of particle partial widths for excited target level μ and the target ground state μ_0 . The inclusion of excited target states in the total reaction rate introduces no additional temperature dependence other than a weak dependence through the quantity G_0^{norm} . For charged-particle reactions and low resonance energies, the penetration factor (and hence the particle partial width) varies strongly with energy (Section 2.5.4). Therefore, we expect in this case a negligible influence of excited target states on the total rate, that is, $\Gamma_{\rho\mu} \ll \Gamma_{\rho\mu_0}$, unless the target excitation energy $E_{0\mu}$ is very small, implying $E_{\rho\mu} \approx E_{\rho\mu_0}$ or $\Gamma_{\rho\mu} \approx \Gamma_{\rho\mu_0}$ (for similar values of the corresponding reduced widths; see Fig. 3.26). It is also clear that in charged-particle reactions at higher resonance energies or in neutron-induced reactions, where the particle partial widths are less sensitive to energy variations, the ratio $\Gamma_{\rho\mu}/\Gamma_{\rho\mu_0}$ can be relatively large. Under such conditions, excited target states may in fact dominate the total stellar reaction rates. See, for example, Vancraeynest et al. (1998) and Schatz et al. (2005).

The second issue concerns photodisintegration rates. Forward and reverse rates are related by Eqs. (3.35) or (3.45). Rewriting that expression by using

the above notation yields

$$\frac{\lambda_{\gamma}(3)}{N_A \langle \sigma v \rangle} = \left(\frac{2\pi}{h^2}\right)^{3/2} \frac{(m_{01}kT)^{3/2}}{N_A} \frac{g_{0\mu_0}g_1}{g_{3\nu_0}} \left(\frac{G_0^{\text{norm}}}{G_3^{\text{norm}}}\right) e^{-Q_{01 \to \gamma 3}/kT}$$
(3.124)

where for simplicity we omitted the Kronecker delta (that is, we assume nonidentical nuclei 0 and 1 and set $G_1^{\text{norm}} = 1$ (we disregard excited states in the light particle 1); $g_{0\mu_0}$, g_1 , and $g_{3\nu_0}$ are the statistical weights of the target ground state, of light particle 1, and of the ground state of the residual nucleus, respectively; $Q_{01\rightarrow\gamma3}$ denotes the *Q*-value for the ground states of nuclei 0, 1, and 3. If the forward reaction proceeds predominantly through isolated and narrow resonances, then the rates for the (reverse) photodisintegration can be found by substitution of Eq. (3.123) into Eq. (3.124),

$$\lambda_{\gamma}(3) = \left(\frac{2\pi}{h^{2}}\right)^{3/2} \frac{(m_{01}kT)^{3/2}}{N_{A}} \frac{g_{0\mu_{0}}g_{1}}{g_{3\nu_{0}}} \frac{1}{G_{3}^{\text{norm}}} e^{-Q_{01\to\gamma3}/kT} \sum_{\rho} N_{A} \langle \sigma v \rangle_{\rho}^{\mu_{0}} \sum_{\mu} \frac{\Gamma_{\rho\mu}}{\Gamma_{\rho\mu_{0}}}$$
$$= \frac{1}{\hbar} \frac{g_{0\mu_{0}}g_{1}}{g_{3\nu_{0}}} \frac{1}{G_{3}^{\text{norm}}} \sum_{\rho} e^{-E_{\rho x}/kT} \frac{g_{\rho}}{g_{0\mu_{0}}g_{1}} \frac{\Gamma_{\rho\mu_{0}}\Gamma_{\rho\gamma}}{\Gamma_{\rho}} \sum_{\mu} \frac{\Gamma_{\rho\mu}}{\Gamma_{\rho\mu_{0}}}$$
$$= \frac{1.519 \times 10^{21}}{G_{3}^{\text{norm}}} \frac{g_{0\mu_{0}}g_{1}}{g_{3\nu_{0}}} \sum_{\rho} e^{-11.605 E_{\rho x}/T_{9}} (\omega\gamma)_{\rho\mu_{0}} \sum_{\mu} \frac{\Gamma_{\rho\mu}}{\Gamma_{\rho\mu_{0}}} \qquad (\text{cm}^{3}\text{mol}^{-1}\text{s}^{-1})$$
(3.125)

where the excitation energy in the compound nucleus 3 (corresponding to resonance ρ) is equal to the sum of the resonance energy for the target ground state and the ground-state *Q*-value, $E_{\rho x} = E_{\rho \mu_0} + Q_{01 \rightarrow \gamma 3}$ (see Fig. 3.26). In the above numerical expression, the resonance energies and strengths are in units of MeV. The photodisintegration rate depends exponentially on the excitation energies $E_{\rho x}$ of the compound levels that correspond to the resonances through which the forward reaction proceeds. The excitation energies also enter implicitly via the particle partial widths $\Gamma_{\rho\mu}$ and $\Gamma_{\rho\mu_0}$, and via the resonance strengths ($\omega \gamma$)_{$\rho\mu_0$} for the target ground state. A numerical example is given in Problem 3.6.

3.2.5

Broad-Resonance Reaction Rates

The results derived in the last section are independent of the precise shape of the resonance cross section. The formalism of narrow-resonance reaction rates is not applicable in cases for which the explicit energy dependence of the cross section is important. As an example, a charged-particle reaction will be discussed in the following. Three common situations are schematically displayed in Fig. 3.27, showing in each case the Maxwell–Boltzmann factor



Fig. 3.26 Energy level diagram for narrow resonances in the $0 + 1 \rightarrow \gamma + 3$ capture reaction, showing thermally excited states in the target and the final nucleus. For clarity, only a single narrow resonance (ρ) and one excited state is shown for the target 0 and the final nucleus 3. All vertical arrows represent γ -ray transitions. In the forward reaction $0 + 1 \rightarrow \gamma + 3$, the level ρ may be populated either from the target ground state or from the excited target state.

(dashed line), the Gamow factor (dashed-dotted line), the Gamow peak (dotted line), and the cross section of a *broad* resonance (upper solid line). The reaction rates are proportional to the area under the lower solid line, that is, the product of the Maxwell–Boltzmann distribution and the cross section. The curves are obtained for the ²⁴Mg(p, γ)²⁵Al reaction at *T* = 0.05 GK. For simplicity, the resonant cross sections are calculated using an arbitrary constant γ -ray partial width; angular momenta are neglected and the penetration factor is approximated by the Gamow factor $e^{-2\pi\eta}$.

Part (a) shows a broad resonance at $E_r = 0.1$ MeV with a width of $\Gamma = 5$ keV, located inside the Gamow peak. Clearly, it can no longer be assumed that the partial widths, the de Broglie wavelength, and the Maxwell–Boltzmann distribution are constant over the width of the resonance. The energy dependence of these quantities has to be taken into account. The product of Maxwell–Boltzmann distribution and cross section is now a complicated function of energy (lower solid line) and can no longer be integrated analytically. Instead, the reaction rates have to be calculated numerically by solving (see Eqs. (3.69) and (3.110))

$$N_A \langle \sigma v \rangle = \sqrt{2\pi} \, \frac{N_A \omega \hbar^2}{(m_{01} kT)^{3/2}} \int_0^\infty e^{-E/kT} \frac{\Gamma_a(E) \Gamma_b(E+Q-E_f)}{(E_r-E)^2 + \Gamma(E)^2/4} \, dE \qquad (3.126)$$

where the partial width for the exit channel, Γ_b , has to be calculated at the energy $E_{23} = E_{01} + Q_{01 \rightarrow 23} - E_f$ available to the pair 2 + 3. The above expression involves a transition from the resonance to a specific final state E_f . If the reaction involves transitions to several final states, then the different contributions to the total cross section add incoherently. As a useful rule, if the

resonance width is small compared to the width of the Gamow peak, $\Gamma \ll \Delta$, then the reaction rates may be calculated by using the narrow resonance formalism (Section 3.2.4). Otherwise, the reaction rates have to be obtained from a numerical integration of Eq. (3.126).

Part (b) shows a resonance at $E_r = 0.25$ MeV with a width of $\Gamma = 0.6$ keV. It can be seen that the resonance is located outside the Gamow peak. We concluded in the last section that in such cases the contribution of this resonance to the total reaction rates is negligible compared to other narrow resonances that are located in the Gamow peak. But suppose that no other narrow resonances exist below E = 0.25 MeV. It is important to realize that it would be incorrect in this case to calculate the reaction rates using the narrow resonance formalism (see Eq. (3.112)). The latter equations have been derived assuming a negligible energy dependence of the partial widths, de Broglie wavelength, and Maxwell-Boltzmann distribution over the total width of the resonance. The value of the Maxwell–Boltzmann distribution at the resonance energy E_r appears in the narrow-resonance reaction rate expression which takes only the reaction rate contribution at the resonance energy into account. However, the product of Maxwell-Boltzmann distribution and cross section (lower solid line) gives rise to another maximum at lower energies which is caused by the low-energy wing of the resonance. For the example chosen it is apparent that this first maximum gives a far larger contribution to the reaction rates than the second maximum at E_r . The reason is that the Maxwell–Boltzmann distribution has a stronger energy dependence than the cross section, as can be seen by comparing the magnitude of both functions at the positions of the two maxima of the lower solid line. As an approximate rule, if a resonance at E_r is located within the energy range between $E_0 - 2\Delta$ and $E_0 + 2\Delta$, then the narrow resonance formalism is applicable (Section 3.2.4). Otherwise, the wing of the resonance has to be taken into account explicitly even if the resonance is narrow in the sense that $\Gamma \ll \Delta$. In the latter case, the reaction rates can be calculated either by numerical integration or, if the S-factor of the resonance wing varies smoothly over the energy range of interest, by expanding the S-factor into a Taylor series and by applying the nonresonant reaction rate formalism.

Part (c) shows a subthreshold resonance, corresponding to a compound nucleus level which is located below the proton threshold. For reasons of clarity, the lower solid line displays the *S*-factor rather than the cross section. It is obvious that in this case the high-energy wing of the resonance has to be taken into account explicitly. The *S*-factor (or cross section) can be calculated by using the one-level Breit–Wigner formula (see Eq. (2.185) and Example 2.1). Again, the reaction rates are then evaluated either by numerical integration or, if the *S*-factor varies smoothly over the energy range of interest, by using the nonresonant reaction formalism.





Fig. 3.27 The influence of broad resonances on reaction rates for (a) a broad resonance located in the Gamow peak; (b) a broad resonance located outside the Gamow peak; and (c) a high-energy wing of a subthreshold resonance. In each panel the Maxwell-Boltzmann factor, Gamow factor, and Gamow peak are shown as dashed, dashed-dotted, and dotted lines, respec-

tively. The Breit-Wigner cross sections are displayed as upper solid lines and the product of cross section and Maxwell-Boltzmann factor as lower solid lines. The latter product determines the reaction rates (see Eqs. (3.69) and (3.126)). The curves are obtained for the ²⁴Mg(p, γ)²⁵Al reaction at T = 0.05 GK. In part (c) the *S*-factor instead of the cross section is shown for clarity.

For the explicit calculation of broad-resonance reaction rates it is of advantage to express the *S*-factor in terms of measured quantities. Using the onelevel Breit–Wigner formula the cross section can be written as

$$\sigma_{\rm BW}(E) = \frac{\pi \hbar^2 \omega}{2m_{01}E} \frac{\Gamma_a(E)\Gamma_b(E+Q-E_f)}{(E_r-E)^2 + \Gamma(E)^2/4}$$
(3.127)

Numerically, we find $\pi \hbar^2 / (2m_{01}) = 0.6566 (M_0 + M_1) / (M_0M_1)$ MeV·b. Suppose first that the partial widths of the broad resonance at E_r , $\Gamma_a(E_r)$ and $\Gamma_b(E_r)$, are known experimentally. We may parametrize the particle partial width by using $\Gamma_i(E) \sim P_i(E)$ (see Eq. (2.176)) and the γ -ray partial width by $\Gamma_\gamma(E_\gamma) \sim E_\gamma^{2L+1}$, where E_γ and L denote the energy and multipolarity, respectively, of the emitted γ -ray (see Eq. (1.21)). Although approximate expressions exist for the calculation of the penetration factors (see, for example, Clayton 1983), it is more reliable to obtain $P_i(E)$ directly from numerical computations of Coulomb wavefunctions (Section 2.5.4 and Appendix A.3). For reactions involving particles with rest mass one finds from Eq. (3.127)

$$\sigma_{\rm BW}(E) = \frac{\pi \hbar^2 \omega}{2m_{01}E} \frac{\frac{P_a(E)}{P_a(E_r)} \Gamma_a(E_r) \frac{P_b(E+Q-E_f)}{P_b(E_r+Q-E_f)} \Gamma_b(E_r+Q-E_f)}{(E_r-E)^2 + \Gamma(E)^2/4}$$
(3.128)

and for reactions involving photon emission

$$\sigma_{\rm BW}(E) = \frac{\pi \hbar^2 \omega}{2m_{01}E} \frac{\frac{P_a(E)}{P_a(E_r)} \Gamma_a(E_r) \left[\frac{E+Q-E_f}{E_r+Q-E_f}\right]^{2L+1} \Gamma_{\gamma}(E_r+Q-E_f)}{(E_r-E)^2 + \Gamma(E)^2/4}$$
(3.129)

For many broad resonances the partial widths Γ_i have not been measured, but only the resonance strength $\omega\gamma$ and the total width Γ , both measured at E_r , are known experimentally. With the definition of the resonance strength

$$\omega\gamma \equiv \omega \, \frac{\Gamma_a(E_r)\Gamma_b(E_r + Q - E_f)}{\Gamma(E_r)} \tag{3.130}$$

the cross section is given by

$$\sigma_{\rm BW}(E) = \frac{\pi\hbar^2}{2m_{01}E} \frac{P_a(E)}{P_a(E_r)} \frac{\Gamma_b(E+Q-E_f)}{\Gamma_b(E_r+Q-E_f)} \frac{\omega\gamma\Gamma(E_r)}{(E_r-E)^2 + \Gamma(E)^2/4}$$
(3.131)

where the ratio of partial widths Γ_b is given as before: either by the ratio of penetration factors, $P_b(E + Q - E_f)/P_b(E_r + Q - E_f)$, for reactions involving particles with rest mass, or by the factor $[(E + Q - E_f)/(E_r + Q - E_f)]^{2L+1}$ for reactions emitting photons. The cross section for subthreshold resonances is calculated similarly (Example 2.1). It can be seen from the Breit–Wigner expression (see Eq. (3.127)) that broad resonances with an energy E_r close to E_0 and with large partial widths (that is, a large value of C^2S and a small value of orbital angular momentum ℓ) will make the largest contributions to the total reaction rates.
Example 3.7

The s-wave resonance at $E_r = 214$ keV ($J^{\pi} = 1/2^+$) in 24 Mg(p,γ) 25 Al has a measured strength of $\omega \gamma = 1.3 \times 10^{-2}$ eV, a proton width of $\Gamma_p = 1.4 \times 10^{-2}$ eV, a γ -ray partial width of $\Gamma_{\gamma} = 1.4 \times 10^{-1}$ eV, and a total width of Γ = 1.5×10^{-1} eV (Powell et al. 1999). All widths are *observed* quantities. No other channels are open, hence $\Gamma = \Gamma_p + \Gamma_\gamma$. Suppose that the resonance decays via a dipole transition (L = 1) to the $E_f = 452$ keV state ($J^{\pi} = 1/2^+$) in 25 Al with a branching ratio of 100%. A level diagram is shown in Fig. 3.28a. Calculate the reaction rate contribution of this resonance at temperatures between T = 0.01 GK and 1 GK by using: (i) the narrow resonance formalism, and (ii) the broad resonance formalism (that is, by explicitly taking the energy dependence of the *S*-factor into account).

To calculate the narrow-resonance reaction rate, only the resonance energy E_r and strength $\omega\gamma$ is needed. The numerical results are shown as the dashed line in Fig. 3.28b and have been obtained directly from Eq. (3.114). The resonant *S*-factor is calculated from Eqs. (3.70) and (3.129),

$$S_{BW}(E) = E\sigma_{BW}(E)e^{2\pi\eta} = \frac{\pi\hbar^2}{2m_{01}}e^{2\pi\eta}\omega \frac{\frac{P_a(E)}{P_a(E_r)}\Gamma_a(E_r)\left[\frac{E+Q-E_f}{E_r+Q-E_f}\right]^{2L+1}\Gamma_{\gamma}(E_r+Q-E_f)}{(E_r-E)^2+\Gamma(E)^2/4}$$

The penetration factors are obtained from numerically computed Coulomb wavefunctions for $\ell = 0$ (s-wave), using a radius parameter of $r_0 = 1.25$ fm. The broad-resonance reaction rates may then be calculated by numerical integration of Eq. (3.69). The results are shown as the solid line in Fig. 3.28b.

It can be seen that above a temperature of T = 0.05 GK the narrow- and broad-resonance reaction rates are in agreement. The result is expected since for this temperature region the $E_r = 214$ keV resonance is located inside the Gamow peak. Below T = 0.05 GK, the resonance is located outside the energy window $E_0 \pm 2\Delta$ and, therefore, the narrow resonance formalism underestimates the reaction rates substantially. The low-energy wing of the resonance provides a much larger reaction rate contribution compared to the contribution at E_r . Figure 3.28b displays at T = 0.05 GK a change in the slope of the reaction rates, reflecting the different temperature dependences of the narrow-resonance reaction rate expression (because of the contribution at the resonance energy E_r alone) and the broad-resonance reaction rate formalism (because of the additional contribution from the Gamow peak near E_0).



Fig. 3.28 (a) Level diagram of 25 Al and (b) reaction rates for 24 Mg(p, γ) 25 Al versus temperature. The solid and dashed curves in part (b) are calculated with the reaction rate formalism for broad resonances and narrow resonances, respectively.

3.2.6

Electron Screening

The formalism discussed so far for calculating thermonuclear reaction rates involving two charged particles is based on the assumption that Coulomb interactions with electrons or with other nuclei are negligible. However, in the fully ionized stellar plasma electrons are attracted to a particular nucleus while other nuclei are repelled. In other words, each nucleus will polarize its neighborhood to some extent. We may imagine that each nucleus is surrounded by an imaginary sphere containing an inhomogeneously charged cloud. Therefore, in a nuclear reaction the potential seen by either one of the colliding nuclei is modified from the simple Coulomb form. The effective barrier for the nuclear fusion reaction becomes thinner and, therefore, both the

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tunneling probability and the reaction rate increase over their values which are obtained for the same reaction taking place in a vacuum. This effect is referred to as *electron screening*. The ratio of the actual reaction rate in the plasma to the vacuum rate is called the screening factor f_s and will be derived in the following.

Consider a nearly perfect gas at relatively low density for which the average Coulomb energy between two neighboring nuclei is much smaller than their thermal energy. In this case, the screened potential for two colliding nuclei 0 and 1 is given by (Salpeter 1954)

$$V_s(r) = \frac{Z_0 Z_1 e^2}{r} e^{-r/R_D}$$
(3.132)

where R_D is the Debye–Hückel radius

$$R_D = \sqrt{\frac{kT}{4\pi e^2 \rho N_A \zeta^2}} = 2.812 \times 10^{-7} \rho^{-1/2} T_9^{1/2} \zeta^{-1} \qquad (\text{cm})$$
(3.133)

and

$$\zeta \equiv \sqrt{\sum_{i} \frac{(Z_i^2 + Z_i \theta_e) X_i}{A_i}}$$
(3.134)

with θ_e being the electron degeneracy factor. The sum is over all types of positive ions present in the plasma and the density in the numerical expression is in units of g/cm³.

The Debye–Hückel radius is a measure for the size of the charged cloud surrounding each nucleus. Beyond a distance of $r = R_D$ the screened potential vanishes quickly. The condition of a weak Coulomb energy compared to the thermal energy defines the *weak screening* regime. It is equivalent to the assumption that the Debye–Hückel radius is much larger than the average distance between neighboring nuclei. This condition, which holds for most thermonuclear reactions in stars, can be numerically expressed as (Clayton 1983)

$$T \gg 10^5 \rho^{1/3} \zeta^2 \tag{3.135}$$

with *T* and ρ in units of K and g/cm³, respectively.

Consider now a nonresonant reaction for which the cross section is given by Eq. (3.70). The energy dependence of $\sigma(E)$ is mainly given by the Gamow factor exp $(-2\pi\eta)$, while the *S*-factor varies smoothly with energy. The s-wave transmission coefficient for the unscreened Coulomb potential can be found from

$$\widehat{T} \approx \exp\left(-\frac{2}{\hbar}\sqrt{2m}\int_{0}^{R_{c}}\sqrt{\frac{Z_{0}Z_{1}e^{2}}{r}-E}\,dr\right)$$

$$= \exp\left(-\frac{2}{\hbar}\sqrt{\frac{2m}{E}}Z_{0}Z_{1}e^{2}\int_{0}^{1}\sqrt{\frac{1}{z}-1}\,dz\right)$$

$$= \exp\left(-\frac{2\pi}{\hbar}\sqrt{\frac{m}{2E}}Z_{0}Z_{1}e^{2}\right) \equiv e^{-2\pi\eta}$$
(3.136)

This derivation is simpler than that in Section 2.4.3 because we assume here that, for low bombarding energies compared to the Coulomb barrier height, the classical turning point is much larger than the radius of the square-well potential, $R_c \gg R_0$ (Fig. 2.17). The lower integration limit for r is then $R_0 \rightarrow 0$, while that for z is $R_0/R_c \rightarrow 0$. Consequently, the Gamow factor is directly obtained without the correction of Eq. (2.124). For the screened Coulomb potential, the classical turning point is defined by

$$E = (Z_0 Z_1 e^2 / R_c) e^{-R_c / R_D}$$
(3.137)

Proceeding in exactly the same manner as in Section 2.4.3, we find for the modified transmission coefficient (Problem 3.7)

$$\widehat{T} \approx e^{x\pi\eta - 2\pi\eta} \tag{3.138}$$

where the variable $x = x(E) = R_c/R_D$ depends explicitly on energy through R_c . Equation (3.138) is derived assuming that x is a small number, $R_D \gg R_c$, which is frequently the case (see below). Substitution of the modified transmission coefficient into the expression for nonresonant reaction rates (see Eq. (3.71)) gives

$$N_A \langle \sigma v \rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty S(E) e^{x\pi\eta} e^{-2\pi\eta} e^{-E/kT} dE$$
(3.139)

Both *x* and η depend on energy but the above expression can be approximated by evaluating the factor $e^{x\pi\eta}$ at the most effective energy of the interaction in the plasma, that is, the Gamow energy E_0 . With $e^{x\pi\eta} \approx e^{(x\pi\eta)}E_0 \equiv f_s$ one obtains

$$N_A \langle \sigma v \rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} f_s \int_0^\infty S(E) e^{-2\pi\eta} e^{-E/kT} dE$$
(3.140)

and consequently the screened reaction rate is simply obtained by multiplying the unscreened reaction rate by the screening factor $f_s = e^{(x\pi\eta)_{E_0}}$. The small corrections introduced by a more rigorous calculation, in which the integral in

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Eq. (3.139) is evaluated by taking the energy dependence of the factor $e^{x\pi\eta}$ into account, are discussed in Bahcall et al. (1998) and Liolios (2000). Numerically one finds

$$x(E_0) = (R_c/R_D)_{E_0} = \frac{Z_0 Z_1 e^2}{E_0} \sqrt{\frac{4\pi e^2 \rho N_A}{kT}} \zeta$$

= 4.197 × 10⁻⁶ (Z_0Z_1)^{1/3} $\left(\frac{M_0 M_1}{M_0 + M_1}\right)^{-1/3} \sqrt{\rho} T_9^{-7/6} \zeta$ (3.141)

$$(x\pi\eta)_{E_0} = (R_c/R_D)_{E_0}(\pi\eta)_{E_0} = \frac{Z_0 Z_1 e^2}{R_D} \frac{(\pi\eta)_{E_0}}{E_0} = \frac{Z_0 Z_1 e^2}{R_D kT}$$

= 5.945 × 10⁻⁶ \sqrt{\rho} Z_0 Z_1 T_0^{-3/2} \zeta \sqrt{2} (3.142)

$$f_s = e^{(x\pi\eta)_{E_0}} = e^{Z_0 Z_1 e^2 / (R_D kT)} = e^{5.945 \times 10^{-6} \sqrt{\rho} Z_0 Z_1 T_9^{-3/2} \zeta}$$
(3.143)

where it is assumed that the classical turning points for the screened and unscreened potentials are approximately equal. All nonresonant rate expressions for charged-particle reactions derived in this chapter have to be multiplied by f_s if the nuclear reaction takes place under the conditions of weak screening.

For increasing densities at a given temperature a point is eventually reached where the average Coulomb energy of neighboring nuclei, $\langle E_c \rangle$, is no longer small compared to the thermal energy kT. The condition $\langle E_c \rangle \approx kT$ defines the intermediate screening regime while strong screening refers to the regime $\langle E_c \rangle \gg kT$. Approximate expressions for the corresponding screening factors can be found, for example, in DeWitt, Graboske and Cooper (1973) and Graboske et al. (1973).

Example 3.8

Calculate the electron screening correction for the nonresonant reaction $p + p \rightarrow e^+ + \nu + d$ (Section 5.1.1) in the region where the Sun's nuclear energy production is at maximum. Assume for the temperature and density values of T = 0.0135 GK and $\rho = 93$ g/cm³, respectively. The mass fractions of hydrogen, helium, and oxygen in this region amount to $X(^{1}H) = 0.52$, $X(^{4}He) = 0.46$, and $X(^{16}O) = 0.01$. For the electron degeneracy factor assume a value of $\theta_e = 0.92$.

First, the parameter ζ is computed

$$\zeta \equiv \sqrt{\sum_{i} \frac{(Z_i^2 + Z_i \theta_e) X_i}{A_i}}$$

= $\sqrt{\frac{(1^2 + 1 \cdot 0.92)0.52}{1}} + \frac{(2^2 + 2 \cdot 0.92)0.46}{4} + \frac{(8^2 + 8 \cdot 0.92)0.01}{16}$
= $\sqrt{0.998 + 0.672 + 0.045} = 1.31$

Since $10^5 \rho^{1/3} \zeta^2 = 8 \times 10^{-4}$ GK $\ll 0.0135$ GK, the condition for the weak screening regime is fulfilled. For the Debye–Hückel radius we find a value of

$$R_D = 2.812 \times 10^{-7} (93)^{-1/2} (0.0135)^{1/2} (1.31)^{-1} = 2.58 \times 10^{-9} \text{ cm} = 25,800 \text{ fm}$$

The parameter $x(E_0)$ amounts to

$$x(E_0) = 4.197 \times 10^{-6} (1 \cdot 1)^{1/3} \left(\frac{1 \cdot 1}{1 + 1}\right)^{-1/3} \sqrt{93} (0.0135)^{-7/6} 1.31 = 0.010$$

It is small compared to unity and thus the linear expansion of Eq. (3.138) is justified in this case. The screening factor is

$$f_s = \exp[5.945 \times 10^{-6} \sqrt{93} \cdot 1 \cdot 1(0.0135)^{-3/2} 1.31] = e^{0.0479} = 1.049$$

Other examples are given in Liolios (2000).

We have discussed so far only nonresonant reactions. The electron screening correction for a narrow resonance depends on the relative magnitude of the incoming (Γ_a) and outgoing (Γ_b) partial widths. Consider for example a capture reaction $A(a,\gamma)B$. If $\Gamma_a \gg \Gamma_\gamma$, then exactly the same screening correction factor as in Eq. (3.143) is obtained, despite the fact that in this case the reaction rate is completely independent of the penetration factor for the incoming channel. This counterintuitive result can be explained by revisiting the derivation of the narrow-resonance reaction rate (Section 3.2.4). The screening potential (see Eq. (3.132)) can be approximated by

$$V_s(r) = \frac{Z_0 Z_1 e^2}{r} e^{-r/R_D} \approx \frac{Z_0 Z_1 e^2}{r} - \frac{Z_0 Z_1 e^2}{R_D} = \frac{Z_0 Z_1 e^2}{r} + U_s$$
(3.144)

The first term is the Coulomb potential while the second term represents a perturbing potential which is caused by the shielding charge density. The latter potential is negative (attractive) and thus effectively increases the kinetic energy of the projectile by an amount $|U_s| = Z_0 Z_1 e^2 / R_D$ (Fig. 2.17). Without electron shielding, only those projectiles with energies near $E = E_r$ will be able to excite the resonance. But in a plasma it is the projectiles with *smaller* energies near $E' = E_r + U_s$ that give rise to the population of the resonance. Hence, for the condition $\Gamma_a \gg \Gamma_{\gamma}$, Eq. (3.112) must be replaced by

$$N_A \langle \sigma v \rangle = N_A \left(\frac{2\pi}{m_{01}kT}\right)^{3/2} \hbar^2 e^{-(E_r + U_s)/kT} \omega \Gamma_b \tag{3.145}$$

Obviously, this expression differs from Eq. (3.112) only by a screening factor $f_s = e^{-U_s/kT} = e^{Z_0 Z_1 e^2/(R_D kT)}$, that is, the same result as obtained in

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Eq. (3.140). An example for the applicability of this result is the 3α reaction where the condition $\Gamma_a \gg \Gamma_b$ holds for each of the two successive interactions (Example 3.4 and Section 5.3.1). For the opposite case, $\Gamma_a \ll \Gamma_b$, the screening factor has a more complicated form. See Salpeter and Van Horn (1969) and Mitler (1977).

We end the discussion by noting that the electrons of target nuclei also introduce screening effects in laboratory measurements of nuclear reactions if the bombarding energy is sufficiently low. As was the case for a plasma, the screened laboratory cross section is larger compared to the unscreened one. In cases where such effects are found to be significant, the measured cross sections have to be multiplied by the appropriate screening factors, which differ from those derived for a stellar plasma, in order to calculate the laboratory cross section for bare nuclei (Assenbaum, Langanke and Rolfs 1987, Raiola et al. 2002, and references therein). In a second step, the latter cross section may then be corrected for plasma screening effects when computing the screened reaction rates.

3.2.7

Total Reaction Rates

For the calculation of the total reaction rates, all processes contributing significantly to the reaction mechanism in the effective stellar energy range have to be taken into account. The effective energy range is given by the Gamow peak or the Maxwell–Boltzmann distribution for reactions induced by charged particles or neutrons, respectively. The details will be different for each nuclear reaction, but some general statements are useful at this point. We will use a capture reaction as an example.

Consider first low stellar temperatures corresponding to effective energies close to the particle threshold. For light target nuclei, the density of resonances in this energy region is relatively small and they can be resolved experimentally. For charged particles, the resonance strengths are usually determined by the small charged-particle partial width Γ_a (since $\omega \gamma \approx \omega \Gamma_a$). For neutrons, on the other hand, $\omega \gamma \approx \omega \Gamma_{\gamma}$. All contributions of narrow resonances have to be measured or estimated, since they may strongly influence the total reaction rates. If the resonances are too weak or if none are located at the effective stellar energies, then other processes, such as high-energy wings of subthreshold resonances, low-energy wings of broad resonances located at higher energies, and nonresonant reaction contributions, are likely to dominate the total rates. As already noted, charged-particle measurements are typically performed down to an energy of E_{\min} . Direct measurements at lower energies are difficult, if not impossible, with present experimental techniques. In this case, any expected narrow resonances have to be investigated indirectly by nuclear structure studies (Fig. 3.25), while nonresonant cross sections

or wings of broad resonances have to be extrapolated from measurements at higher energies. In neutron-induced reactions, on the other hand, an experimental cutoff energy E_{min} does not exist since the Coulomb barrier is absent. Hence, the cross section can in principle be measured directly at the effective stellar energies.

For increasing stellar temperatures, the density of resonances in the effective energy range will become larger. These resonances are located at higher energies so that, for charged-particle reactions, the particle partial width may exceed the γ -ray partial width ($\Gamma_a \gg \Gamma_{\gamma}$), and hence $\omega \gamma \approx \omega \Gamma_{\gamma}$. The strengths or cross sections of narrow or broad resonances with energies of up to a few MeV have been measured for many reactions.

At even higher energies, corresponding to effective energies in excess of a few MeV, the number of resonances and their total widths become so large that they strongly overlap. Note that this situation pertains already at low energies in the case of neutron-induced reactions on heavy target nuclei when the $Q_{n\gamma}$ value is large. In some reactions, individual resonances are no longer resolved and the total cross section gives rise to a continuum that varies smoothly with energy. In other reactions, individual resonances may still be resolved, but their density in the effective stellar energy window is so large that only the energy-averaged cross section is of interest. Cross sections for some reactions have been measured directly in this energy regime.

As will be explained in Chapter 5, the nucleosynthesis in certain burning processes can involve a large number of reactions (from several hundred in the case of silicon burning to several thousands in the case of the p-process), many of which proceed on unstable target nuclei. Clearly, only a small fraction of these reactions has been measured and in the vast majority of cases the cross sections need to be estimated by using theoretical models. The most successful among these is the Hauser-Feshbach statistical model (Section 2.7). It assumes that near the incident energy there is a large number of levels for each J^{π} value in the compound nucleus through which the reaction can proceed. The Hauser-Feshbach formula (see Eq. (2.219)) predicts a cross section reliably if the input parameters, such as transmission coefficients and level densities, are fine-tuned for the reaction of interest. In reality, however, the number of unmeasured reactions is very large and it becomes therefore important to compute the desired cross sections with global instead of local parameters. For proton- and neutron-induced reactions, such global Hauser-Feshbach calculations are found to yield cross sections and reaction rates that are reliable within a factor of \approx 2–3, provided that the level density in the compound nucleus is sufficiently large (say, at least ten compound levels in the effective stellar energy window). For α -particle-induced reactions, however, the theoretical predictions are less reliable due to difficulties in constructing appropriate global optical model potentials. For comparisons of Hauser-Feshbach

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predictions with measured cross sections, see Rauscher et al. (1997), Sargood (1982) and Arnould and Goriely (2003). Another obvious advantage of the Hauser–Feshbach model is that it can include the effects of thermally excited target states in a straightforward manner (Section 3.1.5).

The various contributions to the total reaction rates may be added incoherently if interferences are negligible, so that

$$N_A \langle \sigma v \rangle_{\text{total}} = \sum_i N_A \langle \sigma v \rangle_{\text{narrow}}^i + \sum_k N_A \langle \sigma v \rangle_{\text{broad}}^k$$
$$+ N_A \langle \sigma v \rangle_{\text{nonresonant}} + N_A \langle \sigma v \rangle_{\text{continuum}}$$
(3.146)

To a good approximation, interference effects are negligible for narrow resonances ($\Gamma < 1 \text{ eV}$). No interference effects are expected between two broad resonances of different J^{π} values, or between a resonance and a nonresonant process of different incoming orbital angular momenta. In other situations, interference effects may need to be taken into account in Eq. (3.146).

Examples for measured charged-particle-induced reaction cross sections have already been discussed in connection with the (p,γ) reactions on ¹³C and ¹⁶O (Figs. 3.10 and 3.11). Both of these cross sections have relatively simple energy dependences. In many other charged-particle reactions, however, the total cross section has a complex structure. A schematic example for a typical *S*-factor is displayed in Fig. 3.29, showing nonresonant contributions, narrow and broad resonances, and a continuum at higher energies caused by many



Fig. 3.29 Schematic representation of an *S*-factor versus energy for a charged-particle-induced reaction. At low energies, narrow resonances (NR), wings of subthreshold resonances (SR), tails of broad resonances (TBR), and nonresonant processes (NNR) may typically contribute to the total *S*-factor. At higher energies, the *S*-factor is typically dominated by broad resonances (BR) and by overlapping narrow and broad resonances (OBR + ONR).



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Fig. 3.30 Cross sections for neutron capture on ⁷Li, ³¹P, and ⁹⁰Zr versus energy. The curve in the upper panel shows a 1/v behavior, while resonances are visible in the middle and lower panels.

overlapping contributions. The *S*-factor rather than the cross section is shown in Fig. 3.29 since the latter quantity drops rapidly for decreasing energy. Total reaction rates $N_A \langle \sigma v \rangle$ of charged-particle-induced reactions depend strongly on temperature, as shown in Sections 3.2.1 and 3.2.4. The reaction rates fall rapidly for decreasing temperature in most reactions of astrophysical interest when the effective energies are below the height of the Coulomb barrier. Examples for reaction rates have already been discussed (Fig. 3.28b).

Examples of cross sections for neutron capture on a light, medium and heavy target nucleus are shown in Fig. 3.30. The cross section for the ⁷Li(n, γ)⁸Li reaction ($Q_{n\gamma} = 2.0$ MeV) follows the 1/v law over the entire neutron energy range shown ($E_n = 1$ –100 keV). For ³¹P(n, γ)³²P ($Q_{n\gamma} = 7.9$ MeV) the cross section varies smoothly up to about E = 20 keV where a few nar-

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row isolated resonances start to appear. For 90 Zr(n, γ) 91 Zr ($Q_{n\gamma} = 7.2$ MeV) many narrow and broad resonances are apparent. The density of resonances increases for larger neutron energies and they start to overlap strongly beyond an energy of ≈ 10 keV. The vastly different energy dependence of neutron reaction cross sections compared to charged-particle-induced reactions (Figs. 3.10 and 3.11) is caused by the absence of the Coulomb barrier. The corresponding Maxwellian-averaged cross sections, $\langle \sigma v \rangle / v_T$, versus kT are displayed in Fig. 3.31. It is apparent that neutron reaction rates are far less temperature sensitive compared to charged-particle reaction rates.



Fig. 3.31 Maxwellian-averaged cross sections versus kT for ${}^{7}\text{Li}(n,\gamma){}^{8}\text{Li}$, ${}^{31}\text{P}(n,\gamma){}^{32}\text{P}$, and ${}^{90}\text{Zr}(n,\gamma){}^{91}\text{Zr}$. Data from Bao et al. (2000).

Problems

3.1 Consider a situation where the three species *A*, *B* and *C* achieve equilibrium at elevated temperatures via the reactions $A + a \leftrightarrow B + \gamma$ and $B + b \leftrightarrow C + \gamma$ (Fig. 3.7). In addition to Eqs. (3.55) and (3.56), the two conditions $\lambda_{C \to B} > \lambda_{C \to C'}$ and $\lambda_{B \to C} > \lambda_{B \to B'}$ must be fulfilled in order for such an equilibrium to be established. Derive an expression for $\lambda_{A \to B \to (C \to C' \text{ or } B')}$, that is, the decay constant of species *A* for consumption via the paths $A \to B \to C \to C'$ or $A \to B \to B'$.

3.2 Derive the correction factor $F(\tau)$ for nonresonant charged-particle-induced reaction rates (see Eq. (3.90)). Start by expressing *F* in terms of the new variables $y \equiv \sqrt{\epsilon} - 1$, $\beta \equiv \sqrt{3/\tau}$ and $\zeta \equiv y/\beta$. Then expand $F(\beta)$ into a quadratic Taylor series.

3.3 Derive the thermonuclear rate for nonresonant neutron-induced reactions when $S \equiv \sigma v$ depends on velocity (see Eq. (3.101)).

3.4 For an arbitrary value of ℓ , find the γ -ray energy at which the decay constant for nonresonant (γ , n) reactions (that is, the integrand in Eq. (3.109)) has a maximum.

3.5 Consider the narrow resonances described in Example 3.6. Calculate the reaction rates numerically for T = 0.02 GK and T = 0.08 GK and show that the arguments based on the Gamow peak concept are valid.

3.6 Consider the ²⁰Ne(γ, α)¹⁶O photodisintegration reaction at a temperature of T = 1.5 GK. The lowest lying narrow resonances in the forward ¹⁶O(α, γ)²⁰Ne reaction (Q = 4730 keV) are located at $E_r^{cm} = 891$ keV, 1058 keV, and 1995 keV, corresponding to ²⁰Ne levels at $E_x = 5621$ keV, 5788 keV, and 6725 keV, respectively (Fig. 5.46). Their (ground-state) strengths amount to $\omega\gamma = 1.9 \times 10^{-3}$ eV, 2.3 × 10⁻² eV, and 7.4 × 10⁻² eV, respectively (Angulo et al. 1999). Which level do you expect to dominate the stellar ²⁰Ne(γ, α)¹⁶O reaction rates? Calculate and compare the individual level contributions to the total photodisintegration reaction rates. The spins of ⁴He, ¹⁶O, and ²⁰Ne are all $j_i = 0$; the normalized partition functions for these nuclei are equal to unity at T = 1.5 GK (see Rauscher and Thielemann 2000). Also, the first excited state in ¹⁶O is located at a relatively high energy ($E_x = 6049$ keV; Tilley, Weller and Cheves 1993) and, therefore, the (forward) capture reaction from excited target states is negligible at this temperature.

3.7 Derive the transmission coefficient (see Eq. (3.138)) for the screened Coulomb potential (see Eq. (3.132)). Assume that the variable $x = x(E) = R_c/R_D$ is a small number and use the expansions $e^x \approx 1 + x$ and $\sqrt{1-x} \approx 1 - x/2$. In the derivation retain only terms that are linear in x.

3.8 Calculate the electron screening correction for the ${}^{12}\text{C} + {}^{12}\text{C}$ reaction under typical hydrostatic carbon burning conditions (T = 0.9 GK and $\rho = 10^5$ g/cm³; Section 5.5.1). The mass fractions of carbon, oxygen, and neon are given by $X({}^{12}\text{C}) = 0.25$, $X({}^{16}\text{O}) = 0.73$, $X({}^{20}\text{Ne}) = 0.01$, and $X({}^{22}\text{Ne}) = 0.01$. Assume that the reaction is nonresonant and disregard the electron degeneracy factor ($\theta_e = 1$).

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4 Nuclear Physics Experiments

4.1 General Aspects

In this chapter, experimental techniques and procedures are discussed that are frequently applied in investigations of astrophysically important reactions. A vast number of different experimental procedures are used in the field of nuclear astrophysics. These can be divided into two groups, that is, direct and indirect measurements. A measurement of a cross section or a resonance strength in a given reaction of astrophysical interest is referred to as direct measurement. All other studies that are performed in order to improve the thermonuclear rates of this particular reaction, for example, elastic scattering, particle transfer, charge-exchange, and so on, represent indirect measurements (see also Section 3.2.4 and Fig. 3.25). Here, we will focus our attention on *direct* measurements of nuclear reactions and a number of topics will be discussed in some depth. In most of this chapter, with the exception of Sections 4.8 and 4.9, all quantities are given in the laboratory system, unless mentioned otherwise. Expressions that relate kinematic quantities in the center-of-mass system and the laboratory system can be found in Appendix C.

Figure 4.1 shows schematically some major experimental components involved in nuclear reaction measurements. An accelerator provides collimated beams of well-defined energy. The beam is directed to a target which contains the target nuclei involved in the nuclear reaction. The target has to be stable under beam bombardment. The nuclear reaction takes place in the target. Radiative capture reactions, $A(a,\gamma)B$, are among the most important types of reactions occurring in stars, but reactions involving only particles with rest mass, A(a,b)B, are of importance as well. The reaction products (for example, γ -rays or light particles) emitted from the target are measured by a suitable detector of high efficiency. From the measured energies and intensities the nuclear properties of interest (resonance and excitation energies, cross sections, spins and parities, lifetimes, branching ratios, angular correlations, and so on) are deduced. Frequently, unwanted background will contribute to the signal count rate of interest. It is important to reduce this background through various means to tolerable levels.

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Fig. 4.1 Basic components for the measurement of astrophysically important nuclear reactions. See the text. The shield must surround the detector as completely as possible.

A discussion of sources, accelerators and beam transport systems can be found in Rolfs and Rodney (1988). Here, we will briefly summarize some of the key requirements for beams in nuclear astrophysics measurements.

4.1.1

Charged-Particle Beams

The effective ion energies in a stellar plasma depend both on the temperature and the charges of the projectile and target nucleus involved in the reaction. It has been shown in Section 3.2 that thermonuclear reactions most likely proceed at energies below the Coulomb barrier height (Figs. 3.12 and 3.13). Therefore, accelerators have to cover the energy range below a few MeV for direct measurements of reactions. Indirect measurements, which investigate the structure of astrophysically important nuclei, are typically performed above the Coulomb barrier (that is, in the tens of MeV range). Most measurements of charged-particle reactions in nuclear astrophysics have been carried out using ion beams from electrostatic accelerators. Different types of electrostatic accelerators, such as Van de Graaff, Cockroft–Walton, Dynamitron, and Pelletron, are widely used.

It must be remembered that cross sections of charged-particle reactions in general decrease rapidly with decreasing beam energy because of the Coulomb barrier (Figs. 3.10 and 3.11). Therefore, measurements far below the Coulomb barrier require relatively large ion beam currents up to the mA range in order to initiate a statistically significant number of nuclear reactions. For example, a 1 mA current of singly charged protons or α -particles corresponds to $N_i/t = I/e = (1 \times 10^{-3} \text{ A})/(1.6 \times 10^{-19} \text{ C}) = 6.25 \times 10^{15}$ incident particles per second, where $e = 1.6 \times 10^{-19} \text{ C}$ is the elementary charge. At higher energies (say, above 1 MeV) smaller currents in the 0.1–10 µA range are of advantage if detector count rates become limited by the intense radiation from contaminant reactions. Ion energy spreads of about 1 keV or better are usually necessary to resolve complex resonance structures.

The ion beam energy should be variable in steps of at least a few 100 eV in order to measure precise resonance energies. The beam should also be well collimated. Low-energy measurements far below the Coulomb barrier require a beam spot size of a few square centimeters cross-sectional area (for solid targets) in order to reduce target heating and degradation to tolerable levels. Indirect measurements at higher energies usually require beam spots of much smaller size (a few square millimeters). The beam should also be as free of contaminants as possible.

The absolute energy calibration of the electrostatic accelerator is an important quantity for the determination of thermonuclear reaction rates. The precise energy of narrow resonances enters sensitively in the narrow resonance reaction rate formalism (see Eq. (3.114)). Furthermore, nonresonant cross sections are a steep function of energy below the Coulomb barrier and, thus, systematic shifts in absolute beam energy may cause large errors in the nonresonant reaction rates. We will illustrate this effect with two examples.

Consider first the $E_r = 151$ keV resonance in the ${}^{18}\text{O}(\text{p},\gamma){}^{19}\text{F}$ reaction corresponding to an energy of 143 keV in the center-of-mass system (see Eq. (C.24)). Suppose, that a measurement of E_r yields an erroneous value of 148 keV (or 140 keV in the center-of-mass system). At a temperature of T = 0.06 GK, the resulting narrow resonance reaction rates (see Eq. (3.114)) will then be too high by a factor of

$$\frac{N_A \langle \sigma v \rangle_{E_r - \Delta E}}{N_A \langle \sigma v \rangle_{E_r}} = \frac{e^{-11.605(0.143 - 0.003)/0.06}}{e^{-11.605 \cdot 0.143/0.06}} \approx 1.80$$
(4.1)

corresponding to a variation of 80%. The variation will increase for lower temperatures.

As an example for a nonresonant reaction, consider ${}^{16}O(p,\gamma){}^{17}F$ at 100 keV in the center-of-mass system. If the measurement is erroneously performed at a center-of-mass energy of 103 keV, then the cross section (see Eqs. (2.126) and (3.70)) will be too high by a factor of

$$\frac{\sigma(E+\Delta E)}{\sigma(E)} = \frac{\frac{1}{0.103} \exp\left(-0.9895 \cdot 1 \cdot 8\sqrt{\frac{16\cdot 1}{16+1}\frac{1}{0.103}}\right)}{\frac{1}{0.100} \exp\left(-0.9895 \cdot 1 \cdot 8\sqrt{\frac{16\cdot 1}{16+1}\frac{1}{0.100}}\right)} = 1.40$$
(4.2)

corresponding to a variation of 40%. For this estimate we assumed a negligible energy dependence of the *S*-factor (see Eq. (3.10)).

Frequently, a magnetic analyzer with input and output slits is used to define the beam energy. For an ideal system, the magnetic field strength *B* and the

particle energy *E* are related by (Marion 1966)

$$B = \frac{k}{q}\sqrt{2mc^2E + E^2} \tag{4.3}$$

where mc^2 and q are the rest energy and the charge state of the ion, respectively. The calibration constant k cannot be calculated precisely from the magnet geometry since B is not necessarily constant along the particle trajectory through the magnet and, furthermore, the magnetic field along the trajectory may not be proportional to the field measured at some reference point (for example, using a NMR or a Hall probe). Therefore, k must be obtained through a calibration of the magnet by using energies of well-known nuclear reactions. For this purpose, narrow resonances are frequently used below $E \approx 2$ MeV, while (p,n) threshold energies are utilized at higher energies.

Absolute resonance energies of selected resonances below 1.5 MeV energy are listed in Table 4.1. It is interesting to point out that almost all published resonance energies are directly or indirectly related to the energy of the E_r = 992 keV resonance in ²⁷Al(p, γ)²⁸Si. The table also lists total resonance widths, which should be small (less than 1 keV) for energy calibration standards. The determination of precise resonance energies from measurements of the reaction yield versus energy will be discussed in Section 4.8.

Tab. 4.1 Laboratory energies and widths of narrow resonances commonly used for ion beam calibrations. Data from (a) Uhrmacher et al. (1985), (b) Bindhaban et al. (1994), (c) Becker et al. (1995), and (d) Endt (1998). Errors are given in parentheses and refer to the last significant digit(s). For example, 150.82(9) stands for 150.82 ± 0.09 .

Reaction	$E_{\rm lab}$ (keV)	Γ (eV)
¹⁸ O(p,α) ¹⁵ N	150.82(9)°	130(10)°
¹⁹ F(p,αγ) ¹⁶ O	223.99(7)ª	985(20)ª
	483.91(10)ª	903(30)ª
23 Na(p, γ) 24 Mg	308.75(6) ^a	< 36 ª
24 Mg(p, γ) 25 Al	222.89(8) ^a	< 32 ª
$^{26}Mg(p,\gamma)^{27}Al$	292.06(9) ^a	< 37 ª
27 Al(p, γ) 28 Si	222.82(10)ª	< 34 ª
	293.08(8)ª	59(16)ª
	326.97(5)ª	$< 38^{a}$
	405.44(10)ª	<42ª
	991.756(17) ^b	70(14) ^d
	1316.87(3) ^b	35(4) ^d

4.1.2 Neutron Beams

For measurements of neutron-induced reactions on stable or long-lived target nuclei (Section 5.6.1), neutron beam energies between a fraction of a keV and several hundred keV are of primary interest. Neutrons can be produced using

a variety of techniques, including linear electron or proton accelerators, and electrostatic accelerators.

At linear electron accelerators, neutrons are produced via (γ ,n) reactions by bombarding heavy metal targets with pulsed electron beams of $\approx 50 \text{ MeV}$ energy and repetition rates of ≈ 0.5 kHz. The neutrons are released with energies ranging from the subthermal region up to 50 MeV. They are slowed down by a moderator (Section 4.2.3) and are collimated before they impinge on the sample of interest. The primary electron beam produces a very intense background caused by bremsstrahlung and thus the metal target area needs to be shielded well. The astrophysically important neutron energy range corresponds only to a small window of the entire neutron spectrum. Neutrons with a similar broad energy distribution are produced with high-energy proton beams at linear accelerators. In this case, the primary beam is incident on a suitable target and neutrons are produced via spallation reactions. Fluxes on the order of $\approx 10^6$ neutrons s⁻¹ cm⁻², integrated over an energy range of 1-300 keV, are typically achieved at both kinds of facilities (Koehler 2001). Reaction measurements with these moderated neutron sources are performed by using time-of-flight techniques (Section 4.6.3).

Charged-particle beams from electrostatic accelerators can be utilized to produce neutrons via nuclear reactions (Hanson, Taschek and Williams 1949). For relatively low neutron energies of astrophysical interest, a frequently employed reaction is ${}^{7}\text{Li}(p,n){}^{7}\text{Be}$ (Q = -1.644 MeV). It follows from the kinematics of this endothermic reaction that at the threshold $(E_p^{\text{thresh}} \approx -Q(m_n +$ m_{7Be} / M_{7Be} = 1.881 MeV; Eq. (C.8)) neutrons are released with an energy of 30 keV and they are emitted in the forward direction only. For proton bombarding energies up to $E_p = 1.92$ MeV, neutrons are emitted into a cone of limited angle in the forward direction. At each angle within this cone, two groups of neutrons with different energies are emitted. The cone widens with increasing proton energy until it includes the forward hemisphere. For $E_p >$ 1.92 MeV, neutrons of only one discrete energy are emitted at each angle in the complete sphere about the target. The kinematics of endothermic reactions is discussed in more detail in Appendix C.1. The energy resolution of the released neutrons depends on the energy spread of the incident protons, the finite thickness of the ⁷Li target, and the finite angle subtended by the sample to be irradiated.

An interesting technique has been applied in a number of neutron-induced reactions by bombarding a $\approx 10 \ \mu m$ thick metallic lithium target with protons of energy $E_p = 1912 \ \text{keV}$, only 31 keV above the reaction threshold. The released neutrons are emitted in the forward direction in a cone with an opening angle of 120°. In this case, the angle-integrated energy distribution of emitted neutrons closely resembles a Maxwell–Boltzmann distribution at $kT = 25 \ \text{keV}$, as shown in Fig. 4.2. If the irradiation sample is mounted very close





Fig. 4.2 Angle-integrated neutron energy distribution resulting from the bombardment of $a \approx 10 \,\mu\text{m}$ thick metallic lithium target with $E_p = 1912 \,\text{keV}$ protons. The neutrons are emitted in the forward direction in a cone with an opening angle of 120° . The angle-integrated energy distribution

of emitted neutrons closely resembles a Maxwell–Boltzmann distribution at kT =25 keV. Reprinted with permission from W. Ratynski and F. Käppeler, Phys. Rev. C, Vol. 37, p. 595 (1988). Copyright (1988) by the American Physical Society.

to the lithium target, then the energy distribution of neutrons incident on the sample is given by the same Maxwell–Boltzmann distribution. The measured average cross section gives then directly the Maxwellian-averaged cross section or the reaction rate (Section 3.2.2), as will be shown in Section 4.9.3. With typical proton beam currents of 50–100 µA, integrated yields of $\approx 10^8-10^9$ neutrons/s are achieved (Beer and Käppeler 1980). This technique is useful because the energy of kT = 25 keV is close to the effective energy range of some s-process scenarios (Section 5.6.1). A similar procedure, but using the ³H(p,n)³He or ¹⁸O(p,n)¹⁸F reactions instead of ⁷Li(p,n)⁷Be, yields Maxwell–Boltzmann distributions of neutron energies at kT = 52 keV (Käppeler, Naqvi and Al-Ohali 1987) or 5 keV (Heil et al. 2005), respectively. Direct measurements of Maxwellian-averaged cross sections are frequently performed by using the activation method (Section 4.6.2).

Cross sections of neutron-induced reactions are usually much larger compared to those of charged-particle-induced reactions (Figs. 3.10, 3.11, and 3.30), thus compensating for the fact that neutron beam intensities are much lower compared to the intensities that are available for the study of chargedparticle-induced reactions.

4.2 Interaction of Radiation with Matter

Radiation interacts with matter and thereby loses part or all of its energy. This aspect is important for a number of experimental considerations. First, a particular incident particle may lose energy in the target prior to initiating a nuclear reaction. An exact knowledge of the energy loss is required in order to determine the effective energy and the probability with which the reaction takes place. Second, the energy or intensity of emitted reaction products may be influenced by interactions in the target or the surrounding material. Third, the reaction products have to be detected in order to determine the reaction cross section, that is, the probability with which the reaction occurs. Thus, knowledge of the processes by which radiation interacts with matter is of paramount importance for the design and performance of radiation detectors.

Figure 4.3 indicates schematically some experimental locations where radiation typically interacts with matter: (i) incident particle energy loss in the target, (ii) reaction product energy or intensity loss in the target, target holder, detector dead layer, and so on, and (iii) energy deposition of reaction products in the active volume of the detector. The processes responsible for the interaction of radiation with matter depend on the type of radiation. In the following, interactions of heavy charged particles (for example, protons and α -particles), photons, and neutrons are discussed in more detail. We will refer to the material in which the interactions occur as *absorber*.

A frequently used quantity for the considerations of the present chapter is the number density N of atoms (in units of atoms per cubic centimeter). For a solid absorber with mass density ρ , consisting of atoms with relative atomic mass M (in units of u), there are N_A/M atoms per gram of absorber material. The number density of atoms is then given by

$$N = \rho \frac{N_A}{M} \tag{4.4}$$

For an absorber gas at pressure P and temperature T, the number density of atoms can be calculated from

$$N = \nu L \frac{P}{760 \operatorname{torr}} \frac{273 \operatorname{K}}{T}$$
(4.5)

with the Loschmidt constant $L = 2.68677 \times 10^{19} \text{ cm}^{-3}$ and ν the number of atoms per molecule.

4.2.1

Interactions of Heavy Charged Particles

Heavy charged particles, such as protons or α -particles, interact with matter primarily through: (i) inelastic collisions with atomic electrons of absorber



Fig. 4.3 Schematic setup showing a beam incident on a target. The locations at which the primary (beam) or secondary (emitted particle or photon) radiation typically interacts with matter are circled.

atoms, and (ii) elastic scattering on absorber nuclei. These interactions cause an energy loss of the incident particle and a deflection of the particle from its incident direction. The former interaction occurs much more frequently compared to the latter, except at very low projectile energies where the contribution of elastic scattering on absorber nuclei has to be taken into account.

A heavy (positively) charged particle moving through matter interacts simultaneously with many electrons. Cross sections for these collisions are typically in the 10^{-17} – 10^{-16} cm² range (corresponding to 10^7 – 10^8 b). In each encounter, an electron feels the attractive Coulomb force as the charged particle passes in close vicinity. Energy is transferred from the particle to an absorber atom, causing either excitation of an atomic electron to higher lying shells (soft collision) or complete removal of an electron, that is, ionization (hard collision). The maximum energy that can be transferred in each collision is a small fraction of the particle's total energy, but the number of collisions per path length is very large. At any given time the particle interacts with many electrons, causing an almost continuous energy loss until the particle is stopped. The paths of heavy particles in matter are relatively straight because the particle is not strongly deflected by any one collision. After ionization, the electrons tend to recombine with positive ions. Most types of radiation detectors suppress the recombination process and utilize the number of created electron-ion pairs as a basis for the detector response (see Eq. (4.4)). In certain very close encounters, sufficient energy may be transferred to an electron that it can create electron-ion pairs in subsequent collisions. These high-energy electrons are referred to as δ (or knock-on) electrons.

Stopping power

The collisions of heavy charged particles with absorber atoms are statistical in nature. Since the number of collisions per path length is very large, the fluctuations in the total energy loss are small. Thus, the slowing down process may be described in terms of an average energy loss per unit path length. The ratio of differential energy loss and differential path length is called *linear stopping power*, and is defined by

$$S_L(E) \equiv -\frac{dE}{dx} \tag{4.6}$$

in units, for example, of eV/cm. The linear stopping power depends on the number density of electrons in the absorber or, equivalently, the absorber mass density ρ . The related quantity

$$S_M(E) \equiv -\frac{1}{\rho} \frac{dE}{dx} \tag{4.7}$$

is called *mass stopping power* with units, for example, of $eV cm^2/g$. The stopping power may also be given per absorber atom. For an absorber of number density *N* (in units of atoms per cubic centimeter) we obtain

$$S_A(E) \equiv -\frac{1}{N} \frac{dE}{dx}$$
(4.8)

in units, for example, of $eV \operatorname{cm}^2/\operatorname{atom}$. The quantity $S_A(E)$ is called *stopping cross section*. For a given projectile of energy *E*, it is found that $S_M(E)$ and $S_A(E)$ vary relatively little over a wide range of absorber materials. In numerical calculations, we will mainly be using the quantity $S_A(E)$ and simply refer to it as stopping power.

The theoretical calculation of stopping powers is complicated. For high projectile energies (> 0.6 MeV/u) the Bethe–Bloch formula, with a few empirically determined parameters, describes the energy loss accurately. For nonrelativistic projectile energies, the electronic stopping power (that is, the contribution due to inelastic collisions between projectile and atomic electrons) is given by (Knoll 1989)

$$-\frac{dE}{dx} \approx \frac{4\pi e^4}{m_e} \frac{Z_p^2}{v^2} \left(N_A \rho \frac{Z_t}{M_t} \right) \ln\left(\frac{2m_e v^2}{I}\right)$$
(4.9)

where Z_p , v, Z_t , and M_t are the charge and velocity of the projectile, the atomic number and the relative atomic mass of the absorber, respectively; m_e is the electron rest mass, e the electron charge, and I represents an average excitation and ionization potential of the absorber which is treated as an empirical parameter. The equation holds if the projectile velocity is large compared to

the electron velocities in the absorber atoms. Over a wide energy range, excluding very high energies where the logarithmic term in Eq. (4.9) dominates, the magnitude of the stopping power decreases with increasing projectile energy as $1/v^2$ or 1/E. This behavior can be explained by the fact that the projectile spends a greater time in the vicinity of a given electron if its velocity is small and, consequently, the energy transfer becomes large. The stopping power increases with Z_p^2 . Hence, α -particles will experience a larger energy loss compared to protons in the same absorber medium. The stopping power also depends linearly on the absorber density ρ .

At very low projectile energies (E < 30 keV/u), where the projectile velocity is smaller than the electron velocities in the absorber atoms, the Bethe–Bloch formula is no longer applicable. This situation occurs in the slowing down process of recoil nuclei (for example, implantation or lifetime measurements). In this case, the projectile energy is too small to cause significant ionization of the absorber atoms. Also, the positively charged projectile tends to pick up electrons from the absorber. As a result, its effective charge and the stopping power are reduced. For this energy range, the electronic stopping power is usually calculated by the LSS theory (Lindhard, Scharff and Schiott 1963) which is not as accurate as the Bethe–Bloch formula. The electronic stopping power is given by

$$-\frac{dE}{dx} = k\sqrt{E} \tag{4.10}$$

where the constant *k* is a function of the masses and charges of projectile and absorber atoms. In addition, the contribution from elastic scattering of projectiles on absorber nuclei (nuclear stopping power) has to be taken into account at low energies. The intermediate energy range (30 keV/u < E < 0.6 MeV/u) is poorly covered by theory and a number of different formulas are in use.

The stopping power is shown schematically in Fig. 4.4. At very low energies, it is influenced by the nuclear component (dotted line) and, with increasing energy, follows the \sqrt{E} behavior predicted by the LSS theory (dashed– dotted line). A maximum occurs where the velocities of the projectile and the atomic electrons of the absorber are comparable. For higher energies beyond the maximum, the stopping power is given by the Bethe–Bloch formula (dashed line). For nonrelativistic projectile energies, the stopping power is dominated by the 1/E dependence and decreases until $v \approx 0.96c$, where a minimum is reached. At this point, the projectiles are called *minimum ionizing*. This minimum value of dE/dx is approximately constant for all particles of the same charge Z_p . Beyond this point, the stopping power increases due to the logarithmic term in the Bethe–Bloch formula.

In practice, measured values of stopping powers are fit over a broad energy range by expressions containing the proper low- and high-energy behaviors. These fits may then be used for interpolations to obtain stopping powers for



Fig. 4.4 Schematic representation of total stopping power (solid line) and different components (dashed or dotted lines) versus particle energy. See the text.

absorbers for which no experimental information exists. Tabulated stopping powers, including compilations of experimental values, can be found in Paul and Schinner (2002) or Ziegler (2003). As an example, Fig. 4.5 shows stopping powers in units of eV cm²/atom for hydrogen and helium projectiles in various absorber elements versus energy.

It will become apparent later in this chapter that stopping power values enter in most experimental determinations of charged-particle cross sections and resonance strengths. Therefore, a reliable estimate of stopping power errors is very important. Errors of stopping powers calculated by using the computer code SRIM (Ziegler 2003) amount to a few percent at higher energies where the Bethe-Bloch formula is applicable. However, at lower energies, which are important for direct nuclear astrophysics measurements, the errors are typically larger. The uncertainties of calculated stopping powers for a given projectileabsorber combination can be estimated from the average deviation between calculated and measured values and from the scatter in the measured data. Table 4.2 provides some useful information. It lists stopping power uncertainties for several projectiles and absorbers. The errors are estimated by considering tabulated stopping powers, obtained with the code SRIM, together with all measured data in a projectile energy range that is of primary importance for direct measurements in nuclear astrophysics. The results are given in terms of the quantities Δ and σ (both given in percent), which are defined in Paul and Schinner (2005). In brief, Δ represents the *systematic* difference between tabulated and experimental stopping powers, while σ is the *random* error that provides information regarding the experimental scatter. It can be seen that in most cases Δ is close to zero and thus the SRIM tabulation is quite reliable.



Fig. 4.5 Stopping power (in units of $10^{-15}~{\rm eV\,cm^2/atom})$ of hydrogen (top) and helium (bottom) projectiles versus particle energy in various absorber elements. Both the projectile energy and the stopping power are given in the laboratory system. The curves are calculated by using the computer code SRIM (Ziegler 2003).

Only in the case of heavy projectiles incident on H₂ gas is there indeed a systematic deviation, amounting to $\Delta \approx -3\%$. The experimental scatter is seen to depend on the physical state of the absorber. For protons and α -particles incident on gaseous absorbers one finds values of $\sigma \approx 3 - 4\%$, while for solid absorbers the experimental scatter amounts to $\sigma \approx$ 5 – 8%. Similar values are obtained for heavy projectiles incident on H2 gas. Obviously, the stopping power errors are expected to be larger for projectile-absorber combinations for which no data exist.

Figure 4.6b shows schematically a beam of projectiles with energy E_0 incident on an infinitely thick absorber, that is, the projectiles are stopped in the

Tab. 4.2 Errors of stopping powers. The quantity Δ represents the systematic difference between tabulated and experimental values, while σ is the random error that provides information regarding the experimental scatter (Paul and Schinner 2005). Tabulated and measured stopping power values are considered here only for the indicated projectile energy ranges that are appropriate for direct nuclear astrophysics measurements. Courtesy of Helmut Paul.

Projectile	Energy	Absorber	Number	Δ	σ
	(MeV/u)		of points	(%)	(%)
Protons	0.01–1.0	Al, B, Be, C, Ca, Co, Cr, Fe, Li, Mg, Mn, Ni, Se, Si, Ti, V	2518	0.7	8.0
Protons	0.01-1.0	Ar, Cl_2 , N_2 , Ne, O_2	504	0.1	3.9
α-particles	0.1–1.0	Al, B, Be, C, Ca, Co, Cr, Fe, Li, Mg, Mn, Ni, Si, Ti, V	975	0.7	4.7
α -particles	0.1-1.0	Ar, Cl ₂ , N ₂ , Ne, O ₂	428	-0.1	3.2
Ar, B, C, Cl, Li, Mg, N, Na, Ne, S	0.2–20	H_2 gas	136	-3.0	8.4

medium. Part (a) displays the stopping power versus energy. As the charged projectiles slow down in matter, their stopping power increases, that is, more energy is deposited per path length for an increasing length of the track. This may also be seen in a plot of dE/dx versus distance (part c). Near the end of the track (at energy E_1) the charge of the projectile is reduced due to electron pickup and the curves fall off. The maximum, called the Bragg peak, indicates that projectiles lose the largest part of their energy toward the end of their path. Part (d) shows the intensity of the projectiles in the absorber versus distance. For the largest distance over the projectile path, the intensity is constant. In other words, the projectiles slow down but their number does not change. Toward the end of the path, the intensity does not drop immediately to zero, but slopes down over a certain path length. This phenomenon is known as range straggling. It is caused by the statistical nature of the slowing down process, since two projectiles of the same mass and energy will, in general, not penetrate the absorber to exactly the same distance. The distance at which the projectile intensity falls by 50% is called the *mean range*.

The mean path length traveled by projectiles of incident energy E_0 can be calculated from

$$R = \int_0^{E_0} \frac{dE}{(dE/dx)} \tag{4.11}$$

Note that the value thus obtained will differ in general from the straight-line penetration distance because each projectile is deflected slightly in each of the many collisions with absorber atoms. Figure 4.7 shows ranges versus incident energy for light ions in a silicon absorber. For example, if a silicon counter is used for measuring the total energy of incident charged particles (Sections 4.4.2 and 4.5.1), then its active thickness must be larger than the particle range.



Fig. 4.6 Schematic representation of a beam of projectiles with energy E_0 incident on an infinitely thick absorber (part b). (a) Stopping power versus projectile energy. (c) Stopping power versus distance. (d) Intensity of projectiles in absorber versus distance; the mean range corresponds to the distance at which the projectile intensity

falls to 50% of its initial value. The projectiles lose the largest part of their energy toward the end of the path. The maximum in parts (a) and (c), corresponding to an energy E_1 , is called the Bragg peak. Note that in general the mean range is not equal to the straight-line penetration distance; see the text.

The stopping power is also useful in cases where the projectiles lose only a fraction of their energy in an absorber. The thickness *d* of the absorber (in units of length) is related to the total energy loss of projectiles with incident energy E_0 by

$$d = \int_{E_0 - \Delta E}^{E_0} \frac{dE}{(dE/dx)}$$

$$\tag{4.12}$$

For a very thin absorber (target or detector), the energy lost by the projectile is relatively small and the stopping power is approximately constant over the absorber thickness. We obtain in this case

$$d = \frac{1}{(dE/dx)_{E_0}} \int_{E_0 - \Delta E}^{E_0} dE = \frac{\Delta E}{(dE/dx)_{E_0}}$$
(4.13)

or

$$\Delta E = \left(\frac{dE}{dx}\right)_{E_0} d = \left(\frac{1}{N}\frac{dE}{dx}\right)_{E_0} Nd = \left(\frac{1}{\rho}\frac{dE}{dx}\right)_{E_0} \rho d$$
(4.14)

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Fig. 4.7 Range in silicon versus incident laboratory energy of light ions. Reprinted from D. J. Skyrme, Nucl. Instrum. Methods, Vol. 57, p. 61 (1967). Copyright (1967), with permission from Elsevier.

where *N* denotes the number density of atoms and *Nd* is the number of atoms per square centimeter.

Compounds

We have discussed so far stopping powers for pure elements. For compounds, approximate stopping powers may be obtained by a weighted average over the individual stopping powers according to the fraction of electrons belonging to each element. This approximation is called *Bragg's rule*. For a compound $X_a Y_b$ consisting of two elements X and Y, where a and b are the number of atoms per molecule of element X and Y, respectively, one obtains for the stopping power per molecule

$$\frac{1}{N_c} \left(\frac{dE}{dx}\right)_c = a \frac{1}{N_X} \left(\frac{dE}{dx}\right)_X + b \frac{1}{N_Y} \left(\frac{dE}{dx}\right)_Y$$
(4.15)

where N_c is the number density of molecules and $(1/N_i)(dE/dx)_i$ is in units, for example, of eV cm² per atom (for pure elements) or per molecule (for the compound). Equivalently, we find with Eq. (4.4)

$$\frac{1}{\rho_c} \left(\frac{dE}{dx} \right)_c = \frac{aM_X}{M_c} \frac{1}{\rho_X} \left(\frac{dE}{dx} \right)_X + \frac{bM_Y}{M_c} \frac{1}{\rho_Y} \left(\frac{dE}{dx} \right)_Y$$
(4.16)

where $(1/\rho_i)(dE/dx)_i$ is in units, for example, of eV cm²/g, with $M_c = aM_X + bM_Y$. Caution should be exercised when using Bragg's rule since for certain compounds experimental stopping powers differ by as much as 10–20% from those calculated with Eqs. (4.15) and (4.16) (Knoll 1989).

For projectiles of incident energy E_0 , the energy loss in a thin absorber consisting of a compound $X_a Y_b$ is obtained from Eqs. (4.14) and (4.15) as

$$\Delta E_c = \frac{1}{N_c} \left(\frac{dE}{dx}\right)_{c,E_0} N_c d = a \frac{1}{N_X} \left(\frac{dE}{dx}\right)_X N_c d + b \frac{1}{N_Y} \left(\frac{dE}{dx}\right)_Y N_c d$$
$$= \frac{1}{N_X} \left(\frac{dE}{dx}\right)_X N_X d + \frac{1}{N_Y} \left(\frac{dE}{dx}\right)_Y N_Y d$$
(4.17)

where $N_X = aN_c$ and $N_Y = bN_c$.

Example 4.1

Calculate the energy loss of a 500 keV proton beam moving through a 1 µm thick layer of ice. The stopping powers at $E_0 = 500$ keV, calculated with the computer code SRIM (Ziegler 2003), amount to $(1/N_H)(dE/dx)_H = 1.8 \times 10^{-15}$ eV cm² and $(1/N_O)(dE/dx)_O = 8.1 \times 10^{-15}$ eV cm². Assume that the stopping powers are approximately constant over the thickness of the absorber. All of the above quantities are given in the laboratory system.

The density of ice (H₂O) is about 1 g/cm³, corresponding to 3.3×10^{22} H₂O molecules/cm³ (since 18 g contain 6.022×10^{23} H₂O molecules). Equation (4.17) gives

$$\Delta E = a \frac{1}{N_H} \left(\frac{dE}{dx}\right)_H N_{\text{H}_2\text{O}}d + b \frac{1}{N_O} \left(\frac{dE}{dx}\right)_O N_{\text{H}_2\text{O}}d$$

= 2(1.8 × 10⁻¹⁵ eV cm²)(3.3 × 10²² cm⁻³)(10⁻⁴ cm)
+ 1(8.1 × 10⁻¹⁵ eV cm²)(3.3 × 10²² cm⁻³)(10⁻⁴ cm) = 39 keV

The same result is obtained from the number densities of H and O atoms, $N_{\rm H} = 2 \cdot N_{\rm H_2O} = 6.6 \times 10^{22} \text{ cm}^{-3}$ and $N_{\rm O} = 1 \cdot N_{\rm H_2O} = 3.3 \times 10^{22} \text{ cm}^{-3}$. Hence

$$\begin{split} \Delta E &= \frac{1}{N_{\rm H}} \left(\frac{dE}{dx} \right)_{H} N_{H} d + \frac{1}{N_{\rm O}} \left(\frac{dE}{dx} \right)_{O} N_{\rm O} d \\ &= (1.8 \times 10^{-15} \, {\rm eV} \, {\rm cm}^2) (6.6 \times 10^{22} \, {\rm cm}^{-3}) (10^{-4} \, {\rm cm}) \\ &+ (8.1 \times 10^{-15} \, {\rm eV} \, {\rm cm}^2) (3.3 \times 10^{22} \, {\rm cm}^{-3}) (10^{-4} \, {\rm cm}) = 39 \, {\rm keV} \end{split}$$

Energy straggling

So far, we considered the mean energy loss of a projectile passing through an absorber. Suppose that the projectiles are represented by an initially monoenergetic beam. When a projectile penetrates into the absorber, it will undergo a large number of independent interactions, causing the projectile to slow down. Statistical fluctuations in the number of collisions and in the energy transferred per collision give rise to an energy distribution of the beam, centered around a value of $E_0 - \Delta E$, that is, the incident energy minus the mean energy loss.

The maximum energy that a nonrelativistic heavy charged particle of mass m and kinetic energy E can transfer to a free atomic electron with mass m_e in a single collision is of the order (Problem 4.2)

$$4E[m_{\rm e}m/(m_{\rm e}+m)^2] \approx 4E(m_{\rm e}/m) \approx 4E/(2000\,M) = 2 \times 10^{-3}E/M \quad (4.18)$$

with M in units of u. For example, a 10 MeV proton losing a total energy of 1 MeV in an absorber can transfer a maximum energy of 20 keV to a single electron and undergoes at least, and very likely many more than, (1 MeV)/(20 keV) = 50 collisions. If the number of collisions is large, the energy distribution function will approach a Gaussian shape (Leo 1987). If the number of collisions is not very large, the energy distribution function of projectiles passing through an absorber of a certain thickness will be skewed. A schematic representation of energy distribution functions f(E, x) as the projectiles move through the absorber is shown in Fig. 4.8, where x denotes the path length of the projectiles. A beam of projectiles with a small initial energy spread shows a broad and skewed distribution early on in the slowing down process. For increasing path length, corresponding to an increasing number of collisions between each projectile and the atomic electrons of the absorber, the energy spread increases, but the skewness weakens so that the energy distribution function resembles a Gaussian shape after a certain path length. Further energy loss close to the end of the projectile range results in a decreasing energy spread until all projectiles come eventually to rest in the absorber.

A useful approximation for the width of the Gaussian energy distribution function was derived by Bohr, assuming that the number of collisions between a nonrelativistic projectile and the absorber electrons is very large and that, at the same time, the mean energy loss is small compared to the initial projectile energy (that is, for relatively thin absorbers). The full width at half maximum (FWHM) of the energy distribution function is then given by (Bohr 1915)

FWHM =
$$2\sqrt{2\ln 2}\sqrt{4\pi e^4 Z_p^2 Z_t N d} = 1.20 \times 10^{-12} \sqrt{Z_p^2 Z_t N d}$$
 (MeV)
(4.19)

where in the numerical expression N and d are in units of atoms/cm³ and cm, respectively.



Fig. 4.8 Schematic representation of energy distribution functions f(E, x) for a beam of charged particles with a small initial energy spread as they move through an absorber along a path of length x; E is the particle energy. From G. F. Knoll, Radiation Detection and Measurement, 2nd edn., John Wiley & Sons (1989). Reprinted with permission. Copyright ©1989 John Wiley & Sons, Inc.

4.2.2

Interactions of Photons

Photons interact with matter through processes that are fundamentally different from those involving charged particles. The main interactions of γ -rays in matter are: (i) the photoelectric effect, (ii) Compton scattering, and (iii) pair production. All of these processes may transfer either the entire energy or a substantial fraction of the photon energy to an electron of an absorber atom in a single interaction. Therefore, the photon either disappears or is significantly deflected from its original direction of motion. These considerations have two important consequences. First, γ -rays are far more penetrating in matter compared to charged particles. Second, a beam of photons passing through matter is reduced in intensity depending on the absorber thickness. However, the photons that pass straight through the absorber did not undergo any interactions and hence possess their original energy.

The energetic electron, leaving the atom after the interaction, slows down in the absorber and thereby creates more charge carriers (electron–ion or electron–hole pairs). Gamma-ray detectors take advantage of these charge pairs in order to determine, for example, the total energy deposited by the incident photon in the absorber. Interaction processes involving photons are discussed in more detail below.

Photoelectric effect

In the photoelectric effect, a photon transfers its entire energy to a single electron of an absorber atom and hence it disappears. The electron, called photoelectron, is ejected from the atom with an energy of

$$E_e = E_\gamma - E_b \tag{4.20}$$

where E_{γ} and E_b are the incident photon energy and the binding energy of the photoelectron, respectively. A free electron cannot absorb a photon and at the same time conserve linear momentum. Hence, the photoelectric effect always involves bound electrons, with the whole atom absorbing the recoil momentum. For photon energies above 100 keV the photoelectron most likely originates from the K shell (the most tightly bound shell) of the atom.

The photoelectric effect transforms a neutral atom into an electron–ion pair. The vacancy in the ion is quickly filled through the capture of a free electron originating from other absorber atoms or by the rearrangement of electrons from other shells in the ion. During these secondary processes, characteristic X-ray photons or Auger electrons may be generated. In most cases, the X-rays undergo further photoelectric absorption near the primary photon–electron interaction site and thus their energy is retained in the absorber (Knoll 1989).

The photoelectric effect is difficult to treat theoretically. A useful approximation for the probability per absorber atom of photoelectric absorption for photon energies above 100 keV is given by (Evans 1955, Knoll 1989)

$$p_{\text{photo}} \sim \frac{Z^n}{E_{\gamma}^{7/2}} \tag{4.21}$$

where *Z* is the atomic number of the absorber and *n* varies between 4 and 5 over the γ -ray energy region between 0.1 and 5 MeV. As will be seen later, the photoelectric effect is the predominant interaction process for photons of relatively low energy. The strong *Z*-dependence in Eq. (4.21) is the reason for the fact that high-*Z* materials, such as lead, are used for the shielding against background γ -rays. For the same reason, high-*Z* materials are preferred as the active volume of γ -ray detectors.

A graph of the photoelectric absorption probability (in units of cm^2/g ; see later) for lead is shown in Fig. 4.9. For decreasing photon energies, the probability increases while the photon energy approaches the electron binding energy of the most tightly bound shell (the K shell) in the absorber atom. For slightly smaller photon energies, the probability drops drastically since the K electrons are no longer available for the photoelectric effect. This rapid decrease is called the K absorption edge. For even smaller photon energies, the probability increases again while the next shell (the L shell) is approached.



Fig. 4.9 Mass attenuation coefficient of lead versus γ -ray energy. The solid line corresponds to the total mass attenuation coefficient while the dashed lines show the individual components for Compton effect, photoelectric effect, and pair production. Data from Boone and Chavez (1996).

Compton effect

The scattering of a photon by a free electron is referred to as the Compton effect. Although the absorber electrons are bound to atoms, they can be considered to be nearly free if the γ -ray energy is large compared to the electron binding energy. The process is shown schematically in Fig. 4.10a. The electron is assumed to be initially at rest. A photon of incident energy E_{γ} transfers a fraction of its energy to the electron and is deflected by an angle θ with respect to its original direction, while the recoil electron emerges from the scattering center under an angle ϕ . All scattering angles are possible and, therefore, the transferred energy varies from zero to a large fraction of the incident photon energy.

The energies of the scattered photon and recoil electron can be obtained by solving simultaneously the equations for the conservation of energy and linear momentum. One finds (Leo 1987)

$$E'_{\gamma} = \frac{E_{\gamma}}{1 + \frac{E_{\gamma}}{m_{e}c^{2}}(1 - \cos\theta)}$$
(4.22)

$$K_e = E_{\gamma} - E_{\gamma}' = E_{\gamma} \frac{\frac{E_{\gamma}}{m_e c^2} (1 - \cos \theta)}{1 + \frac{E_{\gamma}}{m_e c^2} (1 - \cos \theta)}$$
(4.23)

where $m_ec^2 = 511$ keV is the electron rest energy. For the special case of a photon scattering angle of $\theta = 0^\circ$ the recoil electron energy is zero and, therefore,

the scattered photon loses no energy. The maximum energy transfer occurs at $\theta = 180^{\circ}$, where the energies of the recoil electron and scattered photon are given by

$$K_e^{\max} = E_{\gamma} \frac{2\frac{E_{\gamma}}{m_e c^2}}{1 + 2\frac{E_{\gamma}}{m_e c^2}}$$
(4.24)

$$(E_{\gamma}')^{\min} = \frac{E_{\gamma}}{1 + 2\frac{E_{\gamma}}{m_e c^2}}$$

$$(4.25)$$

The cross section for Compton scattering is given by the Klein-Nishina formula (see, for example, Leo 1987). A polar plot of the angular distribution of scattered photons is shown in Fig. 4.11a for different energies of incident photons that approach the scattering center from the left. It can be seen that the distribution is symmetric around $\theta = 90^{\circ}$ for small photon energies (E_{γ} < 1 keV), whereas scattering into the forward direction is strongly preferred for large γ -ray energies. The Klein–Nishina formula also predicts the energy distribution of recoil electrons. The result is shown in Fig. 4.11b for incident photons with energies of E_{γ} = 0.5, 1.0, and 1.5 MeV. Each curve displays a maximum recoil electron energy K_e^{\max} , called the *Compton edge*, corresponding to a photon scattering angle of $\theta = 180^{\circ}$. These recoil electrons are usually stopped in the absorber and, therefore, the graphs also represent the distribution of energy deposited by the incident photons in the absorber (for example, a γ -ray detector). For $E_{\gamma} \gg m_e c^2$, the energy difference between incident photon and maximum recoil electron energy amounts to $E_{\gamma} - K_e^{\text{max}} \approx m_e c^2/2 = 256 \text{ keV}.$ It has been assumed so far that the Compton scattering process involves an electron that is initially free. A proper consideration of the electron binding energy prior to scattering results in a rounding-off of the sharp Compton edge displayed in Fig. 4.11b.

The probability per absorber atom of Compton scattering increases with the number of electrons available as scattering targets and is approximately given by

$$p_{\text{Compton}} \sim \frac{Z}{E_{\gamma}}$$
 (4.26)

This probability (in units of cm²/g) is shown for lead in Fig. 4.9. The Compton scattering probability varies moderately with incident photon energy. At an energy of $E_{\gamma} \approx 500$ keV (for lead) it becomes comparable to the photoelectric absorption probability and dominates over the latter at higher energies.

Pair production

The process involving the transformation of a photon into an electronpositron pair is referred to as pair production, and is shown schematically



Fig. 4.10 Representations of the (a) Compton effect, and (b) pair production. Note that in part (b) neither the electron nor the positron existed before the interaction. The atom is a spectator only and facilitates the simultaneous conservation of energy and linear momentum.

in Fig. 4.10b. The photon must have at least an energy of twice the electron rest energy ($2 \cdot 511 \text{ keV} = 1022 \text{ keV}$) for this process to occur. Also, pair production must involve a third body (usually the nucleus of an absorber atom). Otherwise total energy and linear momentum are not simultaneously conserved. The fraction $E_{\gamma} - 2m_{\rm e}c^2$ of the incident photon energy is transferred to the kinetic energies of the electron and positron, that is,

$$E_{\gamma} = (K_{e^{-}} + m_{e}c^{2}) + (K_{e^{+}} + m_{e}c^{2})$$
(4.27)

Both of these particles slow down in the absorber. The positron will subsequently annihilate with another electron. Thus, two annihilation photons of 511 keV energy, emitted into opposite directions, are produced as a byproduct of the interaction.

Theoretical expressions for the probability of the pair production process are rather complicated (Leo 1987). The probability is approximately proportional to Z(Z + 1) and rises with increasing incident photon energy. The pair production probability (in units of cm²/g) for lead is shown in Fig. 4.9. It can be seen that pair production dominates over photoelectric absorption and Compton scattering above an energy of $E_{\gamma} \approx 5$ MeV.

Photon attenuation

So far, we considered individual photon interaction processes. We will now discuss the combined effect of multiple interactions for monoenergetic incident photons. Consider Fig. 4.12 showing the interaction processes discussed in the previous sections for photons incident on an absorber. Cases (a), (b), and (d) correspond to photons which undergo photoelectric absorption, Compton scattering, and pair production, respectively. More complicated interaction



Fig. 4.11 (a) Polar plot of the angular distribution of Compton-scattered photons for different incident energies. The incident photons approach the scattering center from the left. (b) Schematic graph of the energy distribution of recoil electrons after Compton scattering (solid lines). The dashed lines

indicate the energies of the corresponding incident, monoenergetic photons ($E_{\gamma} = 0.5$, 1.0, and 1.5 MeV). The maxima of the distributions are referred to as Compton edges. They correspond to a photon scattering angle of $\theta = 180^{\circ}$.

sequences are possible. For example, the Compton scattered photon in case (b) may in turn undergo photoelectric absorption, producing an X-ray photon which leaves the absorber (case e). Clearly, a complete description of all possible interactions involving the directions and energy distributions of scattered photons and electrons is rather complicated and can only be achieved by a Monte Carlo calculation. However, the most important information frequently of interest is the fraction of monoenergetic photons that traverse the absorber without any interaction (case c). As already noted, these photons

possess their original energy and direction. The fraction of attenuated photons refers to those γ -rays that are either absorbed or scattered in the absorber.

Each process is characterized by the probability for the occurrence of the interaction, or equivalently, by the probability per unit path length in the absorber that a photon is removed from the incident beam by an interaction. This probability is called *linear absorption coefficient*. The total linear absorption coefficient μ is given by the sum of the partial absorption coefficients involving the different photon processes. Thus

$$\mu = \mu_{\text{photo}} + \mu_{\text{Compton}} + \mu_{\text{pair}} \tag{4.28}$$

If a beam of monoenergetic photons is incident perpendicular to the surface of an absorber, the fractional intensity loss, dI/I, in traversing a thickness dx is

$$\frac{dI}{I} = -\mu \, dx \tag{4.29}$$

Hence, we obtain for the ratio of the number of transmitted and incident photons

$$\frac{I}{I_0} = e^{-\mu x}$$
(4.30)

where μ has dimensions of inverse length. The absorption coefficient is related to the mean free path λ , defined as the average distance traversed in the absorber before an interaction takes place, by

$$\lambda = \frac{\int_0^\infty x e^{-\mu x} \, dx}{\int_0^\infty e^{-\mu x} \, dx} = \frac{1}{\mu}$$
(4.31)

Values of λ typically amount to $\approx 10^{-3}$ – 10^{-1} m in solid absorbers for common γ -ray energies.

The probability for any photon interaction to occur depends on the absorber density, for example, whether the absorber is present in solid, liquid, or gaseous form. The density dependence is removed by introducing the quantity μ/ρ , called *mass attenuation coefficient*, which is widely used in the literature. Equation (4.30) can be written as

$$\frac{I}{I_0} = e^{-(\mu/\rho)\rho x}$$
(4.32)

where μ/ρ is in units, for example, of cm²/g. The product ρx is called *mass thickness* and is in units of mass per area.

If the absorber consists of a compound $X_a Y_b$, the mass attenuation coefficient can be calculated from an expression similar to Bragg's rule (see
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Fig. 4.12 Possible interactions of monoenergetic photons incident on an absorber. See discussion in the text.

Eq. (4.16)) by replacing the mass stopping power by the mass attenuation coefficient,

$$\left(\frac{\mu}{\rho}\right)_{c} = \frac{aM_{X}}{M_{c}} \left(\frac{\mu}{\rho}\right)_{X} + \frac{bM_{Y}}{M_{c}} \left(\frac{\mu}{\rho}\right)_{Y}$$
(4.33)

where $M_c = aM_X + bM_Y$. Total mass attenuation coefficients for two common γ -ray shielding materials (Fe and Ta) and two common γ -ray detector crystal materials (NaI and Ge) are displayed in Fig. 4.13.

Certain geometrical considerations require careful consideration when using photon attenuation coefficients. Figure 4.14a shows γ -rays that are emitted from a point source and traverse an absorber. It can be seen that the γ -rays traverse the absorber at different angles. Therefore, the average path length through the absorber, rather than the absorber thickness, must be estimated and used in the calculation of the γ -ray attenuation. Figure 4.14b shows a typical detector arrangement. Gamma rays from a point source are detected in the active volume of a detector. With the knowledge of the crystal size, the distance between source and crystal, and the number of photons emitted from the source, the total number of photons detected in the crystal can be calculated by using the attenuation coefficient. However, massive absorbers such as lead are frequently used in order to shield the detector from unwanted γ ray background. Consequently, the detector may respond not only to γ -rays coming directly from the source, but also to those reaching the detector after scattering in the shielding material, or to other types of secondary radiations induced by the source γ -rays in the shielding material. Thus, the number of photons counted in the detector will be larger compared to a setup without the



Fig. 4.13 Total mass attenuation coefficients for the common γ -ray shielding materials Fe and Ta (top) and the common γ -ray detector crystal materials NaI and Ge (bottom). Data from Boone and Chavez (1996).

shielding material. This effect has to be taken into account when estimating detector efficiencies (Section 4.5.2).

4.2.3 Interactions of Neutrons

Neutrons carry no charge and, therefore, cannot interact in matter via the Coulomb force. Instead, neutrons interact with absorber nuclei via the strong force. The range of this force is short and an interaction can only occur if the neutron comes within $\approx 10^{-15}$ m of a nucleus. As a result, neutron interactions are relatively rare and thus they can penetrate absorbers of many cm thickness without interaction. A neutron may interact with a nucleus via a



Fig. 4.14 (a) Photons emitted by a point source traversing an absorber at different angles. (b) Photons reaching the detector after scattering in nearby shielding material.

number of different processes including: (i) elastic scattering (n,n), (ii) inelastic scattering (n,n'), (iii) radiative capture (n, γ), and (iv) reactions that produce charged particles such as (n,p) or (n, α). The relative importance of the various interaction types depends strongly on the neutron energy. In the following, neutrons with energies above and below ≈ 0.5 eV will be designated as *fast neutrons* and *slow neutrons*, respectively (Knoll 1989).

Slow neutrons that are incident on an absorber may undergo elastic scattering. After many collisions have occurred, the neutrons are in thermal equilibrium with the absorber material before other types of interactions take place. These neutrons are referred to as *thermal neutrons*, corresponding to an average energy of kT = 0.025 eV at room temperature (Section 3.1.1). For many absorber materials, radiative capture is the most likely neutron-induced reaction which has important implications for neutron shielding considerations. The majority of slow neutron detectors, on the other hand, are based on the detection of secondary charged particles that are emitted in reactions of type (n,p), (n, α), and so on.

The cross section for the majority of neutron-induced reactions decreases rapidly with increasing neutron energy. Therefore, elastic scattering becomes the most likely process for fast neutrons. In this case, the neutron can transfer in each interaction a significant amount of energy to the recoil nucleus. As a result of many collisions, the incident neutron slows down. This process is referred to as *moderation*. Hydrogen is the most efficient moderator since, according to scattering kinematics (Appendix C), the neutron can lose all its energy in a single collision. Most fast neutron detectors rely on the detection of the (charged) recoil nuclei. For sufficiently high neutron energies, inelastic scattering may also occur. In this case, the recoil nucleus is left in an excited state and de-excites quickly via emission of secondary γ -rays. Inelastic scattering is an important process for the shielding of high-energy neutrons.

The total cross section for the interaction of neutrons with matter is given by the sum of cross sections for the individual interactions,

$$\sigma_T = \sigma_{(\mathbf{n},\mathbf{n})} + \sigma_{(\mathbf{n},\mathbf{n}')} + \sigma_{(\mathbf{n},\gamma)} + \sigma_{(\mathbf{n},\mathbf{p})} + \sigma_{(\mathbf{n},\alpha)} + \cdots$$
(4.34)

Cross sections versus neutron energy for some reactions of interest to neutron detection are shown in Fig. 4.15a. Below a neutron energy of $E_n \approx 100$ keV, the cross sections follow the 1/v law (see Eq. (2.207)). Total cross sections for common neutron shielding materials are displayed in Fig. 4.15b.

The product of the total cross section and the number density of atoms in the absorber, $N\sigma_T$, has dimensions of inverse length. This quantity represents the probability per path length in the absorber medium that any type of interaction will occur, or equivalently, that a neutron is removed by an interaction from the incident beam. It has the same physical meaning for neutrons as the linear absorption coefficient has for photons (Section 4.2.2). In analogy to photons, a beam of monoenergetic neutrons incident perpendicular to the surface of an absorber of thickness *x* will be attenuated exponentially. The transmission *T* (which is *not* related to the transmission coefficient \hat{T} of Chapter 2) is given by

$$T = \frac{I}{I_0} = e^{-N\sigma_T x} \tag{4.35}$$

where *I* and I_0 are the measured intensities with and without absorber, respectively, between incident neutron beam and detector. The neutron mean free path is accordingly (see Eq. (4.31)

$$\lambda = \frac{1}{N\sigma_T} \tag{4.36}$$

Values of λ in solid absorbers typically amount to \approx cm and \approx 10 cm for slow and fast neutrons, respectively. As was the case for γ -rays, the exponential attenuation (see Eq. (4.35)) only applies to a collimated beam of neutrons. For situations in which neutrons can reach a detector after scattering in the material surrounding the active volume, corrections are necessary in order to predict the true number of transmitted neutrons.

If the incident neutrons are not monoenergetic, but are represented by a distribution where f(E) is the fraction of incident neutrons having energies between *E* and *E* + *dE* per unit energy interval with $\int f(E) dE = 1$, then the transmission is given by

$$T = \int_0^\infty f(E)e^{-n\sigma_T(E)} dE$$
(4.37)

where n = Nx is the number of sample (or absorber) nuclei per unit area. If $\sigma_T(E)$ = const over the neutron distribution, then we obtain again Eq. (4.35).

If, on the other hand, $\sigma_T(E) \neq \text{const}$ and the sample is very thin ($n\sigma_T \ll 1$), then one obtains from an expansion of the integrand in Eq. (4.37)

$$T \approx \int_0^\infty f(E)[1 - n\sigma_T(E)] dE = 1 - n \int_0^\infty f(E)\sigma_T(E) dE$$
$$\approx \exp\left(-n \int_0^\infty f(E)\sigma_T(E) dE\right) = e^{-n\overline{\sigma}_T}$$
(4.38)

where we defined an average total cross section by $\overline{\sigma}_T \equiv \int f(E)\sigma_T(E) dE$.

If the absorber consists of a compound, the transmission is given by the product of the transmissions for hypothetical absorbers made of the individual elements alone, each containing the same number of nuclei per area as are present in the compound. The same procedure applies for pure elements containing more than one isotope.

Example 4.2

Calculate the attenuation of thermal neutrons ($E_n = 0.025 \text{ eV}$) in a ¹⁰BF₃ gas region of 30 cm length. The gas pressure and temperature are P = 600 torr and T = 20 °C, respectively. Assume that the only process absorbing neutrons from the incident beam is the ¹⁰B(n, α)⁷Li reaction, which has a cross section of 3840 b for thermal neutrons.

The number of ¹⁰B atoms per cubic centimeter is calculated from Eq. (4.5),

$$N = 1 \cdot (2.68677 \times 10^{19} \,\mathrm{cm}^{-3}) \cdot \left(\frac{600 \,\mathrm{torr}}{760 \,\mathrm{torr}}\right) \left(\frac{273 \,\mathrm{K}}{293 \,\mathrm{K}}\right) = 1.98 \times 10^{19} \,\mathrm{cm}^{-3}$$

It follows

$$\frac{I}{I_0} = e^{-(1.98 \times 10^{19} \,\mathrm{cm}^{-3})(3840 \times 10^{-24} \,\mathrm{cm}^2)(30 \,\mathrm{cm})} = 0.10$$

Thus, about 90% of the incident neutrons are absorbed in the gas by the (n,α) reaction, while 10% traverse the absorber without undergoing an interaction.

4.3

Targets and Related Equipment

In the laboratory, a nuclear reaction is initiated by a beam bombarding a suitable target and the nuclear reaction takes place at the target position. Targets and associated equipment (target holders, chambers, and backings) have to be prepared and designed carefully for nuclear astrophysics experiments. Solid targets or backings can either be sufficiently thin for an ion beam to





Fig. 4.15 (a) Cross section versus neutron energy for reactions of interest to neutron detection. Below a neutron energy of $E_n \approx$ 100 keV, the cross sections follow the 1/v law (Eq. 2.207). From G. F. Knoll, Radiation Detection and Measurement, 2nd edn., John Wiley & Sons (1989). Reprinted with permission. Copyright ©1989 John

Wiley & Sons, Inc. (b) Total cross section versus neutron energy for common neutron shielding materials. Reprinted from W. R. Leo, Techniques for Nuclear and Particle Physics Experiments, Springer-Verlag (1987). Copyright (1987), with kind permission of Springer Science and Business Media.

pass through, or can be relatively thick and stop the ion beam. These targets are referred to as transmission and beamstop targets, respectively. Targets in gaseous form are also being used. In order to avoid confusion, we will be using the term *sample* instead of target for neutron-induced reaction studies. The type of target or sample to be used in an experiment depends on the nuclear reaction and the observable to be measured. In the following, issues related to targets or samples for nuclear astrophysics experiments will be discussed in more detail.

4.3.1 Backings

The vast majority of targets that are used for charged-particle reaction studies are prepared by depositing the target material on some sort of backing. Exceptions are self-supporting targets and gas targets. There are several requirements for the backing material: (i) the target material should adhere uniformly to the backing, (ii) the backing should not cause unwanted background radiation if exposed to the ion beam, and (iii) for beamstop targets, the backing must provide efficient cooling to prevent target degradation.

Common materials used for beamstop target backings are tantalum, nickel, and copper. They have a high atomic number and thus do not initiate nuclear reactions at low bombarding energies. Their melting points are high and hence they are stable under intense ion bombardment. Before the deposition of the target material, the backing has to be cleaned to reduce surface contaminations. Common procedures are the etching of backings by a suitable mixture of acids to remove part of the surface (Vermilyea 1953), and the subsequent resistive heating to temperatures above ≈ 1200 °C to drive out remaining contaminants. Backings for beamstop targets are typically ≈ 0.5 –2 mm thick. These are especially convenient for the study of capture reactions since they attenuate the capture γ -rays very little. For example, a 0.5 mm thick tantalum sheet has a 90% and 96% transmission for 0.5 MeV and 5 MeV γ -rays, respectively (Problem 4.3). This circumstance simplifies the setup considerably, since the γ -ray detector can be placed outside the vacuum chamber in very close geometry to the target, thus optimizing the counting efficiency.

Beamstop target backings produce a large number of Coulomb-scattered ions and, therefore, are not suitable for measurements of elastic scattering cross sections. Furthermore, at bombarding energies in excess of several MeV, nuclear reactions induced by contaminants in the backing become significant. In such instances, the use of transmission targets can be of advantage. However, these targets are difficult to cool and, therefore, they have to be sufficiently thin so that the heat deposited by the ion beam does not damage either the target or the backing. On the other hand, the backing has to be thick enough to allow for the deposition of the target material without damage. Frequently used backings for transmission targets are carbon foils of \approx 5–40 µg/cm² thickness, mounted on suitable metal frames.

In certain instances the backing may be eliminated altogether. Examples are gas targets or self-supporting transmission targets. However, these can only be prepared for a restricted number of target elements. In the latter case, they are also easily destroyed in experiments at low energies involving high intensity ion beams since they cannot be cooled efficiently.

For studies of neutron-induced reactions, requirements on sample backings are not as stringent since neutrons are far more penetrating than charged par-

ticles (Section 4.2.3) and the number of incident neutrons is much smaller compared to typical charged-particle beam currents. A variety of materials is used to support or contain the sample material, including carbon foils, thinwalled aluminum or stainless steel cans, adhesive tape, and pressurized stainless steel spheres for noble gas samples. The material of the backing or the containment vessel should not contribute significantly to the background and should be sufficiently thin to minimize neutron attenuation, scattering, and absorption of the reaction products. Also, self-supporting samples are more common in neutron work compared to charged-particle measurements.

4.3.2

Target Preparation

The yield of a nuclear reaction induced by charged particles is obtained by integrating the ratio of cross section and stopping power over the thickness of the target, as will be shown in Section 4.8. In most cases the ion beam is not completely stopped, but loses only a fraction of its energy (typically < 10%) moving through the target. Such targets are relatively thin and, as a consequence, the nuclear reaction of interest takes place in a localized region within the target with a rather well-defined interaction energy.

In neutron-induced reaction studies, on the other hand, each depth in a thin sample is exposed to the same neutron energy distribution. In general, samples can be made thicker (see below) to increase count rates. Requirements for the preparation of samples for neutron irradiation are less restrictive, resulting in a larger variety of sample materials in use, including metal foils, powders, compressed tablets and pellets, and implanted samples. If the sample reacts with air, then it may be sealed in a tight thin-walled can.

Evaporated and sputtered targets

Solid targets are frequently prepared by evaporating or sputtering a thin layer of material containing the target nuclei in vacuum onto suitable backings (Holland 1956, Maxman 1967). Target preparation by evaporation or sputtering is an extensive subject and different researchers have different recipes. An ²⁷Al target, for example, is easily prepared by evaporating a thin layer of Al metal onto a backing. Aluminum targets are rather stable under ion bombardment and, furthermore, contain only a single isotope (²⁷Al). The preparation of suitable ²³Na targets, on the other hand, is more complicated. In this case compounds, such as NaCl, NaBr or Na₂WO₄, have to be used for evaporation. All these targets degrade to a certain degree at low bombarding energies with beam currents in excess of 100 µA. Also, the "inactive" atoms in these compounds (that is, the atoms of elements other than the target element of interest) will also contribute to the slowing down process of the beam.

Thus, for the same value of target thickness (in energy units) there will be fewer target atoms present in a chemical compound compared to a pure target. Consequently, the nuclear reaction yield will decrease. This undesirable effect becomes more pronounced with increasing number and charge of inactive atoms in the compound. On the other hand, the smaller the charge of the inactive atoms, the larger the probability that they contribute to unwanted beam-induced background radiation. Depending on the circumstance, a compromise has to be found.

Similar arguments hold for elements with more than one stable isotope. For example, the evaporation of natural magnesium will produce a target containing ²⁴Mg (79%), ²⁵Mg (10%), and ²⁶Mg (11%). Even if a pure Mg target is fabricated, the ion beam will very likely induce nuclear reactions involving the isotopes other than the one of interest. In order to avoid these unwanted contributions to the count rate, targets may be fabricated by using isotopically enriched material which is commercially available. Obviously, such targets will also produce a higher reaction yield compared to a target made from a natural isotopic mixture. It is important to point out that when chemical compounds are evaporated, one may not assume that the composition of the target is the same as that of the original compound. A number of cross sections and resonance strengths reported in the literature are erroneous because of this unjustified assumption, as will be shown in Section 4.8.4.

Implanted targets

Evaporated targets are in certain situations unsuitable for nuclear reaction studies. First, it may be that none of the targets produced by evaporating chemical compounds are sufficiently stable under bombardment with a highintensity ion beam. Second, even if the targets are stable and consist of a single element, isotopes other than that of interest may cause intolerable background radiation. Third, certain elements cannot be evaporated at all (for example, noble gases). These problems are frequently solved by using implanted targets. In this case, the target ions of interest are accelerated and mass separated by using an electromagnetic isotope separator. Only ions of the isotope of interest are directed onto a suitable backing. These target nuclei are hence implanted into the backing. The accelerating voltage determines the range of the ions in the substrate and thus the effective target thickness for the subsequent nuclear reaction study. Both implanted transmission targets and beamstop targets have been used extensively in nuclear reaction studies. In the latter case, the backings are usually directly water cooled if high-intensity ion beams are used during the implantation process. Implanted samples are also employed in studies of neutron-induced reactions.

Several factors limit the number of ions that can be implanted into a substrate. These include the sputtering yield, the range and mobility of the ions

in the substrate, the number of incident ions per unit area, and the substrate temperature. Sputtering, which releases atoms from the substrate upon ion impact, is the dominant limiting mechanism at low substrate temperatures. Tantalum is frequently chosen as a substrate for beamstop targets since it has a relatively low sputtering yield (Almen and Bruce 1961), and since diffusion velocities of various elements are small in tantalum. For transmission targets, carbon foils with thicknesses of $\approx 10-40 \ \mu g/cm^2$ can be used as substrates. During the implantation process the beam power deposited in the carbon foil has to be limited (< 25 mW/cm²; see Smith et al. 1992) in order to avoid rupture of the foil. Several procedures may be applied to extend the lifetime of thin carbon foils during implantation (Fifield and Orr 1990, Smith et al. 1992).

Table 4.3 provides information about some implanted targets and samples used in nuclear astrophysics studies. It lists incident ion energies and doses, as well as the measured number of implanted ions (either as stoichiometry for beamstop targets or in units of atoms per square centimeter for transmission targets or samples). The table also includes a few examples of implanted radioactive targets and samples. Many targets or samples become saturated during implantation. In other words, they reach a stage at which target atoms are lost due to sputtering and diffusion at the same rate as they are implanted into the substrate. On the other hand, for a number of ion species the sputtering ratio (the number of released atoms per incident ion) is small for collisions between ions and substrate atoms, and at the same time it is less than unity for ion–ion collisions (self-sputtering). In such cases, saturation is never reached and a pure layer of target material builds up on the substrate surface (for example, for *C*, Si, and Ca implantation into tantalum; Almen and Bruce 1961).

It has also been shown that for relatively small incident ion energies, the distribution of implanted atoms extends to the front surface of the backing (Selin, Arnell and Almen 1967). Hence, in the nuclear reaction study there is usually no substrate dead layer in which charged projectiles lose energy or neutrons are attenuated before hitting the target material. Implanted targets can also be stored for years without noticeable loss of target material (Selin, Arnell and Almen 1967, Geist et al. 1996).

Gas targets

In certain situations it is desirable to use gaseous instead of solid targets. First, it can be seen from the Table 4.3, that the stoichiometries achieved for implanted noble gas targets are unfavorable since they contain more substrate atoms than target nuclei. Therefore, the reaction yield will be reduced compared to a pure target. Second, the backing may produce intolerable beam-induced background radiation. Third, in nuclear reaction studies that are performed in inverse kinematics (by directing a heavy ion beam onto hydrogen or helium target nuclei), it may prove impossible to prepare targets of sufficient purity other than gaseous targets.

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Tab. 4.3 Properties of implanted targets and samples. (a) For beamstop targets, the range of backing thicknesses was 0.1–0.5 mm; for transmission targets, 30–75 μg/cm² carbon foils were used; the ³³S and ¹³⁵Cs ions were implanted into 0.7 mm and 0.1 mm thick C disks, respectively. (b) Incident dose of singly charged ions in mC/cm²; for ²²Na and ³³S the incident beam current is quoted (in nA). (c) For beamstop targets the stoichiometry is given, while for transmission targets (and for ³³S) the total number of implanted target nuclei per square centimeter is listed. (d) Iliadis (1996). (e) Giesen (1995). (f) Powell et al. (1999). (g) Fifield and Orr (1990). (h) Seuthe et al. (1987). (i) Smith et al. (1992). (j) Schatz et al. (1995). (k) Schmidt et al. (1995). (l) Patronis et al. (2004). (m) Ugalde (2005).

Target	Backing	Dose ^b (mC/cm ²)	Energy (keV)	Stoichiometry or (ions/cm ²) ^c	Ref.
¹² C	Та	400	110	¹² C ₃ Ta ₂	h
¹⁴ N	Та	784	120	¹⁴ N ₃ Ta ₂	h
¹⁹ F	Fe	31	37	¹⁹ F ₁ Fe ₈	m
²⁰ Ne	С		15–40	2.2×10 ¹⁷	i
²² Ne	Та	57	25	²² Ne ₁ Ta ₄	е
²² Na	Ni	25–80 nA	60	5.7×10 ¹⁵	k
²² Na	С	25–80 nA	60	7.6×10 ¹⁶	k
²³ Na	Ni	96	50	²³ Na ₅ Ni ₁	h
²⁴ Mg	Та	426	100	²⁴Mg₃Ta₁	f
²⁸ Si	Та	190	80	²8Si₃Ta₁	d
³¹ P	Та	180	80	³¹ P ₃ Ta ₂	d
³² S	Та	108	80	³² S ₁ Ta ₁	d
³³ S	С	400 nA	300	1.6×10 ¹⁶	j
³⁶ S	С		50	2×10 ¹⁷	g
³⁵ Cl	Та	180	80	^{₃₅} Cl ₁ Ta ₆	d
	С	70	60	1×10 ¹⁷	d
³⁶ Ar	Та	44	80	³⁶ Ar ₁ Ta ₅	d
¹³⁵ Cs	С			1.8×10 ¹⁵	1

Early gas target designs involved small cells containing a pure gas, with thin entrance and exit window foils for transmitting the ion beam. These foils have the undesirable effects of reducing the ion beam energy and of broadening the beam spread. Furthermore, they are sources of unwanted beam-induced background radiation. More sophisticated designs involve windowless gas targets. The ion beam is usually stopped sufficiently far away behind the target and detector region that the beam-induced background is kept small. Windowless gas targets involve several stages of high pumping speed in order to lower the gas pressure from typical target chamber pressures ($\approx 10^{-2}$ to 10 torr) down to 10^{-6} torr in the beamline. Therefore, gas target designs are very complex compared to solid targets. High pumping speeds are achieved by using large roots blowers and turbo pumps (Rolfs and Rodney 1988). Gas targets are either of the extended type (that is, the gas is contained in a differentially pumped chamber) or are nearly point-like (for example, a gas jet of small diameter streaming from a nozzle perpendicular to the beam direction). They have been used in several charged-particle reaction studies that would have been difficult to perform with solid targets (Rolfs and Rodney 1988). Us-

ing gas jets, target thicknesses of $\approx 10^{19}$ atoms/cm² for H, N, and Ar have been achieved (Bittner, Kretschmer and Schuster 1979).

For neutron-capture studies, pressurized gas samples in stainless steel spheres of 2 cm diameter and 0.5 mm wall thickness have been employed, with sample masses amounting to a few grams (Beer 1991).

Target thickness and stability

The choice of target thickness for a particular experiment depends on the type of experiment one wishes to perform. There is no apparent advantage of choosing a target thickness (in energy units) that is smaller than the ion beam resolution (≈ 1 keV). If a charged-particle reaction is measured over an energy range containing previously observed resonances, then the target thickness should be chosen to be smaller than the energy separation of the resonances. Below an ion energy of 1 MeV, typical target thicknesses amount to \approx 5–20 keV, while at *E* = 1–2 MeV the density of resonances increases and target thicknesses are usually smaller (\approx 1–5 keV). In searches for expected, yet unobserved, resonances at low bombarding ion energies, it is frequently of advantage to use thicker targets (\approx 20–40 keV) in order to study the energy range of interest in a reasonable amount of time. Target thicknesses are conveniently determined by measuring yield curves of narrow, well-known resonances in charged-particle-induced reactions, as will be explained in Section 4.8.3. The target thickness should also be uniform if the ion beam spot has a smaller diameter than the target.

The stability of a particular target depends not only on the ion beam intensity, but also on the ion type. Targets that are stable under bombardment with high intensity (> 100 μ A) proton beams will likely degrade to some degree if bombarded with a high intensity α -particle beam. Blistering is a particularly troublesome effect where the beam α -particles are implanted in the target and then move quickly to lattice defects. Eventually, high pressure gas blisters are formed which rupture and thereby degrade the target locally (Cole and Grime 1981). In some cases, there may be no alternative other than replacing the target after substantial degradation.

For neutron-induced reaction studies, the physical stability of the sample is usually of lesser concern. Samples must be thick enough in order to provide sufficiently high count rates, but have to be thin enough to minimize the attenuation and scattering of incident neutrons and the absorption of reaction products. Sample thicknesses for neutron work are typically in the range of $\approx \text{mg/cm}^2\text{-g/cm}^2$, which is significantly thicker than the target thicknesses used in charged-particle reaction studies.

4.3.3 Contaminants

Studies of nuclear reactions are frequently hampered by the presence of contaminants in either the target or the backing. Although the concentration of contaminants is usually very small, their cross section for reactions induced by the incident beam may be very large. Therefore, they may either contribute to, or obscure altogether, the count rate of interest. If both the reaction of interest and a given contaminant reaction proceed through narrow resonances of different energies, then it may be possible to adjust the beam energy so that the contaminant resonance is not excited. Alternatively, if the reaction of interest proceeds through a narrow resonance and the contaminant reaction proceeds either through a broad resonance or a nonresonant process, then it is often possible to measure the count rate just below, on top, and just above the resonance of interest. The difference between the on- and off-resonance spectra can then be used to estimate the contaminant contribution. This procedure is not applicable if both the reaction of interest and the contaminant reaction proceed through broad resonances or nonresonant processes. In such cases it is useful to estimate the background contributions by comparing the count rates from different runs involving the target-plus-backing and the backing alone (or gas in-gas out for gas targets).

For proton-induced reactions, one of the most troublesome contaminants is ¹⁹F which gives rise to γ -rays and α -particles through the ¹⁹F(p, $\alpha\gamma$)¹⁶O reaction. Another common contaminant is ¹¹B which produces α -particles through the ¹¹B(p, α)2 α reaction and γ -rays via ¹¹B(p, γ)¹²C. Experience shows that the concentration of ¹⁹F and ¹¹B, which remains in the backing after common cleaning procedures, varies greatly. Therefore, it is useful to test backing materials from different suppliers for minimum ¹⁹F and ¹¹B concentrations. Proton captures on ¹²C and ¹³C also contribute to γ -ray background. Considerable care needs to be taken in preparing, storing, and handling targets to ensure that no additional contaminants are added to their surface.

An important contaminant for reactions induced by α -particles is ¹³C which produces neutrons via the ¹³C(α ,n)¹⁶O reaction. The neutrons contribute either directly to the detector background count rate, or indirectly by producing secondary γ -rays in the surrounding material via neutron inelastic scattering or neutron capture.

Carbon contamination generally builds up on the target during ion bombardment. Hydrocarbons from organic components of the vacuum system (for example, vacuum sealing O-rings) diffuse into the beam and are subsequently transported onto the target. This carbon layer not only contributes to beam-induced γ -ray background, but will also cause a reduction of the incident beam energy before the projectiles strike the target. Carbon deposition can be reduced substantially by having the beam move through a

liquid-nitrogen cooled metal tube which is placed close to the target (Section 4.3.4). Table 4.4 lists common contaminant reactions induced by lowenergy (E < 1 MeV) proton and α -particle beams and, if present, the energies of their characteristic discrete γ -rays.

In studies of neutron-induced reactions, accurate cross section and transmission measurements require precise knowledge of the sample composition. Oxidization is a potential problem when metal samples are used. The composition may also change as a result of hygroscopy, that is, the absorption of moisture from the air. Increases in sample weight by 16% due to hygroscopy have been observed for powdered samples (Mizumoto and Sugimoto 1989). The water does not only increase the weight of the sample, but also gives rise to an additional energy loss for charged particles that are emitted in a neutroninduced reaction, causing an increased tailing in the pulse height spectrum. In neutron-capture studies, a fraction of the incident neutrons slows down via scattering on hydrogen. For the scattered neutrons, the reaction is induced at lower energies where either the capture cross section is higher, or the cross section fluctuates rapidly because of resonances. In both cases, the capture rate may increase drastically. In some instances the moisture can be removed by heating in vacuum, as indicated by the weight loss of the sample.

Tab. 4.4 Common contaminant reactions and their discrete characteristic γ -ray energies.

Contaminant	Reaction	E_{γ} (keV)
¹⁹ F	¹⁹ F(p,αγ) ¹⁶ O	6130
¹¹ B	${}^{11}B(p,\gamma){}^{12}C$	4439
	$^{11}B(p,\alpha)2\alpha$	
¹⁵ N	$^{15}N(p,\alpha\gamma)^{12}C$	4439
¹² C	${}^{12}C(p,\gamma){}^{13}N$	
¹³ C	${}^{13}C(p,\gamma){}^{14}N$	2313
	${}^{13}C(\alpha,n){}^{16}O$	
¹⁶ O	${}^{16}O(p,\gamma){}^{17}F$	495
²³ Na	23 Na(p, γ) 24 Mg	1369
	23 Na(p, $\alpha\gamma$) 20 Ne	1634
²⁷ AI	27 Al(p, γ) 28 Si	1779

4.3.4

Target Chamber and Holder

Targets are mounted in a target chamber which represents the location where the nuclear reactions take place. The specific design of the chamber depends on the type of target used (beamstop target, transmission target, or irradiation sample) and the type of detector employed (γ -ray detector, charged-particle detector, or neutron counter). For charged-particle-induced reaction studies, target chambers must provide an accurate measurement of the integrated ion beam charge and they also have to accommodate radiation detectors. The chamber has to hold a vacuum of $\approx 10^{-6}$ torr or lower in order to minimize the interaction of the ion beam with residual gas molecules and to reduce the condensation of contaminants on the target surface. Figure 4.16 shows a target chamber designed for (p, γ) and (α , γ) reaction measurements at low energies (E < 1 MeV) with high intensity beams (currents of $I \approx 0.1$ –1 A). The design will be discussed below because it takes several important considerations into account. Examples of experimental setups used for studies of other charged-particle-induced reactions or of neutron-induced reactions will be discussed in later sections.

The target shown in Fig. 4.16 is a beamstop target. The beam loses its entire energy in the target and backing. The beam power (energy per time) deposited by the beam is given by the product of voltage and current, that is, $P = U \cdot I$. For example, for a singly charged ion beam of 100 keV energy and 1 mA current, the power amounts to $P = (0.1 \text{ MV})(1000 \text{ }\mu\text{A}) = 100 \text{ W}$. If the beam spot on the target is too small, say, only a few square millimeters, then the locally produced heat will quickly destroy the target or backing. Therefore, it is important to defocus the beam sufficiently. Even with a defocussed beam, the heat produced by the ion beam will degrade the target, unless efficient cooling is provided. Therefore, the backside of the target backing is directly water cooled. The water reservoir has to be large enough and the water flow strong enough to provide efficient cooling. On the other hand, the target holder thickness should be kept small so that the γ -ray detector can be placed as close as possible to the target in order to maximize counting efficiency. Furthermore, capture γ -rays are not attenuated substantially in a thin target holder before reaching the detector.

The chamber design shows several features that minimize beam-induced γ -ray background. A beam-defining aperture is mounted some distance away from the target and ensures that the beam hits only the target, but not other parts of the target holder or chamber. The buildup of contaminants, such as 12 C and 13 C, on the target is reduced by having the beam move through a metal tube which is directly cooled by liquid nitrogen. Thus, troublesome hydrocarbons that are released by vacuum O-ring seals condense on the cold surface of the metal tube instead of the target. Since the target holder design involves several O-ring seals, it is important that this tube extends as close as possible to the target without touching it.

The target chamber also represents a Faraday cup for integrating the ion beam current. If the charge state of the ion beam, q, is known, the total number of ions incident on the target, N_i , can be easily calculated from $N_i = Q/(qe)$, where Q is the total accumulated charge (or integrated beam current). The most important systematic error in the beam current integration arises from secondary electrons that are emitted from surfaces hit by the beam. For example, a singly charged positive ion hitting the target will deposit one elementary



Fig. 4.16 Typical target chamber design used in radiative capture reaction studies. The beam passes through a defining collimator and is incident on a directly water-cooled beamstop target. A copper tube, cooled to liquid-nitrogen (LN_2) temperature, reduces the buildup of contaminants (such as ¹²C and ¹³C) on the target. The chamber

is electrically insulated from the rest of the beamline and thus acts as a Faraday cup for the integration of the total charge accumulated by the beam on the target. A negative voltage is applied to the copper tube in order to suppress the emission of secondary electrons. The small (full and open) circles show the location of vacuum O-ring seals.

charge on the Faraday cup. At the same time, however, secondary electrons are emitted and these may move away from the target without being collected on the Faraday cup. Thus, the measured current will yield an overestimate for the number of positive ions hitting the target (since removing an electron from the Faraday cup has the same effect as adding a positive charge). For this reason, a reliable target chamber design must account for secondary electron suppression. In Fig. 4.16, a negative voltage of several hundred volts (Rolfs and Rodney 1988) is applied to the metal tube, thus repelling secondary electrons that are emitted from the target or the collimator. Also, possible current losses through the target cooling water need to be checked carefully.

4.4 Radiation Detectors

4.4.1 General Aspects

Nuclear reactions are studied by measuring the reaction products (for example, protons, neutrons, α -particles, or γ -rays) with suitable detectors. Different types of radiation interact differently with matter and, therefore, the type of detector to be used will depend on the identity of the radiation of interest. Most detectors produce, directly or indirectly, a given amount of electric charge as a result of energy deposition by the radiation. The charge is col-

lected by applying an electric field and, as a result, an electric signal is produced. The precise shape of this signal depends, among other things, on how and where the charge is produced in the active volume, how fast the charge is collected, and the characteristics of the electric circuit to which the detector is connected (for example, preamplifier or photomultiplier tube). Although the signal shape varies strongly from one detector type to another, the amplitude of the signal pulse is usually directly proportional to the charge generated within the active volume or, equivalently, the energy deposited by the radiation in the detector. Furthermore, the rate at which such pulses occur depends on the corresponding rate of radiation interactions within the active volume. This rate is directly proportional to the number of nuclear reactions occurring per time interval. If a large number of such pulses is examined, their amplitudes will not all be the same. Variations in pulse heights are caused by a number of effects: (i) the radiation incident on the detector may not be monoenergetic; (ii) even for incident monoenergetic radiation, different amounts of energy may be deposited in the detector; and (iii) fluctuations in the intrinsic detector response.

In practice, the output signal from a preamplifier or a photomultiplier tube is further amplified and shaped by additional electronic circuits (spectroscopy amplifiers) while still preserving the pulse height information. The data are then displayed as a differential pulse height distribution (or *pulse height spectrum*), showing the pulse height on the horizontal axis and the number of pulses observed within a pulse height interval, divided by the interval width, on the vertical axis. Physical interpretations almost always involve areas under the spectrum, or total counts, between two given pulse height values. Through careful (energy and efficiency) calibrations, the information displayed in a differential pulse height distribution can be related to the energies and intensities of the incident radiation. The latter information is then used to determine nuclear reaction cross sections.

A schematic pulse height spectrum (differential number of pulses per pulse height interval, dI/dH, versus pulse height, H) is shown in Fig. 4.17a. The shape of the spectrum can be very complicated and depends on the nature and energy of the incident radiation as well as the intrinsic detector response. The latter has to be well understood in order to relate the spectrum shape to properties of the incident radiation. Relatively narrow peaks in a spectrum frequently indicate that incident α -particles, protons, neutrons, or photons deposited their entire discrete energy in the active detector volume. Suppose that a sharp peak occurs at a pulse height of H_0 , which is proportional to the energy of the incident radiation. The peak is superimposed on a background, representing a relatively flat part of the spectrum. The net intensity of the peak (shaded area labeled \mathcal{N}), which is proportional to the number of radiation quanta of specific incident energy, is calculated by subtracting the





Fig. 4.17 (a) Schematic pulse height spectrum. The net intensity of the peak centered around pulse height H_0 (area labeled \mathcal{N}) is obtained by subtracting the number of background counts (area labeled *B*) from the total number of counts between H_1 and H_2 . The FWHM of the peak, as measured relative to the background level (dashed horizontal line) indicates the detector energy



resolution. (b) Pulse height spectrum illustrating the difference between total and peak efficiencies. The former and latter quantity is calculated by dividing \mathcal{N}_{total} and \mathcal{N}_{peak} , respectively, by the total number of quanta emitted by the source. Below the detection threshold (leftmost vertical dashed line) the spectrum is dominated by noise.

background (area labeled *B*) from the total number of counts in the region of interest between H_1 and H_2 ,

$$\mathcal{N} = T - B \tag{4.39}$$

The background B can be estimated from the number of counts in regions on the left- and right-hand side of the peak. Counts in nuclear physics are distributed according to a Poisson probability function, with the standard deviation given by the square root of the number of counts. Thus, we obtain for the error (one standard deviation) in the number of net counts

$$\Delta \mathcal{N} = \sqrt{(\Delta T)^2 + (\Delta B)^2} = \sqrt{T+B}$$
(4.40)

In more complicated cases, including overlapping peaks and nonlinear background structures, sophisticated fitting programs are usually employed. Elaborate discussions of statistical data analysis in nuclear counting experiments can be found in Leo (1987) and Knoll (1989).

The full width at half maximum of the narrow peak (FWHM), as measured from the background level, is in general determined by the energy distribution of the incident radiation as well as the intrinsic response of the detector. Suppose that the spectrum shown in Fig. 4.17a has been obtained by measuring monoenergetic incident radiation so that the observed FWHM of the peak is a measure for the intrinsic *energy resolution* of the detector. It is highly desirable that this peak width is as small as possible for two reasons. First,



Fig. 4.18 Gamma-ray spectra measured with different detector resolutions (1.75, 5.6, and 10.8 keV). Each spectrum contains a signal of the same net intensity. The loss in detection sensitivity with decreasing resolution is evident. In the bottom spectrum, corresponding to the poorest energy reso-

lution, a statistically significant peak above background can no longer be discerned. Reprinted with permission from G. A. Armantrout, A. E. Bradley, and P. L. Phelps, IEEE Trans. Nucl. Sci. NS-19(1), p. 107 (1972). ©1972 IEEE.

the detector will be able to better separate closely spaced peaks. Second, the detector will have a better sensitivity for observing weak peaks in the presence of a broad (background) continuum. The latter effect is demonstrated in Fig. 4.18, showing three different γ -ray spectra. Each spectrum contains a signal of the same net intensity, superimposed on a continuous background, but with different energy resolution. In the bottom spectrum, corresponding to the poorest energy resolution, a statistically significant peak above background can no longer be discerned.

The energy resolution is quantitatively defined by the ratio of FWHM and the location of the peak centroid H_0 ,

$$R \equiv \frac{\text{FWHM}}{H_0} \tag{4.41}$$

and is frequently expressed in percent. The energy resolution is influenced by a number of factors which are present even if each incident radiation quantum

deposits precisely the same amount of energy in the detector. These include pulse height drifts during the course of the measurement, random noise from the detector and associated electronics, and statistical fluctuations in the number of created charge carriers. The last contribution sets an inherent limit on the detector performance (Knoll 1989). For example, semiconductor detectors generate a very large number of charge carriers per event. Since this implies relatively small statistical fluctuations in the number of charge carriers, these types of detectors have excellent energy resolutions. In general, if several independent factors contribute to the intrinsic detector energy resolution, then the overall detector response function will tend toward a Gaussian shape according to the central-limit theorem of statistics.

Another important detector property, called *detection efficiency*, is related to the probability of detecting a quantum of radiation emitted by a source (for example, a radioisotope or a nuclear reaction). Efficiencies can be determined from the information presented in pulse height spectra. Suppose that the spectrum shown in Fig. 4.17b is obtained by measuring a source that emits N_0 monoenergetic radiation quanta. Some incident quanta deposit their entire energy in the spectrum, corresponding to the observed sharp peak, while others deposit only a fraction of their energy giving rise to a continuum below the full-energy peak. The leftmost vertical dashed line indicates a threshold below which electronic noise dominates the spectrum. The *total efficiency* is then defined by the ratio of total counts recorded in the spectrum above the threshold and the number of radiation quanta emitted by the source,

$$\eta_{\text{tot}} \equiv \frac{\mathcal{N}_{\text{total}}}{\mathcal{N}_0} \tag{4.42}$$

It is assumed that any background contributions unrelated to the source have been subtracted from N_{total} . Furthermore, we can also define a (full-energy) *peak efficiency* as the ratio of counts recorded only in the full-energy peak and the number of quanta emitted by the source,

$$\eta_{\text{peak}} \equiv \frac{\mathcal{N}_{\text{peak}}}{\mathcal{N}_0} \tag{4.43}$$

It is again assumed that any background contributions have been subtracted from N_{peak} . Sometimes efficiencies are obtained by replacing the total number of emitted quanta in Eqs. (4.42) and (4.43) by the number of quanta that are incident on the detector. The resulting quantity is referred to as intrinsic (total or peak) detection efficiency. We write

$$\eta = \eta_{\rm int} \frac{\Omega}{4\pi} \tag{4.44}$$

with Ω being the solid angle of the detector in steradian. Note that η_{tot} and η_{peak} include the effective solid angle subtended by the detector as an implicit factor and, therefore, are of primary interest for our considerations.

In the following, we will briefly address certain detector types that are frequently employed in nuclear astrophysics measurements. Extensive discussions of radiation detectors can be found in Leo (1987) and Knoll (1989).

4.4.2 Semiconductor Detectors

The operating principle of semiconductor detectors relies on the formation of a semiconductor junction. The junction is formed by using doped semiconductors, with silicon and germanium being the most widely used materials. For example, at the interface of a p-type and a n-type semiconductor material, a region is created which is devoid of mobile charge carriers (electrons or holes). If a reverse bias voltage is applied to the junction, for example, a negative voltage on the p-side, the depletion zone representing the active volume of a radiation detector is significantly enlarged. Incident ionizing radiation will deposit a certain amount of its energy in this zone and thereby create electron–hole pairs. These are swept out by the electric field and a current signal proportional to the amount of the deposited energy is produced.

A main advantage of semiconductors over other detector types is the very small average energy needed for the creation of an electron–hole pair. This energy amounts to only 3.8 eV and 3.0 eV for Si and Ge, respectively, at liquidnitrogen temperature (77 K). These values are smaller by more than an order of magnitude compared to other types of radiation detectors, such as gas ionization chambers or scintillators. Therefore, for the same deposited radiation energy, the number of created charge carriers will be much larger in semiconductors and the energy resolution is significantly improved. Furthermore, the average energy needed for the creation of an electron–hole pair is independent of the radiation energy. Thus, the signal pulse height given by the total number of created electron–hole pairs is proportional to the amount of deposited energy and the response of a semiconductor detector is highly linear.

Silicon is the most common semiconductor material used for the detection of charged particles. The intrinsic efficiency amounts to about 100% since few incident particles will fail to produce some ionization in the active detector volume. For measurements of the incident particle energy, the depth of the depletion zone has to be larger than the particle range. For example, α -particles of 10 MeV energy have a range of \approx 70 µm in silicon (Fig. 4.7).

For the detection of photons, germanium is preferred over silicon because of the much larger atomic number. However, the average energy for creating an electron–hole pair is smaller in germanium. As a result, the leakage current through the semiconductor junction is larger at room temperature, thus contributing to electronic noise at the detector output. Therefore, the germanium crystal must be cooled to liquid-nitrogen temperature.

The bias voltage determines the thickness of the depletion zone. Typical values for silicon charged-particle detectors amount to 50–300 V, while bias voltages of a few 1000 V are used for germanium photon detectors.

Semiconductor detectors are subject to radiation damage. Incident ionizing radiation causes lattice defects by knocking atoms out of their normal position. These defects can trap charge carriers, leading to incomplete charge collection. As a result, the leakage current increases and the energy resolution degrades. For example, significant deterioration in the performance of silicon charged-particle detectors has been observed for integrated incident fluxes of $10^{12}-10^{13}$ protons/cm² and $10^{11} \alpha$ -particles/cm² (Knoll 1989). For germanium photon detectors, significant degradation in energy resolution occurs for integrated fast neutron fluxes in excess of 10^7-10^9 /cm², depending on the detector specifications.

Silicon charged-particle detectors

The most widely used silicon detectors for charged-particle measurements are silicon surface barrier detectors. In this case, a junction is formed between a doped semiconductor region and a metal, for example, n-type silicon and gold. Such junctions are called Schottky barriers. The situation is similar to the pn junction described above and a depletion zone extending over the entire semiconductor region is formed. The outer housing and the front surface of a silicon surface barrier detector are grounded, while the output signal is extracted from the back surface of the silicon wafer. Since usually n-type silicon is used for the production of surface barrier detectors, a positive voltage is required for the reverse bias voltage of the junction. Surface barrier detectors of various depletion zone thicknesses (between several micrometers and a few millimeters) and active areas (up to several tens of square centimeters) have been produced. They are of compact size and can be placed conveniently in scattering chambers.

Junctions are also produced by forming heavily doped n- or p-layers in semiconductor material via ion implantation. The depth profile and concentration of the impurity ions are controlled by adjusting the implantation energy and current. Ion-implanted detectors have improved properties for measuring charged particles compared to surface barrier detectors. The former have thinner entrance windows (several tens of nanometers), and the active area is less sensitive to surface contamination.

A typical pulse height spectrum of an implanted silicon detector, obtained with a ²⁴¹Am source, is shown in Fig. 4.19. This source emits several discrete α -particle groups with energies around 5.5 MeV. The detector has an active surface area of 50 mm² and a resolution of 10 keV (for α -particles of 5.5 MeV). It resolves most of the α -particle groups. While energy resolutions of \approx 10 keV (0.2%) are routinely achieved with such small silicon charged-particle detectors, resolutions for larger detectors typically amount to \approx 15–20 keV.



Fig. 4.19 Pulse height spectrum of α -particles from an ²⁴¹Am source, measured by using a high-resolution implanted silicon detector (of 50 mm² active area and 10 keV resolution). The source emits several discrete α -particle groups. Their energies (in keV) are adopted from the National Nuclear Data Center, Brookhaven National Laboratory. Most of the known α -particle groups are resolved in the spectrum. Courtesy of Joseph Newton.

Germanium photon detectors

The depletion depths of the semiconductor detectors discussed above amount at most to a few millimeters and thus are too thin for the detection of more penetrating radiation, such as photons. In this case, much larger active detector volumes are necessary. Gamma-ray detectors are produced from highpurity p- or n-type germanium (HPGe), with impurity concentrations below 10^{10} atoms/cm³. In coaxial, closed-ended HPGe detectors, one of the electrical contacts is produced by forming a heavily doped n-type region of several 100 µm thickness (usually via lithium evaporation and diffusion), while the other contact represents a heavily doped p-type region of less than 1 µm thickness (formed, for example, by ion implantation). The active volume is the entire region between the electrical contacts. The regions of the contacts do not produce charge carriers and, therefore, are called dead layers. The detector capsule includes the germanium crystal and the preamplifier. The germanium crystal is in thermal contact with liquid nitrogen contained in an insulated dewar, keeping the crystal at a temperature of 77 K.

The excellent energy resolution of semiconductor detectors compared to other types of γ -ray spectrometers is demonstrated in Fig. 4.20. The spectra are obtained with a ¹⁵²Eu source by using a HPGe detector (part a) and a NaI(Tl) scintillator (part b). Scintillators will be discussed in Section 4.4.3. The superior energy resolution of the HPGe detector is striking. The ¹⁵²Eu source emits γ -rays of many discrete energies (Table 4.6). Clearly, the excellent energy resolution in the HPGe detector spectrum allows the separation of many closely spaced γ -ray peaks which remain unresolved in the NaI(Tl) spectrum.



Fig. 4.20 Pulse height spectra of γ -rays from a ¹⁵²Eu source, measured by using a (a) HPGe detector, and (b) Nal(TI) detector. This particular γ -ray source emits photons of many discrete energies that are listed in Table 4.6. Far more peaks are visible in the top compared to the bottom spectrum, demonstrating the superior energy resolution of germanium detectors compared to scintillators. Courtesy of Richard Longland.

For this reason, germanium detectors are used in the majority of γ -ray spectroscopy studies. The energy resolution varies with energy and, therefore, the values are specified at a fixed energy. For germanium photon detectors, energy resolutions are usually quoted for a γ -ray energy of 1333 keV (as provided by a ⁶⁰Co source). Measured values of the FWHM typically amount to $\approx 2-3$ keV, corresponding to $\approx 0.2\%$ (see Eq. (4.41)).

The output pulse shape of germanium detectors depends on a number of factors, including the charge collection process and the location in the crystal at which an incident radiation quantum deposits its energy. The latter effect is shown in Fig. 4.21. The results were obtained by a Monte Carlo simulation and indicate, for a germanium detector, the fraction of the full-energy peak contributed by different interaction mechanisms. Above an energy of a few 100 keV, which is of main importance in nuclear astrophysics measurements, the photoelectric effect is much less likely to occur compared to Compton scattering (Section 4.2.2). Thus, events contributing to the full-energy peak in this energy range arise mainly from multiple interactions, for example, one



Fig. 4.21 Fraction of full-energy peak contributed by different interaction mechanisms. The results are obtained from Monte Carlo simulations of photon interactions in a germanium detector. Reprinted with permission from J. Roth, J. H. Primbsch, and R. P. Lin, IEEE Trans. Nucl. Sci. NS-31(1), p. 367 (1984). ©1984 IEEE.

or more Compton scattering events followed by photoelectric absorption of the scattered γ -ray, rather than a single photoelectric interaction. Such effects result in a large variation of the pulse rise time and make germanium detectors sometimes less suitable for measurements in which the precise arrival time difference of two events is of interest.

4.4.3 Scintillation Detectors

Radiation incident on a scintillator deposits energy in the active volume and thereby excites atoms and molecules. The atoms de-excite mainly by prompt emission (within $\approx 10^{-8}$ s) of light, but delayed emission may also occur if some excited states are metastable. These processes are referred to as fluorescence and phosphorescence, respectively. The light strikes a photosensitive surface (photocathode), releasing at most one photoelectron per incident photon. These secondary electrons are accelerated and multiplied through a series of electrodes, called dynodes. They are finally collected on the anode and form the output pulse of the photomultiplier tube. These processes are shown in Fig. 4.22. Scintillator detectors must have a high probability for converting absorbed energy into fluorescent light. They must be transparent to their own light, and the light emission spectrum has to be consistent with the response of the photomultiplier. For many scintillators, the light output, and

hence the amplitude of the electrical output signal, is nearly proportional to the absorbed energy. Therefore, scintillators are suitable as devices for energy measurements, although their energy resolution and linearity is far inferior compared to semiconductor detectors (Fig. 4.20). On the other hand, scintillators have certain advantages over semiconductors. First, they have fast response and recovery times and hence are frequently used if the measurement of the time differences between two events is of interest. Second, scintillators can be produced in a variety of sizes and shapes.

For any scintillator, it is important to collect a large fraction of the light emitted from the track of the ionizing radiation. However, many light photons are reflected one or more times at the scintillator surface before reaching the photomultiplier tube. If the angle of incidence of the light is less than a certain value (called the critical angle), then only partial reflection takes place and some light will escape from the surface. For a given scintillator shape, the fraction of light lost will in general depend on the location of the radiation track with respect to the photomultiplier tube. The uniformity of the light collection determines the variation in signal pulse amplitude and thus the energy resolution of the scintillator. Therefore, scintillator crystals are usually surrounded by a reflecting surface (for example, paint, powder or foil) in order to recapture some of the escaping light. In certain instances, light collection may also be improved by viewing the scintillator with more than one photomultiplier tube. On the other hand, any internal reflection must be minimized at the interface between scintillator and the glass end window of the photomultiplier tube. This is usually achieved by using silicon oil of high viscosity as an optical coupling fluid. Furthermore, scintillation detectors must be shielded from room light for obvious reasons.

Many different types of scintillators in solid, liquid, or gaseous form are used in radiation detection studies. Here, we will focus on inorganic scintillators for photon detection and organic scintillators for counting charged particles and fast neutrons.

Inorganic scintillator photon detectors

The most common inorganic scintillators are single crystals of alkali halides, such a NaI. A polycrystalline scintillator would cause light reflections and absorptions at crystal surfaces and thus a single crystal is needed to achieve light transparency. To increase the probability for light emission and to reduce self-absorption of light, small amounts of impurities, called activators, are added to the crystal, with thallium being a common choice. For NaI(Tl), the high atomic number of thallium ($Z_{TI} = 53$) results in a larger γ -ray efficiency compared to germanium detectors ($Z_{Ge} = 32$). Therefore, inorganic scintillators are preferred in certain types of experiments, even though they have an inferior energy resolution (Fig. 4.20b). A disadvantage of NaI(Tl) is the fact that



Fig. 4.22 Schematic diagram of a scintillation detector. Incident radiation produces light in the scintillator material. The light strikes the photocathode and the few emitted secondary electrons are multiplied in the photomultiplier tube by a series of dynodes. The output pulse is extracted from the anode. See the text.

the crystal degrades quickly in the presence of moisture (hygroscopy). Consequently, these scintillators must be contained in air tight protective closures.

The energy resolution of NaI(Tl) detectors is customarily quoted at an energy of 662 keV as provided by a 137 Cs source. For smaller cylindrical detectors, resolutions of 6–7% can be achieved, while for more complicated crystal shapes the light collection is less uniform and the energy resolution becomes worse.

Two pure inorganic scintillators, that do not require the presence of an activator element to promote the scintillation process, are bismuth germanate (Bi₄Ge₃O₁₂ or BGO) and barium fluoride (BaF₂). The very high atomic number of bismuth ($Z_{Bi} = 83$) in the BGO material has major advantages for the detection of γ -rays above 10 MeV energy compared to other detector types, although the energy and time resolution of BGO detectors is inferior compared to NaI(Tl). The scintillation light of barium fluoride contains a very fast component with a decay time of less than 1 ns, which is smaller than decay times achieved even in the fastest organic scintillators. Therefore, BaF₂ detectors are attractive for applications in which large efficiencies and fast timing are required, for example, for the time-of-flight technique in neutron-capture studies (Section 4.6.3).

Organic scintillator charged-particle and neutron detectors

Organic scintillators consist of aromatic hydrocarbon compounds that contain benzene-ring structures. Their most outstanding feature is a very rapid signal pulse decay time of about 1–2 ns. They can be used in many physical forms, such as liquid or solid solutions, without loss of their scintillating properties. The most widely used organic radiation detectors are organic scintillators in a solid plastic solvent, called *plastic scintillators*. These are easily shaped and

fabricated to desired forms. Plastic scintillators are produced as sheets, blocks, cylinders, and thin films of a few μ g/cm² thickness. They are rugged and resistant to lower alcohols, but not to acetone and body acids and, therefore, must be handled with care. Liquid solutions of organic scintillators in an organic solvent, called *liquid scintillators*, are also widely used. They have the advantage that they can be loaded easily with certain materials in order to increase their efficiency for specific applications. Liquid scintillators are, however, very sensitive to impurities in the solvent.

It is sometimes of advantage if the scintillator is not directly coupled to the photomultiplier tube. This may be, for example, because of geometrical considerations or an unusual shape of the scintillator. The coupling can then be achieved by using a transparent solid with a high refractive index, such as lucite, which acts as a guide for the scintillation light and hence is called *light guide*. In principle, a light guide should transmit all the light that enters at its input but in practice some light loss will occur. Alternatively, optical fibers may be used as light guides, allowing for a flexible connection between scintillator and photomultiplier tube (Longland et al. 2006).

Organic scintillators are not suitable as high-resolution γ -ray spectrometers because their small atomic number results in greatly reduced γ -ray interaction probabilities for photoelectric effect and pair production. Recall, that either of these processes has to occur besides Compton scattering in order to contribute to the full-energy peak count rate. Gamma-ray pulse height spectra of organic scintillators show pronounced Compton edges, but virtually no full-energy peaks. Organic scintillators are very useful for the detection of γ -rays when fast timing rather than pulse height resolution is of primary interest, such as in time-of-flight measurements of neutron-capture reactions (Section 4.6.3).

Figure 4.23 shows a room background spectrum measured with a plastic scintillator detector. The counter is viewed by a single photomultiplier tube. The surfaces of plastic scintillators are highly polished in order to increase internal reflection. The scintillator was wrapped first with aluminum foil to increase external reflection and then with black tape to provide a light-tight layer. A thin layer of air, having a small refractive index, between scintillator and aluminum foil also increases internal reflection. The spectrum shown was measured over a period of 14 h. The structures at low pulse heights represent the Compton edges of various room background γ -ray lines, while the broad peak at large pulse heights is caused by cosmic-ray muons that deposit a small fraction of their energy traversing the scintillator. Plastic scintillators are frequently used as muon anticoincidence shields around a primary detector (for example, a germanium crystal). In such an arrangement, the output of the primary detector is only accepted if, during a certain time window, there is no coincident pulse at the output of the plastic scintillator. With this method, the background induced by cosmic rays in the primary detector can be reduced significantly (Section 4.7).



Fig. 4.23 Room background spectrum measured with a plastic scintillator detector over a period of 14 h at sea level. The detector dimensions for the length, width, and thickness are 30, 20, and 5 cm, respectively. Compton edges of photons dominate the spectrum at low pulse heights, while the broad peak at larger pulse heights is caused by cosmic-ray muons that deposit a small fraction of their energy traversing the scintillator. From Longland et al. (2006).

Liquid organic scintillator detectors are frequently employed for the detection of fast neutrons. Incident neutrons elastically scatter on the hydrogen contained in the active detector volume. In each scattering process a neutron transfers energy to a recoil proton. The latter particle, in turn, is detected in the scintillator like any other energetic proton as it slows down in the active volume. Depending on the scattering angle, the transferred energy can range between zero and the total incident neutron energy. Furthermore, for incident neutron energies below 10 MeV, the elastic scattering from hydrogen is nearly isotropic. As a result, the energy distribution of the recoil protons, and hence the detector response function, should have a rectangular shape. In reality several factors distort this simple rectangular distribution (Knoll 1989). Some organic scintillators consist of special liquids that have the characteristic of producing different pulse shapes in response to different types of incident radiation. For example, a scintillator loaded with NE213 will give rise to different pulse shapes for neutrons and γ -rays (Lynch 1975). Based on the measured pulse shape differences, events from various radiation types can be distinguished electronically. This procedure is referred to as pulse shape dis*crimination*. It allows for a substantial reduction of an unwanted γ -ray background in the detection of fast neutrons. The efficiency of liquid scintillator detectors for MeV neutrons can be as high as 50%.

272 4 Nuclear Physics Experiments 4.4.4 **Proportional Counters**

A proportional counter consists of a vessel with conducting walls, acting as a cathode, and an anode (for example, a metal wire) located inside the vessel. The vessel is filled with a suitable gas and a large positive voltage is applied to the anode. Incident radiation deposits energy in the counter gas and thereby creates a certain number of electron-ion pairs. For most gases, on average about one electron-ion pair is created per 30 eV of energy lost. The mean number of ion pairs created depends then on the energy deposited by the incident radiation quantum in the gas. The electrons and ions are accelerated toward the anode and cathode, respectively. If the electric field strength is sufficiently large, the primary electrons are accelerated toward the anode to energies at which they are also capable of ionizing gas molecules in the detector. The created secondary electrons, in turn, are also accelerated and give rise to still more ionization and so on. The result is an ionization avalanche, with a total number of electron-ion pairs that is directly proportional to the number of primary electron-ion pairs.

A frequently used counter gas is a mixture of 90% Ar and 10% CH_4 (methane). The excited Ar ions in the avalanche de-excite by emission of visible or ultraviolet photons capable of ionizing the cathode and causing further avalanches. This effect is undesirable since it leads to a loss of proportionality. The methane molecules act as a quencher by absorbing the emitted photons and then by dissipating this energy through dissociation or elastic collisions. With such a gas mixture, the factor of proportionality, or multiplication factor, can be as high as 10° . The gas is usually at atmospheric pressure, but higher pressures are sometimes used in order to increase the detection efficiency. A potential problem is the relatively large number of quencher molecules depleted in each detected event, causing changes in the operational characteristics after a certain total number of events has been observed. This problem can be avoided by using a continuous gas flow instead of a sealed vessel.

Proportional counters are used for detecting charged particles and lowenergy X-rays. They are less useful for detecting γ -rays since the probability of interaction between a photon and the detector gas is very small. Proportional counters can also be used for neutron detection by choosing a fill gas with a large cross section for a neutron-induced reaction. The most frequently used fill gases for converting incident neutrons to directly detectable charged particles are ¹⁰BF₃ and ³He. These take advantage of the reactions ¹⁰B(n, α)⁷Li and ${}^{3}\text{He}(n,p){}^{3}\text{H}$, respectively. If slow neutrons are incident on such a detector, the neutron energy is negligible compared to the energy release per reaction. Therefore, the total energy imparted to the charged reaction products (⁷Li + α or ${}^{3}H + p$) for each event is equal to the Q-value, while any information about the incident neutron energy is lost.

4.4.5 Microchannel Plate Detectors

Microchannel plates consist of a lead glass plate with a large number ($\approx 10^7$) of microscopic channels, typically 10–50 µm in diameter, oriented parallel to each other (Wiza 1979). The inner surfaces of the channels are treated so as to act as secondary electron emitters. The front and rear surfaces of the plate are coated with a metallic alloy, such as nichrome ($Ni_7Cr_2Fe_3$), and act as electrodes so that a voltage can be applied along the length of the channels (Fig. 4.24). This device has a direct sensitivity for detecting charged particles (electrons, ions) and energetic photons. A radiation quantum incident on the front face enters one of the microchannels and produces secondary electrons upon impact with the channel wall. The secondary electrons are accelerated along the channel until they eventually strike the wall again, releasing further electrons and so on. Typical electron multiplication factors amount to $\approx 10^4$ for a single microchannel plate. This avalanche of secondary electrons is collected at the anode and results in a large output pulse. Each microchannel acts as an independent electron multiplier. Several plates may be used together to provide a higher overall gain. Figure 4.24 shows two plates in the common "chevron" geometry. Here, the microchannels are oriented at an angle with respect to the plate surface and to each other in order to reduce troublesome feedback effects from positive ions that occasionally form in the channels and that drift back to the plate front face. In the chevron geometry, these ions are made to strike the channel wall before their energy is high enough to create secondary electrons.

Microchannel plate detectors are not very useful for energy measurements because relatively few secondary electrons are emitted upon impact of the incident radiation. A main advantage is their excellent timing property. The total transit time of the secondary electrons through a channel is only a few nanoseconds. The timing performance depends on the spread in transit time and amounts to only ≈ 100 ps, a value that is much smaller compared to even the fastest plastic scintillators. Microchannel plate detectors are very robust and have been used in experiments with count rates of up to $\approx 10^7$ s⁻¹ (Mosher et al. 2001). Their intrinsic efficiency varies according to the energy and type of the incident radiation (Wiza 1979). For ions with masses of A = 3–16 and energies of E = 0.3–10 MeV, measured intrinsic efficiencies amount to 65–90% (Mosher et al. 2001).





Fig. 4.24 Microchannel plate detector. (Left panel) Single lead glass plate with a large number of microscopic channels (each with a diameter of $10-50 \mu$ m). (Right panel) Detector consisting of two microchannel plates in a "chevron" geometry. Incident charged particles (electrons, ions) or energetic photons entering one of the channels give rise to the emission of secondary electrons. Each microchannel acts as an indepen-

dent electron multiplier because of the large voltage applied between the front and back faces of the two plates. The avalanche of secondary electrons is collected on the metal anode and results in a large output pulse. Reprinted from J. L. Wiza, Nucl. Instrum. Methods, Vol. 162, p. 587 (1979). Copyright (1979), with permission from Elsevier.

4.5

Nuclear Spectroscopy

The science and study of spectra is called spectroscopy. We will be mainly concerned with detector pulse height spectra induced by nuclear radiation. Most important is the analysis of relatively sharp peaks in the spectrum. The energy of a discrete line corresponds frequently to the energy difference between the initial and final nuclear states involved in the transition and thus reflects the origin of the measured radiation. The intensity of a discrete line is proportional to the number of decaying nuclear states and hence allows for the determination of nuclear cross sections. The quantitative interpretation of pulse height spectra requires the knowledge of certain detector properties. First, the signal pulse height (or channel number) has to be calibrated and expressed in terms of the radiation energy. Second, the measured peak intensity needs to be corrected for the detector efficiency. These procedures are referred to as energy and efficiency calibrations. In the following, we will discuss some typical experimental situations encountered in charged particle, γ -ray, and neutron spectroscopy.

4.5.1 Charged-Particle Spectroscopy

Energy calibrations

Consider first a radioactive source that emits charged particles. The most common radioactive charged-particle sources emit α -particles since long-lived proton emitting sources do not exist. Suppose further that an α -particle source is placed at some distance from a charged-particle detector, such as a silicon counter. The α -particles from the nuclear source are emitted with discrete energies. If the thickness of the active detector volume is larger than the α particle range ($R < 100 \,\mu\text{m}$ for $E_{\alpha} < 10 \,\text{MeV}$; Fig. 4.7) and since processes that backscatter the incident particle or otherwise result in partial energy deposition are usually negligible, the incident radiation will deposit its entire energy in the silicon detector. As a result, discrete peaks of nearly Gaussian shape appear in the pulse height spectrum, each corresponding to an α -particle group of discrete energy (Fig. 4.25). If the energies of the α -particles are well known from previous measurements, the horizontal axis of the pulse height spectrum can be calibrated by relating channel numbers C_i to energies E_i . As already noted, semiconductor detectors respond nearly linearly to the energy of the incident radiation and, therefore, a useful expression for the calibration is

 $E_i = aC_i + b \tag{4.45}$

where *a* and *b* are empirical constants. Properties of some α -particle calibration sources are listed in Table 4.5. For precise energy calibrations, the α -particle energy loss in the detector dead layer may need to be taken into account. The thickness of the dead layer can be determined by measuring the energy of a monoenergetic charged-particle group at several different angles of incidence (Knoll 1989). The energy loss in the source itself may also need to be considered. Most α -particle calibration sources are prepared by depositing a thin layer of the isotope on the surface of a backing in order to minimize energy losses and α -particle absorption. These sources are also protected with a very thin layer of foil.

It is important to point out that there are small differences in the pulse heights observed for different incident light charged particles (for example, protons and α -particles) of the same energy in semiconductor detectors (Knoll 1989). These differences are on the order of 1% (\approx 30 keV for 3 MeV total deposited energy). Therefore, the pulse height spectrum should be calibrated, if possible, by using the same species of particles as is emitted in the actual reaction measurement. For heavy ions the pulse height differences are much larger. The effect is referred to as *pulse height defect*.





Fig. 4.25 Measurement of charged particles emitted by an α -particle source. (a) Setup showing the source and the charged-particle detector; *d* and *a* are the source–detector distance and the radius of the active detector area, respectively. (b) Typical pulse height spectrum showing peaks at discrete channel numbers. An energy calibration of the spectrum is performed by relating the discrete channel numbers to known energies of the α -particle groups emitted by the source.

Tab. 4.5 Properties of some α -particle calibration sources. (a) From Nichols (1996). Errors are given in parentheses and refer to the last significant digit(s).

Isotope	Half-life ^a	Energies ^a (keV)	Branching ^a (%)
¹⁴⁸ Gd	75(3) y	3182.68(2)	100.0
²⁴¹ Am	432.7(5) y	5442.90(13)	12.8(2)
		5485.60(12)	85.2(8)
²³⁰ Th	7.54(3) y	4620.5(15)	23.4(1)
		4687.0(15)	76.3(3)
²⁴² Cm	162.94(6) d	6069.42(12)	25.0(5)
		6112.72(8)	74.0(5)

Efficiencies

The intrinsic efficiency of silicon counters for detecting charged particles is close to unity and, therefore, the peak efficiency is given by the solid angle Ω subtended by the detector. The efficiency can be measured by using a calibration source of well-known activity. Assuming that the source emits radiation isotropically and that no attenuation takes place between source and detector, we obtain for the peak efficiency (see Eqs. (4.43) and (4.44))

$$\eta_{\text{peak}} = \frac{\Omega}{4\pi} = \frac{\mathcal{N}_{\text{peak}}}{\mathcal{N}_0} = \frac{\mathcal{N}_{\text{peak}}}{AtB}$$
(4.46)

where the solid angle Ω is in units of steradians and $\mathcal{N}_{\text{peak}}$ is the net area of the full-energy peak; *A*, *t*, and *B* are the activity of the source at the time of the measurement, the measuring time, and the branching ratio of the radiation, respectively. Branching ratios, defined as the fraction of a specific transition per nuclear decay, for some common calibration sources are listed

in Table 4.5. As a test of consistency, it is often useful to estimate the efficiency without relying on the activity of radioactive sources. For the common case of a point-like source and a circular detector positioned with its face normal to the source–detector axis, the peak efficiency is given by (Knoll 1989)

$$\eta_{\text{peak}} = \frac{1}{2} \left(1 - \frac{d}{\sqrt{d^2 + a^2}} \right) \tag{4.47}$$

where *d* and *a* are the distance between source and detector and the detector radius, respectively. If the distance *d* is large compared to the radius, $d \gg a$, the peak efficiency reduces to

$$\eta_{\text{peak}} \approx \frac{\pi a^2}{4\pi d^2} = \frac{a^2}{4d^2} \tag{4.48}$$

Elastic scattering studies

An example of a setup for the study of elastic scattering is shown in Fig. 4.26a. A proton beam of energy $E_p = 440$ keV is incident on a transmission target consisting of a thin MgO layer evaporated onto a thin carbon foil. A silicon detector with a resolution of ≈ 10 keV, positioned at an angle of $\theta = 155^{\circ}$ with respect to the beam direction, is used for detecting elastically scattered protons. The measured pulse height spectrum is shown in Fig. 4.26b. Three peaks are observed in the spectrum, corresponding to elastic scattering from Mg, O, and C, the three elements present in the target.

The observed peak centroids can be used for calibrating the proton energy in the spectrum if the target is relatively thin so that energy loss effects are negligible. In this case, the widths of the peaks resulting from proton scattering on Mg and O are given by the detector resolution. The peak centroids correspond to the energy $E_{p'}$ of the elastically scattered protons which, for a given bombarding energy E_p and detector angle θ , are determined by the kinematics of the scattering process (Appendix C.1).

In general, however, the target thickness has to be taken into account and the widths and centroids of the measured peaks are influenced both by the proton energy loss in the target and the detector resolution. Note that the energy calculated from kinematics only applies to those protons that are elastically scattered from the first target layers. For protons scattered from Mg or O deeper inside the target, not only the energy loss of the projectiles has to be considered, but that of the scattered protons on their path through the target as well. The location of the carbon elastic scattering peak is also influenced by energy loss. In order to reach the carbon layer, the projectiles have to traverse the MgO target, while the scattered protons also lose a fraction of their energy moving through the target on their way to the detector.

The measured peak areas may be used to calculate the differential cross sections of elastic scattering, if the number of target nuclei (or the target thick-





Fig. 4.26 Typical elastic scattering study. (a) Setup showing a proton beam, a MgO transmission target evaporated on a thin carbon backing, and a particle detector mounted at a back-angle. Detected protons that are scattered from the backside of the target will have a smaller energy (E_2) compared to those scattered from the front side (E_1) because of energy loss effects.

(b) Measured spectrum of elastically scattered protons at an incident proton energy of $E_{\rm p}$ = 440 keV. The peaks correspond to protons scattered from the target (Mg and O) and the backing (C). Reprinted from D. C. Powell et al., Nucl. Phys. A, Vol. 660, p. 349 (1999). Copyright (1999), with permission from Elsevier.

ness) and the detector efficiency are known. Expressions relating measured yields to cross sections are given in Section 4.8. At sufficiently low bombarding energies, the cross section will be dominated by Coulomb scattering. This circumstance is frequently taken advantage of in order to determine the number of target nuclei from the measured peak intensity and the calculated Rutherford cross section (see Eq. (4.136)). At higher energies, resonances may contribute to the elastic scattering process. In this case, the measured elastic scattering cross section provides information on the resonance parameters, such as resonance energies, partial widths, and quantum numbers (Section 2.5).

Nuclear reaction studies

Figure 4.27a shows an experimental arrangement for measuring the reaction ${}^{31}P(p,\alpha){}^{28}Si$ at a bombarding energy of $E_p = 390$ keV. Cross sections of astrophysical interest at low energies are usually small. The charged-particle detector for measuring the reaction α -particles has to cover as large a solid angle as possible in order to maximize the count rate. The active area of the silicon detector and the distance between detector and target are 450 mm² and 5 cm, respectively. The energy resolution of the detector amounts to \approx 20 keV. A directly water-cooled beamstop target is used to allow for high proton beam currents of several 100 µA. The target is produced by implanting ${}^{31}P$
ions into a tantalum sheet and consists of a ³¹P–Ta layer with a stoichiometry of 3:2 (Table 4.3). For a given beam energy (E_p = 390 keV) and detector angle (θ = 145°), the energy of the emitted α -particles is determined by the kinematics of the nuclear reaction (Appendix C.1) and amounts to \approx 2 MeV.

Since the beamstop target consists of a high-*Z* material (tantalum), the number of elastically scattered protons becomes very large (> 10⁶ s⁻¹). Note that the protons are scattered both from the relatively thin ³¹P–Ta target layer as well as the thick tantalum backing. Hence, their energies range from a maximum of \approx 350 keV (protons scattered by the first target layers without energy loss) down to zero (protons scattered inside the tantalum backing and losing energy on the path to the detector). As a result, the detector count rate in the region of the expected reaction α -particles would be overwhelmed by the pileup of signals caused by the unwanted scattered protons. Therefore, a foil is placed in front of the charged-particle detector, which is sufficiently thick to stop the elastically scattered protons, but at the same time is thin enough to transmit the reaction α -particles.

The measured pulse height spectrum is shown in Fig. 4.27b. The α -particles from the ³¹P(p, α)²⁸Si reaction have lost a significant fraction of their energy in the foil and are observed in the first part of the spectrum ($E_{\alpha} \approx 0.5$ MeV). Furthermore, the α -particle peak is significantly broadened due to energy straggling in the foil ($\Delta E \approx 100$ keV). The steep background at low energies is mainly caused by elastically scattered protons leaking through the foil. At higher energies, the spectrum displays contributions from the (p, α) reactions on ¹¹B, ¹⁵N, and ¹⁸O contaminants in the target and backing (Section 4.3.3). These contaminants give rise to discrete peaks, except in the case of the ¹¹B(p, α)2 α reaction which emits three particles in the exit channel and thus produces a continuous background.

The total number of induced reactions, N_R , can be calculated from the measured peak intensity by using an expression similar to Eq. (4.46). In general, the intensity of the reaction products is not isotropic (Appendix D) and, therefore, the angular correlation W has to be taken into account. One finds

$$\mathcal{N}_R = \frac{\mathcal{N}_{\text{peak}}}{\eta_{\text{peak}}BW} \tag{4.49}$$

The quantities η_{peak} and *W* are usually obtained at a laboratory detection angle of θ and have to be expressed in the center-of-mass system (Appendix C.2). The branching ratio, *B*, is now defined as the fraction of a specific transition per nuclear reaction.





Fig. 4.27 Study of the ³¹P(p,α)²⁸Si reaction. (a) Setup showing the proton beam, the beamstop target consisting of ³¹P implanted into Ta, and the particle detector covered by a foil in order to reduce the large number of elastically scattered protons reaching the detector. (b) Measured pulse height spectrum at a bombarding energy of E_p = 390 keV. The α -particle peak of interest, arising from a resonance in ³¹P(p,α)²⁸Si,

occurs at relatively small pulse heights in a region which is dominated by protons leaking through the foil (caused by foil thickness inhomogeneities). Alpha-particles originating from (p, α) reactions on the contaminants ¹¹B, ¹⁵N, and ¹⁸O are visible at higher energies. Reprinted from C. Iliadis et al., Nucl. Phys. A, Vol. 533, p. 153 (1991). Copyright (1991), with permission from Elsevier.

4.5.2

Gamma-Ray Spectroscopy

Response function

The response of γ -ray detectors to incident radiation is more complicated compared to charged-particle detectors. As already noted, γ -rays interact with matter via the photoelectric effect, Compton scattering and pair production (Section 4.2.2). The influence of these effects on the measured pulse height distribution is shown in Fig. 4.28. In the following, we will assume that monoenergetic photons of energy E_{γ} are incident on the detector.

In case (a), an incident photon undergoes photoelectric absorption. The emitted photoelectron travels typically a distance of at most a few millimeters and loses its energy through ionization and excitation of atoms in the active detector volume and through the emission of bremsstrahlung. For a sufficiently large active volume, the entire energy of the photoelectron is absorbed in the detector and, therefore, the resulting pulse height appears in the region of the full-energy peak (FEP), corresponding to a photon energy of E_{γ} .

In case (b), the incident photon undergoes Compton scattering. The scattered photon escapes from the active volume and thus only a fraction of the incident photon energy is deposited in the detector. The precise energy trans-



Fig. 4.28 Response of a γ -ray detector to monoenergetic incident radiation. (a) Representation of different photon histories. (b) Pulse height spectrum; the meanings of the labels are: full-energy peak (FEP), multiple-site events (MSE), Compton edge (CE), single-escape peak (SEP), double escape peak (DEP), Compton continuum (CC) and back-scattering peak (BSP).

ferred to the recoil electron depends on the scattering angle. All scattering angles are possible and, therefore, the energy distribution of the recoil electrons gives rise to the Compton continuum (CC). The maximum possible value of K_e^{max} , that is, the Compton edge (CE), corresponds to a photon scattering angle of $\theta = 180^{\circ}$ (Section 4.2.2). The continuous Compton background is clearly an unwanted contribution to the pulse height spectrum. It reduces the signal-to-noise ratio for the detection of weak discrete peaks and it also makes the interpretation of complex γ -ray spectra that result from incident photons of different energies more difficult.

In case (c), the incident photon is Compton scattered several times at different locations in the active volume until eventually photoabsorption occurs. The duration of this more complex history amounts to < 1 ns, a value that is smaller than the inherent response time of present day γ -ray detectors. Consequently, the different Compton scattering events and the final photoabsorption occur essentially in time coincidence and exactly the same total energy is deposited in the detector as if the incident photon had undergone a single photoelectric absorption. Such events appear then in the region of the fullenergy peak. In fact, for incident photon energies above a few 100 keV, most events in the full-energy peak are caused by such multiple scattering histories (Fig. 4.21). It has been pointed out in Section 4.2.2 (see also Fig. 4.11) that an energy gap exists between the full-energy peak and the Compton edge. In actual measurements this gap is partially filled in by multiple Compton scattering events (MSE) that are followed by photon escape.

In case (d), an incident photon with an energy of $E_{\gamma} > 2m_ec^2$ undergoes pair production. The created electron and positron lose all their kinetic energy in

the active detector volume. Subsequently, the positron will annihilate with another electron and two photons, each of 511 keV energy, are produced. Again, the annihilation radiation appears virtually in time coincidence with the original pair production event. If both 511 keV photons are absorbed by the detector (for example, via photoelectric effect), then the resulting pulse height will appear in the region of the full-energy peak at E_{γ} . If only one 511 keV photon is absorbed while the other one escapes detection, then the resulting pulse height will give rise to a discrete peak at an energy of $E_{\gamma} - 511$ keV, which is called the single-escape peak (SEP). If both 511 keV photons escape from the detector, a discrete peak appears at an energy of $E_{\gamma} - 1022$ keV, which is called the double-escape peak (DEP). More complicated histories involving Compton scattering of the annihilation quanta occur as well. Such events contribute to a continuum in the pulse height spectrum between the double-escape and full-energy peaks.

Finally, a broad peak is frequently observed in pulse height spectra at an energy of \approx 200–250 keV. It is caused by photons that Compton scatter in material surrounding the active volume before detection. The peak is referred to as back-scattering peak (BSP).

The response of a real γ -ray detector will depend on the size, shape, and composition of the active volume. It can be simulated theoretically by using Monte Carlo calculations which track numerically the histories of many different events taking place in the detector. Calculated response functions for a germanium detector to monoenergetic incident radiation below 3 MeV energy are displayed in Fig. 4.29. Many of the features described above can be observed in the spectra. It is apparent that for increasing γ -ray energy the intensity of the escape peaks becomes gradually larger as the pair production cross section increases.

Energy calibrations

Full-energy peaks are of major interest in γ -ray spectroscopy studies. They correspond to full-energy deposition of the incident photon energy and, therefore, their location in the spectrum is not influenced by any photon energy losses. The pulse heights in a γ -ray spectrum can be calibrated by using absolute energy standards. Some useful γ -ray energy standards are provided in Table 4.6. Note the small errors of the calibration energies which amount to less than 0.001%. The listed radioactive sources are all commercially available and cover an energy range up to ≈ 3.5 MeV. Furthermore, the two γ -ray lines from ⁴⁰K and ²⁰⁸Tl (1460.8 and 2614.5 keV, respectively) are the most prominent room background peaks in γ -ray spectra and thus provide a convenient internal calibration without using radioactive sources. The energy range may be extended up to ≈ 5 MeV by using the radioisotope ⁶⁶Ga (Helmer and van der Leun 2000), although its half-life is rather short ($T_{1/2} = 9.5$ h). Therefore,



Fig. 4.29 Response functions for a germanium detector to monoenergetic incident radiation of energies below 3 MeV, obtained from Monte Carlo simulations of photon histories. All energies are in units of keV. Reprinted from C. Meixner et al., Nucl. Instrum. Methods, Vol. 119, p. 521 (1974). Copyright (1974), with permission from Elsevier.

the region above 3.5 MeV is frequently calibrated by using γ -rays emitted in nuclear reactions. These energies can be calculated precisely from the kinematics of the reaction (Appendix C.1) if certain quantities (such as the bombarding energy, the masses of projectile and target, and the detector angle) are well known. The resulting calibration energies are not as precisely determined as those from radioisotopes. One also has to be careful since many γ -rays emitted in nuclear reactions are Doppler shifted (see Eq. (C.12)). If capture reactions (for example, (p, γ) or (α, γ) reactions) are used for the energy calibration, it is of advantage to position the detector at an angle of $\theta = 90^{\circ}$ where the Doppler shift is zero to first order. Sometimes it may be possible to use the single- and double-escape peaks for calibrations since their energies relative to the location E_{γ} of the corresponding full-energy peak are well known ($E_{\gamma} - m_e c^2$ and $E_{\gamma} - 2m_e c^2$, respectively). However, small systematic shifts in the location of the escape peaks have been observed. The deviations can amount to several 100 eV and seem to depend on the type and geometry of the detector (Endt et al. 1990). This effect has to be considered if γ -ray energies of high precision are of interest.

Tab. 4.6 Properties of common γ -ray calibration sources. (a) From Helmer and van der Leun (2000), unless noted otherwise. (b) From Debertin and Helmer (1988). (c) From Greenwood, Helmer and Gehrke (1979). (d) Half-life and γ -ray yield per disintegration; from Lorenz and Nichols (1996). Errors are given in parentheses and refer to the last significant digit(s).

Source	Half-lifed (days)	Energy ^a (keV)	Branching ratiod (%)
¹⁵² Eu	4933(11)	121.7817(3)	28.37(13)
		244.6974(8)	7.53(4)
		344.2785(12)	26.57(11)
		411.1165(12)	2.238(10)
		778.9045(24)	12.97(6)
		867.380(3)	4.214(25)
		1085.837(10)	10.13(5)
		1089.737(5)	1.731(9)
		1112.076(3)	13.54(6)
		1212.948(11)	1.412(8)
		1299.142(8)	1.626(11)
50 -		1408.013(3)	20.85(9)
⁵⁶ Co	77.31(19)	846.7638(19)	99.933(7)
		1037.8333(24)	14.13(5)
		1175.0878(22)	2.239(11)
		1238.2736(22)	66.07(19)
		1360.196(4)	4.256(15)
		1771.327(3)	15.49(5)
		2015.176(5)	3.029(13)
		2034.752(5)	7.771(27)
		2598.438(4)	16.96(6)
		3201.930(11)	3.13(9)
		3253.402(5)	7.62(24)
		3272.978(6)	1.78(6)
5700	071 70(0)	3451.119(4)	0.93(4)
	271.79(9)	122.00005(12)	05.00(17) 10.69(9)
198	2 6042(8)	130.47350(29)	10.00(0)
¹³⁷ Ce	2.0943(6) 1 102(6) $\times 10^4$	411.80203(17) 661.657(3)	95.57(47)
54Mn	$1.102(0) \times 10$ 312.3(4)	834 838(5)	00.0758(24)
	106 630(25)	808 036(4)	99.9756(24)
1	100.000(20)	1836 052(13)	94.0(3)
⁶⁰ Co	1025 5(5)	1173 228(3)	99.30(3)
00	1923.3(3)	1332 / 02(/)	99.037 (22)
²² Na	950 8/9)	107/ 537(7)	99.905(0)
40 K	330.0(3)	1460 830(5)	33.305(13)
208 TI		2614 533(13)	
		2014.000(10)	

When several energy calibration points have been established over the region of interest, a calibration curve relating energies to channel numbers may be derived from a least-squares fit. With germanium detectors it is frequently sufficient to represent the energy as a linear function of the channel. Deviations from linearity depend primarily on nonlinearities of the electronic amplifier–analyzer system and may amount to several hundred electron volts. It is sometimes even appropriate to use for a linear energy calibration only two well-known peaks with energies E_1 and E_2 and channel centroids C_1 and C_2 ,

$$E_i = aC_i + b = E_1 + \frac{E_2 - E_1}{C_2 - C_1}(C_i - C_1)$$
(4.50)

If higher precision is desired, the energy calibration can be obtained from a cubic least-squares fit. Higher order polynomials may be necessary if NaI(Tl) detectors are used since their response is far less linear compared to germanium detectors.

Efficiency calibrations

Measurements of cross sections require knowledge of detection efficiencies. Furthermore, the accuracy of the derived cross sections depends directly on the uncertainty of the efficiencies. Full-energy peak efficiencies may be calculated with Monte Carlo procedures if the crystal dimensions and the geometry of the setup are precisely known. However, for germanium detectors this is rarely the case. Crystals are not standardized to any degree and, furthermore, crystal dimensions supplied by manufacturers have been found to be inaccurate by several millimeters (Helmer et al. 2003). Also, long-term changes in the charge collection process can cause the detector efficiency to vary with time. Consequently, it is recommended that users perform their own efficiency measurements. It is important that the measurements of the efficiency and of the actual cross section are carried out in the same geometry (that is, the same distance and orientation of the detector with respect to the source or target). The same argument applies to any γ -ray absorbing material between source or target and detector (for example, the target holder). Such effects are expected to be most important at relatively low γ -ray energies (<1 MeV; Section 4.2.2).

Typically, detector efficiencies are needed for an energy range between \approx 100 keV and \approx 15 MeV. Since no single process covers this entire energy region, γ -rays from several different processes have to be used. The peak efficiency is related to the full-energy peak intensity by Eq. (4.46). It is obvious from that expression that a reliable determination of detector efficiencies requires accurate knowledge of the γ -ray branching ratios, B, which are defined as the fraction of a specific γ -ray transition per nuclear decay. Several radioactive sources for which very precise branching ratios are known (with relative errors of < 1% for most transitions) are listed in Table 4.6. Especially useful are the radioisotopes ¹⁵²Eu and ⁵⁶Co which cover the energy ranges of 0.1-1.4 MeV and 0.8-3.5 MeV, respectively. For higher energies, efficiencies have to be obtained by using γ -rays from nuclear reactions. If charged-particle beams are available, frequently used calibration standards are provided by the 655 and 992 keV resonances in ${}^{27}\text{Al}(p,\gamma){}^{28}\text{Si}$, covering the energy range of 1– 11 MeV. The errors in the reported branching ratios are < 3% (Endt 1990). At lower bombarding energies, the 293 and 327 keV resonances in the same re-

action may be used, although the reported branching ratio errors are larger (<10%; Iliadis et al. 1990). The relevant data are summarized in Table 4.7. The 278 keV resonance in the ¹⁴N(p, γ)¹⁵O reaction is also useful in this respect. This particular resonance emits photons in the range of 0.8–7 MeV and has a simple γ -ray decay scheme, with all decays proceeding via cascades that consist of two γ -rays only. Since the number of photons for each transition in a given cascade is the same, the measured intensity ratio of the two γ -rays is equal to the ratio of the corresponding efficiencies. The branching ratio data and the relevant decay scheme are shown in Table 4.8 and Fig. 4.30, respectively. Furthermore, the γ -ray emission from this resonance is isotropic (J=1/2) and hence angular correlation effects are negligible (Appendix D). If thermal neutrons are available, precise γ -ray efficiencies can be obtained from the ¹⁴N(n, γ)¹⁵N capture reaction. The prompt γ -ray emission probabilities per neutron-capture event from this reaction are listed in Table 4.9.

The following strategy is frequently employed in order to determine a composite γ -ray efficiency curve. The data from radioactive calibration sources are analyzed and plotted first. Their well-known activity also provides a normalization of the absolute efficiency scale. In a subsequent step, the reaction data are analyzed. The resulting efficiencies from each reaction measurement are vertically adjusted until they agree with the radioactive source values in the overlap region. An efficiency curve obtained in this manner is shown in Fig. 4.31a. In this case, a HPGe detector of 582 cm³ volume is positioned at a distance of 1.6 cm between source or target and detector. It can be seen that the peak efficiency drops drastically with increasing energy, reflecting the fact that both photoelectric absorption and Compton scattering become less likely at higher energies (Section 4.2.2).

Once the efficiency of a detector has been measured at several energies, a fit to the data can be performed in order to determine efficiency values between measured points by interpolation. Frequently used analytical fitting functions are polynomials, but more complicated functions are also in use (Debertin and Helmer 1988).

The full-energy peak efficiency cannot be calculated analytically since it depends on the γ -ray energy in a complicated way. We already mentioned that Monte Carlo techniques may be used to calculate individual photon histories. Although in the end peak efficiencies should be measured directly for the reasons given above, the Monte Carlo calculations are useful for estimating *relative* peak efficiencies. Uncertainties in crystal dimensions or interaction parameters (that is, cross sections for photoelectric absorption, Compton scattering and pair production) have a relatively small effect on these relative values. Therefore, Monte Carlo calculations can provide the shape of the peak efficiency curve and aid in the interpolation between calibration points. An interesting application of the Monte Carlo method in order to obtain very precise germanium detector efficiencies can be found in Helmer et al. (2003).

Tab. 4.7 Gamma-ray branching ratios (fraction of a specific transition per proton capture
event) of low-energy resonances in ${}^{27}AI(p,\gamma){}^{28}Si$. Branching ratios are given in percent.
(a) Resonance energy in keV. (b) Excitation energy of final state in keV. (c) Excitation energy
of initial state in keV. (d) From Iliadis et al. (1990). (e) From Endt et al. (1990). Errors are given
in parentheses and refer to the last significant digit(s).

	E_r^{a} : 293 ^d	327 ^d	655°	992°
E_{xf}^{b} :	<i>E_{xi}°</i> : 11867	11900	12216	12541
0				
1779			42.1(10)	76.4(4)
4618	60.4(14)	72.2(8)		4.09(12)
6276			4.5(2)	2.15(7)
6879			1.63(9)	0.70(2)
6889	12.4(8)	12.1(5)		0.294(9)
7381				0.187(6)
7416			1.82(10)	0.297(9)
7799				8.5(3)
7933			6.4(2)	3.96(12)
8259			1.60(9)	
8328			1.27(7)	
8413	5.9(4)			
8589		5.3(5)	3.36(13)	0.173(6)
9165	5.1(3)			0.147(5)
9316			2.09(9)	0.047(2)
9382			29.1(9)	
9417		2.8(2)		0.79(3)
9479				1.11(4)
9765	3.2(3)			0.195(7)
10182				0.085(3)
10209				0.146(5)
10311				0.061(3)
10376			0.52(3)	
10540	2.3(2)			
10596			1.39(7)	
10668				0.288(9)
10900			0.63(4)	
11195				0.089(3)
11265				0.082(3)

Tab. 4.8 Branching ratios (in percent) of ¹⁵O levels (from Ajzenberg-Selove 1991 and Runkle et al. 2005); E_{xi} and E_{xf} denote the excitation energies (in keV) of the initial and final states, respectively, involved in the transition. The level at $E_{xi} = 7556$ keV corresponds to a resonance at $E_r = 278$ keV in ¹⁴N(p, γ)¹⁵O. See also Fig. 4.30a. Errors are given in parentheses and refer to the last significant digit(s).

E_{xf} :	E_{xi} :	5183	5241	6176	6793	6859	7276	7556
0 5183 5241		100	100	100 <2.5 <2.5	100 <3 <3	<10 <4 100	3.8(12) <4 96.2(12)	1.70(7) 17.3(2)
6176 6793 6859					<7	<0.4	<2	58.3(5) 22.7(3) <5.8



Fig. 4.30 Energy level diagrams of (a) 15 O and (b) 60 Ni. The γ -ray branching ratios of $^{15}\mathrm{O}$ levels are presented in Table 4.8. The $\beta\text{-decay}$ and γ -ray branching ratios for the ⁶⁰Co \rightarrow ⁶⁰Ni decay are adopted from Firestone and Shirley (1996).

Tab. 4.9 Gamma-ray emission probabilities per neutron-capture event from the ${}^{14}N(n,\gamma){}^{15}N$ reaction induced by thermal neutrons (from Raman et al. 2000). Errors are given in parentheses and refer to the last significant digit(s).

E_{γ} (keV)	B (%)	E_{γ} (keV)	B (%)
1678	7.96(9)	5269	29.94(20)
1885	18.72(20)	5298	21.27(18)
2000	4.05(5)	5533	19.66(21)
2520	5.68(7)	5562	10.66(12)
2831	1.72(3)	6322	18.45(14)
3532	9.09(9)	7299	9.56(9)
3678	14.70(15)	8310	4.17(5)
4509	16.63(17)	10829	14.0(3)

So far, we have only discussed the determination of full-energy peak efficiencies. There are situations where the precise knowledge of total efficiencies becomes also important. For example, total efficiencies are typically needed for estimating coincidence summing corrections for germanium detectors (see below) or coincidence efficiencies for $\gamma\gamma$ -detection techniques (Section 4.7.3). In contrast to peak efficiencies, the calculation of total efficiencies is in principle straightforward. The probability that an incident photon traversing a path length x in the active volume is not detected, that is, does not undergo any interaction in the crystal, is given by $\overline{P} = \mathcal{N}/\mathcal{N}_0 = e^{-\mu x}$ (see Eq. (4.30)), where \mathcal{N} and \mathcal{N}_0 denote the total number of transmitted and incident quanta, respectively. Equivalently, we may calculate the probability that this incident photon will undergo any interaction and deposit any energy in the crystal



Fig. 4.31 Efficiencies for a HPGe detector of 582 cm³ volume, positioned at a distance of 1.6 cm between source or target and detector. (a) Experimental peak efficiencies; the curve is constructed by using calibration sources (⁵⁶Co and ¹⁵²Eu) and calibration resonances [¹⁴N(p, γ)¹⁵O and ²⁷Al(p, γ)²⁸Si]. The displayed efficiencies are corrected for coincidence summing. Data courtesy

of Robert Runkle. (b) Calculated total efficiencies; the solid line is obtained from an expression similar to Eq. (4.52), but additionally taking into account the insensitive detector core. The results agree with those from Monte Carlo simulations (shown as diamonds) using the computer code MCNP (Briesmeister 1993). Data courtesy of Chris Fox and Richard Longland.

from $P = 1 - \overline{P} = 1 - e^{-\mu x}$. In general, the path length *x* will depend on the angle of photon emission with respect to the crystal. Therefore, the total efficiency can be found by integrating over the solid angle Ω subtended by the detector,

$$\eta_{\text{tot}} = \frac{1}{4\pi} \int \left(1 - e^{-\mu x}\right) \, d\Omega \tag{4.51}$$

For the case of a cylindrical detector of radius R and length t, and a point source located on the detector axis at a distance d (Fig. 4.32) we obtain (Debertin and Helmer 1988)

$$\eta_{\text{tot}} = \frac{1}{2} \int_{0}^{\theta_{1}} \left[1 - e^{-(\mu t/\cos\theta)} \right] \sin\theta \, d\theta + \frac{1}{2} \int_{\theta_{1}}^{\theta_{2}} \left[1 - e^{-(\mu R/\sin\theta) + (\mu d/\cos\theta)} \right] \sin\theta \, d\theta$$
(4.52)

with $\theta_1 = \arctan[R/(d+t)]$ and $\theta_2 = \arctan(R/d)$. An unscattered photon emitted at an angle of θ_1 (as measured from the detector axis) passes through the detector backside, while an unscattered photon emitted at angles between θ_1 to θ_2 passes through the detector sides. The above integrals can be solved numerically.

Frequently, the detector crystal geometry is not a simple cylinder. For example, coaxial germanium detectors have an insensitive cylindrical core which reduces the calculated total efficiency. Or, sources may be placed inside annular NaI(Tl) detectors (Section 4.7.3) in order to maximize counting rates. In these cases, total efficiencies can be calculated by using more complicated analytical expressions (Longland et al. 2006). Interactions in any absorbing material located between source or target and detector can additionally be taken into account by calculating the γ -ray attenuation. The total efficiency of a HPGe detector, estimated in this manner, is displayed in Fig. 4.31b. The detector crystal has a length and diameter of 93 mm and 90 mm, respectively, and the insensitive core a length and diameter of 79 mm and 9 mm, respectively. The detector-source distance is 16 mm. The solid line is obtained from an analytical expression, while the diamonds represent the results of a Monte Carlo simulation. It can be seen that η_{tot} varies smoothly beyond a γ -ray energy of \approx 3 MeV. The results from the analytical expression agree with those from the Monte Carlo calculation within 3% (Longland et al. 2006).

The above considerations apply only to an ideal measuring geometry with no photon scattering in the surroundings of the detector. Clearly, photons that are originally emitted in a direction outside the solid angle of the detector could be scattered in the surrounding material and may thus reach the active detector volume. These photons will then contribute to the total efficiency (but not to the full-energy peak efficiency; see also Fig. 4.14b). Although calculated values of η_{tot} are useful for estimates of *relative* total efficiencies, it is preferable to measure absolute total efficiencies in a geometry identical to that used for the actual cross section measurement. Experimental η_{tot} values can be obtained by using single-line γ -ray emitters, such as ¹³⁷Cs or ⁵⁴Mn. In the data analysis, the background intensity (without source) needs to be subtracted appropriately and, in addition, the spectrum has to be extrapolated beyond the discriminator threshold to zero pulse height (Fig. 4.17b). Alternatively, twoline γ -ray emitters, such as ⁶⁰Co, may be used to measure total efficiencies (see below). Multiple-line γ -ray emitters are not as useful for this purpose because of coincidence summing effects, which will be described below.

Coincidence summing

In many cases of practical interest, nuclear levels de-excite to the ground state via the sequential emission of two or more photons, rather than by emitting only a single γ -ray. Suppose that two coincident photons, belonging to the



Fig. 4.32 Geometry for the calculation of the total γ -ray efficiency of a cylindrical detector according to Eq. (4.52); *d* is the distance between source and detector front face, *t* the length of the detector crystal, and *R* the crystal radius.

same γ -ray cascade, interact simultaneously with the detector. The resulting summed pulse will appear in the spectrum in a region which is different from the full-energy peak of either photon. Furthermore, the coincident photons that give rise to the summed signal are missing from the full-energy peaks of the individual photons. The effect is referred to as *coincidence summing* and has to be properly accounted for in order to avoid errors when efficiencies and cross sections are measured. It is particularly severe for a nuclear level with a complicated decay scheme, that is, if the level can decay through a large number of lower lying states.

It is important to point out that coincidence summing is not related to the phenomenon of pulse pileup. The latter effect is also referred to as *random summing* and occurs when photons belonging to different cascades sum their energies randomly because of relatively high pulse rates (Knoll 1989). Coincidence summing, on the other hand, is independent of the pulse rate, but depends on the distance between detector and source. In principle, coincidence summing effects can always be reduced to insignificant levels by increasing the detector–source distance. This procedure, however, may decrease the counting efficiency to intolerable levels, especially in measurements of very weak cross sections. The experimentalist has frequently little choice but to maximize the counting efficiency by minimizing the detector–source distance and, at the same time, to properly account for coincidence summing effects.

As a simple example, consider Fig. 4.33 showing a decay scheme involving three levels in nucleus Y. Level 2 is populated, say, by a capture reaction. It may either decay directly to the ground state $(2 \rightarrow 0)$ via emission of photon γ_{20} with a branching ratio of B_{20} , or to level 1 ($2 \rightarrow 1$) via emission of photon γ_{21} with a branching ratio of B_{21} . Subsequently, level 1 decays to the ground state ($1 \rightarrow 0$) via emission of photon γ_{10} with a branching ratio of $B_{20} + B_{21} = 1$. Angular correlations between γ -rays

will be neglected in the following. With the total number of decaying levels 2 given by N, the number of detected photons γ_{21} in the full-energy peak is equal to

$$\mathcal{N}_{21} = \mathcal{N}B_{21}\eta_{21}^P - \mathcal{N}B_{21}\eta_{21}^P\eta_{10}^T = \mathcal{N}B_{21}\eta_{21}^P(1-\eta_{10}^T)$$
(4.53)

with η_{21}^{P} and η_{10}^{T} the (full-energy) peak and total efficiency of photon γ_{21} and γ_{10} , respectively. The intensity of the full energy peak is *reduced* by the amount $\mathcal{N}B_{21}\eta_{21}^{P}\eta_{10}^{T}$, which corresponds to the probability that photon γ_{21} is fully detected and, at the same time, the coincident photon γ_{10} leaves any measurable amount of energy in the detector (for example, through Compton scattering). Equivalently, the term $\mathcal{N}B_{21}\eta_{21}^{P}(1-\eta_{10}^{T})$ corresponds to the probability that photon γ_{21} is fully detected and, at the same time, photon γ_{10} escapes detection. If both photons γ_{21} and γ_{10} are detected simultaneously, then counts are removed from the full energy peak of photon γ_{21} . This effect, which depends on both the peak and total detector efficiency, is referred to as *summing-out*. Similarly, we obtain for the number of detected photons γ_{10} in the full-energy peak

$$\mathcal{N}_{10} = \mathcal{N}B_{21}\eta_{10}^P - \mathcal{N}B_{21}\eta_{10}^P\eta_{21}^T = \mathcal{N}B_{21}\eta_{10}^P(1-\eta_{21}^T)$$
(4.54)

On the other hand, the number of detected photons γ_{20} in the full energy peak is

$$\mathcal{N}_{20} = \mathcal{N}B_{20}\eta_{20}^P + \mathcal{N}B_{21}\eta_{21}^P\eta_{10}^P \tag{4.55}$$

The intensity of the full energy peak is *increased* by the amount $\mathcal{N}B_{21}\eta_{21}^{P}\eta_{10}^{P}$, which corresponds to the probability that both photons γ_{21} and γ_{10} are fully absorbed in the detector. This effect, which depends only on the peak efficiency of the detector, is referred to as *summing-in*.

Coincidence summing effects become significant for large efficiency values or, equivalently, for close detector–source geometries. For example, if the ground-state transition via emission of photon γ_{20} , is weak ($B_{20} \approx 0$), the measured intensity \mathcal{N}_{20} may arise entirely from summing-in. Consequently, neglecting coincidence summing corrections may cause large systematic errors in the interpretation of the γ -ray decay scheme. If the decay shown in Fig. 4.33 is used for determining the peak efficiencies of photons γ_{21} and γ_{10} , then we obtain from Eqs. (4.53) and (4.54)

$$\eta_{21}^{P} = \frac{\mathcal{N}_{21}}{\mathcal{N}B_{21}(1-\eta_{10}^{T})}, \qquad \eta_{10}^{P} = \frac{\mathcal{N}_{10}}{\mathcal{N}B_{21}(1-\eta_{21}^{T})}$$
(4.56)

A comparison of Eqs. (4.43) and (4.56) shows that, in the presence of coincidence summing, the efficiency expression has to be modified by the total detection efficiency factor $(1 - \eta_{ij}^T)$. Similar arguments hold if this decay is



Fig. 4.33 Coincidence summing of γ -rays. (a) Scheme of three levels (0, 1, 2). Level 2 is populated either in a capture reaction or by β -decay. It can γ -decay either to level 1 or to the ground state 0. The intermediate level 1 can only decay to the ground state. (b) Pulse height spectrum. The peaks labeled " γ_{21} " and " γ_{10} " are affected by summing-out, while the peak labeled " γ_{20} , $\gamma_{21} + \gamma_{10}$ " is influenced by summing-in.

used for determining the number \mathcal{N} of decaying levels 2 (proportional to the source activity or the cross section) from the measured peak intensities of γ_{21} or γ_{10} . A proper account for summing corrections gives (see Eqs. (4.53) and (4.54))

$$\mathcal{N} = \frac{\mathcal{N}_{21}}{B_{21}\eta_{21}^P(1-\eta_{10}^T)} = \frac{\mathcal{N}_{10}}{B_{21}\eta_{10}^P(1-\eta_{21}^T)}$$
(4.57)

In more complicated cases involving different multiple- γ -ray cascades, β -decays to intermediate levels, internal conversion transitions, angular correlations and so on, summing corrections can no longer be calculated analytically. Positron decays to excited levels in the daughter nucleus give rise to annihilation quanta that are coincident with γ -rays from the de-excitation of those levels and thus have to be considered carefully as well (even for single-line γ -ray sources). General numerical methods have been developed for such cases (Debertin and Helmer 1988, Semkow et al. 1990).

Coincidence summing effects are frequently apparent from a visual inspection of the (uncorrected) full-energy peak efficiency curve if data are taken in close geometry. Some efficiency values will lie on a smooth curve, corresponding to noncoincident photons, while other data points that are affected by coincidence summing will lie away from the curve. Based on this information, the experimentalist can decide if corrections need to be applied in order to achieve the desired precision. Note that coincidence summing not only influences peak intensities in the pulse height spectrum, but also total intensities since two (or more) photons are registered by the detector as one pulse.

Sum peak method

The importance of absolute normalization of the peak and total efficiency curves has been stressed in previous sections. For some commercially available γ -ray sources, absolute activities can be quoted to about 1% precision. In many cases, however, a set of absolutely calibrated sources may not be available to the experimentalist. Here we will describe a method that not only provides simultaneously absolute peak and total efficiencies without knowledge of the source activity, but the derived results will also be automatically corrected for coincidence summing effects. The technique is referred to as the *sum peak method* and utilizes the coincidence summing of photons that belong to a two- γ -ray cascade.

Consider again the decay scheme in Fig. 4.33. Level 2 is populated by some process (for example, β -decay of a parent nucleus), but assume now that it decays exclusively to the intermediate level 1 ($B_{21} = 1$, $B_{20} = 0$) and, subsequently, to the ground state 0 ($B_{10} = 1$). The measured intensities of the full-energy peaks (N_{21} , N_{10}), the sum peak (N_{20}), and the total intensity in the spectrum (N_t) are given by

$$\mathcal{N}_{21} = \mathcal{N}\eta_{21}^P \left(1 - W\eta_{10}^T \right) \tag{4.58}$$

$$\mathcal{N}_{10} = \mathcal{N}\eta_{10}^P \left(1 - W\eta_{21}^T\right) \tag{4.59}$$

$$\mathcal{N}_{20} = \mathcal{N}\eta_{21}^{P}\eta_{10}^{P}W \tag{4.60}$$

$$\mathcal{N}_{t} = \mathcal{N} \left(\eta_{21}^{T} + \eta_{10}^{T} - \eta_{21}^{T} \eta_{10}^{T} W \right)$$
(4.61)

These relationships take explicitly the angular correlation W of photons γ_{21} and γ_{10} into account, but otherwise the first three expressions are identical to Eqs. (4.53)–(4.55). The term $\eta_{21}^T \eta_{10}^T W$ in the last expression corresponds to the probability that each of the coincident photons deposits some amount of energy in the detector. In this case, the two photons are registered in the detector as one pulse and, consequently, the total intensity in the spectrum is reduced. As already pointed out, it is assumed that the intensity \mathcal{N}_t has been corrected for the background (without source) and is extrapolated to zero pulse height.

The above equations can be solved iteratively until convergence in the solutions is achieved. However, in certain important cases (for example, ⁶⁰Co; see below) the energies of the two emitted photons are very similar. Hence we can replace in the above expressions the total efficiencies η_{21}^T and η_{10}^T by their average value $\eta^T \approx (\eta_{21}^T + \eta_{10}^T)/2$. With this approximation one obtains after

some algebra

$$\mathcal{N} = \left(\frac{\mathcal{N}_{21}\mathcal{N}_{10}}{\mathcal{N}_{20}} + \mathcal{N}_t\right) W \tag{4.62}$$

$$\eta_{21}^{P} = \frac{1}{W} \sqrt{\frac{\mathcal{N}_{21}\mathcal{N}_{20}^{2}}{\mathcal{N}_{10}\mathcal{N}_{20}\mathcal{N}_{t} + \mathcal{N}_{21}\mathcal{N}_{10}^{2}}}$$
(4.63)

$$\eta_{10}^{P} = \frac{1}{W} \sqrt{\frac{\mathcal{N}_{10}\mathcal{N}_{20}^{2}}{\mathcal{N}_{21}\mathcal{N}_{20}\mathcal{N}_{t} + \mathcal{N}_{10}\mathcal{N}_{21}^{2}}}$$
(4.64)

$$\eta^{T} = \frac{1}{W} - \frac{1}{W} \sqrt{\frac{\mathcal{N}_{21}\mathcal{N}_{10}}{\mathcal{N}_{20}\mathcal{N}_{t} + \mathcal{N}_{21}\mathcal{N}_{10}}}$$
(4.65)

These expressions for the total number of decaying nuclei (\mathcal{N}) and the absolute peak and total efficiencies (η_{21}^{P} , η_{10}^{P} , η^{T}) depend, apart from the factor W, only on the measured intensities \mathcal{N}_{21} , \mathcal{N}_{10} , \mathcal{N}_{20} , and \mathcal{N}_{t} .

As a specific example, consider the decay scheme of the radioisotope ⁶⁰Co (Fig. 4.30b). The β -decay populates the 2506 keV level in the daughter nucleus ⁶⁰Ni. This level, in turn, γ -decays to the first excited state at 1333 keV by emission of a 2506 keV – 1333 keV = 1173 keV photon. Subsequently, this state de-excites to the ground state by emission of a 1333 keV photon. Other β - and γ -decays are very weak and, therefore, this decay represents an almost ideal realization of the schematic case discussed above. The angular correlation for the two coincident photons from the decay of ⁶⁰Co is given by (Example D.1)

$$W(\theta) = 1 + \frac{5}{49} Q_2^{21} Q_2^{10} P_2(\cos\theta) + \frac{4}{441} Q_4^{21} Q_4^{10} P_4(\cos\theta)$$
(4.66)

where $P_n(\cos \theta)$ denotes a Legendre polynomial of order n and Q_n^{ab} is the solid angle attenuation factor for photon γ_{ab} ; θ is the angle between the directions of the two photons. In this case, $\theta = 0^{\circ}$ and hence $P_2(\cos \theta) = P_4(\cos \theta) = 1$ (see Eqs. (A.12) and (A.14)). The factors Q_n^{ab} can be estimated from the efficiency and the detector crystal geometry by using, for example, Monte Carlo simulations (Appendix D.5). Strictly speaking, the factors Q_n^{ab} also depend on the type of event ("full-energy peak efficiency" solid angle attenuation factor versus "total efficiency" solid angle attenuation factor). For example, Eq. (4.58) must in principle be replaced by

$$\mathcal{N}_{21} = \mathcal{N}\eta_{21}^{P} \left[1 - \left(1 + \frac{5}{49} Q_2^{21,P} Q_2^{10,T} + \frac{4}{441} Q_4^{21,P} Q_4^{10,T} \right) \eta_{10}^{T} \right]$$
(4.67)

In practice, the distinction between the factors $Q_n^{ab,P}$ and $Q_n^{ab,T}$ is found to have a negligible effect on the final results if the distance between detector and source is small (< 1% change in the derived efficiencies for distances < 1 cm).

However, at larger distances this distinction may need to be taken into account (Kim, Park and Choi 2003; Longland et al. 2006).

Gamma-ray branching ratios

The γ -ray decay probability for a transition from a given initial state to a specific lower lying final state, normalized to the total γ -ray decay probability of the initial level, is called the γ -ray branching ratio. It is defined by the ratio of the γ -ray partial width for a specific transition and the total γ -ray partial width of the initial state (see Eq. (1.29)). Branching ratios contain important information regarding the nuclear structure of the initial and final nuclear states. They are also needed in order to calculate from the measured intensities of specific transitions the total number of populated compound levels. This number is equal to the total number of eactions that occurred and thus determines the reaction cross section (Sections 4.8 and 4.9).

Consider the schematic level diagram shown in Fig. 4.34. An initial level *i* in nucleus Y is populated by some fusion reaction X + a. The initial state can directly decay either to the ground state (0) or to three lower lying excited states (1, 2, 3). These transitions (thick solid arrows) are referred to as *primary* γ -ray decay branches. The corresponding primary γ -ray branching ratios are experimentally given by

$$B_{ij} \equiv \frac{\mathcal{N}_{ij} / (\eta_{ij}^P W_{ij})}{\sum\limits_{j} \mathcal{N}_{ij} / (\eta_{ij}^P W_{ij})}$$
(4.68)

with \mathcal{N}_{ij} , η_{ij}^{P} , and W_{ij} the measured full-energy peak intensity, peak efficiency, and angular correlation, respectively, for the transition leading from the initial level *i* to a specific final state *j*. It is assumed that \mathcal{N}_{ij} and η_{ij}^{P} have been corrected for coincidence summing effects. There are further decay possibilities involving levels other than the initial state *i*. These transitions (indicated by thin solid and dashed arrows) are referred to as *secondary* γ -ray decay branches.

The total number of compound nuclei created in the fusion reaction, that is, the total number of initially populated levels *i*, can be obtained either from the primary branching ratios (thick solid arrows),

$$\mathcal{N}_{i} = \sum_{j=0,1,2,3} \frac{\mathcal{N}_{ij}}{\eta_{ij}^{P} W_{ij}} = \frac{\mathcal{N}_{ij}}{B_{ij} \eta_{ij}^{P} W_{ij}}$$
(4.69)

or from all transitions leading to the ground state (thin solid arrows plus the primary ground-state branch),

$$\mathcal{N}_{i} = \sum_{j=1,2,3,i} \frac{\mathcal{N}_{j0}}{\eta_{j0}^{P} W_{j0}}$$
(4.70)

The proper interpretation of a γ -ray spectrum can be challenging if the decay scheme is complex. Sometimes it is found that peaks originating from the reaction of interest overlap with escape peaks, room background lines, or peaks from reactions involving target or beam contaminants. For the analysis of reaction data it is frequently of advantage to compare spectra obtained with beam on target with those measured without beam (room background) or with beam on a blank backing.

It is sometimes possible to determine ratios of partial widths from the observed intensity balance of primary and secondary γ -ray transitions. As an example, consider Fig. 4.35 showing a level scheme of ²⁵Al and a germanium γ -ray spectrum measured in the ²⁴Mg(p, γ)²⁵Al reaction. The reaction populates a resonance located at $E_r = 1616$ keV, corresponding to a level at $E_x = 3823$ keV in the compound nucleus. This level decays via several primary γ -ray transitions. One of these primary transitions proceeds to the proton unbound $E_x = 2485$ keV level. This state, in turn, has three possibilities of decay: (i) a γ -ray transition to the $E_x = 452$ keV state; (ii) a γ -ray transition to the $E_x = 452$ keV state; decays the proton unbound $E_x = 2485$ keV level. This state to the 2^{4} Mg ground state via emission of a proton. The partial width ratio Γ_{γ}/Γ of the $E_x = 2485$ keV state is then given by the ratio of the total number of γ -ray transitions decaying from this level (2485 \rightarrow 452, 2485 \rightarrow 945) and the number of γ -ray transitions feeding this level (3823 \rightarrow 2485),

$$\frac{\Gamma_{\gamma}}{\Gamma} = \frac{\Gamma_{\gamma}}{\Gamma_{\rm p} + \Gamma_{\gamma}} = \frac{(\mathcal{N}_{2485 \to 452} / \eta_{2485 \to 452}^P) + (\mathcal{N}_{2485 \to 945} / \eta_{2485 \to 945}^P)}{(\mathcal{N}_{3823 \to 2485} / \eta_{3823 \to 2485}^P)}$$

= 0.91 ± 0.04 (4.71)

Angular correlation effects are negligible in this case. The measured value provides important input information for the extrapolation of the ${}^{24}Mg(p,\gamma){}^{25}Al$ cross section to low energies (Powell et al. 1999).

4π detection of γ -rays

It is obvious that the interpretation of a complex γ -ray spectrum in order to determine branching ratios of individual γ -ray transitions, and eventually cross sections, can be a very time-consuming task. Such investigations are especially tedious if a large number of resonances has to be measured in a specific reaction. Furthermore, if the *Q*-value of the capture reaction is large and if the target nucleus is heavy, then incident charged-particles or neutrons may excite a number of overlapping resonances simultaneously, giving rise to a multitude of capture γ -rays lines in the pulse height spectrum. It must be emphasized that the experimental information on individual γ -ray transitions is not necessarily required in order to determine the number of nuclear reactions





Fig. 4.34 Level scheme showing primary (thick solid arrows) and secondary (thin solid and dashed arrows) γ -ray transitions. The thick solid arrows originate from level *i* which is directly populated in the capture reaction X + a \rightarrow Y. The thin solid arrows correspond to those secondary transitions that proceed to the ground state of nucleus Y.



Fig. 4.35 Study of the ²⁴Mg(p, γ)²⁵Al reaction at a proton bombarding energy of E_p = 1620 keV. (a) Level scheme of ²⁵Al. The capture reaction populates directly a level at E_x = 3823 keV which γ -ray decays to a state at E_x = 2485 keV. The latter state either decays via proton emission or via γ -ray transitions to lower lying levels (E_x = 452 or

945 keV). (b) Measured pulse height spectrum. The γ -ray transitions populating or decaying from the E_x = 2485 keV state are indicated by solid circles. See discussion in the text. Reprinted from D. C. Powell et al., Nucl. Phys. A, Vol. 660, p. 349 (1999). Copyright (1999), with permission from Elsevier.

that took place during the experiment. All that is needed from the astrophysical point of view is the total number of γ -ray cascades initiated by the reaction of interest.

Consider as an example the setup shown in Fig. 4.36. The target or sample is located at the center of a large detector crystal that covers a solid angle of nearly 4π . If every single γ -ray of a specific cascade emitted by the target is fully absorbed by the detector, then each radiative capture would result in a single pulse. The system would have a detection efficiency of unity for each radiative capture, independent of the cascade structure. The pulse height spectrum will show a peak at an energy equal to the sum of the *Q*-value and the center-of-mass bombarding energy, $E_{\gamma}^{\text{sum}} = Q + E_{\text{cm}}$. Not only does such a technique greatly simplify the data analysis, but it has the additional advantage that angular correlation effects are negligible. Furthermore, contaminant reactions will give rise to sum peaks at different locations in the spectrum since their *Q*-values are likely to differ from the *Q*-value of the reaction of interest. This technique, in fact, takes advantage of coincidence summing.

The 4π detection method has been successfully applied in a number of investigations, including charged-particle and neutron-capture reactions (Lyons, Toevs and Sargood 1969; Wisshak et al. 1990; Harissopulos 2004). Complications arise since any crystal of finite dimensions has a total detection efficiency of less than unity for any given photon. Some photons may escape through openings of the detector (for example, the beam pipe). Others may be absorbed by the target chamber or may simply traverse the detector without interaction. Such effects cause incomplete summing and give rise to a continuum of pulses below the sum peak. Also, the efficiency for detecting a cascade (summing efficiency) is no longer constant and will depend on the γ -ray decay scheme. In practice it is found, with sufficiently large detector crystals, that the summing efficiency depends only slightly on the cascade structure and that this dependence can be modeled using Monte Carlo codes (Tsagari et al. 2004). Figure 4.37 shows three pulse height spectra, measured with a summing crystal near the $E_r = 992$ keV resonance in ${}^{27}\text{Al}(p,\gamma){}^{28}\text{Si}$. The on- and off-resonance spectrum is shown in parts (a) and (b), respectively. The difference spectrum, shown in part (c), clearly displays the sum peak resulting from coincidence summing of complete γ -ray cascades emitted in the 27 Al(p, γ)²⁸Si reaction (Q = 11.6 MeV). The continuum and all the other peaks visible below the sum peak in part (c) are caused by incomplete coincidence summing.

4.5.3

Neutron Spectroscopy

Neutrons must be observed through nuclear interactions (reactions or scattering) in the detection medium that result in energetic charged-particles (Section 4.2.3). The cross section for these processes depends in most cases strongly on neutron energy. Consequently, very different devices are in use



Fig. 4.36 4π detection of cascading γ -rays. (a) Level scheme showing a cascade consisting of three photons (γ_1 , γ_2 , and γ_3). (b) Summing crystal covering a solid angle close to 4π . The target is located at the center of the detector. Photons emitted in the direction of the beam pipe escape detection. (c) Schematic pulse height spectrum showing a sum peak corresponding to an energy of $E(\gamma_1) + E(\gamma_2) + E(\gamma_3)$.

for detecting neutrons in different energy regions. A detailed discussion of the various types of neutron detectors can be found in Knoll (1989). It has already been pointed out that astrophysically important reactions frequently have very small cross sections and thus detectors with high efficiencies are required for these measurements. Most neutron measurements in nuclear astrophysics have been performed by using moderated proportional counters. In the following, we will focus on this detector type. Scintillators also have high efficiencies for neutron detection (Section 4.4.3) but are also sensitive to beam-induced and room-background γ -rays. Although it is possible with scintillators to suppress unwanted γ -ray signals via pulse shape discrimination techniques, the remaining γ -ray background is not negligible. For this reason, moderated proportional counters achieve in general higher sensitivities compared to scintillators in measurements of weak cross sections.

Response function

We start with a discussion of the response function of proportional counters. As a specific example, a detector filled with ³He gas is chosen. Similar arguments will hold for ¹⁰BF₃ proportional counters. Suppose that thermal neutrons ($E_n = 0.025$ eV) are incident on such a detector, as shown in Fig. 4.38a.



Fig. 4.37 Pulse height spectra, measured with a summing crystal near the E_r = 992 keV resonance in ²⁷Al(p, γ)²⁸Si. (a) On-resonance spectrum. (b) Off-resonance spectrum. (c) Difference spectrum displaying the sum peak that results from coincidence summing of complete γ -ray cascades emitted in the ²⁷Al(p, γ)²⁸Si reaction (Q =

11.6 MeV). The continuum and all other peaks visible below the sum peak in part (c) are caused by incomplete coincidence summing. Reprinted from S. Harissopulos et al., Eur. Phys. J. A, Vol. 9, p. 479 (2000). Copyright (2000), with kind permission of Springer Science and Business Media.

The *Q*-value of the ³He(n,p)³H reaction amounts to 764 keV. Since the incoming neutron momentum is very small, the reaction products (protons and tritons) are emitted in opposite directions and the total reaction energy is imparted to the fragments as kinetic energy according to the ratio of their masses $(E_p = 573 \text{ keV}, E_t = 191 \text{ keV})$. If both particles are stopped in the counter gas (case a), then the magnitude of the current output pulse corresponds to 764 keV. These events appear in the full-energy peak (FEP) of the pulse height spectrum (Fig. 4.38b). However, if one of the particles strikes the counter wall, then a smaller pulse is produced. This phenomenon is referred to as the *wall effect*. For example, case (b) shows a reaction occurring close to the counter wall. The proton is completely stopped in the gas, while the entire energy of the triton is absorbed by the wall. The corresponding event then appears in the pulse height spectrum at an energy of 573 keV. If the reaction occurs at some distance from the wall (case c), so that the triton can deposit at least a



Fig. 4.38 Measurement of thermal neutrons ($E_n = 0.025 \text{ eV}$) with a ³He gas proportional counter. (a) Histories of neutron interactions in the detector. (b) Pulse height spectrum. The full-energy peak occurs at the *Q*-value of the ³He(n,p)³H reaction (Q = 764 keV). The steps at 191 keV and 573 keV are caused by the wall effect (see the text). The discriminator threshold is indicated by the vertical dashed line.

fraction of its energy in the gas, then a larger pulse is produced. Similar arguments hold for the opposite case (case d), that is, when the energy of the triton is fully absorbed in the gas while only partial proton energy deposition occurs. The wall effect gives rise to steps at 191 keV and 573 keV in the pulse height spectrum, corresponding to the individual energies of the proton and triton fragments. It also depends on the geometry and size of the counter and is less pronounced for larger detectors and for higher gas pressures. The wall effect can be reduced by adding a small amount of a heavier gas (for example, Kr) to the ³He since then the ranges of the charged particles become smaller.

The resolution of the full-energy peak measured with ³He or ¹⁰BF₃ proportional counters amounts typically to several percent for thermal neutrons. It should be clear from the above considerations that for incident thermal neutrons, the pulse height spectrum measured with proportional counters provides no information regarding the neutron energy.

An important property of proportional counters is their ability to discriminate neutrons from room or beam-induced background γ -rays. Photons interact mainly with the counter walls by creating secondary electrons. These electrons have relatively large ranges in gases and thus will deposit only a small fraction of their energy in the active volume before reaching the counter wall. As a result, most γ -rays will produce pulses of much smaller amplitude compared to those induced by neutrons. In practice, a discriminator threshold level (dashed line in Fig. 4.38b) is set just below the structure caused by the wall effect. By accepting only events located above the threshold, all the neutrons are counted while low amplitude events caused by electronic noise, γ -rays and so on, are rejected.

Moderated proportional counters

In analogy to the case of γ -rays (Section 4.5.2), the detection probability for an incident neutron traversing a path length of x in the active detector volume is given by $P = 1 - e^{-N\sigma x}$, with σ being the cross section of the reaction that converts the incoming neutron to charged particles and N being the number density of active detector nuclei (¹⁰B or ³He). For example, for a 30 cm long cylindrical ¹⁰BF₃ proportional counter of 600 torr gas pressure we obtain a total efficiency of about 90% for thermal neutrons that are incident along the detector axis (Example 4.2). However, in astrophysically important reactions the neutrons are typically emitted with energies in the keV to MeV range rather than with thermal energies. The cross sections of the reactions 3 He(n,p) 3 H and 10 B(n, α) 7 Li decrease rapidly for increasing neutron energies, as can be seen from Fig. 4.15a. Consequently, the efficiency of proportional counters for directly detecting fast neutrons is rather small, making such detectors unsuitable for measuring weak cross sections of astrophysical interest. The detection efficiency of proportional counters for fast neutrons can be substantially improved by surrounding the detector with a suitable moderator, such as polyethylene or paraffin. The incident fast neutrons slow down in the moderating medium before reaching the counter and are detected with a much higher efficiency.

Figure 4.39 shows a typical setup of a system with a high efficiency for detecting fast neutrons. It consists of several ³He- or ¹⁰BF₃-filled proportional counters, arranged in a concentric ring around the target chamber. The counters are embedded in a cylindrical polyethylene moderator and are surrounded by layers of boron-paraffin and cadmium. The latter materials act as shields by moderating room background neutrons in the paraffin and by absorbing the moderated neutrons in the boron or cadmium layer before they can reach the active detector volume. The (beam-induced) reactions of interest take place in the target chamber, located close to the center of the entire detector. The total efficiency of such devices can amount to \approx 20–30% for neutron energies between 0.5 and 10 MeV (Section 4.7.4). The choice between ³He and $^{10}\text{BF}_3$ as fill gas is usually governed by considerations of efficiency and γ -ray sensitivity (East and Walton 1969). For cases in which the largest detection efficiency is required, ³He counters are preferable to ¹⁰BF₃ tubes since the former can be operated at much higher pressures. On the other hand, the latter are much less sensitive to γ -ray background because the ${}^{10}B(n,\alpha)^7$ Li reaction has a far higher *Q*-value compared to the 3 He(n,p) 3 H reaction.



Fig. 4.39 Moderated proportional counter. Several ³He or ¹⁰BF₃ proportional detectors (open circles) are arranged in a concentric ring around the target chamber (shown as full circle). The detectors are embedded in polyethylene, which acts as a moderator for beam-induced fast neutrons. After moderation, these neutrons are much more efficiently detected. The inner detector core is surrounded by layers of cadmium and boron-paraffin. These act as shields for unwanted contributions from background neutrons (see the text). The entire assembly may be surrounded by plastic scintillator veto shields (Section 4.4.3) in order to reduce the background from cosmic-ray muons.

Efficiency calibrations

Most efficiency calibrations of neutron detectors are performed with calibrated neutron sources. Radioisotopes that emit neutrons with discrete energies are practically not available as sources and thus laboratory neutron sources are based either on spontaneous fission or on nuclear reactions.

Many transuranium elements decay by spontaneous fission and thereby release fast neutrons, fission fragments, β - and γ -radiation. The material is usually encapsulated in a relatively thick container so that only neutrons and γ -rays emerge from the source. The most common type of this source is ²⁵²Cf ($T_{1/2} = 2.65$ y). The energy spectrum of the neutrons is continuous up to about 10 MeV with a maximum at ≈ 0.6 MeV. The neutron yield amounts to about $2.3 \times 10^6 \text{ s}^{-1} \text{µg}^{-1}$ (Knoll 1989). Compared to other neutron sources, ²⁵²Cf can be made in relatively small sizes.

Neutron sources can also be produced by mixing an α -emitting isotope with a substance such as ⁹Be, which exhibits a relatively large (α ,n) reaction cross section. One of the most common sources of this type consists of a ²³⁹Pu–Be mixture in which α -particles of 5.14 MeV energy from the decay of ²³⁹Pu initiate the reaction ⁹Be(α ,n)¹²C. The neutron transitions occur either to the ground state or to various excited states in ¹²C. The α -particles lose some of their initial energy in the source before reacting with the ⁹Be nuclei and, therefore, the neutron energy spectrum is continuous up to about 11 MeV. Similar argu-

ments hold for other (α ,n) neutron sources. A ²³⁹Pu–Be source yields about 60 neutrons per 10⁶ primary α -particles. The neutron yield of these sources should decay according to the half-life of the α -emitting nuclei. However, this assumption does not necessarily hold if the source contains contaminants that either emit directly α -particles or decay to α -emitting daughter nuclei. Such contaminants can even cause the neutron yield to increase with time (Knoll 1989).

Similarly, γ -ray emitters are sometimes used in order to produce neutrons via the photoneutron reactions ${}^{9}\text{Be}(\gamma,n){}^{8}\text{Be}(Q = -1.66 \text{ MeV}) \text{ or }{}^{2}\text{H}(\gamma,n)p$ (Q = -2.23 MeV). Suitable γ -ray emitters must provide photons of relatively large energies in order to initiate (γ ,n) reactions. An example for such a neutron source is a mixture of ${}^{88}\text{Y}$ -Be. Since the γ -rays are emitted with discrete energies and are not slowed down in the source, the emitted neutrons will also be monoenergetic, apart from a small kinematic energy spread. The disadvantage of (γ ,n) neutron sources is that high γ -ray activities are required to achieve reasonable neutron intensities. As a result, the neutrons are accompanied by a large γ -ray background.

Specific examples for the types of sources discussed above, together with their neutron energy regions, are listed in Table 4.10. Neutrons can also be produced directly at accelerators in reactions such as $D(d,n)^{3}$ He (Q = 3.27 MeV) and 7 Li(p,n) 7 Be (Q = -1.64 MeV). In addition, theoretical calculations of neutron detector efficiencies are routinely performed by using Monte Carlo simulations (Briesmeister 1993).

Source	Туре	Half-life ^a	E_n^a (MeV)
²⁵² Cf	s.f.	2.65 y	$< 10^{b}$
²³⁹ Pu–Be	(α,n)	24000 y	< 11
²⁴¹ Am–Be	(α,n)	433 y	$< 10^{b}$
⁸⁸ Y–Be	(γ, \mathbf{n})	107 d	0.152
			0.949
¹²⁴ Sb–Be	(γ ,n)	60.2 d	0.023

Tab. 4.10 Properties of neutron calibration sources. (a) From Knoll (1989), unless mentioned otherwise. (b) From Lorch (1973).

4.6

Miscellaneous Experimental Techniques

A number of experimental techniques are used in direct measurements of astrophysically important reactions that require special equipment and procedures. In this section, we will focus on three particularly important examples: (i) radioactive ion beams, (ii) the activation method, and (iii) the time-offlight technique. The last two are used both in charged-particle and neutron-

induced reaction studies, but in nuclear astrophysics their main (although not exclusive) application is in the field of neutron-induced reactions. For other interesting techniques, such as accelerator mass spectrometry (Wallner et al. 2006) or the use of etched track detectors (Somorjai et al. 1998), the reader is referred to the literature.

4.6.1

Radioactive Ion Beams

Proton- and α -particle-induced reactions in a stellar plasma at elevated temperatures do not only involve stable nuclides, as will become apparent in the next chapter, but unstable nuclides also participate in the nucleosynthesis. The instability of one of the interacting nuclei represents a significant challenge for the experimentalist. If the half-life exceeds a few days, then it may be possible to fabricate a radioactive target and to measure directly the reaction of interest by bombarding the target with protons or α -particles using the experimental techniques and procedures described so far. Examples for such studies are the measurements of proton capture reactions on the radioactive species ²²Na (Seuthe et al. 1990, Stegmüller et al. 1996) and ²⁶Alg (Buchmann et al. 1984, Vogelaar 1989) with half-lives of $T_{1/2} = 2.6$ y and 7.2×10^5 y, respectively. However, if the half-life of a species amounts to a few minutes or less, then the fabrication of a radioactive target is not feasible. A direct measurement of such reactions is nevertheless possible if the role of target and projectile are interchanged. Consider a reaction between a light particle x (proton or α -particle) and a short-lived heavy nucleus X. The bombardment of target X with projectiles x may not be feasible, but it may be possible to produce a beam of radioactive nuclei X which is then directed onto a stationary target consisting of the light nuclei x. Such measurements are referred to as *inverse* kinematics studies. For example, suppose one would like to measure the proton capture cross section of the p + X reaction at a center-of-mass energy of $E_{\rm cm} = 0.5$ MeV, where the short-lived nucleus *X* has a mass number of A = 20. The laboratory beam energy of *X* must then be $E_{\text{lab}}(^{20}X) = E_{\text{cm}}(m_{\text{p}} + m_X)/m_{\text{p}}$ = 10.5 MeV (see Eq. (C.24)). The time of flight of X over a distance of \approx 100 m amounts only to $\approx 10 \,\mu\text{s}$ and, therefore, a measurement of the reaction p + X is in principle feasible if the half-life of X is not too short (say, in this example, $T_{1/2} > 10 \ \mu s$).

The production, transport, and acceleration of radioactive ion beams suitable for nuclear astrophysics measurements requires substantial resources and efforts. Several different techniques have been developed which are complementary in their capabilities. In the simplest case, the radioactive material of interest is produced offline at a nuclear reactor or accelerator and is then converted into a suitable chemical form before installation in an ion source of a second accelerator capable of accelerating the radioactive heavy ions. This method, called the batch mode technique, is only suitable for beams of relatively long-lived nuclei. It has been applied, for example, in studies of the $^{7}Be(p,\gamma)^{8}B$ and ${}^{44}\text{Ti}(\alpha,p){}^{47}\text{V}$ reactions (Gialanella et al. 2000, Sonzogni et al. 2000). The most direct approach, however, is the online production of radioactive nuclei, their ionization and extraction from an ion source, and their subsequent acceleration. This method, referred to as the *isotope separator online (ISOL) technique*, has been used extensively in nuclear astrophysics measurements. In the following we will briefly describe this method. More specific information can be found in Smith and Rehm (2001) and Blackmon, Angulo and Shotter (2006). Other techniques involving measurements with (low-energy) unaccelerated radioactive ion beams or with high-energy radioactive beams produced via projectile fragmentation are mainly used in indirect studies of important nuclear structure properties. Since they are usually not suitable for direct measurements of low-energy nuclear reactions, they will not be discussed here. The reader can find more information on the latter topics in Kratz (1988) and Mueller and Sherrill (1993).

The isotope separator online (ISOL) technique is shown schematically in Fig. 4.40. A beam of stable nuclei from a production accelerator bombards a thick target and produces radioactive nuclei. These diffuse out of the target, through a transfer tube, and into an ion source where they are ionized and continuously extracted. The radioactive ions are then mass separated from other, undesired, isotopes. At this stage they represent a beam of unaccelerated, low-energy radioactive ions. Subsequently, they are accelerated by a post accelerator which allows a tuning of the beam energy to the desired value. This accelerated radioactive ion beam is finally incident on a hydrogen or helium target. The radiation emitted in the reaction of interest is then observed using suitable detectors. Beams from ISOL facilities, which are operational for example at Louvain-la-Neuve, Oak Ridge National Laboratory, CERN and TRIUMF, have excellent beam qualities (resolution and spread). The success of an experiment depends obviously on the radioactive ion beam intensity, which is limited by the primary production cross section, the diffusion velocity of the radioactive ions in the production target, the effusion of radioactive ions out of the target, and the ionization efficiency in the ion source. Unfortunately, there is no single combination of production beam, thick target, and ion source that can produce all radioactive species of astrophysical interest. More typically, each radioactive ion beam experiment requires an extensive and time-consuming effort of beam development where the composition and chemistry of the production target is varied until the intensity of the radioactive species of interest is maximized. Beams of some elements, for example, noble gases or alkali metals, can be produced with relatively high intensities, while beams of refractory elements are difficult to extract from the production target and are thus available only with much lower intensities. Some of



Fig. 4.40 Basic components of an isotope separator online (ISOL) facility for the production of accelerated radioactive ion beams. The dashed square marks the location of the actual reaction measurement of astrophysical interest. In a reaction of type X(a,b)Y, counters 1 or 3 detect the emitted light particle *b*. Alternatively, the light particle *b* and the corresponding heavy residual

nucleus *Y* may be detected in coincidence. In a capture reaction, $X(a,\gamma)Y$, the heavy residual nucleus *Y* may be detected in a recoil separator (counter 2). Alternatively, the residual nuclei may be detected by counter 2 in coincidence with the corresponding prompt γ -rays in counters 1 or 3. See the text.

the issues related to the production of radioactive beams at ISOL facilities are discussed in Dombsky, Bricault and Hanemaayer (2004). It is important to emphasize that at present, even under favorable circumstances, the radioactive beam intensities delivered to an experiment amount to at most $\approx 10^{10}$ ions/s. Comparison to a value of 6×10^{14} particles/s for a 100 µA proton beam in a typical normal kinematics experiment reveals that a radioactive ion beam facility must be carefully designed and optimized in order to avoid any intensity losses of the precious radioactive beam. In addition, it is imperative that the detection system has a large detection efficiency and large discriminating power against unwanted beam-induced background contributions.

We will first briefly discuss some important components of ISOL facilities before describing a specific experiment in more detail. In a star, most charged-particle reactions involving radioactive ions are induced by protons or α -particles. Therefore, hydrogen and helium are the most important target materials in direct radioactive ion beam measurements. The target requirements differ somewhat from those appropriate for normal kinematics experiments (Section 4.3). In the case of hydrogen, thin polyethylene [(CH₂)_n] foils have been employed successfully in several measurements. They are mechanically stable, even if stretched to thicknesses of 20–1000 µg/cm², and they have been used with beam intensities of up to 10^9 particles/s without significant degradation. However, the carbon content may give rise to an intense beam-induced background dominated by elastic scattering. Gas targets are the obvious choice for helium, and are also advantageous for hydrogen. Gas cells with thin entrance and exit windows are easy to handle but the window foils degrade the beam energy resolution and induce background reactions. Windowless gas targets are the preferred choice (Section 4.3.2) although they are bulky and expensive since they require many pumping stations in order to reduce the pressure to the 10^{-7} torr range.

The fact that radioactive beam experiments are performed in inverse kinematics has for the reaction products the interesting consequence that the solid angle in the center-of-mass system is compressed into a significantly smaller solid angle in the laboratory reference frame (Appendix C.2). Detection systems used in radioactive ion beam experiments take advantage of this circumstance in order to increase the efficiency and sensitivity. Light charged particles from (p,α) or (α,p) reactions have been measured using arrays of silicon strip detectors, arranged to cover a large solid angle around the target. These are highly segmented, with well over 100 elements, and provide excellent energy and angle resolution. Thicknesses for these counters amount to 50–1000 µm and arrays can be stacked to allow for particle identification by measuring both energy loss (ΔE) and total energy (*E*). In some cases the heavy reaction products Y have been detected in coincidence by using additional detectors placed downstream of the target. Such experiments require typically radioactive ion beam intensities in excess of 10⁵ particles/s in order to achieve sufficient counting statistics.

Radiative capture reactions of type (p,γ) or (α,γ) can in principle be studied by the in-beam measurement of γ -rays (Section 4.5.2) or by the activation method (Section 4.6.2). Both of these techniques have drawbacks. The direct detection of γ -rays alone is especially difficult for proton-rich radioactive beams which give rise, after scattering and positron decay in and near the target chamber, to a high background from 511 keV photons. The activation measurement is useful only in those cases where the observed decay is indeed a signature of the reaction of astrophysical interest. The best method of studying capture reactions is by directly detecting the recoil nuclei Y. This technique is particularly well suited to radioactive ion beam measurements in inverse kinematics. The outgoing γ -ray transfers a very small momentum to the compound nucleus which is therefore typically emitted within an angle of $\phi_{\text{lab}} \approx 1^{\circ}$ with respect to the beam direction (Problem 4.8). This allows for an efficient detection of the heavy reaction products provided that they can be separated from the incident radioactive beam which of course moves into the same direction. In fact, the incident projectiles and the heavy reaction products have the same linear momentum and differ in mass and velocity by only

a few percent. In addition, the cross sections of interest are usually small and thus the number of beam projectiles exceeds the number of reaction products by very large factors ($10^{10}-10^{15}$). Recoil separators are sophisticated devices that facilitate the detection of reaction products in the presence of an overwhelming background of beam particles. Mass separation and beam rejection are accomplished by using an arrangement of dipole magnets, electrostatic deflectors, or Wien filters. The reaction products are collected at the focal plane of the device and are dispersed according to their mass-to-charge ratio. A variety of detection schemes may be employed, for example, time-of-flight, *Z* identification or delayed activity detection. The detection sensitivity may be significantly improved by measuring the heavy recoils at the focal plane of the separator in coincidence with prompt γ -rays detected near the target. Typically, radioactive ion beam intensities in excess of $\approx 10^7$ particles/s are required for such experiments in order to accumulate sufficient counting statistics.

The first nuclear astrophysics experiment with an accelerated radioactive ion beam was the measurement of the ${}^{13}N(p,\gamma){}^{14}O$ reaction at Louvain-la-Neuve (Delbar et al. 1993). Since this pioneering study, several astrophysically important reactions have been measured directly or indirectly at a number of different radioactive ion beam facilities worldwide. A discussion of some of these experiments is given in Smith and Rehm (2001) and Blackmon, Angulo and Shotter (2006) and will not be repeated here. Radioactive ion beam facilities have opened a window of previously unavailable capabilities in nuclear astrophysics. The results obtained from these measurements have a crucial impact on predictions of explosive nucleosynthesis. Therefore, it is worthwhile to discuss as an example one particular experiment in more detail.

The ²¹Na(p,γ)²²Mg reaction is important for the production of the longlived γ -ray emitter ²²Na in classical novae (Section 5.2.2). The reaction was directly measured (D'Auria et al. 2004) at the TRIUMF-ISAC facility (see color Fig. 12 on page 642), located in Vancouver, Canada, in the energy range of the nova Gamow peak ($E_0 \pm \Delta/2 = 270 \pm 100$ keV at T = 0.3 GK). A 500 MeV proton beam of \leq 30 μ A intensity from the TRIUMF cyclotron bombarded a thick SiC production target. Spallation reactions on Si produced ²¹Na which diffused from the hot target through a transfer tube and was ionized in a surface ionization source. After mass separation, the low-energy ²¹Na beam was accelerated to energies variable between 0.15 and 1.5 MeV/u by using a radiofrequency quadrupole (RFQ) accelerator and a drift-tube linac. The intensity of the ²¹Na beam delivered to the experiment amounted up to 10⁹ ²¹Na ions per second. The radioactive ²¹Na beam was then incident on a windowless hydrogen gas target. Prompt γ -rays were detected in an array of 30 BGO scintillator detectors, packed tightly around the gas target, with an almost 4π coverage of the solid angle. The ²²Mg nuclei were separated from the intense beam by using the DRAGON recoil separator (Engel et al. 2005) and were de-



Fig. 4.41 Pulse height spectrum of heavy ions detected at the focal plane of a recoil separator in the study of the $E_r^{\rm cm} = 207$ keV resonance in ²¹Na(p, γ)²²Mg. The dashed histogram shows the singles spectrum and is dominated by ²¹Na beam particles leaking through the separator. The shaded spectrum displays those heavy ions which are in coincidence with prompt γ -rays detected

in a BGO array surrounding the hydrogen gas target. These correspond to the reaction products ²²Mg since ²¹Na beam particles are not in coincidence with prompt γ -rays. Reprinted with permission from J. M. D'Auria et al., Phys. Rev. C, Vol. 69, 065803 (2004). Copyright (2004) by the American Physical Society.

tected in the focal plane by a double-sided silicon strip detector. A coincidence requirement between the ²²Mg nuclei detected at the focal plane and the corresponding γ -rays measured near the target by the BGO array allowed for a very high detection sensitivity, even though the ²²Mg nuclei had rather low energies. A pulse height spectrum of heavy ions detected at the focal plane of the recoil separator at a bombarding energy in the region of the lowest lying resonance ($E_r^{cm} = 207 \text{ keV}$) in ²¹Na(p, γ)²²Mg is shown in Fig. 4.41. The dashed histogram displays the singles spectrum and is dominated by unwanted ²¹Na beam particles leaking through the separator. The shaded histogram shows only those heavy ions that are in coincidence with γ -rays ($E_{\gamma} \ge 3 \text{ MeV}$) detected in the BGO array. These correspond to ²²Mg ions since the ²¹Na beam particles are not expected to be in coincidence with prompt γ -rays. The clean identification of ²²Mg recoils allowed for a precise measurement of the energy and strength of this resonance which dominates the total ²¹Na(p, γ)²²Mg reaction rates at typical nova temperatures.

4.6.2

Activation Method

We already discussed the prompt detection of reaction products, that is, the direct detection of the emitted particles or γ -rays in reactions of type X(a,b)Y or $X(a,\gamma)Y$, respectively. From their measured intensity we can infer the to-

tal number of reactions that took place. This information is used to calculate cross sections or resonance strengths (Sections 4.8 and 4.9). There are instances where such a reaction will produce a radioactive nucleus *Y* in its ground state (or a long-lived isomeric state). Instead of detecting the prompt radiation *b*, one could count the number of nuclei *Y* by observing the delayed residual radioactivity after the bombardment of the target or sample with projectiles has stopped. This technique is referred to as the *activation method*. In nuclear astrophysics measurements, it is mainly used for studies of neutron-capture reactions (Käppeler 1999). For applications to charged-particle-induced reaction studies see, for example, Sauter and Käppeler (1997) or Gyuerky et al. (2003).

The activation method has certain advantages over other techniques. For example, consider a capture reaction that gives rise to a complicated γ -ray decay scheme. The determination of prompt γ -ray intensities and branching ratios in order to infer the number of nuclear reactions that took place may become very challenging in this case (Section 4.5.2). However, all these γ -ray cascades will eventually make transitions to the ground state (or a longlived isomeric state) of nucleus Y. Counting the number of radioactive nuclei *Y* via the activity provides directly the information of astrophysical interest, independent of the details of decay branchings or angular correlation effects. Furthermore, since the activity is measured after the irradiation took place, there is no prompt beam-induced background and the counting setup can be optimized more easily for efficiency since geometrical complications, such as target chambers and beam pipes, are absent in the offline measurement. Finally, the activation method is selective for specific reactions, that is, by measuring the energies of the radioactive decay products or the time evolution of the radioactive decay, one can infer the identity of the radioactive nuclei Y. It is sometimes even possible to determine cross sections for several different reactions in a single measurement. The activation method is suitable for radioactive reaction products with half-lives between several years and fractions of a second (Beer et al. 1994).

The rate of change in the number of radioactive nuclei *Y* is given by the difference of production and decay rates,

$$\frac{d\mathcal{N}_{Y}(t)}{dt} = P(t) - \lambda_{Y}\mathcal{N}_{Y}(t)$$
(4.72)

with N_Y and $\lambda_Y = \ln 2/T_{1/2}$ the number and decay constant of nuclei *Y*, respectively. The production rate is given by

$$P(t) = \mathcal{N}_X \int \sigma(E)\phi(E,t) \, dE = \mathcal{N}_X \hat{\sigma} \int \phi(E,t) \, dE = \mathcal{N}_X \hat{\sigma} \phi(t) \tag{4.73}$$

with \mathcal{N}_X the number of target nuclei *X*, σ the cross section for the *X*(*a*,*b*)*Y* reaction, and $\phi(t)$ the incident particle flux (in particles per area per time). A

number of assumptions have been made in the above expression: (i) the number of target or sample nuclei *X* does not change during the irradiation (that is, the target does not deteriorate and the fraction of target nuclei destroyed is negligible), and (ii) the target or sample is sufficiently thin so that the energy loss of incident charged particles in the target, or the attenuation of incident neutrons in the sample, is small. Note that $\hat{\sigma}$ represents a cross section which is averaged over the energy distribution of incident projectiles (and over the target thickness for charged particles). For the general case of a varying incident particle flux, Eq. (4.72) has to be integrated numerically. For the special case of a constant flux, $\phi(t) = \text{const}$, we can solve Eq. (4.72) analytically. For the initial condition $\mathcal{N}_{\Upsilon}(t = 0) = 0$ the solution is

$$\mathcal{N}_{Y}(t) = \frac{\mathcal{N}_{X}\hat{\sigma}\phi}{\lambda_{Y}} \left(1 - e^{-\lambda_{Y}t}\right) \tag{4.74}$$

If $\lambda_Y t \ll 1$, we find $\mathcal{N}_Y(t) \approx \mathcal{N}_X \hat{\sigma} \phi [1 - (1 - \lambda_Y t)] / \lambda_Y = \mathcal{N}_X \hat{\sigma} \phi t$, that is, $\mathcal{N}_Y(t)$ increases linearly for small irradiation times. For $\lambda_Y t \gg 1$, we obtain $\mathcal{N}_Y(t) \approx \mathcal{N}_X \hat{\sigma} \phi / \lambda_Y = \mathcal{N}_X \hat{\sigma} \phi T_{1/2} / \ln 2 = \mathcal{N}_Y^S$, and $\mathcal{N}_Y(t)$ reaches a saturation value, \mathcal{N}_Y^S , where the production rate becomes equal to the destruction rate. At the end of the irradiation period, $t = t_0$, the number of nuclei \mathcal{N}_Y is $\mathcal{N}_Y(t_0) = \mathcal{N}_X \hat{\sigma} \phi (1 - e^{-\lambda_Y t_0}) / \lambda_Y$. Since nuclei Y are no longer produced for $t > t_0$, the production rate is zero and the time evolution of $\mathcal{N}_Y(t)$ is given by

$$\mathcal{N}_{Y}(t) = \mathcal{N}_{Y}(t_{0})e^{-\lambda_{Y}(t-t_{0})}$$

$$= \frac{\mathcal{N}_{X}\hat{\sigma}\phi}{\lambda_{Y}} \left(1 - e^{-\lambda_{Y}t_{0}}\right)e^{-\lambda_{Y}(t-t_{0})} \quad \text{for } t > t_{0}$$
(4.75)

If the sample is counted between t_1 and t_2 , the number of disintegrations in that period is given by the integral over the activity $A_Y(t) = N_Y(t)\lambda_Y$,

$$D(t_1, t_2) = \int_{t_1}^{t_2} \lambda_Y \mathcal{N}_Y(t) dt = \mathcal{N}_X \hat{\sigma} \phi \left(1 - e^{-\lambda_Y t_0}\right) \int_{t_1}^{t_2} e^{-\lambda_Y(t-t_0)} dt$$
$$= \frac{\mathcal{N}_X \hat{\sigma} \phi}{\lambda_Y} \left(e^{\lambda_Y t_0} - 1\right) \left(e^{-\lambda_Y t_1} - e^{-\lambda_Y t_2}\right)$$
(4.76)

Hence, the cross section $\hat{\sigma}$ can be determined from the number of disintegrations, the number of target nuclei, and the total flux of incident particles. Obviously, Eq. (4.76) can also be used to determine an unknown neutron flux from a well-known cross section.

Figure 4.42 shows schematically the time evolution of the number of radioactive nuclei *Y*. In this example, the incident particle flux is constant, $\phi(t)$ = const. The irradiation of the target starts at t = 0 and stops at $t_0 = 6T_{1/2}$, where $\mathcal{N}_Y(t)$ is close to the saturation value [$\mathcal{N}_Y(t)/\mathcal{N}_Y^S = 0.984$]. After some waiting period between t_0 and t_1 , the activity is counted between t_1 and t_2 ,



Fig. 4.42 Evolution of the number of radioactive nuclei *Y* (in units of the saturation value \mathcal{N}_Y^S) versus time (in units of the half-life $T_{1/2}$). Here, the incident particle flux is assumed to be constant. The irradiation of the target starts at time t = 0 and stops at $t_0 = 6T_{1/2}$, when the ratio $\mathcal{N}_Y(t)/\mathcal{N}_Y^S$ is close to unity. After some waiting period, $t_1 - t_0$, the activity is counted between t_1 and t_2 when $\mathcal{N}_Y(t)$ decays exponentially.

when $N_Y(t)$ decays exponentially. The relationship between the number of disintegrations (or the measured number of counts) and the cross section is discussed in Sections 4.8 and 4.9.

Targets or samples must be sufficiently thick in order to achieve reasonable counting statistics. But they should not be too thick or otherwise: (i) incident neutrons may be attenuated significantly or undergo multiple scattering while effects may be difficult to correct for; (ii) the cross section for incident charged particles will be integrated over too large an energy range and cannot be determined with reasonable energy resolution; and (iii) the self-absorption of the emitted delayed radiation (for example, electrons or photons) may become significant. The loss of radioactive nuclei Y due to sputtering or backscattering is another problem when targets are bombarded with intense chargedparticle beams. Such losses can be measured and accounted for by surrounding the target with a catcher foil. Furthermore, one must ensure that the radioactive nuclei Y of interest are not produced during the irradiation via some other nuclear reaction, Z(c,d)Y, involving contaminants in the beam or the target. For example, a measurement of the 27 Al(n, γ) 28 Al reaction may be complicated by the presence of a ²⁸Si contamination in the aluminum sample since the ²⁸Si(n,p)²⁸Al reaction also produces ²⁸Al and hence interferes with the actual measurement.


Fig. 4.43 Results of an activation measurement of the ¹⁷⁵Lu(n, γ)¹⁷⁶Lu^{*m*} capture reaction. The incident neutron energy distribution closely resembles a Maxwell–Boltzmann distribution. Shown are relevant parts of γ -ray spectra in the vicinity of the 88 keV γ -ray line from the (delayed) decay of the isomeric state ¹⁷⁶Lu^{*m*} ($T_{1/2}$ = 3.68 h).

The different spectra were observed in 1 h intervals. The exponential decay, shown in the inset, serves as an additional check on the possible presence of radioactive contaminants. Reprinted with permission from H. Beer and F. Käppeler, Phys. Rev. C, Vol. 21, p. 534 (1980). Copyright (1980) by the American Physical Society.

Results of an activation measurement are shown in Fig. 4.43. The capture reaction ${}^{175}Lu(n,\gamma){}^{176}Lu^m$ was studied by using an incident neutron energy distribution that closely resembled a Maxwell–Boltzmann distribution (Section 4.1.2). The figure shows relevant parts of γ -ray spectra in the vicinity of the 88 keV γ -ray line from the (delayed) decay of the isomeric state ${}^{176}Lu^m$ ($T_{1/2} = 3.68$ h). The spectra were observed in 1 h intervals and clearly reveal a decline in activity. The exponential decay, shown in the inset, serves as an additional check on the possible presence of radioactive contaminants.

4.6.3

Time-of-Flight Technique

The time-of-flight method provides neutron beams with a resolution that is far superior compared to most other techniques. Consider Fig. 4.44 showing a pulsed proton or electron beam incident on a neutron production target, as described in Section 4.1.2. With each pulse, a group of neutrons with a broad energy distribution is produced. The neutrons travel to the irradiation sample which is located at a distance of L from the neutron-producing target.

Neutron-induced reactions take place in the sample and the prompt radiation produced in the interaction is detected by using a suitable counter. For example, detector 1 in Fig. 4.44 is a γ -ray detector for the study of a (n,γ) reaction. The neutron velocity is determined by the measured length of the flight path and the time difference, $t = t_{stop} - t_{start}$, between the arrival time of the primary electron or proton pulse at the neutron-producing target and the time of detection of the prompt reaction products (provided that the latter time is practically simultaneous with the arrival of the neutrons at the sample). The neutron energy is given by

$$E = \frac{1}{2}m_{\rm n}v^2 = \frac{1}{2}m_{\rm n}\left(\frac{L}{t}\right)^2$$
(4.77)

with m_n the neutron mass. The use of this nonrelativistic expression introduces an error of less than 0.2% at E = 1 MeV. In practice, the events are sorted electronically into a histogram displaying the flight time on the horizontal axis (that is, a particular channel corresponds to flight times between t_i and t_{i+1}). Subsequently, the flight time scale is converted to a neutron energy scale. From Eq. (4.77) we find numerically

$$\frac{t}{L} = \frac{72.3}{\sqrt{E}} \tag{4.78}$$

with *t*, *L*, and *E* in units of μ s, m, and eV, respectively. For example, for a flight path of 10 m and neutrons of 1 keV energy the flight time amounts to $\approx 23 \ \mu$ s. For a broad energy distribution of incident neutrons, the time-of-flight technique allows a measurement of the intensity of reaction products as a function of the incident neutron energy in a single experiment.

According to Eq. (4.77), the energy resolution of the neutron beam is given by

$$\frac{\Delta E}{E} = 2\sqrt{\left(\frac{\Delta L}{L}\right)^2 + \left(\frac{\Delta t}{t}\right)^2} \tag{4.79}$$

The uncertainty in the flight path (for example, due to the finite sizes of the neutron-producing target and the detector) can be reduced by increasing L, although the intensity of the neutron beam at the sample position will decrease at the same time. In most cases, the uncertainty in flight time will dominate the energy resolution. One finds numerically

$$\Delta E = 0.028 \frac{\Delta t}{L} E^{3/2} \tag{4.80}$$

The uncertainty Δt is influenced by a number of factors, including the time width of a neutron group after a particular proton or electron burst (< 100 ns),



Fig. 4.44 Neutron time-of-flight technique. A pulsed proton or electron beam is incident on a neutron production target. With each pulse, a neutron group with a broad energy distribution is produced. The neutrons travel a distance of *L* to the irradiation sample. Prompt radiation, for example, γ -rays from an (n, γ) reaction induced in the sample, is detected by counter 1, while counter 2 represents a neutron detector for

measuring the transmission (Section 4.2.3). The incident neutron energy is given by the flight path length *L* and the time difference, $t = t_{stop} - t_{start}$, between the primary electron or proton pulse arrival time at the neutron-producing target and the detection time of the prompt reaction products in counter 1 (or of the transmitted neutrons in counter 2).

the pulse rise time of the detector (< 5 ns), and the uncertainty in the neutron slowing down time if the neutron-producing target is surrounded by a moderator. The primary requirements for the detector are fast timing properties, relatively high efficiency and low sensitivity to neutron-induced background radiation. For neutron-capture studies, organic scintillators or BaF₂ detectors are frequently used, while ionization chambers or solid-state counters are the detectors of choice for (n,p) or (n, α) type experiments. With time-of-flight techniques, energy resolutions of better than 1 eV have been obtained for neutron energies up to a few keV (that is, $\Delta E/E \approx 0.001$). As an example, Fig. 4.45 shows the measured count rate versus time of flight in the vicinity of the 4.9 eV resonance in the ¹⁹⁷Au(n, γ)¹⁹⁸Au reaction. Other examples of transmission and neutron-capture yield curves obtained with the time-of-flight technique are shown in Fig. 4.65.

4.7

Background Radiation

All radiation detectors used in fusion reaction measurements will record a certain number of pulses that are caused by natural radioactivity in the environment or by cosmic radiation. For relatively large reaction cross sections, the background count rate may be negligible compared to the signal count rate. However, astrophysically important reactions have frequently very small





Fig. 4.45 Measured count rate versus time of flight in the vicinity of the 4.9 eV resonance in the ¹⁹⁷Au(n, γ)¹⁹⁸Au reaction. Reprinted from R. L. Macklin, J. Halperin and R. R. Winters, Nucl. Instrum. Methods, Vol. 164, p. 213 (1979). Copyright (1979), with permission from Elsevier.

cross sections at the energies of interest. In such cases, the experiment has to be designed carefully so that the signal is not obscured by the background. It has to be kept in mind that in nuclear counting experiments the sensitivity for detecting a signal above background is approximately directly proportional to the signal count rate, but inversely proportional to the square root of the background count rate (Knoll 1989). For example, a background reduction by a factor of 100 corresponds to an improvement in sensitivity by only a factor of 10. Thus, a substantial effort of reducing the background is required in order to observe very weak cross sections or resonance strengths. In order to develop detection techniques that reduce the background, it is first necessary to understand the nature of the background in more detail. A comprehensive review of these issues can be found in Knoll (1989) and Heusser (1995). Here, we will discuss the influence of natural radioactivity and cosmic rays on detectors that are used in nuclear astrophysics measurements. We are especially interested in methods of background suppression. Other important sources of background, for example, from electronic noise or from beam-induced processes (Section 4.3.3 and Table 4.4), are not a topic of the present section.

4.7.1

General Aspects

Figure 4.46 shows a typical experimental setup consisting of accelerator, target chamber, detector, and shielding. The most important sources of background radiation are indicated. Terrestrial radiation near the Earth's surface and in ordinary construction materials (walls, detector, shielding materials, air, and so on) is caused by naturally occurring radioisotopes. In particular, Th, U,

and Ra are members of radioactive decay series and lead to a large number of daughter nuclei that emit α -, β -, and γ -radiation. Among these daughter products are the short-lived radioactive gases ²²⁰Rn and ²²²Rn that are present in the ambient air. The background from radon and progenitors may be reduced by replacing the air surrounding the detection setup with a radon-free gas (for example, nitrogen). Furthermore, spontaneous fission (in particular of ²³⁸U) contributes to the γ -ray and neutron background. Another important source of terrestrial β - and γ -radiation originates from the decay of ⁴⁰K ($T_{1/2} = 1.3 \times 10^9$ y). The activity of certain fission products that originate from past nuclear weapons testing also contributes to the background (for example, ¹³⁷Cs).

The activity levels in common materials vary substantially (Knoll 1989). In demanding low-background applications, the construction and shielding materials have to be selected carefully. One may ordinarily expect that the background count rate is inversely proportional to the thickness of the detector shield. However, beyond a certain optimum thickness the background will not decrease any further because a more massive shield represents a larger target for cosmic-ray-induced background (see below).

Primary cosmic radiation consists mainly of protons and α -particles with extremely high kinetic energies. They hit the upper atmosphere with an intensity of about $10^3 \text{ m}^{-2} \text{ s}^{-1}$. Through interactions with air molecules, a large number of different (secondary) elementary particles with energies extending into the 100 MeV range are produced. Among the secondary radiation, protons, electrons, and pions are easily absorbed by the concrete floors of a building. The most relevant components for low level background measurements are muons and neutrons.

The muon-induced background arises from direct ionization events in the detector volume, radioisotope production via interactions with nuclei (for example, spallation), muon bremsstrahlung, production of δ electrons, muon decay ($\mu^{\pm} \rightarrow e^{\pm} + \nu + \overline{\nu}$), and electron–positron pair production. The latter three processes also give rise to bremsstrahlung. Secondary neutrons originate from primary cosmic radiation, while tertiary neutrons are produced by slow muons via the capture reaction $p(\mu^{-},\nu_{\mu})n$ and by fast muons via (γ ,n) reactions and photofission. Neutron background not only originates from cosmic-ray interactions but also from terrestrial radioisotopes via (α ,n) reactions and spontaneous fission of ²³⁸U. Fast neutrons react with nuclei via ($n,n'\gamma$) reactions, while thermal neutrons interact via (n,γ) reactions.

Muons are very penetrating particles and large shielding depths (for example, several 100 m of earth) are required to attenuate their intensity substantially. For this reason, low-level detection systems are sometimes operated deep underground. Such a laboratory, dedicated to nuclear astrophysics experiments, is described in Bemmerer et al. (2005). Alternatively, in measure-



Fig. 4.46 Sources of environmental background in a typical nuclear physics counting experiment. Other sources, such as electronic noise or beam-induced background radiation, are not shown. See discussion in the text.

ments at sea level, it is usually possible to surround the primary detector (for example, germanium) with a secondary (guard) counter (for example, plastic scintillator). If both detectors are operated in anticoincidence mode, that is, if events are rejected when both counters respond at the same time, then the background is significantly reduced.

Note that the magnitude of some background components can change with time. Such fluctuations may arise from variations in cosmic-ray intensity or in airborne radioactivity depending on the meteorological conditions. For the analysis of pulse height spectra obtained with beam on target it is therefore helpful to carry out background measurements (without beam) before and after the actual experiment.

4.7.2 Background in Charged-Particle Detector Spectra

We will focus in this section on semiconductor charged-particle detectors. The extreme degree of purity required of semiconductor crystals results in relatively low levels of inherent radioactivity. However, radioactive impurities in the construction materials of the setup (detector holder, target chamber, shielding, and so on) will contribute to the measured background. This background will extend to several MeV, corresponding to typical energies of α -particles from terrestrial radioisotopes. For example, commercial aluminum shows a low-level α -activity of $\approx 0.3 \alpha$ -particles h⁻¹ cm⁻² above 250 keV energy (Knoll 1989). The α -particle emission rate of stainless steel is about an order of magnitude lower. For low-level background measurements it is thus important to select the construction materials carefully.

Terrestrial γ -radiation and cosmic-ray-induced γ -rays, charged particles, and neutrons will also contribute to the background in charged-particle detector spectra. These components can effectively be reduced through the use of appropriate shielding such as low-activity lead or mercury. Cosmicray muons, on the other hand, are only weakly absorbed in the shield. The muons are minimum ionizing (Section 4.2.1) and lose energy in silicon at a rate of $dE/dx \approx -400$ keV/mm. The spatial distribution of the muons is at maximum in a direction perpendicular to the Earth's surface. Therefore, the energy deposited by the muons in the detector is approximately equal to the product of dE/dx and the effective thickness of the active detector volume. Since some muons will pass at oblique angles through the sensitive region, the background peak in the pulse height spectrum will exhibit a high-energy tail. This general behavior is displayed in Fig. 4.47a, showing a background spectrum measured with a silicon detector of 300 µm thickness, with its active surface positioned parallel to the surface of the Earth. In order to reduce terrestrial background, the detector was mounted in a highpurity aluminum oxide insulator and the detector container was made from low-contamination copper. The muon peak in the spectrum is expected to occur at about (400 keV/mm)(0.3 mm) = 120 keV, consistent with observation. The spectrum in Fig. 4.47b was obtained by rejecting all events corresponding to simultaneous signals from the silicon detector and a NaI(Tl) active shield counter that was positioned above the silicon detector. Clearly, the muon background is substantially reduced by using anticoincidence techniques.

Figure 4.48 shows the measured background of a 300 µm thick silicon detector in the energy range of 50–400 keV versus mode of operation. The results are given in units of counts per minute (cpm) and per square centimeter of the active detector surface which is oriented parallel to the Earth's surface. The background count rate amounts to ≈ 3.5 cpm/cm² without passive or active shielding. Surrounding the setup with lead of 2.5 cm thickness



Fig. 4.47 (a) Background spectrum measured with a silicon charged-particle detector of 300 µm thickness. The active detector surface is positioned parallel to the surface of the Earth. The muon peak, which has the same origin as the one shown in Fig. 4.23, occurs at about 120 keV. (b) Same as part (a), but rejection of all events correspond-

ing to simultaneous signals from the silicon detector and a Nal(TI) active shield counter that was positioned above the silicon detector. The substantial reduction of the muon background is evident. Reprinted from F. J. Walter and R. R. Boshart, Nucl. Instrum. Methods, Vol. 42, p. 1 (1966). Copyright (1966), with permission from Elsevier.

shields the detector from terrestrial and cosmic-ray-induced γ -radiation and reduces the background count rate to $\approx 1.4 \text{ cpm/cm}^2$. A further improvement is achieved by using 5.1 cm of low-background lead for a passive shield ($\approx 0.97 \text{ cpm/cm}^2$). Finally, the additional use of an anticoincidence shield reduces the background count rate to $\approx 0.16 \text{ cpm/cm}^2$. We can infer from these results that the muon-induced background rate at sea level amounts to (0.97–0.16) cpm/cm² $\approx 0.8 \text{ cpm/cm}^2$ in the energy range of 50–400 keV.

In certain reactions with relatively large *Q*-values it is sometimes of advantage to use two instead of one silicon detector. The nuclear reaction products of interest deposit a fraction of their energy in a thin front (" ΔE ") detector and are completely stopped in a thick rear ("*E*") counter. By requiring a coincidence between the two detector signals, events that are caused by terrestrial α -, β -, and γ -radiation and that deposit energy in only one of the detectors are rejected. The muon-induced background can then be suppressed either by locating the setup deep underground (Junker et al. 1998) or by surrounding the setup with a suitable active veto counter. This technique is not applicable to the study of reactions with *Q*-values of less than several MeV since in this case the emitted nuclear reaction products will have insufficient energies to penetrate even the thinnest commercially available ΔE detectors.



Fig. 4.48 Measured background (in counts per minute and per square centimeter) of a 300 µm thick silicon detector in the energy range of 50–400 keV versus mode of operation: (1) no passive or active shielding; (2) Passive shield of 2.5 cm thick lead; (3) Passive shield of 5.1 cm thick low-activity

lead; (4) Additional use of an (active) anticoincidence shield. The muon-induced background rate, given by the difference of histogram heights in mode 3 and 4, amounts to ≈ 0.8 cpm/cm² in the energy range of 50– 400 keV at sea level. Data from Walter and Boshart (1966).

4.7.3 Background in γ -Ray Detector Spectra

The background in γ -ray spectra is usually much higher compared to chargedparticle spectra for two reasons. First, γ -ray detectors have a much larger volume and second, the nature of the γ -ray background is rather complex. A typical background γ -ray spectrum, recorded without beam on target with a HPGe detector of 582 cm³ volume for about 15 h, is shown in Fig. 4.49. A large number of discrete peaks can be observed. Most of these originate from radionuclides that occur naturally in the material of the detector and the surroundings. The two most intense room background γ -ray peaks occur at 1461 and 2615 keV and originate from the decays of the radioisotopes ⁴⁰K and ²⁰⁸Tl, respectively (Table 4.6 and Fig. 4.50). The nucleus 40 K β -decays to the 1461 keV level in ⁴⁰Ar which, in turn, decays to the ground state by emission of a single photon. The nucleus 208 Tl β -decays to several 208 Pb levels with excitation energies between 3 and 4 MeV. Subsequently, these states decay through the first excited state at 2615 keV to the ground state via γ -ray cascades (that is, by emission of two or more coincident photons). Compilations of other background peaks can be found in Debertin and Helmer (1988) and Knoll (1989). The analysis of a spectrum obtained with beam on target requires a careful identification of room background peaks. Furthermore, the peaks are superimposed on a continuous background, caused by Compton scattering of room





Fig. 4.49 Background γ -ray spectrum, recorded for about 15 h without beam on target by using a HPGe detector of 582 cm³ volume. Two prominent peaks occur at 1461 and 2615 keV. They originate from the decays of the radioisotopes ⁴⁰K and ²⁰⁸Tl, respectively. All the other peaks originate from the sources shown in Fig. 4.46. Note that the γ -ray background beyond $E_{\gamma} \approx 2.6$ MeV is continuous and shows no discrete peaks.

background photons and by cosmic-ray-induced processes. In nuclear astrophysics measurements the cross sections are frequently very small and, therefore, it is ultimately this continuous background which is the major obstacle in observing the peaks from the nuclear reactions of interest.

As was the case for silicon detectors, the inherent activity of high-purity germanium is very small. However, radioimpurities in construction materials including the aluminum crystal housing, stainless steel and copper cooling rod, electrical solder in the preamplifier, and so on, may contribute to the γ -ray background. In scintillation counters, the glass envelope of the photomultiplier tube and the tube base are a potential source of background. In low-background detectors, the contribution from these sources is substantially reduced by a careful selection of the construction materials.

The vast majority of γ -rays from terrestrial or cosmogenic background sources have energies of less than 3.0 MeV, although some γ -rays with energies of up to 7 MeV are produced in the spontaneous fission of ²³⁸U. Contributions from these sources can be reduced by surrounding the detector with metal shields of high purity. Because of its high density and large atomic number, specially refined lead with a low concentration of the radioisotope ²¹⁰Pb ($T_{1/2} = 22.3$ y) is the best choice for a γ -ray shield. Iron and copper are also used but have higher cross sections for the cosmogenic production of radionuclides compared to lead. Beyond a certain optimum shielding thickness (\approx 10–15 cm for lead), the background is not reduced any further due to an in-



Fig. 4.50 Level schemes of (a) ⁴⁰Ar and (b) ²⁰⁸Pb. The decay of ⁴⁰K produces a single photon (1461 keV), while the decay of ²⁰⁸Tl gives rise to the emission of two or more coincident photons. Data from Firestone and Shirley (1996).

crease of secondary radiation caused by cosmic-ray interactions in the shield. Most γ -rays from neutron inelastic scattering and radiative neutron capture on the detector material have energies of less than about 3 MeV, although neutron capture on iron can produce γ -rays with energies of about 10 MeV. The neutron-induced component is sometimes reduced by an additional neutron absorber in the shield (for example, borated polyethylene). When the detector shield contains hydrogen (for example, concrete), neutron capture sometimes gives rise to a discrete γ -ray of 2.2 MeV energy in the spectrum.

The continuous γ -ray background caused by cosmic-ray muons arises from several different types of interactions and it is not apparent which of the processes discussed in Section 4.7.1 dominates. Monte Carlo simulations have been performed in order to investigate this issue. It is found (Vojtyla 1995) that bremsstrahlung from δ electron production dominates the background at energies below $E \approx 5$ MeV. At higher energies, the background is dominated by direct ionization events that give rise to a broad peak between 10 and 40 MeV, with the exact location depending on the crystal size. The observed broad peaks in Figs. 4.23 and 4.47a are caused by the same process.

Gamma-ray background count rates, measured by specially designed lowbackground germanium detector systems, are compared in Fig. 4.51. The horizontal axis displays the location (shielding depth) of the apparatus in units of meter water equivalent (m w.e.) and the vertical axis shows the measured total count rate in the energy region below 3 MeV in units of counts per hour per 100 cm³ detector volume. The construction materials for all of these detection systems were carefully selected in order to reduce radioimpurities. Passive shields consisted of several layers of different low-activity materials (Pb, Cu,





Fig. 4.51 Comparison of γ -ray background count rates in the energy region below E_{γ} = 3 MeV (in units of counts per hour per 100 cm³ detector volume) versus shielding depth (in units of meter water equivalent). The data indicated by the circles (passive shielding only) and triangles (passive and active shielding) are adopted from Semkow et al. (2002). All these spectrometers are specially designed for ultralow background

measurements. The two dashed lines are to guide the eye. It is obvious that the γ -ray (singles) background in this energy range can be reduced by at least two orders of magnitude by placing the apparatus deep underground. Results from $\gamma\gamma$ -coincidence measurements using conventional detectors (Rowland et al. 2002b) are shown as square (no shielding) and diamond (active shielding).

Fe, and so on). In some cases (shown as triangles) plastic scintillators or multiwire proportional chambers have been used for active cosmic-ray background discrimination. It can be seen that in laboratories located at sea level (shielding depths < 1 m w.e.) the lowest achieved background count rates amount to \approx 1000 counts/h. Note that this represents already an improvement by orders of magnitude compared to the background shown in Fig. 4.49, which was obtained using a conventional detector and setup. At moderate shielding depths of 10–15 m w.e., background count rates of \approx 100 counts/h have been measured. Another order of magnitude can be gained by locating the apparatus deep underground (shielding depths of several 1000 m w.e.), where measured background count rates amount to \approx 10 counts/h.

$\gamma\gamma$ -coincidence techniques

In many nuclear reactions of astrophysical interest, two or more photons are emitted in a cascade. In such cases, the background can be reduced substantially through the use of coincidence techniques. Consider a simple setup consisting of two γ -ray detectors. Most background events will occur in only one detector at a time and, therefore, are eliminated by demanding a coincidence between the signal outputs of both detectors.

An example for a setup is shown in Fig. 4.52a. A HPGe detector is positioned in very close geometry to the target in order to maximize the peak efficiency and a NaI(Tl) annulus surrounds both the target and the germanium crystal. Figure 4.52b shows for coincidence events the energy deposited in the HPGe detector versus the energy deposited in the NaI(Tl) detector. We will first discuss the simple case of a two- γ -ray cascade. Assume that a capture reaction populates an initial state of 9 MeV excitation energy and that this state decays through an intermediate level at 1 MeV to the ground state. The two diagonal lines correspond to a total energy deposition (in both the HPGe and NaI(Tl) detector) of 4 MeV and 9.5 MeV. Events located above the dashed line $(E_{\gamma}^{\text{Ge}} + E_{\gamma}^{\text{NaI}} > 9.5 \text{ MeV})$ can immediately be excluded from the analysis since any event that originates from the capture reaction of interest can at most have a total energy of 9 MeV (apart from a small energy spread caused by the finite detector resolutions). Such high-energy events originate, for example, from cosmic-ray muons that traverse and deposit energy in both detectors. Most room background coincidence events appear in the region below the solid line $(E_{\gamma}^{\text{Ge}} + E_{\gamma}^{\text{NaI}} < 4 \text{ MeV})$. These include 1461 keV photons from ⁴⁰K decay that deposit energy in both detectors via Compton scattering, as well as coincident γ -rays (including 2615 keV photons) from ²⁰⁸Tl decay (Fig. 4.50). By accepting only events located in the region between the solid and the dashed lines, the background is significantly reduced.

Suppose now that the primary 8 MeV γ -ray is observed in the NaI(Tl) detector, while the secondary 1 MeV photon is counted in the HPGe detector. Events corresponding to the full-energy peaks of both incident photons are located in the dark-shaded oval region of the two-dimensional energy spectrum. The oval shape is caused by the far better energy resolution of the HPGe detector. If only these events are accepted in the data analysis, then the background is indeed drastically reduced. However, at the same time the efficiency of the coincidence apparatus, given by the product of HPGe and NaI(Tl) peak efficiencies, is significantly reduced compared to the peak efficiency of a single detector. This result is undesirable in view of the very weak cross sections of astrophysical interest. The problem is solved by accepting all events located between the two diagonal lines, including those caused by Compton scattering and pair production in the NaI(Tl) detector. In this particular mode, the HPGe detector provides the spectroscopic information of interest (peak energies and intensities), while the main function of the NaI(Tl) annulus is to provide a large coincidence efficiency.

The power of the coincidence method is demonstrated in Fig. 4.53. It shows three HPGe pulse height spectra in the energy range between 0.8 and 2.5 MeV, measured at the weak $E_r = 227$ keV resonance in the ${}^{26}Mg(p,\gamma){}^{27}Al$ reaction. Each spectrum was recorded with a proton beam intensity of only 1.5 μ A for a measuring time of 10 h. The vertical dashed lines indicate the locations of

(a)





Fig. 4.52 $\gamma\gamma$ -Coincidence technique. (a) Setup consisting of a HPGe detector and a Nal(TI) annulus surrounding the target chamber. (b) Two-dimensional histogram displaying the energy deposited in the HPGe detector (horizontal axis) and the Nal(TI) detector (vertical axis). The inset shows a simple decay scheme consisting of a two-photon cascade ($E_{\gamma_1} = 9$ MeV – 1 MeV = 8 MeV; $E_{\gamma_2} = 1$ MeV). If two (or

more) of these photons are detected in coincidence, and if only those events located between the solid and dashed lines are accepted, then the environmental background is substantially reduced. The events located below the solid line are mainly caused by room background ($E_{\gamma} < 4$ MeV), while those located above the dashed line originate from cosmic-ray interactions.

expected secondary γ -ray transitions in ²⁷Al at 1014 keV and 2211 keV. The upper spectrum was obtained without detector shielding. All observed γ -ray peaks are caused by environmental background contributions. The middle spectrum was measured by shielding the detector with 5 cm thick lead. The background is reduced by about one order of magnitude, but still no peaks originating from the ²⁶Mg(p, γ)²⁷Al reaction can be identified. The lower spectrum was measured in coincidence with γ -rays observed in a NaI(Tl) annulus. The coincidence requirement was 4 MeV $\langle E_{\gamma}^{Ge} + E_{\gamma}^{NaI} \langle 9 \text{ MeV}$. No shielding was used in this case. Compared to the unshielded singles HPGe spectrum (top), the γ -ray background is reduced by more than three orders of magnitude and all the discrete peaks originating from environmental radioactivity have disappeared. The resonant γ -rays from the decays of the 1014 keV and 2211 keV levels in ²⁷Al are now clearly observed.

The remaining continuous background in the coincidence spectrum is caused by cosmic-ray-induced bremsstrahlung and by muons that directly ionize both detectors. The measured *coincidence* background count rate, shown as a square in Fig. 4.51, compares favorably with *singles* background count rates measured with specially designed low-level detection systems that are located deep underground. The coincidence background count rate



Fig. 4.53 HPGe γ -ray spectra in the energy range between 0.8 and 2.5 MeV, measured at the weak $E_r = 227$ keV resonance in the ²⁶Mg(p, γ)²⁷Al reaction (Q = 8271 keV). Each spectrum was recorded with a proton beam intensity of 1.5 µA for a running time of 10 h. The vertical dashed lines indicate the location of expected secondary γ -ray transitions in ²⁷Al at 1014 keV and 2211 keV. (a) No detector shielding; (b) passive 5 cm thick lead shielding; (c) coincidence requirement of 4 MeV $< E_{\gamma}^{\text{Ge}} + E_{\gamma}^{\text{Nal}} < 9$ MeV (but no

shielding). The background is reduced by more than three orders of magnitude and the expected secondary transitions induced by the weak ²⁶Mg(p, γ)²⁷Al resonance are clearly observed. Note that the peak close to a γ -ray energy of 2211 keV shown in parts (a) and (b) originates from room background (²¹⁴Bi) and not from decays in ²⁷Al. Reprinted from C. Rowland et al., Nucl. Instrum. Methods A, Vol. 480, p. 610 (2002). Copyright (2002), with permission from Elsevier.

can be further reduced by using a muon veto shield (for example, plastic scintillators; see Rowland et al. 2002b), as indicated by the diamond in Fig. 4.51. It is important to point out that coincidence techniques will not improve the detection sensitivity significantly if the background is mainly induced by contaminant reactions that produce two or more photons of sufficient energy in coincidence (as is the case for the troublesome ¹¹B(p, γ)¹²C reaction; Table 4.4). The need for targets and backings that are almost free of contaminants has already been stressed in Section 4.3.3.

The total number \mathcal{N} of compound nuclei created in a fusion reaction can be calculated from the peak intensity measured in a coincidence spectrum by

using an expression similar to Eq. (4.69). The factor $B\eta W$ has to be replaced by the quantity $f(B, \eta, W)$, which is a function of branching ratios, detection efficiencies, and angular correlations. Consider as an example the level diagram displayed in Fig. 4.54a, showing the decay of a compound nucleus to the ground state via several different γ -ray cascades. First, suppose that photon γ_{10} is fully absorbed in a HPGe detector and that other photons belonging to the same cascade are detected in coincidence in a NaI(TI) annulus. Neglecting angular correlation effects (that is, assuming W = 1), the contribution of the two- γ -ray cascade (a) to the peak intensity of γ_{10} in the HPGe coincidence spectrum is given by

$$\mathcal{N}_{10}^{\text{Ge},C,a} = \mathcal{N}B_{31}B_{10}\eta_{10}^{\text{Ge},P}\eta_{31}^{\text{NaI},T} = \mathcal{N}B_{31}B_{10}\eta_{10}^{\text{Ge},P}\left[1 - \left(1 - \eta_{31}^{\text{NaI},T}\right)\right] \quad (4.81)$$

where the product of branching ratios $B_{31}B_{10}$ represents the probability that the compound nucleus decays via the cascade $3 \rightarrow 1 \rightarrow 0$; $\eta_{31}^{\text{NaI},T}$ is the *total* NaI(Tl) efficiency for a particular energy range selected by the gate in the two-dimensional spectrum of E_{γ}^{Ge} versus E_{γ}^{NaI} . Similarly, we obtain for the contribution of the three- γ -ray cascade (b)

$$\mathcal{N}_{10}^{\text{Ge},C,b} = \mathcal{N}B_{32}B_{21}B_{10}\eta_{10}^{\text{Ge},P} \left[\eta_{32}^{\text{NaI},T} \left(1 - \eta_{21}^{\text{NaI},T}\right) + \eta_{21}^{\text{NaI},T} \left(1 - \eta_{32}^{\text{NaI},T}\right) + \eta_{32}^{\text{NaI},T}\eta_{21}^{\text{NaI},T}\right]$$
$$= \mathcal{N}B_{32}B_{21}B_{10}\eta_{10}^{\text{Ge},P} \left[1 - \left(1 - \eta_{32}^{\text{NaI},T}\right) \left(1 - \eta_{21}^{\text{NaI},T}\right)\right]$$
(4.82)

The term $\eta_{32}^{\text{NaI},T}(1-\eta_{21}^{\text{NaI},T})$ corresponds to the probability that photon γ_{32} is observed in the NaI(Tl) annulus with an energy consistent with the selected coincidence gate and, at the same time, photon γ_{21} escapes detection in the NaI(Tl) annulus. The term $(1-\eta_{32}^{\text{NaI},T})(1-\eta_{21}^{\text{NaI},T})$ is equal to the probability that neither photon γ_{32} nor γ_{21} is detected in the NaI(Tl) annulus. Equivalently, the term $1-(1-\eta_{32}^{\text{NaI},T})(1-\eta_{21}^{\text{NaI},T})$ corresponds to the total probability of detecting photon γ_{32} or γ_{21} . Cascade (c) does not contribute to the peak intensity of γ_{10} .

The above expressions can be generalized (Fig. 4.54b) to find the total fullenergy peak intensity of photon γ_{ij} in the HPGe coincidence spectrum,

$$\mathcal{N}_{ij}^{\text{Ge},C} = \mathcal{N}\eta_{ij}^{\text{Ge},P} \sum_{k} \left\{ \left[\prod_{i'>j'} B_{i'j',k} \right] \left[1 - \prod_{\substack{i'>j'\\i\neq j}} \left(1 - \eta_{i'j',k}^{\text{NaI},T} \right) \right] \right\}$$
(4.83)

where the sum is over all cascades *k* containing the transition $i \rightarrow j$. The first product is over the branching ratios of all transitions in cascade *k* and represents the probability that the initial level will decay via this particular

cascade. The second product is over the total NaI(Tl) detector efficiencies for all transitions in cascade *k* except the branch $i \rightarrow j$ which is observed in the HPGe detector. Hence, we obtain for the total number \mathcal{N} of compound nuclei created in the fusion reaction

$$\mathcal{N} = \frac{\mathcal{N}_{ij}^{\text{Ge,P}} f(B_{i'j',k}, \eta_{i'j',k}^{\text{NaL},T})}{\eta_{ij}^{\text{Ge,P}} \sum_{k} \left\{ \left[\prod_{i'>j'} B_{i'j',k} \right] \left[1 - \prod_{\substack{i'>j'\\i \neq j}} \left(1 - \eta_{i'j',k}^{\text{NaL},T} \right) \right] \right\}$$
(4.84)

The total NaI(Tl) efficiency for a particular energy range selected by a gate in the two-dimensional spectrum of E_{γ}^{Ge} versus E_{γ}^{NaI} , $\eta_{i'j',k}^{\text{NaI},T}$, can be obtained in the following way. As an example, consider again a two- γ -ray cascade $3 \rightarrow 1 \rightarrow 0$ (Fig. 4.54a). The intensities of photons γ_{10} and γ_{31} observed in the singles and coincidence HPGe spectrum are given by expressions similar to Eqs. (4.53) and (4.81). Neglecting coincidence summing corrections, we find

$$\mathcal{N}_{10}^{\text{Ge}} = \mathcal{N}B_{31}B_{10}\eta_{10}^{\text{Ge},P}, \qquad \mathcal{N}_{31}^{\text{Ge}} = \mathcal{N}B_{31}\eta_{31}^{\text{Ge},P} \mathcal{N}_{10}^{\text{Ge},C} = \mathcal{N}B_{31}B_{10}\eta_{10}^{\text{Ge},P}\eta_{31}^{\text{NaI},T}, \qquad \mathcal{N}_{31}^{\text{Ge},C} = \mathcal{N}B_{31}B_{10}\eta_{31}^{\text{Ge},P}\eta_{10}^{\text{NaI},T}$$
(4.85)

and thus

$$\eta_{31}^{\text{NaI},T} = \frac{\mathcal{N}_{10}^{\text{Ge},C}}{\mathcal{N}_{10}^{\text{Ge}}} \qquad \text{and} \qquad \eta_{10}^{\text{NaI},T} = \frac{\mathcal{N}_{31}^{\text{Ge},C}}{\mathcal{N}_{31}^{\text{Ge}}} \qquad (4.86)$$

In the derivation of Eq. (4.86) we explicitly assumed that cascades consisting of three or more γ -rays do not contribute to the measured intensities in the HPGe coincidence spectrum (that is, $B_{31} = 1$ and $B_{10} = 1$). It can be seen that the above relations provide absolute *total* NaI(Tl) detector efficiencies without using the activity of calibrated radioactive sources or the cross section of capture reactions. If the γ -ray source or the target is located very close to both detectors, then Eq. (4.86) may become inaccurate due to coincidence summing (Section 4.5.2). In this case, a Monte Carlo simulation of the detection setup must be performed.

4.7.4

Background in Neutron Detector Spectra

The neutron background originates from (α ,n) reactions induced by terrestrial α -particle emitters, spontaneous fission of ²³⁸U, and cosmic-ray-induced processes. Different principles apply to the shielding of neutrons compared to the



Fig. 4.54 (a) Level scheme showing three different γ -ray decays from excited level 3 to the ground state 0. It is assumed that the transition $1 \rightarrow 0$ is observed in a HPGe detector and that other photons of the same cascade are detected in coincidence in a NaI(TI) annulus. (b) General case; the transition $i \rightarrow j$ is observed in a HPGe detector, while coincident photons are detected in a NaI(TI) detector. See the text.

shielding of charged particles or γ -rays. Neutrons need to be quickly moderated and absorbed in a medium of a high absorption cross section. The most effective moderators consist of light nuclei and contain preferably hydrogen (Section 4.2.3). Frequently used materials include paraffin, polyethylene, or water. Mean free paths of fast neutrons amount typically to several tens of centimeter and, therefore, thicknesses of about 1 m are required to moderate fast neutrons effectively. After moderation, the neutrons have to be captured. Since the capture cross section of hydrogen is relatively low, another component with a large neutron absorption cross section (for example, ¹⁰B, ⁶Li, or Cd) is added, either as a homogeneous mixture with the moderator or as an absorbing layer between moderator and detector. The (n,α) reaction on ¹⁰B and the (n, γ) reaction on Cd produce secondary γ -rays, while the (n, α) reaction on ⁶Li proceeds directly to the ground state. Thus, ⁶Li is preferred as neutron absorption material in applications that are sensitive to γ -ray background. The importance of other background sources depends on the type and the constructional details of the neutron detector. For example, for ¹⁰BF₃ or ³He proportional counters (Section 4.5.3) the intrinsic α -radioactivity of the detector itself may cause a significant background counting rate if the construction materials have not been selected carefully.



Fig. 4.55 Neutron fluxes (number of neutrons per second per square centimeter) from various sources (solid lines) versus shielding depth (in units of meter water equivalent). The muon flux is shown as a dashed line. Secondary neutrons are those that are produced by primary cosmic rays. The solid line labeled "Tertiary neutrons"

represents the flux of neutrons produced by muons in a typical lead shield; the label "S.F." refers to neutrons from spontaneous fission. From Heusser (1995). Reprinted with permission from the Annual Reviews of Nuclear and Particle Science, Volume 45. Copyright ©1995 by Annual Reviews.

Figure 4.55 shows the neutron flux (number of neutrons per second per square centimeter) from various sources, together with the muon flux, versus shielding depth in meter water equivalent. Without any shielding, the contribution from secondary neutrons dominates the background. This intensity decreases rapidly with increasing shielding depth and becomes smaller than the neutron intensity from (α ,n) reactions and spontaneous fission beyond a moderate shielding depth of ≈ 10 m w.e. The figure also shows for comparison the intensity of tertiary neutrons produced by muons in a typical lead shield. This background contribution clearly dominates the neutron flux at shielding depths of 2–100 m w.e. Consequently, the use of massive shields should be avoided in this shielding depth region, if possible.

The actual background count rate measured by a neutron detector depends strongly on the location, the measuring geometry, and constructional details. Thus, one has to be very careful when comparing experimental background rates measured by different detectors. Nevertheless, such a comparison is interesting since relatively little information on this subject is given in the literature. Table 4.11 compares background count rates measured in four different studies. Other parameters such as efficiency, location, and the type of shielding are also given. In each case, the setup consisted of ³He proportional counters moderated by either polyethylene or paraffin. At sea level and without active shielding, the measured background count rate is ≈ 10 cpm. With active shielding, the background is substantially suppressed and amounts to

 \approx 2 cpm. It is likely that a further reduction can be achieved by locating the setup underground, although the results listed in the table are inconclusive if adjusted for differences in total efficiencies or the size of the analyzed region in the spectrum.

Finally, a word may be added regarding the suppression of muon-induced neutrons via anticoincidence techniques. With polyethylene or paraffin as moderator, it takes on average $\approx 200 \ \mu s$ for a fast neutron to be thermalized and captured in the neutron detector. Thus, if the anticoincidence counter indicates a muon hit, the signals from the neutron detector have to be vetoed for several 100 μs in order to suppress such events effectively.

Tab. 4.11 Comparison of low-background neutron spectrometers. (a) Full detector consists of seven ¹⁰BF₃ and two ³He counters; the ¹⁰BF₃ counters are not listed here for comparison because of their much higher background count rate. (b) Result of Monte Carlo simulation. (c) At Gran Sasso underground laboratory. (d) On a conventionally powered submarine. (e) Plastic scintillator. (f) Total counts above discriminator threshold. (g) Counts in a region of the spectrum representing 95% of thermal neutron peak intensity. (h) Counts in a region of the spectrum representing 70% of thermal neutron peak intensity.

Reference	Giesen et al.	Wang, Vogelaar and	Stella et al.	Mayer et al.
	(1993)	Kavanagh (1991)	(1995)	(1993)
Detector	³Не	³Не	³Не	³Не
Moderator	Polyethylene	Polyethylene	Polyethylene	Paraffin
Number of counters	31	12	2 ^a	18
Efficiency (%)	20	22	6 ^a	22
Calibration source	Am-Li	²⁵² Cf	M.C. ^b	^{nat} UO
Location (m w.e.)	Sea level	Sea level	3950°	50 ^d
Passive shield	Yes	Yes	Yes	yes
Active shield	No	Yes ^e	No	No
Background (cpm)	11 ^f	2 ^f	0.6 ^f (0.06 ^g)	0.06 ^h

4.8

Yields and Cross Sections for Charged-Particle-Induced Reactions

The calculation of thermonuclear reaction rates requires knowledge of the nuclear reaction cross section. However, what is usually determined experimentally is the total number of nuclear reactions that occurred and the total number of incident beam particles. The ratio of these two quantities,

$$Y \equiv \frac{\text{total number of nuclear reactions}}{\text{total number of incident beam particles}} = \frac{N_R}{N_b}$$
(4.87)

is called the *yield* of the reaction. Comparison to Eq. (2.1) shows that the yield is related, but not equal, to the cross section σ . In this section, we will derive relationships for these two quantities. We will also discuss how to derive resonance strengths (that is, integrated cross sections) from measured yields.

A function of yield versus bombarding energy is referred to as *yield curve* or *excitation function*.

The following definitions will be used for the stopping power (in units of $eV \text{ cm}^2/\text{atom}$),

$$\varepsilon(E) \equiv S_A(E) = -\frac{1}{N} \frac{dE}{dx}$$
(4.88)

and for the concentration of target nuclei,

$$n \equiv Nd = \frac{\mathcal{N}_t}{A} \tag{4.89}$$

The quantities N_t and N denote the total number of target nuclei and the number density of target nuclei (atoms per unit volume), respectively (Section 4.2.1); d is the target thickness (in units of length). Hence, n is the number of target nuclei per unit area. All quantities in this section will be expressed in the center-of-mass system, unless mentioned otherwise.

4.8.1

Nonresonant and Resonant Yields

Suppose that a beam of energy E_0 is incident on a target. The target can be divided into a number of slices of thickness Δx_i , and it can be assumed that the energy lost by the beam in each slice, ΔE_i , is small. In other words, both the cross section, σ_i , and the stopping power, ε_i , are constant over Δx_i . With Eqs. (2.1), (4.87), and (4.89) we obtain for the yield from a particular slice in the target

$$\Delta Y_i = \frac{\mathcal{N}_{R,i}}{\mathcal{N}_b} = \sigma_i \frac{\mathcal{N}_{t,i}}{A} = \sigma_i N_i \Delta x_i \tag{4.90}$$

The total yield is given by integrating over all target slices,

$$Y(E_0) = \int \sigma(x)N(x) \, dx = \int \sigma(x)N(x) \, dx \frac{dE(x)}{dx} \frac{dx}{dE(x)}$$
$$= \int_{E_0 - \Delta E}^{E_0} \frac{\sigma(E)}{\varepsilon(E)} \, dE$$
(4.91)

The quantity ΔE is the total energy lost by the beam in the target (that is, the target thickness in energy units). Note that the above expression neglects the effects of beam resolution and straggling which will be discussed later. The cross section $\sigma = \sigma(E)$ can always be found from the measured yield by solving Eq. (4.91) numerically. In special cases, which occur frequently in practice, the above integral can be evaluated analytically. Such cases will be addressed below.

Constant σ and ε over target thickness

Suppose that the cross section is approximately constant over the target thickness. This may be the case, for example, if the reaction proceeds via a non-resonant mechanism or a broad resonance. Furthermore, we assume that the energy lost by the beam in the target is small so that the stopping power is nearly constant as well. The situation is displayed as case (a) in Fig. 4.56. The yield follows directly from Eqs. (4.14) and (4.91),

$$Y(E_0) = \frac{\sigma(E_{\text{eff}})}{\varepsilon(E_0)} \int_{E_0 - \Delta E}^{E_0} dE = \frac{\Delta E(E_0)}{\varepsilon(E_0)} \sigma(E_{\text{eff}}) = n\sigma(E_{\text{eff}})$$
(4.92)

The mean effective energy in the target is $E_{\rm eff} = E_0 - \Delta E/2$ and, therefore, we may assign this energy to the cross section obtained from Eq. (4.92). Note that the above assumption of a constant cross section implies that reactions will occur over the entire thickness of the target. Furthermore, the shapes of the measured yield curve and the cross section will be similar. The situation is schematically shown as case (a) in Fig. 4.56. The above expression applies to total cross sections σ and total yields Y. For differential cross sections, $(d\sigma/d\Omega)_{\theta}$, and differential yields, $(dY/d\Omega)_{\theta}$, one finds

$$\left[\frac{dY(E_0)}{d\Omega}\right]_{\theta} = \frac{\Delta E(E_0)}{\varepsilon(E_0)} \left[\frac{d\sigma(E_{\text{eff}})}{d\Omega}\right]_{\theta} = n \left[\frac{d\sigma(E_{\text{eff}})}{d\Omega}\right]_{\theta}$$
(4.93)

We assumed so far that the target consists of a pure element. If instead the target consists of a compound $X_a Y_b$ with n_X active nuclei per square centimeter (the target nuclei of interest) and n_Y inactive nuclei per square centimeter (nuclei that do not participate in the reaction of interest), then we obtain from Eqs. (4.17), (4.88), and (4.89)

$$\frac{\Delta E_c}{n_X} = \frac{\varepsilon_X n_X + \varepsilon_Y n_Y}{n_X} = \varepsilon_X + \frac{n_Y}{n_X} \varepsilon_Y \equiv \varepsilon_{\text{eff}}$$
(4.94)

where $n_Y/n_X = b/a$. The quantity ε_{eff} , which is different from the total stopping power for a compound (see Eq. (4.15)), is referred to as the *effective stopping power*. The total and differential yields are given by Eqs. (4.92)–(4.94) as

$$Y(E_0) = n_X \sigma(E_{\text{eff}}) = \frac{\Delta E_c(E_0)}{\varepsilon_{\text{eff}}(E_0)} \sigma(E_{\text{eff}})$$
(4.95)

$$\left[\frac{dY(E_0)}{d\Omega}\right]_{\theta} = n_X \left[\frac{d\sigma(E_{\text{eff}})}{d\Omega}\right]_{\theta} = \frac{\Delta E_c(E_0)}{\varepsilon_{\text{eff}}(E_0)} \left[\frac{d\sigma(E_{\text{eff}})}{d\Omega}\right]_{\theta}$$
(4.96)

As long as the stopping power is constant over the target thickness, it follows that the yield is calculated in the same way as for a pure target (see Eqs. (4.92) and (4.93)), except that the stopping power is replaced by ε_{eff} . Similar arguments apply to yield expressions obtained for other assumptions (for example, resonances) as will be seen below.

Example 4.3

Suppose that a beam of singly charged protons with a laboratory energy of 200 keV and 1 µA intensity is incident on a 5 keV thick (in the laboratory system) natural carbon target for a period of 1 h. Calculate the total number of photons originating from the ¹³C(p, γ)¹⁴N capture reaction, assuming that one photon is emitted per reaction. Assume further that both the cross section and the stopping power are approximately constant over the target thickness. The cross section amounts to $\sigma_{13C(p,\gamma)}(E_{lab} = 200 \text{ keV}) = 10^{-7} \text{ b}$ and the stopping power of protons in carbon, calculated by using the computer code SRIM (Ziegler 2003), is given by $\varepsilon_{p\rightarrow C}(E_{lab} = 200 \text{ keV}) = 11.8 \times 10^{-15} \text{ eV cm}^2/\text{ atom}$.

The target consists of active ¹³C (1.1%) and inactive ¹²C (98.9%) nuclei. If we assume that the stopping power of hydrogen in ¹²C and ¹³C is the same, we obtain for the effective stopping power (see Eq. (4.94))

$$\begin{split} \varepsilon_{\text{eff}} &= \varepsilon_{\text{p}\to^{13}\text{C}} + \frac{98.9}{1.1} \varepsilon_{\text{p}\to^{12}\text{C}} = \varepsilon_{\text{p}\to\text{C}} \left(1 + \frac{98.9}{1.1} \right) \\ &= (11.8 \times 10^{-15} \,\text{eV}\,\text{cm}^2/\text{atom}) \left(1 + \frac{98.9}{1.1} \right) = 1.0 \times 10^{-12} \,\text{eV}\,\text{cm}^2/\text{atom} \end{split}$$

The yield is then given by

$$Y = \frac{\Delta E_c}{\varepsilon_{\text{eff}}} \sigma = \frac{5 \times 10^3 \,\text{eV}}{1.0 \times 10^{-12} \,\text{eV} \,\text{cm}^2/\text{atom}} (10^{-7} \cdot 10^{-24} \,\text{cm}^2)$$
$$= 5.0 \times 10^{-16} = \frac{N_{\gamma}}{N_{\text{p}}}$$

The total number of incident protons can be calculated from the total accumulated charge Q and the elementary charge e (Section 4.3.4). The value of Q is given by the beam intensity and the measuring time,

$$\mathcal{N}_{\rm p} = \frac{Q}{e} = \frac{It}{e} = \frac{(1 \times 10^{-6} \,\mathrm{A})(3600 \,\mathrm{s})}{1.6 \times 10^{-19} \,\mathrm{C}} = 2.25 \times 10^{16}$$

Thus, for a measuring time of 1 h, we obtain for the number of emitted photons at a laboratory bombarding energy of $E_{lab} = 200 \text{ keV}$

$$\mathcal{N}_{\gamma} = Y \mathcal{N}_{p} = (5.0 \times 10^{-16}) (2.25 \times 10^{16}) \approx 11$$

Note that in this example the yield is obtained from the ratio of the two quantities ΔE_c and ε_{eff} that are both given in the laboratory system. Obviously, the ratio $\Delta E_c / \varepsilon_{\text{eff}} = n_X$ is independent of the reference frame. Multiplication of

the numerator and denominator by the center of mass to the laboratory frame conversion factor $M_X/(M_X + M_p)$ (see Eq. (C.24); M_X and M_p are the relative atomic masses of the active target nuclei and the projectiles, respectively) shows that the tabulated effective stopping power has to be multiplied by this factor if the yield is calculated from the target thickness in the center-of-mass system.

Moderately varying σ and constant ε over target thickness

If the stopping power is constant, but the cross section varies over the target thickness, then the yield is given by (see Eq. (4.91))

$$Y(E_0) = \frac{1}{\varepsilon(E_0)} \int_{E_0 - \Delta E}^{E_0} \sigma(E) \, dE$$
(4.97)

We will assume that the cross section does not vary drastically, that is, we exclude narrow-resonance cross sections which will be discussed later. The situation is shown as case (b) in Fig. 4.56. The above integral can be replaced by the product $\sigma(E_{\text{eff}})\Delta E(E_0)$ and we obtain again the expression (see Eq. (4.92))

$$Y(E_0) = \frac{\Delta E(E_0)}{\varepsilon(E_0)} \sigma(E_{\text{eff}})$$
(4.98)

The quantity E_{eff} is now the effective beam energy between $E_0 - \Delta E$ and E_0 at which 50% of the total yield is obtained (that is, the energy which divides the shaded region under the cross section curve into two equal areas). In general, the effective beam energy must be obtained numerically, but in special cases analytical expressions may be used. For example, if the cross section varies linearly between $\sigma_1 = \sigma(E_0)$ and $\sigma_2 = \sigma(E_0 - \Delta E)$, then the effective beam energy is given by (Rolfs and Rodney 1988)

$$E_{\rm eff} = E_0 - \Delta E + \Delta E \left[-\frac{\sigma_2}{\sigma_1 - \sigma_2} + \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2(\sigma_1 - \sigma_2)^2}} \right]$$
(4.99)

As was the case before, reactions occur over the entire thickness of the target, but the number of reaction products emitted from different target depths is no longer constant.

Breit–Wigner resonance σ and constant ε over resonance width

Suppose that a resonant cross section is given by the Breit–Wigner formula (see Eq. (2.185)). It is also assumed that the stopping power ε , the Broglie wavelength λ , and the partial widths Γ_i of the resonance are independent of



Fig. 4.56 Yield for a (a) constant cross section, (b) weakly energy-dependent cross section. The yield is given by the area under the cross section curve (shaded areas in the top part). The integration is performed from the bombarding energy E_0 to an energy of $E_0 - \Delta E$, with ΔE the target thickness in en-

r

ergy units. In situation (a), the shape of the yield and cross section curves are identical. In situation (b), the energy which divides the shaded region under the cross section curve into two equal areas corresponds to the effective beam energy.

energy over the resonance width. Hence, these quantities can be evaluated at the resonance energy E_r . With the substitutions $\omega \equiv (2J + 1)(1 + \delta_{01})/[(2j_0 + 1)(2j_1 + 1)]$ and $\omega \gamma \equiv \omega \Gamma_a \Gamma_b / \Gamma$ (Section 3.2.4), we obtain from Eqs. (3.110) and (4.91) (Fowler, Lauritsen and Lauritsen 1948)

$$Y(E_0) = \int_{E_0 - \Delta E}^{E_0} \frac{1}{\varepsilon(E)} \frac{\lambda^2}{4\pi} \omega \frac{\Gamma_a \Gamma_b}{(E_r - E)^2 + \Gamma^2 / 4} dE$$
$$= \frac{\lambda_r^2}{2\pi} \frac{\omega \gamma}{\varepsilon_r} \frac{\Gamma}{2} \int_{E_0 - \Delta E}^{E_0} \frac{dE}{(E_r - E)^2 + (\Gamma/2)^2}$$
$$= \frac{\lambda_r^2}{2\pi} \frac{\omega \gamma}{\varepsilon_r} \left[\arctan\left(\frac{E_0 - E_r}{\Gamma/2}\right) - \arctan\left(\frac{E_0 - E_r - \Delta E}{\Gamma/2}\right) \right]$$
(4.100)

where λ_r and ε_r denote the de Broglie wavelength and the stopping power at the resonance energy E_r , respectively. With the expressions $\tan(x - y) = [\tan(x) - \tan(y)]/[1 + \tan(x)\tan(y)]$ and $d(\arctan x)/dx = 1/(1 + x^2)$ one

finds after some algebra

$$E_{0,\max} = E_r + \frac{\Delta E}{2} \tag{4.101}$$

$$Y_{\max} = Y(E_{0,\max}) = \frac{\lambda_r^2}{\pi} \frac{\omega\gamma}{\varepsilon_r} \arctan\left(\frac{\Delta E}{\Gamma}\right)$$
(4.102)

$$E_{0,50\%} = E_r + \frac{\Delta E}{2} \pm \frac{1}{2}\sqrt{\Gamma^2 + \Delta E^2}$$
(4.103)

$$FWHM = \sqrt{\Gamma^2 + \Delta E^2} \tag{4.104}$$

with $E_{0,\text{max}}$, Y_{max} , $E_{0,50\%}$, and FWHM the location of the maximum, the maximum yield, the energies corresponding to one half of the maximum yield, and the FWHM of the resonance yield curve, respectively. For the de Broglie wavelength (in the center-of-mass system) we find numerically

$$\frac{\lambda_r^2}{2} = 2\pi^2 \frac{\hbar^2}{2m_{01}E_r} = \left(\frac{M_0 + M_1}{M_1}\right)^2 \frac{4.125 \times 10^{-18}}{M_0 E_r^{\text{lab}}} \qquad (\text{cm}^2) \tag{4.105}$$

with m_{01} , M_0 , and M_1 the reduced mass of the projectile–target system, the projectile mass (in u), and the target mass (in u), respectively; E_r^{lab} is the laboratory resonance energy in units of eV.

These results are illustrated in Fig. 4.57, showing a Breit-Wigner cross section (part a) and the corresponding yield (part b) of a resonance at $E_r = 500$ keV with a total width of Γ = 15 keV for different values of the target thickness ΔE . If the target thickness is much smaller compared to the resonance width, $\Delta E \ll \Gamma$, then the shape of the yield curve corresponds to the shape of the cross section (that is, a Lorentzian shape). The maximum yield is located close to the resonance energy, $E_{0,\max} \approx E_r$, and the width of the yield curve is given by the resonance width, FWHM $\approx \Gamma$. For example, at a bombarding energy of $E_0 = 495$ keV and a target thickness of $\Delta E = 5$ keV (shaded area on the lefthand side in part a) the target integrates only over a small region of the cross section (that is, between 490 and 495 keV). The resulting yield is shown as the open circle at 495 keV in part (b). In this case, reactions occur over the entire thickness of the target. On the other hand, if the target thickness is much larger compared to the total resonance width, $\Delta E \gg \Gamma$, then the shape of the yield curve is determined by the arctan function. The yield curve shows a flat plateau with a maximum located at $E_{0,max} = E_r + \Delta E/2$ and the width is FWHM $\approx \Delta E$. For example, at $E_0 = 550$ keV and $\Delta E = 50$ keV (shaded region on right-hand side in part a) the target integrates over almost half of the entire cross section curve (that is, between 500 and 550 keV). The resulting yield, shown as the open circle at 550 keV in part (b), represents then about 50% of the maximum yield at the plateau height. In this case, the number of reactions varies strongly over the thickness of the target.



Fig. 4.57 (a) Cross section and (b) yield curve for a Breit–Wigner resonance with energy-independent partial widths. The resonance is located at E = 500 keV and has a width of $\Gamma = 15$ keV. The yield depends strongly on the bombarding energy and the target thickness. The shaded areas (part

a) and corresponding open circles (part b) depict the situation for two different sets of conditions ($E_0 = 495 \text{ keV}$, $\Delta E = 5 \text{ keV}$ on the left-hand side and $E_0 = 550 \text{ keV}$, $\Delta E = 50 \text{ keV}$ on the right-hand side). The symbol ∞ in part (b) labels the resonant yield for an infinitely thick target.

For an increasing target thickness ΔE , both the maximum yield Y_{max} and the width of the yield curve will increase since the target integrates the cross section over a larger energy region. In the limit of an infinitely thick target, $\Delta E \rightarrow \infty$, the yield in Eq. (4.100) becomes

$$Y_{\Delta E \to \infty}(E_0) = \frac{\lambda_r^2}{2\pi} \frac{\omega \gamma}{\varepsilon_r} \left[\arctan\left(\frac{E_0 - E_r}{\Gamma/2}\right) + \frac{\pi}{2} \right]$$
(4.106)

and Eqs. (4.102) and (4.103) give

$$Y_{\max,\Delta E \to \infty} = \frac{\lambda_r^2}{2} \frac{\omega \gamma}{\varepsilon_r}$$
(4.107)

$$E_{0,50\%,\Delta E \to \infty} = E_r \tag{4.108}$$

The yield $Y_{\Delta E \to \infty}$ for a resonance at $E_0 = 500$ keV with a total width of $\Gamma = 15$ keV is also shown in Fig. 4.57b. The difference between the energies at which the yield for an infinitely thick target is at 75% and 25% of its maximum



Fig. 4.58 Ratios $Y_{max}/Y_{max,\Delta E \to \infty}$, FWHM/ ΔE and $(E_r - E_{0,50\%})/\Gamma$ as a function of $\Delta E/\Gamma$ for a Breit–Wigner resonance with energy-independent partial widths.

value is equal to the total resonance width, that is,

$$E_{0,75\%,\Delta E \to \infty} - E_{0,25\%,\Delta E \to \infty} = \Gamma$$
(4.109)

It is interesting to investigate the ratios $Y_{\max}/Y_{\max,\Delta E \to \infty}$, FWHM/ ΔE , and $(E_r - E_{0,50\%})/\Gamma$ as a function of $\Delta E/\Gamma$. The results are shown in Fig. 4.58. Obviously, the thicker the target, the closer the yield resembles that of an infinitely thick target. For example, suppose that the target thickness is ten times larger than the total resonance width ($\Delta E/\Gamma = 10$). The maximum yield at the plateau is then 94% of the yield for an infinitely thick target and the FWHM is equal to the target thickness within 0.5%. Furthermore, the difference of energies at which the yield is at 50% of its maximum and the resonance energy amounts to 0.025\Gamma (see Eq. (4.103)). This deviation amounts only to 0.37 keV for a total resonance width of $\Gamma = 15$ keV.

4.8.2

General Treatment of Yield Curves

We neglected so far the influence of certain experimental factors on the measured yield. These include the finite beam energy resolution, beam straggling in the target, target nonuniformities, and the thermal motion of target atoms. In order to account for such effects, we have to replace Eq. (4.91) by the general expression (Gove 1959)

$$Y(E_0) = \int_{E_0 - \Delta E}^{E_0} dE' \int_{E_i = 0}^{\infty} dE_i \int_{E = 0}^{E_i} \frac{\sigma(E)}{\varepsilon(E)} g(E_0, E_i) f(E_i, E, E') dE$$
(4.110)

where $g(E_0, E_i) dE_i$ is the probability that a particle in the incident beam of mean energy E_0 has an energy between E_i and $E_i + dE_i$; $f(E_i, E, E') dE$ is the probability that a particle incident on the target at an energy E_i has an energy between E and E + dE at a depth inside the target corresponding to the energy E' (that is, $E_0 - \Delta E < E' < E_0$). The functions $g(E_0, E_i)$ and $f(E_i, E, E')$ are assumed to be normalized. The cross section $\sigma(E)$ can be found numerically from the measured yield $Y(E_0)$ by using deconvolution procedures (see, for example, McGlone and Johnson 1991).

For a constant cross section and stopping power, the triple integral reduces to our earlier result (see Eq. (4.92)) as can be seen by using the normalizations of *g* and *f*, and by carrying out the integrations in the order *E*, E_i , and E'. It follows that the yield for nonresonant cross sections ($\sigma \approx \text{const}$) and thin targets ($\epsilon \approx \text{const}$) is not affected by the beam resolution and beam straggling. In other words, all projectiles in the beam can in principle contribute to the yield.

In the following, resonance yield curves will be discussed in more detail. We will make a few assumptions that apply frequently in practice: (i) the energy distribution of particles in the beam is a function of $E_0 - E_i$ only, $g(E_0, E_i) = g(E_0 - E_i)$. That is, the beam spread is independent of the mean energy E_0 ; (ii) the distribution describing energy loss and straggling is a function of $E_i - E$ and E' only, $f(E_i, E, E') = f(E_i - E, E')$. That is, the spread in f is independent of the energy E_i ; (iii) the functions g, f, and σ vanish on both sides of their maximum values; (iv) the stopping power is constant over the total width of the resonance and the total width of the target, $\varepsilon(E) = \varepsilon_r$. The latter condition implies that the target is uniform. Otherwise, if the target consists of a compound with changing stoichiometry, the energy and depth dependence of the effective stopping power, $\varepsilon_{\text{eff}}(E)$, has to be taken explicitly into account. With the above assumptions, Eq. (4.110) becomes

$$Y(E_0) = \frac{1}{\varepsilon_r} \int_{E_0 - \Delta E}^{E_0} dE' \int_{E_i = 0}^{\infty} dE_i \int_{E = 0}^{E_i} \sigma(E) g(E_0 - E_i) f(E_i - E, E') dE$$
(4.111)

A typical situation is represented in Fig. 4.59. A beam of initial mean energy E_0 , with an energy distribution given by $g(E_0 - E_i)$, traverses a target of thickness ΔE . The energies E and E' represent the projectile energy at a fixed depth x in the target and the mean energy of the beam, respectively. The vertical thick line indicates the position of a narrow resonance with $E_r < E_0$. At position (a) near the target surface, all projectile energies are too large to excite the resonance and the yield will be negligible. At position (b) inside the target, the beam has slowed down so that the maximum of the projectile energy distribution coincides with the resonance energy and, consequently, the largest contribution to the yield arises from this depth in the target. At position (c) near the back side of the target, most projectiles have slowed to energies be-



Fig. 4.59 Slowing down process of a beam with initial mean energy E_0 and an energy distribution given by $g(E_0 - E_i)$, traversing a target of thickness ΔE . The energies *E* and *E'* represent the projectile energy at a fixed depth *x* in the target and the mean energy of the beam, respectively; *E'* decreases as the target is traversed. The *z*-axis rep-

resents the magnitude of the probability distribution f. The vertical thick line indicates the position of a narrow resonance with $E_r < E_0$. Positions (a), (b), and (c) indicate different depths within the target. The largest contribution to the resonance yield arises from position (b).

low E_r . Only a few projectiles on the high-energy tail of the distribution f can excite the resonance. The contribution of this target depth to the yield is larger than for position (a), but less than for position (b).

Target of infinite thickness

For an infinitely thick target, $\Delta E \rightarrow \infty$, the lower integration limit of E' in Eq. (4.111) is zero. The shape of the yield curve can be obtained using the normalization of the function f since the probability of finding a projectile that experienced a specific energy loss of $E_i - E$ anywhere in the target is unity. Thus

$$Y_{\Delta E \to \infty}(E_0) = \frac{1}{\varepsilon_r} \int_{E'=0}^{E_0} f(E_i - E, E') dE' \int_{E_i=0}^{\infty} dE_i \int_{E=0}^{E_i} \sigma(E) g(E_0 - E_i) dE$$

= $\frac{1}{\varepsilon_r} \int_{E_i=0}^{\infty} g(E_0 - E_i) dE_i \int_{E=0}^{E_i} \sigma(E) dE$ (4.112)

The yield depends on both the cross section (for example, the total resonance width) and the beam spread, but is independent of beam straggling. The maximum yield for an infinitely thick target can be obtained in the limit $E_0 \rightarrow \infty$. In this case the only contribution from the integration over E_i results from

 $E_i \rightarrow \infty$. Using the normalization of the distribution *g* we find

$$Y_{\max,\Delta E \to \infty} = \frac{1}{\varepsilon_r} \int_{E=0}^{\infty} \sigma(E) \, dE \tag{4.113}$$

It follows that the maximum yield for an infinitely thick target is not affected by the beam resolution, beam straggling, or the total resonance width. The value of $Y_{\max,\Delta E\to\infty}$ depends only on the stopping power and the integrated cross section. If the target consists of a compound, then ε_r has to be replaced by $\varepsilon_{\text{eff},r}$ (see Eq. (4.94)) and, consequently, $Y_{\max,\Delta E\to\infty}$ depends on the stoichiometry of the target compound. If the cross section is given by the Breit– Wigner formula with constant partial widths and de Broglie wavelength over the width of the resonance, the integration over σ yields

$$Y_{\max,\Delta E \to \infty} = \frac{1}{\varepsilon_r} \frac{\lambda_r^2}{2} \omega \gamma \tag{4.114}$$

which is identical to our earlier result (see Eq. (4.107)).

Target of finite thickness

For a target of finite thickness the area under the resonance yield curve is obtained by evaluating the expression

$$A_{Y} = \int_{E_{0}=0}^{\infty} Y(E_{0}) dE_{0}$$

= $\frac{1}{\varepsilon_{r}} \int_{E_{0}=0}^{\infty} dE_{0} \int_{E_{0}-\Delta E}^{E_{0}} dE' \int_{E_{i}=0}^{\infty} dE_{i} \int_{E=0}^{E_{i}} \sigma(E)g(E_{0}-E_{i})f(E_{i}-E,E') dE$
(4.115)

The multiple integral can be solved by using the normalizations of g and f and by carrying out the integrations in the order E_0 , E_i , and E'. The integral over E can be taken with an upper limit of infinity since the beam energy varies between 0 and ∞ . Thus

$$A_{Y} = \frac{1}{\varepsilon_{r}} \int_{E_{0}=0}^{\infty} g(E_{0} - E_{i}) dE_{0} \int_{E_{0}-\Delta E}^{E_{0}} dE' \int_{E_{i}=0}^{\infty} dE_{i} \int_{E=0}^{E_{i}} \sigma(E) f(E_{i} - E, E') dE$$
$$= \frac{1}{\varepsilon_{r}} \int_{E_{i}=0}^{\infty} f(E_{i} - E, E') dE_{i} \int_{E_{0}-\Delta E}^{E_{0}} dE' \int_{E=0}^{\infty} \sigma(E) dE$$
(4.116)

The probability that, at a target depth corresponding to E', a projectile of energy E has an initial energy of E_i anywhere between 0 and ∞ is unity and, therefore,

$$A_{Y} = \frac{1}{\varepsilon_{r}} \int_{E_{0} - \Delta E}^{E_{0}} dE' \int_{E=0}^{\infty} \sigma(E) dE$$

$$= \frac{\Delta E}{\varepsilon_{r}} \int_{E=0}^{\infty} \sigma(E) dE = n \int_{E=0}^{\infty} \sigma(E) dE$$
(4.117)

We obtain the important result that the area under a resonance yield curve for a target of finite thickness is independent of beam resolution, straggling, target thickness, stopping power, and total resonance width. The value of A_Y only depends on the total number of target nuclei per square centimeter and the integrated cross section. If the target consists of a compound, then *n* has to be replaced by the number of active target nuclei, $n_X = \Delta E_c / \varepsilon_{\text{eff}}$ (see Eq. (4.94)). It has been shown (Palmer et al. 1963) that the above expression also holds for nonuniform targets (for example, targets of varying stoichiometry). From Eqs. (4.113) and (4.117) we find that the area under the resonance yield curve is equal to the product of maximum yield for an infinitely thick target and the target thickness,

$$A_Y = Y_{\max,\Delta E \to \infty} \Delta E \tag{4.118}$$

For example, for a Breit–Wigner cross section with constant partial widths and de Broglie wavelength over the width of the resonance, one obtains with Eq. (4.114)

$$A_Y = \frac{\Delta E}{\varepsilon_r} \frac{\lambda_r^2}{2} \omega \gamma = n \frac{\lambda_r^2}{2} \omega \gamma \tag{4.119}$$

We will now turn our attention to the influence of beam resolution and straggling on the shape of resonance yield curves for targets of finite thickness. We are specifically interested to investigate by how much the quantities Y_{max} , $E_{0,50\%}$, and FWHM (see Eqs. (4.102)–(4.104)) change because of these effects. In the following, results will be discussed that are obtained by solving Eq. (4.111) numerically, assuming specific distributions for σ , g, and f. For these calculations, the following assumptions will be made: (i) the cross section is given by the Breit-Wigner formula with constant partial widths and de Broglie wavelength over the total resonance width; the resonance is located at an energy of $E_r = 500$ keV and the area under the resonance cross section (that is, the resonance strength) is fixed; (ii) the beam profile is approximated by a Gaussian with a full width at half maximum of Δ_{beam} ; and (iii) the distribution f is also approximated by a Gaussian, an assumption which is appropriate if the number of collisions is large. The full width at half maximum of fcan be approximated by Eq. (4.19), which is applicable for relatively thin absorbers. Assuming a constant stopping power over the total resonance width, one finds from Eqs. (4.14) and (4.19)

$$\Delta_{\text{stragg}} = 1.20 \times 10^{-9} \sqrt{Z_p^2 Z_t (E_0 - E') / \varepsilon} \qquad (\text{keV})$$
$$= \text{const} \sqrt{E_0 - E'} \qquad (\text{keV}) \qquad (4.120)$$

We adopt here arbitrary values of $Z_p = 1$, $Z_t = 10$, and $\varepsilon = 10 \times 10^{-15} \text{ eV cm}^2/\text{atom}$, yielding const = 1.2. The resulting calculated yield curves are shown in Fig. 4.60.

Part (a) shows the effect of varying the beam energy spread Δ_{beam} . For the target thickness, total resonance width, and beam straggling, values of ΔE = 10 keV, $\Gamma = 0$, and $\Delta_{\text{stragg}} = 0$, respectively, are adopted. The curves are obtained for values of $\Delta_{\text{beam}} = 0, 1, 3, 5$, and 8 keV. The rectangular yield curve corresponds to the case of $\Gamma = 0$, $\Delta_{\text{stragg}} = 0$, and $\Delta_{\text{beam}} = 0$. It can be seen that the beam spread causes a decrease in the slope of both the low-energy and the high-energy edge of the yield curve. If the beam spread is small compared to the target thickness, $\Delta_{\text{beam}}/\Delta E < 0.5$, then the beam resolution is equal to the difference of energies at which the yield reaches 12% and 88% of its maximum value (assuming $\Delta_{\text{beam}} \gg \Gamma$), as is appropriate for a Gaussian distribution. For ratios in excess of $\Delta_{\text{beam}} / \Delta E \approx 0.5$, the quantities Y_{max} , $E_{0.50\%}$, and FWHM are all influenced by the beam resolution. The maximum yield decreases, the energy at which the yield reaches 50% of its maximum value shifts below the resonance energy E_r , and the value of FWHM becomes larger than the target thickness. For example, for $\Delta_{\text{beam}} = 8 \text{ keV}$ the energy difference $E_r - E_{0,50\%}$ amounts to ≈ 0.5 keV. The influence of the beam spread on the shape of the yield curve has to be taken into account if the resonance strength $\omega\gamma$ is derived from the observed value of Y_{max} by using Eq. (4.102).

Part (b) demonstrates the influence of beam straggling. For the total resonance width, beam energy resolution, and straggling constant, values of $\Gamma = 0$, $\Delta_{\text{beam}} = 0$, and const = 1.2, respectively, are adopted. The curves are obtained for target thicknesses of $\Delta E = 1, 3, 5$, and 10 keV. The rectangular yield curve corresponds again to the case $\Gamma = 0$, $\Delta_{\text{beam}} = 0$ and $\Delta_{\text{stragg}} = 0$. It can be seen that straggling has no effect on the low-energy edge of the yield curve and, therefore, the energy difference $E_r - E_{0.50\%}$ is negligible (assuming $\Gamma = 0$). However, straggling causes a decrease in the slope of the high-energy edge of the yield curve. The value of Δ_{stragg} (at $E_0 - E' = \Delta E$) is approximately equal to the difference of energies at which the yield reaches 12% and 88% of its maximum value (assuming $\Delta_{\text{stragg}} \gg \Gamma$ and $\Delta_{\text{stragg}} \gg \Delta_{\text{beam}}$). It is obvious that straggling will reduce the maximum yield if the target becomes too thin ($\Delta E < 3$ keV in our specific case). As was the case for the beam spread, the effects of straggling have to be taken into account if $\omega \gamma$ is derived from the observed value of Y_{max} . Note that the value of FWHM is relatively insensitive to straggling.

Part (c) shows the combined effects of the total resonance width, beam resolution, and beam straggling on the shape of a resonance yield curve for specific sets of parameters. For the target thickness a value of $\Delta E = 10$ keV is adopted. The rectangular profile is obtained with the values $\Gamma = 0$, $\Delta_{\text{beam}} = 0$, and $\Delta_{\text{stragg}} = 0$. Since $\Delta E/\Gamma \rightarrow \infty$, the plateau height corresponds to the maximum yield for an infinitely thick target (see Eq. (4.107)). The dotted line is calculated by using the values $\Gamma = 0.5$ keV, $\Delta_{\text{beam}} = 0$, and $\Delta_{\text{stragg}} = 0$. Since $\Delta E/\Gamma = 20$, the plateau height decreases to $0.97Y_{\text{max},\Delta E \rightarrow \infty}$, consistent with the

results shown in Fig. 4.58. In addition, the slopes of the high- and low-energy edges of the yield curve decrease. The solid line obtained with the values of $\Gamma = 0.5 \text{ keV}$, $\Delta_{\text{beam}} = 1.5 \text{ keV}$, and $\Delta_{\text{stragg}} = 0$ includes the effects of a finite beam energy resolution. Since we have $\Delta_{\text{beam}} \ll \Delta E$, the maximum yield Y_{max} is little affected by the beam spread but the low- and high-energy edges become less steep. Finally, the solid line calculated with $\Gamma = 0.5 \text{ keV}$, $\Delta_{\text{beam}} = 1.5 \text{ keV}$, and const = 1.2 for the straggling constant shows the effects of straggling. It causes the high-energy edge of the yield curve to become less steep. Since we have $\Delta_{\text{stragg}} \ll \Delta E$, the value of Y_{max} is little affected by beam straggling. The values of $E_{0.50\%}$ and FWHM are very close to E_r and ΔE , respectively. Note that the areas under all curves shown in Fig. 4.60c have the same value according to Eq. (4.119).

4.8.3

Measured Yield Curves and Excitation Functions

We will now discuss what kind of information may be extracted from the properties of a measured yield curve, that is, its observed width (FWHM), the slope of the low-energy edge, the maximum yield (Y_{max}), the energy at which the yield reaches 50% of its maximum value ($E_{0,50\%}$), and the area under the yield curve (A_Y). In order to properly interpret the experimental information, some information must be known a priori. We will assume in the following that the data represent the yield curve of an isolated, well-resolved resonance and that the total resonance width Γ is known from independent sources.

Consider as a first example Fig. 4.61a, showing a yield curve for the 151 keV resonance in the ¹⁸O(p, γ)¹⁹F reaction. The yield is obtained from the measured intensity of a specific primary transition. This resonance has a total width of Γ = 130 ± 10 eV (Table 4.1). The target was produced by anodizing a tantalum backing in ¹⁸O-enriched water. Such targets are known to consist of a ¹⁸O-Ta compound (Vermilyea 1953).

The yield curve shows a structure with an observed width of FWHM = 34 keV. The beam spread influences the slopes of the low-energy and highenergy edges of the yield curve, while straggling contributes only to the slope of the high-energy edge. It can be seen that both edges extend over energy regions that are smaller than the observed width of the yield curve. In addition, the flat plateau indicates that the plateau height is not influenced by the beam resolution or by straggling. Otherwise the yield maximum would show a round shape (Fig. 4.60). From these arguments it follows that FWHM $\gg \Delta_{\text{beam}}$ and FWHM $\gg \Delta_{\text{stragg}}$. Furthermore, we have FWHM $\gg \Gamma$, and thus we conclude that the observed width is equal to the target thickness, FWHM $= \Delta E = 34$ keV.

The low-energy edge extends over an energy range of several keV. Since Γ is very small, the slope reflects the resolution of the beam. From the difference of energies at which the yield reaches 12% and 88% of its maximum value, we



Fig. 4.60 General shape of resonance yield curves, obtained by solving Eq. (4.111) numerically. See the text for specific assumptions. The curves are obtained for the following conditions of target thickness ΔE , total resonance width Γ , beam straggling Δ_{stragg} , and beam energy spread Δ_{beam} : (a) ΔE = 10 keV, Γ = 0, Δ_{stragg} = 0, Δ_{beam} = 0, 1, 3, 5, 8 keV; (b) Γ = 0, Δ_{beam} = 0,

const = 1.2, ΔE = 1, 3, 5, 10 keV; (c) the target thickness amounts to ΔE = 10 keV for each curve; Γ = 0, Δ_{beam} = 0, Δ_{stragg} = 0; Γ = 0.5 keV, Δ_{beam} = 0, Δ_{stragg} = 0; Γ = 0.5 keV, Δ_{beam} = 1.5 keV, Δ_{stragg} = 0; Γ = 0.5 keV, Δ_{beam} = 1.5 keV, const = 1.2. The areas under all curves shown in part (c) are identical.

find $\Delta_{\text{beam}} = 4.0$ keV. Note that the Doppler effect due to the thermal motion of the target atoms also contributes to the slopes of yield curve edges (Rolfs and Rodney 1988). The beam spread together with the Doppler broadening can be described by a Gaussian with a full width at half maximum of $\Delta_{\text{beam}+\text{Dopp}} = (\Delta_{\text{beam}}^2 + \Delta_{\text{Dopp}}^2)^{1/2}$. In practice, one finds $\Delta_{\text{Dopp}} \leq 100$ eV and, unless beams

of very high resolution are used, we have $\Delta_{\text{beam}+\text{Dopp}} \approx \Delta_{\text{beam}}$. The energy at which the yield reaches 50% of its maximum value is in this case neither influenced by the total resonance width (since $\Delta E \gg \Gamma$; see Eq. (4.103)), nor by the beam spread (since the ratio $\Delta_{\text{beam}}/\Delta E = 4.0 \text{ keV}/34 \text{ keV} = 0.11$ is small; see Fig. 4.60). Thus, we find $E_{0.50\%} = E_r = 150.5 \text{ keV}$.

We concluded that the plateau height is not influenced by beam spread and straggling effects. From the ratio $\Delta E/\Gamma = 34 \text{ keV}/130 \text{ eV} \approx 260$ we find with Eqs. (4.102) and (4.107) that the maximum yield Y_{max} is equal to $0.998Y_{\text{max},\Delta E\to\infty}$ and, therefore, represents the yield of an infinitely thick target to within 0.2%. Furthermore, the area A_Y under the yield curve in general only depends on the number of active target nuclei (¹⁸O) and the resonance strength $\omega\gamma$ (see Eq. (4.119)).

Similar arguments apply to the data displayed in Fig. 4.61b, showing a yield curve of the 918 keV resonance in 36 Ar(p, γ) 37 K. The yield is obtained from the intensity of the primary transition to the 37 K ground state. This resonance has a total width of $\Gamma = 300 \pm 50$ meV (Endt 1998). The target was prepared by implanting 36 Ar ions into a tantalum sheet. Consequently, the target consists of an 36 Ar–Ta compound (Table 4.3). The width of the structure (FWHM = 6.5 keV) is much larger than the total resonance width. The energy region of the leading edge is small compared to the value of FWHM and, therefore, the beam spread neither reduces the maximum yield nor contributes to the observed width of the yield curve. Furthermore, straggling does not influence the leading edge or the observed value of FWHM . Thus, we extract from the yield curve the values $\Delta E = 6.5$ keV, $\Delta_{\text{beam}} = 1.0$ keV, and $E_r = 917.5$ keV.

There is an important difference between the two yield curves shown in Fig. 4.61. In part (b), a flat plateau is not observed and the high-energy edge displays a pronounced tail. This effect, which is partly caused by proton beam straggling, also reflects the range straggling of the implanted ³⁶Ar ions in the tantalum backing. Although these straggling effects will not influence the deduced values of ΔE , Δ_{beam} , and E_r , we may no longer conclude that the influence of straggling on the maximum yield height Y_{max} is negligible.

Narrow resonance yield curves also provide information on the number of active target nuclei and on the stoichiometry of the target compound if the resonance strength is well known. According to Eq. (4.119), the number of active target nuclei per square centimeter for pure targets or compounds is given by $n_X = 2A_Y/(\lambda_T^2 \omega \gamma)$. For a target compound $X_a Y_b$, the effective stopping power can be found from the measured target thickness by using Eq. (4.94). Note that this procedure does not rely on the maximum yield Y_{max} , which may be influenced by straggling effects. The stoichiometry n_Y/n_X can then be derived from the effective stopping power (see Eq. (4.94)).

It is obvious from the above considerations that a yield curve over a narrow resonance provides a wealth of information, including the resonance energy,


Fig. 4.61 (a) Measured yield curve of the $E_r^{\rm lab}$ = 151 keV resonance in the ¹⁸O(p, γ)¹⁹F reaction, obtained from the intensity of the primary transition to the E_x = 3908 keV state. The total resonance width is Γ = 130 \pm 10 eV (Table 4.1). The target was produced by anodizing a tantalum backing in ¹⁸O-enriched water (Vermilyea 1953). (b)

Measured yield curve of the $E_r^{\text{lab}} = 918 \text{ keV}$ resonance in ³⁶Ar(p, γ)³⁷K, obtained from the intensity of the primary transition to the ³⁷K ground state. The total resonance width is $\Gamma = 300 \pm 50 \text{ meV}$ (Endt 1998). The target was prepared by implanting ³⁶Ar ions into a tantalum sheet (Table 4.3).

beam energy resolution, target thickness, number of active target nuclei per square centimeter, and the target stoichiometry. Alternatively, if the beam energy spread is small compared to the total resonance width ($\Delta_{\text{beam}} \ll \Gamma$) and if $\Gamma \ll \Delta E$, then the difference in energies at which the yield reaches 25% and 75% of its maximum value will be equal to Γ (see Eq. (4.109)). Such techniques are frequently applied for measuring the quantities E_r , Δ_{beam} , ΔE , n_X , n_Y/n_X , and Γ .

As a final example, consider the yield curve displayed in Fig. 4.62, which was measured in the ${}^{24}Mg(p,\gamma){}^{25}Al$ reaction at bombarding energies near 1.6 MeV. The yield was obtained from the intensity of a secondary transition.



Fig. 4.62 Measured yield versus laboratory bombarding energy for the $^{24}\text{Mg}(p,\gamma)^{25}\text{Al}$ reaction at energies near 1.6 MeV. The yield is obtained from the intensity of the secondary 945 keV \rightarrow 0 transition in ^{25}Al . The target was produced by evaporating ^{24}Mg onto a tantalum backing. The narrow structure at \approx 1.65 MeV shows the yield curve

over a narrow resonance $(E_r^{\rm lab} = 1654 \text{ keV}, \Gamma = 0.1 \text{ keV})$ while the broad structure at $\approx 1.62 \text{ MeV}$ corresponds to the yield curve of a broad resonance $(E_r^{\rm lab} = 1616 \text{ keV}, \Gamma = 36 \text{ keV})$. Reprinted from D. C. Powell et al., Nucl. Phys. A, Vol. 660, p. 349 (1999). Copyright (1999), with permission from Elsevier.

The measurement was performed using an evaporated, enriched ²⁴Mg target. The narrow structure at ≈ 1.65 MeV shows the yield curve over a narrow resonance ($E_r = 1654$ keV, $\Gamma = 0.1$ keV), similar to the examples discussed above. The full width at half maximum of about 3 keV, as measured with respect to the underlying continuum, reflects the target thickness, since $\Delta E \gg \Gamma$. The broad structure at ≈ 1.62 MeV corresponds to the yield curve of a broad resonance ($E_r = 1616$ keV, $\Gamma = 36$ keV). Since in this case we have $\Delta E \ll \Gamma$, the yield curve reflects the shape of the cross section curve, as discussed in Section 4.8.1.

4.8.4

Determination of Absolute Resonance Strengths and Cross Sections

The importance of absolute cross sections and resonance strengths for the calculation of thermonuclear reaction rates has been stressed in Chapter 3. What is directly measured in experiments are yields rather than cross sections or resonance strengths, as we have seen in previous sections. We will now discuss methods of deriving absolute values of σ and $\omega\gamma$ from measured yields. It is again assumed that the stopping power is approximately constant over the width of the target. The assumption is justified if the target thickness is less than a few tens of keV. In this case, the target thickness and stopping power are related by Eqs. (4.14) or (4.17).

Experimental yields

Yields of nuclear reactions are usually measured with detectors that are located at a certain detection angle θ with respect to the incident beam direction and cover a solid angle Ω . The total yield is experimentally given by Eqs. (4.49), (4.69), and (4.87),

$$Y = \frac{\mathcal{N}_R}{\mathcal{N}_b} = \frac{\mathcal{N}}{\mathcal{N}_b B \eta W}$$
(4.121)

with N_R the total number of reactions that occurred, N_b the total number of incident projectiles, and B, N, η , and W the branching ratio (probability of emission per reaction), the total number of detected particles or photons, the detector efficiency, and the angular correlation, respectively, for a specific nuclear transition. The latter three quantities depend, in general, on the detection angle θ . If the reaction proceeds to only one final state, or if the yield is presented for a specific transition rather than for the total number of reactions, then B = 1.

The differential yield for a nonresonant cross section is usually given for a specific transition (B = 1). With Eqs. (4.44) and (4.121) we write

$$\left(\frac{dY}{d\Omega}\right)_{\theta} = \frac{\mathcal{N}}{\mathcal{N}_{b}\eta_{\text{int}}\Omega}$$
(4.122)

where η_{int} denotes the intrinsic detection efficiency (for example, $\eta_{\text{int}} = 1$ for silicon charged-particle detectors; Section 4.5.1) and Ω is the detector solid angle in steradians.

Absolute resonance strengths and cross sections

With few exceptions, most experimental resonance strengths have been determined using the plateau height of thick targets (see Eq. (4.107)),

$$\omega\gamma = \frac{2\varepsilon_r}{\lambda_r^2} \Upsilon_{\max,\Delta E \to \infty} = \frac{2\varepsilon_r}{\lambda_r^2} \frac{\mathcal{N}_{\max,\Delta E \to \infty}}{\mathcal{N}_b B \eta W}$$
(4.123)

where the subscript r indicates that the corresponding quantities relate to the resonance energy E_r . The quantities B, η , and W are usually constant over a given resonance yield curve. It is important to point out that the resonance strength in the above expression does not depend on the absolute number of target nuclei, but only on the stopping power, and the stoichiometry if the target consists of a compound. When using Eq. (4.123), one has to verify carefully that the maximum observed yield is not affected by the beam spread, straggling, or the total resonance width. Since the area under the yield curve is independent of such effects (see Eq. (4.117)), it is usually more reliable to deduce the resonance strength from Eq. (4.119) instead of Eq. (4.123). From

Eqs. (4.119) and (4.121) one finds

$$\omega\gamma = 2\frac{A_Y}{n\lambda_r^2} = \frac{2}{\lambda_r^2}\frac{\varepsilon_r}{\Delta E}\int_0^\infty Y(E_0)\,dE_0 = \frac{2}{\lambda_r^2}\frac{\varepsilon_r}{\Delta E}\frac{1}{B\eta W}\int_0^\infty \frac{\mathcal{N}(E_0)}{\mathcal{N}_b(E_0)}\,dE_0 \quad (4.124)$$

Note that the subscript *r* is omitted for the target thickness ΔE although this quantity also refers to an energy near E_r . Similarly, we may use Eqs. (4.92) and (4.121) for determining slowly varying absolute cross sections,

$$\sigma(E_{\text{eff}}) = \frac{Y(E_0)}{n} = Y(E_0) \frac{\varepsilon(E_0)}{\Delta E(E_0)} = \frac{\varepsilon(E_0)}{\Delta E(E_0)} \frac{\mathcal{N}(E_0)}{\mathcal{N}_b(E_0)B(E_0)\eta(E_0)W(E_0)}$$
(4.125)

Determinations of absolute $\omega\gamma$ and σ values by using Eqs. (4.124) and (4.125) are difficult. These procedures require knowledge of the absolute number of incident particles, absolute detector efficiencies, absolute branching ratios, and so on. In particular, absolute stopping powers carry relatively large errors (Section 4.2.1). Furthermore, if the target consists of a compound, then the effective stopping power ε_{eff} has to be used in the above expressions and, consequently, the target stoichiometry has to be accurately known (see Eq. (4.94)).

Uncertainties of measured $\omega\gamma$ and σ values are typically in the 15–20% range, where the effective stopping power contributes frequently the major fraction of the error. It is important to realize that in many cases the $\omega\gamma$ values for a given resonance measured by different research groups deviate from each other by factors of 2–4. Similar arguments apply to some cross sections. These deviations reflect the difficulties in measuring the absolute magnitudes of quantities entering into the resonance strength or cross section calculations. For example, the absolute beam intensity is usually determined from the total charge deposited by the beam on a Faraday cup, but systematic errors are likely if secondary electron emission is not properly accounted for (Section 4.3.4).

A major problem for the determination of absolute $\omega\gamma$ and σ values is the incomplete knowledge of the target stoichiometry. If evaporated targets are used (Section 4.3.2) it is frequently assumed that the composition of the target during the nuclear reaction measurement is the same as the composition of the raw material used before the actual evaporation took place. This assumption is rarely valid since the target composition may change either during evaporation or later during ion beam bombardment. For example, Mg targets are frequently prepared by reductive evaporation of MgO (Takayangi et al. 1966) and, therefore, such targets are expected to consist of a pure layer of Mg. However, measurements have shown (Iliadis et al. 1990) that these targets consist of a compound Mg₅O, indicating either incomplete oxygen reduction during target preparation or oxidization in air before the actual experiment. Another striking example is NaCl targets. It has been shown (Paine, Kennett

and Sargood 1978) that such targets change their stoichiometry during proton bombardment from NaCl to Na₁₇Cl₁₀ after an accumulated charge of only $\approx 1 \times 10^{-4}$ C.

Relative resonance strengths and cross sections

It is much simpler and more reliable to obtain resonance strengths and cross sections relative to some absolute, carefully measured, standard resonance strength or cross section. Using the expression for the maximum yield of an infinitely thick target (see Eq. (4.123)) we obtain

$$\frac{\omega\gamma_1}{\omega\gamma_2} = \frac{\varepsilon_{r,1}}{\varepsilon_{r,2}} \frac{\lambda_{r,2}^2}{\lambda_{r,1}^2} \frac{\gamma_{\max,\Delta E \to \infty,1}}{\gamma_{\max,\Delta E \to \infty,2}} = \frac{\varepsilon_{r,1}}{\varepsilon_{r,2}} \frac{\lambda_{r,2}^2}{\lambda_{r,1}^2} \frac{\mathcal{N}_{\max,\Delta E \to \infty,1}}{\mathcal{N}_{\max,\Delta E \to \infty,2}} \frac{\mathcal{N}_{b,2}}{\mathcal{N}_{b,1}} \frac{B_2}{B_1} \frac{\eta_2}{\eta_1} \frac{W_2}{W_1} \quad (4.126)$$

where the subscripts 1 and 2 correspond to the resonance of interest and the standard resonance, respectively. The error of the resonance strength $\omega \gamma_1$ depends on the accuracy of the $\omega \gamma_2$ value of the standard resonance. Otherwise, only ratios of stopping powers, efficiencies, numbers of incident particles, and so on, enter in Eq. (4.126), thus minimizing the influence of potential sources of error. If we use instead the respective areas under the yield curves (see Eq. (4.124)), then

$$\frac{\omega\gamma_{1}}{\omega\gamma_{2}} = \frac{A_{Y,1}}{A_{Y,2}} \frac{\lambda_{r,2}^{2}}{\lambda_{r,1}^{2}} \frac{\varepsilon_{r,1}}{\varepsilon_{r,2}} \frac{\Delta E_{2}}{\Delta E_{1}} = \frac{\lambda_{r,2}^{2}}{\lambda_{r,1}^{2}} \frac{\varepsilon_{r,1}}{\varepsilon_{r,2}} \frac{\Delta E_{2}}{\Delta E_{1}} \frac{\int_{0}^{\infty} Y_{1}(E_{0,1}) dE_{0,1}}{\int_{0}^{\infty} Y_{2}(E_{0,2}) dE_{0,2}}
= \frac{\lambda_{r,2}^{2}}{\lambda_{r,1}^{2}} \frac{\varepsilon_{r,1}}{\varepsilon_{r,2}} \frac{\Delta E_{2}}{\Delta E_{1}} \frac{B_{2}\eta_{2}W_{2}}{B_{1}\eta_{1}W_{1}} \frac{\int_{0}^{\infty} \frac{N_{1}(E_{0,1})}{N_{b,1}(E_{0,1})} dE_{0,1}}{\int_{0}^{\infty} \frac{N_{2}(E_{0,2})}{N_{b,2}(E_{0,2})} dE_{0,2}}$$
(4.127)

Note that the above expression does not depend on the stopping power or the target thickness if $\omega \gamma_1$ is determined relative to a standard resonance in the same reaction and using the same target, since then $\varepsilon_{r,1}/\Delta E_1 = \varepsilon_{r,2}/\Delta E_2 = n$. Similarly, we find from Eq. (4.125) for slowly varying cross sections

$$\frac{\sigma_{1}(E_{\text{eff},1})}{\sigma_{2}(E_{\text{eff},2})} = \frac{\varepsilon_{1}(E_{0,1})}{\varepsilon_{2}(E_{0,2})} \frac{\Delta E_{2}(E_{0,2})}{\Delta E_{1}(E_{0,1})} \frac{Y_{1}(E_{0,1})}{Y_{2}(E_{0,2})}
= \frac{\varepsilon_{1}(E_{0,1})}{\varepsilon_{2}(E_{0,2})} \frac{\Delta E_{2}(E_{0,2})}{\Delta E_{1}(E_{0,1})} \frac{\mathcal{N}_{1}(E_{0,1})}{\mathcal{N}_{2}(E_{0,2})} \frac{\mathcal{N}_{b,2}(E_{0,2})}{\mathcal{N}_{b,1}(E_{0,1})} \frac{B_{2}(E_{0,2})}{B_{1}(E_{0,1})} \frac{\eta_{2}(E_{0,2})}{\eta_{1}(E_{0,1})} \frac{W_{2}(E_{0,2})}{W_{1}(E_{0,1})}
(4.128)$$

where the subscripts 1 and 2 refer to the nonresonant cross section of interest and the standard cross section, respectively. Again, the stopping powers and target thicknesses cancel if both cross sections are measured in the same reaction using the same target.

A nonresonant cross section σ can also be determined relative to a wellknown resonance strength $\omega\gamma$ (or vice versa). We obtain, for example, from Eqs. (4.123) and (4.125)

$$\frac{\sigma_1(E_{\text{eff},1})}{\omega\gamma_2} = \frac{\lambda_{r,2}^2}{2\Delta E_1(E_{0,1})} \frac{\varepsilon_1(E_{0,1})}{\varepsilon_{r,2}} \frac{\mathcal{N}_1(E_{0,1})}{\mathcal{N}_{\max,\Delta E \to \infty,2}} \times \frac{\mathcal{N}_{b,2}}{\mathcal{N}_{b,1}(E_{0,1})} \frac{B_2}{B_1(E_{0,1})} \frac{\eta_2}{\eta_1(E_{0,1})} \frac{W_2}{W_1(E_{0,1})}$$
(4.129)

or from Eqs. (4.124) and (4.125)

$$\frac{\sigma_1(E_{\text{eff},1})}{\omega\gamma_2} = \frac{\lambda_{r,2}^2}{2} \frac{\varepsilon_1(E_{0,1})}{\varepsilon_{r,2}} \frac{\Delta E_2}{\Delta E_1(E_{0,1})} \frac{B_2}{B_1(E_{0,1})} \frac{\eta_2}{\eta_1(E_{0,1})} \frac{W_2}{W_1(E_{0,1})} \\ \times \frac{\mathcal{N}_1(E_{0,1})}{\mathcal{N}_{b,1}(E_{0,1}) \int_0^\infty \frac{\mathcal{N}_2(E_{0,2})}{\mathcal{N}_{b,2}(E_{0,2})} dE_{0,2}}$$
(4.130)

where in the last two expressions the subscripts 1 and 2 refer to the nonresonant cross section and the resonance, respectively.

Example 4.4

For the narrow $E_r^{\text{lab}} = 317$ keV resonance in the ${}^{25}\text{Mg}(p,\gamma){}^{26}\text{Al}$ reaction ($\Gamma < 40 \text{ eV}$), a yield curve is measured by using the intensity of the primary γ -ray transition to the 417 keV state in ${}^{26}\text{Al}$ ($R \rightarrow 417$ keV). An evaporated ${}^{25}\text{Mg}_5\text{O}$ target has a thickness of $\Delta E = 15$ keV. The beam spread amounts to $\Delta_{\text{beam}} = 0.5$ keV. Calculate the resonance strength from the measured values given below. Neglect angular correlation effects ($W_{R\rightarrow 417} = 1$).

$\mathcal{N}_{\max,R\to417} = 3480\pm63$	γ -ray intensity on the yield curve plateau
$Q = (0.090 \pm 0.005) C$	accumulated ion beam charge on target
$\eta_{R\to417} = (7.34\pm0.30)\times10^{-4}$	peak efficiency for $R \rightarrow 417 \text{ keV}$
$B_{R\to417} = (33\pm1)\%$	branching ratio of $R \rightarrow 417 \text{ keV}$

Assume that the stopping power is constant over the thickness of the target. Use the following values (with 10% errors) for protons in Mg and O, as obtained from the computer code SRIM (Ziegler 2003): $\varepsilon_{p \to Mg}(E_r^{lab} = 317 \text{ keV}) = 12.8 \times 10^{-15} \text{ eV cm}^2/\text{atom}, \ \varepsilon_{p \to O}(E_r^{lab} = 317 \text{ keV}) = 10.6 \times 10^{-15} \text{ eV cm}^2/\text{atom}.$

With a stoichiometry of n_{Mg} : $n_{O} = 5$: 1 we obtain for the effective stopping power (see Eq. (4.94))

$$\varepsilon_{\text{eff}} = \frac{M_{25}_{\text{Mg}}}{M_{25}_{\text{Mg}} + M_{\text{H}}} \left[\varepsilon_{25}_{\text{Mg}} + \frac{n_{\text{O}}}{n_{\text{Mg}}} \varepsilon_{\text{O}} \right]$$

= $\frac{24.985}{24.985 + 1.008} \left[(12.8 \times 10^{-15} \,\text{eV} \,\text{cm}^2/\text{atom}) + \frac{1}{5} (10.6 \times 10^{-15} \,\text{eV} \,\text{cm}^2/\text{atom}) \right]$
= $1.43 \times 10^{-14} \,\text{eV} \,\text{cm}^2/\text{atom} \,(\pm 10\%)$

The de Broglie wavelength is given by (see Eq. (4.105))

$$\begin{aligned} \frac{\lambda_r^2}{2} &= \left(\frac{M_p + M_t}{M_t}\right)^2 \frac{4.125 \times 10^{-18}}{M_p E_r^{\text{lab}}} \quad (\text{cm}^2) \\ &= \left(\frac{1.008 + 24.985}{24.985}\right)^2 \frac{4.125 \times 10^{-18}}{(1.008)(317000)} \,\text{cm}^2 = 1.40 \times 10^{-23} \,\text{cm}^2 \end{aligned}$$

The total number of incident protons (assuming a positively charged proton beam, q=1) amounts to

$$\mathcal{N}_{\rm p} = \frac{Q}{qe} = \frac{(0.090 \pm 0.005)\,\rm C}{1 \cdot (1.6 \times 10^{-19}\,\rm C)} = 5.63 \times 10^{17}\,(\pm 6\%)$$

If we neglect the influence of the beam spread (which is small compared to the target thickness), straggling, and the total resonance width ($\Gamma \ll \Delta E$) on the maximum yield, then the observed plateau height corresponds to the maximum yield of an infinitely thick target, $Y_{\max,\Delta E \to \infty}$. With Eq. (4.123) one finds

$$\begin{split} \omega \gamma &= \frac{2\varepsilon_{\text{eff},r}}{\lambda_r^2} \frac{\mathcal{N}_{\text{max},\Delta E \to \infty}}{\mathcal{N}_b B \eta W} \\ &= \frac{1.43 \times 10^{-14} \text{ eV cm}^2}{1.40 \times 10^{-23} \text{ cm}^2} \frac{3480}{(5.63 \times 10^{17})(0.33)(7.34 \times 10^{-4})} \\ &= 2.61 \times 10^{-2} \text{ eV} \, (\pm 13\%) \end{split}$$

Note that all energies in the general yield expression of Eq. (4.91) are given in the center-of-mass system. Obviously, the quantity dx = dE/(dE/dx) is independent of the reference frame. Multiplication of numerator and denominator by the center of mass to laboratory frame conversion factor $M_t/(M_t + M_p)$ (see Eq. (C.24); M_t and M_p are the relative atomic masses of the active target nuclei and the projectiles, respectively) shows that the effective stopping power measured in the laboratory or calculated with SRIM must be multiplied by this factor before it can be used in yield calculations.

Determination of resonance strengths and cross sections relative to Rutherford scattering

It is apparent from the above discussion that measurements of absolute resonance strengths and cross sections are difficult to perform since a variety of experimental artifacts, such as beam spread, straggling, stoichiometries, stopping powers, integrated beam charge, and so on, may lead to substantial systematic errors. Nevertheless, for a number of resonances, careful measurements of their absolute strengths have been performed. The results are given in Table 4.12. This set of recommended $\omega\gamma$ values may be used as an absolute standard for the determination of other resonance strengths or of nonresonant cross sections according to Eqs. (4.126)–(4.130).

Almost all the $\omega\gamma$ values listed in the table have been determined relative to the intensity of Rutherford-scattered projectiles. The experimental details vary from study to study, but such techniques essentially eliminate the influence of at least some artifacts on the $\omega\gamma$ values. Consequently, we expect that these results are more reliable compared to those obtained from Eqs. (4.123)–(4.125). In the following, a method will be discussed which eliminates almost all experimental artifacts. This technique provides absolute resonance strengths and cross sections that depend almost exclusively on measured intensities and, in particular, does not require knowledge of beam or target properties. The method is based on the fact that certain quantities cancel in the determination of $\omega\gamma$ if the nuclear reaction products and Rutherford-scattered beam particles are measured simultaneously.

We start with Eq. (4.124) which relates the resonance strength to the area under the resonance yield curve,

$$\omega \gamma_1 = 2 \frac{A_{Y,1}}{n\lambda_{r,1}^2} = \frac{2}{n\lambda_{r,1}^2} \frac{1}{B_1 \eta_1 W_1} \int_0^\infty \frac{\mathcal{N}_1(E_{0,1})}{\mathcal{N}_{b,1}(E_{0,1})} dE_{0,1}$$
(4.131)

where the subscript 1 is used for all quantities related to the measurement of the resonance. As already noted, the above expression is independent of the beam spread, straggling, and the total resonance width. For the derivation of this result we used the assumptions that the resonant cross section is given by the Breit–Wigner formula with constant partial widths and de Broglie wavelength over the total resonance width, and that the stopping power is approximately constant over the width of the target (Section 4.8.2).

Suppose now that the projectiles that are Rutherford-scattered by the active target nuclei are measured in a second detector located at an angle of θ_2 with respect to the incident beam direction. If the target is sufficiently thin (say, less than 10 keV), so that the variation of the Rutherford-scattering cross section

Tab. 4.12 Recommended resonance strengths. (a) Rowland et al. (2002a); (b) Paine and Sargood (1979); (c) Powell et al. (1999); (d) Powell et al. (1998); (e) Anderson et al. (1980); (f) weighted average of Goosman and Kavanagh (1967) and Mohr et al. (1999); (g) weighted average of Wiescher et al. (1980), Becker et al. (1982), and Vogelaar et al. (1990); (h) weighted average of Becker et al. (1982), Formicola et al. (2004), and Runkle et al. (2005). Absolute errors are given in parentheses and refer to the last significant digit(s).

Reaction	E_r^{lab} (keV)	J^{π}	$\omega \gamma_{\rm cm}$ (eV)	Error (%)	Reference
¹⁴ N(p,γ) ¹⁵ O	278	1/2+	$1.37(7) \times 10^{-2}$	5.1	h
¹⁸ O(p,γ) ¹⁹ F	151	1/2+	$9.7(5) \times 10^{-4}$	5.2	g
²³ Na(p,α) ²⁰ Ne	338	1-	$7.16(29) imes 10^{-2}$	4.0	а
$^{\scriptscriptstyle 23}$ Na(p, $\gamma)^{\scriptscriptstyle 24}$ Mg	512	(1,2 ⁺)	$9.13(125) imes 10^{-2}$	13.7	b
24 Mg(p, γ) 25 Al	223	1/2+	$1.27(9) imes 10^{-2}$	7.1	С
	419	3/2+	$4.16(26) imes 10^{-2}$	6.2	d
25 Mg(p, γ) 26 Al	435	4-	$9.42(65) imes 10^{-2}$	6.9	d
	591	1+	$2.28(17) imes 10^{-1}$	7.4	е
26 Mg(p, γ) 27 Al	338	3/2-	$2.73(16) imes 10^{-1}$	5.9	d
	454	1/2+	$7.15(41) imes 10^{-1}$	5.7	d
	1966	5/2+	5.15(45)	8.7	b
$^{\scriptscriptstyle 27}Al(p,\gamma)^{\scriptscriptstyle 28}Si$	406	4+	$8.63(52) imes 10^{-3}$	6.0	d
	632	3-	$2.64(16) imes 10^{-1}$	6.1	b
	992	3+	1.91(11)	5.7	b
$^{\scriptscriptstyle 30}{}{ m Si}({ m p},\gamma)^{\scriptscriptstyle 31}{ m P}$	620	1/2-	1.95(10)	5.1	b
³¹ Ρ(p,γ) ³² S	642	1-	$5.75(50) imes 10^{-2}$	8.7	b
	811	2+	$2.50(20) imes 10^{-1}$	8.0	b
³⁴ S(p,γ) ³⁵ Cl	1211	7/2-	4.50(50)	11.1	b
${}^{35}Cl(p,\gamma){}^{36}Ar$	860	3-	$7.00(100) imes 10^{-1}$	14.3	b
36 Ar(p, γ) 37 K	918	5/2+	$2.38(19) imes 10^{-1}$	8.0	f
${}^{\scriptscriptstyle 37}Cl(p,\gamma){}^{\scriptscriptstyle 38}Ar$	846	1-	$1.25(16) imes 10^{-1}$	12.8	b
^{₃9} K(p,γ)⁴⁰Ca	2042	1+	1.79(19)	10.6	b
$^{\scriptscriptstyle 40}$ Ca(p, $\gamma)^{\scriptscriptstyle 41}$ Sc	1842	7/2+	$1.40(15) imes 10^{-1}$	10.7	b

over the target thickness is small, then we obtain from Eq. (4.93)

$$\left[\frac{dY_2(E_{0,2})}{d\Omega}\right]_{\theta_2}^{\text{Ruth}} = n \left[\frac{d\sigma_2(E_{\text{eff},2})}{d\Omega}\right]_{\theta_2}^{\text{Ruth}}$$
(4.132)

where the subscript 2 refers to all quantities related to the measurement of the Rutherford-scattered beam particles by the second detector. Solving for n, one finds with Eq. (4.122) (assuming $\eta_{int,2} = 1$ for silicon charged-particle detectors)

$$n = \frac{\left[\frac{dY_2(E_{0,2})}{d\Omega}\right]_{\theta_2}^{\text{Ruth}}}{\left[\frac{d\sigma_2(E_{\text{eff},2})}{d\Omega}\right]_{\theta_2}^{\text{Ruth}}} = \frac{\frac{\mathcal{N}_2(E_{0,2})}{\mathcal{N}_{b,2}(E_{0,2})\Omega_2}}{\left[\frac{d\sigma_2(E_{\text{eff},2})}{d\Omega}\right]_{\theta_2}^{\text{Ruth}}}$$
(4.133)

It follows that the ratio of differential yield and differential cross section for Rutherford scattering is constant (that is, equal to n) and thus may be mea-

sured at any bombarding energy. If the resonant reaction products are measured *simultaneously* with the Rutherford-scattered particles, so that $E_{0,1} = E_{0,2} \equiv E_0$ and $\mathcal{N}_{b,2}(E_{0,2}) = \mathcal{N}_{b,1}(E_{0,1})$, then we obtain from Eqs. (4.131) and (4.133)

$$\omega\gamma_1 = \frac{2}{\lambda_{r,1}^2} \frac{1}{B_1 W_1} \frac{\Omega_2}{\eta_1} \int_0^\infty \frac{\mathcal{N}_1(E_0)}{\mathcal{N}_2(E_0)} \left[\frac{d\sigma_2(E_{\text{eff},2})}{d\Omega} \right]_{\theta_2}^{\text{Ruth}} dE_0$$
(4.134)

The resonance strength in this expression is independent of the properties of the target (stoichiometry, stopping power, uniformity) and the beam (current integration, straggling). It depends on (i) the observed number of resonant reaction products (particles or photons) and Rutherford-scattered particles, $\mathcal{N}_1(E_0)$ and $\mathcal{N}_2(E_0)$; (ii) the calculated Rutherford scattering cross section, $[d\sigma_2(E_{\text{eff},2})/d\Omega]^{\text{Ruth}}_{\partial_2}$; and (iii) the de Broglie wavelength, branching ratio, and angular correlation of the resonant reaction products ($\lambda_{r,1}^2$, B_1 , W_1). Also, note that $\omega\gamma_1$ depends on the ratio Ω_2/η_1 and, consequently, is independent of the knowledge of absolute detection properties.

If a nonresonant reaction cross section is measured relative to Rutherford scattering, we find in complete analogy

$$\sigma_1(E_{\text{eff},1}) = \frac{1}{B_1(E_0)W_1(E_0)} \frac{\Omega_2}{\eta_1} \frac{\mathcal{N}_1(E_0)}{\mathcal{N}_2(E_0)} \left[\frac{d\sigma_2(E_{\text{eff},2})}{d\Omega} \right]_{\theta_2}^{\text{Kuth}}$$
(4.135)

D 1

For (p, α) or (α ,p) type reactions we may substitute in Eqs. (4.134) and (4.135) $\eta_1 = \Omega_1/(4\pi)$, assuming that the intrinsic efficiency for detecting the resonant particles is unity ($\eta_{\text{int},1} = 1$). In general, we have $E_{\text{eff},1} \neq E_{\text{eff},2}$ as can be seen, for example, from Eq. (4.99).

The theoretical Rutherford scattering cross section is given by (Evans 1955)

$$\left[\frac{d\sigma(E)}{d\Omega}\right]_{\theta}^{\text{Ruth}} = \left(\frac{Z_p Z_t e^2}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}$$
$$= 1.296 \left(\frac{Z_p Z_t}{E}\right)^2 \frac{1}{\sin^4(\theta/2)} \qquad (\text{mb/sr})$$
(4.136)

with Z_p and Z_t the atomic numbers of projectile and target, respectively. In the numerical expression, the energy *E* is in units of MeV.

The method of measuring absolute resonance strengths and cross sections described above depends on the assumption that the intensity of elastically scattered beam particles at the energy of the resonance, or in the region of the nonresonant cross section, is well described by the Rutherford formula. However, at higher bombarding energies (E > 0.5 MeV) and for relatively broad resonances ($\Gamma > 1$ keV) the elastic scattering cross section generally deviates

from Rutherford scattering since it is influenced by resonant scattering (Section 2.5.3). The above technique is most useful at low bombarding energies and for relatively narrow resonances, a situation which is frequently of interest for thermonuclear reactions. In any case, one has to verify experimentally through careful measurements that the elastic scattering cross section is indeed given by the Rutherford formula.

As an example, consider the measurement of the $E_r^{\text{lab}} = 338 \text{ keV}$ resonance ($\Gamma = 0.7 \text{ keV}$) in the ²³Na(p, α)²⁰Ne reaction. The setup is shown in Fig. 4.63a. It consists of two silicon charged-particle detectors ($\eta_{\text{int}} = 1$). The first detector, positioned at 140°, covers a relatively large solid angle and is used for measuring the resonant α -particles. A thin metal foil is placed in front of this counter to prevent the large number of elastically scattered protons from interfering with the detection of the α -particles (Section 4.5.1). The second detector, placed at 155°, covers a very small solid angle and is used for measuring the elastically scattered protons. A proton beam of a few 100 nA intensity is incident on a transmission target that was prepared by evaporating NaCl on a thin carbon foil.

Typical α -particle and proton spectra, measured at $E_p^{\text{lab}} = 341$ and 400 keV, respectively, are shown in Figs. 4.63b and c. In the proton spectrum, only the intensity of the peak corresponding to protons elastically scattered from (the active) ²³Na nuclei is of interest here. The resonant α -particle yield curve is displayed in Fig. 4.64a. It can be seen that the target is about 6 keV thick. Figure 4.64b shows the measured yields of elastically scattered protons from ²³Na at a fixed detector angle as a function of bombarding energy over the region of the $E_r^{\text{lab}} = 338$ keV resonance. The solid line represents the calculated Rutherford yield, normalized to the data. It is apparent that the data are well described by the Rutherford formula.

The value of the resonance strength measured with this technique and derived from Eq. (4.134) is listed in Table 4.12. It is significantly smaller compared to previous $\omega\gamma$ values that have been determined by using the maximum yield of thick targets (see Eq. (4.123)). The disagreement is explained by the fact that previous studies have erroneously assumed a target stoichiometry of Na₁Cl₁, whereas the technique described above is independent of the target stoichiometry (Rowland et al. 2002a).

4.9

Transmissions, Yields, and Cross Sections for Neutron-Induced Reactions

In this section, the relationships between directly measured quantities (yields and transmissions) and cross sections or resonance strengths are given for neutron-induced reactions. Examples for measured transmission and yield



Fig. 4.63 Measurement of the E_r^{lab} = 338 keV resonance ($\Gamma = 0.7 \text{ keV}$) in the ²³Na(p, α)²⁰Ne reaction. (a) Setup showing the proton beam (\approx 100 nA), the transmission target prepared by evaporating NaCl (6 keV thick) on a thin carbon foil, and two silicon charged-particle detectors. The first detector is used for measuring resonant α particles and is covered by a thin metal foil. The second detector is used for measuring

elastically scattered protons. (b) α -Particle spectrum measured in the region of the resonance at a laboratory proton energy of 341 keV. (c) Spectrum of elastically scattered protons, measured at a laboratory proton energy of 400 keV. Reprinted with permission from C. Rowland et al., Phys. Rev. C, Vol. 65, 064609 (2002). Copyright (2002) by the American Physical Society.

curves will be presented, and the determination of absolute cross sections is briefly discussed.

4.9.1

Resonance Transmission

Of particular interest are transmission measurements of resolved resonances. In actual measurements, the shape of the transmission curve depends not only on the total cross section, but is influenced by other factors, such as the Doppler effect or the resolution of the neutron detector (Beckurts and Wirtz 1964). The quantity of interest, however, is usually not the energy dependence of the total resonance cross section, but the determination of the resonance properties that enter in the expression for the resonant reaction rate (Section 3.2.4).

Consider the simplest case of an isolated resonance. Suppose that (i) the resonance cross section is given by the Breit-Wigner formula, (ii) only the



Fig. 4.64 (a) Resonant α -particle yield versus proton energy for the $E_r^{\rm lab} = 338$ keV resonance in ²³Na(p, α)²⁰Ne; (b) yield of elastically scattered protons from ²³Na versus energy; the solid line represents the calculated Rutherford yield, normalized to the data. Both yield curves were measured with the setup shown in Fig. 4.63.

neutron and γ -ray channels are open, and (iii) the energy dependence of the partial widths over the total resonance width can be neglected. In the vicinity of an isolated resonance, the *total* neutron-induced cross section can be written as (Section 2.5.5)

$$\sigma_{T,BW}(E) = \frac{\lambda^2}{4\pi} \omega \frac{\Gamma_n \Gamma}{(E_r - E)^2 + \Gamma^2/4} = \sigma_{T,max} \frac{\Gamma}{2} \frac{\Gamma/2}{(E_r - E)^2 + \Gamma^2/4}$$
(4.137)

with $\sigma_{T,\text{max}} = \sigma_{T,\text{BW}}(E = E_r) = (\lambda_r^2 / \pi) \omega \Gamma_n / \Gamma$ the maximum total cross section. For an incident beam of monoenergetic neutrons, the transmission is (see Eq. (4.37))

$$T(E) = \exp\left[-n\sigma_{T,\max}\frac{\Gamma}{2}\frac{\Gamma/2}{(E_r - E)^2 + \Gamma^2/4}\right]$$
(4.138)

with *n* the number of sample nuclei per unit area. For the area *above* the transmission curve one finds

$$A_T = \int_0^\infty \left\{ 1 - \exp\left[-n\sigma_{T,\max} \frac{\Gamma}{2} \frac{\Gamma/2}{(E_r - E)^2 + \Gamma^2/4} \right] \right\} dE$$
(4.139)

which reduces for the limiting case of a thin sample ($n\sigma_{T,max} \ll 1$) to

$$A_T^{n\sigma_{T,\max}\ll 1} = \int_0^\infty n\sigma_{T,\max} \frac{\Gamma}{2} \frac{\Gamma/2}{(E_r - E)^2 + \Gamma^2/4} dE$$

$$= \frac{\pi}{2} n\sigma_{T,\max} \Gamma = \frac{\lambda_r^2}{2} n\omega\Gamma_n$$
(4.140)

Hence, a measurement of the transmission curve provides an estimate of the neutron partial width Γ_n . This expression also holds if the instrumental resolution and the Doppler effect change the shape of the transmission curve. For thin samples, the area above the transmission curve is independent of these effects. More information on transmission curves can be found in Lynn (1968).

4.9.2

Resonant and Nonresonant Yields

The general expression for the yield of a neutron-induced reaction can be derived from the expression for the transmission (see Eq. (4.37))

$$Y = \int_0^\infty f(E) \left[1 - e^{-n\sigma_T(E)} \right] \frac{\sigma(E)}{\sigma_T(E)} dE$$
(4.141)

with $\sigma(E)$ and $\sigma_T(E)$ the cross section for the reaction of interest and the total cross section (see Eq. (4.34)), respectively, and f(E) the fraction of incident neutrons having energies between *E* and E + dE per unit energy interval. For either a monoenergetic incident neutron beam or for constant cross sections $\sigma_T(E)$ and $\sigma(E)$ one finds

$$Y = \left(1 - e^{-n\sigma_T}\right) \frac{\sigma}{\sigma_T} \int_0^\infty f(E) \, dE = \left(1 - e^{-n\sigma_T}\right) \frac{\sigma}{\sigma_T} \tag{4.142}$$

If the cross sections $\sigma_T(E)$ and $\sigma(E)$ are not constant, but if the sample is very thin ($n\sigma_T \ll 1$), then Eq. (4.141) becomes

$$Y_{n\sigma_T \ll 1} = \int_0^\infty f(E) \left[1 - (1 - n\sigma_T(E)) \right] \frac{\sigma(E)}{\sigma_T(E)} dE = n \int_0^\infty f(E)\sigma(E) dE = n\overline{\sigma}$$
(4.143)

where we defined an average reaction cross section by $\overline{\sigma} \equiv \int f(E)\sigma(E) dE$. A few specialized expressions for the *thin-sample* yield are given below.

Constant σ over neutron energy distribution

This situation occurs, for example, for a smoothly varying cross section and a nearly monoenergetic neutron beam. The yield is given by

$$Y_{n\sigma_T \ll 1} = n\sigma \int_0^\infty f(E) \, dE = n\sigma \tag{4.144}$$

This expression also applies to an isolated resonance when the total width is large compared to the neutron beam resolution ($\Gamma \gg \Delta E_n$). The yield is then directly proportional to the cross section and the resulting excitation function has a resonance shape. In the latter case, we may describe the resonance by the Breit–Wigner formula. Suppose that only the neutron and γ -ray channels are open, as is frequently the case, and that the partial widths are energy independent. The area under the resonance neutron-capture yield curve for a thin sample follows from Eq. (4.144),

$$A_{Y}^{n\sigma_{T}\ll1} = n \int_{0}^{\infty} \sigma_{BW}(E) dE = n \int_{0}^{\infty} \frac{\lambda^{2}}{4\pi} \omega \frac{\Gamma_{n}\Gamma_{Y}}{(E_{r}-E)^{2}+\Gamma^{2}/4} dE$$
$$= \frac{\lambda_{r}^{2}}{2\pi} n\omega\gamma \int_{0}^{\infty} \frac{\Gamma/2}{(E_{r}-E)^{2}+\Gamma^{2}/4} dE = n \frac{\lambda_{r}^{2}}{2} \omega\gamma$$
(4.145)

Exactly the same result was obtained for the area under a resonance yield curve in charged-particle-induced reactions (see Eq. (4.119)).

Narrow resonance with $\Gamma \ll \Delta E_n$

If a narrow resonance located at E_r has a small total width compared to the neutron beam resolution, then one finds from the Breit–Wigner formula

$$Y_{n\sigma_T \ll 1} = n \int_0^\infty f(E)\sigma_{BW}(E) dE = nf(E_r) \int_0^\infty \sigma_{BW}(E) dE$$
$$= \frac{\lambda_r^2}{2\pi} nf(E_r)\omega\gamma \int_0^\infty \frac{\Gamma/2}{(E_r - E)^2 + \Gamma^2/4} dE = n\frac{\lambda_r^2}{2}f(E_r)\omega\gamma \quad (4.146)$$

with $f(E_r)$ the fraction of neutrons per unit energy interval at the resonance energy. In contrast to charged-particle reaction studies (see Eq. (4.114)), the narrow resonance yield for neutron-induced reactions depends on the absolute number of sample nuclei per unit area.

4.9.3

Effective Cross Section

If the incident neutrons are not monoenergetic, then an effective cross section is sometimes introduced which is defined in terms of the neutron current density, or neutron flux, instead of the number density of neutrons. If we divide the neutron energy distribution into thin slices, then the number of reactions per volume and per time from each slice is given by Eq. (3.1),

$$\frac{(\mathcal{N}_{R,i}/V)}{t} = \frac{\mathcal{N}_t}{V} \sigma_i v_i \frac{\mathcal{N}_{\nu,i}}{V}$$
(4.147)

with N_t/V and $N_{v,i}/V$ the target density and neutron density, respectively. Integrating over all energies, we find

$$\frac{(\mathcal{N}_R/V)}{t} = \frac{\mathcal{N}_t}{V} \int_0^\infty \sigma(E) v \frac{\mathcal{N}_v(E)}{V} dE = \frac{\mathcal{N}_t}{V} \int_0^\infty \sigma(E) \phi(E) dE$$
(4.148)

The neutron flux is defined by $\phi(E) \equiv v \mathcal{N}_{\nu}(E)/V$ and the total flux for all neutron energies is $\phi = \int \phi(E) dE$ (in units of neutrons per area per time). Alternatively, we may express the total number of reactions per volume and per time in terms of an effective reaction cross section $\hat{\sigma}$ by

$$\frac{(\mathcal{N}_R/V)}{t} = \frac{\mathcal{N}_t}{V}\hat{\sigma}\int_0^\infty v \frac{\mathcal{N}_v(E)}{V} dE = \frac{\mathcal{N}_t}{V}\hat{\sigma}\int_0^\infty \phi(E) dE = \frac{\mathcal{N}_t}{V}\hat{\sigma}\phi$$
(4.149)

Equating the above two expressions, we obtain for the effective cross section

$$\hat{\sigma} = \frac{\int_0^\infty \sigma(E) v \frac{N_\nu(E)}{V} dE}{\int_0^\infty v \frac{N_\nu(E)}{V} dE} = \frac{\int_0^\infty \sigma(E) \phi(E) dE}{\int_0^\infty \phi(E) dE} = \frac{N_n \int_0^\infty \sigma(E) v f(E) dE}{N_n \int_0^\infty v f(E) dE}$$
(4.150)

where we used $N_{\nu}(E)/V = f(E)N_n$, with N_n the total number density of neutrons.

If the energies of the incident neutrons are given by a Maxwell–Boltzmann distribution (Section 4.1.2 and Fig. 4.2), then one finds with Eq. (3.8) for the flux

$$\phi = \int_0^\infty \phi(E) \, dE = N_n \int_0^\infty v f(E) \, dE$$

= $N_n \int_0^\infty \sqrt{\frac{2E}{m_{01}}} \frac{2}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} \sqrt{E} e^{-E/kT} \, dE = N_n \frac{2}{\sqrt{\pi}} \sqrt{\frac{2kT}{m_{01}}} = \frac{2}{\sqrt{\pi}} N_n v_T$
(4.151)

The effective cross section is given by Eqs. (3.8), (3.69), (4.150), and (4.151),

$$\hat{\sigma} = \frac{N_{\rm n} \int_0^\infty \sigma(E) v f(E) \, dE}{N_{\rm n} \frac{2}{\sqrt{\pi}} v_T} = \int_0^\infty \sigma(E) \sqrt{\frac{2E}{m_{01}} \frac{1}{v_T (kT)^{3/2}} \sqrt{E} e^{-E/kT} \, dE} = \frac{1}{(kT)^2} \int_0^\infty \sigma(E) E e^{-E/kT} \, dE = \frac{\sqrt{\pi}}{2} \frac{\langle \sigma v \rangle}{v_T}$$
(4.152)

Hence, the measured effective reaction cross section gives directly the reaction rate (Section 4.1.2).

4.9.4

Measured Yields and Transmissions

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The transmission can be expressed either in terms of intensities (see Eq. (4.35)) or in terms of counting rates,

$$T \equiv \frac{I}{I_0} = \frac{dC/dt}{dC_0/dt} \tag{4.153}$$

where dC/dt and dC_0/dt are the measured counting rates with and without sample, respectively, between incident neutron beam and detector. The transmission is obviously independent of absolute detection efficiencies. The expressions for *T* given above is strictly valid only if neutrons that undergo an interaction in the sample are *not* counted by the detector. However, for any sample and detector of finite size, neutrons scattered forward in the sample toward the detector will be counted as if no interaction had occurred. Corrections for this *in-scattering effect* (Miller 1963) can be obtained most reliably from Monte Carlo simulations.

In terms of experimental quantities, the yield can be expressed by

$$Y = \frac{\mathcal{N}_R}{\mathcal{N}_\nu} = \frac{C}{\eta B f \mathcal{N}_\nu} = \frac{C}{\eta B f \Phi A}$$
(4.154)

where N_{ν} and *C* are the total number of incident neutrons and the measured total number of counts induced by the nuclear reaction of interest, respectively; $\Phi = \int \phi(t) dt$ is the time-integrated neutron flux (in units of particles per area); A is the area of the sample exposed to the beam; η is the detection efficiency; *B* is the branching ratio (probability of emission per nuclear reaction); and the factor f takes any necessary corrections into account (for example, for multiple neutron scattering in the sample, self-absorption of reaction products, and so forth). Depending on the experimental procedure, the yield may also need to be corrected for angular correlation effects (Appendix D). Multiple elastic scattering of neutrons may become a serious problem for thicker samples. Scattered neutrons have a much higher chance of undergoing a reaction than the incident neutrons because of an increase in the average path length in the sample. The situation becomes even more complex when the total and the reaction cross section exhibit a narrow resonance structure. In this case, incident neutrons with energies somewhat higher than the location of the narrow resonance are scattered and thereby lose a fraction of their energy. These scattered neutrons may then undergo reactions in the region of the resonance. Consequently, the measured reaction yield can become much larger than the true reaction yield caused by the incident (unscattered) neutrons. Such effects can be corrected for by using Monte Carlo procedures (Poenitz 1984).

Examples of a yield curve and a transmission curve, both measured in the 144 Sm + n reaction, are shown in Fig. 4.65. The data have been obtained with the time-of-flight technique (Section 4.6.3) using the Oak Ridge Electron Accelerator (ORELA). The neutron-capture data are fitted with Breit–Wigner expressions, while the transmission data are analyzed using an *R*-matrix code to account for the additional complication of potential scattering (Section 2.5). Necessary corrections caused by Doppler broadening, multiple scattering, and instrumental resolution were applied to both data sets. The shapes of the



Fig. 4.65 (Top) Yield curve for the 144 Sm(n, γ) 145 Sm reaction; (bottom) Transmission curve for 144 Sm + n. The data have been obtained with the time-of-flight technique (Section 4.6.3) using the Oak Ridge Electron Accelerator (ORELA). The neutron-capture data (top) are fitted with

Breit–Wigner expressions, while the transmission data (bottom) are analyzed using an *R*-matrix code. Reprinted with permission from R. L. Macklin et al., Phys. Rev. C, Vol. 48, p. 1120 (1993). Copyright (1993) by the American Physical Society.

narrow resonance at the lowest energy are dominated by the instrumental resolution and by Doppler broadening, while the shapes of the broad resonances at higher energies are dominated by their total widths.

4.9.5

Relative and Absolute Cross Sections

We will first discuss the determination of an unknown cross section relative to a standard value. The activation method (Section 4.6.2) will be chosen as an example. The situation is shown in Fig. 4.66. A proton beam is incident on a Li target which is mounted on a water-cooled Cu backing. The irradiation sample is mounted close to a foil consisting of a material relative to which the neutron cross section is being measured (for example, a gold foil). For the sake of simplicity, we will assume that the incident neutron flux is constant, $\phi(t) = \text{const}$ (see Beer and Käppeler (1980) for a time-dependent flux). After the irradiation period is over (at $t = t_0$), the samples are moved to an off-line detector system for the counting of the delayed activity between t_1 and t_2 .

The number of disintegrations between t_1 and t_2 , $D(t_1, t_2)$, is related to the net number of counts, *C*, in the region of interest in the off-line pulse height spectrum by

$$D(t_1, t_2) = \frac{C}{\eta B f} \tag{4.155}$$

with η and *B* the detection efficiency and the branching ratio of a particular transition, respectively; the factor *f* takes any necessary corrections into account (self-absorption of γ -rays in the sample, multiple elastic scattering of neutrons, and so on). Using Eqs. (4.76) and (4.155) and solving for the effective cross section gives

$$\hat{\sigma} = \frac{C\lambda}{\eta B f \phi \mathcal{N} \left(e^{\lambda t_0} - 1 \right) \left(e^{-\lambda t_1} - e^{-\lambda t_2} \right)} \tag{4.156}$$

where λ denotes the decay constant of the residual radioactive nuclei and N is the number of sample nuclei. The ratio of effective cross sections for the sample of interest, *i*, and the standard material, *s*, is then

$$\frac{\hat{\sigma}_i}{\hat{\sigma}_s} = \frac{C_i \lambda_i \eta_s B_s f_s \mathcal{N}_s \left(e^{\lambda_s t_0} - 1\right) \left(e^{-\lambda_s t_1} - e^{-\lambda_s t_2}\right)}{C_s \lambda_s \eta_i B_i f_i \mathcal{N}_i \left(e^{\lambda_i t_0} - 1\right) \left(e^{-\lambda_i t_1} - e^{-\lambda_i t_2}\right)}$$
(4.157)

The relative determination of an effective cross section according to Eq. (4.157) has the advantage that the total neutron flux, $\phi(t) = \Phi/t_0 = \text{const}$, cancels if the sample of interest and the standard sample are irradiated simultaneously (corrections are necessary for a time-dependent flux). Furthermore, only *relative* detection efficiencies are required if the sample of interest and the standard sample are measured with the same experimental setup. However, the number of nuclei in the two samples, N_i and N_s , have to be determined carefully. The number of sample nuclei is given by (see Eq. (1.13))

$$\mathcal{N} = \frac{m_{\text{sample}} N_A}{M} X \tag{4.158}$$

with m_{sample} and M the mass and relative mass of the sample, respectively. If the sample consists of a compound, then m_{sample} , M, and the mass fraction Xrefer to the active sample nuclei, that is, the nuclei participating in the reaction of interest. Masses of self-supporting samples are frequently determined by weighing, whereas masses of deposited samples can be found from the weight difference between the backing and the combined sample-plus-backing. For compounds or samples consisting of more than one isotope, a chemical or isotopic analysis is required to obtain the number of nuclei \mathcal{N} (Wagemans 1989).

The 197 Au $(n,\gamma)^{198}$ Au capture reaction provides one of the most widely used absolute cross section standards in the keV neutron energy range, that is, the

region of astrophysical interest. We will describe in the following an elegant method for determining this cross section standard by using the activation method. For more information, see Ratynski and Käppeler (1988). Suppose that in Fig. 4.66 a proton beam of $E_p = 1912$ keV energy is incident on a thick Li target which is mounted on a water-cooled Cu backing. As pointed out previously (Section 4.1.2), under such circumstances the neutron energy distribution closely resembles a Maxwell–Boltzmann distribution at kT = 25 keV (Fig. 4.2) and all the neutrons are kinematically focussed in the forward direction into a cone with an opening angle of 120°. A gold sample covers the entire solid angle of the neutron emission cone. It consists of a homogeneous spherical segment instead of a flat foil, so that the sample appears equally thick for all neutrons passing through it. The number of Au nuclei is determined by carefully measuring the weight and the thickness of the sample. The half-life of ¹⁹⁸Au amounts to $T_{1/2}$ = 2.6 d and the decay produces a γ -ray of 412 keV energy. We will again assume for simplicity that the neutron flux is constant, that is, $\phi(t) = \text{const.}$ According to Eqs. (4.76) and (4.155), we find for the total number of disintegrations during the measuring interval between t_1 and t_2

$$D_{Au}(t_{1}, t_{2}) = \frac{C_{Au}}{(\eta B f)_{Au}} = \frac{(\mathcal{N}_{Au}/A)\hat{\sigma}_{Au}\mathcal{N}_{\nu}(t_{0})}{\lambda_{Au}t_{0}} \left(e^{\lambda_{Au}t_{0}} - 1\right) \left(e^{-\lambda_{Au}t_{1}^{Au}} - e^{-\lambda_{Au}t_{2}^{Au}}\right)$$
(4.159)

where we used $\phi = \Phi/t_0 = N_{\nu}(t_0)/(At_0)$. The quantity $N_{\nu}(t_0)$ is the total number of incident neutrons after irradiation time t_0 and A denotes the area covered by the sample.

Since the ⁷Li(p,n)⁷Be reaction is used as a source of neutrons, there is one ⁷Be nucleus ($T_{1/2} = 53$ d) produced for each emitted neutron. Hence, the total number of neutrons emitted from the Li target, and that are incident on the Au foil, can be deduced by measuring the 478 keV γ -rays emitted by the radioactive decay of ⁷Be in the Li target. The proton energy loss in the Li production target amounts to about 100 keV. Therefore, the production rate of ⁷Be is given by Eq. (4.110) rather than Eq. (4.73). We will simply assume that the incident proton current, and hence the production rate of ⁷Be, is constant. The total number of neutrons, or the total number of ⁷Be nuclei, produced after a time t_0 is then given by $\mathcal{N}_{\text{Be}}(t_0) = \mathcal{N}_{\nu}(t_0) = P_{\text{Be}}t_0$. For the number of disintegrations during a measuring interval between t_1 and t_2 we obtain from Eqs. (4.76) and (4.155)

$$D_{Be}(t_{1}, t_{2}) = \frac{C_{Be}}{(\eta B f)_{Be}} = \frac{P_{Be}}{\lambda_{Be}} \left(e^{\lambda_{Be}t_{0}} - 1 \right) \left(e^{-\lambda_{Be}t_{1}^{Be}} - e^{-\lambda_{Be}t_{2}^{Be}} \right) = \frac{\mathcal{N}_{\nu}(t_{0})}{\lambda_{Be}t_{0}} \left(e^{\lambda_{Be}t_{0}} - 1 \right) \left(e^{-\lambda_{Be}t_{1}^{Be}} - e^{-\lambda_{Be}t_{2}^{Be}} \right)$$
(4.160)





Fig. 4.66 Example for the measurement of a neutron cross section using the activation technique (Section 4.6.2). A proton beam is incident on a Li target which is mounted on a water-cooled Cu backing. The irradiation sample is mounted close to a foil consisting

of a material relative to which the neutron cross section is measured (for example, a gold foil). After the irradiation period is over, the samples are moved to an offline detector system for the counting of the delayed activity (bottom part).

From Eqs. (4.159) and (4.160) one finds for the 197 Au(n, γ) 198 Au cross section

$$\hat{\sigma}_{Au} = \frac{1}{(\mathcal{N}_{Au}/A)} \frac{C_{Au}(\eta B f)_{Be} \lambda_{Au}}{C_{Be}(\eta B f)_{Au} \lambda_{Be}} \frac{(e^{\lambda_{Be}t_0} - 1) \left(e^{-\lambda_{Be}t_1^{Be}} - e^{-\lambda_{Be}t_2^{Be}}\right)}{(e^{\lambda_{Au}t_0} - 1) \left(e^{-\lambda_{Au}t_1^{Au}} - e^{-\lambda_{Au}t_2^{Au}}\right)}$$
(4.161)

In this expression the number of neutrons cancels and only relative detection efficiencies are needed if the same setup is used for counting the delayed activities of ¹⁹⁸Au and ⁷Be. The measured average cross section amounts to $\hat{\sigma}_{197Au(n,\gamma)}{}^{198}Au = 586 \pm 8$ mb (Ratynski and Käppeler 1988) and corresponds to a (quasi-)Maxwellian neutron energy distribution at kT = 25 keV. The error represents an uncertainty of only 1.4%. This standard has been used for the determination of a large number of astrophysically important neutron-capture cross sections. Other standard cross sections are provided by the ⁶Li(n, α)³H, ¹⁰B(n, α)⁷Li, and ¹⁰B(n, $\alpha\gamma$)⁷Li reactions. More information can be found in Bao et al. (2000).

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Problems

4.1 The energy loss of charged particles is calculated in Example 4.1 by using the "thin-absorber approximation," that is, by assuming that the stopping power is approximately constant over the absorber thickness. If the stopping power is not constant, the energy loss can always be obtained from a numerical integration of Eq. (4.12). If a graph of range versus energy is available, however, a much simpler method can be used by expressing Eq. (4.12) in terms of ranges. Explain this method and use it to estimate from Fig. 4.7 the energy loss of a 10 MeV cosmic-ray proton incident on a 400 µm thick silicon detector.

4.2 Derive Eq. (4.18) from the expressions for energy and linear momentum conservation (see Eqs. (C.1)–(C.3)) in the elastic scattering of an incident particle on an electron at rest. Assume a head-on collision for maximum energy transfer.

4.3 Calculate the attenuation of 0.5 MeV and 5 MeV γ -rays in (i) a tantalum absorber ($\rho_{Ta} = 16.7 \text{ g/cm}^3$) of 0.5 mm thickness, and (ii) a lead absorber ($\rho_{Pb} = 11.4 \text{ g/cm}^3$) of 1.3 cm thickness. Assume the following numerical values for the mass attenuation coefficients: (μ/ρ)_{Ta,0.5 MeV} = 0.13 cm²/g, (μ/ρ)_{Ta,5.0 MeV} = 0.041 cm²/g, (μ/ρ)_{Pb,0.5 MeV} = 0.16 cm²/g, (μ/ρ)_{Pb,5.0 MeV} = 0.041 cm²/g.

4.4 Estimate the thickness of water ($\rho = 1.0 \text{ g/cm}^3$) necessary in order to reduce the intensity of incident neutrons with an energy of 300 keV by a factor of 10^{10} . Assume for the total neutron cross section at this energy a value of 60 b.

4.5 Solve the equations for total energy and linear momentum conservation in α -decay when the decaying nucleus is at rest. Apply the expressions to the α -decay of ²⁴¹Am and calculate the total energy release (or the *Q*-value) by using the information given in the α -particle spectrum shown in Fig. 4.19. Assume that the α -particle group with the largest kinetic energy populates the ground state of the daughter nucleus ²³⁷Np and consider only this particular transition. What is the kinetic energy of the daughter nucleus?

4.6 Suppose that an excited nuclear level (2) decays to the ground state (0) via a two- γ -ray cascade through an intermediate state (1), that is, $B_{21} = B_{10} = 1$ and $B_{20} = 0$ (Fig. 4.33). The energies of the photons are $E_{21} = 1$ MeV and $E_{10} = 2$ MeV. Their measured peak intensities are $\mathcal{N}_{21} = 357$ and $\mathcal{N}_{10} = 237$. The values for the peak and total efficiencies amount to $\eta_{21}^P = 0.043$, $\eta_{21}^T = 0.21$, $\eta_{10}^P = 0.030$, $\eta_{10}^T = 0.17$. (i) Calculate the total number \mathcal{N} of decaying levels 2 with and without coincidence summing corrections. (ii) What do you expect for the intensity of the sum peak at 3 MeV?

4.7 A gold sample with a mass of 10 g is irradiated with a thermal neutron flux of 10^{14} cm⁻² s⁻¹. The cross section for the ¹⁹⁷Au(n, γ)¹⁹⁸Au reaction amounts to 99 b and the half-life of ¹⁹⁸Au is $T_{1/2} = 2.7$ d. (i) Calculate the saturation

value for the number of radioactive ¹⁹⁸Au nuclei. (ii) What is the irradiation time necessary until the number of ¹⁹⁸Au nuclei achieves 90% of the saturation value?

4.8 Consider a measurement of the ²¹Na + p \rightarrow ²²Mg + γ radiative capture reaction in inverse kinematics, that is, by bombarding a stationary hydrogen target with radioactive ²¹Na nuclei. The *Q*-value amounts to *Q* = 5504.2 keV. Suppose that the reaction excites the astrophysically important resonance at $E_r^{\rm cm}$ = 206.8 keV. Calculate for the γ -ray transition to the ground state (branching ratio of 14%): (i) the laboratory bombarding energy disregarding any energy losses in the target; (ii) the energy of the photon which is emitted at a laboratory angle of θ = 0°. What are the magnitudes of the full Doppler and recoil energy shifts? Neglect any energy losses of the ²²Mg recoil nuclei in the target; (iii) the maximum laboratory angle $\phi_{\rm max}$ of the ²²Mg recoil emission direction. Use the following values for the masses: $M(^{1}\text{H})$ = 1.0078250 u, $M(^{21}\text{Na})$ = 20.9976546 u, $M(^{22}\text{Mg})$ = 21.9995706 u (Mukherjee et al. 2004).

4.9 Calculate the number of $p(p,e^+\nu)d$ reactions that occur if a pure hydrogen target with a thickness of 10^{20} protons/cm² is bombarded with a proton beam of 1 MeV laboratory energy and 1 A intensity. The *S*-factor is given by Eq. (5.2).

4.10 An α -particle beam of 15 MeV bombarding energy and 1 μ A intensity is incident on a 1 μ m thick, pure ¹²C target ($\rho = 1.9 \text{ g/cm}^3$) for a duration of 1 h. Each incident α -particle has a charge of 2+ (⁴He²⁺). Neutrons are produced via the ¹²C(α ,n)¹⁵O reaction. The cross section at this bombarding energy amounts to 25 mb. A neutron detector has an efficiency of 1%. Assume that both the cross section and the stopping power are constant over the target thickness. How many neutrons are detected?

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5 Nuclear Burning Stages and Processes

In the previous section, we considered the thermonuclear rate of individual nuclear reactions and the relationship of forward and reverse reactions. A particular reaction destroys particles in the incoming channel and creates new particles in the exit channel. In general, however, a number of different nuclear processes take place simultaneously in the stellar plasma. Nuclei that are created by some fusion reactions are destroyed by other reactions. Thus, when discussing stellar nucleosynthesis, it is more appropriate to consider a network of nuclides which are linked by different, and frequently competing, nuclear processes. In this chapter, we will discuss this interplay of different nuclear processes in the stellar plasma.

It was pointed out in Section 1.4.3 that the energy released by thermonuclear reactions supplies the power that is radiated from the stellar surface. At the same time, these nuclear reactions provide the necessary internal pressure that prevents stars from collapsing gravitationally. In fact, all stable stars maintain a hydrostatic equilibrium between the internal pressure and the force of gravity. We showed earlier (Fig. 3.14) that, for a given temperature and composition of the stellar plasma, those reactions with the smallest Coulomb barrier will proceed most rapidly and will account for most of the nuclear energy generation. Consequently, we expect nuclear reactions involving hydrogen and helium to be the main energy sources in most stars. It is to consider the simplest processes among these nuclei, for example, $p + p \rightarrow {}^{2}\text{He}$, $p + {}^{4}\text{He} \rightarrow {}^{5}\text{Li}$, and ${}^{4}\text{He} + {}^{4}\text{He} \rightarrow {}^{8}\text{Be}$, as the most likely nuclear reactions. However, the newly created nuclei ${}^{2}\text{He}$, ${}^{5}\text{Li}$, and ${}^{8}\text{Be}$ are particle unstable and decay back into the entrance channel after very short time periods. Therefore, we have to consider more complicated processes.

It is important to realize that thermonuclear reactions change the composition of the stellar gas. When the nuclei with the smallest Coulomb barrier have been consumed, a star will contract under the influence of gravity. The temperature steadily increases to a point where nuclei with the next lowest Coulomb barrier, which were previously inactive, are consumed. The nuclear energy produced by the burning of the new fuel stabilizes the star against further contraction. Depending on its total mass, a star may experience several

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of these nuclear burning stages, which are referred to as hydrogen burning, helium burning, carbon burning, neon burning, oxygen burning, and silicon burning (Section 1.4.3). When the ignition of the new fuel gives rise to a more advanced burning stage in the stellar core, the previous burning stage may not completely disappear, but may still continue in a shell surrounding the core. We will discuss in the next sections the nuclear physics aspects of the most important burning stages in detail. The advanced burning stages, carbon through silicon burning, will be described with the aid of reaction network calculations performed at constant temperature-density conditions representative of the hydrostatic core burning stages for a star with an initial mass of $M = 25 M_{\odot}$. Stars with this mass have been shown to produce elemental abundances similar to the observed solar system abundances. Hydrostatic shell burning or the explosive burning of carbon, neon, oxygen, and silicon are quantitatively different, although the nucleosynthesis products frequently resemble those of the corresponding hydrostatic core burning stage. In any case, the physical principles will become apparent in the discussion of the respective hydrostatic core burning stages.

The different nuclear burning stages have a profound influence on the structure and evolution of the entire star. The temperature–density evolution for the center of a star with solar initial composition and an initial mass of $M = 25 M_{\odot}$ is shown in Fig. 5.1a. The circles indicate the *T*- ρ conditions that are representative of a particular burning stage in the core. Most of the burning occurs near the location of the circles, since this is where the star spends most of its time during a particular burning stage. It can be seen that the temperature and density vary by about two and eight orders of magnitude, respectively, between hydrogen and silicon burning.

Hydrogen burning releases far more energy per unit fuel consumed ($\approx 6 \times$ 10^{24} MeV/g or $\approx 10^{19}$ erg/g) compared to helium burning ($\approx 6 \times 10^{23}$ MeV/g or $\approx 10^{18}$ erg/g) or more advanced burning stages ($\approx 3 \times 10^{23}$ MeV/g or $\approx 5 \times 10^{17}$ erg/g for carbon and oxygen burning). Thus, a star will consume its hydrogen fuel much more slowly than other fuel in order to balance both gravity and the energy radiated from its surface. There is also a fundamental difference in the manner in which the nuclear energy generated in the stellar interior is transformed and radiated from the surface. For hydrogen and helium burning, nuclear energy is almost exclusively converted to light. In later burning stages, when the temperature exceeds T = 0.5 GK, the thermonuclear energy released is almost entirely radiated as neutrino-antineutrino pairs and the light radiated from the star's surface represents only a very small fraction of the total energy release. Neutrino energy losses rise strongly with temperature (Clayton 1983). Since the temperature increases from one advanced burning stage to the next (Fig. 5.1a), the fuel consumption rapidly accelerates during carbon, neon, oxygen, and silicon burning. This can be seen in Fig. 5.1b,

showing the duration of various burning stages in the stellar core versus initial stellar mass for solar metallicity models. For example, silicon burning in the core of a 25 M_{\odot} star lasts only for ≈ 1 d. In fact, the advanced burning stages in the stellar core proceed so quickly that the stellar surface cannot keep pace with the evolution of the interior. Frequently, the appearance of the massive star—luminosity and effective emission temperature—does not change until the end of hydrostatic silicon burning. It also follows from these considerations that hydrogen burning lasts much longer than helium burning or any of the advanced burning stages. Thus, perhaps as many as $\approx 90\%$ of the observed stars are burning hydrogen. In other words, there is only a small probability of observing stars in their advanced stages of evolution. Although most of a star's life is spent in the hydrogen burning stage, it is the later burning stages that account for the synthesis of the majority of heavy elements in the A = 16-64 mass range.

The synthesis of the majority of the heavier nuclei (A > 60) requires a drastically different mechanism. Their observed abundances cannot be explained by charged-particle fusion reactions since the transmission probability through the Coulomb barrier becomes negligibly small at these higher nuclear charges. Such nuclei are synthesized instead, unhindered by the Coulomb repulsion, via the capture of neutrons. Two distinct neutron capture processes, the s- and the r-process, are discussed toward the end of this chapter. A subsequent section describes the synthesis of those heavy nuclides (p-nuclides) that cannot be accounted for by neutron capture processes. The last section contains information on the origin of the solar system nuclides.

Numerical values of the reaction rates used in this chapter are adopted from the literature. The survey of the literature was concluded in 2006. Many of the reaction rates used here are adopted from published evaluations (Caughlan and Fowler 1988, Angulo et al. 1999, Iliadis et al. 2001). They are based on all available experimental information (cross sections, resonance energies and strengths, excitation energies, spectroscopic factors). These evaluations do not only present reaction rates versus stellar temperature, but also report *uncertainties* for each individual rate. It is important to emphasize that rate uncertainties of some reactions may strongly influence certain stellar model predictions and, therefore, significant experimental efforts are underway to improve the accuracy of many important reaction rates. An investigation of how reaction rate uncertainties influence predictions of isotopic abundances or energy generation is beyond the scope of this book (see, for example, Bahcall et al. 1982, The et al. 1998, Hoffman, Woosley and Weaver 2001, Iliadis et al. 2002, Jordan, Gupta and Meyer 2003). Unless noted otherwise, we are not concerned here with reaction rate errors, but will use the latest published rec*ommended* reaction rates in order to illustrate how the different burning stages influence nuclidic abundances and the nuclear energy generation.

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Fig. 5.1 (a) Central temperature-density evolution of an $M = 25 M_{\odot}$ star with solar initial composition. The circles indicate conditions that are representative of a particular hydrostatic burning stage in the stellar core. (b) Durations of various hydrostatic burning

stages in the stellar core versus initial stellar mass for solar metallicity models. Reprinted with permission from S. E. Woosley, A. Heger and T. A. Weaver, Rev. Mod. Phys., Vol. 74, p. 1015 (2002). Copyright (2002) by the American Physical Society.

Values for mass differences, Q-values, and particle separation energies used in this chapter are adopted from Audi, Wapstra and Thibault (2003). Those reaction rates, stellar enhancement factors, and normalized partition functions that are derived from the Hauser-Feshbach statistical model are adopted from Rauscher and Thielemann (2000), unless noted otherwise.

5.1

Hydrostatic Hydrogen Burning

Hydrogen is the most abundant isotope in the Universe. The fusion of four ¹H nuclei to the tightly bound ⁴He nucleus is called hydrogen burning. Indepen-

dently from the details of this transformation, the process releases an energy (Section 1.5.3) of

$$Q = 4(M.E.)_{H} - (M.E.)_{^{4}He} = 4 \cdot (7288.97 \text{ keV}) - (2424.92 \text{ keV})$$

= 26.731 MeV (5.1)

The obvious question arises as to precisely how this fusion process takes place. Early estimates showed that the probability for the simultaneous interaction of four protons in the stellar plasma is far too small to account for the observed luminosity of stars. Instead, sequences of interactions involving two particles in the entrance channel are much more likely to occur. The two principle ways by which hydrogen is converted to helium in hydrostatic hydrogen burning are called the proton-proton chains and the CNO cycles. These processes were first suggested more than 60 years ago (Atkinson 1936, Bethe and Critchfield 1938, von Weizsäcker 1938, Bethe 1939) and are described in this section. It is useful for the following discussion to keep in mind that, depending on the stellar mass and metallicity, typical temperatures in core hydrogen burning are in the range of $T \approx 8-55$ MK, while the hydrogen burning shells in AGB stars achieve temperatures of $T \approx 45$ –100 MK. The central temperature of the Sun, for example, is T = 15.6 MK (Bahcall 1989). On the other hand, far higher temperatures are attained in explosive hydrogen burning, which will be discussed in later sections. As will be seen, the details of the nuclear processes depend sensitively on the temperature.

5.1.1 pp Chains

The following three sequences of nuclear processes are referred to as protonproton (or pp) chains:

pp1 chain	pp2 chain	pp3 chain
p(p,e ^{+ν})d d(p, γ) ³ He	$p(p,e^+\nu)d$ $d(p,\gamma)^3He$	$p(p,e^+\nu)d$ $d(p,\gamma)^3He$
°He(°He,2p)α	[°] He(α,γ) [°] Be ⁷ Be(e ⁻ , ν) ⁷ Li	°He(α,γ)'Be ⁷ Be(p,γ) ⁸ B
	⁷ Li(p, α) α	⁸ B(β ⁺ ν) ⁸ Be ⁸ Be(α)α
	T _{1/2} : ⁸ B (770 ms)	

The different pp chains are also displayed in Fig. 5.2. Each of these chains starts from hydrogen and converts four protons to one ⁴He nucleus (or α -particle). The first two reactions are the same for each chain. Other nuclear reactions involving the light nuclei ¹H, ²H, ³He, and so on, are less likely to occur in stars (Parker, Bahcall and Fowler 1964).





Fig. 5.2 Representation of the pp chains in the chart of the nuclides. Each arrow represents a specific nuclear interaction connecting the initial with the final nucleus. For example, the reaction ${}^{3}\text{He}(\alpha,\gamma){}^{7}\text{Be}$ is represented by an arrow extending from ${}^{3}\text{He}$ to ${}^{7}\text{Be}$ (middle and bottom panel). Each of the pp chains effectively fuses four protons to one ${}^{4}\text{He}$ nucleus. Stable nuclides are shown as shaded squares.

The p(p, $e^+\nu$)d reaction

The first reaction of each pp chain, ${}^{1}\text{H} + {}^{1}\text{H} \rightarrow {}^{2}\text{H} + e^{+} + \nu$, fuses two protons to one deuterium nucleus. The reaction releases an energy of Q = 1.442 MeV, including the annihilation energy of the positron with another electron from the environment (Example 1.3). The p(p,e^+\nu)d reaction represents a special case since it converts a proton into a neutron, a process that closely resembles a β -decay. Thus, unlike almost all other stellar fusion reactions which are governed exclusively by the strong nuclear force and the Coulomb force, the p(p,e^+\nu)d reaction is influenced by the weak nuclear force as well. Since this

process involves two charged particles in the entrance channel, the overall energy dependence of the cross section is mainly determined by the transmission through the Coulomb barrier. The absolute magnitude of the cross section, however, is relatively small due to the influence of the weak nuclear force. A calculation of the $p(p,e^+\nu)d$ cross section is presented, for example, in Bahcall and May (1969) and is not repeated here. The theoretical *S*-factor is varying smoothly with energy and is given by (Angulo et al. 1999)

$$S(E) = 3.94 \times 10^{-25} + 4.61 \times 10^{-24}E + 2.96 \times 10^{-23}E^2 \qquad (\text{MeV b})$$
(5.2)

For example, at a center-of-mass energy of 0.5 MeV (corresponding to a laboratory proton bombarding energy of 1 MeV) the above *S*-factor translates into a cross section of about $\sigma_{pp} = 8 \times 10^{-48}$ cm². With this cross section, a 1 MeV proton beam of 1 A intensity (6.3×10^{18} protons/s) incident on a dense proton target (10^{20} protons/cm²) will produce a single p + p reaction in about 6 years (Problem 4.9). Such a small event rate seems immeasurably small in the forseeable future and, therefore, the *S*-factor is based entirely on theory. Nevertheless, the different factors that determine the *S*-factor can be calculated with substantial confidence. The quoted reaction rate errors (Angulo et al. 1999) amount to a few percent only and are significantly smaller compared to the rate errors of most measured stellar fusion reactions.

The energy of 1.442 MeV released in the p + p reaction is shared among the reaction products. The neutrino, however, has a large probability for escaping from the star and, hence, its energy is carried away and is not converted into heat. From the detailed shape of the neutrino energy spectrum one obtains an average neutrino energy of about 265 keV (Bahcall 1989). The nuclear energy available from this reaction for conversion into heat is thus 1442 keV - 265 keV = 1177 keV.

Alternative processes to the $p(p,e^+\nu)d$ reaction have been proposed. For example, the reaction ${}^{1}\text{H} + {}^{1}\text{H} + e^- \rightarrow {}^{2}\text{H} + \nu$, which is referred to as the *pep reaction*, also fuses two hydrogen nuclei to one deuterium nucleus. Calculations have shown that this process can compete with $p(p,e^+\nu)d$ only at stellar densities in excess of 10^4 g/cm^3 (Bahcall and May 1969) and, therefore, the pep reaction plays no significant role in hydrostatic hydrogen burning. However, it may contribute to the energy production in the early stages of explosive hydrogen burning (Section 5.2).

The d(p, γ)³He reaction

The deuterium produced in the $p(p,e^+\nu)d$ reaction may, in principle, be destroyed by a number of different interactions. The $d(p,\gamma)^3$ He reaction is by far the most important among these. Other reactions, such as $d + d \rightarrow p + t$ or $d + d \rightarrow n + {}^{3}$ He, may have higher cross sections. However, recall that the reaction rate does not only depend on the cross section but also on the

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abundances of the interacting nuclei (see Eq. (3.6)). Since there are far more protons available compared to the few deuterium nuclei that are produced by the very slow p + p reaction, the d + p interaction is much more likely to occur compared to the d + d interaction.

The d(p, γ)³He reaction (Q = 5.493 MeV) has been measured at center-ofmass energies above $E_{\rm cm} \approx 10$ keV. For the calculation of the reaction rates at all temperatures of practical interest, the data can be reliably extrapolated to zero energy by using, for example, the direct capture model of nuclear reactions. The *S*-factor is given by (Angulo et al. 1999)

$$S(E) = 0.20 \times 10^{-6} + 5.60 \times 10^{-6}E + 3.10 \times 10^{-6}E^2 \qquad (MeV b)$$
(5.3)

This reaction depends only on the electromagnetic and the strong nuclear force. Consequently, the *S*-factor and the reaction rate are larger by many orders of magnitude compared to the p + p reaction. Reaction rate uncertainties amount to about 30–40%. Such errors are typical for many stellar fusion reactions.

pp1 chain

The ³He nuclei created by the two processes discussed above may, in principle, fuse with the abundant protons to form ⁴He via the process ³He + p $\rightarrow \gamma + {}^{4}\text{Li} \rightarrow \gamma + {}^{4}\text{He} + e^{+} + \nu$. However, the ⁴Li nucleus is particle unstable, with a proton separation energy of about -2.5 MeV, and decays back to ³He after a very short time period. It turns out that the ³He(³He,2p)⁴He reaction is the most likely ³He destroying process and it completes the conversion of four protons to one ⁴He nucleus in the pp1 chain. It is not so obvious to see why the ³He(³He,2p)⁴He reaction is much more important compared to another ³He destroying reaction, ³He(d,p)⁴He, especially in view of the fact that both of these reactions have similar cross sections. This question will be addressed, among other issues, in this section. The *S*-factor of the ³He(³He,2p)⁴He reaction is given by (Angulo et al. 1999)

$$S(E) = 5.18 - 2.22 E + 0.80 E^2$$
 (MeV b) (5.4)

Although this *S*-factor is much larger compared to the one for the $d(p,\gamma)^3$ He reaction (see Eq. (5.3)), the ³He(³He,2p)⁴He reaction rate per particle pair is actually much smaller because of the larger value of the product Z_pZ_t , resulting in a much reduced transmission through the Coulomb barrier. This circumstance has an important consequence for the mean lifetimes of deuterium and ³He in the stellar plasma, as will be shown below.

In the following, we will investigate how the ²H and ³He abundances evolve in the pp1 chain. The isotope ²H is created by the p + p reaction and is destroyed via the d + p reaction, while ³He is created by the d + p reaction and

destroyed via the ${}^{3}\text{He} + {}^{3}\text{He}$ reaction. Disregarding at first other reactions, we find by using Eqs. (3.20) and (3.26) for the time dependence of the ${}^{2}\text{H}$ and ${}^{3}\text{He}$ abundances the differential equations

$$\frac{dD}{dt} = r_{\rm pp} - (1 + \delta_{\rm dp})r_{\rm dp} = \frac{H^2 \langle \sigma v \rangle_{\rm pp}}{(1 + \delta_{\rm pp})} - (1 + \delta_{\rm dp})\frac{HD \langle \sigma v \rangle_{\rm dp}}{(1 + \delta_{\rm dp})}$$
$$= \frac{H^2}{2} \langle \sigma v \rangle_{\rm pp} - HD \langle \sigma v \rangle_{\rm dp}$$
(5.5)

$$\frac{d(^{3}He)}{dt} = r_{\rm dp} - (1 + \delta_{^{3}\rm{He}^{3}\rm{He}})r_{^{3}\rm{He}^{3}\rm{He}}$$
$$= DH\langle \sigma v \rangle_{\rm dp} - (^{^{3}}He)^{2} \langle \sigma v \rangle_{^{3}\rm{He}^{3}\rm{He}}$$
(5.6)

In order to avoid confusion, we use italic symbols H, D, and ${}^{3}He$ for the number densities of the isotopes ${}^{1}H$ (or p), ${}^{2}H$ (or d), and ${}^{3}He$, respectively. Note that no Kronecker symbol occurs in front of the first term on the right-hand side of the above equations since a single p + p reaction or a single d + p reaction creates only one ${}^{2}H$ or ${}^{3}He$ nucleus, respectively.

We start with the abundance of ²H. If there is initially no deuterium present in the stellar plasma, then the second term on the right-hand side of Eq. (5.5) is zero. With increasing time the deuterium abundance builds up because of the p + p reaction. The more deuterium is created, the larger the second term describing the destruction of deuterium via the d + p reaction will become. Eventually, an equilibrium, dD/dt = 0, is established. Alternatively, if for some reason the initial deuterium abundance is very large, then the second term on the right-hand side of Eq. (5.5) will dominate over the first term. The deuterium abundance, and hence the second term, will decrease. The more deuterium is depleted, the smaller the second term will become. This continues until an equilibrium, dD/dt = 0, is established. The above equation has been called *self-regulating* (Clayton 1983) since the deuterium abundance always seeks an equilibrium value. The equilibrium ratio $(D/H)_e$, which is obtained for the condition dD/dt = 0, is given by

$$\left(\frac{D}{H}\right)_{e} = \frac{\langle \sigma v \rangle_{\rm pp}}{2\langle \sigma v \rangle_{\rm dp}} = \frac{N_A \langle \sigma v \rangle_{\rm pp}}{2N_A \langle \sigma v \rangle_{\rm dp}} = \frac{\tau_{\rm p}({\rm d})}{2\tau_{\rm p}({\rm p})}$$
(5.7)

The quantity $(D/H)_e$ is determined by the ratio of the p + p and d + p reaction rates and is shown in Fig. 5.3a versus stellar temperature. It can be seen that the $(D/H)_e$ value amounts to about $(D/H)_e \approx 10^{-18}$ – 10^{-17} over most of the relevant temperature range.

We can be more specific and ask how long it takes for the deuterium abundance to achieve equilibrium. It has been shown above that the deuterium lifetime against destruction via the d + p reaction is very short compared to the hydrogen lifetime against destruction via that p + p reaction. Thus, the

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Fig. 5.3 (a) Equilibrium abundance ratio $(D/H)_{\ell}$ versus stellar temperature. (b) Time evolution of the abundance ratio (D/H) for the conditions T = 15 MK, $\rho = 100$ g/cm³, and $X_{\rm H} = 0.5$. The dashed and solid lines are obtained for initial deuterium abundances of $(D/H)_0 = 0$ and $(D/H)_0 = 10^{-5}$, respectively. In either case, the deuterium abundance reaches equilibrium in a time which is negligible compared to the lifetime of stars.

deuterium abundance will change much more quickly compared to the hydrogen abundance. In fact, the difference in the respective lifetimes is so large that it is safe to assume that the deuterium abundance achieves equilibrium in a time too short for the hydrogen abundance to change significantly. With this approximation of a constant hydrogen abundance, Eq. (5.5) can be solved and we obtain

$$\frac{d(D/H)}{dt} = \frac{H}{2} \langle \sigma v \rangle_{\rm pp} - H\left(\frac{D}{H}\right) \langle \sigma v \rangle_{\rm dp}$$
(5.8)

With the substitutions x = (D/H), $a = (H/2)\langle \sigma v \rangle_{pp}$, and $b = H \langle \sigma v \rangle_{dp}$, we write

$$\frac{dx}{dt} = a - bx \tag{5.9}$$

With y = a - bx and dy/dx = -b we obtain, assuming $y = y_0$ at t = 0,

$$\frac{dy}{y} = -b\,dt \qquad \text{and} \qquad y = y_0 e^{-bt} \tag{5.10}$$

Thus, with Eqs. (3.23) and (5.7) one finds

$$H\langle \sigma v \rangle_{\rm dp} \left(\frac{D}{H} \right)_t = \frac{H}{2} \langle \sigma v \rangle_{\rm pp} - \left[\frac{H}{2} \langle \sigma v \rangle_{\rm pp} - H \langle \sigma v \rangle_{\rm dp} \left(\frac{D}{H} \right)_0 \right] e^{-H \langle \sigma v \rangle_{\rm dp} t}$$

$$\left(\frac{D}{H} \right)_t = \left(\frac{D}{H} \right)_e - \left[\left(\frac{D}{H} \right)_e - \left(\frac{D}{H} \right)_0 \right] e^{-t/\tau_{\rm p}({\rm d})}$$
(5.11)

The deuterium abundance approaches its equilibrium value exponentially with a 1/*e* time of $\tau_p(d)$. The time evolution of the quantity (D/H) for the conditions T = 15 MK, $\rho = 100$ g/cm³, and $X_H = 0.5$ is shown in Fig. 5.3b. The two lines are obtained for (i) a zero initial deuterium abundance, $(D/H)_0 = 0$, and (ii) a value of $(D/H)_0 = 10^{-5}$. Clearly, the deuterium abundance reaches equilibrium in a time which is negligible compared to the lifetime of stars.

The very small deuterium-to-hydrogen ratio of $(D/H)_e = 10^{-18} - 10^{-17}$ that is established in the hydrogen burning cores of stars has interesting astrophysical implications. Any significant deuterium abundance that might be present when a star forms will be quickly depleted during the hydrogen burning stage. Since there are no other stellar sites that produce deuterium in significant amounts, deuterium is destroyed as the Universe evolves and interstellar gas is cycled through generations of stars. Observations of the deuterium abundance in the Universe will thus provide lower limits on the primordial deuterium abundance which was established before stellar formation took place. Recent observations indicate a primordial deuterium abundance of about $(D/H)_{\text{prim}} \approx 3 \times 10^{-5}$ (Kirkman et al. 2003). It is commonly assumed that the primordial deuterium was produced during the nucleosynthesis in the early Universe and, therefore, the observed $(D/H)_{prim}$ value provides an important test of standard Big Bang nucleosynthesis. Furthermore, if stars are born from interstellar matter with a (D/H) ratio on the order of 10^{-5} , then the deuterium abundance is sufficiently large for initiating the $d(p,\gamma)^3$ He reaction already at relatively low temperatures during the stellar contraction phase, that is, before the actual hydrogen burning stage. Therefore, it appears that the $d(p,\gamma)^3$ He reaction is in fact the first thermonuclear energy source of some stars. This process is referred to as *deuterium burning* and will not only slow the contraction of the newly forming star, but may also provide an important source of ³He in the young star.

We will next discuss the evolution of the 3 He abundance. Since the deuterium abundance achieves equilibrium in a negligible amount of time, Eq. (5.6) can be simplified using Eq. (5.7),

$$\frac{d(^{3}He)}{dt} = \frac{H^{2}}{2} \langle \sigma v \rangle_{\rm pp} - (^{3}He)^{2} \langle \sigma v \rangle_{^{3}\rm He^{3}\rm He}$$
(5.12)

This expression is also self-regulating in the sense that the ³He abundance will seek an equilibrium value. The equilibrium ratio $({}^{3}He/H)_{e}$ is again obtained for the condition $d({}^{3}He)/dt = 0$, with the result

$$\left(\frac{{}^{3}He}{H}\right)_{e} = \sqrt{\frac{\langle \sigma v \rangle_{\rm pp}}{2\langle \sigma v \rangle_{^{3}{\rm He}^{3}{\rm He}}}} = \sqrt{\frac{N_{A}\langle \sigma v \rangle_{\rm pp}}{2N_{A}\langle \sigma v \rangle_{^{3}{\rm He}^{3}{\rm He}}}}$$
(5.13)

The quantity $({}^{3}He/H)_{e}$ is determined by the ratio of the p + p and ${}^{3}He + {}^{3}He$ reaction rates and is shown in Fig. 5.4a versus stellar temperature. Since

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Fig. 5.4 (a) Equilibrium abundance ratio $({}^{3}He/H)_{e}$ versus stellar temperature. (b) Time required for 3 He to reach 99% of its equilibrium abundance versus temperature. The curve is calculated for the conditions $\rho = 100 \text{ g/cm}^{3}$ and $X_{\rm H} = 0.5$.

the ${}^{3}\text{He}({}^{3}\text{He},2p){}^{4}\text{He}$ reaction rate is much smaller compared to the $d(p,\gamma){}^{3}\text{He}$ reaction rate, the ${}^{3}\text{He}$ abundance builds up to a much larger value compared to the deuterium abundance in order to achieve equilibrium.

The time it takes for the ³He abundance to achieve equilibrium can be calculated assuming that the hydrogen abundance stays nearly constant. It can be seen from Fig. 5.4a that this is a reasonable assumption for temperatures above T = 6 MK where $({}^{3}He/H)_{e} < 0.01$. With the approximation of a constant hydrogen abundance, Eq. (5.12) can be solved and we obtain

$$\frac{d(^{3}He/H)}{dt} = \frac{H}{2} \langle \sigma v \rangle_{\rm pp} - H \left(\frac{^{3}He}{H}\right)^{2} \langle \sigma v \rangle_{^{3}\rm He^{3}\rm He}$$
(5.14)

Using the substitutions $x = ({}^{3}He/H)$, $a = (H/2)\langle \sigma v \rangle_{pp}$, and $b = H \langle \sigma v \rangle_{{}^{3}He^{3}He}$ we write

$$\frac{dx}{dt} = a - bx^2 \qquad \text{or} \qquad \frac{1}{a}\frac{dx}{dt} = 1 - \frac{b}{a}x^2 \tag{5.15}$$

From $y = x\sqrt{b/a}$ and $dy/dx = \sqrt{b/a}$, we obtain, assuming y = 0 at t = 0,

$$\frac{dy}{1-y^2} = a\sqrt{\frac{b}{a}}\,dt \qquad \text{or} \qquad y = \tanh(t\sqrt{ab}) \tag{5.16}$$

From Eqs. (3.23) and (5.13) we find

$$\left(\frac{{}^{3}He}{H}\right)_{t} = \sqrt{\frac{\langle \sigma v \rangle_{\rm pp}}{2\langle \sigma v \rangle_{\rm 3He^{3}He}}} \tanh\left(t\sqrt{\frac{H}{2}\langle \sigma v \rangle_{\rm pp}}H\langle \sigma v \rangle_{\rm 3He^{3}He}}\right)$$

$$= \left(\frac{{}^{3}He}{H}\right)_{e} \tanh\left(\frac{t}{[\tau_{\rm 3He}({}^{3}He)]_{e}}\right)$$
(5.17)

Note that we have explicitly assumed that the initial ³He abundance is zero (y = 0 at t = 0). The quantity $[\tau_{^{3}\text{He}}(^{^{3}}He)]_{e}$ denotes the mean lifetime of ³He against destruction via the ³He(³He,2p)⁴He reaction after ³He has reached its equilibrium value. The time required for the ³He abundance to achieve a fraction $f = (^{^{3}}He/H)_{t}/(^{^{3}}He/H)_{e}$ of its equilibrium abundance is obtained from

$$t_f = [\tau_{^3\text{He}}(^3\text{He})]_e \operatorname{arctanh}(f) = \frac{\operatorname{arctanh}(f)}{\rho \frac{X_{\text{H}}}{M_{\text{H}}} N_A \langle \sigma v \rangle_{^3\text{He}^3\text{He}} \left(\frac{^3\text{He}}{H}\right)_e}$$
(5.18)

This time is shown in Fig. 5.4b at temperatures above T = 6 GK (where the hydrogen abundance is approximately constant) for the conditions f = 0.99, $\rho = 100$ g/cm³, and $X_{\rm H} = 0.5$. It can be seen that the value of t_f exceeds 10^9 years below $T \approx 8$ MK and becomes comparable to the lifetime of some stars. For sufficiently small temperatures, the ³He abundance will never reach equilibrium. For a temperature of T = 15 MK, on the other hand, the ³He abundance increases gradually and reaches an equilibrium of $({}^3He/H)_e = 10^{-5}$ after about 10^6 years.

It is interesting to compare the mean lifetimes $\tau_p(p)$, $\tau_p(d)$, $\tau_\alpha({}^{3}He)$, $[\tau_d(d)]_e$, $[\tau_d({}^{3}He)]_e$, $[\tau_{3He}(d)]_e$, and $[\tau_{3He}({}^{3}He)]_e$. The first three quantities are given by the usual relation (see Eq. (3.22)), while, for example, the fifth quantity denotes the mean lifetime of 3 He against destruction via the 3 He(d,p)⁴He reaction *after the deuterium abundance has reached an equilibrium value*. From Eqs. (3.22), (5.7), and (5.13) we find

$$[\tau_{\rm d}({\rm d})]_e = \left(\frac{N_A \langle \sigma v \rangle_{\rm pp}}{2N_A \langle \sigma v \rangle_{\rm dp}} \rho \frac{X_{\rm H}}{M_{\rm H}} N_A \langle \sigma v \rangle_{\rm dd}\right)^{-1}$$
(5.19)

$$[\tau_{\rm d}({}^{3}He)]_{e} = \left(\frac{N_{A}\langle\sigma v\rangle_{\rm pp}}{2N_{A}\langle\sigma v\rangle_{\rm dp}}\rho\frac{X_{\rm H}}{M_{\rm H}}N_{A}\langle\sigma v\rangle_{{}^{3}{\rm He}\,{\rm d}}\right)^{-1}$$
(5.20)

$$[\tau_{^{3}\text{He}}(d)]_{e} = \left(\sqrt{\frac{N_{A}\langle\sigma v\rangle_{\text{pp}}}{2N_{A}\langle\sigma v\rangle_{^{3}\text{He}^{^{3}\text{He}}}}}\rho\frac{X_{\text{H}}}{M_{\text{H}}}N_{A}\langle\sigma v\rangle_{^{3}\text{He}\,d}\right)^{-1}$$
(5.21)

$$[\tau_{^{3}\mathrm{He}}(^{^{3}}\mathrm{He})]_{e} = \left(\sqrt{\frac{N_{A}\langle\sigma v\rangle_{\mathrm{pp}}}{2N_{A}\langle\sigma v\rangle_{^{3}\mathrm{He}^{^{3}}\mathrm{He}}}}\rho\frac{X_{\mathrm{H}}}{M_{\mathrm{H}}}N_{A}\langle\sigma v\rangle_{^{3}\mathrm{He}^{^{3}}\mathrm{He}}\right)^{-1}$$
(5.22)

The subscripts pp, dp, dd, ³Hed, and ³He³He denote the reactions $p(p,e^+\nu)d$, $d(p,\gamma)^{3}He$, $d(d,n)^{3}He$, ³He(d,p)⁴He, and ³He(³He,2p)⁴He, respectively. The mean lifetimes are calculated for the conditions $\rho = 100 \text{ g/cm}^{3}$, $X_{H} = X_{He} = 0.5$ and are shown in Fig. 5.5a. Several important points can be made. First, it can be seen that $\tau_p(d) \ll [\tau_{3He}(d)]_e \ll [\tau_d(d)]_e$ and hence the assumption of Eq. (5.5) that deuterium is predominantly destroyed via the $d(p,\gamma)^{3}He$




Fig. 5.5 Mean lifetimes versus stellar temperature, calculated for the conditions $\rho = 100 \text{ g/cm}^3$ and $X_{\rm H} = X_{\rm He} = 0.5$. In (a) the operation of only the pp1 chain is considered, while in (b) all three pp chains are assumed to operate simultaneously. Note that for hydrostatic hydrogen burning only temperatures below T = 0.1 GK are of interest.

reaction is justified. Second, we have $[\tau_{^{3}\text{He}}(^{^{3}}\text{He})]_{e} \ll [\tau_{d}(^{^{3}}\text{He})]_{e}$ and thus ³He is predominantly destroyed via the ³He(³He,2p)⁴He reaction, while the ³He(d,p)⁴He reaction plays no significant role after ³He has reached its equilibrium value. We suspect that the ³He(d,p)⁴He reaction is more likely to occur than the ³He(³He,2p)⁴He reaction only before ³He reaches equilibrium when its abundance is still very small. Under such conditions, however, the production rate of ³He via the two reactions $p(p,e^{+}\nu)d$ and $d(p,\gamma)^{3}$ He is much larger compared to the destruction rate so that the latter can be neglected. Therefore, the assumption of Eq. (5.6) that ³He is predominantly destroyed in the pp1 chain via the ³He(³He,2p)⁴He reaction is justified.

The energy production rate in the pp1 chain may be expressed as a sum of two parts. The first step involves the $p(p,e^+\nu)d$ and $d(p,\gamma)^3$ He reactions. Their cumulative effect is to convert three protons to one ³He nucleus at a rate that

is given by the much slower $p(p,e^+\nu)d$ reaction. The energy produced is calculated according to Eq. (1.10) from the atomic mass excesses and amounts to 6.936 MeV. Subtracting an average neutrino energy of 0.265 MeV (see above) yields an energy of 6.671 MeV, that is available to the star. In the second step, the ³He(³He,2p)⁴He reaction releases an energy of 12.861 MeV. The total energy production rate in the pp1 chain is then given by (see Eq. (3.63))

$$\varepsilon_{\rm pp1} = \frac{6.671 \,\,\mathrm{MeV}}{\rho} r_{\rm pp} + \frac{12.861 \,\,\mathrm{MeV}}{\rho} r_{^3\mathrm{He}^3\mathrm{He}}$$
$$= \frac{6.671 \,\,\mathrm{MeV}}{2\rho} H^2 \langle \sigma v \rangle_{\rm pp} + \frac{12.861 \,\,\mathrm{MeV}}{2\rho} H^2 \left(\frac{{}^3\mathrm{He}}{\mathrm{H}}\right)^2 \langle \sigma v \rangle_{^3\mathrm{He}^3\mathrm{He}} \qquad (5.23)$$

The energy generation rate depends on whether the ³He abundance has achieved equilibrium or not. In the general case, when equilibrium has not been reached yet, the ³He abundance and the corresponding energy generation rate are changing with time and both quantities have to be computed numerically. Alternatively, for the conditions of a constant temperature, for $({}^{3}He/H)_{e} < 0.01$ (that is, a constant hydrogen abundance) and a zero initial ³He abundance, the ratio $({}^{3}He/H)$ can be approximated by Eq. (5.17). The expression for the energy generation rate simplifies considerably after the ³He abundance has achieved equilibrium. From Eqs. (5.13) and (5.23) we find

$$\varepsilon_{\rm pp1}^{e} = \frac{6.671 \,\,{\rm MeV}}{2\rho} H^{2} \langle \sigma v \rangle_{\rm pp} + \frac{12.861 \,\,{\rm MeV}}{2\rho} \frac{H^{2}}{2} \frac{N_{A} \langle \sigma v \rangle_{\rm pp}}{N_{A} \langle \sigma v \rangle_{^{3}{\rm He}^{3}{\rm He}}} \,\langle \sigma v \rangle_{^{3}{\rm He}^{3}{\rm He}}$$
$$= 6.551 \,N_{A} \langle \sigma v \rangle_{\rm pp} \left(\frac{X_{\rm H}}{M_{\rm H}}\right)^{2} \rho N_{A} \qquad ({\rm MeV} \,{\rm g}^{-1} \,{\rm s}^{-1})$$
(5.24)

The energy generation rate in the pp1 chain at ³He equilibrium is determined by the p + p reaction rate. The temperature dependence of ε_{pp1}^{e} is thus given by Eq. (3.87). For example, near $T_0 = 15$ MK we obtain $\tau = 13.6$ for the p(p,e⁺ ν)d reaction, implying

$$\varepsilon_{\text{pp1}}^{e}(T) = \varepsilon_{\text{pp1}}^{e}(T_0) \left(T/T_0\right)^{(\tau-2)/3} = \varepsilon_{\text{pp1}}^{e}(T_0) \left(T/T_0\right)^{3.9}$$
(5.25)

The quantity ε_{pp1}^{e} will be presented versus temperature in Section 5.1.2 and will be compared to the energy generation rate from the CNO cycles.

pp2 and pp3 chains

So far we have neglected reactions other than ${}^{3}\text{He}({}^{3}\text{He},2p){}^{4}\text{He}$ that destroy ${}^{3}\text{He}$. Figure 5.5a also compares the quantity $[\tau_{3}_{\text{He}}({}^{3}\text{He})]_{e}$ with the lifetime of ${}^{3}\text{He}$ against destruction via the ${}^{3}\text{He}(\alpha,\gamma){}^{7}\text{Be}$ reaction, $\tau_{\alpha}({}^{3}\text{He})$. It can be seen that the ${}^{3}\text{He}(\alpha,\gamma){}^{7}\text{Be}$ reaction becomes the dominant destruction mechanism for ${}^{3}\text{He}$ if the temperature and the ${}^{4}\text{He}$ abundance are sufficiently large. The

⁴He may either be produced during hydrogen burning or may be of primordial origin. Following the 3 He(α, γ)⁷Be reaction, the ⁷Be nucleus may β -decay to ⁷Li and the subsequent ⁷Li(p, α)⁴He reaction completes the conversion of four protons to one ⁴He nucleus. This reaction sequence is referred to as the pp2 chain (Fig. 5.2). The β -decay of ⁷Be has interesting properties. In the laboratory, ⁷Be has a half-life of $T_{1/2}$ = 53 d and decays by capture of an atomic electron, $^{7}\text{Be} + e^{-} \rightarrow ^{7}\text{Li} + \nu$. In the stellar plasma, on the other hand, ^{7}Be is partially ionized and the decay can occur either by capture of one of the remaining atomic electrons or of a free electron from the surrounding continuum (Section 1.8.4). A calculation of the electron capture rate of 7 Be in the stellar environment can be found in Bahcall and Moeller (1969). In order to calculate the decay constant, the tabulated values for the rate have to be multiplied by n_{e^-}/N_A , where n_{e^-} denotes the electron density (see Eq. (1.65)). It will be shown below that this decay rate depends only weakly on temperature, in contrast to the strong temperature dependence of charged-particle reactions. In particular, at sufficiently high temperatures, the $^{7}Be(p,\gamma)^{8}B$ reaction instead of ⁷Be(e⁻, ν)⁷Li becomes the dominant ⁷Be destruction mechanism. The β^+ decay of ⁸B to ⁸Be and the subsequent breakup of the particle-unstable ⁸Be nucleus, ⁸Be $\rightarrow \alpha + \alpha$, complete the so-called pp3 chain (Fig. 5.2).

The pp2 and pp3 chains have an output of two α -particles, but require an input of one α -particle. The net effect is the fusion of one ⁴He nucleus per reaction sequence and hence one of the α -particles acts only as a catalyst which allows the ³He(α , γ)⁷Be reaction to take place. The total energy released in any of the chains is the same (26.731 MeV) but the amount of energy carried away by the neutrinos will be different in each case. The nuclear energy available to the star for conversion to thermal energy for each chain is given by

$$Q_{\rm pp1} = 26.73 \,\mathrm{MeV} - 2\overline{E}_{\nu}^{\rm pp} = 26.19 \,\mathrm{MeV}$$
 (5.26)

$$Q_{\rm pp2} = 26.73 \,\mathrm{MeV} - \overline{E}_{\nu}^{\rm pp} - \overline{E}_{\nu}^{\rm Be} = 25.65 \,\mathrm{MeV}$$
 (5.27)

$$Q_{\rm pp3} = 26.73 \,\mathrm{MeV} - \overline{E}_{\nu}^{\rm oB} - \overline{E}_{\nu}^{\rm oB} = 19.75 \,\mathrm{MeV}$$
 (5.28)

The average neutrino energies \overline{E}_{ν}^{l} are adopted from Bahcall (1989). The neutrino losses in the pp1, pp2, and pp3 chains amount to 2%, 4%, and 26%, respectively.

In a hydrogen-burning star that contains a significant ⁴He abundance, all three pp chains will operate simultaneously. The contributions of the different chains to the energy production and the nucleosynthesis depend on the conditions of temperature, density, and composition. If we consider a situation where convection, expansion and mixing in the stellar plasma can be disregarded, then nuclear transformations are the only source of abundance changes. In this case one obtains the following set of nonlinear coupled differential equations

$$\frac{dH}{dt} = 2\frac{({}^{3}He)^{2}\langle\sigma v\rangle_{{}^{3}\text{He}^{3}\text{He}}}{2} - 2\frac{H^{2}\langle\sigma v\rangle_{\text{pp}}}{2} - HD\langle\sigma v\rangle_{\text{pd}} - H({}^{7}Be)\langle\sigma v\rangle_{\text{p}^{7}\text{Be}} - H({}^{7}Li)\langle\sigma v\rangle_{\text{p}^{7}\text{Li}}$$
(5.29)

$$\frac{dD}{dt} = \frac{H^2}{2} \langle \sigma v \rangle_{\rm pp} - HD \langle \sigma v \rangle_{\rm pd}$$
(5.30)

$$\frac{d(^{3}He)}{dt} = HD\langle\sigma v\rangle_{\rm pd} - 2\frac{(^{3}He)^{2}\langle\sigma v\rangle_{^{3}\rm He^{^{3}\rm He}}}{2} - (^{3}He)(^{4}He)\langle\sigma v\rangle_{a^{3}\rm He}$$
(5.31)

$$\frac{d({}^{4}He)}{dt} = \frac{({}^{6}He)^{2}\langle\sigma v\rangle_{^{3}\text{He}^{3}\text{He}}}{2} + 2H({}^{7}Be)\langle\sigma v\rangle_{p^{7}\text{Be}} + 2H({}^{7}Li)\langle\sigma v\rangle_{p^{7}\text{Li}} - ({}^{3}He)({}^{4}He)\langle\sigma v\rangle_{\alpha^{3}\text{He}}$$
(5.32)

$$\frac{d(^{7}Be)}{dt} = (^{3}He)(^{4}He)\langle\sigma v\rangle_{\alpha^{3}He} - (^{7}Be)\lambda_{e^{7}Be} - H(^{7}Be)\langle\sigma v\rangle_{p^{7}Be}$$
(5.33)

$$\frac{d(^{7}Li)}{dt} = (^{7}Be)\lambda_{e^{7}Be} - H(^{7}Li)\langle\sigma v\rangle_{p^{7}Li}$$
(5.34)

Note the factor of 2, for example, in the numerator of the first term on the right-hand side of Eq. (5.29) because *two* protons are created in *one* 3 He(3 He,2p)⁴He reaction. The term $\lambda_{e^{7}Be}$ denotes the decay constant for 7 Be electron capture. The 8 B and 8 Be abundances have been eliminated because both decays have very short mean lifetimes (1.1 s and 4×10^{-22} s, respectively). Therefore, the sequence 7 Be(p,γ) 8 B($\beta^{+}\nu$) 8 Be(α) α can be considered as a single step, 7 Be + p $\rightarrow 2\alpha + \nu$. This set of equations can be solved numerically. It is instructive, however, to calculate analytical solutions by using certain approximations. We have seen in the discussion of the pp1 chain that several important results can be expressed in terms of the 3 He equilibrium abundance. Therefore, we will first focus on this quantity in the following and then estimate the overall energy generation in the pp chains.

It is again safe to assume that the deuterium abundance will achieve equilibrium in a negligible amount of time (seconds to hours) compared to the evolution of the star. Hence, dD/dt = 0 in Eq. (5.30) and $HD\langle \sigma v \rangle_{pd}$ can be replaced in Eq. (5.31) by $H^2 \langle \sigma v \rangle_{pp}/2$,

$$\frac{d(^{3}He)}{dt} = \frac{H^{2}}{2} \langle \sigma v \rangle_{\rm pp} - 2 \frac{(^{3}He)^{2} \langle \sigma v \rangle_{^{3}\mathrm{He}^{3}\mathrm{He}}}{2} - (^{3}He)(^{4}He) \langle \sigma v \rangle_{^{a}\mathrm{3}\mathrm{He}}$$
(5.35)

We will also assume that the ³He abundance has reached an equilibrium. The $({}^{3}He)_{e}$ abundance will be smaller compared to what it was in the operation of the pp1 chain alone because now there is an additional ³He destroying reaction. With $d({}^{3}He)/dt = 0$ we find

$$({}^{3}He)_{e}^{2}\langle\sigma v\rangle_{{}^{3}\mathrm{He}^{3}\mathrm{He}} = \frac{H^{2}}{2}\langle\sigma v\rangle_{\mathrm{pp}} - ({}^{3}He)_{e}({}^{4}He)\langle\sigma v\rangle_{\alpha^{3}\mathrm{He}}$$
(5.36)





Fig. 5.6 Values of $({}^{3}He)_{e}$ obtained from the operation of all three pp chains, divided by $({}^{3}He)_{e}$ values derived from the operation of the pp1 chain only. The curve is obtained for a composition of $X_{\rm H} = X_{\alpha} = 0.5$. The ratio of $({}^{3}He)_{e}$ values shown amounts to unity below T = 10 MK and decreases rapidly for increasing temperatures due to the operation of the pp2 and pp3 chains.

Solving for $({}^{3}He)_{e}$ yields the expression

$$({}^{3}He)_{e} = \frac{-({}^{4}He)\langle\sigma v\rangle_{\alpha^{3}\mathrm{He}} + \sqrt{({}^{4}He)^{2}\langle\sigma v\rangle_{\alpha^{3}\mathrm{He}}^{2} + 2H^{2}\langle\sigma v\rangle_{\mathrm{pp}}\langle\sigma v\rangle_{\mathrm{3He^{3}He}}}{2\langle\sigma v\rangle_{\mathrm{3He^{3}He}}}$$

$$(5.37)$$

It is apparent that for a zero hydrogen abundance $(H \rightarrow 0)$, the $({}^{3}He)_{e}$ abundance also vanishes. Furthermore, for a zero 4 He abundance, implying no destruction of 3 He via the 3 He $(\alpha, \gamma)^{7}$ Be reaction, the above expression reduces to Eq. (5.13). The ratio of $({}^{3}He)_{e}$ values obtained from Eq. (5.37) to that resulting from the operation of the pp1 chain alone (see Eq. (5.13)) is shown in Fig. 5.6 for a composition of $X_{\rm H} = X_{\alpha} = 0.5$. The ratio amounts to unity for temperatures below T = 10 MK and decreases rapidly for increasing temperature because of the operation of the pp2 and pp3 chains (see below).

We are now in a position to investigate the competition between the pp1, pp2 and pp3 chains. The pp2 and pp3 chains will dominate over the pp1 chain when the ${}^{3}\text{He}(\alpha,\gamma)^{7}\text{Be}$ reaction becomes more likely than the ${}^{3}\text{He}({}^{3}\text{He},2p)^{4}\text{He}$ reaction. Similarly, the pp3 chain will dominate over the pp2 chain when the ${}^{7}\text{Be}(p,\gamma)^{8}\text{B}$ reaction becomes more likely than the competing electron capture ${}^{7}\text{Be}(e^{-},\nu)^{7}\text{Li}$. According to Eqs. (3.22) and (5.37), the corresponding mean life-

times are

$$\tau_{\alpha}({}^{3}He) = \left(\rho \frac{X_{\alpha}}{M_{\alpha}} N_{A} \langle \sigma v \rangle_{\alpha^{3}\mathrm{He}}\right)^{-1}$$

$$[\tau_{^{3}\mathrm{He}}({}^{3}He)]_{e} = \left(-\frac{\rho}{2} \frac{X_{\alpha}}{M_{\alpha}} N_{A} \langle \sigma v \rangle_{\alpha^{3}\mathrm{He}} + \frac{\rho}{2} \sqrt{\frac{X_{\alpha}^{2}}{M_{\alpha}^{2}} (N_{A} \langle \sigma v \rangle_{\alpha^{3}\mathrm{He}})^{2} + 2\frac{X_{\mathrm{H}}^{2}}{M_{\mathrm{H}}^{2}} N_{A} \langle \sigma v \rangle_{\mathrm{pp}} N_{A} \langle \sigma v \rangle_{^{3}\mathrm{He}^{3}\mathrm{He}}}\right)^{-1}$$

$$(5.38)$$

$$T_{\rm p}(^{7}Be) = \left(\rho \frac{X_{\rm H}}{M_{\rm H}} N_A \langle \sigma v \rangle_{\rm p^{7}Be}\right)^{-1}$$
(5.39)
(5.40)

$$\tau_{\rm e^-}(^7Be) = (\lambda_{\rm e^7Be})^{-1} \tag{5.41}$$

The mean lifetimes are calculated from the above expressions for the conditions $\rho = 100 \text{ g/cm}^3$ and $X_{\text{H}} = X_{\text{He}} = 0.5$. In order to calculate $\lambda_{e^7\text{Be}}$, the approximation $n_{\text{e}^-}/N_A \approx \rho(1 + X_{\text{H}})/2$ is used (Fowler, Caughlan and Zimmerman 1975), which is applicable for a fully ionized gas. The results are shown in Fig. 5.5b. We find that, for the conditions assumed, $\tau_{\alpha}({}^{3}\text{He}) \approx [\tau_{3\text{He}}({}^{3}\text{He})]_{e}$ at $T \approx 18$ MK. Above this temperature, the pp2 and pp3 chains will dominate over the pp1 chain. Furthermore, we obtain $\tau_p({}^{7}\text{Be}) \approx \tau_{\text{e}^-}({}^{7}\text{Be})$ at $T \approx$ 25 MK, implying that the pp3 chain will dominate over the pp2 chain at temperatures in excess of this value. Note that these two temperature values are independent of density as can be seen from Eqs. (5.38)–(5.41).

Finally, the nuclear energy generated by all the pp chains operating together is estimated under the assumption that ³He has achieved an equilibrium abundance. Remember that the neutrino losses are different in each chain. The energy generation rate, corrected for neutrino losses, can be written as the product

$$\varepsilon_{\rm pp} = \frac{Q_{4\rm H} + 4\rm He}{\rho} \frac{d(^{4}\rm He)}{dt} \left(f_{\rm pp1}F_{\rm pp1} + f_{\rm pp2}F_{\rm pp2} + f_{\rm pp3}F_{\rm pp3} \right) \qquad ({\rm MeV}\,{\rm g}^{-1}\,{\rm s}^{-1})$$
(5.42)

with $Q_{4H\rightarrow^{4}He} = 26.73$ MeV the energy release per ⁴He nucleus produced and $d(^{4}He)/dt$ the production rate of ⁴He. The factor f_{ppi} is the fraction of the total energy $Q_{4H\rightarrow^{4}He}$ which is retained in the star if the ⁴He nucleus is produced in the ppi chain ($f_{pp1} = 0.98$, $f_{pp2} = 0.96$, $f_{pp3} = 0.74$; see above). The quantity F_{ppi} denotes the fraction of ⁴He nuclei produced by the ppi chain ($F_{pp1} + F_{pp2} + F_{pp3} = 1$). The production rate of ⁴He is given by Eq. (5.32). The mean lifetimes of ⁷Be and ⁷Li amount to less than a year at most temperatures and densities of interest. Thereafter, both abundances will follow the

buildup of ³He. With $d(^{7}Be + ^{7}Li)/dt \approx 0$ one finds from Eqs. (5.33) and (5.34)

$$H(^{7}Be)\langle \sigma v \rangle_{p^{7}Be} + H(^{7}Li)\langle \sigma v \rangle_{p^{7}Li} = (^{3}He)(^{4}He)\langle \sigma v \rangle_{a^{3}He}$$
(5.43)

This expression is satisfied long before 3 He achieves equilibrium. Substitution of Eq. (5.43) into Eq. (5.32) yields a much simplified expression for the 4 He production rate,

$$\frac{d(^{4}He)}{dt} = \frac{(^{3}He)^{2}\langle\sigma v\rangle_{^{3}\text{He}^{3}\text{He}}}{2} + (^{3}He)(^{4}He)\langle\sigma v\rangle_{\alpha^{3}\text{He}}$$
(5.44)

When 3 He achieves equilibrium, we obtain with Eq. (5.36)

$$\frac{d(^{4}He)}{dt} = \frac{H^{2}}{4} \langle \sigma v \rangle_{\rm pp} + \frac{1}{2} (^{3}He)_{e} (^{4}He) \langle \sigma v \rangle_{\alpha^{3}\rm He}$$
(5.45)

The fraction of 4 He nuclei produced by the pp1 chain can be written as the ratio of reaction rates (see Eq. (3.6)),

$$F_{\rm pp1} = \frac{r_{^{3}\rm He^{^{3}\rm He}}}{r_{^{3}\rm He^{^{3}\rm He}} + r_{\alpha^{^{3}\rm He}}} = \frac{(^{^{3}\rm He})_{e}\langle\sigma v\rangle_{^{3}\rm He^{^{3}\rm He}}}{(^{^{3}\rm He})_{e}\langle\sigma v\rangle_{^{3}\rm He^{^{3}\rm He}} + 2(^{^{4}\rm He})\langle\sigma v\rangle_{\alpha^{^{3}\rm He}}}$$
(5.46)

Similarly, one finds for the fraction of ⁴He nuclei produced in the pp2 chain

$$F_{\rm pp2} = (1 - F_{\rm pp1}) \frac{r_{\rm e^7Be}}{r_{\rm e^7Be} + r_{p^7Be}} = (1 - F_{\rm pp1}) \frac{\lambda_{\rm e^7Be}}{\lambda_{\rm e^7Be} + H \langle \sigma v \rangle_{\rm p^7Be}}$$
(5.47)

with $(1 - F_{pp1})$ the probability that the ⁴He nucleus is *not* produced in the pp1 chain. Furthermore, the fraction of ⁴He nuclei produced by the pp3 chain is given by $F_{pp3} = 1 - F_{pp1} - F_{pp2}$. The fractions F_{ppi} , which are independent of density, are shown in Fig. 5.7a assuming a composition of $X_H = X_{\alpha} = 0.5$ and a fully ionized gas. It is again apparent that for temperatures below T = 18 MK the ⁴He nuclei are mainly produced via the pp1 chain. The pp2 chain takes over above this temperature value, while for T > 25 MK the pp3 chain is the main producer of ⁴He nuclei.

The energy production rate of the pp chains after ³He has achieved an equilibrium abundance, ε_{pp}^{e} , can now be calculated from Eqs. (5.42), (5.45), (5.46), and (5.47) as a function of temperature and composition. The ratio of the energy generation rate by the pp chains to that by the pp1 chain alone, $\varepsilon_{pp}^{e} / \varepsilon_{pp1}^{e}$, is displayed in Fig. 5.7b for a composition of $X_{\rm H} = X_{\alpha} = 0.5$ and a fully ionized gas. The ratio is independent of density and amounts to unity at temperatures below T = 10 MK where the pp1 chain is the dominant process. Recall that in the pp1 chain two p + p reactions are necessary for the production of one ⁴He nucleus. On the other hand, in the pp2 and pp3 chains, the creation of one ⁴He nucleus requires only one p + p reaction, causing an increase in





Fig. 5.7 (a) Fraction of ⁴He nuclei produced by the pp1, pp2, and pp3 chains. The pp1, pp2, and pp3 chains are the main producers of ⁴He at temperatures of T < 18 MK, T = 18-25 MK, and T > 25 MK, respectively. (b) Ratio of the energy generation rate by all three pp chains to that by the pp1 chain alone versus temperature. The ratio amounts to unity for T < 10 MK where the

pp1 chain dominates. The maximum at $T \approx$ 23 MK is caused by the dominant operation of the pp2 chain. About 90% of the Sun's energy is produced by the pp1 chain. All curves shown in parts (a) and (b) are independent of density and are calculated for a composition of $X_{\rm H} = X_{\alpha} = 0.5$ and a fully ionized gas.

 $d({}^{4}He)/dt$ (by a factor of 2) and in ε_{pp}^{e} (by a factor of 2 minus neutrino losses) compared to the operation of the pp1 chain alone. This can clearly be seen at temperatures above T = 40 MK, where the pp3 chain dominates, yielding a ratio of $\varepsilon_{pp}^{e}/\varepsilon_{pp1}^{e} = 2(f_{pp3}/f_{pp1}) = 2(0.74/0.98) = 1.51$. The maximum at $T \approx 23$ MK is caused by the dominant operation of the pp2 chain, for which the neutrino losses are much less compared to those of the pp3 chain. In the center of the Sun, the temperature amounts to T = 15.6 MK. Averaged over the entire hydrogen burning region, it turns out that about 90% of the Sun's energy is produced in the pp1 chain.

We conclude the discussion of the pp chains with a few final remarks. The evolution of the ³He abundance is much more complicated than that of deuterium. We already discussed that any initial deuterium nuclei are quickly converted inside stars to ³He, thus increasing the ³He abundance. Compared to the deuterium destroying $d(p,\gamma)^{3}$ He reaction, the ³He consuming reactions ³He(³He,2p)⁴He and ³He(α,γ)⁷Be involve higher Coulomb barriers and, therefore, have smaller cross sections. In the cooler outer layers of most stars, and specifically throughout most of the volume of cooler low-mass stars, ³He will thus survive. However, in the hotter stellar regions, ³He is converted to ⁴He via the pp chains. The situation becomes more complex because the outer cooler layers of a star may be mixed to the hotter interior regions, a process that will contribute to the destruction of ³He. Clearly, there is a delicate balance between stellar ³He production and destruction. Whether or not this ³He will survive and, after ejection, enrich the interstellar medium is controversial (see the review by Tosi 2000).

The isotope ⁷Li is produced in the pp2 chain. However, the cross section of the ⁷Li(p, α) α reaction is very large and hence the ⁷Li abundance at any time during the operation of the pp chains becomes very small $[(^7Li/H)_{pp} \approx 2 \times 10^{-9};$ Parker, Bahcall and Fowler 1964]. There is evidence that a large fraction of the ⁷Li abundance observed in the solar system $[(^7Li/H)_{\odot} \approx 2 \times 10^{-9}]$ is not produced in stars, but originates from high-energy spallation reactions involving cosmic rays and the interstellar medium, and from primordial nucleosynthesis. Nevertheless, models of Galactic chemical evolution require a stellar source that produces the remaining, unexplained fraction of the ⁷Li abundance (Romano et al. 2001). In these sources, ⁷Be is produced by the ³He(α , γ)⁷Be reaction and is transported via convection from the hot burning zone to the outer, cooler layers where it decays by electron capture to ⁷Li. This beryllium transport process is referred to as the *Cameron–Fowler mechanism* (Cameron and Fowler 1971) and explains the lithium enrichments observed in certain red giants and AGB stars.

Finally, we comment on the cross sections of the reactions that are part of the pp chains. All these reactions exhibit nonresonant cross sections in the energy range important for hydrostatic hydrogen burning. Direct cross section measurements for d(p, γ)³He, ³He(³He,2p)⁴He, ³He(α , γ)⁷Be, ⁷Be(p, γ)⁸B, and 7 Li(p, α) α have been performed down to center-of-mass energies of 10 keV, 15 keV, 100 keV, 70 keV, and 10 keV, respectively (Angulo et al. 1999). In comparison, the centers of the solar Gamow peaks (T_{\odot} = 15.6 MK; see Eq. (3.74)) for these reactions are located at 7 keV, 22 keV, 23 keV, 18 keV, and 15 keV, respectively. Thus, measurements of the $d(p,\gamma)^{3}$ He, 3 He(3 He,2p) 4 He, and ⁷Li(p, α) α reactions cover directly the energy range important for hydrogen burning in stars with masses of $M \ge M_{\odot}$. In other cases (for example, for 3 He(α, γ)⁷Be and 7 Be(p, γ)⁸B at solar temperature, or for all the above reactions at lower temperatures that are typical of stars with $M < M_{\odot}$) the astrophysical S-factor has to be extrapolated down to the energy range of interest, either by a polynomial expansion or by using a suitable nuclear reaction model (Section 3.2.1).

5.1.2 CNO Cycles

If a star consists exclusively of hydrogen and helium, then significant energy can only be generated during the hydrogen burning stage via the operation of the pp chains. Most stars, however, consist of gas that contains heavier nuclides, particularly those in the C, N, and O mass region. Hence, these nuclei can participate in hydrogen burning. The resulting four sets of reactions through which hydrogen can be converted to helium are referred to as the

<u>CNO1</u>	<u>CNO2</u>	CNO3	CNO4	
$^{12}C(p,\gamma)^{13}N$	$^{14}N(p,\gamma)^{15}O$	$^{15}N(p,\gamma)^{16}O$	$^{16}O(p,\gamma)^{17}F$	
$^{13}C(p,\gamma)^{14}N$	¹⁵ N(p,γ) ¹⁶ O	$O(\rho, \gamma) = 1^{17} F(\beta^+ \nu)^{17} O$	$^{17}O(p,\gamma)^{18}F$	
¹⁴ N(p,γ) ¹⁵ O	¹⁶ O(p,γ) ¹⁷ F	¹⁷ O(p,γ) ¹⁸ F	${}^{18}{\sf F}(\beta^+\nu){}^{18}{\sf O}$	
¹⁵ Ο(β ⁺ ν) ¹⁵ Ν	¹⁷ F(β ⁺ ν) ¹⁷ O	¹⁸ F(β ⁺ ν) ¹⁸ O	$^{18}\mathrm{O}(\mathrm{p},\gamma)^{19}\mathrm{F}$	
¹⁵ N(p,α) ¹² C	¹⁷ O(p,α) ¹⁴ N	¹⁸ O(p,α) ¹⁵ N	¹⁹ F(p,α) ¹⁶ O	
T1/2: ¹³ N (9.965 min); ¹⁵ O (122.24 s); ¹⁷ F (64.49 s); ¹⁸ F (109.77 min)				

CNO cycles. The reactions of the CNO cycles are listed below (together with the β -decay half-lives) and are shown in Fig. 5.8.

These cycles have interesting properties. The end result of each process is the same as for the pp chains, that is, $4H \rightarrow {}^{4}He + 2e^{+} + 2\nu$. In each cycle, C, N, O, or F nuclei act only as catalysts, in the sense that the total abundance of the heavy nuclei is not altered while only hydrogen is consumed. Therefore, a substantial amount of nuclear energy can be generated even if the total abundance of the heavy nuclei is relatively low. Of course, the operation of a particular cycle will change the abundance of the individual heavy nuclei. Consider as an example the CNO1 cycle. If there are initially only ${}^{12}C$ nuclei present in the stellar gas, then some of these will be converted to other CNO nuclei and the individual abundances will evolve depending on the magnitude of the reaction rates involved. The energy generation rate depends on the abundance of the catalysts and the time it takes to complete the cycle.

The various CNO cycles exist because for the proton-induced reactions on the nuclei ${}^{15}N$, ${}^{17}O$, ${}^{18}O$, and ${}^{19}F$ both the (p,γ) and (p,α) channels are energetically allowed, in contrast to the proton-induced reactions on the nuclei 12 C, 13 C, 14 N, and 16 O that can only proceed via the (p, γ) reaction. The (p, α) reaction will convert a heavier nucleus back to a lighter one, thereby giving rise to a cycle of nuclear processes. At each of the branch point nuclei ¹⁵N, ¹⁷O, ¹⁸O, and ¹⁹F, the (p, α) reaction will compete with the (p, γ) reaction. The branching ratio, or the ratio of probabilities for the occurrence of the (p, α) and (p,γ) reaction, is then given by the ratio of the corresponding reaction rates, $B_{p\alpha/p\gamma} = N_A \langle \sigma v \rangle_{(p,\alpha)} / N_A \langle \sigma v \rangle_{(p,\gamma)}$. The branching ratios versus temperature are displayed in Fig. 5.9. The solid lines show the upper and lower limits of $B_{p\alpha/p\gamma}$, caused by presently unknown contributions to the reaction rates (for example, unobserved resonances). Despite the rate uncertainties, it is obvious that for the target nuclei ^{15}N , ^{17}O , ^{18}O , and ^{19}F the (p, α) reaction is faster than the (p, γ) reaction over the entire temperature range (except perhaps for ¹⁷O and ¹⁸O at very low temperatures of T < 20 MK). An impression on the relative likelihood of the various CNO reactions can be obtained from





Fig. 5.8 Representation of the four CNO cycles in the chart of the nuclides. Stable nuclides are shown as shaded squares. Each reaction cycle fuses effectively four protons to one ⁴He nucleus.

Fig. 5.10, showing the reaction rates normalized to the rate of the slowest reaction, ${}^{16}O(p,\gamma){}^{17}F$.

A few important points need to be stressed before continuing the discussion. First, at relatively low temperatures characteristic of hydrostatic hydrogen burning ($T \leq 55$ MK), β^+ -decays of unstable nuclei in the CNO mass range proceed on much faster time scales compared to the competing protoninduced reactions. Thus, reactions involving unstable nuclei are unimportant under such conditions. At temperatures above T = 100 MK, additional reactions not listed above (those involving unstable target nuclei) take place in the CNOF mass region and the characteristics of the cycles change substantially. In this section we will concentrate on the temperature range T < 100 MK, while hydrogen burning in the CNOF mass region at higher temperatures is discussed in Section 5.2.1. Second, the relative initial abundance of the various CNOF isotopes is obviously important in order to describe the detailed operation of the CNO cycles. These seed nuclei are produced at the helium burning stage in a previous generation of stars. The most abundant nuclides produced during helium burning (see Section 5.3.2) are ¹²C, ¹⁶O, and, to a lesser extent, ¹⁴N. For example, the solar ratio of these isotopes is ¹²C:¹⁴N:¹⁶O = 10:3:24. Hence, the CNO cycles will most likely operate with 12 C and 16 O as seed nuclei. Third, consider now the different fate of these two nuclides. The ¹²C nuclei will initiate the CNO1 sequence of reactions. At ¹⁵N, there is a small chance of about 1:1000, according to Fig. 5.9, that catalytic material leaks



Fig. 5.9 Branching ratio $B_{p\alpha/p\gamma} = N_A \langle \sigma v \rangle_{(p,\alpha)} / N_A \langle \sigma v \rangle_{(p,\gamma)}$ versus temperature for the reactions (a) ¹⁵N + p, (b) ¹⁷O + p, (c) ¹⁸O + p, and (d) ¹⁹F + p. The two solid lines in each panel represent the currently accepted upper and lower limits of $B_{p\alpha/p\gamma}$. The area between the solid lines represents the uncertainty in $B_{p\alpha/p\gamma}$ that is caused by unknown contributions to the (p, γ) and (p, α) reaction rates.



Fig. 5.10 Reaction rates in the CNO cycles versus temperature. For a better comparison, the values of $N_A \langle \sigma v \rangle$ are normalized to the rate of the slowest reaction, ¹⁶O(p, γ)¹⁷F.

into the CNO2 cycle via the ${}^{15}N(p,\gamma){}^{16}O$ reaction. However, most of the catalytic material will be transformed back to ${}^{12}C$ via the dominant (p,α) reaction. On the other hand, ${}^{16}O$ is transformed to ${}^{17}O$, but the subsequent processing is more complicated. A large fraction of ${}^{17}O$ nuclei will be destroyed by the (p,α) reaction, leading to the formation of ${}^{14}N$ and the further operation of the CNO1 and CNO2 cycles. But another fraction, depending on the stellar temperature, will be converted to ${}^{18}F$, thus initiating the CNO3 and CNO4 cycles.

In order to gain some insight into the operation of the CNO cycles, we will proceed as follows. It is first assumed that only ¹²C, ¹³C, ¹⁴N, or ¹⁵N seed nuclei are present in the stellar plasma and that the CNO1 cycle is closed, that is, the (p,γ) reaction on ¹⁵N is negligible compared to the competing (p,α) reaction. The corresponding set of differential equations describing the evolution of isotopic abundances will then be solved for the equilibrium operation of the CNO1 cycle. In a second step, the interplay of all CNO cycles is considered for different assumptions of initial seed abundances by solving numerically the equations describing the abundance changes.

Steady-state operation of the CNO1 cycle

10

Assuming that nuclear transformations are the only source of abundance changes, the following set of coupled differential equations can be obtained for a closed CNO1 cycle (also called *CN cycle*):

$$\frac{d({}^{12}C)}{dt} = H({}^{15}N)\langle\sigma v\rangle_{{}^{15}N(\mathbf{p},\alpha)} - H({}^{12}C)\langle\sigma v\rangle_{{}^{12}C(\mathbf{p},\gamma)}$$
(5.48)

$$\frac{d({}^{15}N)}{dt} = H({}^{12}C)\langle\sigma v\rangle_{12}{}_{C(\mathbf{p},\gamma)} - ({}^{13}N)\lambda_{13}{}_{N(\beta^+\nu)}$$
(5.49)

$$\frac{d({}^{13}C)}{dt} = ({}^{13}N)\lambda_{13}{}_{N(\beta^+\nu)} - H({}^{13}C)\langle\sigma v\rangle_{13}{}_{C(\mathbf{p},\gamma)}$$
(5.50)

$$\frac{d({}^{14}N)}{dt} = H({}^{13}C)\langle\sigma v\rangle_{13}C(\mathbf{p},\gamma) - H({}^{14}N)\langle\sigma v\rangle_{14}N(\mathbf{p},\gamma)$$
(5.51)

$$\frac{d(^{15}O)}{dt} = H(^{14}N)\langle \sigma v \rangle_{^{14}N(\mathbf{p},\gamma)} - (^{15}O)\lambda_{^{15}O(\beta^+\nu)}$$
(5.52)

$$\frac{d({}^{15}N)}{dt} = ({}^{15}O)\lambda_{15}O(\beta^+\nu) - H({}^{15}N)\langle\sigma\nu\rangle_{15}N(\mathbf{p},\alpha)$$
(5.53)

At the temperatures of interest here (T < 0.1 GK), the β -decay lifetime of ¹³N is much shorter compared to the lifetime of ¹²C versus destruction by the (p, γ) reaction (the preceding step). For all practical purposes, the abundances of H and ¹²C will be constant over the short time it takes ¹³N to reach steady state. Hence, Eq. (5.49) can be solved with the same method used to derive Eq. (5.8).

Using $({}^{13}N)_{t=0} = 0$ we obtain

$${}^{(13}N)_{t} = \frac{H\langle \sigma v \rangle_{^{12}C(\mathbf{p},\gamma)}}{\lambda_{^{13}N(\beta^{+}\nu)}} {}^{(12}C) - \left[\frac{H\langle \sigma v \rangle_{^{12}C(\mathbf{p},\gamma)}}{\lambda_{^{13}N(\beta^{+}\nu)}} {}^{(12}C)\right] e^{-\lambda_{^{13}N(\beta^{+}\nu)}t}$$

$$= \frac{\tau_{\beta} {}^{(13}N)}{\tau_{\mathbf{p}} {}^{(12}C)} {}^{(12}C) \left[1 - e^{-t/\tau_{\beta} {}^{(13}N)}\right]$$
(5.54)

The result shows that the ¹³N abundance approaches its steady-state value $({}^{13}N/{}^{12}C)_e = \tau_{\beta}({}^{13}N)/\tau_{\rm p}({}^{12}C)$ in times on the order of $\tau_{\beta}({}^{13}N)$, that is, a few minutes. The same arguments hold for the ¹⁵O abundance. Therefore, we may set Eqs. (5.49) and (5.52) equal to zero and eliminate ¹³N and ¹⁵O from the system of equations. After a few minutes, the nuclear burning in the CNO1 cycle is then described by the system of equations

$$\frac{d(^{12}C)}{dt} = H(^{15}N)\langle \sigma v \rangle_{^{15}N(\mathbf{p},\alpha)} - H(^{12}C)\langle \sigma v \rangle_{^{12}C(\mathbf{p},\gamma)}$$
(5.55)

$$\frac{d({}^{13}C)}{dt} = H({}^{12}C)\langle\sigma v\rangle_{12}C(\mathbf{p},\gamma) - H({}^{13}C)\langle\sigma v\rangle_{13}C(\mathbf{p},\gamma)$$
(5.56)

$$\frac{d({}^{14}N)}{dt} = H({}^{13}C)\langle\sigma v\rangle_{13}C(\mathbf{p},\gamma) - H({}^{14}N)\langle\sigma v\rangle_{14}N(\mathbf{p},\gamma)$$
(5.57)

$$\frac{d({}^{15}N)}{dt} = H({}^{14}N)\langle\sigma v\rangle_{{}^{14}N(\mathbf{p},\gamma)} - H({}^{15}N)\langle\sigma v\rangle_{{}^{15}N(\mathbf{p},\alpha)}$$
(5.58)

Several observations are immediately apparent. First, it is obvious that $d({}^{12}C)/dt + d({}^{13}C)/dt + d({}^{14}N)/dt + d({}^{15}N)/dt = 0$ and, consequently, the sum of CNO1 abundances is constant, $\sum CNO1 = \text{const.}$ Second, after the CNO1 cycle has reached steady state, all time derivatives in the above expressions are zero. As a result, the rates of all CNO1 reactions become equal, while the ratio of any two nuclidic abundances is simply given by the inverse ratio of their reaction rates (or the ratio of mean lifetimes). For example,

$$\left(\frac{{}^{14}N}{{}^{12}C}\right)_e = \frac{\langle \sigma v \rangle_{{}^{12}C(\mathbf{p},\gamma)}}{\langle \sigma v \rangle_{{}^{14}N(\mathbf{p},\gamma)}} = \frac{\tau_{\mathbf{p}}({}^{14}N)}{\tau_{\mathbf{p}}({}^{12}C)}$$
(5.59)

The fractional abundance, for example, for ¹²C is

$$\frac{\binom{1^{2}C}{c}}{\sum CNO1} = \frac{\binom{1^{2}C}{e}}{\binom{1^{2}C}{e} + \binom{1^{3}C}{e} + \binom{1^{4}N}{e} + \binom{1^{5}N}{e}} = \left(1 + \frac{\langle \sigma v \rangle_{^{12}C(p,\gamma)}}{\langle \sigma v \rangle_{^{13}C(p,\gamma)}} + \frac{\langle \sigma v \rangle_{^{12}C(p,\gamma)}}{\langle \sigma v \rangle_{^{14}N(p,\gamma)}} + \frac{\langle \sigma v \rangle_{^{12}C(p,\gamma)}}{\langle \sigma v \rangle_{^{15}N(p,\alpha)}}\right)^{-1} = \frac{\tau_{p}(^{12}C)}{\tau_{p}(^{12}C) + \tau_{p}(^{13}C) + \tau_{p}(^{14}N) + \tau_{p}(^{15}N)}$$
(5.60)



Fig. 5.11 (a) Abundance ratios and (b) fractional abundances versus temperature. The curves are calculated by assuming steady-state operation of a closed CNO1 cycle.

The CNO1 abundance ratios and fractional abundances are shown in Fig. 5.11 versus temperature.

The net effect of the CNO1 cycle operation is the conversion of carbon and nitrogen seed nuclei to ¹⁴N, which becomes by far the most abundant heavy nuclide when steady state is reached. This result is a consequence of the fact that the ¹⁴N destroying reaction ¹⁴N(p, γ)¹⁵O is the slowest process in the CNO1 cycle for temperatures of *T* < 0.1 GK, as can be seen from Fig. 5.10. Note that not all abundances in the CNO1 cycle are constant with time, even under steady-state conditions, since hydrogen is continuously converted to helium [*dH*/*dt* < 0 and *d*(⁴*He*)/*dt* > 0].

The energy generation rate from the operation of the CNO1 cycle at constant temperature and density can be expressed as (see Eq. (3.63))

$$\varepsilon_{\text{CNO1}} = \sum_{i \to j} \varepsilon_{i \to j} = \frac{1}{\rho} \sum_{i \to j} (Q_{i \to j} - \overline{E}_{\nu}^{i \to j}) r_{i \to j}$$
(5.61)

where the sum is over all relevant processes $i \rightarrow j$ and \overline{E}_{ν} denotes the average energy of the neutrinos released in the β -decays. Since the β^+ -decays of ¹³N and ¹⁵O occur on negligible small time scales, we can consider them together with the preceding reactions ${}^{12}C(p,\gamma){}^{13}N$ and ${}^{14}N(p,\gamma){}^{15}O$, respectively. The reaction and decay energies available to the star are given by

$$Q_{^{12}C(p,\gamma)}^{13}N(\beta^+\nu) - \overline{E}_{\nu}^{^{13}N(\beta^+\nu)} = (1.944 + 2.220 - 0.706) \,\text{MeV}$$

$$Q_{^{13}\mathrm{C}(\mathrm{p},\gamma)} = 7.551\,\mathrm{MeV} \tag{5.63}$$

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$$Q_{^{14}N(p,\gamma)^{15}O(\beta^+\nu)} - \overline{E}_{\nu}^{^{15}O(\beta^+\nu)} = (7.297 + 2.754 - 0.996) \text{ MeV}$$

= 9.055 MeV (5.64)

$$Q_{15}_{N(p,\alpha)} = 4.966 \,\mathrm{MeV} \tag{5.65}$$

The average neutrino energies \overline{E}_{ν}^{i} are adopted from Bahcall (1989) (see also Eq. (1.45) and Problem 1.9). For the equilibrium operation of the CNO1 cycle we obtain from Eq. (5.61)

$$\rho \varepsilon_{\text{CNO1}}^{e} = (3.458 \,\text{MeV}) H(^{12}C)_{e} \langle \sigma v \rangle_{^{12}C(p,\gamma)} + (7.551 \,\text{MeV}) H(^{13}C)_{e} \langle \sigma v \rangle_{^{13}C(p,\gamma)} + (9.055 \,\text{MeV}) H(^{14}N)_{e} \langle \sigma v \rangle_{^{14}N(p,\gamma)} + (4.966 \,\text{MeV}) H(^{15}N)_{e} \langle \sigma v \rangle_{^{15}N(p,\alpha)} = (3.458 \,\text{MeV}) \frac{(^{12}C)_{e}}{\tau_{p}(^{12}C)} + (7.551 \,\text{MeV}) \frac{(^{13}C)_{e}}{\tau_{p}(^{13}C)} + (9.055 \,\text{MeV}) \frac{(^{14}N)_{e}}{\tau_{p}(^{14}N)} + (4.966 \,\text{MeV}) \frac{(^{15}N)_{e}}{\tau_{p}(^{15}N)}$$
(5.66)

From Eq. (5.60) we obtain

$$\frac{\binom{12C}{r_{p}(^{12}C)}}{\tau_{p}(^{12}C)} = \frac{\binom{13C}{r_{p}(^{13}C)}}{\tau_{p}(^{14}N)} = \frac{\binom{15N}{r_{p}(^{15}N)}}{\tau_{p}(^{15}N)}$$
$$= \frac{\sum CNO1}{\tau_{p}(^{12}C) + \tau_{p}(^{13}C) + \tau_{p}(^{14}N) + \tau_{p}(^{15}N)}$$
(5.67)

Hence, the energy generation rate at equilibrium can be written as

$$\varepsilon_{\text{CNO1}}^{e} = \frac{25.030 \,\text{MeV}}{\rho} \frac{\sum \text{CNO1}}{\tau_{\text{p}}^{(12}C) + \tau_{\text{p}}^{(13}C) + \tau_{\text{p}}^{(14}N) + \tau_{\text{p}}^{(15}N)}$$
(5.68)

The sum of the lifetimes in the denominator is called the cycle time, and is almost entirely dominated by the long 14 N lifetime. Hence

$$\varepsilon_{\text{CNO1}}^{e} \approx \frac{25.030 \,\text{MeV}}{\rho} \frac{\sum \text{CNO1}}{\tau_{\text{p}}(^{14}N)} = \frac{25.030 \,\text{MeV}}{\rho} (\sum \text{CNO1}) H \langle \sigma v \rangle_{^{14}\text{N}(\text{p},\gamma)}$$
$$= 25.030 \,N_A \langle \sigma v \rangle_{^{14}\text{N}(\text{p},\gamma)} \left(\sum_i \frac{X_i}{M_i}\right) \frac{X_{\text{H}}}{M_{\text{H}}} \rho N_A \quad (\text{MeV g}^{-1} \,\text{s}^{-1}) \quad (5.69)$$

where the sum is over all CNO1 isotopes. Clearly, the energy generation rate in the CNO1 cycle at steady state is determined by the ¹⁴N(p, γ)¹⁵O reaction rate. This reaction is nonresonant for temperatures below T = 0.1 GK and, therefore, the temperature dependence of the energy generation rate is obtained from Eq. (3.87). For example, at T = 25 MK, which is roughly characteristic of CNO burning on the upper main sequence, we obtain $\tau = 51.96$ and hence

$$\varepsilon_{\text{CNO1}}^{e}(T) = \varepsilon_{\text{CNO1}}^{e}(T_0) \left(T/T_0\right)^{(\tau-2)/3} = \varepsilon_{\text{CNO1}}^{e}(T_0) \left(T/T_0\right)^{16.7}$$
(5.70)

We are now in a position to compare the equilibrium energy generation rates of the pp1 chain and the CNO1 cycle. These processes will compete with each other in hydrogen-burning stars that contain initial CN seed nuclei. In Fig. 5.12, the quantities $\varepsilon_{pp1}^{e}/(\rho X_{H}^{2})$ from Eq. (5.24) and $\varepsilon_{CNO1}^{e}/(\rho X_{H}^{2})$ from Eq. (5.69) are displayed versus temperature. The former expression is a function of temperature only (through the $p(p,e^+\nu)d$ reaction rate), while the latter depends both on temperature (through the $^{14}\mathrm{N}(\mathrm{p},\gamma)^{15}\mathrm{O}$ reaction rate) and on the mass fractions of the CNO1 isotopes relative to the hydrogen mass fraction. For illustration purposes, values of $X_{\rm H}$ = 0.711, $X_{12\rm C}$ = 2.46 × 10⁻³, $X_{14\rm N}$ = 7.96 \times 10⁻⁴, and X_{13C} = 2.98 \times 10⁻⁵ are chosen which are representative of the solar system and population I stars (Lodders 2003). Note that these initial seed abundances can be used in Eq. (5.69) since we assumed a closed CNO1 cycle [$\sum (X/M)$ = const]. For other values of X_i/M_i , the CNO1 curve shown in Fig. 5.12 will shift vertically. For the conditions chosen, the pp1 chain generates most of the nuclear energy for temperatures below T = 20 MK. At higher temperatures most of the energy is produced in the CNO1 cycle. The temperature in the stellar interior depends on the stellar mass. Therefore, we conclude that the pp chains dominate the energy production in all hydrogen burning stars with insignificant CNO seed abundances. In stars with significant CNO seed abundances, the pp chains will dominate in low-mass stars, while in stars of higher mass (slightly more massive than the Sun; Section 1.4.3) the CNO cycles are the dominant source of energy.

The very different temperature dependence for the energy generation rate of the pp chains compared to the CNO cycles has a profound influence on the internal structure of a star. For example, if helium is mainly synthesized by the pp chains, then energy is transported through the central regions by radiation. In contrast, the rate of the CNO cycles is so sensitive to temperature that, when it is the dominant process, the energy-generating regions are unstable to convection, which becomes the main energy transport mechanism to the outer regions of the star.

Approach to steady state in the CNO cycles

We have so far considered only the steady state operation of the CNO1 cycle. We will now investigate nonequilibrium situations. Two aspects are of special interest: (i) the approach to steady state in the CNO1 cycle, and (ii) the simultaneous operation of all CNO cycles. The system of coupled differential equations describing the abundance changes of all CNO nuclei is similar in structure to Eqs. (5.48)–(5.53), but it is more complex because of the inclusion of oxygen and fluorine isotopes. Such a system of equations represents an example of a nuclear reaction network (Section 3.1.3). It can only be solved analytically if a number of simplifying assumptions are made (Clayton 1983). With one exception, we will not make such assumptions but will instead com-



Fig. 5.12 Equilibrium energy generation rates of the pp1 chain and the CNO1 cycle. The curve for the CNO1 cycle is calculated for a solar system composition (Lodders 2003). For a different composition, the CNO1 curve shifts vertically. The rate of

the ¹⁴N(p, γ)¹⁵O reaction is adopted from Runkle et al. (2005). The pp1 chain and the CNO1 cycle dominate for temperatures below and above T = 20 MK, respectively. The pp1 chain is the primary energy source in the Sun.

pute the time evolution of CNO abundances numerically. For the numerical calculations described in this section, the assumption of constant temperature and density conditions is made. It is important to emphasize that the internal temperature of a real star is actually changing during its evolution on the main sequence. However, in hydrostatic burning environments these changes occur slowly over long time periods. Therefore, the assumption of constant *T* and ρ , although not correct for a real star, is quite useful for obtaining physical insight into the nucleosynthesis and energy production.

We first consider the approach to steady state in the CNO1 cycle. The temperature and density are assumed to be T = 25 MK and $\rho = 100$ g/cm³. Such values are typical of CNO burning on the upper main sequence. For the initial composition we assume $X_H^0 = 0.70$, $X_{4He}^0 = 0.28$, and $X_{12C}^0 = 0.02$, that is, only 12 C is initially present as a CNO seed nucleus. The reaction network, including all four CNO cycles, is solved until hydrogen is exhausted, that is, until the hydrogen concentration falls below $X_H = 0.001$. The time evolution of abundances is shown in Fig. 5.13a. As expected from the operation of the CNO cycles, the hydrogen abundance declines from its initial value, while the helium abundance increases. Hydrogen is exhausted after 30 million years. The initial carbon abundance is steadily depleted and converted to other nuclides. It can be seen that, for the chosen temperature and density conditions, steady state in the CNO1 cycle is reached after only 4000 years. From then on until the end of the calculation, the abundances of 12 C, 13 C, 14 N, and 15 N remain constant. The most abundant CNO isotope in equilibrium is 14 N, while the

least abundant one is ¹⁵N because of its small lifetime versus destruction by the (p, α) reaction. For example, from the numerical results shown in Fig. 5.13a one obtains $(X_{12C}/X_{14N})_e = 0.008$, and thus the ratio of number abundances is $({}^{12}C/{}^{14}N)_e = 0.008(M_{14N}/M_{12C}) \approx 0.01$, in agreement with the results obtained analytically (Fig. 5.11a).

The conversion of initial ¹²C seed nuclei to ¹⁴N must proceed at a rate determined by the ¹²C lifetime, which, for the chosen conditions, amounts to $\tau_p(^{12}C) = 350$ y. It is apparent from Fig. 5.13a that the ¹²C abundance decays away with a 1/*e* time that is approximately equal to $\tau_p(^{12}C)$, while the CNO1 cycle reaches steady state after several ¹²C half-lives. There is indeed a small leakage of material from the CNO1 to the CNO2 cycle, as can be seen from the increasing ¹⁶O abundance. However, it remains insignificant compared to the ¹⁴N abundance. The time evolution of the nuclear energy generation rate is shown in Fig. 5.13b. The energy generation rate drops by more than an order of magnitude until equilibrium is reached after about 4000 years. For example, at $t = 10^4$ y we obtain from the numerical results presented in Fig. 5.13b a value of $\varepsilon_{CNO} \approx 2.2 \times 10^{10}$ MeV g⁻¹ s⁻¹, in agreement with the analytical steady state value calculated from Eq. (5.69). For times beyond $t = 3 \times 10^5$ y the energy production rate drops because the abundance of the hydrogen fuel decreases.

We consider next the effects caused by a change in composition. The temperature and density are the same as before (T = 25 MK, $\rho = 100$ g/cm³). For the initial composition we assume $X_{\rm H}^0 = 0.70$, $X_{4\rm He}^0 = 0.28$, $X_{12\rm C}^0 = 0.01$, and $X_{16O}^0 = 0.01$, that is, both ¹²C and ¹⁶O are now present with equal concentrations as seed nuclei. The reaction network is again solved until hydrogen is exhausted ($X_{\rm H}$ < 0.001). The results are displayed in Fig. 5.13c. The hydrogen and helium abundance evolves similar as before. The abundances of ¹²C, ¹³C, and ¹⁴N reach again steady state after about 4000 years, with ¹⁴N being by far the most abundant species. The ¹⁵N abundance is omitted in Fig. 5.13c since it is very small. At $t = 10^4$ y, the ratio of ¹²C to ¹⁴N mass fractions is the same as in the previous network calculation $[(X_{12C}/X_{14N})_e = 0.008]$. At this point, only vanishingly small amounts of ¹⁶O have been consumed and little else has changed due to the presence of ¹⁶O as seed nucleus. Recall that the ${}^{16}O(p,\gamma){}^{17}F$ reaction is one of the slowest processes in the CNO cycles (Fig. 5.10). Therefore, it takes a significant time for ¹⁶O to be depleted. Small, but noticeable, changes occur after $t = 10^4$ y. The ¹⁶O abundance starts to decline, while at the same time the ¹⁷O, ¹⁸O, and ¹⁹F abundances start to increase. After $t = 10^5$ y the ¹²C, ¹³C, and ¹⁴N abundances increase, indicating a transfer of catalytic material from the CNO2 cycle to the CNO1 cycle by means of the strong ${}^{15}N(p,\alpha){}^{12}C$ reaction. After $t = 10^7$ y, individual CNO abundances stay constant and steady state has been achieved in all CNO cycles. At this point,

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Fig. 5.13 Time evolution of abundances and nuclear energy generation rate for two different compositions: (a), (b) $X_{\rm H}^0 = 0.70$, $X_{\rm 4He}^0 = 0.28$, $X_{\rm 12C}^0 = 0.02$, and (c), (d) $X_{\rm H}^0 = 0.70$, $X_{\rm 4He}^0 = 0.28$, $X_{\rm 12C}^0 = X_{\rm 16O}^0 = 0.01$. For the temperature and density, constant

values of T = 25 MK and $\rho = 100$ g/cm³ are assumed in both cases. All curves shown are obtained by solving the reaction network numerically. The calculations are terminated when the hydrogen mass fraction falls below $X_{\rm H} = 0.001$.

the abundance ratios obtained from Fig. 5.13c agree with those calculated analytically from the ratio of lifetimes (see Eq. (5.59)). For example, one finds $({}^{17}O/{}^{16}O)_e = \langle \sigma v \rangle_{^{16}O(p,\gamma)} / [\langle \sigma v \rangle_{^{17}O(p,\gamma)} + \langle \sigma v \rangle_{^{17}O(p,\alpha)}] = 0.025$, consistent with the value derived from Fig. 5.13c. Note that the ¹⁸O and ¹⁹F abundances are very small, indicating a small leakage from the CNO2 cycle to the CNO3 and CNO4 cycles at T = 0.025 GK. The presence of ¹⁶O seed nuclei has changed the final ¹²C, ¹³C, and ¹⁴N abundances by less than 20% compared to the earlier network calculation with only ¹²C as seed nucleus (Fig. 5.13a). These results show that, in spite of the simultaneous operation of all CNO cycles, the initial ¹²C and ¹⁶O seed nuclei are essentially transformed to ¹⁴N if there is sufficient time to achieve steady state.

The time evolution of the energy generation rate is shown in Fig. 5.13d. Only about half of the energy per unit time is produced compared to the results shown in Fig. 5.13b since only half of the initial ¹²C seed nuclei are present. At times of $t = 10^5$, 10^6 , and 10^7 y, the energy generation rate displayed in Fig. 5.13d amounts to $\varepsilon_{CNO} = 1.2 \times 10^{10} \text{ MeV g}^{-1} \text{ s}^{-1}$, $1.3 \times 10^{10} \text{ MeV g}^{-1} \text{ s}^{-1}$ and $2.7 \times 10^{9} \text{ MeV g}^{-1} \text{ s}^{-1}$, respectively. Interestingly, these values are very close (within 10%) to those calculated analytically from Eq. (5.69) assuming a closed CNO1 cycle in steady state, with only ¹²C as initial seed nuclei. Hence, after the CNO1 cycle reaches steady state, the energy generation rate of the simultaneous operation of all CNO cycles is approximately equal to that of the CNO1 cycle alone, that is, $\varepsilon_{\text{CNO}} \approx \varepsilon_{\text{CNO1}}^{e}$. This circumstance is explained by several factors. First, the (p,α) reactions on the branching point nuclei ¹⁵N, ¹⁷O, ¹⁸O, and ¹⁹F are much faster compared to the competing (p, γ) reactions (Fig. 5.9) which tends to increase the CNO1 abundances, and specifically that of ¹⁴N, at the expense of the abundances in the other cycles. Second, the CNO1 cycle reactions involve lower Coulomb barriers and, therefore, are generally faster compared to the reactions of the other cycles (Fig. 5.10). Third, the Q-values in the CNO1 cycle are much larger compared to the energy released by processes that complete the CNO2 cycle ($Q_{16O(p,\gamma)} = 0.600 \text{ MeV}$; $Q_{17F(\beta^+\nu)} = 2.761 \text{ MeV}$; $Q_{17O(p,\alpha)} = 1.192 \text{ MeV}$). Only in special cases when the initial oxygen abundance is overwhelmingly larger than the initial carbon or nitrogen abundance, and if oxygen has not yet reached steady state, will the assumption of $\varepsilon_{\text{CNO}} \approx \varepsilon_{\text{CNO1}}^{e}$ be invalid.

The results of two network calculations, performed for different constant temperatures of *T* = 20 and 55 MK, a constant density of $\rho = 100 \text{ g/cm}^3$ and a solar initial composition ($X_{\rm H}^0 = 0.706$; $\sum X_{\rm CNO}^0 = 0.0137$; ¹²C:¹⁴N:¹⁶O = 10:3:24), are displayed in Fig. 5.14. The abundance evolutions are shown versus the amount of hydrogen consumed, $\Delta X_{\rm H} = X_{\rm H}^0 - X_{\rm H}(t)$, with time increasing from left to right. Some similarities between the two calculations can be noticed. All CNO abundances reach steady state at the end of the calculations (when $X_{\rm H} < 10^{-3}$), although this is not readily apparent in Fig. 5.14a if the ¹⁶O, 17 O, 18 O, and 19 F abundances are plotted versus $\Delta X_{\rm H}$. The 14 N abundance increases steadily, first as a result of ¹²C to ¹⁴N conversion in the operation of the CNO1 cycle, and at a later time because of ¹⁶O to ¹⁴N conversion in the CNO2 cycle. Thus, ¹⁴N is enhanced while ¹²C and ¹⁶O are depleted at the end of the calculation. The abundances of ¹⁸O and ¹⁹F are also depleted during the nuclear burning, while the final ¹³C abundance changes by less than a factor of 2 compared to its initial abundance. The evolution of 17 O is interesting. At *T* = 20 MK, the nuclear burning strongly enhances the ¹⁷O abundance by the time hydrogen is exhausted, while at T = 55 MK its abundance is depleted. The two calculations predict final ¹⁷O abundances that differ by more than three orders of magnitude, that is, the ¹⁷O abundance is very sensitive to the hy-



Fig. 5.14 Abundance evolutions versus the amount of hydrogen consumed for two different constant temperatures: (a) T = 20 MK, and (b) T = 55 MK. The density ($\rho = 100$ g/cm³) and the initial composition (solar) is the same for both cases. All curves shown are obtained by solving the reaction network numerically. The calculations are terminated when the hydrogen mass fraction falls below $X_{\rm H} = 0.001$.

drogen burning temperature. Note that the higher the temperature, the larger the reaction rates and hence the time it takes to reach hydrogen exhaustion is much shorter at T = 55 MK compared to T = 20 MK.

There is significant observational evidence for the operation of the CNO cycles in hydrogen burning. In many stars, the products of the nucleosynthesis have been carried by turbulent convection from the stellar interior to the surface. Consider as an example the isotopes of carbon. Stars that form from matter with a solar composition will initially exhibit an abundance ratio of $({}^{13}C/{}^{12}C)_{\odot} = 0.011$ at their surface. Once CNO steady state has been achieved, we expect, according to Fig. 5.11a, an abundance ratio of $({}^{13}C/{}^{12}C)_e$ = 0.25 in the hydrogen burning region. Note that the latter value is insensitive to temperature below T < 0.1 GK. Many stars show $({}^{13}C/{}^{12}C)$ surface abundance ratios between these two values, indicating that a fraction of the hydrogen burning matter has been transported to the stellar surface. Some stars even display $({}^{13}C/{}^{12}C)$ surface abundance ratios close to the steady state value (Sneden, Pilachowski and Vandenberg 1986). Such observations not only provide evidence for CNO burning, but also demonstrate that most of the matter at the stellar surface must have been cycled through the hydrogen burning region in the stellar interior. The operation of the CNO cycles in AGB stars is believed to be the main source of ¹³C and ¹⁴N in the Universe.

We will now briefly summarize the experimental situation regarding measurements of CNO-cycle reactions. At the upper temperature range characteristic of hydrostatic hydrogen burning ($T \approx 55$ MK), the Gamow peaks for the ¹²C + p and ¹⁹F + p reactions are located at $E_0 \pm \Delta/2 = 60 \pm 20$ keV and

 80 ± 24 keV, respectively. From the experimental point of view, the CNO reactions can be divided into two groups, depending on whether measured cross sections exist in the Gamow peak or not. For example, the reactions $^{13}C(p,\gamma)^{14}N$, $^{14}N(p,\gamma)^{15}O$, $^{15}N(p,\gamma)^{16}O$, and $^{16}O(p,\gamma)^{17}F$ have been measured down to center-of-mass energies of 100, 93, 130, and 130 keV, respectively (Angulo et al. 1999). In these cases, no data exist in the Gamow peak for hydrostatic hydrogen burning ($T \le 55$ MK) and hence the S-factor has to be extrapolated down to the energy range of interest, either by a polynomial expansion or by using a suitable nuclear reaction model (Section 3.2.1). The Sfactors for the above reactions are determined by nonresonant contributions (tails of broad resonances or direct capture) at E < 100 keV. Recent studies of the important ${}^{14}N(p,\gamma){}^{15}O$ reaction are reported in Formicola et al. (2004) and Runkle et al. (2005). The ${}^{17}O(p,\gamma){}^{18}F$, ${}^{18}O(p,\gamma){}^{19}F$, and ${}^{19}F(p,\gamma){}^{20}Ne$ reactions have also not been measured down to the relevant Gamow-peak region. In these cases, the reaction rates at $T \leq 55$ MK are expected to be dominated by unobserved (low-lying) narrow resonances. These contributions have been estimated by using all the available nuclear structure information on the corresponding states in the compound nuclei. On the other hand, the ${}^{12}C(p,\gamma){}^{13}N$, $^{15}N(p,\alpha)^{12}C$, and $^{18}O(p,\alpha)^{15}N$ reactions have all been measured down to energies of about 70 keV, covering at least part of the Gamow peak for the higher temperatures near $T \approx 55$ MK. The first two reactions are nonresonant, while the latter process is influenced both by resonant and nonresonant contributions to the reaction mechanism. The ${}^{17}O(p,\alpha){}^{14}N$ reaction represents an exceptional case. At temperatures of T = 18-55 MK, the most important contribution to the reaction rates originates from a narrow resonance located at $E_r^{\rm cm} = 65$ keV. This particular resonance has been observed and, in fact, represents the weakest resonance measured to date in the laboratory (with $\omega \gamma_{p\alpha}$ = $(4.7 \pm 0.8) \times 10^{-9}$ eV; Blackmon et al. 1995). For errors in CNO reaction rates, the reader is referred to Angulo et al. (1999). A discussion of the influence of reaction rate uncertainties on the evolution of CNO abundances is presented in Arnould, Goriely and Jorissen (1999).

5.1.3

Hydrostatic Hydrogen Burning Beyond the CNO Mass Region

The nucleosynthesis in hydrostatic hydrogen burning not only involves nuclei in the CNO mass range, but heavier nuclei as well. The most likely reactions to occur in the mass region above A = 20 are shown in Fig. 5.15. In the following, we will discuss some general properties of hydrostatic hydrogen burning involving heavier nuclei and will explain why some processes are more likely to occur than others. It is important to stress that there is no connection between the CNO and $A \ge 20$ mass ranges. In other words, pre-existing CNO seed nuclei will be transformed to other nuclei in the CNO mass range only. In principle, the ¹⁹F(p, γ)²⁰Ne reaction could provide a link between the CNO and $A \ge 20$ mass ranges. However, its reaction rate is at least three orders of magnitude smaller compared to the competing ¹⁹F(p, α)¹⁶O reaction (Fig. 5.9). Consequently, hydrostatic hydrogen burning beyond the CNO mass range must start from pre-existing seed nuclei with masses of $A \ge 20$.

These nuclei are transformed by β -decays, (p, γ) , and (p, α) reactions and the competition between these processes defines the resulting nucleosynthesis paths in the nuclidic chart. As was the case in the previous section, protoninduced reactions involving unstable target nuclei play no significant role in hydrostatic hydrogen burning since the competing β -decays are much faster (with τ_{β} of seconds to minutes in most cases). This conclusion applies even to long-lived nuclei, such as ²²Na with a half-life of $T_{1/2} = 2.6$ y. The mean lifetime of ²²Na versus destruction by the (p,γ) reaction is compared in Fig. 5.16 with the mean lifetime of the ²²Na β^+ -decay (Example 3.1). A density of ρ = 100 g/cm³ and $X_{\rm H}/M_{\rm H}$ = 1 are assumed. The quantity $\tau_{\beta}(^{22}{\rm Na})$ is independent of temperature and density for the conditions considered here (Section 1.8.4), while $\tau_p(^{22}Na)$ decreases with increasing T and ρ (see Eq. (3.22)). The ²²Na(p,γ)²³Mg reaction dominates over the competing β^+ -decay only at T > 0.065 GK, well above the temperature range characteristic of most hydrostatic hydrogen burning environments. The nucleus ²⁶Al represents an important exception. The half-life of the ground state amounts to $T_{1/2} = 7.2 \times 10^5$ y, a time sufficiently long for proton capture to compete with the β^+ -decay. The mean lifetimes $\tau_{\beta}({}^{26}\text{Al}^g)$ and $\tau_{p}({}^{26}\text{Al}^g)$ are also shown in Fig. 5.16 (the superscript *g* labels the ground state). It is apparent that below T = 37 MK the ²⁶Al^g nucleus is mainly destroyed by β^+ -decay, while the (p, γ) reaction dominates at higher temperatures. In other words, both processes will be important in hydrostatic hydrogen burning. An additional complication arises due to the existence of an isomeric state in ²⁶Al at $E_x = 228$ keV (Fig. 1.14). This level, with a half-life of $T_{1/2}$ = 6.3 s, is also produced in hydrogen burning. As explained in Section 1.7.5, the ground state, ²⁶Al^g, and the isomeric state, ²⁶Al^m, do not come into equilibrium at temperatures below T = 0.4 GK and hence have to be treated as two separate species in the reaction network describing hydrostatic hydrogen burning.

For a number of nuclei in the mass A = 20-40 range, most notably ²³Na, ²⁷Al, ³¹P, and ³⁵Cl, the (p, α) reaction channel is energetically allowed and the (p, γ) and (p, α) reactions on these nuclei will compete. If the reaction rate branching ratio $B_{p\alpha/p\gamma} = N_A \langle \sigma v \rangle_{(p,\alpha)} / N_A \langle \sigma v \rangle_{(p,\gamma)}$ is sufficiently large, then reaction cycles similar to the CNO cycles may develop. These processes are sometimes referred to in the literature as NeNa, MgAl, SiP, and SCl cycles. However, the current (p, γ) and (p, α) reaction rate uncertainties have to be considered carefully before drawing such conclusions. The quantity $B_{p\alpha/p\gamma}$ is



Fig. 5.15 Nuclear interactions in the mass $A \ge 20$ region during hydrostatic hydrogen burning. Stable nuclides are shown as shaded squares. The key relates an arrow to a specific interaction (proton capture, (p, α) reaction, or β^+ -decay). The nuclide ²⁶Al can be formed either in its ground state or in its isomeric state ($E_x = 228$ keV).



Fig. 5.16 Mean lifetimes of ²²Na (solid lines) and ²⁶Al^g (dashed lines) versus temperature. The curves are calculated for the conditions $\rho = 100 \text{ g/cm}^3$ and $X_{\rm H}/M_{\rm H} = 1$. The mean lifetimes for the β^+ -decays, $\tau_{\beta}(^{22}{\rm Na})$ and $\tau_{\beta}(^{26}{\rm Al}^g)$, are independent of temperature and density for the conditions of hydrostatic hydrogen burning.

displayed in Fig. 5.17 for the branching point nuclei ²³Na, ²⁷Al, ³¹P, and ³⁵Cl. The solid lines in each panel indicate the upper and lower limits of $B_{p\alpha/p\gamma}$ caused by unobserved narrow resonances in the (p,γ) and (p,α) reactions. Below T = 55 MK, the (p,α) reaction on ²³Na dominates over the competing (p,γ) reaction and hence a NeNa cycle may develop (but only if the cycling time is shorter than the duration of the hydrogen burning stage). The situation is



Fig. 5.17 Branching ratio $B_{p\alpha/p\gamma} = N_A \langle \sigma v \rangle_{(p,\alpha)} / N_A \langle \sigma v \rangle_{(p,\gamma)}$ versus temperature for the reactions (a) ²³Na + p, (b) ²⁷Al + p, (c) ³¹P + p, and (d) ³⁵Cl + p. The two solid lines in each panel represent the currently accepted upper and lower limits of $B_{p\alpha/p\gamma}$. The area between the solid lines represents the uncertainty in $B_{p\alpha/p\gamma}$ that is caused by unknown contributions to the (p, γ) and (p, α) reaction rates.

not as clear for the other branching point nuclei. For ²⁷Al the quantity $B_{p\alpha/p\gamma}$ ranges from about 0.04 to 100 below T = 55 MK and, therefore, current reaction rate uncertainties do not permit an unambiguous conclusion regarding the existence of a MgAl cycle. For the nuclei ³¹P and ³⁵Cl, on the other hand, we obtain $B_{p\alpha/p\gamma} < 1$ at temperatures characteristic of hydrostatic hydrogen burning and hence closed SiP and SCl cycles do not exist.

The rates of various reactions in the $A \ge 20$ mass range are compared in Fig. 5.18 to the ${}^{16}\text{O}(p,\gamma){}^{17}\text{F}$ reaction rate. Recall that the latter process represents the slowest reaction in the CNO mass region (Fig. 5.10). It can be seen that below T = 55 MK, as a result of the increasing Coulomb barrier, most reactions involving heavier target nuclei are much slower than the ${}^{16}\text{O}(p,\gamma){}^{17}\text{F}$ reaction. The only two exceptions are the proton captures on ${}^{21}\text{Ne}$ and ${}^{22}\text{Ne}$. Therefore, we expect reactions in the $A \ge 20$ mass range to be insignificant contributors to the overall nuclear energy generation rate in hydrostatic hydrogen burning. Nevertheless, an understanding of the nucleosynthesis is important for the interpretation of certain abundance observations that are discussed below.



Fig. 5.18 Rates of proton-induced reactions versus temperature in two mass regions: (a) NeNa, and (b) MgAlSi. For a better comparison, the values of $N_A \langle \sigma v \rangle$ are normalized to the rate of the ¹⁶O(p, γ)¹⁷F reaction.

The evolution of abundances in the mass range $A \ge 20$ versus the amount of hydrogen consumed is displayed in Fig. 5.19. The results are obtained from two network calculations that are performed for constant temperatures of *T* = 25 and 55 MK, a constant density of ρ = 100 g/cm³, and a solar initial composition ($X_{\rm H}^0 = 0.706$; $\sum X_{A=20-40}^0 = 0.00375$). The most abundant seed nuclei in the A = 20-40 mass range are ²⁰Ne, ²⁸Si, ²⁴Mg, ³²S, and ²²Ne. We will first discuss the situation at T = 25 MK. At this temperature, hydrogen is exhausted (X_H < 10^{-3}) after $t \approx 5 \times 10^7$ y. According to Fig. 5.18, the fastest reactions in the $A \ge 20$ mass range are ${}^{22}\text{Ne}(p,\gamma){}^{23}\text{Na}$ and ${}^{25}\text{Mg}(p,\gamma){}^{26}\text{Al}$. Therefore, the abundances of ²²Ne and ²⁵Mg are depleted, while those of ²³Na and ²⁶Al^g increase with time. At this temperature, the nuclide ²⁶Al^g is mainly destroyed via β^+ -decay (Fig. 5.16). Thus, its abundance, after reaching a maximum, starts to decline. As a result, the abundance of the daughter nucleus ²⁶Mg increases. The ²⁰Ne abundance stays almost constant during the calculation since the 20 Ne(p, γ) 21 Na reaction is relatively slow. Nevertheless, a small amount of ²⁰Ne is depleted, giving rise to a noticeable increase in the abundance of the rare isotope ²¹Ne, which is produced via ²¹Na($\beta^+\nu$)²¹Ne. Other reactions, including ${}^{24}Mg + p$ and ${}^{27}Al + p$, are too slow to cause any abundance changes. The same applies to the 2^{3} Na + p reactions. At T = 25 MK the mean lifetime of ²³Na versus destruction by the (p,α) reaction is $\tau_{\rm p}(^{23}{\rm Na}) \approx 2 \times 10^9$ y. This lifetime far exceeds the time after which hydrogen is exhausted. Hence, ²⁰Ne is not produced and, in particular, no NeNa cycle develops despite the fact that the 23 Na(p, α) 20 Ne reaction is much faster than the 23 Na(p, γ) 24 Mg reaction (Fig. 5.17). Also note that under the assumed conditions, no significant nuclear transformations occur in the $A \ge 28$ mass range. At the end of the network calculation, ²¹Ne, ²³Na, and ²⁶Mg are overproduced, while ²²Ne and ²⁵Mg have been depleted.

We will now discuss the situation at T = 55 MK. At this temperature, hydrogen is exhausted after only $t \approx 510$ y. The temperature is sufficiently high for more nuclear reactions to take part in the nucleosynthesis. As was the case before, ²²Ne is converted to ²³Na. It can be seen that, contrary to the results obtained at T = 25 MK, the ²²Ne abundance is not entirely destroyed. This is explained by the fact that the ${}^{20}Ne(p,\gamma){}^{21}Na$ reaction, although still the slowest process in the NeNa region (Fig. 5.18), is now fast enough to initiate the chain 20 Ne $(p,\gamma)^{21}$ Na $(\beta^+\nu)^{21}$ Ne $(p,\gamma)^{22}$ Na $(\beta^+\nu)^{22}$ Ne. Indeed, the 20 Ne abundance is slightly depleted, as can be seen in Fig. 5.19b. The mean lifetime of ²³Na versus destruction by protons is $\tau_p(^{23}Na) \approx 100$ y. A fraction of ²³Na nuclei is transformed to ²⁰Ne, although the total ²⁰Ne abundance declines because of the destruction via 20 Ne(p, γ) 21 Na. Nevertheless, a closed NeNa cycle does not develop since the mean lifetime of ²⁰Ne is $\tau_p(^{20}\text{Ne}) \approx$ 600 y, close to the time at which hydrogen is exhausted. The leakage out of the NeNa mass region via 23 Na(p, γ) 24 Mg is clearly seen in Fig. 5.19b as an increase in the ²⁴Mg abundance. The isotope ²⁴Mg is not destroyed, because the ²⁴Mg(p, γ)²⁵Al reaction is the slowest process in the $A \leq 27$ range, with a mean lifetime of $\tau_p(^{24}\text{Mg}) \approx 75000 \text{ y}$. On the other hand, ^{25}Mg is converted to ²⁶Al^g via the ²⁵Mg(p, γ)²⁶Al reaction. At this temperature, ²⁶Al^g is mainly destroyed by the ${}^{26}Al^{g}(p,\gamma){}^{27}Si$ reaction. However, the mean lifetime of ${}^{26}Al^{g}$ amounts to $\tau_p(^{26}\text{Al}^g) \approx 1000 \text{ y}$ and, therefore, it has little time to decay to ²⁶Mg. The ²⁶Mg(p,γ)²⁷Al reaction is now fast enough to cause the depletion of 26 Mg and the production of 27 Al, as can be seen in Fig. 5.19b. The 27 Al + p reactions play only a minor role $[\tau_p(^{27}\text{Al}) \approx 10000 \text{ y}]$. As was the case at *T* = 25 MK, nuclear transformations in the $A \ge 27$ mass range are unimportant. In summary, ²³Na, ²⁶Al^g, and ²⁷Al are enhanced, while ²¹Ne, ²²Ne, ²⁵Mg, and ²⁶Mg are depleted at the end of the network calculation.

Hydrostatic hydrogen burning in the mass $A \ge 20$ range is important for the interpretation of Ne, Na, Mg, and Al abundance observations in stars. The relative isotopic and elemental abundances depend, as shown above, on the conditions of temperature and density in the hydrogen burning region. In order for these species to be observed, either in stellar atmospheres or in presolar grains, they have to be transported from the hydrogen burning region to the stellar surface. Hence, such abundance observations provide important clues regarding stellar evolution and stellar mixing processes. It is obvious that accurate thermonuclear reaction rates are required when comparing abundances from stellar models with those from observations. Hydrostatic hydrogen burning in the mass $A \ge 20$ range is also of interest for the Galactic origin of the radioisotope ²⁶Al. It seems likely that a fraction of the observed ²⁶Al originates from Wolf–Rayet stars where it is synthesized during hydrostatic core hydrogen burning at temperatures of T = 35-45 MK (Section 1.7.5).



Fig. 5.19 Abundance evolutions in the $A \ge 20$ mass region versus the amount of hydrogen consumed for two different constant temperatures: (a) T = 25 MK, and (b) T = 55 MK. The density ($\rho = 100$ g/cm³) and the initial composition (solar) is the same for both cases. All curves shown are obtained by solving the reaction network numerically. The calculations are terminated when the hydrogen mass fraction falls below $X_{\rm H} = 0.001$.

Finally, we will summarize the experimental situation regarding measurements of reactions in the NeNa and MgAl region. At the upper temperature range characteristic of hydrostatic hydrogen burning (T \approx 55 MK), the Gamow peaks for the ²⁰Ne + p and ²⁷Al + p reactions are located at $E_0 \pm \Delta/2$ = 80 ± 23 keV and 95 ± 25 keV, respectively. None of the NeNa or MgAl reactions have been measured directly down to such low energies. In order to estimate the total reaction rates at low temperatures, it becomes therefore important to measure directly at higher energies as many different resonant and nonresonant reaction components as possible (narrow resonances, broad resonances, and direct processes). In addition, indirect reaction studies populating compound levels which are located between the proton threshold and the lowest lying observed resonance are of crucial importance in order to estimate reaction rate contributions of still undetected narrow resonances. Despite these time-consuming experimental efforts, it must be realized that the rates of certain reactions in the mass A = 20-40 range still have appreciable errors. The 20 Ne(p, γ)²¹Na reaction rate is determined by the tail of a subthreshold state (Example 2.1) and by direct radiative capture. The present reaction rate errors range from 40% to a factor of 2 below T = 55 MK (Angulo et al. 1999). The ${}^{24}Mg(p,\gamma){}^{25}Al$ reaction proceeds mainly through an observed narrow resonance at E_r^{cm} = 214 keV (Example 3.7 and Fig. 3.28) and via direct radiative capture. With an error of \leq 20% below *T* = 55 MK, the rate of this reaction is among the most precisely known in the A = 20-40 region (Powell et al. 1999). All other reaction rates in the NeNa and MgAl region are strongly

influenced by unobserved narrow resonances, with rate errors amounting in some cases to orders of magnitude. Significant efforts are at present underway to detect the most important of these unobserved resonances. Their resonance strengths are expected to be far smaller compared to the strength of the $E_r^{cm} = 149$ keV resonance in ${}^{26}\text{Mg}(p,\gamma){}^{27}\text{Al}$, which represents the weakest measured (p,γ) resonance to date $[\omega\gamma_{p\gamma} = (8 \pm 3) \times 10^{-8} \text{ eV};$ Iliadis et al. 1990]. Such experiments are clearly a challenge to the nuclear experimentalist (Chapter 4). An evaluation of reaction rates and their associated errors in the A = 20–40 mass range can be found in Iliadis et al. (2001). A discussion of the influence of reaction rate uncertainties on the evolution of abundances in the NeNa and MgAl regions is given in Arnould, Goriely and Jorissen (1999).

5.2

Explosive Hydrogen Burning

In Section 5.1 we discussed hydrogen burning in the stellar temperature range of T < 0.06 GK. If, under such conditions, the stellar gas consists of pure hydrogen, then hydrogen burning must proceed via the pp chains (with perhaps a contribution from the pep reaction; Section 5.1.1). On the other hand, if a significant fraction of CNO nuclei is present in the stellar gas, then the CNO cycles will generate most of the energy above a certain value of the temperature (for example, above 20 MK for a solar mass fraction of CNO nuclei; see Fig. 5.12). There are two important points that need to be kept in mind regarding hydrogen burning at temperatures below 0.06 GK. First, a specific radioactive nucleus that is produced during the burning will be destroyed by its relatively fast β -decay rather than by the much slower competing protoninduced reaction (with the exceptions of ⁷Be in the pp3 chain and ²⁶Al in the region of $A \ge 20$; see Fig. 5.2 and Section 5.1.3). Second, in the reaction network of the pp chains or the CNO cycles, all the radioactive decays are much faster compared to the slowest proton-induced reaction and, consequently, the energy generation rate does not depend on the half-lives of the radioactive decays. At elevated temperatures typical of explosive hydrogen burning, the situation described above changes substantially. In the following we will discuss the explosive nucleosynthesis in the A < 20 and $A \ge 20$ mass regions at temperatures of T = 0.1-0.4 GK. Another important point needs to be stressed. So far, we explored the nucleosynthesis in hydrostatic burning environments analytically by considering equilibrium burning conditions or numerically by performing reaction network calculations assuming a constant temperature and density. These considerations provide a qualitative picture of the interplay between different nuclear processes. However, the above assumptions are not necessarily valid for explosive events. First, the time to approach equi-

librium conditions is often comparable to the macroscopic hydrogen burning time scale. Second, temperatures and densities in an explosive event change dramatically with time. The time evolution of *T* and ρ depends strongly on the properties of the exploding star. In this section, the reaction networks for explosive hydrogen burning are solved numerically, first with the assumption of constant *T*– ρ conditions, and then by using temperature and density evolutions that describe specific models of stellar explosions.

5.2.1

Hot CNO Cycles

If a star consists of gas that contains a significant fraction of nuclei in the CNO mass region, then at elevated temperatures (T = 0.1-0.4 GK) most of the nuclear energy is generated by the *hot CNO cycles* (or HCNO cycles). The reactions of the HCNO cycles are listed below and are shown in the Fig. 5.20.

Hot CNO1	Hot CNO2	Hot CNO3
¹² C(p,γ) ¹³ N	$^{15}O(eta^+ u)^{15}N$	$^{15}O(\beta^+ u)^{15}N$
¹³ N(p,γ) ¹⁴ O	¹⁵ N(p,γ) ¹⁶ O	¹⁵ N(p,γ) ¹⁶ O
$^{14}\text{O}(\beta^+ u)^{14}\text{N}$	$^{16}O(p,\gamma)^{17}F$	$^{16}O(p,\gamma)^{17}F$
¹⁴ N(p,γ) ¹⁵ O	¹⁷ F(β ⁺ ν) ¹⁷ O	$^{17}F(p,\gamma)^{18}Ne$
$^{15}O(\beta^+ u)^{15}N$	$^{17}O(p,\gamma)^{18}F$	${}^{\scriptscriptstyle 18}{\sf Ne}(eta^+ u){}^{\scriptscriptstyle 18}{\sf F}$
¹⁵ N(p,α) ¹² C	¹⁸ F(p,α) ¹⁵ O	¹⁸ F(p,α) ¹⁵ O
$T_{1/2}: {}^{14}O$	(70.61 s); ¹⁵ O (122.24 s); ¹⁷	F (64.49 s)

The hot CNO cycles have a number of important properties in common with the CNO cycles discussed in Section 5.1.2: (i) each of the hot CNO cycles converts four hydrogen nuclei to one helium nucleus; (ii) the CNOF nuclei involved in the hot CNO cycles act as catalysts and their total number is nearly constant; and (iii) the energy generation rate of the hot CNO cycles depends on the abundances of the catalysts. It will be shown in Section 5.4.1 that, above a certain temperature ($T \ge 0.4$ GK), catalytic material is lost from the hot CNO cycles by various breakout reactions. In this section, we will discuss the operation of the hot CNO cycles in the temperature region of T = 0.1-0.4 GK.

We will start our discussion by considering the CNO1 cycle (Fig. 5.8) and how that cycle is modified when the temperature gradually increases. The ¹³N nucleus has the longest half-life ($T_{1/2} = 9.96$ min) among all the β^+ -decays in the CNO1 cycle. For increasing temperature, a point will be reached where the destruction of ¹³N by proton-capture competes favorably with the β^+ -decay



Fig. 5.20 Representation of the three hot CNO cycles in the chart of the nuclides. Stable nuclides are shown as shaded squares. Each reaction cycle fuses effectively four protons to one ⁴He nucleus. Note that in explosive hydrogen burning the CNO2 cycle (Fig. 5.8) is more likely to occur than the HCNO2 cycle since the ¹⁷O(p, α)¹⁴N reaction rate dominates over the ¹⁷O(p, γ)¹⁸F rate (Fig. 5.9).

of 13 N. Hence, instead of the sequence that occurs in the CNO1 cycle,

$${}^{13}N(\beta^+\nu){}^{13}C(p,\gamma){}^{14}N \tag{5.71}$$

the alternative path

$${}^{13}\mathrm{N}(\mathrm{p},\gamma){}^{14}\mathrm{O}(\beta^+\nu){}^{14}\mathrm{N} \tag{5.72}$$

becomes more likely. The half-life of ¹⁴O ($T_{1/2}$ = 70.6 s) is less than that of ¹³N. Thus ¹³N is converted to ¹⁴N on a faster time scale for the latter path. Furthermore, it is shown in Section 5.1.2 that the ¹⁴N(p, γ)¹⁵O reaction is the slowest process in the CNO1 cycle and, therefore, determines the energy generation rate. For increasing temperature, all the rates for proton-induced reactions

will increase strongly. Eventually a point is reached at which all the (p,γ) and (p,α) reactions, including the ¹⁴N $(p,\gamma)^{15}$ O reaction, are faster compared to the β^+ -decays of ¹⁴O and ¹⁵O. As a result of these two modifications, the CNO1 cycle transforms at higher temperatures into the hot CNO1 cycle (Fig. 5.20). It was mentioned earlier that the energy generation rate of the CNO1 cycle is highly sensitive to the temperature (see Eq. (5.69)). The HCNO1 cycle, on the other hand, has the interesting property that the energy generation rate depends on the β^+ -decays of ¹⁴O and ¹⁵O (that is, the slowest links in the cycle) and hence is independent of temperature. For this reason, the HCNO1 cycle is also referred to as β -limited CNO cycle. The time around one HCNO1 cycle is then at least $\tau_{\beta}(^{14}O) + \tau_{\beta}(^{15}O) = T_{1/2}(^{14}O) / \ln 2 + T_{1/2}(^{15}O) / \ln 2 \approx 278$ s. It follows that a significant fraction of the CNO nuclei will be transformed into ¹⁴O and ¹⁵O. Note that proton captures on ¹⁴O and ¹⁵O are unlikely to occur since the corresponding compound nuclei ¹⁵F and ¹⁶F are unstable by proton emission.

The transition from the CNO1 cycle to the HCNO1 cycle can be represented in a temperature-density diagram (Fig. 5.21). The lifetimes of the ¹³N, ¹⁴O and ¹⁵O β^+ -decays are given by $\tau_{\beta} = T_{1/2}/\ln 2$, while the lifetimes of ¹³N and ¹⁴N versus destruction by protons are $\tau_{\rm p} = [\rho(X_{\rm H}/M_{\rm H})N_A\langle\sigma v\rangle]^{-1}$. The solid curve represents the *T*- ρ conditions at which the ¹³N β^+ -decay lifetime is equal to the lifetime of ¹³N destruction via proton capture, that is

$$\frac{1}{\rho(X_{\rm H}/M_{\rm H})N_A \langle \sigma v \rangle_{^{13}N({\rm p},\gamma)}} = \frac{T_{1/2}(^{13}{\rm N})}{\ln 2}$$
(5.73)

The dashed (or dashed-dotted) curve is obtained for the condition that the ^{13}N (or ^{14}N) lifetime versus destruction by proton capture is equal to the sum of ^{14}O and ^{15}O lifetimes,

$$\frac{1}{\rho(X_{\rm H}/M_{\rm H})N_A\langle\sigma v\rangle_{x_N(\mathbf{p},\gamma)}} = \frac{T_{1/2}({}^{14}O)}{\ln 2} + \frac{T_{1/2}({}^{15}O)}{\ln 2}$$
(5.74)

All curves are calculated for a solar value of $X_{\rm H}/M_{\rm H} = 0.70$. On the left-hand side of each curve, the β^+ -decays are more likely to occur than the proton-induced reaction, while the opposite situation prevails on the right-hand side. The CNO1 cycle operates in region 1 where ¹³N β^+ -decays [$\tau_{\beta}(^{13}N) < \tau_{\rm p}(^{13}N)$] and where $^{14}N({\rm p},\gamma)^{15}O$ is the slowest link in the cycle [$\tau_{\beta}(^{14}O) + \tau_{\beta}(^{15}O) < \tau_{\rm p}(^{14}N)$]. Suppose we start out in region 1 and slowly increase the temperature by keeping the density constant, for example, at $\rho = 500 \text{ g/cm}^3$. When the solid curve is crossed at $T \approx 0.100 \text{ GK}$, we have $\tau_{\beta}(^{13}N) > \tau_{\rm p}(^{13}N)$ and the slow $^{13}N \beta^+$ -decay is bypassed by the sequence $^{13}N({\rm p},\gamma)^{14}O(\beta^+\nu)^{14}N$ (region 2a). When the dashed curve is crossed at $T \approx 0.113 \text{ GK}$, the ^{13}N proton-capture reaction becomes faster than the ^{14}O and $^{15}O \beta^+$ -decays [$\tau_{\beta}(^{14}O) +$

 $\tau_{\beta}(^{15}O) > \tau_{p}(^{13}N)$]. At this stage (region 2b), the $^{14}N(p,\gamma)^{15}O$ reaction is still the slowest link in the cycle and determines the energy generation rate. Finally, when the dashed-dotted curve is crossed at $T \approx 0.128$ GK, the proton capture on ^{14}N becomes faster than the ^{14}O and $^{15}O \beta^+$ -decays $[\tau_{\beta}(^{14}O) + \tau_{\beta}(^{15}O) > \tau_{p}(^{14}N)]$. We have now reached region 3, where the β -limited HCNO1 cycle operates.

For other densities, the situation is similar when increasing the temperature, although the curves may be crossed in different order. For example, at a lower density of $\rho = 5 \text{ g/cm}^3$ the solid curve is crossed at $T \approx 0.161 \text{ GK}$ (region 2a), while the dashed-dotted curve is crossed at $T \approx 0.174 \text{ GK}$. At this stage (region 2c) the sequence ${}^{13}\text{N}(\text{p},\gamma){}^{14}\text{O}(\beta^+\nu){}^{14}\text{N}$ dominates over the ${}^{13}\text{N}\beta^+$ -decay and the proton capture reaction on ${}^{14}\text{N}$ is faster than the ${}^{14}\text{O}$ and ${}^{15}\text{O}\beta^+$ -decays. However, the dashed curve has not been crossed yet, that is, the ${}^{13}\text{N}(\text{p},\gamma){}^{14}\text{O}$ reaction is slower than the ${}^{14}\text{O}$ and ${}^{15}\text{O}\beta^+$ -decays. In fact, the proton-capture reaction on ${}^{13}\text{N}$ is now the slowest link in the cycle and determines the energy generation rate. Finally, the dashed curve is crossed at $T \approx 0.185 \text{ GK}$ and the CNO cycle becomes again β -limited (region 3).

The HCNO1 cycle discussed above represents a nearly closed reaction sequence, in the sense that very little catalytic material is lost. This comes about since the branching ratio $B_{p\alpha/p\gamma}$ at ¹⁵N exceeds a factor of 1000 (Fig. 5.9). Hence, if ¹²C seed nuclei are present in the gas, they will be converted mostly to ¹⁴O and ¹⁵O assuming that hydrogen is not near exhaustion (see later). We already pointed out in Section 5.1.2 that besides ¹²C other seed nuclei, such as ¹⁶O, may be present in the stellar gas. The ¹⁶O nuclei are processed in a number of different, competing reaction cycles. One possibility of processing is the CNO2 cycle which was introduced in Section 5.1.2,

$${}^{16}O(p,\gamma){}^{17}F(\beta^+\nu){}^{17}O(p,\alpha){}^{14}N(p,\gamma){}^{15}O(\beta^+\nu){}^{15}N(p,\gamma){}^{16}O$$
(5.75)

Inspection of Fig. 5.9 reveals that the ¹⁷O(p, α)¹⁴N reaction dominates over the competing ¹⁷O(p, γ)¹⁸F reaction at temperatures of *T* = 0.1–0.4 GK by a factor of \approx 200. A small fraction of the ¹⁶O seed nuclei will be processed via the ¹⁷O(p, γ)¹⁸F reaction, giving rise to the HCNO2 cycle,

$${}^{16}O(p,\gamma){}^{17}F(\beta^+\nu){}^{17}O(p,\gamma){}^{18}F(p,\alpha){}^{15}O(\beta^+\nu){}^{15}N(p,\gamma){}^{16}O$$
(5.76)

The branching ratio $B_{p\alpha/p\gamma}$ at ¹⁸F is shown in Fig. 5.22a. In the temperature range of T = 0.1-0.4 GK, the ¹⁸F(p, α)¹⁵O reaction is faster than the competing ¹⁸F(p, γ)¹⁹Ne reaction by more than a factor of 1000. Under conditions of explosive hydrogen burning, the ¹⁸F(p, α)¹⁵O reaction is also much faster than the ¹⁸F β^+ -decay. This is demonstrated in Fig. 5.22b. The dashed line shows the $T-\rho$ conditions (for $X_H/M_H = 0.70$) at which the ¹⁸F β^+ decay lifetime is equal to the lifetime of ¹⁸F destruction via the (p, α) reaction [$\tau_{p\alpha}$ (¹⁸F) = τ_{β} (¹⁸F)]. For example, at a density of $\rho = 500$ g/cm³ the



Fig. 5.21 Temperature–density diagram showing the transition from the CNO1 cycle (region 1) to the HCNO1 cycle (region 3). The solid curve represents the $T-\rho$ conditions at which the ¹³N β^+ -decay lifetime is equal to the lifetime of ¹³N destruction via proton capture. The dashed (or dashed-dotted) curve is obtained for the condition that the ¹³N (or ¹⁴N) lifetime versus destruc-

tion by proton capture is equal to the sum of the ¹⁴O and ¹⁵O β^+ -decay lifetimes. All curves are calculated for the solar value $X_{\rm H}/M_{\rm H}$ = 0.70. On the left-hand side of each curve, the β^+ -decay is more likely to occur than the proton-induced reaction, while the opposite applies on the right-hand side.

¹⁸F(p, α)¹⁵O reaction dominates over the competing β^+ -decay for temperatures of T > 0.058 GK. Consequently, once the nucleosynthesis path in explosive hydrogen burning reaches ¹⁸F, the (p, α) reaction is the dominant destruction mode. The solid line in Fig. 5.22b shows the $T-\rho$ conditions at which the ¹⁷F β^+ -decay lifetime is equal to the lifetime of ¹⁷F destruction via the (p, γ) reaction [τ_p (¹⁷F) = τ_β (¹⁷F)]. Considering again, as an example, a density of $\rho = 500$ g/cm³, it can be seen that the ¹⁷F(p, γ)¹⁸Ne reaction dominates over the competing β^+ -decay at temperatures of T > 0.23 GK. Hence, the HCNO3 cycle develops, bypassing the isotope ¹⁷O,

$${}^{16}O(p,\gamma){}^{17}F(p,\gamma){}^{18}Ne(\beta^+\nu){}^{18}F(p,\alpha){}^{15}O(\beta^+\nu){}^{15}N(p,\gamma){}^{16}O$$
(5.77)

Network calculations at constant temperature and density

In order to gain a better understanding of the nucleosynthesis, we will first solve the reaction network of the HCNO cycles numerically for constant temperature and density conditions. The extra complications that arise from the much more realistic assumptions of time-dependent temperatures and densities will be dealt with later. We assume for the initial abundances values of $X_{\rm H}^0 = 0.60$, $X_{\rm 4He}^0 = 0.20$, and $X_{\rm 12C}^0 = 0.20$, that is, only ¹²C is initially





Fig. 5.22 (a) Branching ratio $B_{p\alpha/p\gamma} = N_A \langle \sigma v \rangle_{(p,\alpha)} / N_A \langle \sigma v \rangle_{(p,\gamma)}$ versus temperature for the ¹⁸F + p reactions. The two solid lines represent the currently accepted upper and lower limits of $B_{p\alpha/p\gamma}$ (from de Séréville, Berthoumieux and Coc 2005). The area between the solid lines represents the uncertainty in $B_{p\alpha/p\gamma}$ that is caused by unknown contributions to the (p, γ) and (p, α) reaction rates. (b) Temperature–density

diagram showing the competing destruction modes of 17 F and 18 F. The dashed line represents the conditions at which the 18 F β^+ -decay lifetime is equal to the lifetime of 18 F destruction via the (p, α) reaction. The solid line corresponds to the conditions at which the 17 F β^+ -decay lifetime is equal to the lifetime of 17 F destruction via the (p, γ) reaction. The curves in part (b) are calculated assuming X_H/M_H = 0.70.

present as seed for the hot CNO cycles. The reaction network is solved until hydrogen exhaustion ($X_{\rm H} < 0.001$). The time evolution of ¹H and the HCNO1 abundances (¹²C, ¹³N, ¹⁴O, ¹⁴N, ¹⁵O) is shown in Fig. 5.23 for temperatues of T = 0.15 GK and T = 0.3 GK. For both calculations, a density of $\rho = 200$ g/cm³ has been chosen. Although all reactions of the HCNO cycles have been included in the network, the graphs represent mainly the operation of the HCNO1 cycle, since the leakage to the other cycles via the ¹⁵N(p, γ)¹⁶O reaction is very small. For the initial conditions chosen, the abundances of ¹⁶O, ¹⁷O, ¹⁷F, and ¹⁸F never exceed a value of $X_i = 10^{-4}$. Also, the abundance of ¹⁵N is very small because of its strong destruction via the (p, α) reaction and is not displayed in Fig. 5.23.

For a temperature of T = 0.15 GK (Fig. 5.23a), ¹²C is initially converted to ¹³N via the ¹²C(p, γ)¹³N reaction. The ¹³N abundance reaches a maximum after t = 20 s. The subsequent ¹³N(p, γ)¹⁴O reaction causes the ¹⁴O abundance to peak after about t = 80 s. The slow ¹⁴O β^+ -decay increases the ¹⁴N abundance, while the subsequent ¹⁴N(p, γ)¹⁵O reaction is responsible for the growing abundance of ¹⁵O. At a time around t = 500 s the hydrogen abundance has dropped to $X_{\rm H} = 0.5$ and all CNO isotopes achieve equilibrium abundances. The ratio of any two number abundances is then given by Eq. (5.59),

$$\left(\frac{A}{B}\right)_{e} = \left(\frac{X_{A}}{X_{B}}\right)_{e} \left(\frac{M_{B}}{M_{A}}\right) = \frac{\tau_{A}}{\tau_{B}}$$
(5.78)
At this stage, the sum of mean lifetimes in the HCNO1 cycle amounts to

$$\sum \tau_{\text{CNO1}} \equiv \tau_{\text{p}}(^{12}C) + \tau_{\text{p}}(^{13}N) + \tau_{\beta}(^{14}O) + \tau_{\text{p}}(^{14}N) + \tau_{\beta}(^{15}O)$$

= (13 + 63 + 102 + 91 + 176) s = 445 s (5.79)

The nuclide ¹⁵O is the most abundant species in the CNO mass range because its mean lifetime has the largest value. Although the ¹⁴O and ¹⁵O β^+ -decays represent the slowest links in the HCNO1 cycle, the contribution from protoninduced reactions to the sum of mean lifetimes is substantial. Further processing of matter is influenced by the fact that the hydrogen abundance decreases substantially until exhaustion. As a result, all the lifetimes for proton-induced reactions increase (see Eq. (3.22)). For example, at a time of t = 3000 s, the hydrogen abundance has dropped to a value of $X_{\rm H} = 0.18$ and we obtain

$$\sum \tau_{\text{CNO1}} = (34 + 168 + 102 + 252 + 176) \,\text{s} = 732 \,\text{s} \tag{5.80}$$

with ¹⁴N(p, γ)¹⁵O representing the slowest link in the cycle. With decreasing hydrogen abundance, the ¹⁴O and ¹⁵O abundances also decline since their β^+ -decays are now faster than their production via ¹³N(p, γ)¹⁴O and ¹⁴N(p, γ)¹⁵O. The nuclide ¹⁴N becomes the most abundant species and its abundance increases further until the end of the calculation is reached. It is remarkable that hydrogen is exhausted after only t = 8400 s, a time period that is significantly shorter compared to the situation prevailing in hydrostatic hydrogen burning environments. This result is a direct consequence of the strong temperature sensitivity of charged-particle reaction rates.

At a higher temperature of T = 0.3 GK (Fig. 5.23b), the abundances evolve initially similar to the previous case. The nuclide ¹²C is first converted to ¹³N, then further processed to ¹⁴O. The β^+ -decay of the latter nucleus feeds the increasing ¹⁵O abundance via the sequence ¹⁴O($\beta^+\nu$)¹⁴N(p, γ)¹⁵O. Equilibrium CNO abundances are reached after t = 300 s. At that stage, the hydrogen abundance has dropped to $X_{\rm H} = 0.5$ and the sum of all mean lifetimes in the HCNO1 cycle amounts to

$$\sum \tau_{\text{CNO1}} = (0.035 + 0.15 + 102 + 0.016 + 176) \text{ s}$$

$$\approx \tau_{\beta}(^{14}O) + \tau_{\beta}(^{15}O) = 278 \text{ s}$$
(5.81)

with a negligible contribution from proton-capture reactions. The reaction cycle is β -limited and the most abundant species are ¹⁵O and ¹⁴O. For their number abundance ratio, we obtain

$$\left(\frac{^{15}O}{^{14}O}\right)_{e} = \left(\frac{X_{15_{O}}}{X_{14_{O}}}\right)_{e} \left(\frac{M_{14_{O}}}{M_{15_{O}}}\right) = \frac{\tau_{\beta}(^{15}O)}{\tau_{\beta}(^{14}O)} = \frac{176\,\mathrm{s}}{102\,\mathrm{s}} = 1.7\tag{5.82}$$

in agreement with the numerical results displayed in Fig. 5.23b. This situation prevails almost until the end of the calculation. Only at times very close to hydrogen exhaustion do the mean proton-capture lifetimes sufficiently increase

to cause a slight drop in the ¹⁴O and ¹⁵O abundances, with a corresponding rise in ¹²C, ¹³N, and ¹⁴N abundances. Nevertheless, even at hydrogen exhaustion ($X_{\rm H} = 0.001$) we have

$$\sum \tau_{\text{CNO1}} = (18 + 77 + 102 + 8 + 176) \,\text{s} = 381 \,\text{s} \tag{5.83}$$

and the ¹⁴O and ¹⁵O β^+ -decays are still the slowest links in the HCNO1 cycle. Hydrogen is exhausted after a time of t = 2400 s. This value is significantly shorter compared to the result obtained in the previous network calculation since the proton-capture reactions become much faster with increasing temperature.

We will now consider the nucleosynthesis resulting from a change in initial composition. We assume values of $X_{\rm H}^0 = 0.60$, $X_{\rm 4He}^0 = 0.20$ and $X_{\rm 16O}^0 = 0.20$, that is, only ¹⁶O instead of ¹²C is initially present as seed for the hot CNO cycles. For the temperature and density we have again assumed values of T = 0.3 GK and $\rho = 200$ g/cm³, respectively. The results of a network calculation are displayed in Figs. 5.23c and d. The ${}^{16}O(p,\gamma){}^{17}F$ reaction quickly destroys the ¹⁶O seed nuclei and converts them to ¹⁷F, whose abundance peaks after t = 8 s. For the $T - \rho$ conditions chosen, the ${}^{17}F(p,\gamma){}^{18}Ne$ reaction dominates over the competing 17 F β -decay (see Fig. 5.22b). Hence, the HCNO3 cycle operates, as can be seen from the rising ¹⁸Ne abundance. Subsequent to 18 Ne($\beta^+\nu$) 18 F, the fast 18 F(p, α) 15 O reaction feeds the abundance of 15 O which increases steadily. Note that a small abundance flow also proceeds through the CNO2 and HCNO2 cycles, as indicated by the evolution of the ¹⁷O abundance. Further processing of matter is similar to the previous case of T = 0.3 GK and only ¹²C present as seed (see Fig. 5.23b). All HCNO1 abundances achieve equilibrium after $t \approx 300$ s, with ¹⁵O and ¹⁴O being the most abundant species. In effect, the HCNO3 cycle feeds the HCNO1 cycle and most of the ¹⁶O seed nuclei are transformed to ¹⁴O and ¹⁵O. Hydrogen is exhausted after t = 3170 s, which is longer than the result obtained in the previous calculation. The delay is caused by the additional initial processing of matter through the HCNO3 cycle. Finally, it can be seen that the ¹⁶O, ¹⁷F, and ¹⁷O abundances increase toward the end of the calculation, indicating that a small fraction of matter leaks out of the HCNO1 cycle via the ${}^{15}N(p,\gamma){}^{16}O$ reaction.

The total energy generated per HCNO1 cycle that is available to the star is

$$Q_{4H\to^{4}He} - \overline{E}_{\nu}^{14}O(\beta^{+}\nu) - \overline{E}_{\nu}^{15}O(\beta^{+}\nu) = 24.827 \,\text{MeV}$$
(5.84)

with $Q_{4H\rightarrow^4He} = 26.731$ MeV and \overline{E}_{ν}^i the average neutrino energies released in the β^+ -decays (see Eq. (1.45) and Problem 1.9). Since four hydrogen atoms have a mass of $4 M_H/N_A$ g, the total energy generated per gram of consumed



Fig. 5.23 Time evolution of abundances during the operation of the hot CNO cycles for different conditions: (a) T = 0.15 GK, $X_{\rm H}^0 = 0.60$, $X_{4\rm He}^0 = 0.20$ and $X_{12\rm C}^0 = 0.20$; (b) T = 0.30 GK, $X_{\rm H}^0 = 0.60$, $X_{4\rm He}^0 = 0.20$ and $X_{12\rm C}^0 = 0.20$; (c), (d) T = 0.30 GK, $X_{\rm H}^0 = 0.60$, $X_{4\rm He}^0 = 0.20$ and $X_{12\rm C}^0 = 0.20$; (c), (d) T = 0.30 GK, $X_{\rm H}^0 = 0.60$, $X_{4\rm He}^0 = 0.20$ and $X_{12\rm C}^0 = 0.20$; (c), (d) T = 0.30 GK, $X_{\rm H}^0 = 0.60$, $X_{4\rm He}^0 = 0.20$ and $X_{16\rm O}^0 = 0.20$. For the density a constant value of $\rho = 200$ g/cm³ is assumed in all panels. All curves shown are obtained by solving the reaction network numerically. The calculations are terminated when the hydrogen mass fraction falls below $X_{\rm H} = 0.001$.

hydrogen is

$$\frac{Q_{4\mathrm{H}\to^{4}\mathrm{He}} - \overline{E}_{\nu}^{^{14}\mathrm{O}(\beta^{+}\nu)} - \overline{E}_{\nu}^{^{15}\mathrm{O}(\beta^{+}\nu)}}{4M_{\mathrm{H}}/N_{A}\,\mathrm{g}} = \frac{24.827\,\mathrm{MeV}}{4\cdot1.0078/6.022\times10^{23}\,\mathrm{g}}$$
$$= 3.71\times10^{24}\,\mathrm{MeV/g} \tag{5.85}$$

For the previously discussed network calculations we assumed $X_{\rm H}^0=0.60$ and, therefore, the total energy generated until hydrogen exhaustion is $0.60\cdot(3.71\times10^{24}\,{\rm MeV/g})=2.2\times10^{24}\,{\rm MeV/g}$ (or $3.5\times10^{18}\,{\rm erg/g}$). The same value

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Fig. 5.24 Time until hydrogen exhaustion ($X_{\rm H} < 0.001$) during the operation of the HCNO cycles. The solid lines correspond to different assumptions for the initial CNO abundances. For the initial hydrogen abundance and the density, values of $X_{\rm H}^0 = 0.60$ and $\rho = 200$ g/cm³ are used. The curves are obtained by performing a series of numerical reaction network calculations, with *T* and ρ are held constant in each calculation.

is directly obtained from Eq. (3.68). This result does not depend on the values assumed for the density ρ or the initial CNO mass fraction, as long as most of the hydrogen is converted to helium via the HCNO1 cycle. The latter quantities do, however, influence the time it takes until hydrogen is exhausted. This time is displayed in Fig. 5.24 as a function of temperature, with the density held constant at a value of $\rho = 200 \text{ g/cm}^3$. The curves correspond to different assumptions for the initial CNO abundances. In each case, the time until hydrogen exhaustion increases with decreasing temperature since the contribution of proton-induced reactions to the sum of mean lifetimes in the HCNO1 cycle becomes increasingly important, as explained in the previous examples. All curves are approximately constant at temperatures in excess of T = 0.25 GK where the β^+ -decays of ¹⁴O and ¹⁵O alone determine the time scale for the HCNO1 cycle. Consider now a fixed value of temperature, say, T = 0.3 GK. The longest time until hydrogen exhaustion (t = 3170 s) is obtained when only ¹⁶O is initially present ($X_{12C}^0 = 0.00$, $X_{16O}^0 = 0.20$). This is explained by the fact that it takes some additional time until ¹⁶O is consumed and its abundance can feed the HCNO1 cycle. Adding even a small amount of ¹²C $(X_{12C}^0 = 0.01, X_{16O}^0 = 0.20)$ decreases noticeably the time until hydrogen exhaustion (t = 2965 s). The rate of hydrogen consumption increases substantially (t= 2398 s) when only ¹²C instead of ¹⁶O is initially present ($X_{12C}^0 = 0.20$, $X_{16O}^0 =$ 0.00). Finally, hydrogen is consumed even faster (t = 1294 s) if equal amounts of ¹²C and ¹⁶O are initially present ($X_{12C}^0 = X_{16O}^0 = 0.20$).

Network calculations with temperature-density profiles

We will now discuss the more realistic situation of changing temperature and density during the nucleosynthesis. Classical novae (Section 1.4.4) represent an example for explosive hydrogen burning in the temperature region of T = 0.1–0.4 GK. Figure 5.25a shows a temperature and density profile which is adopted from hydrodynamic studies (José and Hernanz 1998) of a thermonuclear runaway caused by the accretion of solar-like matter onto the surface of a $1.0 M_{\odot}$ white dwarf of CO composition. The curves represent the temperature and density evolution of the hottest hydrogen-burning zone, that is, the region in which most of the nucleosynthesis takes place. This particular nova model achieves a maximum temperature of T = 0.17 GK after a time of $t \approx$ 360 s. At t = 1700 s, the temperature has fallen to a value of $T \approx 0.12$ GK. The density evolves from $\rho = 870 \text{ g/cm}^3$ before the outburst to a value of ρ = 21 g/cm³ at t = 1700 s. The reaction network of the HCNO cycles will be solved numerically by using this $T-\rho$ profile. For the initial composition, values of $X_{\rm H}^0 = 0.35$, $X_{4\rm He}^0 = 0.15$, $X_{12\rm C}^0 = X_{16\rm O}^0 = 0.25$ are assumed. These are similar to those used for the calculation of the $T-\rho$ profile displayed in Fig. 5.25a. The network calculation is terminated after t = 1700 s.

The abundance evolutions are displayed in Figs. 5.26a and b. The results are more complicated compared to the earlier calculations since the rapidly changing temperature and density keep the CNO abundances far from equilibrium. Nevertheless, the operation of the HCNO1 cycle is apparent. The isotope ¹²C is first transformed into ¹³N and further processed to ¹⁴O, ¹⁴N, and ¹⁵O. In contrast to the previous calculations for constant $T-\rho$ conditions, a ¹³C abundance builds up since for t > 1000 s the temperature and density evolve in a region in which the decay ${}^{13}N(\beta^+\nu){}^{13}C$ is more likely to occur than the ${}^{13}N(p,\gamma){}^{14}O$ reaction (Figs. 5.21 and 5.25a). For the peak temperature achieved in this nova model, the ${}^{16}O(p,\gamma){}^{17}F$ reaction is rather slow and, therefore, only a small fraction of ¹⁶O is converted, first to ¹⁷F and then to ¹⁷O. Note that for the adopted $T-\rho$ profile the HCNO3 cycle never operates since the decay ${}^{17}F(\beta^+\nu){}^{17}O$ is always faster than the ${}^{17}F(p,\gamma){}^{18}Ne$ reaction (Figs. 5.22b and 5.25a). At the end of the calculation, the hydrogen abundance has fallen to $X_{\rm H}$ = 0.24 and the most abundant CNO isotopes are ¹⁴N, ¹⁶O, ¹³N, ¹²C, ¹⁷O, and ¹³C with mass fractions of 0.21, 0.20, 0.046, 0.030, 0.026, and 0.017, respectively. The results agree qualitatively with observations of large nitrogen and oxygen abundances in the shells of several classical novae (Warner 1995).

It is important to realize that the final CNO abundances differ substantially from the steady state values achieved in hydrostatic hydrogen burning. If we assume that short-lived nuclides present at the end of the network calculation decay to their stable daughter nuclei (13 N to 13 C, 14 O to 14 N, and so on), then





Fig. 5.25 Temperature and density evolution of the hottest hydrogenburning zone during the thermonuclear runaway on the surface of a white dwarf with (a) $M = 1.0 M_{\odot}$ and CO composition, and (b) $M = 1.25 M_{\odot}$ and ONe composition. The curves are adopted from hydrodynamic simulations of classical nova explosions (José and Hernanz 1998).



Fig. 5.26 Explosive hydrogen burning during the thermonuclear runaway on the surface of a CO white dwarf. The results show the operation of the HCNO cycles and are obtained by performing a numerical reaction network calculation using the temperature and density evolution for the hottest zone

displayed in Fig. 5.25a. (a), (b) Abundance evolutions in the A < 20 mass region; (c) ratios of final mass fractions, after all β^+ -decays have been completed, and the corresponding solar system mass fractions; (d) time evolution of the energy generation rate.

we obtain, for example,

$$\binom{13C}{12C} = \frac{X_{13C} + X_{13N}}{X_{12C}} \frac{12}{13} = \frac{0.017 + 0.046}{0.030} \frac{12}{13} = 1.9$$
(5.86)

$$\left(\frac{{}^{15}N}{{}^{14}N}\right) = \frac{X_{15}{}_N + X_{15}{}_O}{X_{14}{}_N + X_{14}{}_O}\frac{14}{15} = \frac{4.6 \times 10^{-6} + 0.0019}{0.21 + 0.00076}\frac{14}{15} = 0.0085$$
(5.87)

compared to $({}^{13}C/{}^{12}C)_e \approx 0.25$ and $({}^{15}N/{}^{14}N)_e \approx (1-5) \times 10^{-5}$ for the equilibrium values in hydrostatic hydrogen burning at temperatures of T < 0.1 GK (Fig. 5.11a). It is also interesting to point out that certain nuclides are strongly overproduced compared to their solar values. Ratios of final mass fractions, after all β^+ -decays have been completed, and the corresponding solar mass fractions, (X/X_{\odot}) , are shown in Fig. 5.26c. The three most overproduced nuclides are ${}^{13}C$, ${}^{15}N$, and ${}^{17}O$, with overproduction factors in the range of $(X/X_{\odot}) \approx 500\text{-}6000$. It has been suggested (Kovetz and Prialnik 1997) that classical novae are the predominant source of the latter two nuclides in the Universe. Significant amounts of ${}^{18}F(X_{18F} = 1.4 \times 10^{-5})$ are produced as well. The decay of ${}^{18}F(T_{1/2} = 110 \text{ min})$ produces photons of 511 keV energy at a time when the expanding nova shell becomes transparent to γ -rays (Hernanz et al. 1999). This signature from nearby classical novae may be detectable in the future with detectors onboard satellites.

The time evolution of the energy generation rate is shown in Fig. 5.26d. It is characterized by a continuous increase, a maximum of $\varepsilon = 1.3 \times 10^{21} \text{ MeV g}^{-1} \text{ s}^{-1}$ close to the time of peak temperature ($t \approx 360 \text{ s}$), and afterward a steady decline as the temperature drops.

The experimental situation regarding reactions involving CNOF nuclei is summarized below. We already pointed out in Section 5.1.2 that the reactions $^{13}C(p,\gamma)^{14}N$, $^{14}N(p,\gamma)^{15}O$, $^{15}N(p,\gamma)^{16}O$, and $^{16}O(p,\gamma)^{17}F$ have been measured down to center-of-mass energies of 100, 93, 130, and 130 keV, respectively. Compared to the situation in hydrostatic hydrogen burning, the Gamow peaks in explosive hydrogen burning are obviously located at higher energies. For example, for the ¹⁴N + p reaction we obtain $E_0 \pm \Delta/2 = 149 \pm 59$ keV near $T \approx 0.2$ GK. Hence, for reactions involving stable CNO nuclei, data do generally exist in the Gamow peak and, as a consequence, the reaction rates at T = 0.1-0.4 GK have relatively small errors (typically < 30%; Angulo et al. 1999). The situation is different for reactions involving unstable nuclei. We have seen that the HCNO cycles are initiated by the reactions ${}^{13}N(p,\gamma){}^{14}O$, 18 F(p, α) 15 O, and 17 F(p, γ) 18 Ne. Our knowledge of the corresponding reaction rates has improved substantially as a result of experiments with radioactive ion beams (Section 4.6.1). The proton capture on ¹³N was the first astrophysically important reaction that was directly measured with a radioactive ion beam (Delbar et al. 1993). In that study, the strength of the broad E_r^{cm} = 528 keV resonance was obtained. However, the $^{13}N + p$ Gamow peak for temperatures of $T \leq 0.4$ GK is located far below this resonance. Hence, the S-factor has to be extrapolated to astrophysically important energies. Current reaction rate uncertainties in the range of T = 0.1-0.4 GK amount to a factor of \approx 2. For the ¹⁸F(p, α)¹⁵O reaction, at least some data exist in the Gamow peak at T = 0.3-0.4 GK since the strength of a low-lying resonance at $E_r^{\rm cm} = 330$ keV has been measured directly (Graulich et al. 1997, Bardayan et al. 2002). These studies represent the first direct measurements in the nova Gamow peak using radioactive ion beams. Nevertheless, the current reaction rate uncertainties are relatively large and amount to factors of 6-30 at temperatures of T = 0.1-0.4 GK because of additional contributions from unobserved resonances (Fig. 5.22a). The ${}^{17}F(p,\gamma){}^{18}Ne$ reaction, on the other hand, has not been directly measured yet. In this case, experiments have been performed (some involving radioactive beams) that measure nuclear quantities, like excitation energies, level widths, and J^{π} -values, from which the reaction rates are partially inferred. The current reaction rate uncertainties in the range of T = 0.1-0.4 GK amount to at least a factor of two.

5.2.2

Explosive Hydrogen Burning Beyond the CNO Mass Region

Hydrogen burning at elevated temperatures also involves nuclei in the $A \ge 20$ mass range. As was the case in hydrostatic hydrogen burning environments (Section 5.1.3), there is almost no leakage of material from the CNO region to the $A \ge 20$ mass range at temperatures of T = 0.1–0.4 GK. Reactions which may provide a link between both regions at $T \ge 0.4$ GK will be discussed in Section 5.4. Therefore, the nucleosynthesis must start from pre-existing seed nuclei with masses of $A \ge 20$. The character of the burning, however, changes drastically since proton-induced reactions on short-lived nuclei may successfully compete with their β^+ -decays. Reactions that may occur in the mass region $A \ge 20$ are shown in Fig. 5.27. Comparison to Fig. 5.15 clearly shows that far more proton-capture reactions and β^+ -decays have to be taken into account at elevated temperatures. The most likely nucleosynthesis paths will depend on the detailed temperature–density history of the explosion.

At temperatures of T = 0.1-0.4 GK, reaction cycles play a less prominent role in the $A \ge 20$ range compared to their outstanding importance in the CNO mass region. From Fig. 5.17 it is apparent that the branching ratios $B_{p\alpha/p\gamma}$ for ²⁷Al, ³¹P, and ³⁵Cl are less than unity and, hence, closed MgAl, SiP, and SCl cycles do not exist for this temperature range. For ²³Na, the branching ratio amounts to $B_{p\alpha/p\gamma} \approx 30$ at $T \approx 0.1$ GK, but it is only $B_{p\alpha/p\gamma} \approx 1$ in the range of $T \approx 0.2-0.4$ GK. Thus, a closed NeNa cycle may develop only at the lower temperature end. It will be shown in the following that reactions in the mass $A \ge 20$ range may also contribute substantially to the energy generation



Fig. 5.27 Nuclear interactions in the mass $A \ge 20$ region during explosive hydrogen burning. Stable nuclides are shown as shaded squares. The key relates an arrow to a specific interaction (proton capture, (p, α) reaction, or β^+ -decay). The nuclide ²⁶Al can be formed either in its ground state

or in its isomeric state ($E_x = 228$ keV). Many unstable nuclei may undergo proton-induced reactions at elevated temperatures. The most likely nucleosynthesis path depends on the detailed temperature–density history of the explosion.

rate. This energy is generated by building up heavier nuclei from lighter seed nuclei via proton-captures and β^+ -decays, rather than by the conversion of four protons to one ⁴He nucleus which takes place in the HCNO cycles.

The nucleosynthesis in the $A \ge 20$ mass range will be explored by considering again classical novae as an example (Section 1.4.4). Heavier white dwarfs are likely to consist of oxygen and $A \ge 20$ nuclei, while the carbon abundance is relatively small since it was consumed in the progenitor star during core carbon burning (Section 1.4.3 and Fig. 1.4). Thermonuclear runaways involving such white dwarfs achieve in general higher peak temperatures compared to those involving CO white dwarfs because the strength of the explosion scales with the surface gravity and the amount of accreted material. Obviously, the heavier mass nuclei, mainly Ne, Na, and Mg, will participate in hydrogen burning. Figure 5.25b shows temperature and density profiles that are adopted from hydrodynamic studies (José and Hernanz 1998) of a classical nova explosion caused by the accretion of solar-like matter onto the surface of a $1.25 M_{\odot}$ white dwarf of ONe composition. The curves represent again the temperature and density evolution in the hottest hydrogen-burning zone. This nova model achieves a maximum temperature of T = 0.25 GK after a time of $t \approx 420$ s. Temperature and density evolve from T = 0.10 GK and

 $\rho = 2800 \text{ g/cm}^3$ before the outburst to values of T = 0.12 GK and $\rho = 38 \text{ g/cm}^3$ at t = 1140 s. The reaction network is much more extensive compared to those described in earlier sections since many more reactions and β^+ -decays participate in the nucleosynthesis. The network is solved numerically for the temperature–density evolution shown in Fig. 5.25b and is terminated after t = 1140 s. For the initial composition, values of $X_{\text{H}}^0 = 0.35$, $X_{4\text{He}}^0 = 0.15$, $X_{16O}^0 = 0.26$, $X_{20\text{Ne}}^0 = 0.16$, $X_{23\text{Na}}^0 = 0.04$, $X_{24\text{Mg}}^0 = 0.03$, $X_{25\text{Mg}}^0 = 0.01$ are assumed. These are similar to those adopted in hydrodynamic studies (José and Hernanz 1998).

We will summarize only briefly the nucleosynthesis in the CNO region and then discuss in more detail the hydrogen burning in the $A \ge 20$ region. Since there are no ¹²C seed nuclei, hydrogen burning in the A < 20mass range has to start with ¹⁶O. The temperatures achieved in this nova model are sufficiently high for ¹⁶O to be destroyed by the ¹⁶O(p, γ)¹⁷F reaction. Further processing via ¹⁷F($\beta^+\nu$)¹⁷O(p, α)¹⁴N, ¹⁷F($\beta^+\nu$)¹⁷O(p, γ)¹⁸F(p, α)¹⁵O and ¹⁷F(p, γ)¹⁸Ne($\beta^+\nu$)¹⁸F(p, α)¹⁵O initiates quickly the HCNO1 cycle. Although there are quantitative differences in the evolution of A < 20 nuclei compared to the CO nova model, the overall final results are qualitatively similar. At the end of the calculation, the hydrogen abundance has fallen to $X_{\rm H} = 0.19$, and the most abundant CNO isotopes are ¹⁴N, ¹³N, ¹²C, ¹⁵O, and ¹⁷O with mass fractions of 0.081, 0.052, 0.041, 0.020, and 0.014, respectively. For the carbon and nitrogen isotopic ratios we obtain values of (¹³C/¹²C) = 1.3 and (¹⁵N/¹⁴N) = 0.22. The most overproduced isotopes are ¹³C, ¹⁵N, and ¹⁷O, with overproduction factors of (X/X_{\odot}) \approx 1600, 4500, and 3600, respectively.

The abundance evolutions in the $A \ge 20$ range are displayed in Figs. 5.28a and b. For the following discussion, it is useful to keep in mind that a peak temperature of $T \approx 0.25$ GK is maintained for about 50 s before the temperature starts to decrease again (Fig. 5.25b). During this time, the hydrogen abundance is about $X_{\rm H} \approx 0.30$, while the density amounts to $\rho \approx 300$ g/cm³. It can be seen in Fig. 5.28 that ²³Na seed nuclei are quickly destroyed. Protoninduced reactions start to deplete ²³Na noticeably at t = 200 s when the temperature amounts to $T \approx 0.1$ GK, that is, long before peak temperature is reached. At $T \approx 0.1$ GK, the ²³Na(p, α)²⁰Ne reaction dominates over the competing 23 Na(p, γ) 24 Mg reaction (Fig. 5.17). Therefore, the largest fraction of the ²³Na seed nuclei is converted to ²⁰Ne. Its abundance increases from an initial value of $X_{20_{Ne}}^0 = 0.16$ to 0.19, giving rise to the small bump seen at t = 400 s in Fig. 5.28a. Around peak temperature, a fraction of the ²⁰Ne abundance is destroyed by the ²⁰Ne(p,γ)²¹Na reaction and the flow reaches again ²³Na. At *T* = 0.25 GK, the branching ratio for ²³Na is about $B_{p\alpha/p\gamma} \approx 1$ (Fig. 5.17) and hence about one half of the ²³Na nuclei are transformed to ²⁴Mg. Note that very little material is processed via ${}^{23}Mg(p,\gamma){}^{24}Al(\beta^+\nu){}^{24}Mg$ since the ${}^{23}Mg\beta^+$ -decay is

far more likely to occur at $T \approx 0.25$ GK than the competing (p, γ) reaction $[\tau_{\beta}(^{23}Mg) = T_{1/2}/\ln 2 = 16 \text{ s versus } \tau_{p}(^{23}Mg) = [\rho(X_{H}/M_{H})N_{A}\langle\sigma v\rangle]^{-1} \approx$ 1370 s]. Once ²⁴Mg is reached, there is no process that can feed this material back to the NeNa mass region. At the end of the calculation most of the ²³Na seed nuclei have effectively been transformed to $A \ge 24$ nuclei, while the ²⁰Ne abundance has not changed from its initial value. This is a consequence of the fact that the ${}^{20}\text{Ne}(p,\gamma){}^{21}\text{Na}$ reaction is relatively slow $[\tau_p({}^{20}Ne) \approx 200 \text{ s at}$ T = 0.25 GK]. The other seed nuclei, ²⁴Mg and ²⁵Mg, are rapidly transformed via proton captures and β^+ -decays to heavier nuclides. We can estimate the nucleosynthesis path by considering the competition between β^+ -decays and proton captures of certain key nuclei. For the nuclei ²⁵Al and ²⁷Si, the mean lifetimes versus β^+ -decay amount to $\tau_\beta \approx 10$ and 6 s, respectively. The mean lifetimes versus proton capture at T = 0.25 GK are $\tau_p = 60$ and 24 s, respectively. Thus both ²⁵Al and ²⁷Si will preferentially β^+ -decay rather than undergo proton captures. Consequently, around peak temperature, the most likely nucleosynthesis path is

²⁴Mg(p,
$$\gamma$$
)²⁵Al($\beta^+ \nu$)²⁵Mg(p, γ)²⁶Al(p, γ)²⁷Si($\beta^+ \nu$)²⁷Al(p, γ)²⁸Si (5.88)

The sum of mean lifetimes for this sequence is

$$\tau_{\rm p}(^{24}Mg) + \tau_{\beta}(^{25}Al) + \tau_{\rm p}(^{25}Mg) + \tau_{\rm p}(^{26}Al^g) + \tau_{\beta}(^{27}Si) + \tau_{\rm p}(^{27}Al) = (0.014 + 10.4 + 0.39 + 0.43 + 6.0 + 1.2) \,\rm s = 18.4 \,\rm s$$
(5.89)

and is dominated by the β^+ -decays of ²⁵Al and ²⁷Si. The sequence does not delay the transformation from ²⁴Mg to ²⁸Si significantly since the sum of mean lifetimes is much smaller than the duration for which hydrogen burning takes place at peak temperature (50 s). The mean lifetime of ²⁸Si versus proton capture at T = 0.25 GK, however, amounts to $\tau_p(^{28}Si) = 69$ s and represents a significant delay. Nevertheless, there is a substantial abundance flow beyond ²⁸Si. The flow ends at ³²S since the proton capture on ³²S is a very slow process $[\tau_p(^{32}S) = 11100$ s at T = 0.25 GK]. At the end of the calculation, most of the ²³Na, ²⁴Mg, and ²⁵Mg seed nuclei have been converted to ²⁸Si, ³²S, ³⁰Si, and ³¹P with final mass fractions of 0.056, 0.024, 0.013, and 0.0084, respectively. The large ²⁰Ne, ²⁸Si, and ³²S final abundances are a consequence of the fact that the 20 Ne(p, γ) 21 Na, 28 Si(p, γ) 29 P, and 32 S(p, γ) 33 Cl reactions are the slowest proton captures involving stable target nuclei in the A = 20-32 mass range. These results agree qualitatively with observations of large neon, silicon, and sulfur abundances in the shells of several classical novae (Warner 1995). Ratios of final mass fractions, after all β^+ -decays have been completed, and the corresponding solar mass fractions are shown in Fig. 5.28c for A = 20-33 nuclei. The two most overproduced isotopes in this mass range are ³¹P and ³⁰Si, with overproduction factors of $(X/X_{\odot}) \approx 1000$ and 590, respectively.



Fig. 5.28 Explosive hydrogen burning during the thermonuclear runaway on the surface of a ONe white dwarf. The results show the operation of nuclear processes in the $A \ge 20$ region and are obtained from a numerical reaction network calculation using the temperature and density evolution for the

hottest zone displayed in Fig. 5.25b. (a), (b) abundance evolutions in the $A \ge 20$ mass region; (c) ratios of final mass fractions, after all β^+ -decays have been completed, and the corresponding solar system mass fractions; (d) time evolution of the energy generation rate.

The explosive burning of hydrogen also produces interesting amounts of the radioisotopes ²²Na ($T_{1/2} = 2.6$ y) and ²⁶Al^g ($T_{1/2} = 7.4 \times 10^5$ y), with mass fractions of $X_{22_{Na}} = 8 \times 10^{-5}$ and $X_{26_{Alg}} = 2 \times 10^{-4}$. The decay of ²²Na produces γ -rays with an energy of $E_{\gamma} = 1275$ keV and this signature from nearby classical novae may be observed in the future with detectors onboard satellites. Novae may also contribute to the abundance of Galactic ²⁶Al^g</sup>.

We will next discuss the energy generation. The total energy produced amounts to 5.4×10^{23} MeV/g. About 70% of the total energy is generated via reactions and decays involving CNOF nuclei while the $A \ge 20$ mass range contributes $\approx 30\%$. The evolution of the energy generation rate is shown in Fig. 5.28d. Its shape is more complex compared to the case of accretion onto a CO white dwarf (Fig. 5.26d). The heavy solid line represents the total energy generation rate, while the thinner solid and dotted lines correspond to the en-

ergy generated per time by processes in the CNOF and $A \ge 20$ mass regions, respectively. Although the CNOF mass range generates the largest fraction of the total energy, the $A \ge 20$ mass region gives rise to a larger energy generation rate before and near peak temperature. In fact, before peak temperature is achieved (t < 360 s), most of the energy is produced by the reactions 23 Na(p, α) 20 Ne, 23 Na(p, γ) 24 Mg, and 24 Mg(p, γ) 25 Al. These processes are rather fast and, in particular, their reaction rates at T = 0.1-0.4 GK are larger than the rate for ${}^{16}O(p,\gamma){}^{17}F$, as shown in Fig. 5.18. At later times, two maxima are visible near peak temperature (t = 360-430 s). The first one is narrow and high, indicating a rapid release of energy within a short period of time. It is caused by the sequence ${}^{25}Mg(p,\gamma){}^{26}Al^g(p,\gamma){}^{27}Si(\beta^+\nu){}^{27}Al(p,\gamma){}^{28}Si$ which consists of relatively fast processes. The second one, at later times, is broader and lower in magnitude. It is caused by reactions and decays in the NeNa mass region. These are significantly delayed by the slow ${}^{20}Ne(p,\gamma){}^{21}Na$ reaction. Around $t \approx 460$ s, similar amounts of energy are produced in the CNOF and $A \ge 20$ mass regions. At later times, reactions and decays in the CNOF mass range generate most of the energy. Note that the half-lives of ¹⁴O and ¹⁵O sensitively influence the evolution of the energy generation rate in the CNOF mass region. If both half-lives were shorter, the shape of the energy generation rate curve would become narrower and higher.

The nova model discussed above achieves a peak temperature of $T_{\text{peak}} = 0.25$ GK. In this case, the β^+ -decays of ²³Mg, ²⁵Al, and ²⁷Si are faster than the competing proton-capture reactions and, therefore, the nucleosynthesis path runs close to the line of stable nuclei. Some models of classical novae involve white dwarfs of higher masses and achieve larger peak temperatures. For example, at $T_{\text{peak}} = 0.35$ GK, $\rho = 300$ g/cm³, and $X_H = 0.3$, we obtain for the ²³Mg(p, γ)²⁴Al, ²⁵Al(p, γ)²⁶Si, and ²⁷Si(p, γ)²⁸P reactions mean lifetimes of $\tau_p(^{23}Mg) = 4.3$ s, $\tau_p(^{25}Al) = 0.35$ s, and $\tau_p(^{27}Si) = 0.44$ s. Consequently, the nucleosynthesis path will run closer to the proton dripline. Clearly, the exact path depends on the temperature history during the explosion.

We will now comment on the experimental situation. For a representative peak temperature of 0.25 GK the Gamow peaks for the ²⁰Ne + p and ³²S + p reactions are located at $E_0 \pm \Delta/2 = 220 \pm 80$ keV and 304 ± 94 keV, respectively. The ²⁰Ne(p, γ)²¹Na reaction has already been discussed in Section 5.1.3. It was measured down to an energy of about $E^{\rm cm} = 350$ keV and hence the data do not quite reach the Gamow peak. The reaction rates are determined by the tail of a subthreshold state and by direct capture. The reaction rate error at T = 0.25 GK amounts to about $\pm 50\%$ (Angulo et al. 1999). The lowest lying resonance in the ²⁸Si(p, γ)²⁹P reaction occurs at $E_r^{\rm cm} = 358$ keV. It is located in the Gamow peak and determines the reaction rates at nova temperatures. Reaction rate errors amount to about $\pm 15\%$ at T = 0.25 GK (Angulo et al. 1999). No resonances are expected to occur in the nova Gamow

peak of the ³²S(p, γ)³³Cl reaction and, therefore, this reaction is very slow. The three lowest lying resonances, located at $E_r^{cm} = 409$, 563, and 570 keV, dominate the reaction rates at nova temperatures. Reaction rate errors amount to ±15% at T = 0.25 GK (Iliadis et al. 2001). With one exception, none of the reactions involving unstable target nuclei in the A = 20–40 range have been measured directly. Their reaction rates are estimated indirectly from nuclear structure information. Hence, rate errors for reactions such as ²³Mg(p, γ)²⁴Al, ²⁵Al(p, γ)²⁶Si, and ²⁷Si(p, γ)²⁹P may amount to an order of magnitude or more. The exception is the ²¹Na(p, γ)²²Mg reaction which influences the production of ²²Na in novae. It is the first (and at present only) radiative capture reaction that has been measured directly in the nova Gamow peak by using radioactive ion beams (see Section 4.6.1). All the important resonances in the energy range $E_r^{cm} \ge 206$ keV have been observed and the reaction rate errors amount to about ±20% at T = 0.25 GK.

5.3

Hydrostatic Helium Burning

The second most abundant nuclide in the Universe is ⁴He. In Section 5.1 we discussed how ⁴He is synthesized during the hydrogen burning phase. When all the hydrogen is consumed in the core, the star will contract and the central temperature will increase. At some point, the helium in the core is ignited and undergoes nuclear transformations. The end products of these processes are ¹²C and ¹⁶O which represent the fourth and third most abundant nuclides, respectively, in the Universe. How exactly the transformation from ⁴He to ¹²C and ¹⁶O comes about has not been understood for some time. The fact that no stable nuclides with mass numbers of A = 5 and A = 8 exist represented a major hurdle in this regard (Section 1.1). For example, we have seen in Section 5.1.1 that the ³He(α , γ)⁷Li reaction may bridge the *A* = 5 instability in the pp2 and pp3 chains, giving rise to the synthesis of small amounts of ⁷Be, ⁷Li, ⁸B, and ⁸Be. But at typical hydrogen burning temperatures none of these nuclei survive since they are all transformed back into ⁴He (Fig. 5.2). Other ideas involved the formation of ¹²C as a result of the simultaneous fusion of three α -particles. However, it was shown that such a many-particle collision has a very small probability and cannot account for the fusion of ⁴He to ¹²C and ¹⁶O. The problem was solved by taking into account some curious nuclear properties, as will be seen in this section.



Fig. 5.29 Representation of helium-burning reactions in the chart of the nuclides. Stable nuclides are shown as shaded squares. The key relates an arrow to a specific interaction. The 3α reaction and the (α, γ) reactions on ¹²C and ¹⁶O are displayed as thick arrows. Other helium-burning reactions are shown as thinner arrows. The reaction ¹⁴N (α, γ) ¹⁸F is represented by an arc for reasons of clarity.

The following reactions take place during helium burning:

4 He($\alpha\alpha$, γ) 12 C	$(Q = 7274.7 \mathrm{keV})$	(5.90)
$^{12}\mathrm{C}(\alpha,\gamma)^{16}\mathrm{O}$	$(Q = 7161.9 \mathrm{keV})$	(5.91)
$^{16}\mathrm{O}(\alpha,\gamma)^{20}\mathrm{Ne}$	$(Q=4729.8{\rm keV})$	(5.92)
20 Ne $(\alpha, \gamma)^{24}$ Mg	$(Q = 9316.6 \mathrm{keV})$	(5.93)

These processes are shown schematically in Fig. 5.29 and will be discussed in more detail in the following. It is worth keeping in mind that, depending on the stellar mass and metallicity, the ranges of temperature and density during hydrostatic helium burning in massive stars amount to T = 0.1-0.4 GK and $\rho = 10^2-10^5$ g/cm³, respectively. The last reaction listed above only plays a role at the higher temperatures. Helium burning in massive stars is believed to be the main origin of ¹⁶O and ¹⁸O in the Universe, while helium burning in massive stars and AGB stars contributes similar amounts to the cosmic ¹²C abundance.

5.3.1 Helium-Burning Reactions

The triple- α reaction

Helium burning starts with the 3α reaction which we already encountered in the discussion of reaction rate equilibria (Example 3.4). The 3α reaction represents a (sequential) two-step process (Salpeter 1952). In the first step, two α -particles interact to form ⁸Be in its ground state. This nucleus is unstable by an energy of only 92 keV and disintegrates back into two α -particles with a half-life of $T_{1/2} = 6.7 \times 10^{-17}$ s (Audi et al. 2003). Over time, a small concentration of ⁸Be will build up until the rate of ⁸Be formation becomes equal to its decay rate,

$${}^{4}\text{He} + {}^{4}\text{He} \leftrightarrow {}^{8}\text{Be} \tag{5.94}$$

In the second step, a third α -particle interacts with the ⁸Be nucleus to form ¹²C via

$${}^{8}\text{Be}(\alpha,\gamma){}^{12}\text{C} \tag{5.95}$$

It was pointed out (Hoyle 1954) that the overall conversion of three α -particles to one ¹²C nucleus during helium burning would be too slow unless the second step proceeds via an s-wave resonance ($J^{\pi} = 0^+$) corresponding to a compound level near the α -particle threshold in ¹²C ($S_{\alpha} = 7367$ keV). This prediction and the subsequent experimental verification of this resonance (Dunbar et al. 1953, Cook et al. 1957) represents a remarkable interplay of astrophysics theory and nuclear experimental work. The 3α reaction bypasses the stable nuclides in the mass A = 6–11 region. Therefore, these nuclei are not synthesized in stars through thermonuclear reactions. Their extremely low observed abundances are the result of other processes, such as the Big Bang or cosmicray spallation.

The energy level diagram for this reaction sequence is shown in Fig. 5.30. The *Q*-value for the $\alpha + \alpha \rightarrow {}^{8}$ Be reaction amounts to -91.84 ± 0.04 keV and, therefore, 8 Be is unstable to α -particle emission. The *Q*-value of the 8 Be(α, γ)^{12}C reaction amounts to 7366.59 ± 0.04 keV. With a value of $E_x = 7654.20 \pm 0.15$ keV (Ajzenberg-Selove 1990) for the excitation energy of the astrophysically important 12 C level, we obtain a center-of-mass energy of $E_r = E_x - Q = 287.6 \pm 0.2$ keV for the corresponding resonance in 8 Be(α, γ)^{12}C. The resonance is formed by α -particle capture and decays to the 12 C ground state either by emission of γ -rays or by internal pair formation (Section 1.7.1 and Example B.4). The partial widths for these processes are given by $\Gamma_{\alpha} = 8.3 \pm 1.0$ eV and $\Gamma_{rad} = \Gamma_{\gamma} + \Gamma_{pair} = (3.7 \pm 0.5) \times 10^{-3}$ eV (Ajzenberg-Selove 1990). With $J({}^{12}C) = j_0(\alpha) = j_1({}^{8}Be) = 0$, we obtain for the resonance strength

(Section 3.2.4)

$$\omega \gamma_{^{8}\text{Be}(\alpha,\gamma)} \equiv \frac{(2J+1)}{(2j_{0}+1)(2j_{1}+1)} \frac{\Gamma_{\alpha}\Gamma_{\text{rad}}}{\Gamma} \approx \Gamma_{\text{rad}} = (3.7 \pm 0.5) \times 10^{-3} \,\text{eV} \ (5.96)$$

In order to derive the decay constant for the 3α reaction, we start by using the expression from Example 3.4,

$$\lambda_{\alpha+\alpha+\alpha\to^{12}C} = \lambda_{3\alpha} = 3N_{\alpha} \left(\frac{h^2}{2\pi}\right)^{3/2} \frac{1}{(m_{\alpha\alpha}kT)^{3/2}} e^{Q_{\alpha+\alpha\to^{8}Be}/kT} \lambda_{^{8}Be(\alpha,\gamma)}$$
(5.97)

The decay constant for the second step, $\lambda_{^{8}Be(\alpha,\gamma)}$, can be expressed by using Eq. (3.23),

$$\lambda_{^{8}\mathrm{Be}(\alpha,\gamma)} = \lambda_{\alpha}(^{^{8}}Be) = N_{\alpha}\langle\sigma v\rangle_{^{8}\mathrm{Be}(\alpha,\gamma)}$$
(5.98)

where $\langle \sigma v \rangle_{^{8}Be(\alpha,\gamma)}$ is given by the expression for the reaction rate of a narrow resonance (see Eq. (3.112))

$$\langle \sigma v \rangle_{^8\text{Be}(\alpha,\gamma)} = \left(\frac{2\pi}{m_{\alpha}{}^8\text{Be}kT}\right)^{3/2} \hbar^2 e^{-E_r/kT} \omega \gamma_{^8\text{Be}(\alpha,\gamma)}$$
(5.99)

From Eqs. (5.97)-(5.99) it follows

$$\lambda_{3\alpha} = 3N_{\alpha} \left(\frac{h^2}{2\pi}\right)^{3/2} \frac{e^{Q_{\alpha+\alpha-\beta_{Be}}/kT}}{(m_{\alpha\alpha}kT)^{3/2}} N_{\alpha} \left(\frac{2\pi}{m_{\alpha}^{8}Be}kT\right)^{3/2} \hbar^{2} e^{-E_{r}/kT} \omega \gamma_{^{8}Be(\alpha,\gamma)}$$
$$= 3N_{\alpha}^{2} 3^{3/2} \left(\frac{2\pi}{m_{\alpha}kT}\right)^{3} \hbar^{5} e^{-E'/kT} \omega \gamma_{^{8}Be(\alpha,\gamma)}$$
(5.100)

where we defined $E' \equiv E_r - Q_{\alpha + \alpha \rightarrow ^8Be}$. Numerically, one finds

$$\lambda_{3\alpha} = 0.23673 \frac{(\rho X_{\alpha})^2}{T_9^3} e^{-11.6048E'/T_9} \omega \gamma_{^8\text{Be}(\alpha,\gamma)} \quad (s^{-1})$$

= 8.7590 × 10⁻¹⁰ $\frac{(\rho X_{\alpha})^2}{T_9^3} e^{-4.4040/T_9} \quad (s^{-1})$ (5.101)

where we used E' = 287.6 keV - (-91.84 keV) = 379.4 keV and $\omega \gamma_{^8\text{Be}(\alpha,\gamma)} = \Gamma_{\text{rad}} = 3.7 \times 10^{-3} \text{ eV}$. Note that this expression is valid only for temperatures of $0.1 \le T_9 \le 2$. For lower and higher temperatures, additional contributions to the reaction rates have to be taken into account (Angulo et al. 1999).

The temperature dependence of the decay constant for the 3α reaction can be found from a calculation similar to the one described in Section 3.2.1. One obtains (see Problem 5.2)

$$(\lambda_{3\alpha})_T = (\lambda_{3\alpha})_{T_0} (T/T_0)^{(4.4040/T_9)-3}$$
(5.102)



Fig. 5.30 Energy level diagrams for the most important nuclides participating in helium burning. The numbers in square brackets represent the energy of the ground state of the system ${}_{Z}^{A}X_{N} + {}_{2}^{4}He_{2}$ with respect to the ground state of the nucleus ${}_{Z+2}^{A+4}Y_{N+2}$ (that is, the *Q*-value of the (α, γ) reaction on ${}_{Z}^{A}X_{N}$ or the α -particle separation energy of ${}_{Z+2}^{A+4}Y_{N+2}$). All information is adopted from Ajzenberg-Selove (1990) and Audi, Wapstra and Thibault (2003).

The energy generation rate of the 3α process is given, according to Eq. (3.63), by the product of the reaction rate (the number of reactions per second and per cubic centimeter) and the energy released per reaction, $Q_{3\alpha} = (3m_{4}_{He} - m_{12}_{C})c^2 = 7.275$ MeV. Each 3α reaction consumes three α -particles and, therefore, the decay constant (the number of α -particles disappearing each second) is related to the reaction rate by $r_{3\alpha} = N_{\alpha}\lambda_{3\alpha}/3$. It follows

$$\varepsilon_{3\alpha} = \frac{Q_{3\alpha}}{\rho} r_{3\alpha} = \frac{Q_{3\alpha}}{\rho} \frac{N_{\alpha} \lambda_{3\alpha}}{3}$$

= $\frac{7.275 \text{ MeV}}{\rho} \frac{1}{3} 8.7590 \times 10^{-10} \left(\rho N_A \frac{X_{\alpha}}{M_{\alpha}} \right) \frac{(\rho X_{\alpha})^2}{T_9^3} e^{-4.4040/T_9}$
= $3.1771 \times 10^{14} \frac{\rho^2 X_{\alpha}^3}{T_9^3} e^{-4.4040/T_9}$ (MeV g⁻¹s⁻¹) (5.103)

The 3α reaction has a remarkable temperature dependence. For example, near $T_0 = 0.1$ GK, we obtain for the energy generation rate

$$\varepsilon_{3\alpha}(T) = \varepsilon_{3\alpha}(T_0) \left(T/T_0\right)^{(4.4040/T_9)-3} = \varepsilon_{3\alpha}(T_0) \left(T/T_0\right)^{41.0}$$
(5.104)

Therefore, energy generation via the 3α reaction in a helium-burning star occurs predominantly in the regions of highest temperature. Furthermore, if the helium gas is electron degenerate, then a small rise in temperature will cause a large release of energy. As a result, the temperature rises faster, producing even more energy. The cycle continues until the degeneracy is lifted in a thermonuclear explosion. This so-called *helium flash* is believed to occur at the onset of hydrostatic helium burning in some stars (Section 1.4.3).

We will briefly comment on the experimental situation. The 3α reaction represents a two-step sequential process and has not been measured yet directly in the laboratory. Even the second step, the ${}^{8}\text{Be}(\alpha, \gamma){}^{12}\text{C}$ reaction, has not been measured directly since the ⁸Be half-life is very short ($T_{1/2} \approx 10^{-16}$ s). Furthermore, the reverse reaction, ${}^{12}C(\gamma,\alpha)^8Be$, cannot be measured either because the direct γ -ray transition from the ¹²C ground state ($J^{\pi} = 0^+$) to the level at $E_x = 7654$ keV ($J^{\pi} = 0^+$) is forbidden (Fig. 5.30 and Example B.4). However, the quantities E' and $\omega \gamma_{^8\text{Be}(\alpha,\gamma)} = \Gamma_{\text{rad}}$ entering the expression for the 3α reaction decay rate (see Eq. (5.100)) have been measured by indirect studies (see, for example, Rolfs and Rodney 1988). In the temperature range important for hydrostatic helium burning, $0.1 \le T_9 \le 0.4$, the total reaction rate of the 3α reaction (or, equivalently, the decay constant) has an error of only 15% (Angulo et al. 1999). This precision is remarkable for a process that cannot be measured directly in the laboratory. The error is mainly caused by the present uncertainty in the partial width $\Gamma_{rad} = \Gamma_{\gamma} + \Gamma_{pair} = (3.7 \pm 0.5) \times 10^{-3}$ eV. Although the quantity $E' = [E_x({}^{12}C) - Q_{^8Be(\alpha,\gamma)}] - Q_{\alpha+\alpha \to ^8Be} = 379.4 \pm 0.2 \text{ keV}$ enters exponentially in Eq. (5.100), its uncertainty has a negligible effect on the total decay constant.

The ${}^{12}C(\alpha,\gamma){}^{16}O$ and ${}^{16}O(\alpha,\gamma){}^{20}Ne$ reactions

If the subsequent ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction would be sufficiently rapid, then most α -particles would be converted to ${}^{16}O$ or perhaps heavier nuclei, with no ${}^{12}C$ left at the end of helium burning. However, the fact that the ratio of number abundances of ${}^{12}C$ and ${}^{16}O$ in the Universe amounts to about $N({}^{12}C)/N({}^{16}O) \approx 0.4$ suggests that the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction is rather slow and that, as a result, some ${}^{12}C$ and ${}^{16}O$ also implies that the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction gives rise to a sensitive balance of these two species, or equivalently, the precise magnitude of the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction rate will have a strong influence on the relative production of ${}^{12}C$ and ${}^{16}O$.

At a typical temperature of T = 0.2 GK, the location and width of the Gamow peak for the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction are $E_0 = 315$ keV and $\Delta = 170$ keV, respectively. The lowest lying resonance occurs at $E_r^{cm} \approx 2.4$ MeV and corresponds to the broad $E_x = 9585$ keV ($J^{\pi} = 1^-$) level in ${}^{16}O$ (Fig. 5.30). Although a lower lying level exists in ${}^{16}O$ at $E_x = 8872$ keV ($J^{\pi} = 2^-$), it cannot be excited as a resonance in the ${}^{12}C + \alpha$ reaction because it has unnatural parity (Example B.1). Thus, no narrow resonance is located in the Gamow peak and the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction must proceed through other reaction mechanisms that will be necessarily slower. These mechanisms include the capture into the high-energy wings of the subthreshold resonances (Example 2.1) at $E_r^{cm} = -45$ keV and $E_r^{cm} = -245$ keV, corresponding to ${}^{16}O$ levels at $E_x = 7117$ keV ($J^{\pi} = 1^-$) and 6917 keV ($J^{\pi} = 2^+$), respectively (Fig. 5.30). Another contribution arises from the wings of distant levels (also sometimes referred to as the direct-capture process).

The ${}^{12}C(\alpha, \gamma){}^{16}O$ reaction has been measured down to a center-of-mass energy of \approx 1 MeV. The Gamow peak for most situations of astrophysical interest is located far below this energy (for example, $E_0 \approx 0.3$ MeV for T =0.2 GK). It has been estimated (Kunz et al. 2002) that the cross section of the $^{12}C(\alpha,\gamma)^{16}O$ reaction at energies important for helium burning is of the order of $\sigma \approx 10^{-17}$ b, that is, orders of magnitude below present experimental observation thresholds. Therefore, the cross section measured at higher energies needs to be extrapolated down to the astrophysically important energy range by using a suitable nuclear reaction model (usually an R-matrix description; Section 2.5.5). This extrapolation is not straightforward because several different amplitudes contribute to the reaction mechanism, as pointed out already. Some of these amplitudes can interfere with each other, further complicating the picture. More reliable cross section extrapolations are obtained if the directly measured data are supplemented with other information for the ${}^{12}C +$ α system. This includes, for example, α -particle reduced widths (or α -particle spectroscopic factors; Section 1.6.2) of the important ¹⁶O levels that are populated in α -particle transfer studies, or phase shifts measured in ${}^{12}C(\alpha,\alpha){}^{12}C$ elastic scattering. An overview of some of the techniques can be found in Rolfs and Rodney (1988), and Wallerstein et al. (1997). At present, different rates of the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction are in use by stellar modelers. Current reaction rate uncertainties amount typically to about $\pm 35\%$ (Kunz et al. 2002) at temperatures of T = 0.12–0.35 GK. The magnitude of the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction rate determines the relative amounts of ¹²C and ¹⁶O at the end of helium burning, as will be shown below. Subsequent advanced burning stages rely on the ¹²C and ¹⁶O fuel. Consequently, the ¹²C(α , γ)¹⁶O reaction has a profound influence on the abundances of many elements up to iron and even on the evolution of massive stars that explode as supernovae (Weaver and Woosley 1993). Therefore, a more precise rate for this reaction is highly desirable.

If the subsequent ${}^{16}O(\alpha,\gamma){}^{20}Ne$ reaction would be fast, then ${}^{16}O$ is converted to ²⁰Ne or heavier nuclei and little ¹⁶O would survive during hydrostatic helium burning. Since ¹⁶O is relatively abundant in the Universe, however, we suspect that this reaction must be rather slow. For example, the Gamow peak for a temperature of *T* = 0.2 GK is located at $E_0 \pm \Delta/2 = 390 \pm 90$ keV. The *Q*-value of ${}^{16}O(\alpha, \gamma)^{20}$ Ne is *Q* = 4.73 MeV. The lowest lying resonance, located at E_r^{cm} = 893 keV (Tilley et al. 1998), is formed via incoming f-waves (ℓ = 3) and corresponds to the ²⁰Ne level at $E_x = 5621$ keV ($J^{\pi} = 3^{-}$). A lower lying compound level exists at E_x = 4967 keV ($J^{\pi} = 2^{-}$), but this state cannot be excited as a ${}^{16}O + \alpha$ resonance because it has unnatural parity (Example B.1). The subthreshold resonance located closest to the α -particle threshold occurs at $E_r^{\rm cm} = -482$ keV, and is formed via incoming g-waves ($\ell = 4$). These resonances are not only located far away from the Gamow peak for T = 0.2 GK, but their formation is also inhibited by the centripetal barrier. The cross section contributions arising from the wings of these resonances are so small that the direct-capture process, although inherently slow for (α, γ) reactions on even–even N = Z target nuclei, dominates the ¹⁶O(α, γ)²⁰Ne reaction rates for temperatures of T < 0.25 GK. At higher temperatures, resonances with $E_r^{\rm cm} \ge 893$ keV move into the Gamow peak and dominate the total reaction rates.

The lowest lying resonance in the ${}^{16}O(\alpha,\gamma){}^{20}Ne$ reaction has been observed at $E_r^{cm} = 893$ keV. As already mentioned above, the reaction rates at T < 0.25 GK are determined by direct capture. However, this process has not been measured yet in the ${}^{16}O(\alpha,\gamma){}^{20}Ne$ reaction, neither at energies below $E_{\alpha}^{cm} = 1$ MeV nor at higher energies. Therefore, the reaction rates at these temperatures are largely based on theoretical model calculations. At temperatures below T = 0.2 GK, the present ratio of upper and lower reaction rate limits amounts to about an order of magnitude (Angulo et al. 1999). Above this temperature, the reaction rate uncertainties are less than 30%.

Comparison of mean lifetimes

The mean lifetimes of ⁴He, ¹²C, and ¹⁶O versus destruction by α -particles as a function of temperature are displayed in Fig. 5.31 for values of $\rho X_{\alpha} = 500 \text{ g/cm}^3$ and $\rho X_{\alpha} = 10^4 \text{ g/cm}^3$. Although we have not discussed explicitly the next α -particle capture reaction, ²⁰Ne(α , γ)²⁴Mg, the corresponding mean lifetime τ_{α} (²⁰Ne) is also included in Fig. 5.31. The curves are obtained from

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Fig. 5.31 Mean lifetimes of ⁴He, ¹²C, ¹⁶O, and ²⁰Ne versus destruction by α -particles as a function of temperature for (a) $\rho X_{\alpha} = 500$ g/cm³, and (b) $\rho X_{\alpha} = 10^4$ g/cm³. The mean lifetime $\tau_{3\alpha}(^4\text{He})$ depends on $(\rho X_{\alpha})^{-2}$ while the mean lifetimes of ¹²C, ¹⁶O, and ²⁰Ne depend on $(\rho X_{\alpha})^{-1}$.

the expressions

$$\tau_{3\alpha}(^{4}He) = 1/\lambda_{3\alpha}(^{4}He) = \left[8.7590 \times 10^{-10} \frac{(\rho X_{\alpha})^{2}}{T_{9}^{3}} e^{-4.4040/T_{9}}\right]^{-1}$$
(5.105)

$$\tau_{\alpha}({}^{12}C) = \left[\frac{(\rho X_{\alpha})}{M_{\alpha}} N_A \langle \sigma v \rangle_{{}^{12}C(\alpha,\gamma)}\right]^{-1}$$
(5.106)

$$\tau_{\alpha}({}^{16}O) = \left[\frac{(\rho X_{\alpha})}{M_{\alpha}} N_A \langle \sigma v \rangle_{{}^{16}O(\alpha,\gamma)}\right]^{-1}$$
(5.107)

$$\tau_{\alpha}({}^{20}Ne) = \left[\frac{(\rho X_{\alpha})}{M_{\alpha}} N_A \langle \sigma v \rangle_{{}^{20}Ne(\alpha,\gamma)}\right]^{-1}$$
(5.108)

Note that the mean lifetime $\tau_{3\alpha}(^{4}\text{He})$ depends on $(\rho X_{\alpha})^{-2}$, while the mean lifetimes of ^{12}C , ^{16}O , and ^{20}Ne depend on $(\rho X_{\alpha})^{-1}$. Hence, carbon production via the 3α reaction is favored by higher density or, equivalently, by lower stellar mass, as will be shown later. It can be seen in Fig. 5.31 that the mean lifetime $\tau_{3\alpha}(^{4}\text{He})$ is the shortest for a wide range of temperatures. Only at very low temperatures (T < 0.14 GK for $\rho X_{\alpha} = 500 \text{ g/cm}^{3}$, or T < 0.12 GK for $\rho X_{\alpha} = 10^{4} \text{ g/cm}^{3}$) is the mean lifetime $\tau_{\alpha}(^{12}\text{C})$ shorter than $\tau_{\alpha}(^{4}\text{He})$. It is also apparent that $\tau_{3\alpha}(^{4}\text{He})$ and $\tau_{\alpha}(^{12}\text{C})$ are much smaller than $\tau_{\alpha}(^{16}\text{O})$ and $\tau_{\alpha}(^{20}\text{Ne})$ at temperatures of T < 0.3 GK. Consequently, the destruction of ^{16}O is very slow and most ^{16}O nuclei will survive under these conditions. At relatively high temperatures of T > 0.3 GK all four mean lifetimes become comparable in magnitude.

5.3.2

Nucleosynthesis During Hydrostatic He Burning

In this section, we discuss the evolution of abundances at hydrostatic helium burning conditions. Early in the burning, helium will be consumed by the 3α process. As the helium abundance decreases, and since the mean lifetime $\tau_{3\alpha}(^{4}\text{He})$ depends on $(\rho X_{\alpha})^{-2}$, α -particle captures on ^{12}C and beyond will become increasingly important. The differential equations for the abundances are

$$\frac{d({}^{4}He)}{dt} = -3r_{3\alpha} - ({}^{4}He)({}^{12}C)\langle\sigma v\rangle_{{}^{12}C(\alpha,\gamma)} - ({}^{4}He)({}^{16}O)\langle\sigma v\rangle_{{}^{16}O(\alpha,\gamma)}$$
(5.109)

$$\frac{d({}^{12}C)}{dt} = r_{3\alpha} - ({}^{4}He)({}^{12}C)\langle\sigma v\rangle_{{}^{12}C(\alpha,\gamma)}$$
(5.110)

$$\frac{d({}^{16}O)}{dt} = ({}^{4}He)({}^{12}C)\langle\sigma v\rangle_{12}C(\alpha,\gamma) - ({}^{4}He)({}^{16}O)\langle\sigma v\rangle_{16}O(\alpha,\gamma)$$
(5.111)

$$\frac{d({}^{20}Ne)}{dt} = ({}^{4}He)({}^{16}O)\langle\sigma v\rangle_{{}^{16}O(\alpha,\gamma)} - ({}^{4}He)({}^{20}Ne)\langle\sigma v\rangle_{{}^{20}Ne(\alpha,\gamma)}$$
(5.112)

$$\frac{d(^{24}Mg)}{dt} = (^4He)(^{20}Ne)\langle\sigma v\rangle_{^{20}Ne(\alpha,\gamma)}$$
(5.113)

The factors of 3 and 1 in the first term on the right-hand sides of the first two equations are explained by the fact that each 3α reaction consumes three ⁴He nuclei and creates one ¹²C nucleus. The reaction rate $r_{3\alpha}$ (in units of reactions per second and per cubic centimeter) is related to the decay constant and the mean lifetime by $3r_{3\alpha} = ({}^{4}He)\lambda_{3\alpha} = ({}^{4}He)/\tau_{3\alpha}$. We included here the ${}^{20}\text{Ne}(\alpha,\gamma)^{24}\text{Mg}$ reaction to account for the destruction of ${}^{20}\text{Ne}$. As will be seen below, this reaction plays a minor role in most hydrostatic helium burning environments.

The above network is numerically solved for constant temperatures and densities of: (i) T = 0.15 GK, $\rho = 5000 \text{ g/cm}^3$, and (ii) T = 0.2 GK, $\rho = 800 \text{ g/cm}^3$. Such conditions occur typically in core helium burning of stars with initial masses of $5 M_{\odot}$ and $20 M_{\odot}$, respectively, and are fairly independent of the initial metallicity of the star (Schaller et al. 1992). It is important to point out that our calculations do not represent the situation in real stars. As the helium fuel is consumed, the energy production rate would also decrease with time if the burning would take place under constant temperature and density conditions. In order to maintain a certain luminosity, the stellar core contracts gravitationally and, consequently, the temperature and density must increase from the start to the end of helium burning in a realistic stellar model. Nevertheless, a reasonable qualitative estimate of helium burning nucleosynthesis can be obtained by reducing a complex situation to its simplest form by assuming constant temperatures and densities. Furthermore, we will assume

a pure ⁴He gas ($X_{4He}^0 = 1$) at the beginning of helium burning. The reaction network is solved until helium exhaustion ($X_{4He} < 0.001$).

The abundance evolutions of ${}^{12}C$ and ${}^{16}O$ at T = 0.15 GK and $\rho = 5000$ g/cm³ versus the amount of helium consumed, $\Delta X_{4He} = X_{4He}^0 - X_{4He}(t)$, are shown in Fig. 5.32a as solid lines (time is increasing from left to right). Initially, as ⁴He is depleted by the 3α reaction, the ¹²C abundance increases linearly. Eventually, the ¹²C abundance reaches a maximum and then declines because of the increasing importance of α -particle captures on ¹²C. At the same time, the ¹⁶O abundance increases toward the end of the calculation. The end products are ¹²C and ¹⁶O, with a number abundance ratio of $({}^{12}C/{}^{16}O) =$ $(X_{12_C}/X_{16_O})(M_{16_O}/M_{12_C}) \approx 0.89$. The final mass fractions of heavier nuclei, such as ²⁰Ne and ²⁴Mg, amount to $\approx 10^{-6}$ and $\approx 10^{-14}$, respectively, emphasizing the very slow destruction of ¹⁶O via α -particle capture. The total nuclear energy generated amounts to 4.8×10^{23} MeV/g (or 7.6×10^{17} erg/g). The relative contributions of the 3α reaction and the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction to the total energy production are 66% and 34%, respectively. It is interesting to consider how the abundances change when current reaction rate errors of the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction ($\pm 35\%$; Kunz et al. 2002) are taken into account. The results are indicated by the dotted and dashed lines, obtained by using the lower and upper limit for the ${}^{12}C(\alpha,\gamma){}^{16}O$ reaction rate, respectively. At helium exhaustion, the number abundance ratio $({}^{12}C/{}^{16}O)$ varies significantly, between 0.65 and 1.23. The current errors in the rates of the 3α reaction ($\pm 15\%$; Angulo et al. 1999) have a much smaller influence on the ${}^{12}C/{}^{16}O$ ratio.

The abundance evolutions of ¹²C and ¹⁶O at T = 0.2 GK and $\rho = 800$ g/cm³ versus the amount of helium consumed, $\Delta X_{^4\text{He}}$, are shown in Fig. 5.32b as solid lines. Again, the ¹²C abundance increases, reaches a maximum, and then starts to decline, while at the same time the ¹⁶O abundance grows steadily. However, at this higher burning temperature and lower density, more ¹⁶O and less ¹²C is produced compared to the previous case. By the end of the calculation, we obtain a value of $({}^{12}C/{}^{16}O) = 0.57$. The 3 α reaction and the ¹²C(α,γ)¹⁶O reaction provide again all the nuclear energy, with relative contributions of 62% and 38%, respectively. The influence of the ¹⁶O(α,γ)²⁰Ne and ²⁰Ne(α,γ)²⁴Mg reactions is small. The final mass fractions of ²⁰Ne and ²⁴Mg amount only to $\approx 10^{-5}$ and $\approx 10^{-11}$, respectively. The dotted and dashed lines display the abundance evolutions when the ¹²C(α,γ)¹⁶O reaction rate errors are taken into account. At helium exhaustion, the abundance ratio (¹²C/¹⁶O) varies between 0.39 and 0.85. Much smaller variations are caused by current errors in the 3 α reaction rate.



Fig. 5.32 Evolution of ¹²C and ¹⁶O versus the amount of helium consumed during hydrostatic helium burning for constant temperatures and densities of (a) T = 0.15 GK and $\rho = 5000$ g/cm³, and (b) T = 0.2 GK and $\rho = 800$ g/cm³. The results are obtained by solving the reaction network numerically, assuming a pure ⁴He gas at the beginning

of helium burning. The calculation is terminated when the helium mass fraction falls below $X_{4\text{He}} = 0.001$. The solid lines are obtained by adopting recommended ${}^{12}\text{C}(\alpha,\gamma){}^{16}\text{O}$ reaction rates, while the dotted and dashed lines result from using the lower and upper limit of the ${}^{12}\text{C}(\alpha,\gamma){}^{16}\text{O}$ reaction rates, respectively.

5.3.3

Other Helium-Burning Reactions

The previous sections showed that the end products of helium burning are mainly ¹²C and ¹⁶O. The precise abundance ratio depends on the heliumburning conditions (for example, temperature and density) which, in turn, are determined by the stellar mass. The more massive the star, the more ¹⁶O is produced relative to ¹²C. The precise abundance ratio (¹²C/¹⁶O) is influenced by the rate of the ¹²C(α , γ)¹⁶O reaction. We assumed so far that helium burning starts exclusively with ⁴He as fuel in the stellar core. However, at the end of hydrogen burning most stars contain a small, but significant, fraction of ¹⁴N as a result of CNO-cycle operation (Section 5.1.2). During helium burning, ¹⁴N will be consumed via the reaction sequence (Cameron 1960)

¹⁴N(
$$\alpha, \gamma$$
)¹⁸F($\beta^+ \nu$)¹⁸O(α, γ)²²Ne (5.114)

as shown in Fig. 5.29. Subsequently, some of the ²²Ne nuclei will be converted by the competing reactions ²²Ne(α, γ)²⁶Mg (Q = 10.62 MeV) and ²²Ne(α, n)²⁵Mg (Q = -0.48 MeV). The latter reaction has a negative *Q*-value and is rather slow in the lower temperature region of $T \approx 0.1$ –0.2 GK. However, toward the end of helium burning, when the temperature exceeds $T \approx 0.25$ GK, the ²²Ne(α, n)²⁵Mg reaction provides an important source of neutrons. These neutrons undergo reactions and influence sensitively the synthesis of neutron-rich nuclei in the mass A = 60–90 range. The resulting

network of neutron-induced reactions and β -decays will be discussed in Section 5.6. The above reaction sequence is also important because it significantly increases the neutron excess parameter η during core helium burning (Section 1.8). Helium burning in massive stars and in AGB stars is the main source of ²²Ne production in the Universe. It is also an important contributor to the cosmic production of ²⁵Mg and ²⁶Mg.

During shell helium burning in massive stars, the sequence of Eq. (5.114) does not go to completion. The surviving ¹⁸O is the main source of ¹⁸O in the Universe. Helium burning also contributes to the synthesis of fluorine via the sequence (Fig. 5.29)

$${}^{18}\mathrm{O}(\mathrm{p},\alpha){}^{15}\mathrm{N}(\alpha,\gamma){}^{19}\mathrm{F}$$
(5.115)

with the protons supplied by the ${}^{14}N(n,p){}^{14}C$ reaction (Meynet and Arnould 2000).

5.4 Explosive Hydrogen-Helium Burning

It was argued in Section 3.2.1 that if a mixture of different nuclei is present in the stellar plasma, then usually those reactions involving the nuclear fuel with the smallest Coulomb barrier will account for most of the nuclear energy generation and nucleosynthesis. Therefore, we considered in previous sections burning stages that were characterized by the consumption of one particular type of fuel (either hydrogen or helium). Interesting situations arise, however, if the stellar temperature is sufficiently large for two different types of fuels, for example, hydrogen and helium, to burn at the same location. In this section we will discuss the burning of a mixture of hydrogen and helium fuel at temperatures in excess of those considered so far (T > 0.4 GK). At such high temperatures several effects, that played no important role in previous sections, are shown to have a sensitive influence on the nucleosynthesis. Foremost among these are photodisintegration reactions and the precise location of the proton dripline, the line that separates proton-bound ($S_p \ge 0$) from proton-unbound ($S_p < 0$) nuclei. It will become apparent that the presence of two different types of fuels, together with photodisintegration reactions and the location of the proton dripline, will change the character of the nucleosynthesis entirely.

The nucleosynthesis in hydrogen–helium burning at elevated temperatures involves many nuclear processes and is very complex. First, we will discuss how, with increasing temperature, certain reaction sequences (*breakout sequences*) convert nuclei from the HCNO cycle region to the A = 20-21 mass range. Second, by performing reaction network calculations for a number of different constant temperatures, we will investigate the location of nucleosyn-

thesis paths between the group of stable nuclei and the proton dripline. Of interest are the nucleosynthesis time scales, the heaviest nuclei synthesized *(endpoints of nucleosynthesis)* and the dependence of the nuclear energy generation rate on the assumed initial composition. Finally, as an example for a more realistic situation, we will discuss the results of a postprocessing network calculation that is representative for the nucleosynthesis occurring during a type I X-ray burst (Section 1.4.4).

5.4.1

Breakout from the HCNO Cycles

For stellar temperatures of $T \leq 0.4$ GK, very little material is lost from the operation of either the cold or the hot CNO cycles (Sections 5.1.2 and 5.2.1, respectively). This is explained by the fact that the heaviest nuclei synthesized in the CNO and the HCNO cycles are ¹⁹F and ¹⁸F, respectively. As can be seen from Figs. 5.9d and 5.22a, the branching ratios $B_{p\alpha/p\gamma}$ of these two isotopes amount to factors of 10^3 – 10^4 in the temperature ranges of the CNO and HCNO cycles (T < 0.1 GK and T = 0.1–0.4 GK, respectively). Thus, both ¹⁹F and ¹⁸F are predominantly converted to lighter nuclei via (p, α) reactions. Since α -particle-induced reactions are unlikely to occur at temperatures of $T \leq 0.4$ GK in the presence of hydrogen fuel, the above reaction cycles are essentially closed.

The situation changes substantially at higher temperatures. At T > 0.5 GK, a number of reaction sequences involving α -particle-induced reactions convert ¹⁴O or ¹⁵O to nuclei in the mass range A = 20–21. These nuclei are permanently lost as catalysts for the HCNO cycles since there are no nuclear processes that can transform them back to the HCNO mass range. The three main breakout sequences are listed below and are displayed in Fig. 5.33.

Sequence 1	Sequence 2	Sequence 3
¹⁵ Ο(α,γ) ¹⁹ Ne	¹⁴ O(α,p) ¹⁷ F	¹⁴ O(α,p) ¹⁷ F
¹⁹ Ne(p,γ) ²⁰ Na	$^{17}F(p,\gamma)^{18}Ne$	$^{17}F(\gamma,p)^{16}O$
	¹⁸ Ne(α,p) ²¹ Na	¹⁶ O(α,γ) ²⁰ Ne

In order to obtain a first impression, consider Fig. 5.34a showing the rates for these reactions, normalized to the ${}^{16}O(p,\gamma){}^{17}F$ reaction rates. It is immediately apparent that the ${}^{19}Ne(p,\gamma){}^{20}Na$ reaction rates exceed those of the preceding ${}^{15}O(\alpha,\gamma){}^{19}Ne$ reaction by orders of magnitude. Hence, we suspect that the time scale for breakout sequence 1 is determined by the slower ${}^{15}O(\alpha,\gamma){}^{19}Ne$ reaction. Furthermore, both the ${}^{18}Ne(\alpha,p){}^{21}Na$ and ${}^{16}O(\alpha,\gamma){}^{20}Ne$ reaction rates are smaller than those of the preceding ${}^{14}O(\alpha,p){}^{17}F$ reaction. Thus, the former reactions will be more important compared to the latter reaction for setting the time scale of breakout sequence 2 or 3. Fig-



Fig. 5.33 Representation of the three breakout sequences (BOS) from the A< 20 mass region (thick arrows) during hydrogen-helium burning. Nuclear interactions that are part of the HCNO cycles are displayed as thin arrows. Stable nuclides

are shown as shaded squares. Once a nucleus has been transformed to a species beyond the dotted line (A = 20) it is permanently lost for the HCNO cycles since there are no processes that can transform the species back to the A < 20 region.

ure 5.34a also displays reaction rates of some alternative breakout sequences. It can be seen that the ¹⁴O(α, γ)¹⁸Ne reaction rate is negligible compared to the rate of the competing ¹⁴O(α, p)¹⁷F reaction. Similarly, the rates of the ¹⁷F(α, p)²⁰Ne reaction are far smaller compared to those of the competing ¹⁷F(p, γ)¹⁸Ne reaction, except at high temperatures (T > 1.2 GK) where both rates become comparable. However, even at T > 1.2 GK, the ¹⁷F(α, p)²⁰Ne reaction will play a role only in the extreme case when the helium mass fraction substantially exceeds the hydrogen mass fraction ($X_{4}_{He}/M_{4}_{He} > X_{H}/M_{H}$). For the considerations of the present section, none of these alternative breakout sequences are of importance.

Next, consider Fig. 5.34b showing the temperature–density conditions for the competition between β^+ -decay and nuclear reaction for those nuclides



Fig. 5.34 (a) Rates that are part of the three breakout sequences versus temperature. For a better comparison, the values $N_A \langle \sigma v \rangle$ are normalized to the rate of the ¹⁶O(p, γ)¹⁷F reaction. Rates for the ¹⁴O(α , γ)¹⁸Ne and ¹⁷F(α ,p)²⁰Ne reactions are also shown for comparison, but are negligible in the present context. (b) Temperature–density conditions for the competition between β^+ -decay and nuclear reaction for unstable nuclides that participate in the three breakout sequences (solid and dashed lines). The β^+ -decay domi-

nates in the region to the left of a solid or dashed line, while on the right-hand side the competing proton- or α -particle-induced reaction is more likely to occur. The dotted line shows the conditions at which the decay constants for the competing ¹⁷F(p, γ)¹⁸Ne and ¹⁷F(γ ,p)¹⁶O reactions are of equal magnitude. The ¹⁷F(p, γ)¹⁸Ne reaction dominates on the left-hand side of the dotted curve, while on the right-hand side, the ¹⁷F(γ ,p)¹⁶O reaction is more likely to occur. All curves are calculated assuming $X_{\rm H} = 0.7$ and $X_{^4{\rm He}} = 0.3$.

that participate in the three breakout sequences. The solid and dashed curves represent $T-\rho$ conditions for which the decay constants (or mean lifetimes) of the two competing processes are of equal magnitude, $\lambda_1(0) = \lambda_\beta(0)$. They are calculated from the expression (see Eq. (5.73))

$$\rho = \frac{\ln 2}{T_{1/2}(0)(X_1/M_1)N_A \langle \sigma v \rangle_{01}}$$
(5.116)

with 0 the nuclide of interest and 1 denoting either hydrogen or helium, depending on the type of reaction. The relevant β^+ -decay half-lives are given

by $T_{1/2}(^{14}\text{O}) = 70.61 \text{ s}$, $T_{1/2}(^{15}\text{O}) = 122.24 \text{ s}$, $T_{1/2}(^{17}\text{F}) = 64.49 \text{ s}$, $T_{1/2}(^{18}\text{Ne}) = 1.67 \text{ s}$, and $T_{1/2}(^{19}\text{Ne}) = 17.22 \text{ s}$ (Audi et al. 2003). The β^+ -decay dominates in the region to the left of a solid or dashed line, while on the right-hand side the competing proton- or α -particle-induced reaction is more likely to occur. Furthermore, the dotted line in Fig. 5.34b shows the $T-\rho$ conditions at which the decay constants for the competing ${}^{17}\text{F}(\rho,\gamma){}^{18}\text{Ne}$ and ${}^{17}\text{F}(\gamma,p){}^{16}\text{O}$ reactions are of equal magnitude, $\lambda_p({}^{17}\text{F}) = \lambda_\gamma({}^{17}\text{F})$. The dotted line is obtained from the expression (see Eqs. (3.23) and (3.45))

$$\rho = 9.8685 \times 10^9 T_9^{3/2} \frac{(2j_{16O} + 1)(2j_p + 1)}{(2j_{17F} + 1)} \left(\frac{G_{16O}^{\text{norm}} G_{17F}^{\text{norm}}}{G_{17F}^{\text{norm}}}\right) \left(\frac{M_{16O} M_{\text{H}}}{M_{17F}}\right)^{3/2} \times \left(\frac{X_{\text{H}}}{M_{\text{H}}}\right)^{-1} e^{-11.605Q_{16O(p,\gamma)}/T_9} \frac{N_A \langle \sigma v \rangle_{^{16}O(p,\gamma)}}{N_A \langle \sigma v \rangle_{^{17}F(p,\gamma)}}$$
(5.117)

with $Q_{16O(p,\gamma)} = 0.600$ MeV. The ${}^{17}F(p,\gamma){}^{18}Ne$ reaction dominates on the lefthand side of the dotted curve, while on the right-hand side the competing ${}^{17}F(\gamma,p){}^{16}O$ reaction is faster. All the curves in Fig. 5.34b are calculated by using the values of $X_{\rm H} = 0.7$ and $X_{^4{\rm He}} = 0.3$. For the following considerations, it is assumed that ${}^{14}O$ and ${}^{15}O$ are the most abundant CNO nuclei as a result of prior HCNO cycle operation (Section 5.2.1). A density of $\rho = 10^4$ g/cm³ is chosen as a representative value and we are particularly interested in the fate of ${}^{14}O$ and ${}^{15}O$ when the temperature is slowly increased.

First, consider only breakout sequence 1 (the two solid curves in Fig. 5.34b). We start at $T \approx 0.1$ GK, where the HCNO cycles operate (Fig. 5.21), and slowly increase the temperature. The line corresponding to the condition $\tau_p(^{19}\text{Ne}) = \tau_{\beta}(^{19}\text{Ne})$ is crossed at T = 0.23 GK. Beyond this temperature, the $^{19}\text{Ne}(p,\gamma)^{20}$ Na reaction becomes more likely than the competing $^{19}\text{Ne} \beta^+$ -decay. However, no breakout from the hot CNO cycles occurs yet because the $^{15}\text{O} \beta^+$ -decay still dominates over the competing $^{15}\text{O}(\alpha,\gamma)^{19}$ Ne reaction [$\tau_{\alpha}(^{15}\text{O}) > \tau_{\beta}(^{15}\text{O})$]. Further increasing the temperature, the line corresponding to the condition $\tau_{\alpha}(^{15}\text{O}) = \tau_{\beta}(^{15}\text{O})$ is crossed at $T \approx 0.46$ GK. Beyond this temperature, ^{15}O nuclei are lost from the HCNO cycles as a result of the operation of the breakout sequence 1.

Next, the breakout sequences 2 and 3 (dashed and dotted lines in Fig. 5.34b) will be discussed. For T > 0.18 GK, the nuclide ¹⁷F is preferentially destroyed by the (p, γ) reaction rather than by β^+ -decay (that is, the HCNO3 cycle starts to operate beyond $T \approx 0.18$ GK; see Fig. 5.22b). Increasing the temperature, we cross the line corresponding to the condition $\tau_{\alpha}(^{14}O) = \tau_{\beta}(^{14}O)$ at $T \approx 0.43$ GK. Beyond this point the ¹⁴O(α ,p)¹⁷F reaction dominates over the competing ¹⁴O β^+ -decay. Although the breakout sequences 2 and 3 do not operate yet, the ¹⁴O(α ,p)¹⁷F reaction is already important at this stage because it provides an additional link between the HCNO1

and HCNO3 cycles (Fig. 5.20). In other words, ¹⁴O is converted to ¹⁵O via the sequence ${}^{14}O(\alpha,p){}^{17}F(p,\gamma){}^{18}Ne(\beta^+\nu){}^{18}F(p,\alpha){}^{15}O$. Consequently, at T >0.46 GK the ${}^{14}O(\alpha,p){}^{17}F$ reaction increases the fraction of CNO nuclei that is lost through breakout sequence 1. Further increasing the temperature, we next cross the dotted line, which is defined by the condition $\tau_p({}^{17}F) = \tau_{\gamma}({}^{17}F)$, at *T* \approx 0.5 GK. For *T* > 0.5 GK, ¹⁷F is preferentially destroyed by the (γ ,p) reaction and one is tempted to assume that the breakout sequence 3 starts to operate at this point. This is not the case, however, since the subsequent breakout reaction ${}^{16}O(\alpha,\gamma){}^{20}Ne$ is much slower than the competing ${}^{16}O(p,\gamma){}^{17}F$ reaction (Fig. 5.34a). The low Q-value of the ${}^{16}O(p,\gamma){}^{17}F$ reaction (Q = 0.600 MeV) ensures that an equilibrium is quickly established between the abundances of ¹⁷F and ¹⁶O. The breakout sequences 2 and 3 must then proceed from these equilibrium abundances. For example, the rate at which ¹⁴O nuclei are lost from the HCNO cycles through breakout sequence 3 depends on the equilibrium number of 16 O nuclei, given by the temperature (see Eq. (3.49)), and the magnitude of the rate for the subsequent ${}^{16}O(\alpha,\gamma){}^{20}Ne$ reaction. Therefore, breakout sequence 3 will become increasingly important for higher temperatures. It will be shown in the next subsection that the breakout sequence 3 operates at T > 1.0 GK (for $\rho = 10^4$ g/cm³).

Finally, the line corresponding to the condition $\tau_{\alpha}(^{18}\text{Ne}) = \tau_{\beta}(^{18}\text{Ne})$ is crossed at $T \approx 0.81$ GK. Beyond this point the $^{18}\text{Ne}(\alpha,p)^{21}\text{Na}$ reaction dominates over the competing $^{18}\text{Ne} \beta^+$ -decay. Consequently, ^{14}O nuclei are lost from the HCNO cycles as a result of the operation of the breakout sequence 2. Qualitatively similar results are obtained for other values of the density, although the various lines are crossed at different temperature values.

5.4.2

Network Calculations at Constant Temperature and Density

We will now turn our attention to the nucleosynthesis that results after breakout from the HCNO cycles has occurred. A representative value of $\rho = 10^4$ g/cm³ is again chosen for the density. Numerical network calculations are performed for three different temperatures (T = 0.5, 1.0, and 1.5 GK) and the results will be discussed below. In order to account properly for the nuclear activity at such high temperatures, the network has to be expanded dramatically in size compared to our earlier calculations. It now consists of ≈ 520 nuclides, including all stable and proton-rich β^+ -unstable (but proton-stable) nuclei up to the element palladium. Some proton-unstable nuclei are also included in order to account for sequential two-proton captures involving nuclei at and beyond the proton dripline (Section 3.1.6). Neutron-rich β^- -unstable nuclei are not included in the network since they cannot be synthesized via hydrogen or helium-induced reactions on stable or proton-rich β^+ -unstable nuclei. The different nuclides in the network are linked by ≈ 5500 nuclear processes, including β^+ -decays, (p,γ) , (p,α) , (α,γ) reactions and inverse processes such as photodisintegrations, (α,p) reactions, and so on. For the initial composition, values of $X_{\rm H}^0 = 0.73$, $X_{4\rm He}^0 = 0.25$, $X_{14O}^0 = 0.01$, and $X_{15O}^0 = 0.01$ are assumed. This assumption is consistent with the earlier result that during the rise to temperatures of $T \ge 0.5$ GK, ¹⁴O and ¹⁵O are the most abundant products as a result of HCNO cycle operation (Section 5.2.1). The network is solved numerically until a time of t = 100 s is reached. This is much shorter compared to our previous network calculations, but it is consistent with the assumption that stellar explosions at elevated temperatures have rather short durations.

Since there are so many different nuclear processes taking part in the nucleosynthesis, it is useful to visualize the nucleosynthesis paths by introducing a quantity called the *time-integrated net abundance flow* between two specific nuclei *i* and *j*,

$$F_{ij} = \int f_{ij} dt = \int \left[\left(\frac{dN_i}{dt} \right)_{i \to j} - \left(\frac{dN_j}{dt} \right)_{j \to i} \right] dt$$
(5.118)

with $(dN_i/dt)_{i\rightarrow j}$ the partial rate of change of the number density N_i induced by all processes converting nucleus *i* to *j* (Section 3.1.3). For example, if we are interested in the nuclear activity due to transformations between ²⁴Mg and ²⁵Al (Fig. 3.4), then we only need to take into account the ²⁴Mg(p, γ)²⁵Al capture reaction and the (reverse) ²⁵Al(γ ,p)²⁴Mg photodisintegration. Hence

$$F_{24}{}_{Mg^{25}Al} = \int \left[\left(\frac{d^{24}Mg}{dt} \right)_{24}{}_{Mg(p,\gamma)} - \left(\frac{d^{25}Al}{dt} \right)_{25}{}_{Al(\gamma,p)} \right] dt$$
(5.119)

Large values of F_{ij} indicate an enhanced nuclear activity between two species and, therefore, help in identifying important links. It is of advantage to express Eqs. (5.118) and (5.119) in terms of mole fractions, $Y_i = N_i / (\rho N_A)$, rather than number densities, N_i , if the density ρ changes during the nucleosynthesis. Variations in Y_i (or X_i) are independent of density and reflect only nuclear transformations, as pointed out in Section 1.5.4. In this and the following sections we will mostly consider net abundance flows that are integrated over the entire duration of the network calculation. Such flows represent gross properties of the nucleosynthesis, but do not reveal details at any particular instant in time. Nevertheless, they are very useful for providing an overview over the nucleosynthesis.

In the $A \le 40$ mass range, we adopt for the majority of reactions the experimental rates of Angulo et al. (1999) and Iliadis et al. (2001). Above A = 40, however, very few nuclear reactions have been measured directly or indirectly. For most reactions in the latter mass region, there is no alternative

but to use theoretically estimated reaction rates. The ⁵⁶Ni(p,γ)⁵⁷Cu reaction represents an exception (see below). We will adopt in the A > 40 range theoretical rates that are calculated by using the Hauser–Feshbach statistical model of nuclear reactions (Section 2.7). This is the first nucleosynthesis scenario we encounter that employs theoretical rather than experimental rates for the vast majority of reactions in the network. It must be emphasized that, except in special cases like $p(p,e^+\nu)d$, reaction rates based on theory carry much larger uncertainties compared to rates based on experimental input. It is frequently even difficult to assign a value to the uncertainty of a specific theoretical reaction rate.

Before discussing the results of the reaction network calculations, it is instructive to consider Fig. 5.35, showing the chart of the nuclides from Sc (Z = 21) to Sr (Z = 38). The heavy solid line represents the proton dripline. Nuclides that are shaded grey are β^+ -unstable and have half-lives in excess of $T_{1/2} \approx 10$ s. All other nuclides shown have half-lives of less than $T_{1/2} \approx 3$ s. In general, we expect the nucleosynthesis paths to be located somewhere between the dripline and the group of grey shaded nuclei, with the exact locations determined by the relative probability of various processes, such as (p, γ) reactions, photodisintegrations, β^+ -decays, and so on. If, for some reason, the abundance flow reaches one of the grey shaded nuclei and can only proceed via a slow β^+ -decay, presumably because proton capture is inhibited, then the nucleosynthesis will be significantly delayed, or in extreme cases, may halt altogether. The most obvious reason for an inhibited proton capture is a small $Q_{p\gamma}$ -value since the reverse photodisintegration process will then quickly remove the proton that was just added to the target nucleus. Nuclei with negative or small positive $Q_{p\gamma}$ -values and relatively long β^+ -decay half-lives are referred to as *waiting point nuclei* and are marked by circles in Fig. 5.35.

Interestingly, the proton dripline runs very close to the group of grey shaded nuclei in the Ge–Rb mass region. In particular, the abundance flow must pass through the potential waiting point nuclei ⁶⁴Ge, ⁶⁸Se, ⁷²Kr, and ⁷⁶Sr ($T_{1/2} = 64, 36, 17, \text{ and } 9 \text{ s}$, respectively). Their slow β^+ -decays may, however, be bypassed via sequential two-proton captures. For example, the negative Q-value for ⁶⁴Ge(p, γ)⁶⁵As ensures that an equilibrium is quickly established between ⁶⁴Ge and ⁶⁵As. The relative probability of the two alternative paths, ⁶⁴Ge($\beta^+\nu$)⁶⁴Ga and ⁶⁴Ge(p, γ)⁶⁵As(p, γ)⁶⁶Se, will then depend on the magnitude of the quantities $T_{1/2}(^{64}\text{Ge})$, $\rho \exp[Q_{^{64}\text{Ge}(p,\gamma)}/kT]$, and $N_A \langle \sigma v \rangle_{^{65}\text{As}(p,\gamma)}$, as explained in Section 3.1.6. For the following reaction network calculations, all reverse (γ ,p) reaction rates are calculated from experimental or theoretical forward (p, γ) rates by using $Q_{p\gamma}$ -values from Audi, Wapstra and Thibault (2003).

For nuclides marked with a solid triangle in Fig. 5.35, both the (p,γ) and the (p,α) reaction channels are open $(Q_{p\gamma} > 0 \text{ and } Q_{p\alpha} > 0)$. At elevated



Fig. 5.35 Section of the chart of the nuclides between Sc (Z = 21) and Sr (Z = 38) on the proton-rich side of the stability valley. The proton dripline according to Audi, Wapstra and Thibault (2003) is marked by a thick solid line. All displayed nuclides are unstable. Those represented by shaded squares have half-lives in excess of $T_{1/2} \approx 10$ s,

while all other nuclides have half-lives of less than $T_{1/2} \approx 3$ s. Nuclides with negative or small positive $Q_{p\gamma}$ -values and relatively long β^+ -decay half-lives are marked by circles (waiting point nuclei). The solid triangles indicate nuclides for which both the (p,γ) and (p,α) reaction channels are open.

temperatures, (p,α) reactions play a much smaller role due to the increasing Coulomb barrier in the A > 40 mass region compared to the CNO range where they give rise to reaction cycles. Consider the ⁷¹Br $(p,\alpha)^{68}$ Se reaction $(Q_{p\alpha} = 2020 \text{ keV})$ as an example. At T = 1.5 GK, the Gamow peak is located at $E_0 = 1700$ keV. Furthermore, suppose that a fictitious resonance is located in the middle of the Gamow peak, $E_r \approx E_0$. Reaction α -particles from the decay of this resonance have energies of $E_{\alpha} = E_r + Q_{p\alpha} \approx 3720$ keV. For an orbital angular momentum of $\ell_{\alpha} = 0$, the single-particle α -width amounts to a value of $\Gamma_{\ell_{\alpha}=0}^{68}(E_{\alpha} = 3720 \text{ keV}) \approx 10^{-5}$ eV. This is smaller than typical γ -ray partial widths and, hence, we find $\Gamma_{\gamma} \gg \Gamma_{\alpha}$ or $B_{p\alpha/p\gamma} \ll 1$. On the other hand, at very high temperatures $T \geq 2$ GK, the α -particle widths Γ_{α} increase and may become comparable to, or even exceed, typical values of Γ_{γ} .

Nucleosynthesis at T = 0.5 GK, $\rho = 10^4$ g/cm³, and t = 100 s

Net abundance flows, integrated over a time of t = 100 s, are displayed in Fig. 5.36. Major flows (those with $F^{\text{max}} \ge F_{ij} > 0.1F^{\text{max}}$) are shown as thick solid arrows, while minor flows $(0.1F^{\text{max}} \ge F_{ij} > 0.01F^{\text{max}})$ are indicated as thin solid arrows. The direction of the arrows corresponds to the direction

of the abundance flows. The heavy solid line represents the proton dripline, while stable nuclides are shown as shaded squares. Under these conditions, the breakout from the HCNO cycles proceeds via ${}^{15}O(\alpha,\gamma){}^{19}Ne(p,\gamma){}^{20}Na$ (sequence 1; Fig. 5.33). After the initiation of breakout, sequences of (p,γ) reactions and β^+ -decays transform CNO nuclei within t = 100 s to the Fe–Co region. The resulting network is referred to as the $r(apid)p(roton \ capture)$ -process (Wallace and Woosley 1981). Recall that the amount of energy generated by the HCNO cycles is independent of temperature since it is limited by the slow ${}^{14}O$ and ${}^{15}O \beta^+$ -decays (beyond a certain value of temperature and for a given composition; see Fig. 5.24). The rp-process is important since it circumvents these slow β^+ -decays. It will be shown below that the processing of CNO seeds to much heavier nuclei can lead to a significantly larger energy generation rate than given by the HCNO cycles alone.

The most likely nucleosynthesis path in the rp-process is defined by the competition between β^+ -decays, (p, γ), and (γ , p) reactions. During explosive burning, a specific nucleus will add progressively more protons. With each proton addition, a nucleus is synthesized that is located closer to the proton dripline. Eventually an isotope is reached that β^+ -decays rather than undergoing another proton capture. The process of proton addition and β^+ -decay repeats itself until the end of the network calculation. Why does the probability of β^+ -decay increase compared to proton capture when the proton dripline is approached? First, by approaching the proton dripline for a fixed neutron number, one moves away from the valley of stability and, therefore, the β^+ -decay half-lives become progressively shorter. Eventually, a β^+ -decay becomes more likely than another proton capture, that is, $\lambda_{\beta} > \lambda_{p\gamma}$. Second, nuclei right at the proton dripline have, per definition, negative Q-values, while some (but not the majority of) nuclei close to the dripline have small positive Q-values. In either case, photodisintegration will inhibit the addition of another proton. The nucleosynthesis must then proceed with a β^+ -decay, even if the condition $\lambda_{\beta} < \lambda_{p\gamma}$ applies.

It can also be seen that below Ti the nucleosynthesis path reaches a number of nuclei that are located right at the proton dripline (²⁴Si, ²⁹S, ³³Ar, ³⁷Ca, ³⁸Ca, ⁴¹Ti, and so on). Above Ti, however, the major abundance flows do not reach the dripline anymore. This is a consequence of decreasing proton capture rates as the Coulomb barrier increases. The interplay of β^+ -decays, (p, γ), and (γ ,p) reactions near ³⁰S is addressed in Problem 5.3. See also Example 3.3.

The evolution of the most abundant nuclides is shown in Fig. 5.37a. Only a small amount of hydrogen is consumed over a period of t = 100 s. The protons are used to produce heavier nuclei via capture reactions, while the fusion of protons to helium via reaction cycles plays only a minor role under these burning conditions. Hence, the helium abundance stays constant. The ¹⁵O abundance increases until about t = 20 s due to the oper-

ation of the ¹⁴O(α ,p)¹⁷F(p, γ)¹⁸Ne($\beta^+\nu$)¹⁸F(p, α)¹⁵O sequence. Around this time, a significant fraction of material breaks out of the CNO region via the ¹⁵O(α , γ)¹⁹Ne(p, γ)²⁰Na sequence. The abundance flow quickly reaches the $A \approx 50$ region. The most abundant nuclides at the end of the calculation in the A > 20 region are ⁵²Fe, ⁵⁶Ni, and ⁵⁵Co. These nuclides have long laboratory β^+ -decay half-lives ($T_{1/2} = 8.3$ h, 6.1 d and 17.5 h, respectively), although their *stellar* β^+ -decay half-lives are expected to be somewhat smaller (Section 1.8.4). At the same time, their proton-capture rates are relatively small, yielding mean lifetimes of $\tau_{p\gamma} = 120$ s, 24 s, and 14 s, respectively, for the burning conditions adopted here. The latter values are significant in magnitude compared to the total burning time and, therefore, these three nuclides represent endpoints for the nucleosynthesis.

Nucleosynthesis at T = 1.0 GK, $\rho = 10^4$ g/cm³, and t = 100 s

Flows and time evolutions of the most abundant species are displayed in Figs. 5.36 and 5.37b. The hydrogen abundance declines slightly with progressing time whereas the ⁴He abundance stays almost constant. The breakout from the CNO mass region proceeds through both the sequence 1 and sequence 2 (Fig. 5.33). In fact, ¹⁴O and ¹⁵O are transformed so fast (within a fraction of a second) to ²¹Na and ²⁰Na, respectively, that the operation of the HCNO cycles is not discernible anymore in Fig. 5.36. At this higher temperature, the proton capture rates become much faster and, consequently, nuclei up to the $A \approx 80$ region are synthesized via the rp-process. Compared to the previous case, the abundance flow reaches the dripline at a number of locations over the entire mass region shown. All the material initially in the form of ¹⁴O and ¹⁵O is converted to heavier mass nuclei, with ⁶⁰Zn, ⁶⁴Ge, and ⁶⁸Se being the most abundant nuclei in the A > 20 region at the end of the calculation.

It is apparent from Fig. 5.36 that the network of nuclear processes is quite complex. However, it should be pointed out that the time scale of the nucleosynthesis is mainly determined by processes involving only a handful of waiting point nuclei: ²²Mg, ²⁶Si, ³⁰S, ³⁴Ar, ⁵⁶Ni, ⁶⁰Zn, ⁶⁴Ge, and ⁶⁸Se. What these nuclei have in common are relatively long β^+ -decay half-lives and small $Q_{p\gamma}$ -values (between -450 keV for ⁶⁸Se and 861 keV for ²⁶Si). In fact, their $Q_{p\gamma}$ -values are so small that the proton-capture rate is much smaller than the reverse photodisintegration rate. Hence, photodisintegration will inhibit the proton capture reaction and the abundance flow is significantly delayed. The abundance of each waiting point nucleus increases until some maximum is reached. Eventually the abundance flow continues via a β^+ -decay (although the competing (α ,p) and (p, γ) reactions are also important for ²²Mg and ²⁶Si, respectively) until the next waiting point nucleus reached. The isotope ⁶⁰Zn has the largest abundance, apart from hydrogen and helium, at the end of the
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5.4 Explosive Hydrogen-Helium Burning 461

Fig. 5.36 Results of numerical reaction network calculations for hydrogen-helium burning at a constant temperature of T = 0.5 GK (top), 1.0 GK (middle) and 1.5 GK (bottom). The same constant density ($\rho = 10^4$ g/cm³) and initial composition (see text) is used in the calculations. The reaction network consists of all nuclides shown as squares. The arrows represent net (forward minus reverse) abundance flows, integrated over the entire computation time of t = 100 s. Thick arrows show the strongest time-integrated net flows, $F^{\text{max}} \geq F_{ij} > 0.1F^{\text{max}}$, where

 F^{\max} corresponds to the reaction with the maximum net flow. Thin arrows represent flows that are weaker by an order of magnitude $(0.1F^{\max} \ge F_{ij} > 0.01F^{\max})$. The key indicates the type of reaction represented by a specific arrow. The heavy solid line marks the proton dripline (Audi, Wapstra and Thibault 2003). Stable nuclides are shown as shaded squares. A particular nuclide can be identified from the element symbol (vertical axis) and neutron number (horizontal axis).

calculation because it possesses the longest half-life ($T_{1/2} = 2.4 \text{ min}$) among the heaviest waiting point nuclei (Fig. 5.35). The nucleus ⁵⁶Ni represents an exception. Since its half-life is so long compared to the burning time, the flow must continue exclusively via the (p, γ) reaction. This case is discussed in detail below. It can be seen in Fig. 5.37b that it takes about 10 s until ⁶⁰Zn becomes the most abundant isotope. This time is approximately equal to the sum of the mean lifetimes of the waiting point nuclei below ⁶⁰Zn. For $t \ge 40$ s, the abundances of ²²Mg, ²⁶Si, ³⁰S, ³⁴Ar, and ⁵⁶Ni stay constant. This is caused by the operation of the 3 α reaction as the abundance flows between ⁴He and ⁵⁶Ni attain equilibrium.

Nucleosynthesis at T = 1.5 GK, $\rho = 10^4$ g/cm³, and t = 100 s

Results of a network calculation at this higher temperature are shown in Figs. 5.36 and 5.37c. The hydrogen and helium abundances are almost constant over the duration of the burning. It can be seen that all three breakout sequences operate under these conditions (Fig. 5.33). Once breakout from the $A \leq 20$ region is initiated, the abundance flow initially follows two sequences of (α_r p) and (p, γ) reactions,

$${}^{20}\text{Na}(p,\gamma){}^{21}\text{Mg}(\alpha,p){}^{24}\text{Al}(p,\gamma){}^{25}\text{Si}(\alpha,p){}^{28}\text{P}$$
(5.120)

²¹Na(p,
$$\gamma$$
)²²Mg(α , p)²⁵Al(p, γ)²⁶Si(α , p)²⁹P(p, γ)³⁰S(α , p)³³Cl (5.121)

This part of the network is referred to as the *ap-process* (Wallace and Woosley 1981). In the previously discussed network calculation, the abundance flow in the $A \leq 30$ region had to wait for slow β^+ -decays of waiting point nuclei since photodisintegration impeded further proton captures. Therefore, the mean lifetime of the waiting point nuclei, and hence the overall time scale of the nucleosynthesis in this mass region, was independent of temperature and density. The significance of the *ap*-process lies in the fact that it bypasses the slow β^+ -decays. The nucleosynthesis in the $A \leq 33$ region becomes now sensitive to temperature and, consequently, the burning of hydrogen and he-





Fig. 5.37 Abundance evolutions during hydrogen–helium burning at a constant temperature of (a) T = 0.5 GK, (b) T = 1.0 GK, and (c) T = 1.5 GK. The same constant density ($\rho = 10^4$ g/cm³) and initial composition is used in all the three cases. The results are extracted from the same numerical reaction network calculations that are displayed in Fig. 5.36.

lium proceeds at an accelerated pace and larger energy generation rates can be achieved. For the burning conditions adopted here, the α p-process switches to the rp-process above mass *A* = 33 where the Coulomb barrier impedes reactions induced by α -particles.

The most important waiting point nuclei are ³⁴Ar, ³⁹Ca, ⁵⁶Ni, and ⁶⁰Zn. In each case, with the exception of ⁵⁶Ni, the reverse photodisintegration rate dominates over the forward rate and the abundance flow must proceed via a slow β^+ -decay. At the end of the calculation the most abundant nuclei, besides hydrogen and helium, are ⁵⁶Ni, ⁶⁰Zn, and ⁶⁴Ge. The large final abundance of ⁵⁶Ni is striking, especially when compared to the previous calculation where only a small ⁵⁶Ni abundance was left over at *t* = 100 s. Related to this observation is the fact that the main abundance flow extends only to ⁶⁰Zn, whereas in the previous calculation at *T* = 1.0 GK the main flow reached much farther (to ⁶⁸Se; see Fig. 5.36). This issue will be discussed in detail below. It is also apparent that for *t* ≥ 20 s the abundances of ³⁰S, ³⁴Ar, ³⁹Ca, and ⁵⁶Ni stay constant because of the operation of the 3 α reaction (see discussion at *T* = 1.0 GK).

It should be pointed out that the nucleosynthesis in the $A \ge 20$ region depends neither on the precise values for the rates nor the identity of the breakout reactions. If we would remove entirely the reactions ${}^{19}N(p,\gamma){}^{20}Na$, ${}^{18}Ne(\alpha,p){}^{21}Na$, and ${}^{16}O(\alpha,\gamma){}^{20}Ne$ from the network, that is, those reactions which complete the breakout sequences shown in Fig. 5.33, then the A < 20 and $A \ge 20$ regions would be bridged by slower reactions, such as ${}^{19}Ne(\alpha,p){}^{22}Na$, ${}^{19}Ne(\alpha,\gamma){}^{23}Mg$, ${}^{18}Ne(\alpha,\gamma){}^{22}Mg$ and ${}^{17}F(\alpha,p){}^{20}Ne$, and the abundance evolutions in the $A \ge 20$ region would closely resemble the results shown in Figs. 5.37b and c.

The ⁵⁶Ni bottleneck

To understand why the abundance flow is significantly delayed at ⁵⁶Ni, we need to consider the unique nuclear properties of this waiting point nucleus. It has a relatively long half-life of $T_{1/2} = 6.1$ d in the laboratory and decays with 100% probability by electron capture. At elevated temperatures and densities, the β -decay half-life will change somewhat (Section 1.8.4; see also Fuller, Fowler and Newman 1982). Values of $Q_{p\gamma}$ and $T_{1/2}$ for nuclei in the vicinity of ⁵⁶Ni are given in Fig. 5.38a. The *Q*-value for the ⁵⁶Ni(p,γ)⁵⁷Cu reaction amounts to only $Q_{p\gamma} = 695$ keV. The subsequent ${}^{57}Cu(p,\gamma){}^{58}Zn$ reaction has a Q-value of $Q_{p\gamma} = 2280$ keV. At a density of $\rho = 10^4$ g/cm³, the abundance flows will pass through the ${}^{56}\text{Ni}(p,\gamma){}^{57}\text{Cu}$ reaction (Fig. 5.36). We are interested in the effective mean lifetime (or decay constant) of ⁵⁶Ni. For temperatures below T = 0.77 GK, the photodisintegration of ⁵⁷Cu is less likely to occur than the competing β^+ -decay of 57 Cu, that is, $\lambda_{{}^{57}Cu(\gamma,p)} < \lambda_{{}^{57}Cu(\beta^+\nu)}$, as can be calculated from numerical values of $N_A \langle \sigma v \rangle_{^{56}\text{Ni}(\text{p},\gamma)}$ and $T_{1/2}(^{57}\text{Cu})$ (Fig 5.38a). Since photodisintegration plays only a minor role in this temperature range, we obtain the effective mean lifetime of ⁵⁶Ni, $\tau_{eff} = 1/\lambda_{eff}$, simply from (see Eq. (3.23))

$$\lambda_{\rm eff}({}^{56}\rm Ni) = \rho \frac{X_{\rm H}}{M_{\rm H}} N_A \langle \sigma v \rangle_{{}^{56}\rm Ni(p,\gamma)}$$
(5.122)

Values of the effective mean lifetime are plotted in Fig. 5.38b versus temperature for $\rho = 10^4 \text{ g/cm}^3$ and $X_{\rm H} = 0.73$. At T = 0.4 GK, we obtain $\tau_{\rm eff}({}^{56}{\rm Ni}) = 185$ s. This value is much shorter than the laboratory lifetime of ${}^{56}{\rm Ni}$, but is large compared to typical macroscopic explosion time scales $(t \le 100 \text{ s})$. The ${}^{56}{\rm Ni}(p,\gamma){}^{57}{\rm Cu}$ reaction rate increases for higher temperatures and thus the effective mean lifetime decreases. For example, at T = 0.77 GK, one obtains a value of $\tau_{\rm eff}({}^{56}{\rm Ni}) = 1.7$ s. At temperatures of T = 0.77–1.27 GK, the photodisintegration of ${}^{57}{\rm Cu}$ cannot be neglected anymore. In fact, the ${}^{57}{\rm Cu}(\gamma,p){}^{56}{\rm Ni}$ reaction is now faster than the competing processes ${}^{57}{\rm Cu}(p,\gamma){}^{58}{\rm Zn}$ and ${}^{57}{\rm Cu}(\beta^+\nu){}^{57}{\rm Ni}$ [$\lambda_{57}{\rm Cu}(\gamma,p) > \lambda_{57}{\rm Cu}(p,\gamma) + \lambda_{57}{\rm Cu}(\beta^+\nu)$]. The conditions of Eqs. (3.55) and (3.56) are fulfilled and, consequently, the abundances of ${}^{56}{\rm Ni}$ is then given by Eq. (3.61),

$$\lambda_{\rm eff}({}^{56}\rm Ni) = \frac{\lambda_{56}\rm Ni(p,\gamma)}{\lambda_{57}\rm Cu(\gamma,p)} [\lambda_{57}\rm Cu(p,\gamma) + \lambda_{57}\rm Cu(\beta^+\nu)]$$
(5.123)

As the temperature increases from 0.77 to 1.27 GK, the ratio of decay constants in Eq. (5.123) becomes smaller (see Eq. (3.61)). Simultaneously, the ⁵⁷Cu(p, γ)⁵⁸Zn reaction rate increases and, as a result, the effective mean lifetime is approximately constant, τ_{eff} (⁵⁶Ni) \approx 3.0 s (Fig. 5.38b). The use of Eq. (5.123) implies that the photodisintegration of ⁵⁸Zn plays a minor role compared to the decay ⁵⁸Zn($\beta^+\nu$)⁵⁸Cu (see condition $\lambda_{C\to C'} > \lambda_{C\to B}$; Section 3.1.6). At temperatures in excess of T = 1.27 GK, however, the ⁵⁸Zn(γ ,p)⁵⁷Cu reaction becomes faster than the competing β^+ -decay of ⁵⁸Zn. Furthermore, the ⁵⁷Cu(p, γ)⁵⁸Zn reaction is faster than the competing β^+ -decay ⁵⁷Cu($\beta^+\nu$)⁵⁷Ni. For these conditions, the abundances of ⁵⁶Ni, ⁵⁷Cu, and ⁵⁸Zn achieve quickly equilibrium. The effective mean lifetime of ⁵⁶Ni is then obtained from (Problem 3.1)

$$\lambda_{\rm eff}({}^{56}\rm Ni) = \frac{\lambda_{56}{\rm Ni}(p,\gamma)}{\lambda_{57}{\rm Cu}(\gamma,p)} \left(\frac{\lambda_{57}{\rm Cu}(p,\gamma)}{\lambda_{58}{\rm Zn}(\gamma,p)} \lambda_{58}{\rm Zn}(\beta^+\nu) + \lambda_{57}{\rm Cu}(\beta^+\nu) \right)$$
(5.124)

The decay constants for the two β^+ -decays are constant with temperature, but both ratios of decay constants for forward and reverse reaction are proportional to $e^{Q_i/kT}$ and hence decrease rapidly with increasing temperature values. Therefore, the effective mean lifetime of ⁵⁶Ni increases steeply beyond T = 1.27 GK, as can be seen in Fig. 5.38b. For example, one obtains $\tau_{eff}({}^{56}\text{Ni}) \approx 246$ s at T = 1.5 GK. It is interesting that a window exists at intermediate temperatures (T = 0.77–1.27 GK) where ${}^{56}\text{Ni}$ does not represent a major waiting point. Consequently, the abundance flow reaches far beyond the Ni region in the middle panel of Fig. 5.36. At lower and higher temperatures, the Coulomb barrier of ${}^{56}\text{Ni} + p$ and two sequential photodisintegration reactions, respec-



Fig. 5.38 (a) Section of the chart of nuclides in the vicinity of the bottleneck ⁵⁶Ni. Nuclides that eventually may reach equilibrium are shown as shaded squares. Values of $Q_{\rm p\gamma}$ (left-hand side) and $T_{1/2}$ (right-hand side) are adopted from Audi, Wapstra and Thibault (2003) and Audi et al. (2003), re-



spectively. (b) Effective mean lifetime of ⁵⁶Ni versus temperature for the conditions $\rho = 10^4$ g/cm³ and $X_{\rm H} = 0.73$. A window exists at intermediate temperatures (T = 0.77-1.27 GK) where ⁵⁶Ni does not represent a major waiting point for the abundance flow. See the text.

tively, are responsible for a substantial increase in the value of $\tau_{eff}(^{56}Ni)$ and the abundance flow does not reach as far (Fig. 5.36; top and bottom panels).

Energy generation

The rp- and α p-processes generate energy in a completely different manner compared to the HCNO cycles. The former processes consist of *sequences* of capture reactions and β^+ -decays. Note that an (α ,p) reaction followed by (p, γ) has the same product as a single (α , γ) reaction. Reaction cycles play only a minor role, and, therefore, none of the nuclides involved in the nucleosynthesis will act as catalysts. Energy is generated not by the fusion of four protons to one ⁴He nucleus, but by using protons and α -particles to build up heavier nuclides starting from CNO seed nuclei. Also, at these higher temperatures, the 3 α reaction operates and supplies a fraction of the CNO seeds. The energy generation rate is sensitive to the total mass fraction of CNO seed nuclei and the initial hydrogen-to-helium abundance ratio ($X_{\rm H}^0/X_{\rm 4He}^0$), but is relatively insensitive to the exact initial CNO composition or the manner by which the breakout from the CNO region proceeds.

Energy generation rates for the previously discussed network calculations at T = 0.5, 1.0, and 1.5 GK are shown in Fig. 5.39. The density ($\rho = 10^4 \text{ g/cm}^3$) and initial composition are the same for each calculation. The solid lines are obtained with the full reaction network. The final hydrogen abundances

amount to $X_{\rm H} = 0.70$, 0.67 and 0.69 at T = 0.5, 1.0, and 1.5 GK, respectively. The main abundance flow eventually reaches the waiting point nucleus ⁵²Fe (T = 0.5 GK), ⁶⁰Zn (T = 1.0 GK), or ⁵⁶Ni (T = 1.5 GK). The flow slows significantly down and material accumulates at the waiting point nucleus (Fig. 5.37). As a result, the energy generation rate drops, giving rise to the broad maxima displayed in Fig. 5.39. Furthermore, the higher the temperature the faster the CNO seed nuclei are transformed to the final, most abundant, waiting point nucleus. As a result, the maximum in the energy generation rate occurs at earlier times ($t_{\rm peak} = 33$, 7.0, and 4.3 s at T = 0.5, 1.0, and 1.5 GK, respectively). The dashed lines in Fig. 5.39 correspond to the energy generation rate if the reaction rates of all possible breakout processes are set equal to zero so that the HCNO cycles and the 3 α reaction are the sole sources of nuclear energy. It is apparent that the rp- and α p-processes enhance the energy generation rate substantially. For T = 0.5, 1.0, and 1.5 GK, the maximum enhancement amounts to a factor of 6, 33, and 25, respectively.

If one would repeat the above calculations at T = 1.0 or 1.5 GK by setting all initial CNO abundances equal to zero, then the nucleosynthesis must start with the 3α reaction. The newly created CNO nuclei are the seeds for the subsequent rp- and α p-processes and the resulting abundance flow patterns in the $A \ge 20$ region would closely resemble those shown in the middle and bottom panels of Fig. 5.36. Eventually, the energy generation rate will stay constant with time as the abundance flows between ⁴He, ⁵⁶Ni, and heavier nuclei attain equilibrium. Of course, the energy generation rate would be much smaller in magnitude compared to the results shown as solid lines in Fig. 5.39.

For more information on the rp- or α p-processes at constant temperature and density conditions see, for example, Schatz et al. (1998).

5.4.3

Nucleosynthesis for Temperature–Density Profiles

We will now consider the more realistic situation of changing temperature and density during the nucleosynthesis. Type I X-ray bursts (Section 1.4.4) represent examples for explosive (thermally unstable) hydrogen–helium burning at temperatures in excess of T = 0.5 GK. Figure 5.40 shows a temperature and density profile that is similar to those obtained in stellar model studies of a thermonuclear runaway caused by the accretion of hydrogen and helium onto the surface of a $1.3 M_{\odot}$ neutron star with a radius of 8 km (Koike et al. 2004). The curves represent the temperature and density evolutions in the hottest nuclear burning zone. In this particular example, the nuclear burning starts with temperature and density values of T = 0.4 GK and $\rho = 10^6$ g/cm³. At t = 4 s, a maximum temperature of $T_{\text{peak}} = 1.36$ GK and a minimum density of $\rho_{\text{peak}} = 5 \times 10^5$ g/cm³ are achieved. After t = 100 s, the temperature has fallen



Fig. 5.39 Evolution of energy generation rate during hydrogen-helium burning at a constant temperature of (a) T = 0.5 GK and (b) T = 1.0 GK. The same constant density ($\rho = 10^4$ g/cm³) and initial composition is used in each case. The results are extracted from the same numerical reaction network calculations that are displayed in Fig. 5.36. Results for T = 1.5 GK are very

similar to those shown in part (b) and are not displayed in the figure. The solid lines are obtained with the full reaction network. The dashed lines correspond to the energy generation rate if the reaction rates of all possible breakout processes are set equal to zero so that the HCNO cycles and the 3α reaction are the sole sources of nuclear energy generation.

to T = 0.7 GK and the density increased to $\rho = 1.4 \times 10^6$ g/cm³. The density is about two orders of magnitude larger compared to the constant value assumed in the previous section. Recall that the forward reaction rates depend on the density, but the photodisintegration rates are independent of ρ (see Eqs. (3.23) and (3.45)). The reaction network is solved numerically by using this $T-\rho$ profile. For the initial composition, values of $X_{\rm H}^0 = 0.73$, $X_{\rm 4He}^0 = 0.25$, $X_{\rm 14O}^0 = X_{\rm 15O}^0 = 0.01$ are assumed. The network calculation is terminated after t = 100 s.

The major abundance flows extend from helium all the way up to the end of the network (palladium), as can be seen from Fig. 5.40. Breakout from the CNO mass region proceeds via sequences 1 and 2 (Fig. 5.33). The latter sequence is more important since the ¹⁴O abundance is fed by $\alpha(2\alpha)^{12}C(p,\gamma)^{13}N(p,\gamma)^{14}O$. After breakout, matter is processed via the α pprocess (below the chlorine region) and the rp-process. The abundance flow reaches the dripline at many locations over the entire mass region shown. The major flow then has to wait in most cases for a β^+ -decay before continuing. The waiting point nuclei ⁶⁴Ge, ⁶⁸Se, ⁷²Kr, and ⁷⁶Sr represent interesting cases. Their $Q_{p\gamma}$ -values are predicted to be negative (Audi, Wapstra and Thibault 2003) while their half-lives amount to $T_{1/2} = 64$ s, 36 s, 17 s, and 9 s, respectively. It can be seen that at ⁶⁴Ge the major flow continues via sequential two-proton capture instead of the very slow β^+ -decay which otherwise would

terminate the nucleosynthesis. For the other three waiting point nuclei, the β^+ -decay is more likely to occur, for the conditions assumed here, than the competing sequential two-proton capture. Hence, the abundance flow will be delayed significantly and we expect an accumulation of material especially at ⁶⁸Se and ⁷²Kr toward the end of the calculation.

A significant fraction of ¹H and ⁴He nuclei is consumed during the thermonuclear explosion. Their abundances decrease gradually with time until they reach values of $X_{\rm H}$ = 0.16 and $X_{\rm ^4He}$ = 0.02 at the end of the calculation. Figure 5.41a shows the abundance evolution of the most important waiting point nuclei, that is, for those nuclides that are most abundant at any given time. It is evident how the flow reaches nuclei such as ¹⁸Ne, ²⁴Si, ²⁵Si, and so on, in sequence. In each case, the abundance flow is delayed by a slow process that consumes the waiting point nucleus (for example, the (α, p) reaction on ¹⁸Ne; β^+ -decays of ²⁴Si, ²⁵Si, and so on; sequential two-proton captures on ⁶⁴Ge). As a result, the abundance of a particular waiting point nucleus increases until a maximum is reached and then decreases with time. At t = 4 s, when the peak temperature is attained, the most abundant nuclei (besides ¹H and ⁴He) are ⁶⁰Zn, ⁵⁵Ni, ³⁸Ca, ⁵⁹Zn, and ⁶⁴Ge, with similar mass fractions of $X \approx 0.03$. Note that ⁵⁶Ni is not a major waiting point nucleus. For the densities adopted here, one finds from Eq. (5.124) a mean lifetime of only $\tau_{\rm eff}({}^{56}{\rm Ni}) = 0.02$ s. Therefore, the ${}^{56}{\rm Ni}$ abundance stays relatively small throughout the calculation. Also, ⁵⁶Ni cannot be bypassed via the sequence 55 Ni(p, γ) 56 Cu(p, γ) 57 Zn, as is sometimes erroneously assumed (Forstner et al. 2001), since ⁵⁷Zn decays preferentially (Audi et al. 2003) by β -delayed proton emission [${}^{57}Zn(\beta^+\nu p){}^{56}Ni$] rather than by β^+ -decay [${}^{57}Zn(\beta^+\nu){}^{57}Cu$]. At t = 10 s, ⁶⁸Se has by far the largest abundance ($X_{68Se} = 0.35$) among all nuclides except ¹H since the abundance flow must wait for its slow β^+ -decay, as noted earlier. With progressing time, ⁶⁸Se is slowly depleted and the abundances of a number of nuclides in the A > 68 region are building up. At t = 100 s, the most abundant nuclides (besides ¹H) are ⁶⁸Se, ⁷²Kr, ⁷⁶Sr, and ⁶⁴Ge. These nuclei will quickly decay to 68 Ge ($T_{1/2}$ = 271 d), 72 Se ($T_{1/2}$ = 8.4 d), 76 Kr ($T_{1/2}$ = 14.8 h), and ⁶⁴Zn (stable), respectively, after the thermonuclear explosion has ceased. A significant fraction of matter ($\Sigma X_i = 0.20$) has been converted to nuclei in the Zr-Ru mass region. The total mass fraction of nuclides located at the end of the network (Rh and Pd) amounts to $\Sigma X_i = 0.16$. This material would be converted to even heavier nuclei if we would not have truncated the network artificially. For a discussion of abundance evolutions in the mass range above Pd, see, for example, Schatz et al. (2001) or Koike et al. (2004).

It is important to point out that the nucleosynthesis in the mass range $A \ge 64$ depends sensitively on the *Q*-values for the (p, γ) reactions on ⁶⁴Ge, ⁶⁸Se, and ⁷²Kr, and on the reaction rates for the (p, γ) reactions on ⁶⁵As, ⁶⁹Br, and ⁷³Rb. Consider the waiting point ⁶⁴Ge as an example. We adopted a value

of $Q_{p\gamma} = -80 \pm 300 \text{ keV}$ (Audi, Wapstra and Thibault 2003) for ${}^{64}\text{Ge}(p,\gamma){}^{65}\text{As}$ and the ${}^{65}\text{As}(p,\gamma){}^{66}\text{Se}$ reaction rate from Goriely (1998). With these values, the mean lifetime of ${}^{64}\text{Ge}$ versus destruction by sequential two-proton capture at T = 1.34 GK and $\rho = 5.9 \times 10^5 \text{ g/cm}^3$ (when the ${}^{64}\text{Ge}$ abundance reaches a maximum; Fig. 5.41a) amounts to $\tau_{2p}({}^{64}\text{Ge}) = 1.5 \text{ s.}$ Clearly, two-proton capture is much more likely to occur than the competing β^+ -decay [$\tau_{\beta}({}^{64}\text{Ge}) = T_{1/2}({}^{64}\text{Ge})/\ln 2 = 92 \text{ s}$] and the relatively short effective lifetime of ${}^{64}\text{Ge}$ allows for a significant production of nuclei in the A > 64 range, as discussed above. Repeating the calculation by using instead a value of $Q_{p\gamma} = -380 \text{ keV}$ yields $\tau_{2p}({}^{64}\text{Ge}) = 21 \text{ s}$ and ${}^{64}\text{Ge}$ rather than ${}^{68}\text{Se}$ would be the most abundant nuclide at the end of the network calculation, with a much-reduced total abundance of nuclei in the A > 80 range. See also Problem 5.4.

The time evolution of the energy generation rate is displayed in Fig. 5.41b. Two narrow and two broad maxima are clearly visible. They are correlated with the abundance evolution of waiting point nuclei. The first narrow maximum ($t \approx 0.29$ s) is caused by the evolution of ¹⁸Ne. Shortly after the ¹⁸Ne abundance increases most rapidly, the flow is temporarily delayed and the ¹⁸Ne abundance peaks. Consequently, the energy generation rate decreases, giving rise to the first maximum. The second ($t \approx 0.33$ s) and third ($t \approx 0.74$ s) maxima are similarly caused by the abundance evolutions of ²⁴Si and ²⁹S, respectively. The transformation of the bulk material from ²⁹S to ³⁸Ca via the rp-process takes only ≈ 1.6 s, while the major abundance flow reaches the isotope ⁵⁵Ni after an additional ≈ 1.3 s. Subsequently, matter starts to accumulate at the major waiting point nuclei ⁶⁴Ge and ⁶⁸Se. The energy generation rate decreases and, as a result, the fourth maximum ($t \approx 3.0$ s) is produced.

For an extensive investigation of type I X-ray burst nucleosynthesis, see Woosley et al. (2004). A discussion of thermally stable hydrogen–helium burning on accreting neutron stars can be found in Schatz et al. (1999). For a different site of the α p-process (sub-Chandrasekhar white dwarf explosions), see Goriely et al. (2002).

Experimental nuclear physics information

Among the processes that are part of the breakout sequences (Section 5.4.1 and Fig. 5.33), the ¹⁹Ne(p, γ)²⁰Na and ¹⁸Ne(α ,p)²¹Na reactions have been measured directly using radioactive ion beams (Groombridge et al. 2002, Couder et al. 2004). These difficult experiments provided only partial information, however, and thus the present errors in the reaction rates amount to 1–2 orders of magnitude at *T* = 0.5–1.0 GK (with the larger error at the lower temperature value). The ¹⁴O(α ,p)¹⁷F reaction rate has been estimated by measuring the reverse ¹⁷F(p, α)¹⁴O reaction (Harss et al. 2002, Blackmon et al. 2003), but the reaction rate was obtained indirectly by using experimental nuclear structure



Fig. 5.40 Time-integrated net abundance flows during a thermonuclear runaway caused by the accretion of hydrogen and helium onto the surface of a 1.3 M_{\odot} neutron star with a radius of 8 km. The evolution of temperature and density in the nuclear burning zone during explosive hydrogen–helium burning, shown in the inset, is similar to the result obtained from hydrodynamic

simulations of type I X-ray bursts (Koike et al. 2004). The reaction network calculation is terminated after t = 100 s. The arrows, shaded squares, and thick solid line have the same meanings as in Fig. 5.36. The abundance flows are defined here in terms of mole fractions rather than number densities since the mass density varies.

information (Davids et al. 2003 and references therein). The current reaction rate uncertainties at T = 0.5 and 1.0 GK amount to a factors of 280 and 3, respectively. The experimental situations for the ${}^{17}F(p,\gamma){}^{18}Ne$ and ${}^{16}O(\alpha,\gamma){}^{20}Ne$ reactions have already been described in Sections 5.2.1 and 5.3.1, respectively.

After breakout from the HCNO cycles has been achieved, a large number of nuclear processes (several thousand) takes part in the nucleosynthesis (rp- and α p-processes). The nuclear physics information necessary to describe quantitatively the nuclear burning consists of: (i) reaction *Q*-values; (ii) thermonuclear reaction rates; and (iii) β -decay half-lives. Precise *Q*-values are especially needed for pairs of nuclei that achieve an equilibrium between forward and backward reaction at elevated stellar temperatures. For example, the *Q*-value for ⁵⁶Ni(p, γ)⁵⁷Cu is known to reasonable precision ($Q_{p\gamma} = 695 \pm 19$ keV), but the *Q*-values for the (p, γ) reactions on the waiting point nuclei ⁶⁴Ge, ⁶⁸Se, ⁷²Kr, and ⁷⁶Sr carry large uncertainties. We adopt here

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Fig. 5.41 Abundance evolutions of the most important waiting point nuclei, and (b) time evolution of the energy generation rate during explosive hydrogen-helium burning. The results are extracted from the same numerical reaction network calculation that is displayed in Fig. 5.40. The narrow and broad maxima of the energy generation rate shown in part (b) are correlated with the abundance evolution of the waiting point nuclei displayed in part (a).

the values of $Q_{p\gamma} = -80 \pm 300 \text{ keV}$, $-450 \pm 100 \text{ keV}$, $-600 \pm 150 \text{ keV}$, and $-50 \pm 50 \text{ keV}$, respectively. The present errors are substantial, especially since these *Q*-values enter exponentially in Eq. (3.62). It should also be pointed out that the errors quoted above do not represent experimental uncertainties, but have been derived from systematic trends of measured masses (see Audi, Wapstra and Thibault 2003 for details). Hence, the true uncertainties are expected to be somewhat larger than the quoted values. With the exception of the ²¹Na(p, γ)²²Mg reaction (D'Auria et al. 2004), none of the thermonuclear rates for reactions along the rp- or α p-process paths have been measured directly. Some of the rates have been estimated by using experimental nuclear structure information (see, for example, Iliadis et al. 2001 or Forstner et al. 2001), but the vast majority of reaction rates are based on the Hauser–

Feshbach statistical model (Rauscher and Thielemann 2000, Goriely 1998). It must be pointed out that not all the reactions that are part of the network have an influence on the nucleosynthesis (Iliadis et al. 1999). Of particular importance are (α ,p) reactions on waiting point nuclei, for example, ${}^{22}Mg(\alpha,p){}^{25}Al$, 25 Si(α ,p) 28 P, 30 S(α ,p) 33 Cl, and second-step (p, γ) reactions in sequential twoproton capture, for example, 57 Cu(p, γ) 58 Zn, 65 As(p, γ) 66 Se, 69 Br(p, γ) 70 Kr, and 73 Rb(p, γ) 74 Sr. The half-lives of the (most-likely proton-unbound) nuclei 65 As, 69 Br, and 73 Rb are predicted to be very short (128 ms, < 24 ns, < 30 ns, respectively; Audi et al. 2003) and, therefore, direct measurements of the protoncapture reactions on these target nuclei seem very challenging. All these reaction rates carry at present large uncertainties. With very few exceptions, labo*ratory* half-lives of all proton-rich β -unstable nuclei up to mass A = 100 have been measured (Audi et al. 2003) and their current uncertainties have no major influence on the nucleosynthesis in explosive hydrogen-helium burning. For an investigation of the impact of *stellar* β -decay half-lives on the nucleosynthesis during type I X-ray bursts, see Woosley et al. (2004).

5.5

Advanced Burning Stages

5.5.1 Carbon Burning

When the helium is consumed in the center of the star, the core contracts gravitationally and the central temperature simultaneously rises until the next nuclear fuel begins to burn. The ashes of helium burning consist overwhelmingly of ¹²C and ¹⁶O (Section 5.3.2). Of all the possible fusion reaction involving these two nuclei, ¹²C + ¹²C, ¹²C + ¹⁶O, and ¹⁶O + ¹⁶O, the first process has the smallest Coulomb barrier and, therefore, initiates the next burning stage. We mentioned already in Section 5.3.1 that the precise abundance ratio of ¹²C to ¹⁶O that is obtained at the end of core helium burning sensitively influences the future evolution of the star.

The ¹²C + ¹²C fusion reaction is the first process we encounter that involves two heavy nuclei in the entrance channel, as opposed to one reaction partner being a light species such as a proton, neutron, or α -particle. The ²⁴Mg compound nucleus formed in the fusion of two ¹²C nuclei is highly excited, with the mass difference between ¹²C + ¹²C and ²⁴Mg amounting to \approx 14 MeV. At such high excitation energies, the reaction will proceed through a large number of overlapping ²⁴Mg states, and we expect that for these levels the particle partial widths (for proton, neutron, and α -particle emission) dominate over the γ -ray partial width. In other words, the excess energy of the highly excited ¹²C + ¹²C system is most effectively removed by emission of light massive particles. The most likely *primary* reactions are (Salpeter 1952, Hoyle 1954)

¹²C(¹²C, p)²³Na
$$(Q = 2241 \text{ keV})$$
 (5.125)

¹²C(¹²C,
$$\alpha$$
)²⁰Ne (0 = 4617 keV) (5.126)

$$^{12}C(^{12}C,n)^{23}Mg$$
 (Q = -2599 keV) (5.127)

with the other possibilities, such as ${}^{12}C({}^{12}C,\gamma){}^{24}Mg$ or ${}^{12}C({}^{12}C,{}^{8}Be){}^{16}O$, being much less important at energies of astrophysical interest (see, for example, Patterson, Winkler and Zaidins 1969). Note that the ${}^{12}C({}^{12}C,n){}^{23}Mg$ reaction is endothermic, that is, it can only occur above a threshold energy of $E_{cm} \approx$ 2.6 MeV. The liberated protons, α -particles, and neutrons will be quickly consumed at elevated temperatures by initiating *secondary* reactions involving, for example, the ashes of helium burning (${}^{12}C$ and ${}^{16}O$) and the heavy product nuclei of the primary reactions (${}^{23}Na$ and ${}^{20}Ne$). This network of primary and secondary reactions is referred to as *carbon burning*. Typical temperatures in core carbon burning amount to T = 0.6-1.0 GK, depending on the mass of the star, while slightly higher temperatures are achieved in hydrostatic shell carbon burning. Explosive carbon burning takes place in the range of T =1.8-2.5 GK.

The total S-factor for ${}^{12}C + {}^{12}C$ is shown in Fig. 5.42. The reaction has been measured down to a center-of-mass energy of $E_{\rm cm} \approx 2.5$ MeV. The height of the Coulomb barrier is ≈ 8 MeV (Section 2.4.3). Since the measured energies are not far below the Coulomb barrier, the Gamow factor (see Eq. (2.125)) will not remove entirely the energy dependence of the cross section and, consequently, the S-factor shown in Fig. 5.42 varies rather strongly (see also Eq. (2.124)). The open bar indicates the location of the Gamow peak for $T \approx 0.85$ GK, a temperature that is typical of core carbon burning ($E_0 \pm \Delta/2 = 2169 \pm 460$ keV). It can be seen that data for the ${}^{12}C + {}^{12}C$ reaction barely touch the Gamow peak region. However, for explosive carbon burning ($T \approx 2.0$ GK) data exist throughout the Gamow peak. The cross section at the lowest measured energy amounts only to a few nanobarns and hence measurements at even lower energies represent an experimental challenge. It was already mentioned that the ${}^{12}C + {}^{12}C$ reaction will most likely proceed through many overlapping levels of the highly excited ²⁴Mg compound nucleus. It is reasonable to expect that the cross section or S-factor varies smoothly with energy. However, this behavior is not reflected in the data which show a strongly fluctuating cross section up to an energy of $E_{\rm cm} \approx 6$ MeV. The origin of this structure remains obscure although many suggestions have been made (see the summary given in Rolfs and Rodney 1988). It must also be pointed out that the various measurements are in poor agreement at the lowest energies. The average trend of the data, disregarding the fluctuations, has been described by a variety of methods in order to extrapolate the cross section to energies important



Fig. 5.42 Total *S*-factor for the ¹²C + ¹²C reaction. The data are adopted from Patterson, Winkler and Zaidins (1969), Mazarakis and Stephens (1973), and High and Cujec (1977). The data of Becker et al. (1981) are not shown. The open bars indicate the location of the Gamow peaks for $T \approx 0.85$ GK

(core carbon burning) and $T \approx 2.0$ GK (explosive carbon burning). The origin of the strong fluctuation in the *S*-factor (or cross section) remains obscure. The solid line shows the fitted total *S*-factor adopted by Caughlan and Fowler (1988).

for core carbon burning. The fitted total *S*-factor adopted by Caughlan and Fowler (1988) is shown in Fig. 5.42 as a solid line.

The ${}^{12}C + {}^{12}C$ reactions populate not only the ground states of the residual ²³Na, ²⁰Ne, and ²³Mg nuclei but proceed to several excited states as well. Hence, the various reaction channels may be studied by applying several different experimental techniques, including the direct measurement of the emitted light massive particles (Patterson, Winkler and Zaidins 1969, Mazarakis and Stephens 1973, Becker et al. 1981), the detection of γ -rays emitted from excited levels in the residual nuclei (High and Cujec 1977, Kettner, Lorenz-Wirzba and Rolfs 1980), and the activation method (Dayras, Switkowski and Woosley 1977). The data reveal that the ${}^{12}C({}^{12}C,p){}^{23}Na$ and ${}^{12}C({}^{12}C,\alpha){}^{20}Ne$ reactions dominate the total ${}^{12}C + {}^{12}C$ fusion cross section, with about equal probabilities for the proton and α -particle channels to occur. The branching ratios amount to $B_p \approx B_\alpha \approx (1 - B_n)/2$, with B_n being a small number. The measured neutron branching ratios B_n amount to 2–10% at energies of $E_{\rm cm}$ = 3.5–5.0 MeV. For lower energies the B_n values decrease, as predicted by an extrapolation of the data using the Hauser-Feshbach model (Dayras, Switkowski and Woosley 1977).

The thermonuclear reaction rates for the various ${}^{12}C + {}^{12}C$ reaction channels (Caughlan and Fowler 1988, Dayras, Switkowski and Woosley 1977) are



Fig. 5.43 Reaction rates for various ¹²C + ¹²C, ¹²C + ¹⁶O, and ¹⁶O + ¹⁶O reaction channels (from Caughlan and Fowler 1988, Dayras, Switkowski and Woosley 1977). For better comparison, the $N_A \langle \sigma v \rangle$ values are given relative to the ¹²C(¹²C, α)²⁰Ne reaction rate. The displayed results disregard electron screening corrections.

displayed in Fig. 5.43 where, for better comparison, the results are normalized to the ${}^{12}C({}^{12}C,\alpha){}^{20}Ne$ reaction rates. The ${}^{12}C({}^{12}C,\alpha){}^{20}Ne$ and ${}^{12}C({}^{12}C,p){}^{23}Na$ rates are approximately equal while the ${}^{12}C({}^{12}C,n){}^{23}Mg$ reaction rate is far smaller and declines rapidly for decreasing temperatures. The latter behavior is expected since the lower integration limit of zero in Eq. (3.69) must be replaced by the threshold energy for endothermic reactions ($E_t = 2.6$ MeV in this case). The reaction rates displayed in Fig. 5.43 neglect corrections for electron screening (Section 3.2.6). Such corrections can be significant at temperature and density conditions of advanced burning stages. The present uncertainties in the rates of the primary carbon burning reactions near $T \approx 0.85$ GK are difficult to quantify. A crude estimate is a factor of \approx 3. Note that the rates per particle pair, $N_A \langle \sigma v \rangle$, of most (secondary) proton-, neutron- or α particle-induced reactions exceed the rates of all the primary carbon burning reactions by many orders of magnitude. For comparison, the rates for the various ${}^{12}C + {}^{16}O$ reaction channels are also shown in Fig. 5.43. Due to the larger Coulomb barrier, these are much smaller compared to those of the primary carbon burning reactions and are thus of interest only in special circumstances (Arnett 1996).

The secondary reactions contribute significantly to the nuclear energy released by the primary carbon burning reactions. It can be estimated that each ${}^{12}\text{C} + {}^{12}\text{C}$ reaction liberates on average an energy of $\overline{Q}_C \approx 10$ MeV (see later). The energy generation rate during hydrostatic carbon burning is then given

by Eq. (3.63),

$$\varepsilon_{\rm C} = \frac{\overline{Q}_{\rm C}}{\rho} r_{^{12}{\rm C}+^{12}{\rm C}} = \frac{\overline{Q}_{\rm C}}{\rho} \frac{(N_{^{12}{\rm C}})^2 \langle \sigma v \rangle_{^{12}{\rm C}+^{12}{\rm C}}}{2} = \frac{N_A \overline{Q}_{\rm C}}{288} X_{^{12}{\rm C}}^2 \rho N_A \langle \sigma v \rangle_{^{12}{\rm C}+^{12}{\rm C}}$$
$$= 2.09 \times 10^{22} X_{^{12}{\rm C}}^2 \rho N_A \langle \sigma v \rangle_{^{12}{\rm C}+^{12}{\rm C}} \qquad (\text{MeV g}^{-1} \text{ s}^{-1})$$
(5.128)

with $N_A \langle \sigma v \rangle_{^{12}C+^{12}C}$ the total $^{12}C + ^{12}C$ reaction rate. The temperature dependence of the rates for the $^{12}C + ^{12}C$ reaction and of the energy generation during carbon burning is obtained by using the expression for nonresonant reactions (see Eq. (3.87)). From Eq. (3.88) we find near a typical temperature of $T_0 = 0.9$ GK a value of $\tau = 87$ and thus, neglecting electron screening,

$$\varepsilon_{C}(T) = \varepsilon_{C}(T_{0}) \left(T/T_{0}\right)^{(87-2)/3} = \varepsilon_{C}(T_{0}) \left(T/T_{0}\right)^{28}$$
(5.129)

The total energy released during carbon burning can be found from Eq. (3.68),

$$\int \varepsilon_{\rm C}(t) \, dt = \frac{N_A \overline{Q}_{\rm C}}{2M_{^{12}\rm{C}}} \Delta X_{^{12}\rm{C}} = 2.51 \times 10^{^{23}} \Delta X_{^{12}\rm{C}} \qquad ({\rm MeV/g}) \tag{5.130}$$

where ΔX_{12} is the mass fraction of the consumed carbon fuel.

We will discuss in the following the results obtained by solving an appropriate reaction network for constant temperature and density. In a given star, carbon burning will take place over a range of temperatures and densities, but during most of the carbon consumption the variations in temperature and density are relatively small (Fig. 5.1a). This simplification will provide a reasonable estimate of the nucleosynthesis (see also Arnett and Truran 1969). Values of *T* = 0.9 GK and $\rho = 10^5$ g/cm³ are chosen for the temperature and density, respectively. These are close to the results obtained by stellar model calculations for core carbon burning in stars with an initial mass of $M = 25 M_{\odot}$ and with initial solar metallicity (Woosley, Heger and Weaver 2002). The initial abundances at the beginning of carbon burning are given by the composition of the ashes of the preceding core helium helium-burning stage (Section 5.3). We expect mainly ¹²C and ¹⁶O, with smaller amounts of ²⁰Ne (Section 5.3.2) and ²²Ne (Section 5.3.3). Small traces of other elements may also be present but will be neglected in the following for the sake of simplicity. We assume values of $X_{12C}^0 = 0.25$, $X_{16O}^0 = 0.73$, $X_{20Ne}^0 = 0.01$, and $X_{22Ne}^0 = 0.01$ that are similar to those reported in Arnett (1996). The network is solved until the carbon fuel is exhausted ($X_{12C} < 10^{-3}$). The electron screening correction factor for the ${}^{12}C + {}^{12}C$ reaction amounts to ≈ 1.2 for the *T*- ρ conditions adopted here (Problem 3.8).

Net abundance flows and the abundance evolutions of selected nuclides are shown in Fig. 5.44. The dominant abundance flows are due to the primary ${}^{12}C({}^{12}C,p){}^{23}Na$ and ${}^{12}C({}^{12}C,\alpha){}^{20}Ne$ reactions and a large fraction of the liberated protons and α -particles are consumed by the secondary ${}^{23}Na(p,\alpha){}^{20}Ne$ and ${}^{16}O(\alpha,\gamma){}^{20}Ne$ reactions. Weaker, but still substantial, flows are due to the (p,γ) reactions on ²¹Ne, ²²Ne, ²³Na, ²⁵Mg, ²⁶Mg, the (α,γ) reaction on 20 Ne, the (α ,n) reactions on 13 C, 21 Ne, 22 Ne, the (n,p) reaction on 22 Na, and the β^+ -decay of ${}^{26}\text{Al}^m$. The primary ${}^{12}\text{C}({}^{12}\text{C},n)^{23}\text{Mg}$ reaction is too weak to influence the nucleosynthesis significantly. Removing this link from the network has only minor effects on the abundances of the major isotopes. However, this reaction may become important at higher temperatures typical of shell carbon burning. The most important source of neutrons is the 22 Ne(α ,n) 25 Mg reaction, with a smaller contribution coming from ²¹Ne(α ,n)²⁴Mg. The liberated neutrons initiate a number of neutron-induced processes, including (n,γ) reactions on ¹²C, ²⁰Ne, ²³Na, ²⁴Mg, and ²⁵Mg. On the other hand, the ${}^{13}C(\alpha,n){}^{16}O$ reaction is not a net producer of neutrons since the species ¹³C is mainly produced via ${}^{12}C(n,\gamma){}^{13}C$ and hence one neutron is consumed for each neutron liberated by the (α,n) reaction on ¹³C. Neutron-induced nucleosynthesis will be discussed in Section 5.6. The neutron excess parameter η (Section 1.8) increases slightly because of the sequence 20 Ne(n, γ) 21 Ne(p, γ) 22 Na(n,p) 22 Ne(α ,n) 25 Mg(p, γ) 26 Al($\beta^+\nu$) 26 Mg. Note that even in a star with zero initial metallicity the neutron excess will increase during core carbon burning because of the sequence ${}^{12}C({}^{12}C,n){}^{23}Mg(\beta+\nu){}^{23}Na$.

The number of free protons, α -particles, and neutrons is very small during carbon burning. At maximum, their mass fraction is only $X_{\rm H} = 5 \times 10^{-16}$, $X_{4\text{He}} = 2 \times 10^{-11}$, and $X_n = 2 \times 10^{-19}$. This circumstance has important implications. First, nuclear reactions involving radioactive target nuclei are unimportant during the nucleosynthesis (Fig. 5.44) despite the fact that the temperature is relatively high and, thus, the rates $N_A \langle \sigma v \rangle$ for many proton and α -particle-induced reactions are rather large. The decay constant for the destruction of nucleus 0 by a reaction with light particle 1 depends on the mass fraction X_1 (see Eq. (3.23)). Since X_1 is a very small number, a radioactive nucleus will β -decay rather than undergo a reaction, that is, $\lambda_{\beta}(0) \gg \lambda_1(0)$. Second, for the $T-\rho$ conditions adopted here, the photodisintegration of ¹³N prevents ¹³C production via the sequence ${}^{12}C(p,\gamma){}^{13}N(\beta^+\nu){}^{13}C$ ($Q_{12}C+p$ = 1944 keV). The decay constants for the β^+ -decay and the photodisintegration of ¹³N amount to $\lambda_{\beta}(^{13}N) = 1.2 \times 10^{-3} \text{ s}^{-1}$ and $\lambda_{\gamma}(^{13}N) = 5.2 \times 10^{1} \text{ s}^{-1}$, respectively. Hence, an equilibrium between ¹²C and ¹³N is quickly established. The equilibrium abundance ratio and the decay constant $\lambda_{12} \longrightarrow \lambda_{13} \longrightarrow \lambda_{13}$ are directly proportional to the mass fraction of protons (see Eqs. (3.49) and (3.62)). Since the proton mass fraction is very small at all times during the nucleosynthesis, the flow through ${}^{12}C(p,\gamma){}^{13}N(\beta^+\nu){}^{13}C$ becomes negligible. For lower core carbon burning temperatures typical of stars with smaller masses, however, the photodisintegration of ¹³N is less important and the sequence ${}^{12}C(p,\gamma){}^{13}N(\beta^+\nu){}^{13}C(\alpha,n)$ may become the dominant neutron source and may give rise to a significant increase in the neutron excess parameter η (Arnett and Thielemann 1985).

While the ¹²C fuel is consumed, most of the initially present ¹⁶O nuclei survive until the end of the calculation. The ²²Ne abundance also declines but the abundances of many other isotopes increase steadily (Fig. 5.44). With progressing time, the number of liberated protons and α -particles that are available for capture by various nuclides decreases and the nucleosynthesis slows down. Beyond $t = 10^{10}$ s, the abundances of the major nuclides change little. The ¹²C fuel is exhausted ($X_{12C} < 0.001$) after ≈ 1600 y ($t = 5 \times 10^{10}$ s). The total nuclear energy generated amounts to 6.3×10^{22} MeV/g, in agreement with the value obtained from Eq. (5.130) where we assumed an average energy release of ≈ 10 MeV per primary ${}^{12}C + {}^{12}C$ reaction. The most abundant nuclei at the end of the calculation are ${}^{16}O(X_f = 0.60)$, ${}^{20}Ne(X_f = 0.35)$, ${}^{24}Mg$ $(X_f = 0.023)$ and ²³Na $(X_f = 0.014)$. Many other isotopes with A < 20 and $A \ge 28$ are produced with mass fractions less than $X = 5 \times 10^{-4}$ and their abundance evolutions are not displayed in Fig. 5.44. Hydrostatic and explosive carbon burning are major sources of the nuclides ^{20,21}Ne, ²³Na, ^{24,25,26}Mg, ^{26,27}Al, ^{29,30}Si, and ³¹P in the Universe (Table 5.2). See also Arnett (1996), and Chieffi, Limongi and Straniero (1998).

We already commented on the experimental situation for the primary ¹²C + ¹²C reactions. Some information regarding important secondary reactions is summarized below. The important neutron sources ${}^{13}C(\alpha,n){}^{16}O$ and 22 Ne(α ,n) 25 Mg will be discussed Section 5.6.1. We need to consider first the location of the Gamow peaks. At a temperature of T = 0.9 GK one obtains, for example, $E_0 \pm \Delta/2 = 555 \pm 240$ keV and $E_0 \pm \Delta/2 = 1250 \pm 360$ keV for the 23 Na + p and 20 Ne + α reactions, respectively. Similar values are found for other proton or α -particle-induced reactions. The rates for the proton-induced reactions on ²¹Ne, ²²Ne, ²³Na, ²⁵Mg, and ²⁶Mg are displayed in Fig. 5.18. The branching ratio $B_{p\alpha/p\gamma}$ for ²³Na is shown in Fig. 5.17. The lowest lying observed resonances in the 21 Ne $(p,\gamma){}^{22}$ Na, 23 Na $(p,\gamma){}^{24}$ Mg, 23 Na $(p,\alpha){}^{20}$ Ne, 25 Mg(p, γ) 26 Al, and 26 Mg(p, γ) 27 Al reactions are located at $E_r^{cm} = 120, 241, 170,$ 190, and 149 keV. Hence, direct measurements cover entirely the region of the Gamow peak at $T \approx 0.9$ GK. (See also the information given at the end of Section 5.1.3.) The ²²Ne(p,γ)²³Na reaction has only been measured directly down to a resonance energy of $E_r^{cm} = 417$ keV but the expected lower lying resonances do no influence the reaction rates at T = 0.9 GK (Hale et al. 2001). The lowest lying resonances in the ${}^{16}O(\alpha, \gamma){}^{20}Ne$, and ${}^{20}Ne(\alpha, \gamma){}^{24}Mg$ reactions are located at E_r^{cm} = 893 keV and 799 keV, respectively (Section 5.1.3 and Angulo et al. 1999), and the region of the Gamow peak has been covered by direct measurements. At T = 0.9 GK, typical errors for the rates of the above protonand α -particle-induced reactions amount to \approx 10–30%, in contrast to the situation at much lower temperatures where reaction rate errors can amount to several orders of magnitude (Section 5.1.3).

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Fig. 5.44 Time-integrated net abundance flows (top) and abundance evolutions (bottom) for a constant temperature and density of T = 0.9 GK and $\rho = 10^5$ g/cm³, respectively. Such conditions are typical of core carbon burning in stars with an initial mass of $M = 25 M_{\odot}$ and with initial solar metallicity. The reaction network is solved numerically until the carbon fuel is exhausted $(X_{12C} < 10^{-3} \text{ after} \approx 1600 \text{ y})$. The magnitude of the abundance flows is represented

by arrows of three different thicknesses: $F^{\max} \ge F_{ij} > 0.1 F^{\max}$ (thick arrows), $0.1 F^{\max} \ge F_{ij} > 0.01 F^{\max}$ (intermediate arrows), and $0.01 F^{\max} \ge F_{ij} > 0.001 F^{\max}$ (thin arrows). The key indicates the type of reaction represented by a specific arrow. Stable nuclides are shown as shaded squares. The ¹⁶O(α, γ)²⁰Ne reaction is obscured by ¹²C(¹²C, α)²⁰Ne in the top part of the figure.

5.5.2 Neon Burning

At the end of core carbon burning, when most of the ¹²C nuclei have been consumed, the core consists mainly of ¹⁶O, ²⁰Ne, ²³Na, and ²⁴Mg. Other nuclides will be present as well, but with much smaller abundances (X_i < 5×10^{-3} ; see Fig. 5.44). The core contracts gravitationally and the temperature and density both increase (Fig. 5.1a). It is reasonable to assume that the next nuclear fuel to ignite is oxygen via the ${}^{16}O + {}^{16}O$ fusion reaction. However, before this happens the temperature has risen to values where photodisintegration reactions will become important (T > 1 GK). The proton, neutron, and α -particle separation energies of the above nuclei are in the range of \approx 7–17 MeV and, therefore, they are rather inert against photodisintegration even at high temperatures. The exception is ²⁰Ne, which has a relatively small α -particle separation energy of 4.73 MeV. For a typical temperature of T = 1.5 GK, the photodisintegration decay constant of ²⁰Ne can be calculated from Eq. (3.45) by using the rate of the (forward) ${}^{16}O(\alpha,\gamma)^{20}Ne$ reaction (Angulo et al. 1999). The result is $\lambda_{\gamma}(^{20}\text{Ne}) = 1.5 \times 10^{-6} \text{ s}^{-1}$ and hence the ^{20}Ne nuclei will photodisintegrate. The liberated α -particles, in turn, induce secondary reactions involving any of the more abundant nuclei. The rates for the most important α -particle-consuming reactions are shown in Fig. 5.45. Recall that the decay constant of α -particles for destruction by a reaction with nucleus 1 is given by $\lambda_1(\alpha) = \rho(X_1/M_1)N_A \langle \sigma v \rangle_{\alpha 1}$ (see Eq. (3.23)). For typical values of temperature and density (*T* = 1.5 GK and ρ = 5 × 10⁶ g/cm³; see below) and assuming an isotopic composition obtained at the end of carbon burning (Fig. 5.44), one finds decay constants of $\lambda_{16O(\alpha,\gamma)}(\alpha) = 2.3 \times 10^4 \text{ s}^{-1}$, $\lambda_{20}_{Ne(\alpha,\gamma)}(\alpha) = 1.6 \times 10^4 \text{ s}^{-1}, \lambda_{23}_{Na(\alpha,p)}(\alpha) = 5.7 \times 10^3 \text{ s}^{-1}, \text{ and } \lambda_{24}_{Mg(\alpha,\gamma)}(\alpha) = 1.6 \times 10^4 \text{ s}^{-1}, \lambda_{23}_{Na(\alpha,p)}(\alpha) = 1.6 \times 10^4 \text{ s}^{-1}, \lambda_{23}_{Na$ 4.1×10^2 s⁻¹. Hence, some of the α -particles will be captured by ¹⁶O, synthesizing again ²⁰Ne. But there is also a good chance that the liberated α -particles will be consumed by reactions such as ${}^{20}Ne(\alpha,\gamma){}^{24}Mg$, ${}^{23}Na(\alpha,p){}^{26}Mg$, or 24 Mg(α, γ)²⁸Si. A number of other α -particle-induced reactions will occur that release protons and neutrons, and these light particles will also participate in the nucleosynthesis. Details will be discussed below.

To summarize, the network of reactions consisting of the primary reaction

²⁰Ne(
$$\gamma, \alpha$$
)¹⁶O (Q = -4730 keV) (5.131)

and subsequent secondary reactions, among which we expect prominantly

²⁰Ne(
$$\alpha, \gamma$$
)²⁴Mg(α, γ)²⁸Si ($Q_{20}_{Ne(\alpha, \gamma)} = 9316 \text{ keV}$)
($Q_{24}_{Mg(\alpha, \gamma)} = 9984 \text{ keV}$) (5.132)

²³Na(
$$\alpha$$
, p)²⁶Mg(α , n)²⁹Si ($Q_{23}Na(\alpha,p) = 1821 \text{ keV}$)

$$(Q_{26}Mg(\alpha,n) = 34 \,\text{keV}) \tag{5.133}$$



Fig. 5.45 Rates for α -particle-induced reactions on ¹⁶O, ²⁰Ne, ²³Na and ²⁴Mg versus temperature. The vertical dashed line indicates a temperature of *T* = 1.5 GK which is typical for core neon burning in massive stars.

is referred to as *neon burning*. The primary reaction is endothermic, that is, it consumes energy. In combination with the subsequent secondary reactions, however, there is a net production of energy for each ²⁰Ne nucleus destroyed by photodisintegration, as will be shown below. Typical temperatures during core neon burning are in the range of T = 1.2–1.8 GK, with somewhat higher values during hydrostatic shell neon burning. Explosive neon burning takes place in the range of T = 2.5–3.0 GK.

The two most important energy-generating reactions are ${}^{20}\text{Ne}(\gamma,\alpha){}^{16}\text{O}$ and ${}^{20}\text{Ne}(\alpha,\gamma){}^{24}\text{Mg}$. An energy level diagram is displayed in Fig. 5.46. At T = 1.5 GK, the ${}^{20}\text{Ne}(\gamma,\alpha){}^{16}\text{O}$ reaction proceeds mainly through ${}^{20}\text{Ne}$ levels at $E_x = 5621$ keV and 5788 keV (see Problem 3.6), while the most important ${}^{24}\text{Mg}$ levels for the ${}^{20}\text{Ne}(\alpha,\gamma){}^{24}\text{Mg}$ reaction are located at $E_x = 10680$ keV, 10917 keV, and 11016 keV (Endt 1990). The rearrangement effectively converts two ${}^{20}\text{Ne}$ nuclei to ${}^{16}\text{O}$ and ${}^{24}\text{Mg}$. Thus, we have

$$^{20}\text{Ne} + ^{20}\text{Ne} \to {}^{16}\text{O} + {}^{24}\text{Mg} + 4586 \text{ keV}$$
 (5.134)

where the value for the energy release is obtained either from Eq. (1.10) or from $Q_{20}_{Ne(\gamma,\alpha)} + Q_{20}_{Ne(\alpha,\gamma)}$. Other secondary reactions contribute to the energy production as well. It can be estimated from network calculations that each $^{20}Ne + ^{20}Ne$ conversion liberates on average an energy of $\overline{Q}_{Ne} \approx 6.2$ MeV near $T \approx 1.5$ GK (see below). For the total energy release during neon burning



Fig. 5.46 Energy level diagrams for the most important nuclides participating in neon burning. Numbers in square brackets represent reaction Q-values (Audi, Wapstra and Thibault 2003; see also caption to Fig. 5.30). Excitation energies and quantum numbers are from Tilley et al. (1998) or Endt (1990). Levels that are not important for neon burning are omitted from the figure.

one finds from Eq. (3.68)

$$\int \varepsilon_{\rm Ne}(t) dt = \frac{N_A \overline{Q}_{\rm Ne}}{2M_{20}_{\rm Ne}} \Delta X_{20}_{\rm Ne} = 9.32 \times 10^{22} \Delta X_{20}_{\rm Ne} \qquad ({\rm MeV/g}) \qquad (5.135)$$

where ΔX_{20}_{Ne} is the mass fraction of the consumed ²⁰Ne fuel. Compared to carbon burning, the total energy release is a factor of \approx 3 smaller for the same amount (by mass) of consumed fuel.

An approximate analytical expression for the energy generation rate during hydrostatic neon burning can be found by assuming an ¹⁶O + $\alpha \leftrightarrow$ ²⁰Ne + γ equilibrium. Note that the values of $\lambda_{16O(\alpha,\gamma)}(\alpha)$, $\lambda_{20Ne(\alpha,\gamma)}(\alpha)$, and $\lambda_{23Na(\alpha,p)}(\alpha)$ quoted above were obtained with $X_{16O} = 0.60$, $X_{20Ne} = 0.35$, and $X_{23Na} = 0.014$, respectively. The ²⁰Ne and ²³Na abundances, however, decline during neon burning while the ¹⁶O abundance increases (see later). Hence, ¹⁶O(α,γ)²⁰Ne will be the dominant α -particle consuming reaction and the assumption of an ¹⁶O + $\alpha \leftrightarrow$ ²⁰Ne + γ equilibrium is justified. The energy

generation rate is then given by (see Problem 5.5)

$$\varepsilon_{\rm Ne} \approx 6.24 \times 10^{33} \frac{(X_{20\rm Ne})^2}{X_{16\rm O}} T_9^{3/2} e^{-54.89/T_9} N_A \langle \sigma v \rangle_{20\rm Ne(\alpha,\gamma)} \qquad ({\rm MeV\,g^{-1}\,s^{-1}})$$
(5.136)

and is independent of the density. The reaction rate for ²⁰Ne(α,γ)²⁴Mg above T = 1 GK can be described by the analytical expression $N_A \langle \sigma v \rangle_{^{20}\text{Ne}(\alpha,\gamma)} = 3.74 \times 10^2 T_9^{2.229} \exp(-12.681/T_9)$ (Angulo et al. 1999). For the temperature dependence of the energy generation rate during neon burning one finds from Eq. (5.136)

$$\varepsilon_{\rm Ne} \sim T_9^{1.5} T_9^{2.229} e^{-54.89/T_9} e^{-12.681/T_9} \sim T_9^{3.729} T_9^{67.57/T_9}$$
 (5.137)

where the term $\exp(-67.57/T_9)$ is derived according to the method described by Eqs. (3.82)–(3.87). Near $T_0 \approx 1.5$ GK we find

$$\varepsilon_{\rm Ne}(T) = \varepsilon_{\rm Ne}(T_0) \left(T/T_0\right)^{49} \tag{5.138}$$

and thus neon burning is very temperature sensitive.

Network calculations for neon burning are performed for a constant temperature of T = 1.5 GK and density of $\rho = 5 \times 10^6$ g/cm³. These values are close to those obtained from stellar model calculations for core neon burning in stars with an initial mass of $M = 25 M_{\odot}$ and with initial solar metallicity (Woosley, Heger and Weaver 2002). For the initial abundances at the beginning of core neon burning we adopt the final abundances obtained at the end of core carbon burning, that is, mainly ¹⁶O ($X_i = 0.60$) and ²⁰Ne ($X_i = 0.35$), with smaller contributions from nuclides in the ²¹Ne–²⁸Si range (Fig. 5.44). The network is solved until the neon fuel is exhausted ($X_{20Ne} < 0.0015$).

Net abundance flows are shown in Fig. 5.47. The dominant flows are due to the reactions ${}^{20}\text{Ne}(\gamma,\alpha){}^{16}\text{O}$ and ${}^{20}\text{Ne}(\alpha,\gamma){}^{24}\text{Mg}(\alpha,\gamma){}^{28}\text{Si}$, consistent with our earlier discussion. Smaller, but substantial, flows are caused by ${}^{24}\text{Mg}(\alpha,p){}^{27}\text{Al}(\alpha,p){}^{30}\text{Si}$, and ${}^{23}\text{Na}(\alpha,p){}^{26}\text{Mg}$. The released protons initiate a number of different reactions, most notably ${}^{26}\text{Mg}(p,\gamma){}^{27}\text{Al}$, ${}^{23}\text{Na}(p,\alpha){}^{20}\text{Ne}$, and ${}^{25}\text{Mg}(p,\gamma){}^{26}\text{Al}(\beta^+\nu){}^{26}\text{Mg}$. Neutrons are produced by the ${}^{21}\text{Ne}(\alpha,n){}^{24}\text{Mg}$, ${}^{25}\text{Mg}(\alpha,n){}^{28}\text{Si}$, and ${}^{26}\text{Mg}(\alpha,n){}^{29}\text{Si}$ reactions. The liberated neutrons undergo (n,γ) reactions involving mainly ${}^{20}\text{Ne}$, ${}^{24}\text{Mg}$, and ${}^{28}\text{Si}$. At maximum, the mass fractions of the light particles amount to $X_{\rm H} = 2 \times 10^{-17}$, $X_{4\rm He} = 1 \times 10^{-12}$, and $X_{\rm n} = 1 \times 10^{-21}$. Changes in the neutron excess parameter η are relatively small during neon burning (Thielemann and Arnett 1985). Note that for the adopted temperature and density conditions the stellar decay constants for some β -decays differ significantly from their terrestrial values (Section 1.8.4). For example, the laboratory half-life for ${}^{24}\text{Na}(\beta^-\nu){}^{24}\text{Mg}$ amounts to $T_{1/2} =$

15 h compared to $T_{1/2}$ = 0.52 h at neon-burning conditions (Fuller, Fowler and Newman 1982).

The evolution of the most abundant nuclides, except for ²¹Ne, ²²Ne, and ²³Na, is also displayed in Fig. 5.47. The latter three nuclides are quickly depleted from their initial abundance values. While the ²⁰Ne fuel is gradually consumed, the ¹⁶O abundance increases with time. Most of the other nuclides displayed in Fig. 5.47 also increase in abundance, except ²⁵Mg and ²⁶Mg whose abundances change little during the nucleosynthesis. After t = 2×10^6 s, the number of liberated protons, α -particles, and neutrons that are available for capture by various nuclides decreases and the nucleosynthesis slows down. Beyond $t = 1.8 \times 10^7$ s the abundances of the major nuclides change little. The 20 Ne fuel is exhausted (X_{20}_{Ne} < 0.0015) after 280 d (t = 2.4×10^7 s). The total nuclear energy generated amounts to 3.3×10^{22} MeV/g. The most abundant nuclei at the end of the calculation are 16 O (X_f = 0.77), 24 Mg ($X_f = 0.11$), and 28 Si ($X_f = 0.083$), while nuclides in the 25 Mg $-^{32}$ S region have final mass fractions in the range of $X_f = 0.002-0.01$. All other nuclides that are not shown in the figure have mass fractions of $X \le 10^{-4}$ throughout the calculation. Similar results are obtained from a more involved stellar model simulation (Arnett 1996). Explosive neon burning is a major source of the nuclides ²⁶Al and ³³S in the Universe (Table 5.2).

At a temperature of T = 1.5 GK, the Gamow peaks for the reactions ${}^{20}\text{Ne}(\alpha,\gamma){}^{24}\text{Mg}$, ${}^{24}\text{Mg}(\alpha,\gamma){}^{28}\text{Si}$, ${}^{23}\text{Na}(p,\alpha){}^{20}\text{Ne}$, and ${}^{25,26}\text{Mg}(p,\gamma){}^{26,27}\text{Al}$ are located at $E_0 \pm \Delta/2 = 1760 \pm 550$ keV, 2010 ± 590 keV, 780 ± 370 keV, and 830 ± 380 keV, respectively. For the ${}^{16}\text{O}(\alpha,\gamma){}^{20}\text{Ne}$ reaction (that is, the reverse of ${}^{20}\text{Ne}(\gamma,\alpha){}^{16}\text{O}$; see Problem 5.5), we obtain $E_0 \pm \Delta/2 = 1500 \pm 510$ keV at T = 1.5 GK. All these reactions have been measured directly over the Gamow peak region. Near this temperature, the reaction rates for ${}^{16}\text{O}(\alpha,\gamma){}^{20}\text{Ne}$, ${}^{23}\text{Na}(p,\alpha){}^{20}\text{Ne}$, ${}^{25}\text{Mg}(p,\gamma){}^{26}\text{Al}$, and ${}^{26}\text{Mg}(p,\gamma){}^{27}\text{Al}$ have errors of < 20% (Angulo et al. 1999, Iliadis et al. 2001). The important ${}^{20}\text{Ne}(\alpha,\gamma){}^{24}\text{Mg}$ and ${}^{24}\text{Mg}(\alpha,\gamma){}^{28}\text{Si}$ reaction rates, however, may be subject to systematic errors on the order of a factor of ≈ 2 , as can be seen from the different results reported by Caughlan and Fowler (1988), Angulo et al. (1999), and Rauscher et al. (2000).

5.5.3 Oxygen Burning

After the neon fuel has been consumed, the most abundant nuclei in the stellar core are ¹⁶O, ²⁴Mg, and ²⁸Si (Fig. 5.47). The core contracts and the temperature increases until the burning of the next fuel starts to generate energy. Among the particle-induced reactions induced by combinations of the above nuclei, the ¹⁶O + ¹⁶O fusion reaction is the most likely process to occur since it has the

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Fig. 5.47 Time-integrated net abundance flows (top) and abundance evolutions (bottom) for a constant temperature and density of T = 1.5 GK and $\rho = 5 \times 10^6$ g/cm³, respectively. Such conditions are typical of core neon burning in stars with an initial mass of $M = 25~M_{\odot}$ and with initial solar metallicity. The reaction network is solved numerically until the neon fuel is exhausted ($X_{20}_{\rm Ne} < 0.0015$ after ≈ 280 d). The arrows in the top part have the same meaning as in Fig. 5.44.

16 .16

lowest Coulomb barrier (Hoyle 1954, Cameron 1959). The situation resembles carbon burning, in the sense that a reaction induced by two heavy nuclei in the incoming reaction channel ($^{16}O + {}^{16}O$) is the primary process sustaining the nuclear burning. The ^{32}S compound nucleus formed in the fusion of two ^{16}O nuclei is highly excited, with the mass difference between $^{16}O + {}^{16}O$ and ^{32}S amounting to ≈ 16.5 MeV. The fusion reaction will then involve many overlapping ^{32}S compound levels. Excess energy is most effectively removed by emission of light massive particles (as opposed to the emission of primary γ -rays). In contrast to the ${}^{12}C + {}^{12}C$ reaction, there are many more exit channel possible for the ${}^{16}O + {}^{16}O$ reaction since the ${}^{32}S$ compound nucleus achieves much higher excitation energies (Spinka and Winkler 1974). The most likely *primary* reactions are

$^{10}O(^{10}O, p)^{31}P$	$(Q = 7678 \mathrm{keV})$	(5.139)

(Q = 381 keV) (5.14)	$O(^{16}O, 2p)^{30}Si$	$(Q = 381 \mathrm{keV})$	(5.140)
-----------------------	------------------------	--------------------------	---------

¹⁶O(¹⁶O,
$$\alpha$$
)²⁸Si (Q = 9594 keV) (5.141)
¹⁶O(¹⁶O, 2α)²⁴Mg (Q = -390 keV) (5.142)

$${}^{16}O({}^{16}O, 2\alpha)^{-1}Mg \qquad (Q = -390 \text{ keV}) \qquad (5.142)$$

$${}^{16}O({}^{16}O, d)^{30}P \qquad (Q = -2409 \text{ keV}) \qquad (5.143)$$

$$(Q = 240 \text{ keV})$$
 (5.143)
 $(Q = 1499 \text{ keV})$ (5.144)

The ¹⁶O(¹⁶O,d)³⁰P and ¹⁶O(¹⁶O,2 α)²⁴Mg reactions are endothermic, that is, they can only occur above a threshold energy of $E_{\rm cm} = -Q$. Note that the deuterons released in the ¹⁶O(¹⁶O,d)³⁰P reaction will be immediately photodisintegrated (d + $\gamma \rightarrow$ p + n) at elevated stellar temperatures. The liberated light particles are quickly consumed by *secondary* reactions involving, for example, the ashes of neon burning and the heavy product nuclei of the primary reactions. This network of primary and secondary reactions is referred to as *oxygen burning*. Typical temperatures during core oxygen burning are in the range of *T* = 1.5–2.7 GK, depending on the stellar mass, with somewhat higher values during shell oxygen burning. In explosive oxygen burning, temperatures of *T* = 3–4 GK are achieved.

It should be pointed out that the photodisintegrations of the nuclei ¹⁶O, ²⁴Mg, and ²⁸Si do not contribute significantly to the nuclear energy generation during hydrostatic oxygen burning. Their proton, neutron, and α -particle separation energies exceed \approx 9 MeV, except the α -particle separation energy of ¹⁶O which amounts to 7.2 MeV. Hence, the ¹⁶O(γ , α)¹²C reaction is the most likely process to occur among these photodisintegrations. The decay constants λ_i (¹⁶O) for the reactions ¹⁶O + ¹⁶O and ¹⁶O(γ , α)¹²C are shown in Fig. 5.48 versus temperature. The decay constant for ¹⁶O + ¹⁶O is obtained from λ_{16O} (¹⁶O) = $\rho(X_{16O}/M_{16O})N_A\langle\sigma v\rangle$ (see Eq. (3.23)), assuming values of ρ = 3 × 10⁶ g/cm³ and X_{16O} = 0.5, whereas λ_{γ} (¹⁶O) is calculated from the forward ¹²C(α , γ)¹⁶O



Fig. 5.48 Decay constants $\lambda_i({}^{16}\text{O})$ for the ${}^{16}\text{O} + {}^{16}\text{O}$ and ${}^{16}\text{O}(\gamma,\alpha){}^{12}\text{C}$ reactions versus temperature. The curve for ${}^{16}\text{O} + {}^{16}\text{O}$ is calculated assuming $\rho = 3 \times 10^6$ g/cm³ and $X_{16}{}_{O} = 0.5$, while the one for ${}^{16}\text{O}(\gamma,\alpha){}^{12}\text{C}$ is derived from the ${}^{12}\text{C}(\alpha,\gamma){}^{16}\text{O}$ reaction rate and is independent of ρ and $X_{16}{}_{O}$. The dashed vertical line indicates a temperature of T = 2.2 GK, which is typical for core oxygen burning in massive stars.

reaction rate and is independent of ρ and X_{16O} (see Eq. (3.45)). Under these conditions, the ¹⁶O + ¹⁶O fusion is more likely to occur at temperatures below T = 4 GK and thus will be the dominant ¹⁶O depleting process during hydrostatic oxygen burning. During explosive oxygen burning, however, the ¹⁶O + ¹⁶O and ¹⁶O(γ , α)¹²C reactions can occur at comparable rates.

The total ${}^{16}\text{O} + {}^{16}\text{O}$ S-factor is shown in Fig. 5.49. The reaction has been measured down to a bombarding energy of $E_{\rm cm} \approx 6.8$ MeV. The height of the Coulomb barrier is \approx 13 MeV and, as was the case in Section 5.5.1, the total *S*factor varies strongly with energy because the Gamow factor does not remove the energy dependence of the cross section entirely at the measured energies (see Eq. (2.124)). The two shaded bars indicate the locations of the Gamow peaks at temperatures typical for core oxygen burning ($T \approx 2.2 \text{ GK}$; $E_0 \pm \Delta/2$ = 6600 \pm 1290 keV) and explosive oxygen burning (*T* = 3.6 GK; *E*₀ \pm $\Delta/2$ = 9170 ± 1950 keV). It can be seen that the data reach down to the center of the Gamow peak ($E_{\rm cm} \approx E_0$) for T = 2.2 GK, whereas the data cover entirely the Gamow peak region for T = 3.6 GK. As expected, the total S-factor varies smoothly with energy since the ${}^{16}O + {}^{16}O$ reaction proceeds through many overlapping resonances at each bombarding energy. In particular, the unexplained cross-section fluctuations observed in the total S-factor of $^{12}\mathrm{C}$ + $^{12}\mathrm{C}$ (Fig. 5.42) are absent in the ${}^{16}O + {}^{16}O$ data. The various measurements are in poor agreement at the lower energies ($E_{cm} < 8$ MeV). The data have been fitted by a number of methods (see, for example, Wu and Barnes 1984). The



Fig. 5.49 Total *S*-factor for the ¹⁶O + ¹⁶O reaction. The data are adopted from Spinka and Winkler (1974), Hulke, Rolfs and Trautvetter (1980), Wu and Barnes (1984), and Thomas et al. (1986). The data from Thomas et al. (1986) are extracted

from their Figs. 9 and 10. The open bars indicate the location of the Gamow peaks for $T \approx$ 2.2 GK (core oxygen burning) and $T \approx$ 3.6 GK (explosive oxygen burning). The solid line shows the fitted total *S*-factor adopted by Caughlan and Fowler (1988).

fitted total *S*-factor adopted by Caughlan and Fowler (1988) at the lower energies is shown in Fig. 5.49 as a solid line.

The ¹⁶O + ¹⁶O reaction populates many levels in the residual nuclei. The different reaction channels have been investigated by using a variety of techniques, including the direct detection of emitted light particles (Spinka and Winkler 1974), detection of γ -rays emitted from excited levels in the residual nuclei (Spinka and Winkler 1974, Wu and Barnes 1984, Thomas et al. 1986) and the activation method (Spinka and Winkler 1974, Wu and Barnes 1984). The cross-section data suggest significant contributions from reactions involving three particles in the exit channel, for example, ¹⁶O(¹⁶O,2p)³⁰Si or ${}^{16}O({}^{16}O,2\alpha)^{24}Mg$. Such three-particle exit channel contributions may account perhaps for a fraction of $\approx 20\%$ of the total cross section at the lowest measured energy of $E_{\rm cm} \approx 6.8$ MeV (Spinka and Winkler 1974). The available data on the partial cross sections are in poor agreement. Furthermore, little information is available on the competition between two- and threeparticle exit channels that produce the same kind of particles, for example between ${}^{16}O({}^{16}O,p){}^{31}P$ and ${}^{16}O({}^{16}O,2p){}^{30}Si$, or between ${}^{16}O({}^{16}O,\alpha){}^{28}Si$ and $^{16}O(^{16}O,2\alpha)^{24}Mg$. The average values of the reported branching ratios at E_{cm} ≈ 6.8 MeV amount to $\approx 60\%$ for ${}^{16}\text{O}({}^{16}\text{O},p){}^{31}\text{P}$ and ${}^{16}\text{O}({}^{16}\text{O},2p){}^{30}\text{Si}$, $\approx 25\%$ for ${}^{16}\text{O}({}^{16}\text{O},\alpha){}^{28}\text{Si}$ and ${}^{16}\text{O}({}^{16}\text{O},2\alpha){}^{24}\text{Mg}$, $\approx 10\%$ for ${}^{16}\text{O}({}^{16}\text{O},d){}^{30}\text{P}$, and $\approx 5\%$ for $^{16}O(^{16}O.n)^{31}S.$

The thermonuclear reaction rates for different ${}^{16}O + {}^{16}O$ exit channels are shown in Fig. 5.43 (Caughlan and Fowler 1988). The individual rates are normalized to the yields of the respective emitted light particles (p, α, n) rather than the actual number of reactions. For example, the curve labeled "¹⁶O(¹⁶O, α)" represents the reaction rate for the production of as many α -particles as are produced by the actual reactions ${}^{16}O({}^{16}O,\alpha){}^{28}Si$ and ${}^{16}O({}^{16}O,2\alpha){}^{24}Mg$ combined. Similar arguments hold for the protonand neutron-producing reaction channels. In particular, the curve labeled "¹⁶O(¹⁶O,n)" represents the reaction rate for neutron production from both ${}^{16}O({}^{16}O,n){}^{31}S$ and from deuteron breakup after ${}^{16}O({}^{16}O,d){}^{30}P$. It can be seen in Fig. 5.43 that the proton exit channel dominates at all temperatures. At T =2.2 GK, the reaction rate contributions for the emission of protons, α -particles, and neutrons are \approx 62%, \approx 21%, and \approx 17%, respectively (Caughlan and Fowler 1988). The present uncertainties in the rates of the primary oxygen burning reactions are difficult to quantify. Considering the poor agreement in the reported total ${}^{16}O + {}^{16}O$ cross section below $E_{cm} = 8$ MeV and our incomplete knowledge of the branching ratios for the different exit channels, one can conclude that the rates of the primary oxygen burning reactions at temperatures below *T* = 3 GK are uncertain by at least a factor of \approx 3.

As was the case in carbon burning (Section 5.5.1), the secondary reactions contribute significantly to the nuclear energy released by the primary oxygen burning reactions. The result of a reaction network calculation near T = 2.2 GK (see below) yields an average energy release of $\overline{Q}_{\rm O} \approx 17.2$ MeV for each ${}^{16}{\rm O} + {}^{16}{\rm O}$ reaction (see also Woosley, Heger and Weaver 2002). The energy generation rate in hydrostatic oxygen burning is then given by Eq. (3.63),

$$\varepsilon_{O} = \frac{\overline{Q}_{O}}{\rho} r_{16O+16O} = \frac{\overline{Q}_{O}}{\rho} \frac{(N_{16O})^{2} \langle \sigma v \rangle_{16O+16O}}{2} = \frac{N_{A} \overline{Q}_{O}}{512} X_{16O}^{2} \rho N_{A} \langle \sigma v \rangle_{16O+16O}$$

= 2.03 × 10²² X_{16O}^{2} \rho N_{A} \langle \sigma v \rangle_{16O+16O} (MeV g⁻¹ s⁻¹) (5.145)

with $N_A \langle \sigma v \rangle_{^{16}O+^{16}O}$ the total $^{16}O + ^{16}O$ reaction rate. The temperature dependence of the $^{16}O + ^{16}O$ rate and of the energy generation rate during oxygen burning can be found from the expression for nonresonant reactions (see Eq. (3.87)). Near a typical temperature of $T_0 = 2.2$ GK one finds a value of $\tau = 104.5$ (see Eq. (3.88)) and thus, neglecting electron screening,

$$\varepsilon_{\rm O}(T) = \varepsilon_{\rm O}(T_0) \left(T/T_0\right)^{(104.5-2)/3} = \varepsilon_{\rm O}(T_0) \left(T/T_0\right)^{34}$$
(5.146)

The total energy released during oxygen burning can be found from Eq. (3.68),

$$\int \varepsilon_{\rm O}(t) \, dt = \frac{N_A \overline{Q}_O}{2M_{16O}} \Delta X_{16O} = 3.24 \times 10^{23} \Delta X_{16O} \qquad ({\rm MeV/g}) \tag{5.147}$$

where ΔX_{16O} is the mass fraction of the consumed oxygen fuel. This value exceeds the total amount of energy released during either carbon or neon burning (see Eqs. (5.130) and (5.135)).

The results of a network calculation for a constant temperature T = 2.2 GK and density $\rho = 3 \times 10^6$ g/cm³ are shown in Fig. 5.50. These values are similar to those obtained from stellar model calculations for core oxygen burning in stars with an initial mass of $M = 25 M_{\odot}$ and initial solar metallicity (Woosley, Heger and Weaver 2002). For the initial abundances at the start of core oxygen burning: ¹⁶O ($X_i = 0.77$), ²⁴Mg ($X_i = 0.11$), and ²⁸Si ($X_i = 0.083$), with smaller contributions from nuclides in the ²⁵Mg–³²S range (Fig. 5.47). The network is solved until oxygen exhaustion ($X_{16O} < 0.001$). The electron screening correction factor for the primary ¹⁶O + ¹⁶O reaction amounts to ≈ 1.3 for the $T-\rho$ conditions adopted here.

It can be seen from Fig. 5.50 that many different nuclear processes occur during oxygen burning. First, those links with the largest net abundance flows (represented by the thickest arrows) will be described. The primary ${}^{16}O + {}^{16}O$ reactions produce ${}^{28}Si$ and ${}^{32}S$ via different sequences: (i) ${}^{16}O({}^{16}O,p){}^{31}P(p,\gamma){}^{32}S$, (ii) ${}^{16}O({}^{16}O,p){}^{31}P(p,\alpha){}^{28}Si$, (iii) ${}^{16}O({}^{16}O,\alpha){}^{28}Si$, and (iv) ${}^{16}O({}^{16}O,n){}^{31}S(\gamma,p){}^{30}P(\gamma,p){}^{29}Si(\alpha,n){}^{32}S$. The two (γ,p) reactions occur because the proton separation energies of 31 S and 30 P are relatively small (S_p = 6133 and 5595 keV, respectively) and, consequently, the photodisintegrations dominate over the competing β^+ -decays. The decay ${}^{31}S(\beta^+\nu){}^{31}P$, although weaker than ${}^{31}S(\gamma,p){}^{30}P$, is nevertheless significant, as will be seen below. Some of the ²⁸Si nuclei are converted to ³²S via ²⁸S(α, γ)³²S. A fraction of the ³²S nuclei is either transformed back to ³¹P via ³²S(n, γ)³³S(n, α)³⁰Si(p, γ)³¹P or is converted to heavier nuclei via ${}^{32}S(\alpha,p){}^{35}Cl(p,\gamma){}^{36}Ar$, and so on. Some of the liberated α -particles deplete the initially abundant ²⁴Mg nuclei via the reactions ${}^{24}Mg(\alpha,\gamma){}^{28}Si$ and ${}^{24}Mg(\alpha,p){}^{27}Al$. Reactions such as ${}^{16}O(p,\gamma){}^{17}F$, ${}^{16}O(\alpha,\gamma){}^{20}Ne$, ${}^{28}Si(p,\gamma){}^{29}P$, ${}^{32}S(p,\gamma){}^{33}Cl$, and ${}^{36}Ar(p,\gamma){}^{37}K$ do not give rise to significant net flows. Their Q-values are so small (Q = 600, 4730, 2749, 2277, and 1858 keV, respectively) that in each case the forward rate is much smaller compared to the reverse photodisintegration rate.

The evolution of the most abundant nuclides is also shown in Fig. 5.50. For reasons of clarity, the nuclides ¹⁶O, ^{24,25,26}Mg, and ²⁷Al are not displayed in the figure. They are quickly depleted with progressing time. While the oxygen fuel is being consumed, the abundances of ²⁸Si and ³²S increase with time. The abundances of ³⁴S, ³⁵Cl, ³⁶Ar, ³⁸Ar, ³⁹K, ⁴⁰Ca, and ⁴²Ca also increase, while those of ^{29,30}Si and ³¹P decrease from their initial values. The ¹⁶O fuel is exhausted after about 162 days ($t = 1.4 \times 10^7$ s). The total nuclear energy generated amounts to 2.5×10^{23} MeV/g. The most abundant nuclei at the end of the calculation are ²⁸Si ($X_f = 0.54$), ³²S ($X_f = 0.28$), ³⁸Ar ($X_f = 0.084$),

³⁴S ($X_f = 0.044$), ³⁶Ar ($X_f = 0.027$), and ⁴⁰Ca ($X_f = 0.021$), while nuclides in the ²⁹Si–⁴²Ca region have final mass fractions in the range of $X_f = 10^{-4}-10^{-3}$ (see also Arnett 1996, Chieffi, Limongi and Straniero 1998). All other nuclides not shown in the figure have mass fractions of $X < 6 \times 10^{-5}$ throughout the calculation.

The neutron excess increases significantly (by a factor of 5 in the above calculation) during core oxygen burning. The most important weak interactions that influence η are the positron decays ${}^{31}S(e^+\nu){}^{31}P$ and ${}^{30}P(e^+\nu){}^{30}Si$ and the electron captures ${}^{33}S(e^-,\nu){}^{33}P$, ${}^{35}Cl(e^-,\nu){}^{35}S$ and ${}^{37}Ar(e^-,\nu){}^{37}Cl$. In fact, the neutron excess becomes so large ($\eta \approx 0.007$) that the composition of the core matter deviates strongly from a solar system abundance distribution. The products of hydrostatic oxygen burning are completely reprocessed in the subsequent explosive oxygen (and explosive silicon) burning phase before being ejected into the interstellar medium at the end of the massive star evolution. Explosive oxygen burning is believed to be a major source of ${}^{28}Si$, ${}^{32,33,34}S$, ${}^{35}Cl$, ${}^{36,38}Ar$, ${}^{39,41}K$, and ${}^{40,42}Ca$ in the Universe (Table 5.2).

It is also interesting that some of the weak interactions compete with nuclear reactions that link the same pair of nuclei. For instance, the *net* abundance flow between ³³S and ³³P is determined by the individual flows from ³³S(e^{-}, ν)³³P, ³³S(n, p)³³P, and ³³P(p, n)³³S. In the above network calculation, the latter reaction gives rise to the largest *individual* flow among these processes, but the first two processes have a larger *combined* flow. Hence, the arrow in Fig. 5.50 points from ³³S to ³³P.

At temperatures typical of core oxygen burning, the influence of thermally excited levels on the rates of the majority of reactions is relatively small. Most reactions involve stable (or long-lived) target nuclei (Fig. 5.50) for the reasons given in Section 5.5.1 and, with few exceptions, their stellar enhancement factors R_{tt} and normalized partition functions G_i^{norm} are close to unity at $T \approx 2$ GK (see also Section 3.1.5). The situation is quite different for the weak interactions. At T = 2.2 GK and $\rho = 3 \times 10^6$ s the half-life, for example, of $^{30}\text{P}(\text{e}^+\nu)^{30}\text{Si}$ is reduced from a laboratory value of $T_{1/2} = 150$ s to a stellar value of $T_{1/2} = 84$ s. As expected, even more drastic changes occur for electron captures. The stellar half-lives for $^{33}\text{S}(\text{e}^-,\nu)^{33}\text{P}$, $^{35}\text{Cl}(\text{e}^-,\nu)^{35}\text{S}$, and $^{37}\text{Ar}(\text{e}^-,\nu)^{37}\text{Cl}$ at the above conditions amount to $T_{1/2} = 4 \times 10^5$ s, 2×10^5 s, and 2×10^4 s, respectively, while in the laboratory ^{33}S and ^{35}Cl are stable and ^{37}Ar is long-lived ($T_{1/2} = 3.0 \times 10^6$ s).

It was already pointed out that oxygen burning resembles carbon burning in the sense that the nucleosynthesis is mainly driven by the fusion of two heavy nuclei. There is, however, a fundamental difference between these two hydrostatic burning stages that is caused by the significantly higher temperature achieved in oxygen burning. In carbon burning, the number of protons, neutrons and α -particles that can be captured by various nuclei decreases toward



Fig. 5.50 Time-integrated net abundance flows (top) and abundance evolutions (bottom) for a constant temperature and density of T = 2.2 GK and $\rho = 3 \times 10^6$ g/cm³, respectively. Such conditions are typical of core oxygen burning in stars with an initial mass of $M = 25 M_{\odot}$ and with initial solar metal-

licity. The reaction network is solved numerically until the oxygen fuel is exhausted ($X_{16_{\rm O}} < 0.001$ after ≈ 162 d). The arrows in the top part have the same meaning as in Fig. 5.44. The ²⁴Mg(α, γ)²⁸Si reaction is obscured by ¹⁶O(¹⁶O, α)²⁸Si in the top part of the figure.

the end as the ¹²C fuel is consumed and, consequently, the nucleosynthesis ceases. In hydrostatic oxygen burning, on the other hand, the temperature is sufficiently high that photodisintegrations of nuclei with the smallest particle separation energies provide another source of light particles, even when the ¹⁶O fuel has been consumed. In the above network calculation, the proton and α -particle abundances are approximately constant ($X_{\rm H} \approx 10^{-13}$, $\bar{X}_{4_{\rm He}} \approx$ 10^{-11}) throughout the nucleosynthesis. There is a steady supply of light particles that undergo reactions so that less tightly bound nuclei are transformed to more stable species, as discussed in Section 3.1.4. This aspect is reflected in Fig. 5.50 where, toward the end of the calculation, the abundances of most nuclides do not stay constant but change as a result of the nuclear rearrangement. As oxygen burning proceeds, many pairs of nuclei achieve an equilibrium between forward and reverse photodisintegration rate. Several of such pairs eventually come into mutual equilibrium, giving rise to a quasiequilibrium cluster (Section 3.1.6). It has been demonstrated in stellar model calculations that, for progressing time and increasing temperature, more species join this group of nuclei (Woosley, Arnett and Clayton 1973, Chieffi, Limongi and Straniero 1998). After oxygen exhaustion and before the ignition of the next nuclear fuel, the nuclei in the A = 24-46 form one large quasiequilibrium cluster. A second cluster consisting of iron peak nuclei also starts to form at the end of oxygen burning. It originates from much heavier nuclei initially present in the star that were disregarded in the above discussion. Most of these nuclei take part in neutron-induced reactions, especially during core helium burning, but also during carbon and neon burning (Section 5.6.1). At the temperatures attained in core oxygen burning, all these heavy nuclei are destroyed by (γ, p) , (γ, α) , and (γ, n) reactions and are transformed to the most tightly bound nuclei, that is, those in the region of the iron peak (Sections 1.5.1). The physics of quasiequilibrium clusters will be described in more detail in the next section. See also Woosley, Arnett and Clayton (1972).

The experimental information for the primary ${}^{16}\text{O} + {}^{16}\text{O}$ reaction has already been presented. The secondary reactions are too numerous to be discussed in detail here. We will focus on a few secondary reactions that give rise to the largest net abundance flows (Fig. 5.50). The following discussion will at least provide an impression on the sources and the reliability of the nuclear physics information entering oxygen burning calculations. The rates of reactions such as ${}^{31}\text{P}(p,\gamma){}^{32}\text{S}$, ${}^{31}\text{P}(p,\alpha){}^{28}\text{Si}$, ${}^{35}\text{Cl}(p,\gamma){}^{36}\text{Ar}$, ${}^{30}\text{Si}(p,\gamma){}^{31}\text{P}$, ${}^{32}\text{S}(\alpha,p){}^{35}\text{Cl}$, ${}^{24}\text{Mg}(\alpha,p){}^{27}\text{Al}$, and ${}^{30}\text{P}(\gamma,p){}^{29}\text{Si}$ near T = 2.2 GK are based on directly measured resonance energies and strengths (see Iliadis et al. 2001). The rates of the latter three reverse reactions are calculated from the corresponding forward rates. Branching ratios for ${}^{31}\text{P} + p$ and ${}^{35}\text{Cl} + p$ are displayed in Fig. 5.17. Typical errors of the above reaction rates at T = 2.2 GK amount to $\pm 25\%$, except for the ${}^{32}\text{S}(\alpha,p){}^{35}\text{Cl}$ reaction for which the rates are uncertain by a factor

of ≈ 2 . The rates for the α -capture reactions ${}^{24}Mg(\alpha,\gamma){}^{28}Si$, ${}^{28}Si(\alpha,\gamma){}^{32}S$, and ${}^{32}S(\alpha,\gamma){}^{36}Ar$ are also based on direct experimental information, but may be subject to systematic errors on the order of factors of ≈ 2 –3, as can be seen from the differences in the results reported by Caughlan and Fowler (1988), and Rauscher et al. 2000. Somewhat larger errors are expected for the rates of reactions such as ${}^{31}S(\gamma,p){}^{30}P$, ${}^{33}S(n,\alpha){}^{30}Si$ and ${}^{29}Si(\alpha,n){}^{32}S$ that are based on Hauser–Feshbach statistical model calculations (Goriely 1998, Rauscher and Thielemann 2000).

5.5.4

Silicon Burning

Near the conclusion of core oxygen burning, when the ¹⁶O fuel is depleted, the most abundant nuclei are ²⁸Si and ³²S (Fig. 5.50). The stellar core contracts and the temperature increases. Fusion reactions such as ${}^{28}\text{Si} + {}^{28}\text{Si}$ or ${}^{28}\text{Si} +$ ³²S are too unlikely to occur because of Coulomb barrier considerations, even at the elevated temperatures achieved at the end of the evolution of a massive star. Instead, the nucleosynthesis proceeds via photodisintegrations of less tightly bound nuclei and the capture of the liberated light particles (protons, neutrons, and α -particles) to create gradually heavier and more tightly bound species, as described in Section 3.1.4. In the process, many forward and reverse reactions achieve equilibrium and with increasing temperature and progressing time several pairs of nuclei link together to form quasiequilibrium clusters. The overall result is another photodisintegration rearrangement process, similar to neon burning, but on a much more extensive scale. We will describe below how ²⁸Si, ³²S and other nuclei in the A = 24-46 region are gradually transformed to the most tightly bound species, that is, the iron peak nuclides (Section 1.5.1). This process provides the star with another source of energy and is referred to as *silicon burning*. Temperatures during core silicon burning are in the range of T = 2.8-4.1 GK, depending on the stellar mass, with somewhat higher values during hydrostatic shell silicon burning. Explosive silicon burning takes place in the range of T = 4-5 GK.

In this section, some of the fundamental concepts of silicon burning are discussed. For more information, the reader is referred to the pioneering work of Bodansky, Clayton and Fowler (1968) and Woosley, Arnett and Clayton (1973). Suppose first that ²⁸Si and ³²S are the only nuclear species present near the conclusion of oxygen burning. The decay constants for the photodisintegrations of both nuclei are displayed in Fig. 5.51. The curves shown are calculated from Eq. (3.45) by using the rates of the corresponding forward reactions. The photodisintegration decay constant depends strongly on the particle separation energy (or the *Q*-value of the forward reaction), as explained in Section 3.1.4 (Fig. 3.6). The proton, neutron, and α -particle separation energies of



Fig. 5.51 Decay constants for the photodisintegrations of ²⁸Si (solid lines) and ³²S (dashed lines) versus temperature. The curves are calculated from the rates of the corresponding forward reactions.

³²S and ²⁸Si amount to $S_p = 8.90$ MeV, $S_n = 15.00$ MeV, $S_\alpha = 6.95$ MeV and $S_p = 11.60$ MeV, $S_n = 17.20$ MeV, $S_\alpha = 9.98$ MeV, respectively. Hence, ³²S is the more fragile nucleus and is destroyed first. As the core temperature increases above $T \approx 2$ GK, ³²S will be consumed via ³²S(γ, α)²⁸Si and ³²S(γ, p)³¹P. The latter reaction is quickly followed by sequences, such as ³¹P(γ, p)³⁰Si(γ, n)²⁹Si(γ, n)²⁸Si, converting effectively ³²S to ²⁸Si. The destruction of ³²S already starts near the end of oxygen burning, as can be seen from Fig. 5.50.

The temperature increases further until the photodisintegration of ²⁸Si becomes substantial. That the separation energy is not the only factor determining the photodisintegration rate is clearly seen in Fig. 5.51. The decay constants for ²⁸Si(γ ,p)²⁷Al and ²⁸Si(γ , α)²⁴Mg have comparable magnitudes although the separation energy for the (γ , α) reaction is much smaller than for the competing (γ ,p) reaction. Other factors that influence sensitively the photodisintegration rate are the transmission probabilities of the photoejected charged particles through the Coulomb barrier and the reduced particle widths of the resonances through which the photodisintegration process proceeds.

The resulting nucleosynthesis that transforms Si and other intermediate mass nuclei to iron peak elements is quite complex. In order to obtain a first impression, the results of a reaction network calculation, performed at constant temperature and density, will now be discussed. Subsequently, several analytical expressions are derived in order to gain a deeper understanding of silicon burning. For the network calculation, we chose a temperature and density of T = 3.6 GK and $\rho = 3 \times 10^7$ g/cm³, respectively. These values are similar to those obtained from stellar evolution calculations for core silicon burning
in stars with an initial mass of $M = 25 M_{\odot}$ and with initial solar composition (Chieffi, Limongi and Straniero 1998, Woosley, Heger and Weaver 2002). The stellar evolution models also predict significant abundance variations at elevated temperatures (T > 2.2 GK) between the termination of oxygen burning and the ignition of silicon burning in the stellar core (Chieffi, Limongi and Straniero 1998). In particular, the ³²S abundance decreases while the abundances of ³⁰Si and ³⁴S, and hence the neutron excess parameter η , increase. It will be seen below that it is the initial value of η at the beginning of silicon burning rather than the exact initial abundance distribution that sensitively influences silicon burning nucleosynthesis. For the network calculation, initial abundances of $X_i(^{28}\text{Si}) = 0.70$ and $X_i(^{30}\text{Si}) = 0.30$ are chosen. These translate into a value of $\eta_i = 0.02$ for the initial neutron excess parameter, in approximate agreement with the results presented by Thielemann and Arnett (1985) and Chieffi, Limongi and Straniero (1998). Thermally excited levels have a profound effect at these elevated temperatures, not only on weak interaction decay constants, as already mentioned in Section 5.5.3, but also on the rates of many forward and reverse reactions through stellar enhancement factors and normalized partition functions that differ significantly from unity (Section 3.1.5). The network is solved until silicon exhaustion (X_{28} _{Si} < 0.001). The results are shown in Fig. 5.52.

The time-integrated net abundance flows F_{ij} show an interesting pattern. Recall that the flows F_{ij} are integrated over the entire time until silicon exhaustion and thus present only the gross properties of the nucleosynthesis (Section 5.4.2). Nevertheless, some of the most outstanding features of silicon burning are reflected in the global flow pattern. The fuel consists initially only of ²⁸Si and ³⁰Si. These nuclei are photodisintegrated, producing a net downward flow from ²⁴Mg to ⁴He. The recapture of the liberated protons, α -particles and neutrons gives rise to a net upward flow via a multitude of secondary reactions. A dense flow pattern in the A = 25-40 mass range, consisting of reactions such as (p,γ) , (α,γ) , (α,γ) , (α,p) , (α,n) , (n,p) and their reverses, is apparent. Nuclei in the region A = 46-64 are also linked by numerous processes, giving rise to another dense flow pattern. There is much less nuclear activity between these two groups of nuclei in the A = 40-46 region. The reader may already suspect that the two groups of nuclei referred to above (A = 25-40 and A = 46-64) represent quasiequilibrium clusters, which are linked by reactions involving nuclei in the A = 40-46 region.

The evolution of the most abundant nuclides is shown in Fig. 5.52. It is apparent how the abundances of nuclei in the A < 40 range gradually decrease (dashed lines), while at the same time the abundances of nuclei in the iron peak region increase (solid lines). Clearly, heavier and more tightly bound nuclei build up as a result of a relatively small leakage of abundance flows from the intermediate mass region toward the iron peak. The silicon fuel is



Fig. 5.52 Time-integrated net abundance flows (top) and abundance evolutions (bottom) for a constant temperature and density of T = 3.6 GK and $\rho = 3 \times 10^7$ g/cm³, respectively. Such conditions are typical of core silicon burning in stars with an initial mass of $M = 25 M_{\odot}$ and with initial solar metallicity. The reaction network is solved numerically

until the silicon fuel is exhausted ($X_{28}_{Si} < 0.001$ after ≈ 4000 s). The arrows in the top part have the same meaning as in Fig. 5.44. The abundance flows in the top part of the figure reflect the existence of two quasiequilibrium clusters in the A = 25–40 and A = 46–64 mass ranges.

exhausted after t = 4000 s ($X_{28}_{Si} < 0.001$). At the end of the calculation, most of the matter ($\approx 94\%$ by mass) has been converted to ⁵⁶Fe ($X_f = 0.56$), ⁵²Cr ($X_f = 0.19$), ⁵⁴Fe ($X_f = 0.11$), ⁵⁵Fe ($X_f = 0.050$), and ⁵³Mn ($X_f = 0.034$). Recall, that ⁵⁶Fe is one of the most tightly bound nuclei (Section 1.5.1). Similar values of final abundances have been obtained in stellar evolution calculations (Chieffi, Limongi and Straniero 1998). The abundances of free protons, α -particles and neutrons amount to $X_p \approx 10^{-7}$, $X_{\alpha} \approx 10^{-6}$, and $X_n \approx 10^{-11}$, respectively, during most of the burning.

The neutron excess (Section 1.8) remains initially constant (until $t \approx 200$ s), but increases significantly afterward, as can be seen in Fig. 5.52 from the transition of ⁵⁴Fe to ⁵⁶Fe as the most abundant species. The behavior of η reflects the fact that the weak interactions are relatively slow. They become mainly important when the iron peak nuclei are reached. The electron captures ⁵³Mn(e⁻, ν)⁵³Cr, ⁵⁴Fe(e⁻, ν)⁵⁴Mn, ⁵⁵Fe(e⁻, ν)⁵⁵Mn, ⁵⁵Co(e⁻, ν)⁵⁵Fe and 56 Co(e⁻, ν) 56 Fe have the largest impact on the evolution of η . The final neutron excess amounts to $\eta_f = 0.067$. Exactly the same values of X_f are obtained if the initial abundances are placed in sulfur or argon instead of silicon isotopes, as long as η_i is kept constant. On the other hand, a variation of η_i strongly influences the resulting composition of the iron peak species. In any case, the neutron excess becomes so large that the composition of the core matter deviates strongly from a solar abundance distribution. The products of hydrostatic silicon burning are completely reprocessed by the subsequent explosive burning phase before being (partially) ejected into the interstellar medium at the end of the massive star evolution. The core collapse and the subsequent supernova explosion depend critically on the composition, and hence the neutron excess, of the matter resulting from core silicon burning. Furthermore, we already pointed out in the discussion of previous advanced burning stages that the released thermonuclear energy is almost entirely radiated as neutrinoantineutrino pairs which are produced by thermal processes. During silicon burning, however, weak interactions contribute significantly to the neutrino losses.

The net abundance flows F_{ij} shown in Fig. 5.52 are integrated over the entire running time of the network calculation. Figure 5.52 provides us neither with information regarding abundance flows at a particular instant in time nor does it tell us which pairs (or groups) of nuclei are in equilibrium. Instead of showing the time-integrated net abundance flows, one can gain further insight into the nucleosynthesis by plotting the quantity $\phi_{ij} \equiv |r_{i\rightarrow j} - r_{j\rightarrow i}| / \max(r_{i\rightarrow j}, r_{j\rightarrow i})$ (see Eq. (3.54)). Recall, that a value of $\phi_{ij} \approx 0$ characterizes an equilibrium between a pair of nuclei *i* and *j*. On the other hand, for a pair of nuclei that is far from equilibrium we obtain $\phi_{ij} \approx 1$. Figure 5.53 shows the flows ϕ_{ij} at different instants in time (t = 0.01 s, 1 s, and 100 s) for the same reaction network calculation that is displayed in Fig. 5.52.

In each panel, the thickest lines show flows with $\phi_{ii} \leq 0.01$ (approximate equilibrium), those of intermediate thickness represent flows with $0.01 < \phi_{ij} \le 0.1$, and the thinnest lines correspond to flows with $0.1 < \phi_{ij} \leq 1$ (no equilibrium). At early times (t = 0.01 s), we see a dense pattern of the thickest lines in the A = 28-43 range. For each of these pairs of nuclei, the forward reaction is partially balanced by the reverse reaction. The *net* abundance flow is much smaller compared to the corresponding total flows or, in other words, the net abundance flow represents a very small difference between two large and nearly equal opposing reaction rates (hence $\phi_{ij} \approx 0$). These pairs of nuclei in the A = 28-43 range are linked together. They are in mutual equilibrium and form a quasiequilibrium cluster (Section 3.1.6). At later times (t = 1 s), the first cluster has grown in size (A = 24-43), while a second cluster appears in the iron peak region (A = 50-67). These two quasiequilibrium clusters are clearly not in mutual equilibrium (they are not linked by the thickest lines). Closer to the end of the calculation (t = 100 s), the two groups have merged and form one large quasiequilibrium cluster in the A = 24-67 region. A discussion of reactions linking the two clusters for a range of temperature and density conditions or in stellar evolution models can be found in Hix and Thielemann (1996) or Chieffi, Limongi and Straniero (1998), respectively. Explosive silicon burning is a major contributor to the cosmic abundances of ^{48,49}Ti, ^{50,52,53}Cr, and ^{54,56,57}Fe (Woosley, Heger and Weaver 2002, Clayton 2003).

Reaction network calculations similar to those just discussed provide a reliable description of silicon burning nucleosynthesis. In order to gain further insight, we will now derive a number of analytical expressions for constant temperature and density conditions by focussing our attention mainly on the reaction links between the even–even N = Z nuclei (or α -nuclei), such as ¹²C, ¹⁶O, ²⁰Ne, ²⁴Mg, and so on. Although the following considerations are very helpful, the reader should be aware that any truncation of the complex problem of silicon burning (that is, the restriction to certain nuclides and reactions) will inevitably give rise to oversimplifications and deviations from the real situation.

Suppose first, that the fuel consists only of ²⁸Si and that (α, γ) and (γ, α) reactions are the only interactions in the ensuing nuclear rearrangement. The reaction links between ¹²C and ⁴⁰Ca are shown in Fig. 5.54b. The numbers next to the arrows indicate the decay constants λ_{α} or λ_{γ} (in units of s⁻¹) at T = 3.6 GK for (α, γ) or (γ, α) reactions, respectively. The quantity λ_{α} is calculated from the reaction rates $N_A \langle \sigma v \rangle$ assuming $\rho = 3 \times 10^7$ g/cm³ and $X_{\alpha} = 10^{-6}$ (see Eq. (3.23)). The latter value is adopted from the network calculation shown in Fig. 5.52. Of course, λ_{α} and λ_{γ} for a pair of forward and reverse reactions are related by Eq. (3.45). An interesting point becomes apparent here. The decay constant for ²⁸Si(γ, α)²⁴Mg is much smaller than the λ_{γ} values for all other α -



Fig. 5.53 Normalized net abundance flows, $\phi_{ij} \equiv |r_{i \rightarrow j} - r_{j \rightarrow i}| / \max(r_{i \rightarrow j}, r_{j \rightarrow i})$, at three different times (t = 0.01 s, 1 s, 100 s) for the same reaction network calculation that is shown in Fig. 5.52. In each panel, the thick lines show flows with $\phi_{ij} \leq 0.01$ (approximate equilibrium), those of inter-

mediate thickness represent flows with $0.01 < \phi_{ij} \le 0.1$, and the thin lines correspond to flows with $0.1 < \phi_{ij} \le 1$ (no equilibrium). Note that the flows ϕ_{ij} are not integrated over time, but provide instead a snapshot for the evolution of quasiequilibrium clusters during the nuclear burning.

nuclei shown. Some of the liberated α -particles will be captured by ²⁴Mg, a process that is more likely to occur than the competing photodisintegration of ²⁴Mg [$\lambda_{\alpha}(^{24}Mg) \gg \lambda_{\gamma}(^{24}Mg)$]. Hence, the ²⁴Mg and ²⁸Si abundances will quickly seek an equilibrium. Another fraction of the liberated α -particles is captured by ²⁸Si. The subsequent photodisintegration of ³²S is more likely to occur than the competing ³²S(α, γ)³⁶Ar reaction [$\lambda_{\alpha}(^{32}S) \ll \lambda_{\gamma}(^{32}S)$]. As a result, the ²⁸Si and ³²S abundances will also seek quickly an equilibrium. The number densities of ²⁴Mg and ²⁸Si or of ²⁸Si and ³²S are related by the Saha equation (see Eq. (3.49)),

$$\frac{N_3}{N_0 N_1} = \frac{1}{N_1} \frac{\lambda_1(0)}{\lambda_\gamma(3)} = \frac{1}{\theta} \left(\frac{M_0 + M_1}{M_0 M_1}\right)^{3/2} \frac{g_3}{g_0 g_1} \frac{G_3^{\text{norm}}}{G_0^{\text{norm}} G_1^{\text{norm}}} e^{Q_{01 \to \gamma^3}/kT}$$
(5.148)

with $\theta \equiv (2\pi m_u kT/h^2)^{3/2} = 5.943 \times 10^{33} T_9^{3/2} \text{ cm}^{-3}$; g_i denotes the statistical weights, m_u is the atomic mass unit, and the index 1 refers to α -particles for the pairs of nuclei quoted above (see Problem 5.6).

Similar equilibria are established between pairs of heavier α -nuclei since in each case the (γ, α) reaction is more likely to occur than the competing (α, γ) reaction (Fig. 5.54b). Hence, the number densities of ²⁴Mg, ³²S, ³⁶Ar, and so on, are all in quasiequilibrium with ²⁸Si and the free α -particles. Photodisintegration reactions of the type (γ, p) and (γ, n) do also occur and they give rise to the synthesis of non- α -nuclei that also come into equilibrium with the α -nuclei (and with ²⁸Si in particular) and the free nucleons. As a result, a quasiequilibrium group of nuclei comes into existence which is build around the tightly bound ²⁸Si nucleus. This conclusion remains unchanged if we take the ${}^{28}Si(\gamma,p){}^{27}Al$ reaction into account, which has been disregarded so far. According to Fig. 5.51, the ${}^{28}\text{Si}(\gamma, p){}^{27}\text{Al}$ reaction is even more likely to occur above T = 2.2 GK than ${}^{28}\text{Si}(\gamma, \alpha){}^{24}\text{Mg}$. Nevertheless, ${}^{28}\text{Si}$ has by far the smallest *total* decay constant, $\lambda = \lambda_{\gamma\alpha} + \lambda_{\gamma p} + \lambda_{\gamma n}$, among all nuclei in the A = 24-67 range. Furthermore, the fact that ²⁴Mg comes into equilibrium with ²⁸Si greatly slows the disintegration of ²⁸Si (see below). In summary, the quasiequilibrium with respect to the residual ²⁸Si can be maintained because the intermediate-mass nuclei capture and emit α -particles, protons or neutrons at rates much larger than the small net rate of ²⁸Si disintegration. The time scale of the process is thus determined by the rate at which ²⁸Si can be decomposed.

The quasiequilibrium abundance of a nucleus ${}^{A}_{Z}Y_{N}$ relative to 28 Si is given by (Bodansky, Clayton and Fowler 1968; see also Problem 5.7)

$$\frac{N_{Y}}{N_{28}_{Si}} = N_{\alpha}^{\delta_{\alpha}} N_{p}^{\delta_{p}} N_{n}^{\delta_{n}} \left(\frac{M_{Y}}{M_{28}_{Si} M_{\alpha}^{\delta_{\alpha}} M_{p}^{\delta_{p}} M_{n}^{\delta_{n}}} \right)^{3/2} \frac{G_{Y}^{\text{norm}}}{G_{28}^{\text{norm}}} \frac{g_{Y}}{2^{\delta_{p}+\delta_{n}}} \frac{1}{\theta^{\delta_{\alpha}+\delta_{p}+\delta_{n}}} \times e^{[B(Y)-B(^{28}Si)-\delta_{\alpha}B(\alpha)]/kT}$$
(5.149)



Fig. 5.54 Reaction chains in silicon burning. (a) The reaction chain ${}^{28}Si \leftrightarrow {}^{32}S \leftrightarrow {}^{33}S \leftrightarrow {}^{34}S$ in equilibrium. (b) $(\alpha, \gamma) \leftrightarrow (\gamma, \alpha)$ reaction links between ¹²C and ⁴⁰Ca. The numbers next to the arrows indicate values of the decay constants λ_{α} and λ_{γ} (in units of s⁻¹) for (α, γ) and (γ, α) reactions, respectively, assuming a temperature of T = 3.6 GK. The quantity

 λ_{α} is calculated by using $\rho = 3 \times 10^7$ g/cm³ and $X_{\alpha} = 10^{-6}$. The latter value is adopted from the network calculation displayed in Fig. 5.52. Nuclides located within the region demarked by the dashed lines are in quasiequilibrium. (c) The closed reaction chain ${}^{28}Si \leftrightarrow {}^{32}S \leftrightarrow {}^{31}P \leftrightarrow {}^{30}Si \leftrightarrow {}^{29}Si \leftrightarrow {}^{28}Si$ in equilibrium.

where A = Z + N; N_{α} , N_{p} and N_{n} are the number abundances of α -particles, protons and neutrons, respectively; δ_{α} , δ_{p} and δ_{n} specify the number of α particles and nucleons of nucleus ${}^{A}_{Z}Y_{N}$ in excess of their number in 28 Si. They are computed relative to the heaviest α -nucleus contained within ${}^{A}_{7}Y_{N}$. If this heaviest α -nucleus contains N' = Z' protons and neutrons, then the integers δ_i are given by $\delta_{\alpha} = (N' + Z' - 28)/4$, $\delta_{p} = Z - Z'$, $\delta_{n} = N - N'$. For example, ³⁴S may be considered as being composed of ³²S plus two neutrons, hence $\delta_{\alpha} = (16 + 16 - 28)/4 = 1, \ \delta_{p} = 16 - 16 = 0, \ \delta_{n} = 18 - 16 = 2.$ The exponent $B(Y) - B(^{28}\text{Si}) - \delta_{\alpha}B(\alpha)$ is the energy required to decompose $^{A}_{7}\text{Y}_{N}$ into $^{28}\text{Si} +$ $\delta_{\alpha}{}^{4}$ He + nucleons, with B(Y) the binding energy of ${}^{A}_{Z}$ Y_N. For example, for the ratio N_{56} _{Ni} / N_{28} _{Si} we obtain from Eq. (5.149)

$$\frac{N_{56}_{\rm Ni}}{N_{28}_{\rm Si}} = N_{\alpha}^{7} \left(\frac{2}{4^{7}}\right)^{3/2} \frac{1}{\theta^{7}} e^{\left[B^{(56}_{\rm Ni}) - B^{(28}_{\rm Si}) - 7B(\alpha)\right]/kT}$$
(5.150)

where $B({}^{56}\text{Ni}) - B({}^{28}\text{Si}) - 7B(\alpha) = 49.385$ MeV. This result will be used later in the discussion of the energy generation rate. The free α -particles, protons and

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neutrons maintain an equilibrium via many different closed reaction chains, such as ${}^{28}Si \leftrightarrow {}^{32}S \leftrightarrow {}^{31}P \leftrightarrow {}^{30}Si \leftrightarrow {}^{29}Si \leftrightarrow {}^{28}Si$ (Fig. 5.54c). The light-particle abundances are related by (Problem 5.8)

$$N_{\alpha} = \frac{1}{16} N_{\rm n}^2 N_{\rm p}^2 \frac{1}{\theta^3} \left(\frac{M_{\alpha}}{M_{\rm p}^2 M_{\rm n}^2} \right)^{3/2} e^{B(\alpha)/kT}$$
(5.151)

with $B(\alpha) = 28.295$ MeV the binding energy of the α -particle. From Eqs. (5.149) and (5.151) one can see that the equilibrium abundance of each nucleus relative to ²⁸Si is determined by the abundances of any two light particles. We also conclude that the quasiequilibrium abundance of nucleus ${}^{A}_{Z}$ Y_N is uniquely specified by the four parameters N_{28Si} , N_{α} , N_{p} , and *T*.

The net downward flow from ²⁴Mg to ⁴He will be considered next (Fig. 5.52). According to Fig. 5.53, the lower bound of the silicon quasiequilibrium cluster is ²⁴Mg. Nuclei lighter than ²⁴Mg are generally *not* in equilibrium with ²⁸Si. This can also be seen from the decay constants given in Fig. 5.54b. The α -captures on ²⁰Ne and ¹²C are *less* likely to occur than the competing (γ , α) reactions [$\lambda_{\alpha\gamma}$ (²⁰Ne) $\ll \lambda_{\gamma\alpha}$ (²⁰Ne) and $\lambda_{\alpha\gamma}$ (¹²C) $\ll \lambda_{\gamma3\alpha}$ (¹²C)] and hence the abundances of the pairs ²⁰Ne–²⁴Mg or ¹²C–¹⁶O will not seek quickly an equilibrium. The effective rate of ²⁸Si destruction is then determined by the photodisintegration of ²⁴Mg. Net flows *f_i* between pairs of the light α -nuclei are given by Eqs. (3.23) and (3.52),

$$f_{24}{}_{Mg \to 20}{}_{Ne} = N_{24}{}_{Mg}\lambda_{\gamma\alpha}({}^{24}Mg) - N_{20}{}_{Ne}\lambda_{\alpha\gamma}({}^{20}Ne)$$
(5.152)

$$f_{20}_{\rm Ne \to ^{16}O} = N_{20}_{\rm Ne} \lambda_{\gamma \alpha} (^{20} \rm Ne) - N_{^{16}O} \lambda_{\alpha \gamma} (^{16}O)$$
(5.153)

$$f_{16O \to 12C} = N_{16O} \lambda_{\gamma \alpha} ({}^{16}O) - N_{12C} \lambda_{\alpha \gamma} ({}^{12}C)$$
(5.154)

$$f_{^{12}C \to ^{4}He} = N_{^{12}C}\lambda_{\gamma 3\alpha}(^{12}C) - r_{3\alpha}$$
(5.155)

with $r_{3\alpha} = N_{\alpha}\lambda_{3\alpha}/3$ the rate of the 3α reaction which depends on N_{α}^3 (see Eq. (5.100)); $\lambda_{\gamma_{3\alpha}}(^{12}\text{C})$ is the decay constant for the disintegration $^{12}\text{C} \rightarrow \alpha + \alpha + \alpha$. Since the decomposition of ^{28}Si is so slow, and thus determines the overall time scale of the process, we conclude that the abundances of the light α -nuclei are small compared to the ^{28}Si abundance. This also means that the abundances of the light α -nuclei achieve a steady state, that is, the abundance flow into each of the nuclei ^{20}Ne , ^{16}O , and ^{12}C is balanced by the flow out. Therefore, the net flows f_i are equal, $f_{24}_{\text{Mg}\rightarrow^{20}\text{Ne}} = f_{20}_{\text{Ne}\rightarrow^{16}\text{O}} = f_{16}_{\text{O}\rightarrow^{12}\text{C}} = f_{12}_{\text{C}\rightarrow^4\text{He}} \equiv f_{\text{an}}$. With this assumption the above system of equations can be solved for f_{an} , with the result (Problem 5.9)

$$f_{\rm an} = \frac{N_{24}{\rm Mg}\lambda_{\gamma\alpha}(^{24}{\rm Mg}) - \frac{\lambda_{\alpha\gamma}(^{20}{\rm Ne})}{\lambda_{\gamma\alpha}(^{20}{\rm Ne})}\frac{\lambda_{\alpha\gamma}(^{16}{\rm O})}{\lambda_{\gamma\alpha}(^{16}{\rm O})}\frac{\lambda_{\alpha\gamma}(^{12}{\rm C})}{\lambda_{\gamma3\alpha}(^{12}{\rm C})}r_{3\alpha}}{1 + \frac{\lambda_{\alpha\gamma}(^{20}{\rm Ne})}{\lambda_{\gamma\alpha}(^{20}{\rm Ne})}\left[1 + \frac{\lambda_{\alpha\gamma}(^{16}{\rm O})}{\lambda_{\gamma\alpha}(^{16}{\rm O})}\left(1 + \frac{\lambda_{\alpha\gamma}(^{12}{\rm C})}{\lambda_{\gamma3\alpha}(^{12}{\rm C})}\right)\right]}$$
(5.156)

The above analytical expression gives the effective photodisintegration rate of 24 Mg, and hence the effective rate for the conversion of 28 Si into heavier nuclei, as a function of temperature and α -particle abundance. The 24 Mg abundance, $N_{^{24}Mg}$, can be obtained from Eq. (5.148).

Figure 5.55 compares the total ²⁸Si photodisintegration rate, $r_{28Si+\gamma}$ = $N_{28}_{Si}[\lambda_{\gamma\alpha}({}^{28}Si) + \lambda_{\gamma p}({}^{28}Si)]$, where $\lambda_{\gamma n}({}^{28}Si)$ is negligible, with the effective rate f_{an} of ²⁸Si consumption. The curves are computed for the conditions T =3.6 GK and $\rho = 3 \times 10^7$ g/cm³ as a function of the remaining ²⁸Si abundance. The α -particle abundance is adopted from the numerical results of a network calculation (Fig. 5.52). A few interesting points are apparent here. First, it can be seen that the effective rate of ²⁸Si consumption is 2–3 orders of magnitude smaller than the total photodisintegration rate of ²⁸Si. This is caused by the ²⁸Si photodisintegration flow being almost exactly balanced by the flow upward from ²⁴Mg (so that the *net* flow is small) and supports the above arguments regarding the very slow conversion of ²⁸Si. Second, it is apparent that the downward flow f_{an} from ²⁸Si decreases with time as ²⁸Si burns. Since the liberated light particles are used to build up iron peak nuclei, the upward flow from ²⁸Si also decreases with time as ²⁸Si burns. Third, the long-dashed line shows the flow $f_{\text{num}} = f_{^{24}Mg \rightarrow ^{20}\text{Ne}}$ which is directly obtained from a net-work calculation according to Eq. (5.152). The curves for f_{an} and f_{num} are in good agreement. Hence, Eq. (5.156) provides indeed a reliable approximation for the effective rate of ²⁸Si consumption. This also means that only (γ, α) or (α, γ) reactions between pairs of α -nuclei are important for the downward flow from ²⁸Si to ⁴He. However, for different temperature and density conditions a number of other reactions play an important role as well (Hix and Thielemann 1996). Finally, for the temperature adopted here, the curve for $f_{\rm an}$ is almost indistinguishable from the values obtained with the approximation $f \approx N_{24}Mg\lambda_{\gamma\alpha}(^{24}Mg)$. For increasing temperature, the α -particle capture on ²⁰Ne becomes important (see Eq. (5.156)) and, consequently, the above approximation deviates from f_{an} .

The total energy release during hydrostatic silicon burning can be estimated approximately from Eq. (3.68) if we assume that for each two ²⁸Si nuclei that are destroyed, one ⁵⁶Fe nucleus is produced (see also Fig. 5.52). The photo-disintegration of the first ²⁸Si nucleus provides a free α -particle which is then captured by the second ²⁸Si nucleus. With $\overline{Q}_{Si} \approx Q_{2^{28}Si \rightarrow 5^{6}Fe} = 17.62$ MeV, we find

$$\int \varepsilon_{\rm Si}(t) \, dt = \frac{N_A \overline{Q}_{\rm Si}}{2M_{^{28}\rm Si}} \Delta X_{^{28}\rm Si} = 1.90 \times 10^{23} \Delta X_{^{28}\rm Si} \qquad ({\rm MeV}/g) \tag{5.157}$$

where ΔX_{28Si} is the mass fraction of the consumed silicon fuel. This value is smaller than what is expected from either carbon or oxygen burning, but ex-



Fig. 5.55 Comparison of total ²⁸Si photodisintegration rate ($r_{28}_{Si+\gamma}$; short-dashed line) with the effective rate of ²⁸Si consumption (f_{an} ; solid line). The curves are computed as a function of the remaining ²⁸Si abundance for the conditions T = 3.6 GK and $\rho = 3 \times 10^7$ g/cm³. The α -particle abun-

dance is adopted from the numerical results of the network calculation displayed in Fig. 5.52. The long-dashed line shows the flow $f_{24}{}_{Mg \rightarrow {}^{20}\rm{Ne}} = f_{\rm num}$ that is directly obtained from the numerical network calculation.

ceeds the total energy released during neon burning (see Eqs. (5.130), (5.135), and (5.147)).

The nuclear energy generation *rate* cannot be described precisely by an analytical expression since the nuclear transformations during silicon burning are very complex. Reaction network calculations show that the energy generation rate is sensitive to the temperature and density conditions, but also to the neutron excess (Hix and Thielemann 1996). In the simplest case, an order-ofmagnitude estimate can be found if one assumes that the initial neutron excess is very small ($\eta \approx 0$), that weak interactions are negligible, and that for each two ²⁸Si nuclei that are destroyed one ⁵⁶Ni nucleus is produced (Bodansky, Clayton and Fowler 1968). As explained above, the rate of ²⁸Si consumption is mainly determined by the ²⁴Mg(γ, α)²⁰Ne reaction. Starting from Eq. (3.63) one finds

$$\epsilon_{\rm Si} = \frac{Q_{2^{28}Si \to 56}Ni}{\rho} r_{2^{28}Si \to 56}Ni} \approx \frac{Q_{2^{28}Si \to 56}Ni}{\rho} r_{24}Mg(\gamma,\alpha)^{20}Ne$$

= $\frac{Q_{2^{28}Si \to 56}Ni}{\rho} N_{24}Mg \lambda_{\gamma\alpha}(^{24}Mg)$ (5.158)

The quantity N_{24}_{Mg} can be replaced by the ²⁸Si abundance using Eq. (5.148) (see also Problem 5.6). The decay constant $\lambda_{\gamma\alpha}(^{24}Mg)$ can be expressed in terms of the corresponding forward reaction rate by using Eq. (3.45). The normalized partition functions for ²⁰Ne, ²⁴Mg, and ²⁸Si in these two expressions are close to unity for $T \leq 5$ GK. Equation (5.148) contains the α -particle abun-

dance, which is derived from Eq. (5.150). Substitution of these three expressions into Eq. (5.158) gives

$$\begin{aligned} \varepsilon_{\rm Si} &= 9.8685 \times 10^9 \frac{2^{3/14}}{8} \frac{Q_{2^{28}\rm Si \to ^{56}\rm Ni}}{\rho} \left(\frac{M_{^{20}\rm Ne}M_{\alpha}^2}{M_{^{28}\rm Si}}\right)^{3/2} N_{^{28}\rm Si} \left(\frac{N_{^{28}\rm Si}}{N_{^{56}\rm Ni}}\right)^{1/7} \\ &\times e^{11.605[B(^{56}\rm Ni) - B(^{28}\rm Si) - 7B(\alpha)]/(7T_9)} e^{-11.605[Q_{^{20}\rm Ne}(\alpha,\gamma)} + Q_{^{24}\rm Mg}(\alpha,\gamma)]/T_9} \\ &\times T_9^{3/2} N_A \langle \sigma v \rangle_{^{20}\rm Ne}(\alpha,\gamma)} \qquad ({\rm MeV\,g^{-1}\,s^{-1}}) \end{aligned}$$
(5.159)

If one assumes in addition that most of the matter resides in either ²⁸Si or ⁵⁶Ni, then $X_{28Si} + X_{56Ni} = 1$. Inserting the numerical values of $Q_{2^{28}Si \rightarrow 5^{6}Ni} = 10.918 \text{ MeV}$, $Q_{20}_{Ne(\alpha,\gamma)} = 9.317 \text{ MeV}$, $Q_{24}_{Mg(\alpha,\gamma)} = 9.984 \text{ MeV}$ and $[B(^{56}Ni) - B(^{28}Si) - 7B(\alpha)] = 49.385 \text{ MeV}$ and replacing the number abundances by mass fractions (see Eq. (1.13)) yields

$$\varepsilon_{\rm Si} = 1.2985 \times 10^{34} X_{28\rm Si} \left(\frac{2X_{28\rm Si}}{1 - X_{28\rm Si}}\right)^{1/7} e^{-142.12/T_9} T_9^{3/2} N_A \langle \sigma v \rangle_{^{20}Ne(\alpha,\gamma)}$$
(MeV g⁻¹ s⁻¹) (5.160)

where $[2X_{28Si}/(1 - X_{28Si})]^{1/7} \approx 1$ within a factor of 2 between $X_{28Si} = 0.01 - 0.99$. Equation (5.160) is independent of the density. An analytical expression for the ${}^{20}Ne(\alpha,\gamma)^{24}Mg$ reaction rate is given in Section 5.5.2. The temperature dependence of the energy generation rate during silicon burning is then

$$\varepsilon_{\rm Si} \sim T_9^{2.229} e^{-12.681/T_9} e^{-142.12/T_9} T_9^{1.5} \sim T_9^{3.729} T_9^{154.80/T_9}$$
(5.161)

where the term $\exp(-154.80/T_9)$ is derived according to the method described by Eqs. (3.82)–(3.87). For example, near $T_0 = 3.6$ GK we find

$$\varepsilon_{\rm Si}(T) = \varepsilon_{\rm Si}(T_0) \left(T/T_0\right)^{47} \tag{5.162}$$

Since so many nuclides achieve quasiequilibrium, the thermonuclear rates of most reactions are not important for the nucleosynthesis and energy production during silicon burning. What is mainly needed in terms of nuclear physics input are binding energies (or *Q*-values), nuclear masses, spins (see Eqs. (5.148) and (5.149)) and stellar weak interaction rates. Binding energies and masses of nuclei close to stability are well known. Where the thermonuclear rates are important, however, are for those reactions that are not in quasiequilibrium for a significant amount of time during the burning. This applies to reactions that determine the net downward flow from ²⁴Mg and to those that mediate between the two quasiequilibrium clusters built around ²⁸Si and the iron-peak nuclei. For the reaction network calculation discussed above, the downward flow from ²⁴Mg is governed by ²⁴Mg(γ , α)²⁰Ne, while

 ${}^{42}Ca(\alpha,\gamma){}^{46}Ti$ and ${}^{45}Sc(p,\gamma){}^{46}Ti$ are among the reactions which link the two clusters. (Detailed lists of reactions are given in Hix and Thielemann (1996) and Chieffi, Limongi and Straniero (1998).) The reverse ${}^{24}Mg(\gamma,\alpha){}^{20}Ne$ reaction rate can be calculated from the forward ${}^{20}\text{Ne}(\alpha,\gamma){}^{24}\text{Mg}$ rate, which has already been discussed in connection with carbon and neon burning (Sections 5.5.1 and 5.5.2). Near $T \approx 3.6$ GK, the 20 Ne(α, γ) 24 Mg reaction rate may be subject to systematic errors of the order of a factor of \approx 2, as can be seen from the different results reported by Caughlan and Fowler (1988), Angulo et al. (1999), and Rauscher et al. (2000). Several, but not all, of the reactions linking the two quasiequilibrium clusters have been measured directly in the Gamow peaks appropriate for hydrostatic and explosive silicon burning. Among the measured reactions are ${}^{42}Ca(\alpha,\gamma){}^{46}Ti$, ${}^{42}Ca(\alpha,p){}^{45}Sc$ (Mitchell et al. 1985), ${}^{42}Ca(\alpha,n){}^{45}Ti$ (Cheng and King 1979), ${}^{41}K(\alpha,p){}^{44}Ca$ (Scott et al. 1991) and ${}^{45}Sc(p,\gamma){}^{46}Ti$ (Solomon and Sargood 1978). Typical reaction rate errors amount to about $\pm 20\%$ where direct data exist. Other mediating reactions, some of which involve radioactive target nuclei such as ⁴¹Ca, ⁴⁴Sc, and ⁴⁴Ti, have not been measured yet. In these cases, the Hauser–Feshbach statistical model is used to estimate the reaction rates theoretically (Goriely 1998, Rauscher and Thielemann 2000).

5.5.5

Nuclear Statistical Equilibrium and Freeze-Out

As the ²⁸Si disappears at the end of silicon burning, the temperature in the stellar core increases steadily (Section 1.4.3 and Fig. 5.1). At some point, the previously nonequilibrated reactions in the A < 24 region come into equilibrium as well (Figs. 5.53 and 5.54). The last link to achieve equilibrium is $3\alpha \leftrightarrow^{12}$ C. Every nuclide in the network is now in equilibrium via strong and electromagnetic interactions and one large quasiequilibrium group stretches from p, n, α to the iron peak nuclei. This situation is referred to as *nuclear* statistical equilibrium. For a distinction between nuclear statistical equilibrium and the related e-process (Burbidge et al. 1957), see Wallerstein et al. (1997). Note that weak interactions do not participate in the equilibrium. For example, the reverse link of electron capture on some parent nucleus is neutrino capture on the corresponding daughter nucleus. Neutrinos normally escape from the star without interaction since their mean free path exceeds the stellar radius. Hence, a true equilibrium involving weak interactions is not achieved. In nuclear statistical equilibrium, the abundance of any isotope ${}^{A}_{\pi}Y_{\nu}$ can be determined by repeated application of the Saha equation (see Eq. (5.148)). The result is (Clifford and Tayler 1965; see also Problem 5.10)

$$N_{Y} = N_{p}^{\pi} N_{n}^{\nu} \frac{1}{\theta^{A-1}} \left(\frac{M_{Y}}{M_{p}^{\pi} M_{n}^{\nu}} \right)^{3/2} \frac{g_{Y}}{2^{A}} G_{Y}^{\text{norm}} e^{B(Y)/kT}$$
(5.163)

with θ defined as in Eq. (5.148), B(Y) the binding energy of ${}^{A}_{\pi}Y_{\nu}$ and $A = \pi + \nu$. The symbols π and ν are used instead of Z and N for the number of protons and neutrons, respectively, in order to avoid confusion with the number density N_i . The abundance of any isotope is hence given in terms of its nuclear properties (binding energy, mass, spin, and so on) and the free nucleon abundances N_p and N_n . The above equation is by itself inadequate to yield the equilibrium abundance N_Y since N_p and N_n are not given. But two additional constraints can be applied. One unknown quantity (say N_p) can be eliminated by using conservation of mass (see Eq. (1.12))

$$\sum_{i} X_i = \frac{\sum_i N_i M_i}{\rho N_A} = 1 \tag{5.164}$$

where the sum *i* is over all nuclei in the network, including free protons, neutrons, and α -particles. Recall that the strong and electromagnetic interactions occur much more rapidly than weak interactions. Thus, nuclei and photons come into equilibrium in a relatively short time while the total numbers of free and bound protons and neutrons are essentially constant. Conservation of total charge is frequently expressed by the requirement that the total number densities of (free and bound) protons and neutrons must preserve the neutron excess

$$\sum_{i} \frac{(\nu_i - \pi_i)}{M_i} X_i = \frac{\sum_i N_i (\nu_i - \pi_i)}{\rho N_A} \equiv \eta$$
(5.165)

It follows immediately from Eqs. (5.163)–(5.165) that the abundance of any nuclide in nuclear statistical equilibrium is uniquely specified by only three independent parameters: temperature, density, and neutron excess. Of course, weak interactions may also occur. They are assumed to be sufficiently slow so that nuclear statistical equilibrium at a specific value of η is established on a much shorter time than the time required for a significant change in the value of η to occur. Weak interactions must be monitored carefully because the composition of the iron peak depends sensitively on η , as will be shown later.

In the following, some interesting properties of Eq. (5.163) will be explored. First, consider the simplest case when $\eta \approx 0$, that is, assume that weak interactions are negligible, and that, in the iron peak, the decomposition of ²⁸Si during the preceding silicon burning stage has mainly produced ⁵⁶Ni. By combining two equations of the form of Eq. (5.163), one for ⁴He and the other for ⁵⁶Ni, one easily finds

$$\frac{N_{4\text{He}}^{14}}{N_{56\text{Ni}}} = \theta^{13} \frac{2^{42}}{56^{3/2}} e^{[14B(^{4}\text{He}) - B(^{56}\text{Ni})]/kT}$$
(5.166)

where $g_{^{4}\text{He}} = g_{^{56}\text{Ni}} = 1$ and $G_{^{4}\text{He}}^{\text{norm}} = G_{^{56}\text{Ni}}^{\text{norm}} = 1$. The latter equality holds within 10% up to T = 5 GK. Furthermore, $14B(^{4}\text{He}) - B(^{56}\text{Ni}) = -87.853$ MeV

is the energy required to separate ⁵⁶Ni into 14 α -particles. Now, assume that the stellar plasma consists entirely of ⁴He and ⁵⁶Ni. We would like to know the *T*- ρ conditions for which the mass fractions of these two nuclides are equal ($X_{4\text{He}} = X_{56\text{Ni}} = 0.5$). This boundary can be obtained easily by rewriting Eq. (5.166) in terms of mass fractions and by solving for the density ρ . The numerical result is

$$\rho = 3.80 \times 10^{11} T_9^{3/2} e^{-78.42/T_9} \qquad (g/cm^3) \tag{5.167}$$

Similarly, another boundary can be obtained assuming instead that the matter consists entirely of α -particles and (free) nucleons to equal amounts (see Problem 5.11). These two boundaries are displayed in Fig. 5.56. They reflect the competition between ⁵⁶Ni, ⁴He, and nucleons in a plasma at nuclear statistical equilibrium for $\eta \approx 0$. In the lower temperature region (to the left of the solid line), ⁵⁶Ni dominates the composition. At intermediate temperatures (between the solid and the dotted lines), ⁴He is the dominant nucleus. At higher temperatures (to the right of the dotted line), the composition consists mainly of protons and neutrons. It is apparent that with rising temperatures and given density, an increasing fraction of the composition resides in light particles (α , n, p). This circumstance is important both for triggering the collapse of the core of an evolved massive star and for causing energy losses to the shock wave generated by the core bounce (Section 1.4.3). Also, for decreasing densities and given temperature, an increasing fraction of the composition resides in light particles.

Consider now temperature–density conditions at which most of the matter resides in iron peak nuclei (the region to the left of the solid line in Fig. 5.56). We would like to find the dominant constituents favored by nuclear statistical equilibrium if the neutron excess parameter is $\eta > 0$. Clearly, a value of $\eta > 0$ will allow the dominant nucleus to be one with a neutron excess. If the plasma would consist of only one species, then it is obvious that η must be equal to the individual neutron excess, (N - Z)/A, of the nuclide in question. It is then reasonable to assume that the abundance of each nuclide will be at maximum close to its individual neutron excess η is then, in general, the one with an individual neutron excess of $(N - Z)/A \approx \eta$ and the largest binding energy (see Eq. (5.163)).

The abundances of the dominant nuclides versus neutron excess parameter η in a nuclear statistical equilibrium composition at T = 3.5 GK and $\rho = 10^7$ g/cm³ are shown in Fig. 5.57. The results are calculated from Eqs. (5.163)– (5.165) by taking into account individual binding energies, spins and normalized partition functions for a large set of nuclei (from H to Zr). As expected, for $\eta = 0$ the dominant nucleus is ⁵⁶Ni [(N - Z)/A = (28 - 28)/56 = 0], which in fact is important for the nucleosynthesis and the light curves of supernovae.



Fig. 5.56 Temperature–density conditions in a plasma at nuclear statistical equilibrium with $\eta\approx 0$ for $X_{^{56}\rm Ni}$ = $X_{^4\rm He}$ = 0.5 (solid line) and $X_{^4\rm He}$ = 0.5, X_p = X_n = 0.25 (dotted line). The lines define regions of the dominant nuclear constituents. The two boundaries are not sharp since a distribution of nuclei and nucleons exists at all $T{-}\rho$ conditions. The above assumptions are

schematic because a strip exists above the solid line where nuclear statistical equilibrium in fact favors ⁵⁴Fe + 2p over ⁵⁶Ni as the dominant constituent (Clayton 1983). The point here is that, with rising temperatures at a given density, or with decreasing densities at a given temperature, an increasing fraction of the composition resides in light particles (α ,p,n).

Around $\eta = 0.04$, ⁵⁴Fe dominates [(28 - 26)/54 = 0.037], while ⁵⁶Fe is the most abundant nucleus at $\eta \approx 0.07$ [(30 - 26)/56 = 0.071]. For larger values of η the equilibrium composition shifts to still heavier and more neutron-rich nuclei. Note that the most tightly bound nucleus with $(N - Z)/A \approx \eta$ is not always the most abundant one. For example, consider the isotopes ⁵⁴Fe and ⁵⁸Ni which have similar values of (N - Z)/A. The binding energies per nucleon, B/A, are almost identical. This means that the binding energy, B, is in fact much larger for ⁵⁸Ni. Nevertheless, at $\eta \approx 0.04$ the mass fraction of ⁵⁴Fe exceeds the ⁵⁸Ni mass fraction by four orders of magnitude in Fig. 5.57. Clearly, binding energy is not the only factor influencing the abundance. In the above example, the A dependences of both θ and ρ also play an important role (see Eq. (5.163)).

It should be remarked that a system at any temperature and density will come into equilibrium provided it is maintained long enough. When it is stated that, at a particular temperature of *T*, the nuclear reactions are in equilibrium, what is meant is that it is believed that this temperature exists long enough for a good approximation to equilibrium to occur. Clearly, the nuclear gas requires a finite amount of time to adjust to equilibrium. An expression for the approximate time to reach nuclear statistical equilibrium for given values of *T* and ρ has been calculated by Khokhlov (1991). The time is displayed in Fig. 5.58 versus temperature for two values of the density ($\rho = 10^4$ g/cm³



Fig. 5.57 Abundances of the dominant species versus neutron excess parameter η in a nuclear statistical equilibrium composition at T = 3.5 GK and $\rho = 10^7$ g/cm³. From Hartmann, Woosley and El Eid (1985). Reproduced by permission of the American Astronomical Society.

and $\rho = 10^{10}$ g/cm³). At T = 4 GK, for example, nuclear statistical equilibrium is established in about 1 h, while at T = 6 GK the time is only $\approx 10^{-3}$ s. At the higher temperatures (say, above T = 6 GK) nuclear statistical equilibrium is achieved even in explosive events. At lower temperatures, however, if thermodynamic conditions vary sufficiently rapidly, nuclear statistical equilibrium may provide a poor approximation for the abundances.

An extensive discussion of nuclear statistical equilibrium is given in Clifford and Tayler (1965). It is found that abundances vary rapidly with η , fairly rapidly with temperature *T* and very slowly with density ρ . Furthermore, at lower temperatures there are fewer nuclides with relatively large abundances, whereas the abundances are spread more evenly at higher temperatures (which is also apparent from the exponential factor $e^{B(Y)/kT}$ in Eq. (5.163)).

Until now we have considered what will happen to matter when the temperature and density slowly increase during the evolution in the core of a massive star. Quasiequilibrium is achieved for sufficiently high values of *T* and ρ and, at even more extreme conditions, the matter is described by nuclear statistical equilibrium. In massive star explosions (Section 1.4.3), however, the series of events is reversed: the outgoing shock wave heats the inner layers of the star outside the core to high temperatures and this matter at nuclear sta-



Fig. 5.58 Approximate time to reach nuclear statistical equilibrium versus temperature for two different densities ($\rho = 10^4$ g/cm³ and $\rho = 10^{10}$ g/cm³). The curves are calculated from the expression $\tau_{\rm NSE} = \rho^{0.2} e^{179.7/T_9 - 40.5}$ (s), where ρ is in units of g/cm³ (see Khokhlov 1991).

tistical equilibrium subsequently experiences a rapid decrease in temperature and density while the shock moves outward. As the matter expands and cools, certain reaction links will fall out of equilibrium. The subsequent nuclear evolution depends then critically on the density of free light particles (α , n, p) and the time scale of the expansion. In the following discussion, let us denote the temperature at which the first reactions begin to fall out of equilibrium by T_{α} .

Suppose first that the density at T_{α} is sufficiently large so that the α -particle abundance is small. Representative values for such a situation are, for example, $T_{\alpha} \approx 3-4$ GK, $\rho \approx 10^7$ g/cm³, and $X_{\alpha} < 0.01$. Near T_{α} the (γ, α) rates are much more temperature sensitive than the forward (α, γ) rates (see Eq. (3.45)). Therefore, when the temperature falls below T_{α} , the photodisintegration reactions are not efficient enough in the time available to produce the α -particles necessary to maintain nuclear statistical equilibrium. As a result, there is a lack of α -particles which are almost all bound in nuclei. The first reactions to drop out of equilibrium are $3\alpha \leftrightarrow^{12}C$ and the α -particle links among the lighter α -nuclei (¹²C, ¹⁶O, ²⁰Ne, ²⁴Mg, and so on). Since the equilibrium is terminated by a lack of α -particles, the process is referred to as *particle-poor* freeze-out (Woosley, Arnett and Clayton 1973, Hartmann, Woosley and El Eid 1985). The abundances of free light particles (especially α -particles) is so low that their subsequent capture during freeze-out does not alter the composition significantly, that is, the abundances ejected from such environments are close to those derived from nuclear statistical equilibrium at temperature T_{α} . Such abundance distributions produce iron peak nuclei, with the dominant species determined by the neutron excess. For example, for $\eta \approx 0, 0.035$, or 0.071

the most abundant species are ⁵⁶Ni, ⁵⁴Fe, or ⁵⁶Fe, respectively. No significant production of nuclides beyond the iron peak occurs.

Now consider a situation where the density at T_{α} is sufficiently small so that the α -particle abundance is large. As the temperature falls toward T_{α} , the free α -particles have the tendency to merge into the iron peak (we are moving from right to left in Fig. 5.56). Most of the α -particles are transformed by the sequence $3\alpha \rightarrow {}^{12}C(\alpha,\gamma){}^{16}O...{}^{52}Fe(\alpha,\gamma){}^{56}Ni$ where the 3α reaction represents the slowest link. When the temperature falls below T_{α} , and if the expansion time scale is sufficiently rapid, the α -particles cannot be converted fast enough, in the time available, to iron-peak nuclei through the slow helium burning reactions in order to maintain nuclear statistical equilibrium. The equilibrium is terminated by an excess of α -particles and, therefore, the process is called α -rich freeze-out (Woosley, Arnett and Clayton 1973). Since the material cools in the presence of a large abundance of free light particles (especially α -particles), their interactions with nuclei during freeze-out alters the composition. The abundances ejected from such environments are different from those derived from nuclear statistical equilibrium at temperature T_{α} . For example, if the neutron excess is small ($\eta \approx 0$), the most abundant species in the ejecta is still ⁵⁶Ni. On the other hand, material that would have been in the form of the trace constituent ⁵⁴Fe ends up mainly as ⁵⁸Ni. Traces of Zn isotopes are also formed, but again, the nucleosynthesis does not proceed beyond the iron peak.

The neutron excess in the silicon and oxygen shells of a massive presupernova star is rather small ($\eta \approx 0$; see for example Section 5.5.3). When the supernova shock wave heats this matter to nuclear statistical equilibrium, the subsequent freeze-out will produce mainly iron peak nuclides, with ⁵⁶Ni as the main constituent. The radioactive decay of this nuclide, first to ⁵⁶Co and then to the stable species ⁵⁶Fe, gives rise to the tail in the light curves of core-collapse supernovae. The amount of ejected ⁵⁶Ni that is predicted by current massive star explosion models agrees with the empirical values of $0.07 \pm 0.01 \, M_{\odot}$ for SN 1987A and $0.08 \pm 0.02 \, M_{\odot}$ for SN 1993J (Woosley, Heger and Weaver 2002). The α -rich freeze-out is also a major source of the nuclides ⁴⁴Ca (made chiefly as radioactive ⁴⁴Ti in supernovae), ⁴⁵Sc and ^{58,60}Ni in the Universe. For the effects of reaction rate uncertainties on the nucleosynthesis during the α -rich freeze-out, see Jordan, Gupta and Meyer (2003). Furthermore, the light curves of type Ia supernovae are also powered by the decay of ⁵⁶Ni, which represents the main product of nuclear statistical equilibrium at low neutron excess (Section 1.4.4).

5.6 Nucleosynthesis Beyond the Iron Peak

The transmission through the Coulomb barrier decreases drastically with increasing nuclear charges. For this reason, charged-particle cross sections are far too small at moderate stellar temperatures to explain the observed solar system abundances of nuclides with masses beyond $A \approx 60$. At very high temperatures, on the other hand, charged-particle reactions will give rise to abundances that are described by nuclear statistical equilibrium, either favoring nuclei of the iron peak group or lighter species (see Fig. 5.56). The situation is quite different when considering neutron-induced reactions as the mechanism for the synthesis of the heavy nuclides. There is no Coulomb barrier for neutrons and thus the neutron capture cross sections, even at moderate stellar energies, are frequently quite large. In fact, the cross sections for most neutron-induced reactions even increase with decreasing incident neutron energies (Fig. 3.30). It is therefore reasonable to assume that heavy nuclides can be synthesized by exposing lighter seed nuclei to a source of neutrons. There is unambiguous evidence for such a mechanism. As will be seen, it provides a natural explanation for the fact that the solar system abundance curve peaks near the mass numbers $A \approx 84$, 138, and 208 (Fig. 5.59), corresponding to the neutron magic numbers of N = 50, 82, and 126, respectively (Section 1.6.1). It should be remembered that neutrons are unstable, with a half-life of $T_{1/2}$ = 614 s. The interstellar medium does not contain a significant concentration of free neutrons. They must be produced in stars. In fact, we have already encountered some neutron-producing reactions in helium burning (Section 5.3.3) and carbon burning (Section 5.5.1). We will first concentrate on the properties of neutron capture nucleosynthesis and afterward discuss the sources of neutrons in various stellar environments. It should also be noted that, unlike previously discussed processes, the neutron capture processes do not generate any significant amount of energy as can be seen from the decline of the binding energy per nucleon beyond the iron peak (Fig. 1.8).

Consider the nuclear transformations that may occur if a stable nucleus, for example ¹⁵⁶Gd, is exposed to a flux of neutrons (Fig. 5.60). Successive stable isotopes of the same element (Gd) will capture neutrons, initiating the sequence ¹⁵⁶Gd(n, γ)¹⁵⁷Gd(n, γ)¹⁵⁸Gd(n, γ)¹⁵⁹Gd. The last nuclide, ¹⁵⁹Gd, is radioactive ($T_{1/2} = 18.5$ h). Further suppose that the neutron flux is sufficiently small so that the β -decay constant of any unstable nucleus created after neutron capture is large compared to the decay constant of the competing (n, γ) reaction ($\lambda_{\beta} \gg \lambda_{n\gamma}$). The path will then continue via ¹⁵⁹Gd($\beta^{-}\nu$)¹⁵⁹Tb(n, γ)¹⁶⁰Tb. The last nuclide, ¹⁶⁰Tb, is radioactive ($T_{1/2} = 72.3$ d). The process repeats itself, giving rise to the sequence ¹⁶⁰Tb($\beta^{-}\nu$)¹⁶⁰Dy(n, γ)¹⁶¹Dy(n, γ)¹⁶²Dy, and so on. In summary, successive neutron captures by a chain of isotopes occur until a radioactive isotope is



Fig. 5.59 Solar system abundances (relative to 10^6 Si atoms) of the heavy nuclides (adopted from Lodders 2003). Abundances of different isobars have been added together. Narrow peaks occur at mass numbers of $A \approx 84$, 138, and 208, corresponding to the neutron magic numbers of N = 50, 82, and 126, respectively. Broader peaks are located approximately 10 mass units below the narrow peaks.

reached at which point a β^- -decay takes place and another successive chain of neutron captures is initiated. The resulting path is shown as the solid line in Fig. 5.60. This mechanism is referred to as s(low neutron capture)-process (Burbidge et al. 1957). Clearly, the s-process path must run close to the group of stable nuclides. More specifically, it will reach only those stable nuclides that are labeled "s" in Fig. 5.60. It will neither reach very neutron-deficient stable nuclei (such as ¹⁵⁸Dy) nor very neutron-rich stable nuclei (such as ¹⁶⁰Gd). The abundances synthesized by the s-process will in general depend on the magnitude of the neutron-capture cross sections involved in the chain. Nuclei with very small neutron-capture cross sections are expected to pile up in abundance, while those with large cross sections will be quickly destroyed and achieve only small abundances. Maxwellian-averaged neutron-capture cross sections on stable and long-lived nuclides at a thermal energy of kT = 30 keV versus mass number A are shown in Fig. 5.61. Recall that nuclides with a magic neutron number (N = 50, 82, and 126) have energetically favorable configurations (Section 1.6.1). The capture of another neutron produces a product nucleus with a relatively small neutron separation energy and, therefore, the compound nucleus is formed at a relatively small excitation energy in a region with a small level density. The reaction must then proceed through a reduced number of compound levels and the cross section becomes relatively small, as can be seen from the location of the minima in Fig. 5.61. In other





Fig. 5.60 The s-process path through the elements Gd, Tb and Dy (solid line). Shaded squares indicate stable nuclides. Nuclides reached by the s-process are labeled "s." Stable nuclides that are reached via the r-process (dotted arrows) through β^- -decay chains along A = const after termination of the neutron flux are labeled "r." Neither process can explain the synthe-

sis of the stable nuclide labeled "p." Notice that some stable nuclides can be synthesized only in the s-process or the r-process, but not by both processes. These are referred to as s-only or r-only nuclides. The s-process branchings in this mass region are weak and have been omitted in the figure.

words, we expect that the s-process will produce these very same nuclei with increased abundances. This is precisely the reason for the narrow peaks at the neutron magic numbers N = 50, 82, and 126 in the solar system abundance curve (Fig. 5.59).

Consider now the other extreme, that is, a neutron flux so large that the decay constant of an unstable nucleus created after neutron capture is small compared to the decay constant of the competing (n, γ) reaction $(\lambda_{\beta} \ll \lambda_{n\gamma})$. In this case, the nucleosynthesis path will run close to the neutron dripline. When the neutron flux terminates, all neutron-rich radioactive nuclei will undergo successive β^- -decays (dashed arrows in Fig. 5.60) along isobaric chains until the most neutron-rich, stable (or very long-lived) isobar is reached. This nucleosynthesis process is called the r(apid neutron capture)-process and will be discussed in more detail in Section 5.6.2. In the example of Fig. 5.60, the r-process synthesizes all nuclides labeled "r." It is interesting that certain nuclides (for example, ¹⁵⁶Gd, ¹⁵⁷Gd) can be produced by both the s- and the r-process. Other nuclei, such as ¹⁶⁰Gd, are never reached in the s-process and are referred to as *r*-only nuclides. The latter nuclide does not undergo a β^{-} decay since it is stable. Hence, ¹⁶⁰Dy which is less neutron-rich than ¹⁶⁰Gd, cannot be reached in the r-process. It is called an *s-only* nuclide because it is shielded from the r-process.



Fig. 5.61 Maxwellian-averaged neutron-capture cross sections on stable and long-lived nuclides at a thermal energy of kT = 30 keV versus mass number A (from Bao et al. 2000). Nuclides with a magic neutron number (N = 50, 82, and 126) have energetically favorable configurations and give rise to relatively small neutron capture cross sections (see the text).

Some of the most neutron-deficient stable nuclides, such as 158 Dy in Fig. 5.60, cannot be synthesized by either the s-process or the r-process. They are shielded from both neutron-capture processes and are referred to as *p*-*nuclei*. The mechanism responsible for their synthesis is called the *p*-process and will be discussed in Section 5.6.3. It is sufficient to remark here that the abundances of almost all p-nuclei are much smaller compared to those of the s- and r-nuclei of the same mass number.

Crude estimates for the number densities of neutrons in the s- and r-process can be obtained by considering typical cross sections for neutron capture. According to Fig. 5.61, the mean value for the Maxwellian-averaged neutroncapture cross section of nuclei in the A = 60–210 region at a thermal energy of kT = 30 keV is $\langle \sigma \rangle_T = \langle \sigma v \rangle / v_T \approx 100 \text{ mb}$. Since average neutron capture cross sections do not vary drastically with thermal energy (Fig. 3.31), this value will be adopted as an order-of-magnitude estimate. For the s-process, typical β^- -decay lifetimes of radioactive nuclei near the valley of stability range from minutes to years. Since $\tau_\beta \ll \tau_{n\gamma}$, the mean lifetime for neutron capture must then typically be $\tau_{n\gamma} \approx 10$ y or more. With $v_T = (2kT/m_{01})^{1/2} \approx$ $[2 \cdot 30 \text{ keV} \cdot c^2 / (m_n c^2)]^{1/2} \approx 2.4 \times 10^8 \text{ cm/s}$, we find from Eq. (3.22) a value of $N_n = (\tau_{n\gamma} \langle \sigma v \rangle_{n\gamma})^{-1} \approx 10^8 \text{ cm}^{-3}$ for the neutron number density in the sprocess. In the r-process, β^- -decay lifetimes for radioactive nuclei far from the valley of stability range from milliseconds to seconds. Since $\tau_\beta \gg \tau_{n\gamma}$, the mean lifetime for neutron capture must then typically be $\tau_{n\gamma} \approx 10^{-4}$ s or

less. For these conditions, a value of $N_n \approx 10^{21} \text{ cm}^{-3}$ or more is obtained as an order-of-magnitude estimate of the neutron number density in the r-process. It is interesting that the gross properties of the solar system abundances in the A > 60 range can be accounted for in terms of two extreme pictures, that is, by relatively low neutron densities achieved in the s-process and by very high neutron exposures characteristic of the r-process. Intermediate exposures between these two extremes seem to play only a minor role for the solar system abundance distribution.

5.6.1

The s-Process

Starting from some seed nuclei, the s-process path runs close to the group of stable nuclei. The majority of neutron captures involves stable target nuclei and all these reactions are accessible in the laboratory (Chapter 4). The heaviest nuclei synthesized by charged-particle reactions are those of the iron peak. Hence, these nuclei will most likely form the seeds for the s-process. Since ⁵⁶Fe is by far the most abundant nucleus in the iron peak (Fig. 1.2), we will assume for the sake of simplicity that it is the sole seed for the neutron captures. The s-process will eventually reach ²⁰⁹Bi, which is the most massive stable nucleus. Further neutron captures produce radioactive nuclei that decay by α -particle emission. Thus, heavier nuclei cannot be synthesized by the s-process and ²⁰⁹Bi represents the termination point.

Consider Fig. 5.62, showing the basic building blocks of the s-process path. In part (a), the stable nucleus with mass number A, shown as shaded square, is destroyed by neutron capture. The same nucleus is produced by neutron capture on nucleus A - 1. The same holds true if nucleus A is radioactive, but has such a long half-life that it can be regarded as being stable for all practical purposes concerning the s-process (that is, if $\lambda_{\beta} \ll \lambda_{n\gamma}$). In part (b), nucleus A is again destroyed by neutron capture, but it is also produced by neutron capture on nucleus A - 1 and the subsequent β^- -decay. We will initially assume that the β^- -decay is so fast that the abundance of the radioactive species can be neglected since it decays immediately to the stable (or very long-lived) nucleus A. Under these assumptions, the abundance at each value of the mass number A resides in precisely one particular nuclide and thus the s-process path is uniquely defined by the mass number. The abundance evolution of any stable (or very long-lived) nuclide with mass number A is then given by

$$\frac{dN_s(A)}{dt} = -N_n N_s(A) \langle \sigma v \rangle_A + N_n N_s(A-1) \langle \sigma v \rangle_{A-1}$$
(5.168)

where $N_s(A)$ and N_n are the number densities of nucleus A and of free neutrons, respectively; $\langle \sigma v \rangle_A$ is the neutron-capture reaction rate per particle pair of nucleus A. The free neutron density may vary with time, $N_n = N_n(t)$, de-



Fig. 5.62 Basic building blocks of the sprocess path. Stable (or very long-lived) nuclei are shown as shaded squares, shortlived nuclei as open squares. In part (a), the nucleus with mass number A is destroyed by neutron capture and is produced by neutron capture on nucleus A - 1. In part (b),

nucleus A is again destroyed by neutron capture, but is produced by neutron capture on nucleus A-1 and the subsequent β^- -decay. In the s-process, it is generally assumed that the β^- -decay is fast compared to neutron capture. Part (c) shows a simple example for an s-process branching.

pending on the details of the stellar model. The reaction rate depends on the time only through variations of the stellar temperature *T*. As a further simplification, we will assume that the temperature is constant during a given neutron irradiation episode, so that $\langle \sigma v \rangle_i = \text{const}$, until the neutron source turns off.

The reaction rate per particle pair can be substituted by the Maxwellianaveraged cross section, $\langle \sigma v \rangle_A = \langle \sigma \rangle_A v_T$ (see Eq. (3.11)). For the heavy target nuclei participating in the s-process the reduced mass is nearly equal to the neutron mass ($m_{01} \approx m_n$). Therefore, the thermal velocity, $v_T = (2kT/m_{01})^{1/2}$, is almost independent of the target mass. Hence

$$\frac{dN_s(A)}{dt} = -N_n(t)N_s(A)\langle\sigma\rangle_A v_T + N_n(t)N_s(A-1)\langle\sigma\rangle_{A-1}v_T$$
$$= v_T N_n(t)[-N_s(A)\langle\sigma\rangle_A + N_s(A-1)\langle\sigma\rangle_{A-1}]$$
(5.169)

In Section 4.9.3 it was found that for a Maxwell–Boltzmann distribution of neutron energies, the flux is given by the product of neutron number density and thermal velocity, $\phi = (2/\sqrt{\pi})N_n v_T$. We introduce the *neutron exposure* (with units of neutrons per area),

$$\tau = v_T \int N_n(t) dt$$
 or $d\tau = v_T N_n(t) dt$ (5.170)

which, apart from a factor $2/\sqrt{\pi}$, is equal to the time-integrated neutron flux $\Phi = \int \phi(t) dt$ (Section 4.9.4). Rewriting Eq. (5.169) by replacing the variable *t*

with τ yields

$$\frac{dN_s(A,\tau)}{d\tau}N_n(t)v_T = v_T N_n(t) \left[-N_s(A,\tau)\langle\sigma\rangle_A + N_s(A-1,\tau)\langle\sigma\rangle_{A-1}\right]$$
$$\frac{dN_s(A,\tau)}{d\tau} = -N_s(A,\tau)\langle\sigma\rangle_A + N_s(A-1,\tau)\langle\sigma\rangle_{A-1}$$
(5.171)

with the boundary conditions $N_s(56, 0) = f N_s^{\text{seed}}(56)$ and $N_s(A > 56, 0) = 0$. The quantity *f* is the fraction of the number of ⁵⁶Fe seed nuclei, $N_s^{\text{seed}}(56)$, that are subjected to an exposure of neutrons. It is clear that $N_s(A, \tau)$ decreases if it becomes too large with respect to $N_s(A - 1, \tau)$ and vice versa,

$$dN_{s}/d\tau < 0 \quad \text{for} \quad N_{s}(A,\tau) > [\langle \sigma \rangle_{A-1}/\langle \sigma \rangle_{A}]N_{s}(A-1,\tau)$$

$$dN_{s}/d\tau > 0 \quad \text{for} \quad N_{s}(A,\tau) < [\langle \sigma \rangle_{A-1}/\langle \sigma \rangle_{A}]N_{s}(A-1,\tau) \quad (5.172)$$

The coupled equations (see Eq. (5.171)) are self-regulating in the sense that they attempt to minimize the difference $N_s(A - 1, \tau)\langle\sigma\rangle_{A-1} - N_s(A, \tau)\langle\sigma\rangle_A$. In the mass regions between the magic neutron numbers the Maxwellianaveraged cross sections are relatively large (Fig. 5.61) so that the difference $N_s(A - 1, \tau)\langle\sigma\rangle_{A-1} - N_s(A, \tau)\langle\sigma\rangle_A$ becomes much smaller than the magnitude of either product $N_s(A, \tau)\langle\sigma\rangle_A$ or $N_s(A - 1, \tau)\langle\sigma\rangle_{A-1}$. In other words, for any nucleus with a mass number removed from closed neutron shells the abundance builds up until the destruction rate approximately equals the production rate. In these mass regions, a steady flow is achieved along the sprocess path, $dN_s/d\tau \approx 0$, and we find

$$N_s(A,\tau)\langle\sigma\rangle_A \approx N_s(A-1,\tau)\langle\sigma\rangle_{A-1}$$
 or $N_s(A,\tau)\langle\sigma\rangle_A \approx \text{const}$ (5.173)

This result is called the *local (equilibrium) approximation* since it is only satisfied locally in regions between magic neutron numbers.

The prediction of Eq. (5.173) can be tested by considering isotopes of the element tellurium (Z = 52). Of eight stable isotopes, three belong to the s-only category (¹²²Te, ¹²³Te, ¹²⁴Te). Two can be synthesized by both the s- and r-process (¹²⁵Te, ¹²⁶Te), two are r-only isotopes (¹²⁸Te, ¹³⁰Te), and ¹²⁰Te is a p-nucleus. The product of solar system abundance, $N_{\odot}(A)$ (Lodders 2003), and Maxwellian-averaged cross section at kT = 30 keV, $\langle \sigma \rangle_A$ (Bao et al. 2000), is shown in Fig. 5.63 versus mass number *A*. It is apparent that for the s-only nuclides

$$N_{\odot}(122)\langle\sigma\rangle_{122} \approx N_{\odot}(123)\langle\sigma\rangle_{123} \approx N_{\odot}(124)\langle\sigma\rangle_{124}$$
(5.174)

thus confirming the local approximation for the s-process. It is also clear that the product $\langle \sigma \rangle_A N_{\odot}(A)$ is not constant for ¹²⁸Te and ¹³⁰Te which are both synthesized by the r-process only. Furthermore, ¹²⁵Te and ¹²⁶Te are overabundant since both the s- and the r-process contribute to their synthesis, that is,

 $N_{\odot}(A) = N_s(A) + N_r(A)$. If the averaged neutron capture cross sections are known, one can use the local approximation in order to estimate the separate contributions of the s- and r-process to the total observed solar system abundances (see Problem 5.12).

The local approximation is most useful for nuclides with adjacent mass numbers in regions between closed neutron shells, but does not hold over the entire A = 56-209 mass range. This is clearly seen in Fig. 5.64, where the symbols show the product $N_{\odot}(A)\langle\sigma\rangle_A$ versus mass number A for the s-only isotopes. The $N_{\odot}(A)\langle\sigma\rangle_A$ values vary in magnitude by a factor of ≈ 100 . They decrease monotonically with increasing mass number, with particularly large variations occurring at $A \approx 84$, 138, and 208, corresponding to closed neutron shells. In the following, an expression for $N_s(A)\langle\sigma\rangle_A$ is derived as a function of neutron exposure. We will again assume a constant temperature. It was found (Clayton et al. 1961) that a single neutron exposure τ would not suffice to explain the observed $N_{\odot}(A)\langle\sigma\rangle_A$ values. Seeger, Fowler and Clayton (1965) showed that much better agreement could be obtained by adopting an exponential distribution of neutron exposures. Such a distribution reflects the physically reasonable assumption of decreased probabilities for increasing neutron exposures, that is, the total exposure experienced by some fraction of material relates to the number of times that material had been processed through successive generations of stars (Clayton 1983) or through successive burning episodes in a specific star (Ulrich 1973).

Suppose that *f* is the fraction of the number of ⁵⁶Fe seed nuclei, $N_s^{\text{seed}}(56)$, that has been subjected to an exponential distribution of neutron exposures, given by

$$p(\tau) = \frac{f N_s^{\text{seed}}(56)}{\tau_0} e^{-\tau/\tau_0}$$
(5.175)

where $p(\tau) d\tau$ is the fraction of ⁵⁶Fe seed nuclei having received an exposure in the range between τ and $\tau + d\tau$. The parameter τ_0 is the mean neutron exposure and determines how rapidly the exposure distribution falls off. The total number of irradiated seed nuclei is

$$\int_0^\infty p(\tau) \, d\tau = f N_s^{\text{seed}}(56) [-e^{-\tau/\tau_0}]_0^\infty = f N_s^{\text{seed}}(56) \tag{5.176}$$

The resulting abundances are

$$\overline{N_s(A,\tau_0)} = \frac{\int_{0}^{\infty} N_s(A,\tau) p(\tau) d\tau}{\int_{0}^{\infty} p(\tau) d\tau} = \int_{0}^{\infty} \frac{N_s(A,\tau)}{\tau_0} e^{-\tau/\tau_0} d\tau$$
(5.177)





Fig. 5.63 The product $N_{\odot}(A) \langle \sigma \rangle_A$ (in units of millibarn per 10⁶ Si atoms) versus mass number A for nuclides of the element tellurium (Z = 52); ¹²²Te, ¹²³Te, ¹²⁴Te are s-only nuclides; ¹²⁵Te, ¹²⁶Te are s,r-nuclides; and ¹²⁸Te, ¹³⁰Te are r-only nuclides. ¹²⁰Te is not synthesized via neutron capture (p-nucleus). The Maxwellian-averaged cross sections $\langle \sigma \rangle_A$ (in units of millibarn and appropriate

for a thermal energy of kT = 30 keV) are adopted from Bao et al. (2000) and the solar system abundances $N_{\odot}(A)$ are from Lodders (2003) (these are relative to 10⁶ Si atoms). Most of the error bars are smaller than the size of the symbols. It is apparent that $N_s(A) \langle \sigma \rangle_A \approx \text{const for the s-only}$ nuclides (dashed line).

For the first two nuclides on the s-process path, 56 Fe and 57 Fe, the abundance evolutions are given by (see Eq. (5.171))

$$\frac{dN_s(56,\tau)}{d\tau} = -N_s(56,\tau)\langle\sigma\rangle_{56}$$
(5.178)

$$\frac{dN_s(57,\tau)}{d\tau} = -N_s(57,\tau)\langle\sigma\rangle_{57} + N_s(56,\tau)\langle\sigma\rangle_{56}$$
(5.179)

For an exponential exposure distribution (see Eq. (5.175)) the solutions can be found analytically. The results are (see Problem 5.13)

$$\langle \sigma \rangle_{56} \overline{N_s(56,\tau_0)} = \frac{f N_s^{\text{seed}}(56)}{\tau_0} \frac{1}{\left[1 + \frac{1}{\tau_0 \langle \sigma \rangle_{56}}\right]}$$
(5.180)

$$\langle \sigma \rangle_{57} \overline{N_s(57,\tau_0)} = \frac{f N_s^{\text{seed}}(56)}{\tau_0} \frac{1}{\left[1 + \frac{1}{\tau_0 \langle \sigma \rangle_{56}}\right]} \frac{1}{\left[1 + \frac{1}{\tau_0 \langle \sigma \rangle_{57}}\right]}$$
(5.181)

and so on. The general solution of Eq. (5.171) is easily deduced from these results. We find (see also Clayton and Ward 1974)

$$\langle \sigma \rangle_A \overline{N_s(A,\tau_0)} = \frac{f N_s^{\text{seed}}(56)}{\tau_0} \prod_{i=56}^A \frac{1}{\left[1 + \frac{1}{\tau_0 \langle \sigma \rangle_i}\right]}$$
(5.182)



Fig. 5.64 The product $N_{\odot}(A) \langle \sigma \rangle_A$ (in units of millibarn per 10⁶ Si atoms) of solar system s-process abundance and Maxwellianaveraged neutron-capture cross section (at a thermal energy of kT = 30 keV) versus mass number A. The symbols correspond to s-only nuclides. The solid curves are obtained by fitting the data to an expression similar to Eq. (5.182) but which includes the effects of significant s-process branchings. The thick solid line is calculated by using a single exponential distribution of neutron

exposures (main s-process component). For $A \leq$ 90, the main component falls below the data points and a second distribution (weak s-process component) must be included in the fit (thin solid line). The sharp structures result from s-process branchings. At these mass numbers the solid lines split into two parts, one corresponding to the more neutron-rich nuclide and the other one to the less neutron-rich nuclide. Courtesy of Franz Käppeler.

Once the capture cross sections $\langle \sigma \rangle_A$ are known, a fit of this expression to the observed solar system values of $N_{\odot}(A)\langle \sigma \rangle_A$ for the s-only nuclides yields the parameters f and τ_0 . The magnitude of these parameters, in turn, is important for identifying the sites and the history of s-process nucleosynthesis. It is interesting that, according to Eq. (5.182), the relative $\langle \sigma \rangle_A \overline{N_s(A, \tau_0)}$ values for any two nuclei (beyond the last seed nucleus) on the s-process path are independent of the true distribution of seed nuclei (Clayton and Ward 1974). Hence, the particular choice of pure ⁵⁶Fe as seed material is as good as any other distribution of iron peak nuclei. On the other hand, this also means that the observed solar system $N_{\odot}(A)\langle \sigma \rangle_A$ values for the s-only nuclides are not a sensitive probe of the initial seed distribution. A useful quantity is the average number of neutrons captured per ⁵⁶Fe seed nucleus,

$$n_{c} = \frac{\sum_{A=56}^{209} (A-56)\overline{N_{s}(A,\tau_{0})}}{fN_{s}^{\text{seed}}(56)} = \frac{1}{\tau_{0}} \sum_{A=56}^{209} \frac{(A-56)}{\langle \sigma \rangle_{A}} \prod_{i=56}^{A} \frac{1}{\left[1 + \frac{1}{\tau_{0}\langle \sigma \rangle_{i}}\right]}$$
(5.183)

Its magnitude provides another constraint on the physical environment. For

two nuclides of adjacent mass numbers one finds immediately from Eq. (5.182)

$$\langle \sigma \rangle_A \overline{N_s(A,\tau_0)} = \frac{\langle \sigma \rangle_{A-1} \overline{N_s(A-1,\tau_0)}}{\left[1 + \frac{1}{\tau_0 \langle \sigma \rangle_A}\right]}$$
(5.184)

Between closed neutron shells the capture cross section $\langle \sigma \rangle_A$, and hence the product $\tau_0 \langle \sigma \rangle_A$, is large. Therefore, we find from Eq. (5.184) $\langle \sigma \rangle_A \overline{N_s(A, \tau_0)} \approx \langle \sigma \rangle_{A-1} \overline{N_s(A-1,\tau_0)}$, consistent with the local approximation discussed above. Near closed neutron shells the cross section $\langle \sigma \rangle_A$, and thus $\tau_0 \langle \sigma \rangle_A$, is relatively small. Consequently, the denominator in the above expression becomes relatively large, producing a step in the distribution of $\langle \sigma \rangle_A \overline{N_s(A,\tau_0)}$ values. In other words, the small capture cross sections of the neutron magic nuclei represent bottlenecks for a continuous abundance flow. The resulting steps are clearly seen in Fig. 5.64 at mass numbers of $A \approx 84$, 138, and 208, corresponding to closed neutron shells. Obviously, the height and shape of the steps are sensitive to the magnitude of the mean neutron exposure τ_0 , while the fraction *f* acts as an overall scaling factor.

The solid curves in Fig. 5.64 are obtained by fitting the data for $N_{\odot}(A)\langle\sigma\rangle_A$ to an expression similar to Eq. (5.182). The sharp structures result from sprocess branchings that will be discussed later. The thick solid line is calculated by using a single exponential distribution of neutron exposures. It describes all the observed $N_{\odot}(A)\langle\sigma\rangle_A$ values for s-only nuclides in a wide range from A = 90 to A = 205 and is called the *main s-process component*. The mean square deviation between the thick solid line and the data points in Fig. 5.64 amounts to only 3% (Käppeler et al. 1990). This excellent agreement is remarkable in view of the fact that the main component is represented by a single exponential distribution of neutron exposures with only the scaling factor and the mean neutron exposure as fitting parameters. The fit gives values of $f \approx 0.06\%$, where it is assumed that the number of seed nuclei is equal to the solar system abundance of ⁵⁶Fe, $\tau_0 \approx 0.3 \text{ mb}^{-1}$ (for cross sections $\langle \sigma \rangle_A$ at kT = 30 keV), and $n_c \approx 10$ (Käppeler et al. 1990). These results imply that the main s-process component was produced by irradiating only 0.06% of the solar system ⁵⁶Fe nuclei with neutrons, while each ⁵⁶Fe seed nucleus captured on average about 10 neutrons. For mass numbers of A < 90, the thick solid line falls below the data points. Therefore, a second component is required in order to explain the synthesis of the s-process nuclides in this lower mass range. It is called the weak s-process component and is shown as the thin solid line in Fig. 5.64. Käppeler et al. (1990) find for this component values of f \approx 1.6%, $\tau_0 \approx 0.07 \text{ mb}^{-1}$ and $n_c \approx$ 3, that is, a much lower mean neutron exposure and a much higher fraction of irradiated seed nuclei compared to the main component. Only in the Pb-Bi mass region, close to the termination point of the s-process, does the two-component model give an unsatisfactory description. In particular, more than 50% of the solar system ²⁰⁸Pb abundance cannot be accounted for in this way. Therefore, a third component has been postulated (Clayton and Rassbach 1967). It is called the *strong s-process component*, for which parameters of $f \approx 10^{-4}$ %, $\tau_0 \approx 7 \text{ mb}^{-1}$, and $n_c \approx 140$ have been reported in Käppeler et al. (1990). In this case, the mean neutron exposure is so large that on average about 140 neutrons are captured per seed nucleus in order to convert a very small fraction of ⁵⁶Fe nuclei to nuclides in the mass region between ²⁰⁶Pb and ²⁰⁹Bi. As will be seen below, it is unlikely that these three vastly different neutron exposures can be obtained in a single astrophysical site. It is more reasonable to assume that different sites are required to explain each of the observed s-process components.

Both the observed $N_{\odot}(A)\langle\sigma\rangle_A$ values and the calculated solid lines in Fig. 5.64 are obtained for a constant s-process temperature of kT = 30 keV (or T = 0.35 GK). This particular value is traditionally used in discussions of the phenomenological s-process model described here. It is important to realize, however, that a precise value of the s-process temperature cannot be deduced easily by matching observed $N_{\odot}(A)\langle\sigma\rangle_A$ values with calculated $\langle\sigma\rangle_A \overline{N_s(A, \tau_0)}$ curves (except when analyzing branching ratios; see later) because most of the neutron-capture cross sections vary in a similar manner with temperature (Section 3.2.2 and Fig. 3.31). Rather, the shape of the $\langle\sigma\rangle_A \overline{N_s(A, \tau_0)}$ curve will give information about the mean neutron exposure τ_0 when the temperature has been selected by another means (see also Seeger, Fowler and Clayton 1965).

In the derivation of Eq. (5.182) it was explicitly assumed that all neutron capture rates on unstable nuclei are either much faster ($\lambda_{\beta} \ll \lambda_{n\gamma}$) or much slower ($\lambda_{\beta} \gg \lambda_{n\gamma}$) than the competing β^- -decay rates so that the s-process path is uniquely defined at each mass number *A*. At certain locations along the s-process path, however, the abundance flow encounters unstable nuclei with unusual decay constants (or half-lives) that are comparable in magnitude to the competing neutron capture rates, $\lambda_{\beta} \approx \lambda_{n\gamma}$. At these locations the s-process path splits into two branches. These s-process *branchings* can also be incorporated into the phenomenological s-process model described above if one assumes that the neutron density $N_n(t)$, in addition to the temperature, is constant with time. In this case, the s-process branchings can be described analytically (Ward, Newman and Clayton 1976). Otherwise, the abundance evolutions must be solved by numerical integration.

Consider as a simple example the situation shown in Fig. 5.62c. At the unstable nucleus of mass number A', the abundance flow splits into two parts because its β^- -decay rate is comparable in magnitude to the rate of the competing neutron capture. The unstable nucleus A' becomes a branch point for the s-process path. Only a fraction of the flow passes through stable nucleus A. But the entire flow passes through *stable* nucleus A + 1 since we assume that the β^- -decay of *unstable* nucleus A' + 1 is much faster than the competing neu-

tron capture. If the branch point is located in a mass region between closed neutron shells, then Eq. (5.173) has to be replaced by

$$N_s(A,\tau)\langle\sigma\rangle_A + N_s(A',\tau)\langle\sigma\rangle_{A'} \approx N_s(A+1,\tau)\langle\sigma\rangle_{A+1}$$
(5.185)

The ratio $N_s(A, \tau)\langle \sigma \rangle_A / N_s(A + 1, \tau)\langle \sigma \rangle_{A+1}$ defines a branching ratio, *B*, which can also be expressed in terms of the decay constants of nucleus *A*' as

$$B \equiv \frac{N_s(A,\tau)\langle\sigma\rangle_A}{N_s(A+1,\tau)\langle\sigma\rangle_{A+1}} = \frac{\lambda_\beta(A')}{\lambda_\beta(A') + \lambda_{n\gamma}(A')}$$
$$= \frac{\ln 2/T_{1/2}(A')}{\ln 2/T_{1/2}(A') + N_n\langle\sigma\rangle_{A'}v_T}$$
(5.186)

With $N_n \langle \sigma v \rangle_{A'} = N_n \langle \sigma \rangle_{A'} v_T$, we obtain

$$N_{n} = \left[\frac{N_{s}(A+1,\tau)\langle\sigma\rangle_{A+1}}{N_{s}(A,\tau)\langle\sigma\rangle_{A}} - 1\right] \frac{1}{\langle\sigma\rangle_{A'}v_{T}} \frac{\ln 2}{T_{1/2}(A')}$$
$$= \left(\frac{1-B}{B}\right) \frac{1}{\langle\sigma\rangle_{A'}v_{T}} \frac{\ln 2}{T_{1/2}(A')}$$
(5.187)

Hence, the analysis of branchings gives the neutron density which is an important parameter for determining the physical conditions during the s-process. A precise value of N_n provides a strong constraint for stellar models of sprocess sites.

Equation (5.187) describes the simplest case of an s-process branching. In reality, more extensive expressions are required for most branchings since each one has its own complications, for example, the interplay of several branchings or isomeric states. Nevertheless, Eq. (5.187) contains the important physics and, in particular, emphasizes the input data that are needed for a reliable extraction of the physical conditions from branching analyses. The first term, (1 - B)/B, depends on the ratio of abundances for the stable nuclides A and A + 1 and on the ratio of their neutron-capture cross sections. Depending on the value of *B*, these input values have to be known to about $\pm 1\%$ for many branchings so that the neutron density can be extracted with an error of, say, $\pm 10\%$. It is of obvious advantage if A and A + 1 are s-only nuclides since in this case their abundances do not have to be corrected for r-process contributions. Also, they are isotopes of the same element and, consequently, their relative abundances are accurately known (Lodders 2003). Very precise capture cross section measurements involving these stable target nuclei are crucial as well (Sections 4.6.2 and 4.6.3; see also Käppeler 1999). The second term in Eq. (5.187) contains the Maxwellian-averaged capture cross section for the radioactive branching point nucleus A'. In the past, no data existed for these reactions and the cross sections had to be estimated by using the

Hauser–Feshbach theory. Recently, however, a number of cross section measurements involving radioactive branching point nuclei have been performed (see, for example, Jaag and Käppeler 1995, Reifarth et al. 2003, Abbondanno et al. 2004). It should be noted that measured capture cross sections have to be corrected for (theoretical) stellar enhancement factors since the quantity $\langle \sigma \rangle_{A'}$ in Eq. (5.186) refers to the *stellar* cross section (Section 3.1.5). The third term represents the *stellar* half-life of the branching point nucleus A' and the corresponding stellar enhancement factors are, again, usually based on nuclear theory (Takahashi and Yokoi 1987). In some cases, there is no difference between the terrestrial and the stellar half-life value. For other branching point nuclei, however, the stellar half-life is very sensitive to the precise temperature or density conditions in the plasma (Section 1.8.4).

There are about 15–20 significant branchings on the s-process path. The following strategy is then employed in order to derive estimates for the physical conditions of the s-process. First, the mean neutron density is deduced by analyzing those branchings that are nearly independent of temperature and density. With this information, the stellar β -decay half-lives are determined from other branchings that depend sensitively on temperature (or density). Finally, the known temperature (or density) dependence of these half-lives yields estimates for the mean s-process temperature (or electron density). By considering several different branchings together, one may then attempt to derive a set of parameters that characterizes the average physical conditions during the s-process. The results thus obtained from the study of s-process branchings (N_n , T, ρ) and from the global fit to the observed $N_{\odot}(A) \langle \sigma \rangle_A$ distribution for s-only nuclides (f, τ_0 or n_c) represent important constraints for stellar models and the identification of the astrophysical sites of the s-process. For more information, see Käppeler (1999).

The empirical s-process described above is called the *classical s-process model*. It is very simple since it disregards the time dependence of s-process parameters, such as neutron density and stellar temperature. It provides a satisfactory description of most observed $N_{\odot}(A)\langle\sigma\rangle_A$ values for s-only nuclides over the entire mass region of interest, requiring only a relatively small number of adjustable parameters. The classical s-process model makes no assumption on the stellar site or the specific reactions which act as neutron sources. In view of these restrictions, the classical model offers remarkable insight into the s-process.

We already pointed out that the shape of the $N_{\odot}(A)\langle\sigma\rangle_A$ distribution for sonly nuclides is a measure for the total number of neutron captures to which seed nuclei have been subjected and, therefore, it contains the global history of the s-process. It must be emphasized that the composition of the interstellar gas out of which the solar system formed reflects a mixture of the ejecta of countless stars. The composition has been homogenized by interstellar mixing

to the degree where it represents the average rate of nucleosynthesis up to the time of solar system formation. The stars that provided the sites for the s-process had certainly a range of masses and metallicities. It is clear from these arguments that a single set of average parameters (f, τ_0 , N_n , T, ρ) derived from the $N_{\odot}(A)\langle\sigma\rangle_A$ distribution does not correspond directly to the properties of any single model star. For the same reasons, one must be careful when using such average parameters to constrain stellar s-process models.

The limitations of the classical s-process model became apparent with the availability of precisely measured neutron-capture cross sections (Käppeler 1999, Bao et al. 2000). It was shown, for example, that the classical model significantly overproduces ¹⁴²Nd (Arlandini et al. 1999). Such results imply that the distribution of neutron exposures during the s-process differs from a simple exponential function (see Eq. (5.175)). Further evidence came from branchings. Analyses of the branching point nuclei ¹⁴⁷Pm, ¹⁸⁵W, and ¹⁹²Ir with the classical model gave for the neutron density values of $N_{\rm n} = (4.94^{+0.60}_{-0.50}) \times 10^8 \text{ cm}^{-3}$ (Reifarth et al. 2003), $(4.7^{+1.4}_{-1.1}) \times 10^8 \text{ cm}^{-3}$ (Mohr et al. 2004), and $(7.0^{+0.5}_{-0.2}) \times 10^7$ cm⁻³ (Koehler et al. 2002), respectively. Similarly, classical analyses of the temperature-sensitive branching point nuclei 176 Lu, 151 Sm, and 128 I yielded values of $T = 0.30 \pm 0.05$ GK (Doll et al. 1999), \approx 0.4 GK (Abbondanno et al. 2004), and \approx 0.093 GK (Reifarth 2002), respectively. Clearly, the classical s-process model provides neither a consistent solution for the neutron density nor for the temperature. A more sophisticated approach, based on realistic stellar models, is required in order to reproduce all the observed s-process abundances.

We now turn to a discussion of stellar models that currently best reproduce the observed s-process abundance pattern. The main s-process component is thought to originate from thermally pulsing, low-mass (1.5–3 M_{\odot}) AGB stars (Section 1.4.3; see also Busso, Gallino and Wasserburg 1999). In this scenario, some protons are mixed below the H-rich envelope into the top layers of the intershell which consists mainly of ⁴He (\approx 75% by mass) and ¹²C (\approx 25% by mass). This occurs after the termination of a thermal pulse at a time when the He-burning shell becomes almost extinct. The star contracts again and the H shell ignites. The protons that are mixed downward initiate the sequence

$$^{12}C(p,\gamma)^{13}N(\beta^{+}\nu)^{13}C(p,\gamma)^{14}N$$
 (5.188)

giving rise to two separate regions in the intershell that are rich in ¹³C and ¹⁴N and are referred to as ¹³C pocket and ¹⁴N pocket, respectively. When the temperature reaches $T \approx 0.09$ GK (or $kT \approx 8$ keV), the mean lifetime of ¹³C versus destruction by the ¹³C(α ,n)¹⁶O reaction becomes smaller than the time between the two thermal pulses. Hence, neutrons are released within the ¹³C pocket and are captured by pre-existing seed nuclei (mainly Fe and s-processed material from the previous pulse) to produce most of the nu-

clides in the main component of the s-process. The neutron flux lasts typically ≈ 20000 y and produces locally high neutron exposures ($\approx 0.1 \text{ mb}^{-1}$). Since the time scale is long, however, the neutron density remains low $(N_n \approx 10^7 \text{ cm}^{-3})$. Only a small number of reaction branchings occur since the β^- -decay constant exceeds the neutron-capture decay constant in most cases. During this time, ¹³C is entirely consumed in the thin ¹³C pocket. Note that the temperature achieved at this stage of the evolution is not sufficient for initiating the ¹⁴N(α, γ)¹⁸O reaction. During H-shell burning, the mass of the intershell increases steadily (and so do temperature and density), up to a point where the He at the bottom of the intershell ignites. This *thermal He pulse* grows outward until it almost reaches the H-burning shell. The large energy release also causes the stellar envelope to expand and extinguishes the H-burning shell. The thermal pulse engulfs the ashes of H-shell burning. It gives rise to higher temperatures ($T \approx 0.27$ GK or $kT \approx 23$ keV), initiating the sequence

¹⁴N(
$$\alpha, \gamma$$
)¹⁸F($\beta^+ \nu$)¹⁸O(α, γ)²²Ne (5.189)

As a consequence, the ²²Ne(α ,n)²⁵Mg neutron source is (marginally) activated and a second neutron burst occurs. Here, the time scale amounts to a few years, with neutron exposures of $\approx 0.01 \text{ mb}^{-1}$ and a peak neutron density of $N_n \approx 10^{10} \text{ cm}^{-3}$. This second neutron burst does not contribute much to the overall production of s-process nuclides. It does however influence significantly the s-process branchings which are operating more efficiently at the higher temperatures. After the thermal pulse, the He shell becomes inactive, the envelope contracts, and the H shell ignites again. The cycle may repeat tens to hundreds of times. For more information, see Busso, Gallino and Wasserburg (1999) or Habing and Olofsson (2004).

Figure 5.65 demonstrates how well current stellar models of thermally pulsing AGB stars reproduce the solar system abundance distribution of s-process nuclides. The results were obtained for a model star with a mass of $1.5 M_{\odot}$ and a metallicity of Z = 0.01 (Arlandini et al. 1999). Abundances are shown as overproduction factors, that is, as ratios of predicted abundances and the corresponding solar system values. The solid circles represent s-only nuclides. The agreement is remarkable, especially in view of the fact that the solar system s-process abundances of the main component must be the products of countless low-mass AGB stars with a range of masses and metallicities. It is also evident that these stars cannot account for the weak s-process component (A < 90).

Stellar model studies of thermally pulsing, low-mass AGB stars (Gallino et al. 1998) revealed that variations in stellar metallicity have a strong effect on the resulting total neutron exposure. In this scenario, the ${}^{13}C(\alpha,n){}^{16}O$ or ${}^{22}Ne(\alpha,n){}^{25}Mg$ reactions are referred to as *primary* neutron sources because the



Fig. 5.65 Abundance distribution resulting from s-process studies of a thermally pulsing AGB star of mass 1.5 M_{\odot} and metallicity Z = 0.01. Abundances are shown as overproduction factors, that is, as ratios of predicted abundances and the corresponding solar system values, normalized to ¹⁵⁰Sm. It is evident that the stellar model reproduces the solar system abundances for the s-only nuclides (solid circles) of the main

s-process component (A > 90). Even the abundances of those s-only nuclides are reproduced that are partially bypassed by the flow due to nearby branchings. Crosses represent all the other heavy nuclides produced in the s-process. Their overproduction factors are less than unity since they are also synthesized by the r-process. From Arlandini et al. (1999). Reproduced by permission of the American Astronomical Society.

¹³C or ¹⁴N (and, hence, ²²Ne) are produced in the star itself from the available hydrogen and ¹²C. For decreasing metallicity, more neutrons per iron seed nuclei are available from these sources and, consequently, heavier nuclides can be synthesized. The increased neutron exposure during s-processing in early generation, metal-poor AGB stars causes an accumulation of material at the end of the s-process path (²⁰⁸Pb and ²⁰⁹Bi). These objects provide a natural explanation for the strong s-process component (Gallino et al. 1998, Travaglio et al. 2001).

The weak s-process component is believed to originate from the core He burning stage in massive stars with $M \ge 13 M_{\odot}$ (Sections 1.4.3 and 5.3.3; see also Peters 1968). The ¹⁴N nuclei produced by the CNO cycles during the preceding H burning stage are rapidly transformed to ²²Ne via ¹⁴N(α,γ)¹⁸F($\beta^+\nu$)¹⁸O(α,γ)²²Ne at the beginning of the He burning stage. But only near helium exhaustion in the core has the temperature risen sufficiently ($T \ge 0.25$ GK or $kT \ge 22$ keV) to ignite the ²²Ne(α,n)²⁵Mg neutron source. More massive stars burn at higher core temperature and they consume a larger quantity of ²²Ne. Therefore, they give rise to a more efficient s-process compared to less massive stars. Total consumption of ²²Ne occurs only in very massive stars. There are three aspects that are especially important when discussing the neutron economy during s-process nucleosynthesis: (i) the abundance of the neutron source nuclei (^{22}Ne) ; (ii) the abundance of the seed nuclei (⁵⁶Fe and other iron-peak species); and (iii) the abundances of any neutron poisons. The latter expression refers to nuclei that capture, and hence remove, neutrons while not contributing to the production of sprocess nuclei. For example, the neutron source ${}^{22}Ne(\alpha,n){}^{25}Mg$ is sometimes called *self-poisoning* because the product nucleus ²⁵Mg has a relatively high cross section for neutron capture. In fact, a significant fraction of the produced neutrons is removed in this way without synthesizing nuclei in the A = 65-90 region, thus constraining the s-process efficiency. In this scenario, the ${}^{22}Ne(\alpha,n){}^{25}Mg$ reaction is also referred to as a *secondary* neutron source since ¹⁴N, and hence ²²Ne, is not produced in the star itself. Both the number of neutrons released by 22 Ne(α ,n) 25 Mg and the amount of iron-peak seed nuclei (mainly ⁵⁶Fe) scale with stellar metallicity, while the neutron-to-seed ratio is metallicity independent. On the other hand, ¹²C and ¹⁶O are *primary* nuclides since they are produced within the star itself. Their neutron capture cross sections are relatively small but their abundances become large during helium burning. Hence, the ${}^{12}C(n,\gamma){}^{13}C$ and ${}^{16}O(n,\gamma){}^{17}O$ reactions may represent important sinks of neutrons, especially if the stellar metallicity is small. Detailed calculations have shown that, independent of metallicity, ¹²C does in fact not represent an important neutron poison because the lost neutrons are recycled, and thus recovered, by the sequence ${}^{12}C(n,\gamma){}^{13}C(\alpha,n){}^{16}O$. The situation is different for ¹⁶O where the sequence ${}^{16}O(n,\gamma){}^{17}O(\alpha,n){}^{20}Ne$ competes with ${}^{16}O(n,\gamma){}^{17}O(\alpha,\gamma){}^{21}Ne$. The neutrons are recovered in the former case, but are lost for the s-process in the latter case. Hence, ¹⁶O most likely represents an important neutron poison in low metallicity massive stars (Rayet and Hashimoto 2000). The strong metallicity dependence of the weak s-process component is important because it may be used to study the role of massive stars in the early phase of galactic chemical evolution.

Some important aspects of the weak s-process component derived from massive stars will be illustrated in the following. The evolution of central temperature and density from the end of core hydrogen burning to the end of core helium burning for a 25 M_{\odot} star with initial solar system composition is shown in Fig. 5.66 (The, El Eid and Meyer 2000). At the end of core hydrogen burning, the most abundant isotopes are ⁴He ($X_{\alpha} = 0.982$), ¹⁴N ($X_{14N} = 0.0122$), ²⁰Ne ($X_{20Ne} = 0.0016$), and ⁵⁶Fe ($X_{56Fe} = 0.00117$). Most of the other abundances are given by their respective solar system values (The, El Eid and Meyer 2000). Using this temperature–density profile and the initial abundances, a core helium burning (postprocessing) reaction network calculation is performed and the results are presented in Fig. 5.66. The neutron capture rates are adopted from the compilation of Bao et al. (2000), while temperature- and density-dependent weak interaction rates are taken from Raiteri et al. (1993). Energy
is produced via the helium burning reactions $\alpha(2\alpha)^{12}C(\alpha,\gamma)^{16}O$ (Section 5.3.2). At the end of the calculation, the mass fractions of ¹²C and ¹⁶O amount to 0.22 and 0.75, respectively. We will now discuss processes related to the production and consumption of neutrons. As already noted, the nuclide ¹⁴N is converted via the sequence ${}^{14}N(\alpha,\gamma){}^{18}F(\beta^+\nu){}^{18}O(\alpha,\gamma){}^{22}Ne$ (Section 5.3.3) while, subsequently, the ²²Ne(α ,n)²⁵Mg neutron source competes with the ²²Ne(α , γ)²⁶Mg reaction. The most important neutron poison reaction is ${}^{25}Mg(n,\gamma){}^{26}Mg$, followed by ²²Ne(n, γ)²³Ne. The sequence ¹²C(n, γ)¹³C(α ,n)¹⁶O shows a significant abundance flow but, in terms of the neutron economy, it is neither a net producer nor a net destroyer of neutrons (see also Section 5.5.1). Moving up in mass, a network of (n, γ) , (n, α) , (n, p) reactions and β^- -decays stretches from Al to the iron-peak group. Although the s-process in massive stars is usually interpreted as a way to produce the weak component, a number of lighter nuclei in the A = 35-45 mass range is also synthesized. In fact, the s-process in massive stars has been suggested to be the major source of ³⁶S, ³⁷Cl, ⁴⁰Ar, and ⁴⁰K in the Universe (Woosley, Heger and Weaver 2002). An increased nuclear activity is seen in the iron peak region. Starting mainly from ⁵⁶Fe seed nuclei, sequences of neutron captures and β^- -decays give rise to a typical s-process flow pattern and synthesize nuclei in the A = 60-90 region, that is, the weak component of the s-process. Smaller abundance flows extend beyond A = 90and are not shown in Fig. 5.66. The neutron exposure and peak neutron density typically amount to $\approx 0.2 \text{ mb}^{-1}$ and $N_n \approx 10^7 \text{ cm}^{-3}$, respectively. Note that most of the ²²Ne consumption occurs at the end of the burning, with less than 10% of helium remaining in the core, when the temperature increases from $T \approx 0.27$ to 0.30 GK and the density climbs from $\rho = 1800$ to 2600 g/cm³ (The, El Eid and Meyer 2000). In Fig. 5.66, this temperature-density range is labeled "S."

For stars with masses of $M < 30 M_{\odot}$, the partial survival of ²²Ne at the end of core helium burning opens the possibility of another episode of sprocessing during carbon burning. The ²²Ne(α ,n)²⁵Mg neutron source is reactivated by α -particles which are released by the primary ¹²C + ¹²C reaction (Section 5.5.1). Core carbon burning is not a promising s-process site since, first, the core matter will not be ejected in the subsequent supernova explosion and, second, any s-process nuclei will be destroyed later via photodisintegration reactions during core oxygen burning (Section 5.5.3). However, the situation is quite different for shell carbon burning. The higher temperatures achieved here ($kT \approx 90$) give rise to high peak neutron densities ($N_n \approx 10^{11} \text{ cm}^{-3}$) which may significantly alter the weak s-process abundance pattern obtained after core helium burning. Figure 5.67 shows overabundances of nuclides relative to their solar system values. The results are obtained from a stellar model calculation of a 25 M_{\odot} star with solar initial metallicity after the completion of core helium and shell carbon burning. The



Fig. 5.66 Time-integrated net abundance flows during core helium burning. The evolution of central temperature and density, shown in the inset, from the end of core H burning (A) to the end of core He burning (B) is adopted from stellar model studies of a 25 M_{\odot} star with initial solar system composition (The, El Eid and Meyer 2000). The numerical network calculation is terminated after $t = 6 \times 10^{12}$ s (the time it takes the star

to evolve from A to B). The arrows have the same meaning as in Figs. 5.36 and 5.40 except that four different thicknesses are used, with each thickness representing a flow range of two orders of magnitude. The flow pattern in the A = 60-90 region reflects the weak component of the s-process. Most of the s-processing occurs toward the end of He burning for $T-\rho$ conditions that are marked by "S" in the inset.

diamonds indicate s-only nuclides. The large overproduction values in the A = 60-90 mass range are evident. The efficiency of the s-process in massive stars declines rapidly beyond A = 90. Some lighter nuclides (A < 50) are also overproduced.

The experimental information for reactions important to s-process nucleosynthesis is briefly described below. The ${}^{13}C(\alpha,n){}^{16}O$ reaction (Q = 2216 keV), responsible for the synthesis of the main s-process component in low-mass AGB stars, has been measured down to a center-of-mass energy of $E_{\alpha}^{cm} =$ 280 keV (Drotleff et al. 1993). The Gamow peak for $T \approx 0.09$ GK is located at $E_0 \pm \Delta/2 = 190 \pm 40$ keV. The reaction rates in the astrophysically important temperature range are found by extrapolating the existing low-energy data, including the high-energy wing of a subthreshold resonance ($E_r^{cm} =$ -3 keV). The present uncertainty in this rate amounts to a factor of ≈ 4 at T = 0.09 GK (Angulo et al. 1999). This uncertainty seems to have a negligible

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Fig. 5.67 Overabundances of heavy nuclides relative to their solar system values after the completion of core helium and shell carbon burning. The results are obtained from a stellar model calculation of a 25 M_{\odot} star with solar initial metallicity. The diamonds indicate s-only nuclides. The

large overproduction values in the A = 60-90 mass range are evident. The efficiency of the s-process in massive stars declines rapidly beyond A = 90. From Raiteri et al. (1991b). Reproduced by permission of the American Astronomical Society.

influence on models of low-mass AGB stars (Cristallo, Straniero and Gallino 2005). The situation is quite different for the ${}^{22}Ne(\alpha,n){}^{26}Mg$ neutron source (Q = -478 keV). Here, the Gamow peak near $T \approx 0.25 \text{ GK}$ (the lower temperature limit where this neutron source becomes operational) is located at $E_0 \pm \Delta/2 = 540 \pm 120$ keV, while the lowest lying measured resonance occurs at E_r^{cm} = 704 keV. Experiments have focussed on the possible contribution from an undetected natural-parity resonance near $E_r^{cm} = 538$ keV, but recent work (Koehler et al. 2002) revealed the existence of several other naturalparity states in ²⁶Mg between the reaction threshold and the lowest lying measured resonance. The rates for this reaction are controversial at present. While Jaeger et al. (2001) and Karakas et al. (2006) estimate a factor of \approx 2 uncertainty near $T \approx 0.25$ GK, the error given in Angulo et al. (1999) amounts to a factor of \approx 170. The *recommended* rates reported by these authors, however, agree within 50%. Even a factor of 2 uncertainty in the rate has a strong influence on the nucleosynthesis both in low-mass AGB stars (Pignatari et al. 2005) and massive stars (The, El Eid and Meyer 2000). The ${}^{22}Ne(\alpha,\gamma){}^{26}Mg$ reaction may also be important in this respect since it competes with the (α, n) reaction in the destruction of ²²Ne without producing neutrons. The recommended rates for the ${}^{22}\text{Ne}(\alpha,n){}^{25}\text{Mg}$ and ${}^{22}\text{Ne}(\alpha,\gamma){}^{26}\text{Mg}$ reactions are of similar magnitude at $T \approx 0.25$ GK (Angulo et al. 1999, Karakas et al. 2006), but the present errors are too large to determine which reaction channel dominates near this temperature value. The ratio of rates for ${}^{17}O(\alpha, n){}^{20}Ne$ and ${}^{17}O(\alpha, \gamma){}^{21}Ne$ is also poorly known at present. These reactions are important for defining the role of ¹⁶O as a neutron poison in massive stars (Rayet and Hashimoto 2000).

For neutron-induced reactions in various s-process scenarios, Maxwellianaveraged cross sections have to be known at energies ranging from $kT \approx 8$ keV in low-mass AGB stars to $kT \approx 90$ keV during shell carbon burning in massive stars. The averaged (n,γ) cross reactions for the neutron poisons ¹²C, ¹⁶O, ²²Ne, and ²⁵Mg are experimentally known to better than ±10% (Bao et al. 2000). The most important neutron poison reaction in low-mass AGB stars is ¹⁴N(n,p)¹⁴C (Lugaro et al. 2003). In this case, the present errors in the averaged cross sections are somewhat larger (see Wagemans et al. 2000 and references therein) and more accurate values are desirable.

For a large number of nuclides in the mass region $A \leq 210$, Maxwellianaveraged neutron-capture cross sections are compiled in Bao et al. (2000) for s-process conditions. The data needs in terms of accuracy are quite different for charged-particle-induced reaction rates compared to neutron-capture reaction rates. In the former case, few reaction rates have been determined experimentally with errors of less than 10%. In the latter case, however, cross sections with uncertainties of \leq 5% are essential for modeling s-process scenarios. Recall that nuclei near the neutron magic numbers N = 50, 82, and 126act as bottlenecks for the abundance flow. In this case, the desired accuracy in the neutron capture cross section is \leq 3%. Even more accurate capture cross sections ($\leq 1\%$) are required for s-only isotopes. These nuclides represent crucial normalization points for the s-process abundance distribution and are also important for the analysis of s-process branchings (see above). For many of the important neutron-capture reactions the required level in cross section accuracy has been reached and the reliability of current cross section data sets for modeling s-process scenarios is quite impressive (Bao et al. 2000). Nevertheless, additional and more accurately measured cross sections are needed for a number of reactions. This also applies to (n, γ) reactions on short-lived branching point nuclei which have only recently become accessible to measurements (Jaag and Käppeler 1995, Reifarth et al. 2003, Abbondanno et al. 2004). Theoretical reaction rates are also indispensible for s-process calculations. First, the (n, γ) rates for many short-lived branching point nuclei are currently based on the Hauser–Feshbach theory (Section 2.7). It is worth mentioning that these rates can be calculated using local nuclear model parameters that are obtained via interpolation from neighboring nuclei. Such results are regarded as more reliable compared to neutron capture rates for nuclei far from the stability valley where *global* parameter sets must be employed. Second, there is little alternative but to estimate stellar enhancement factors (Section 3.1.5) using theoretical nuclear models. The calculations indicate that (n,γ) reactions on 25% of all nuclides involved in the s-process have stellar enhancement factors in the range of 2–40% at kT = 30 keV (Bao et al. 2000). Corrections at this level are quite significant for stellar models of the s-process.

5.6.2 The r-Process

In the previous section it was shown how well models of the s-process reproduce the solar system abundances of s-only nuclides (Figs. 5.65 and 5.67). For the majority of the heavy nuclides both the s-process and the r-process contribute to the observed abundance. Therefore, one may subtract the well-known s-process contribution from the total solar system abundance of a given nuclide ${}^{A}_{Z}X$ in order to find the corresponding solar system r-process abundance,

$$N_r(A,Z) = N_{\odot}(A,Z) - N_s(A,Z) = N_{\odot}(A,Z) - \frac{\langle \sigma \rangle_{A,Z} N_s(A,Z)}{\langle \sigma \rangle_{A,Z}}$$
(5.190)

The resulting N_r values versus mass number A are displayed in Fig. 5.68a. The s-process contributions are calculated by using the classical model according to Eq. (5.182) (see also Problem 5.12). It is apparent that the distribution of solar system r-process abundances is rather smooth and that it is also consistent with the abundances of the r-only nuclides which are shown as solid circles. Interestingly, a very similar solar system r-process abundance distribution is obtained if the s-process abundances are calculated by using stellar models instead of the classical approach (Arlandini et al. 1999). The most outstanding features in Fig. 5.68a are the two pronounced peaks at mass numbers A = 130and 195, which are about 10 mass units removed from the s-process peaks near A = 138 and 208. The existence of the r-process abundance peaks and of the long-lived radioisotopes ²³²Th ($T_{1/2} = 1.4 \times 10^{10}$ y), ²³⁵U ($T_{1/2} = 7.0 \times 10^{8}$ y), and 238 U ($T_{1/2} = 4.5 \times 10^9$ y), located beyond the endpoint of the s-process, provide the strongest evidence for the occurrence of a neutron-induced process that is quite different from the s-process discussed in the previous section. The solar system r-process abundance distribution represents a strong constraint for models of the r-process. Elemental solar system r-process abundances can be obtained easily by summation over isotopic values. These are most useful for comparison to results from stellar spectroscopy which, in most cases, provide only information on elemental abundances. The resulting elemental solar system s- and r-process abundances are displayed in Fig. 5.68b. It is remarkable that two processes so vastly different as the s- and r-process provide abundances of similar magnitude.

The most straightforward explanation of the r-process abundance peaks in Fig. 5.68a is that they are caused, like the s-process abundance maxima (Fig. 5.59), by the neutron magic numbers N = 50, 82, and 126 (Section 1.6.1). The large neutron flux drives the matter to the neutron-rich side, far away from the stability valley, where for reasons to be discussed later the abundances of the neutron magic nuclides accumulate. These neutron magic nuclides are proton deficient compared to their counterparts produced in the



Fig. 5.68 (a) Solar system r-process abundances for A > 90, obtained by subtracting the s-process contribution from the total solar system abundance. The s-process abundance is calculated by using the classical s-process model (Arlandini et al. 1999). The full circles show abundances of r-only nuclides, defined here as those species for

which the s-process contribution amounts to \leq 3%. The influence of the p-process on the displayed abundances is negligible and has been disregarded. The error bars are largest in those regions where the s-process contribution dominates. (b) Elemental solar system s- and r-process abundances; from Burris et al. (2000).

s-process which are located close to the valley of stability. After termination of the neutron flux in the r-process, these neutron magic nuclei undergo sequences of β^- -decays along isobaric chains (A = const) until the most neutron-rich stable (or very long-lived) isobar is reached (Fig. 5.60). Consequently, the r-process produces abundance maxima in mass regions located *below* the cor-

responding s-process abundance peaks. It is important to point out that the observed abundances of the r-process nuclides are not correlated with their neutron-capture cross sections, contrary to the case for the s-process abundances (Fig. 5.63). Rather, the observed r-process abundances reflect the nuclear properties of radioactive progenitors on the neutron-rich side far away from the stability valley.

We will now discuss a simple model for the r-process. Consider seed nuclei, say iron, which are exposed to a constant temperature of $T \ge 1$ GK and a constant neutron density of $N_n \ge 10^{21}$ cm⁻³. In such a hot and neutron-rich environment, both (n,γ) and (γ,n) reactions are much faster than β^- -decays. The abundance evolution of species $\frac{T}{2}X$ is then given by

$$\frac{dN(Z,A)}{dt} = -N_{\rm n}N(Z,A)\langle\sigma v\rangle_{Z,A} + N(Z,A+1)\lambda_{\gamma}(Z,A+1)$$
(5.191)

where N(Z, A) is the number density of nuclide ${}_Z^A X$; $\langle \sigma v \rangle_{Z,A}$ and $\lambda_{\gamma}(Z, A + 1)$ are the neutron-capture reaction rate per particle pair for ${}_Z^A X$ and the photodisintegration decay constant of ${}^{A+1}_Z X$, respectively. For sufficiently large values of N_n and T, the rates for neutron capture and reverse photodisintegration are large enough to ensure thermal equilibrium along the isotopic chain $[dN(Z, A)/dt \approx 0 \text{ for } Z = \text{const}]$. Under such conditions, the abundance ratios for two adjacent isotopes ${}^{A+1}_Z X$ and ${}^A_Z X$ are given by the Saha equation (see Eq. (3.49))

$$\frac{N(Z,A+1)}{N(Z,A)} = N_{\rm n} \left(\frac{h^2}{2\pi m_{A\rm n}kT}\right)^{3/2} \frac{(2j_{Z,A+1}+1)}{(2j_{Z,A}+1)(2j_{\rm n}+1)} \frac{G_{Z,A+1}^{\rm norm}}{G_{Z,A}^{\rm norm}} e^{Q_{\rm n\gamma}/kT}$$
(5.192)

where $Q_{n\gamma}$ is the reaction *Q*-value for the (forward) ${}^{A}_{Z}X(n,\gamma){}^{A+1}_{Z}X$ reaction or, equivalently, the neutron separation energy of ${}^{A+1}_{Z}X$.

It follows from Eq. (5.192) that the abundance ratio N(Z, A + 1)/N(Z, A) depends mainly on the *Q*-value (or neutron separation energy) and is a function only of the temperature *T* and the neutron density N_n during the r-process. The situation is shown in Fig. 5.69a. Within a given isotopic chain, $(n,\gamma) \leftrightarrow (\gamma,n)$ equilibria are established. The number abundance of any isotope in the chain can be found by successive application of the Saha equation, similar to the methods described in Section 5.5.4. If N_{x_m} is the number density of isotope x_m which is produced after *m* neutron captures on (an arbitrary)

species N_{x_0} , then (Problem 5.14)

$$N_{x_m} = N_{x_0} \frac{N_n^m}{\theta^m} \left(\frac{M_{x_m}}{M_{x_0} M_n^m}\right)^{3/2} \frac{g_{x_m}}{g_{x_0} g_n^m} \frac{G_{x_m}^{\text{norm}}}{G_{x_0}^{\text{norm}}} \exp\left[\frac{1}{kT} \sum_{j=0}^{m-1} Q_{x_j(n,\gamma)}\right]$$
$$\approx N_{x_0} \left(\frac{N_n}{1.188 \times 10^{34} T_9^{3/2}}\right)^m \exp\left[\frac{11.605}{T_9} \sum_{j=0}^{m-1} Q_{x_j(n,\gamma)}\right]$$
(5.193)

where the symbols have the same meanings as in Section 5.5.4. In the above numerical approximation, the number densities and Q-values are in units of cm⁻³ and MeV, respectively, while the normalized partition functions and the spins of the heavy nuclei are set equal to unity. Suppose first that all values of $Q_{n\gamma}$ in a given isotopic chain are the same. For a specific temperature and neutron density we can then solve Eq. (5.192) for that value of $Q_{n\gamma}$ which gives rise to the same abundances throughout the chain, $N(Z, A + 1) \approx N(Z, A)$. For example, with T = 1.25 GK and $N_n = 10^{22}$ cm⁻³, and again neglecting spins and normalized partition functions, we find a value of $Q_{n\gamma} \approx 3.0$ MeV. Of course, the neutron capture Q-values are not all equal but decrease on average when moving away from the stability valley toward the neutron dripline. In other words, closer to the stability valley, where $Q_{n\gamma} > 3$ MeV, we have N(Z, A + 1) > N(Z, A), while closer to the neutron dripline, where $Q_{n\gamma} < N(Z, A + 1) > N(Z, A)$ 3 MeV, we obtain N(Z, A + 1) < N(Z, A). Consequently, the equilibrium abundances are not all the same but will show a maximum close to that isotope for which the $Q_{n\gamma}$ -value amounts to about ≈ 3 MeV. Note that for a given temperature and neutron density the abundance maxima in all chains will occur at the *same* neutron capture *Q*-value ($Q_{n\gamma} \approx 3$ MeV for the conditions chosen above). According to Eq. (5.192), an increase in N_n shifts the abundance maxima in all isotopic chains toward the neutron-rich side (to smaller $Q_{n\gamma}$ -values), while a higher temperature moves the abundance maxima toward the less neutron-rich side (to larger $Q_{n\gamma}$ -values). Obviously, T and N_n are correlated in the sense that a variation in temperature can always be compensated for by a corresponding adjustment in neutron density in order to keep the location of the abundance maxima unchanged.

The situation just discussed represents an oversimplification because evenodd effects in nuclear binding energies caused by the pairing effect (Section 1.6.2) have been disregarded so far. Nuclides with an even number of neutrons have relatively small values of $Q_{n\gamma}$ and those with an odd number of neutrons have relatively large $Q_{n\gamma}$ -values. Thus, according to Eq. (5.193), the abundance maximum in each isotopic chain is identified with nuclides of even neutron number. A specific example is in order. Equation (5.193) is used to calculate the abundance distribution for neutron-rich selenium isotopes (A = 92-99) with the conditions T = 1.25 GK and $N_n = 10^{22}$ cm⁻³. The results are shown in Fig. 5.70a. The horizontal line represents a constant *Q*-value of 3 MeV. The actual $Q_{n\gamma}$ values (from Möller, Nix and Kratz 1997) are shown



Fig. 5.69 Basic building blocks of the rprocess path. Part (a) shows an isotopic chain in $(n,\gamma) \leftrightarrow (\gamma,n)$ equilibrium (waiting point approximation). For reasons of clarity, it is assumed that most of the abundance resides in a single isotope (shaded square). Part (b) shows how β^- -decays of the waiting point nuclei transfer matter from one

isotopic chain to the next. The steady flow approximation assumes that the abundance of each element Z is inversely proportional to the total β -decay constant of the chain. Part (c) shows the special case when the r-process path encounters a neutron magic number (see the text).

as a dashed line and display a pronounced odd-even structure because of the pairing effect. The abundance distribution (solid line) peaks at that even-N isotope where the average $Q_{n\gamma}$ -curve falls below 3 MeV, in this case ${}_{34}^{96}Se_{62}$ and, to a lesser degree, ${}^{94}_{34}Se_{60}$. A more quantitative criterion will be derived in Problem 5.15. In practice it is found that the abundance distributions are relatively sharp for given T and N_n . Only one or two even-N isotopes exist in any significant amounts. On the other hand, if the r-process is characterized by some spread in temperature and neutron density, then the abundance distributions will be broadened to include more values of A. It should also

be noted that thermal equilibrium may not be achieved throughout the entire isotopic chain. In particular, closer to the valley of stability the *Q*-values are large and, therefore, the photodisintegration rates become smaller. The reverse (γ ,n) reactions cannot balance the forward (n, γ) reactions so that these lighter isotopes are being rapidly destroyed. In general, this does not represent a problem for the above model since it is found that the (n, γ) \leftrightarrow (γ ,n) equilibrium condition holds for all isotopes with any significant abundance at equilibrium (Seeger, Fowler and Clayton 1965). Hence, the next step is clear. The even-*N* isotopes with significant abundances in each isotopic chain represent waiting points for the abundance flow. At these locations, the r-process path must continue via β^- -decays which are sufficiently slow as not to affect the equilibrium distribution in the isotopic chains (Fig. 5.69b). For this reason, the (n, γ) \leftrightarrow (γ ,n) equilibrium condition is also referred to as *waiting point approximation*.

The β^- -decays transfer matter from one isotopic chain to the next, where again an independent equilibrium within the chain is established (Fig. 5.69b). This repetitive sequence of events gives rise to the r-process path. The total β^- -decay probability of an isotopic chain with given value of *Z* can be defined by

$$\lambda_Z \equiv \sum_A p(Z, A) \lambda_\beta(Z, A) \tag{5.194}$$

where $p(Z, A) = N(Z, A)/N_Z$ is the abundance distribution in the chain for given values of *T* and N_n , normalized to the total abundance $N_Z \equiv \sum_A N(Z, A)$ belonging to element *Z*. Note that λ_Z depends explicitly on *T* and N_n through the equilibrium abundances p(Z, A). The time evolution of the total abundance N_Z is given by

$$\frac{dN_Z}{dt} = -\lambda_Z N_Z + \lambda_{Z-1} N_{Z-1} \tag{5.195}$$

where the first term describes the destruction of element Z via β^- -decay to element Z + 1, while the second term represents the creation of element Zvia β^- -decay from element Z - 1. The above expressions (see Eqs. (5.194) and (5.195)) determine the *elemental* abundance of each isotopic chain, while Eq. (5.193) determines the *isotopic* equilibrium abundances within each isotopic chain. For the boundary conditions of Eq. (5.195) one can assume that initially all nuclei are in a specific isotopic chain Z_0 : $N_Z(t = 0) = N_0$ for $Z = Z_0$ and $N_Z(t = 0) = 0$ for $Z \neq Z_0$. The general solution of the above set of differential equations is given by (Bateman 1910)

$$N_{Z_0}(t) = N_0 e^{-\lambda_{Z_0} t} (5.196)$$

$$N_Z(t) = N_0 \sum_{i=Z_0}^{Z} e^{-\lambda_i t} \frac{\lambda_i}{\lambda_Z} \prod_{\substack{j=Z_0\\j\neq i}} \frac{\lambda_j}{\lambda_j - \lambda_i} \quad \text{for } Z \neq Z_0$$
(5.197)



Fig. 5.70 Neutron capture *Q*-values (dashed lines) and abundance distributions (solid lines) for neutron-rich isotopes of (a) selenium, (b) palladium, and (c) indium. The $Q_{n\gamma}$ -values are adopted from Möller, Nix and Kratz (1997) and display a pronounced odd–even structure caused by the pairing effect. (For some of the isotopes shown, experimental values exist; see Audi, Wap-

stra and Thibault 2003). The horizontal lines represent a constant *Q*-value of 3 MeV. The abundance distributions are calculated using Eq. (5.193) assuming the conditions *T* = 1.25 GK and $N_n = 10^{22}$ cm⁻³. They peak at those even-*N* isotopes where the average $Q_{n\gamma}$ -curve falls below 3 MeV (see the text). Circles mark isotopes with neutron magic numbers.

provided that all values of λ_i are different which is a good assumption if these values are computed precisely. One sees from Eq. (5.197) that the abundance N_Z varies inversely with the corresponding total β -decay constant λ_Z . As was

the case with Eq. (5.171) in the discussion of the s-process, the above coupled equations (see Eq. (5.195)) are self-regulating, in the sense that they attempt to achieve a constant β^- -decay flow from one isotopic chain to the next, $dN_Z/dt \approx 0$. Hence, after a sufficient time has passed we obtain

$$\lambda_Z N_Z \approx \lambda_{Z-1} N_{Z-1}$$
 or $\lambda_Z N_Z \approx \text{const}$ (5.198)

This condition is referred to as *steady flow approximation*.

The importance of nuclides with neutron magic numbers N = 50, 82, and 126 for the r-process path will now be addressed. The situation is sketched in Figs. 5.69c and 5.71. Suppose the abundance flow reaches the isotopic chain with the neutron magic nucleus x as a member. This nucleus has an energetically favorable neutron shell configuration. As a result, the Q-value for the $x(n,\gamma)$ reaction is relatively small, while the Q-value for the preceding $(n,\gamma)x$ reaction is relatively large. The element palladium (Z = 46) is an example for this situation. The neutron capture Q-values in the A = 125-131region are shown in Fig. 5.70b. It is apparent that the neutron magic nucleus $^{128}_{46}$ Pd₈₂ coincides with the location at which the average $Q_{n\gamma}$ -curve falls below the 3-MeV line. Consequently, ${}^{128}_{46}Pd_{82}$ is by far the most abundant species in the chain and represents a waiting point. After the subsequent β^- -decay the process repeats itself in the next isotopic chain: the average $Q_{n\gamma}$ -value curve crosses the 3-MeV line at the location of the neutron magic nucleus (in this specific case, ${}^{129}_{47}Ag_{82}$) which becomes another waiting point. Therefore, a sequence of waiting points is encountered at the same magic neutron number N. The r-process path has no choice but to move vertically upward in Z toward the stability valley (Fig. 5.69c). Moreover, the closer the path approaches the group of stable nuclei the longer the β^- -decay half-lives of the neutron magic nuclei become. Typical half-lives along the r-process path amount to $T_{1/2} \approx$ 0.01–0.05 s, but near neutron magic waiting point nuclei close to the stability valley (for example, ${}^{130}_{48}$ Cd₈₂) they are much longer. Hence, the abundance flow is significantly delayed and these isotopes will build up to relatively large abundances. An interesting situation occurs when the neutron magic nucleus y is reached (Fig. 5.69c). A specific example is the element indium and the corresponding $Q_{n\gamma}$ -values are shown in Fig. 5.70c. As was the case before, the neutron-capture Q-values drop significantly at the location of the neutron magic nucleus $\binom{131}{49}$ In₈₂). However, this isotope is located closer to the stability line compared to the lighter neutron magic nuclei. The extra stability is reflected in the larger overall neutron-capture Q-values. In fact, the average $Q_{n\gamma}$ -curve now drops below the 3-MeV line at a location beyond the neutron magic nucleus ${}^{131}_{49}$ In₈₂ (in this case, at ${}^{133}_{49}$ In₈₄). In other words, the r-process path overcomes the group of isotones with neutron magic number N at a location sufficiently close to the region of the stable nuclei (Fig. 5.69c).

At the cessation of the neutron flux, the neutron-rich nuclei β^- -decay along lines of constant *A* to their stable isobars. Thus, the r-process produces one stable (or long-lived) isotope for each value of *A*. In the simplest case we may assume that the neutron flux and temperature fall instantly to zero after some time τ . Then, knowing the isotopic abundances for each *Z* existing at a time τ when the r-process is halted, one can find the final r-process abundance at each *A* by summing

$$N_{r,A} = \sum_{Z} N_Z(\tau) p(Z, A)$$
(5.199)

To summarize, for given values of N_n , T, and τ the r-process path (and hence the abundances $N_{r,A}$) can be calculated precisely if the nuclear properties are known. The neutron-capture Q-values determine the isotopic equilibrium abundances for each element (see Eq. (5.193)), while the relative amount of material at a given element Z depends only on the total β^{-} -decay probabilities of the isotopic chains (see Eq. (5.197)). Cross sections for neutron captures or photodisintegrations are unimportant since we adopted the waiting point approximation. Furthermore, the time it takes to establish an $(n,\gamma) \leftrightarrow (\gamma,n)$ equilibrium is negligible compared to the β^- -decay half-lives which determine the time delay of the r-process flow toward heavier nuclei. The closer the r-process path is located to the neutron dripline, the shorter the β^- -decay half-lives become, resulting in a faster r-process flow. The longest delays for the abundance flow are expected near the neutron magic waiting point nuclei that are located closest to stability: ${}^{80}_{30}$ Zn₅₀, ${}^{130}_{48}$ Cd₈₂, and ${}^{195}_{69}$ Tm₁₂₆. The abundances near these locations accumulate and, after termination of the neutron flux, give rise to the abundance peaks at A = 80, 130, and 195 in the distribution of observed $N_{r,A}$ values (see Fig. 5.68a).

We assumed in Eq. (5.199) that β^- -decays are the only processes responsible for the decay of nuclei from the r-process path back to the stability valley after freeze-out of the neutron exposure. There are a number of other processes, however, that also need to be taken into account. First, β^- -decays may not always populate neutron bound states in the daughter nucleus. If neutron unbound states are populated, then β -delayed neutron emission may occur (Section 1.8.2). This process has the tendency to smooth out the strong oddeven signatures in the equilibrium abundances, caused by the dependence of $Q_{n\gamma}$ on the neutron number, that would otherwise be present in the final rprocess distribution. Second, in the higher mass range (Z > 80), spontaneous and β -delayed fission may become faster than β^- -decay and will thus influence the final abundances. Third, beyond mass A = 210 the decay toward the stability valley reaches (β -stable) radioactive α -particle emitters. The transmutation of these nuclei along α -decay chains gives rise to the production of the very long-lived nuclides ²³²Th, ^{235,238}U, and ²⁴⁴Pu which are important for nucleochronology (see, for example, Truran et al. 2002).

The termination of the r-process depends, among other things, on the duration of the neutron exposure and thus on the astrophysical environment. For relatively short neutron exposures, the r-process terminates due to a lack of free neutrons sometime before the mass region $A \approx 260$ is reached. For longer neutron exposures, the successive addition of neutrons will continue until the Coulomb barrier, which is proportional to Z^2 , becomes so large that the heavy nuclei decay via neutron-induced or β -delayed fission. Calculations indicate that this happens near $A_{\text{max}} \approx 260$ and $Z_{\text{max}} \approx 94$. The precise location depends sensitively on (yet unmeasured) fission barriers for nuclei far from stability (Panov et al. 2005). After the fission of a heavy nucleus with mass A_{max} , two fragments with masses of roughly $A_{max}/2$ are produced, thereby feeding two seed nuclei back into the neutron-capture chain and giving rise to a fission cycle. The number of r-process nuclei is doubled with each cycle. The cycle time, τ_{cvcle} , required to build an average fission fragment back up to A_{max} may only take a few seconds or less. If the neutron supply lasts sufficiently long, $\tau \gg \tau_{\rm cycle}$, the abundances will grow exponentially as nuclei pass around the fission cycle and large abundances of heavy nuclei can be build up in this way. Fission can also be incorporated into the phenomenological r-process model described above by adding a term to Eq. (5.195). Analytical solutions are given in Seeger, Fowler and Clayton (1965).

The nuclear properties required to describe the r-process include neutron capture Q-values (or neutron separation energies), β^- -decay half-lives, branching ratios for β -delayed neutron emission, normalized partition functions, fission probabilities, and α -decay half-lives. Nuclear masses play a central role for the r-process since they determine directly or indirectly most of the properties listed above. Recall also that the $Q_{n\gamma}$ -values (which are given by mass differences; see Eq. (1.6)) enter exponentially in the determination of the equilibrium abundances (see Eq. (5.192)) and, therefore, must be known rather accurately. It is obvious that the nuclear properties are needed for nuclides that are located far away from the valley of stability. The experimental information will be summarized later in this section. At this point it is sufficient to mention that, with few exceptions, the required information is not known from experiments since most of the nuclei on the r-process path cannot be produced at present in the laboratory. There is little alternative but to estimate the required nuclear properties by using nuclear models. The various models will not be discussed here (see, for example, Cowan, Thielemann and Truran 1991). In practice, attempts are made to derive semiempirical formulas from the known properties of nuclei close to stability that can be extrapolated into the region covered by the r-process path. Such extrapolation procedures are subject to significant uncertainties even for the most sophisticated models. For example, Möller, Pfeiffer and Kratz (2003) estimate an uncertainty of ± 0.5 MeV in calculated values of $Q_{n\gamma}$ and Q_{β} , while half-lives and branching

ratios for β -delayed neutron emission can only be predicted within a factor of 2–3 for nuclei far from stability. It remains to be seen if new *global* mass models can be developed that are not subject to these limitations. Obviously, any deficiencies in current nuclear models will have a direct impact on r-process predictions. The associated nuclear physics uncertainties affect most discussions of the r-process.

The phenomenological model discussed above is referred to as the *classical r*-process model. It is rather simple because it assumes: (i) a constant temperature and neutron density, (ii) an instantaneous termination of the neutron flux after a duration τ , and (iii) the waiting point and steady flow approximations. Clearly, the waiting point approximation will only hold for sufficiently large values of *T* and *N*_n (Goriely and Arnould 1996, Rauscher 2004). Otherwise, the abundance flow in each isotopic chain of given element *Z* is steadily depleted by β^- -decays before the actual waiting point in the chain is reached. The steady flow approximation is only valid if the duration of the neutron exposure exceeds the β^- -decay half-lives of nuclei on the r-process path. Finally, the assumption of a sudden termination in the neutron density disregards the fact that (n, γ) and (γ ,n) reactions will drop out of equilibrium if N_n decreases over a short, but finite, time.

How the classical model can provide insight into the astrophysical conditions of the r-process is demonstrated below. The discussion follows the arguments given in Kratz et al. (1988) and Kratz et al. (1993) to which the reader is referred for details. Consider as an example Fig. 5.71 showing the r-process path near the neutron magic number N = 82. If the waiting point approximation holds, then the path moves vertically upward through $\frac{127}{45}$ Rh₈₂, $\frac{128}{46}$ Pd₈₂, $^{129}_{47}$ Ag₈₂, and $^{130}_{48}$ Cd₈₂ before it branches off horizontally toward heavier nuclei. As discussed earlier, these nuclides are by far the most abundant species in their respective isotopic chains because of the sudden drop of the $Q_{n\gamma}$ -value near neutron magic numbers. For the Z = 49 chain, however, most of the abundance resides in ${}^{131}_{49}In_{82}$ and ${}^{133}_{49}In_{84}$ (Fig. 5.70c). After termination of the neutron flux, the decays of the nuclides near N = 82 give rise to the observed A = 130 solar r-abundance peak. The nuclides ¹³⁰Cd and ¹³¹In will β^- -decay to the stable isobars $^{130}\mathrm{Te}$ and $^{131}\mathrm{Xe},$ respectively. The isotope $^{133}\mathrm{In},$ on the other hand, has a large probability for β -delayed neutron decay and thus decays mainly to the stable nuclide ¹³²Xe. Assuming in addition a steady flow approximation (see Eq. (5.198)) for the Z = 48 (cadmium) and 49 (indium) isotopic chains, we can calculate a value for the ¹³⁰Cd half-life by using the observed solar system r-abundances of ¹³⁰Te, ¹³¹Xe, and ¹³²Xe and the measured half-lives of 131 In and 133 In (Problem 5.16). The result is $T_{1/2}^{\text{calc}}(^{130}\text{Cd}) \approx 187 \text{ ms}$ which is close to the experimental value of $T_{1/2}^{exp} = 162 \pm 7$ ms (Dillmann et al. 2003). Hence, it appears that the solar system r-process peak at A = 130was indeed formed under the conditions of an $(n,\gamma) \leftrightarrow (\gamma,n)$ equilibrium and

a steady flow equilibrium. Similar arguments can be applied to the A = 80r-process peak. We may also estimate the conditions of temperature and neutron density that gave rise to the observed r-abundances. Earlier we calculated equilibrium abundances from given $Q_{n\gamma}$ -values assuming values for T and N_n (Fig. 5.70). The argument can be turned around in order to derive constraints on T and N_n from known equilibrium abundances in a specific isotopic chain. As an example, consider again the pair ¹³¹In and ¹³³In. From the observed solar system r-abundances of ¹³¹Xe and ¹³²Xe (Arlandini et al. 1999) one may derive, after correcting for β -delayed neutron decays (see Problem 5.16), the equilibrium abundances of the precursors ¹³¹In and ¹³³In on the r-process path. These determine, according to Eq. (5.193), the temperature and neutron density if the $Q_{n\gamma}$ -values are known either from experiment or theory. Results of such a procedure for isotopes near $N = 82 (^{131,\overline{1}33}$ In) are shown as dashed lines in Fig. 5.72. For example, one finds that a neutron density of $N_{\rm n} \approx 10^{22} {\rm ~cm^{-3}}$ corresponds to a temperature near $T \approx 1.35 {\rm ~GK}$. As already noted, T and N_n are correlated and thus the possible solutions are located anywhere on the dashed lines.

Attempts to describe the entire observed distribution of solar system rabundances by using the classical r-process model with a single set of $T-N_n-\tau$ conditions were unsuccessful (Kratz et al. 1993). Such global descriptions reproduce three r-abundance peaks, but neither at the correct mass number location nor with the correct magnitude. It is also found that at least three different sets of $T-N_n-\tau$ conditions, each corresponding to a specific r-process path, are required in different mass ranges in order to reproduce the observed solarsystem abundance distribution. Furthermore, the steady flow approximation applies *locally* in each of these mass ranges, but not globally over the entire mass region. Each of the components proceeds up to one of the r-abundance peaks (A = 80, 130, or 195) and achieves a local steady flow equilibrium. However, the steady flow equilibrium breaks down beyond the maximum of each peak where the half-lives of the r-process progenitors are relatively long (\approx seconds). This may indicate that the duration of the neutron exposure is large compared to most of the relatively short β^- -decay half-lives on the r-process path, but is comparable to the longer half-lives of the neutron magic nuclei that come closest to stability. The overall implication is that the solar system r-abundance distribution results from the superposition of components representing different r-process conditions. This may be caused by several different astrophysical r-process sites or by a single site with varying conditions in different zones. An example for the comparison of observed solar system r-abundances and the results of the classical r-process model is shown in Fig. 5.73. The model predictions are obtained from a superposition of three different r-process components: (i) T = 1.35 GK, $N_n = 3 \times 10^{20}$ cm⁻³, $\tau = 1.5$ s for $A \approx 80$; (ii) T = 1.2 GK, $N_n = 1 \times 10^{21}$ cm⁻³, $\tau = 1.7$ s for A = 90-130; and



Fig. 5.71 Schematic r-process path near $A \approx 130$ and $N \approx 82$. Numbers near diagonal arrows represent β^- -decay half-lives (in seconds) and those near horizontal arrows show branching ratios (in percent) for β -delayed neutron decay. The quoted values are adopted from experiment or, when preceded by "~", from nuclear model calculations. Stable end products of the r-process (after freeze-out) are shown in circles and their observed solar system r-abundances are given in square boxes. More recent information on nuclear properties and abundances can be found in Audi et al. (2003),

Möller, Nix and Kratz (1997), and Lodders (2003). Note that ¹³⁰Cd is the neutron magic waiting point nucleus with N = 82 that is located closest to stability. At the next element (indium), the r-process path branches off horizontally toward heavier nuclei (see the text). The nuclide ¹³⁰Cd is the progenitor of the stable isobar ¹³⁰Te which is situated at the maximum of the A = 130 peak in the solar system r-process abundance distribution. Reprinted with permission from K.-L. Kratz et al., J. Phys. G, Vol. 14, p. 331 (1988). Copyright (1988) by IOP Publishing Ltd.

(iii) T = 1.2 GK, $N_n = 3 \times 10^{22}$ cm⁻³, $\tau = 2.5$ s for A = 135-195. The weights of the components are 10 : 2.6 : 1. Note that the $T-N_n$ values of each component do not represent a unique set, but similar r-abundances are obtained for all values that are located on an extended boundary in the $T-N_n$ diagram (see Fig. 5.72 and Fig. 12 of Kratz et al. 1993). Some of the deviations between observed and calculated abundances (lower part of Fig. 5.73) originate from



Fig. 5.72 Conditions for temperature and neutron density in the r-process. The dashed lines are derived from the equilibrium abundance ratio of ¹³³In-¹³¹In. The abundance ratio of these isotopes, which are located on the r-process path, is deduced from the observed solar system r-abundances of ¹³²Xe and ¹³¹Xe. The two

dashed lines are obtained from two different mass formulas. The region between the two solid lines shows the conditions at which all N = 82 species between ¹²⁷Rh and ¹³⁰Cd represent waiting point nuclei. Reprinted with permission from K.-L. Kratz et al., J. Phys. G, Vol. 14, p. 331 (1988). Copyright (1988) by IOP Publishing Ltd.

systematic defects of the mass model used to compute the nuclear properties (Freiburghaus et al. 1999).

There is no obvious reason why the observed solar system r-abundance distribution should be the result of a superposition of only three components. In fact, if either a number of astrophysical sites or different zones representing different conditions in the same site contribute to the observed r-abundances, then it would be reasonable to assume a superposition of many different components. Following this line of thought, some researchers employ many components, each with its associated fitting parameters, and thereby achieve an almost perfect agreement between predicted and observed r-abundances (see, for example, Goriely and Arnould 1996). However, such results mask deficiencies in calculated nuclear properties and, at the same time, may compromise the predictive power of the model beyond the mass range of fitted abundances. Other researchers employ a continuous superposition of r-process components, assuming constant temperature and power-law distributions for component weights and exposure times as a function of neutron density. This procedure requires only a small number of fitting parameters and yields a slight improvement in the predicted r-abundances compared to the results shown in Fig. 5.73 (Freiburghaus et al. 1999). However, neither method seems to reflect directly the physical properties of a realistic r-process site.

Figure 5.74 shows some results obtained with the second procedure, that is, by assuming a continuous superposition of r-process components. Each component is characterized by constant values of T, N_n , and τ . The component



Fig. 5.73 Distribution of observed solar system r-abundances (data points) compared to predictions of the classical r-process model (solid line). The solid line is calculated from Eqs. (5.193) and (5.197) and depends only on neutron separation energies, β^- -decay half-lives, β -delayed neutron decay probabilities, and so on, but not on

cross sections for neutron capture or photodisintegration. The model prediction is obtained from a superposition of three different r-process components (see the text). From Kratz et al. (1993). Reproduced by permission of the American Astronomical Society.

weights and neutron exposure time scales are given by $\omega(N_n) = a_1 N_n^{a_2}$ and $\tau(N_n) = a_3 N_n^{a_4}$, respectively, where the a_i are fitting parameters. The temperature remains constant at T = 1.35 GK. Since different $T-N_n$ conditions correspond to different r-process paths, the overall distribution of waiting point nuclei (large open or solid squares) in each isotopic chain is broadened compared to the use of a single component. The resulting r-process abundance flow pattern represents more appropriately a *boulevard* rather than a narrow path (Kratz 2006). Nevertheless, it can be seen that the abundance flow for all components is funneled through the isotones with neutron magic numbers (N = 82 and 126) before reaching the $A \approx 260$ region. At the cessation of the neutron flux, the short-lived nuclei on the r-process path decay via β^- -decay, β -delayed neutron emission, α -decay and fission (the latter two decays occur only in the region A > 210) and transmute into stable or long-lived isotopes (small solid squares).

The phenomenological model described above makes no assumption regarding the site of the r-process. It is nevertheless quite useful and provides insight into several aspects. As we have seen, the classical r-process model de-



Fig. 5.74 Results of a classical r-process calculation assuming a continuous superposition of many components. The components are determined by power-law distributions of component weights and exposure durations as functions of neutron density (see the text). The large open and solid squares show all waiting point nuclei that, after instant freeze-out of the neutron exposure, contribute more than 1% to the abundance of any stable or long-lived nuclide (small solid squares). The large solid

squares are a subset of waiting point nuclei that decay along the path marked with grey squares and contribute to the production of the long-lived chronometers ²³⁸U and ²³²Th. Nuclei that fission after production via β^- -decay are shown as triangles. The calculation is based on the ETFSI-Q mass model (Pearson, Nayak and Goriely 1996). From Schatz et al. (2002). Reproduced by permission of the American Astronomical Society.

scribes the gross behavior of the solar system r-abundance distribution. It has also been applied for reproducing or predicting abundance ratios of neighboring nuclei, for example, for chronometer nuclides or isotopic anomalies in primitive meteorites. Such abundance ratios are most likely influenced by nuclear properties rather than by the details of the astrophysical r-process site. The classical r-process model also provides a simple framework for studying the impact of nuclear physics uncertainties on predicted r-abundances. But it is also clear from the preceding discussion that the classical model does not account for the observations in terms of a realistic astrophysical site.

It is worthwhile at this point to visualize the results of a dynamic r-process calculation as opposed to the static models described up to now. Clearly, in reality the temperature and neutron density will change with time. Early during the r-process, T and N_n will be sufficiently large to ensure that a $(n,\gamma) \leftrightarrow (\gamma,n)$ equilibrium holds in all isotopic chains. Now suppose that the temperature and neutron density decrease with time. The abundance flow will continuously adjust to the new conditions according to the waiting point approximation. This means that the r-process path, which is defined by T and N_n , must continuously move, starting from a location closer to the neutron dripline to one that is located closer to stability. For each location of the path, the β^- -

decay half-lives are different. Just before termination of the neutron exposure, when the r-process path is still about 15–35 mass units away from stability, the waiting point nuclei have neutron capture *Q*-values of $Q_{n\gamma} \approx 2-4$ MeV. When the neutron flux disappears, the r-process nuclei decay toward stability. But it is apparent that only the r-process path just before freeze-out, and in particular the sections near neutron magic nuclei, matters for the observed final distribution of r-abundances. In other words, at freeze-out the r-process has mostly forgotten its earlier history.

Before discussing possible locations of the r-process, we briefly mention some of the observational evidence. Up to now, we have only focused on reproducing the observed solar system r-abundance distribution by using the classical r-process model. In addition, important conclusions can be drawn from stellar spectroscopy. Figure 5.75 displays the total heavy element abundances (data points) for an extremely metal-poor Galactic halo giant star, compared to the solar system r-abundances (solid line). This star is among the oldest in the Galaxy. The remarkable agreement between the two abundance distributions for elements above barium ($A \ge 135$) provides strong evidence that most of the heavy elements observed in this star were synthesized by the r-process early during the evolution of the Galaxy, with no apparent contribution from the s-process. Similar results have been found for other ultra-metal poor halo giants (Truran et al. 2002). The r-process elements were clearly not synthesized in the halo stars themselves. They must have been produced by progenitors that evolved very rapidly and that ejected the matter into the interstellar medium before the formation of the currently observed halo giants. The most likely candidates seem to be associated with massive stars since lowmass and intermediate-mass stars (the site of the s-process) evolve on much longer time scales. Note that the abundance pattern in these very old Galactic halo stars may have received contributions from only one or at most a few r-process events. The agreement with the solar system r-abundances above mass $A \approx 135$ supports the conclusion that the r-process mechanism is robust in the sense that a similar abundance pattern is produced in each r-process event. It can also be seen that the agreement between stellar and solar system r-abundances does not extend to the lighter elements below barium (A < 135). This may be explained in different ways. Either there are (at least) two different sites for the r-process or a single site with different sets of conditions may account for the observations. See also Wasserburg, Busso and Gallino (1996) and Cowan and Sneden (2004).

A major goal of r-process studies is the identification of the astrophysical site(s) and, by using r-abundances observed in the solar system and in stars, to draw conclusions regarding its detailed properties. This procedure has been immensely successful, for example, in the case of the s-process. However, the site of the r-process remains a mystery. Many different objects have been sug-

gested (see the summary in Cowan, Thielemann and Truran 1991) but only a few of these seem promising. We already mentioned that observations place the beginning of r-processing very early in the evolution of the Galaxy and, therefore, the r-process site is most likely related to massive stars. It is also clear that the site must provide very high neutron densities ($N_n \ge 10^{21}$ cm⁻³) over short time scales (\approx seconds). On the other hand, the temperature should not be too high. Otherwise, the heavy nuclei will either be destroyed by photodisintegration (see Sections 5.5.4 and 5.5.5) or the waiting point abundances shift too close to the stability valley (see Eq. (5.192)) where the β^- -decay halflives are far too slow to allow for efficient r-processing.

One possible site is the merger of two neutron stars (Rosswog et al. 1999). Calculations show that the matter ejected in such events has a solar system composition in the mass range of $A \ge 140$. A problem with this source is that the event rate is perhaps too low and hence the required mass of r-process matter ejected per event would be too large to be consistent with observations (Qian 2000). A second possibility is the ejection of neutronized material in magnetized jets from asymmetric massive star explosions (Cameron 2003). Unfortunately, the thermodynamic conditions for this model are at present poorly determined. The third proposed site, which received the largest attention in recent years, involves the neutrino-powered wind from a neutron star resulting from a type II supernova (Woosley et al. 1994, Takahashi, Witti and Janka 1994).

Although the nature of these sites varies significantly, the basic nuclear rearrangements are similar. We already pointed out that any successful r-process site requires very large neutron densities, sufficiently high temperatures, and rapid expansion time scales. It is worthwhile to discuss briefly the nucleosynthesis in the last scenario in order to better understand the origin of the neutron flux that may give rise to an r-process. The neutrino-powered wind at late times has a relatively large neutron excess η . According to nuclear statistical equilibrium, matter at very high temperatures consists mainly of neutrons and protons (Fig. 5.56). As the wind expands and cools below $T \approx 10$ GK, nucleons start to combine to α -particles. A few α -particles also reassemble into nuclei via the strongly density dependent processes

$$2\alpha + n \to {}^{9}\text{Be}(\alpha, n)^{12}\text{C}$$
(5.200)

$$3\alpha \to {}^{12}\mathrm{C}$$
 (5.201)

At $T \approx 7$ GK, α -particles become the dominant constituent, leaving behind an excess of neutrons. By this time, nuclear statistical equilibrium breaks down because the expansion time scale becomes shorter than the time required to maintain nuclear statistical equilibrium under conditions of high temperature, low density, and large α -particle abundance. In fact, the above two sequences

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Fig. 5.75 Total heavy element abundances (data points) of the metal-poor Galactic halo star CS 22892-052. The solar system r-abundances (solid line) are shifted to compensate for the difference in metallicities. The abundance of element *x* is defined as $\log \varepsilon(x) \equiv \log(N_x/N_{\rm H}) + 12.0$. Residuals of the data points and the solid line are shown in the lower part. The remarkable agreement between the two abundance

distributions for elements above barium $(A \ge 135)$ provides strong evidence that most of the heavy elements observed in this star were synthesized by the r-process early during the evolution of the Galaxy. This figure originally appeared in the Publications of the Astronomical Society of the Pacific (Truran et al. 2002). Copyright 2004, Astronomical Society of the Pacific; reproduced with permission of the Editors.

are the first reactions to drop out of equilibrium. The situation is very similar to the α -rich freeze-out that was discussed in Section 5.5.5. There is again an excess of α -particles and these cannot be consumed fast enough by the slow helium-induced reactions in the time available. The important difference, however, is that the neutron excess is now relatively large. Both the abundant α -particles and the neutrons will participate in the buildup of heavier nuclei, starting with the sequence

$$^{12}C(n,\gamma)^{13}C(\alpha,n)^{16}O(\alpha,\gamma)^{20}Ne...$$
 (5.202)

Recall from the discussion in Section 5.5.5 that without the presence of neutrons, the α -rich freeze-out produces mainly the N = Z isotope ⁵⁶Ni, while the flows beyond the iron peak are negligible. This is so because the Q-values for (α, γ) reactions beyond ⁵⁶Ni along the N = Z line are relatively small. These nuclei are located on the proton-rich side of the stability valley and little binding energy is gained by adding another α -particle. As a consequence of the relatively small Q-values, photodisintegration prevents the synthesis of species beyond the iron peak. The capture of neutrons has the important effect of shifting the abundance flow toward the stability valley where the nuclei are more neutron rich and the α -particle separation energies are larger. Hence, photodisintegration does not terminate the nucleosynthesis anymore and the abundance flow may extend far beyond the iron peak up to $A \approx 100$. The most important processes in the buildup of these heavy nuclei are (α, n) and (n,γ) reactions. This *neutron rich* α *-rich freeze-out* is sometimes referred to as the α -process. The same name was originally given by Burbidge et al. (1957) to a process that in modern terminology is referred to as neon burning (Section 5.5.2). At a temperature of $T \approx 2$ GK, the α -induced reactions become too slow to change the composition of the matter. If there are enough neutrons left, then an r-process can be launched with nuclei in the $A \leq 100$ region, instead of iron peak species, as seeds. The r-process model described above has certain advantages. First, the properties of the neutrino wind are determined by the neutron star, not by the pre-supernova evolution, and thus r-processing in this site may produce similar abundances for events involving neutron stars of the same mass. Second, starting from seeds with $A \approx 100$ implies that the r-process does not need to overcome the waiting point nuclei near the closed neutron shell at N = 50 and, therefore, the overall time scale for r-processing shortens. Furthermore, the presence of these heavier seed nuclei reduces the neutron-to-seed ratio that is necessary to reproduce the solar system r-abundance distribution. For example, in order to explain the abundance peaks at A = 130, 195 and the synthesis of the elements Th, U one requires about 30, 100, and 140 neutrons per seed nucleus, respectively. Third, the physical properties of the neutrino wind predict an ejected amount of rprocess material per supernova that is about consistent with the total mass of r-processed material presently existing in the Galaxy ($\approx 10^4 M_{\odot}$). It is important for a successful r-process that the preceding α -process is not too efficient. Otherwise, too many heavy seed nuclei are produced and too many neutrons are consumed, resulting in insufficient neutron-to-seed ratios. This requirement translates into relatively low densities in the neutrino wind so that the $\alpha \alpha n \rightarrow {}^{9}\text{Be}(\alpha, n){}^{12}\text{C}$ and $3\alpha \rightarrow {}^{12}\text{C}$ reactions are less efficient in converting α particles to heavy nuclei. A fast expansion time scale limits the duration over which the freeze-out operates and is also helpful for reaching a high neutronto-seed ratio. A severe problem with the neutrino-powered wind model of

the r-process is the present difficulty to achieve the necessary low densities (or high entropies) and short time scales in the ejecta in order to reproduce the solar system r-abundances in the range of A > 135 (Thompson, Burrows and Meyer 2001).

It should be clear from the discussion in this section that r-process nucleosynthesis simulations require a very large set of nuclear physics quantities, including nuclear masses, β^- -decay half-lives, branching ratios for β -delayed neutron decay, fission properties, partition functions, and so on. If the freezeout from equilibrium is followed explicitly (that is, if the waiting point and steady flow approximations are not applied and the network is solved numerically) then reaction rates for neutron captures and photodisintegrations are required as well. All of this information is needed for neutron-rich isotopes that are located far away from stability. If the α -process needs to be followed explicitly, then another large data set consisting of rates for chargedparticle reactions and neutron-induced processes, such as (n,α) and (n,p), is required. Needless to say that almost all of this information must be obtained from global, semi-empirical, models for nuclear masses, β^- -decays, and fission, and from Hauser-Feshbach calculations. Information on directly measured properties of nuclei located on the r-process path has only been obtained in a few exceptional cases near neutron magic numbers where the rprocess path comes closest to the stability valley. For example, pioneering experiments leading to the identification of the neutron magic waiting point nuclei ⁸⁰Zn (Lund et al. 1986, Gill et al. 1985) and ¹³⁰Cd (Kratz et al. 1986) provided the first evidence for the existence of a local steady flow equilibrium in the r-process. Results from experiments on unstable nuclei off the r-process path are also important since the information gathered can be used to test current nuclear models from which properties of nuclei on the r-process path are derived. Many experimental r-process studies focus on three mass regions: (i) the neutron-rich Fe, Co, and Ni isotopes up to the double-magic nucleus ⁷⁸Ni, since these species represent the seed nuclei for the classical r-process model; (ii) isotopes of Zr and Pd near $A \approx 115$ where most r-process calculations underpredict the observed solar system r-abundances (see Fig. 5.73), an effect that is possibly caused by deficiencies in present nuclear models; and (iii) the region near the N = 82 neutron magic number (Fig. 5.71) which gives rise to the second r-process abundance peak near $A \approx 130$. A review of experimental and other issues related to the r-process can be found in Pfeiffer et al. (2001).

5.6.3

The p-Process

The very neutron-deficient, stable nuclides with mass numbers of $A \ge 74$ (between ⁷⁴Se and ¹⁹⁶Hg) are bypassed by the s- and r-process. These species are referred to as *p*-nuclei, where the letter *p* designates the fact that they contain more protons relative to other stable isotopes of the same element. (Recall, however, that all stable nuclei above ⁴⁰Ca consist of more neutrons than protons.) The mechanism responsible for the synthesis of the p-nuclei is called *p-process*. A list of the p-nuclei and their associated abundances is given in Table 5.1. The solar system abundances of the p-nuclei are displayed in Fig. 5.76 where they are compared to the abundances that originate from the s- and r-processes. It is apparent that as a group the p-nuclei are the rarest among the stable nuclides. Their abundances are typically a factor of ≈ 100 smaller compared to those of adjacent s- and r-nuclei. In fact, no single element has a p-process isotope as a dominant component. This implies that all knowledge of abundance systematics of these species is entirely based on measurements of solar system material. It is generally accepted that the much more abundant s- and r-nuclei serve as seeds for the p-process.

Important clues regarding the mechanism of the p-process can be obtained from the general nuclear structure of the about 30 p-nuclei. Almost all of these have even numbers of protons and neutrons (Table 5.1). The only exceptions are ${}^{113}_{49}In_{64}$, ${}^{115}_{50}Sn_{65}$, ${}^{138}_{57}La_{81}$, and ${}^{180}_{73}Ta_{107}$ but their abundances are, except for ¹¹⁵Sn, considerably smaller compared to those of the adjacent p-nuclei. It is also apparent from Fig. 5.76 that the p-nuclei abundance distribution has maxima at ${}^{92}_{42}Mo_{50}$, ${}^{112}_{50}Sn_{62}$, and ${}^{144}_{62}Sm_{82}$. The first and third of these have closed neutron shells while the second has a closed proton shell. Therefore, the pprocess seems to favor the production of more strongly bound nuclei, that is, those that have paired protons and neutrons.

There are two kinds of reactions that allow for the production of neutrondeficient nuclei starting from s- or r-process seeds: (p,γ) reactions and (γ,n) photodisintegrations. (Note that (p,n) reactions also produce neutron deficient nuclei but their Q-values on the proton-rich side of the stability valley are negative and thus their reaction rates are much smaller compared to the competing (p, γ) reactions.) Early models placed the p-process in the hydrogen-rich layers of type II supernovae. During the passage of the supernova shock, a combination of (p, γ) and (γ, n) reactions would produce nuclei that are shielded from the neutron-capture processes at temperatures and densities of *T* \approx 2.5 GK and $\rho \approx$ 100 g/cm³ over an explosive expansion time scale of 10–100 s (Burbidge et al. 1957). However, it was pointed out by Woosley and Howard (1978) that the required high densities, temperatures, and relatively long time scales are unlikely to exist in any hydrogen-rich zones of common stars. An exception are type I X-ray bursts (Section 1.4.4). These objects have indeed been proposed (Schatz et al. 1998) to produce some of the lighter p-nuclei via proton captures during the rp-process (Section 5.4). A major obstacle with this scenario is the fact that it is very unlikely for any significant amount of accreted and processed matter to escape the large gravitational potential of the neutron star.



Fig. 5.76 Decomposition of the observed solar system abundances of the heavy nuclides into components that are synthesized by the s-, r- and p-processes. The contributions from the s- and r-processes are adopted from Arlandini et al. (1999), while the p-process abundances are from Anders and Grevesse (1989). The abundance of the rare species ¹⁸⁰Ta is off scale and is omitted from the figure.

Before describing specific sites for the production of the p-nuclei, it is instructive to discuss the generally accepted mechanism of the p-process. Instead of a hydrogen-rich zone it involves a *hot photon* environment with temperatures in the range of $T \approx 2-3$ GK. Starting from some seed nuclei, the most likely interactions to occur at elevated temperatures in *hydrogen-exhausted* stellar zones are photodisintegrations. The decay constants for the photoejection of neutrons, protons or α -particles can be calculated from Eq. (3.45). At a given temperature, the decay constants depend strongly on the *Q*-value for the forward reaction $0 + 1 \rightarrow \gamma + 3$, or equivalently, on the particle separation energy of nucleus 3 (see also the discussion in Section 5.5.4).

As an example, consider the chain of tellurium isotopes at a temperature of T = 2.5 GK, as shown in Fig. 5.77a. Their photodisintegration decay constants are displayed in Fig. 5.77b. The seed isotope ¹²²Te, which is synthesized by the s-process (Fig. 5.63), is most likely destroyed by the (γ ,n) reaction. The next isotope, ¹²¹Te, will also most likely undergo a (γ ,n) reaction. As we move along the isotopic chain toward more neutron-deficient nuclei, the (γ ,n) decay constants fluctuate strongly. The photoejection of a neutron is far more likely for an odd-*N* isotope compared to an even-*N* one. This behavior is mainly caused by the pairing effect (Section 1.6.2) which leads to pronounced odd–even fluctuations in the corresponding neutron separation energies (see also Fig. 5.70). The (γ ,n) decay constants also decrease on average because more

Nuclide	Ζ	Ν	Abundance ^a	Error (%) ^ь
⁷⁴ Se	34	40	0.58	10
⁷⁸ Kr	36	42	0.20	20
⁸⁴ Sr	38	46	0.13124	10
⁹² Mo	42	50	0.386	10
⁹⁴ Mo	42	52	0.241	10
⁹⁶ Ru	44	52	0.1053	20
⁹⁸ Ru	44	54	0.0355	20
¹⁰² Pd	46	56	0.0146	7
¹⁰⁶ Cd	48	58	0.01980	7
¹⁰⁸ Cd	48	60	0.01410	7
¹¹³ ln	49	64	0.0078	7
¹¹² Sn	50	62	0.03625	10
114Sn	50	64	0.02460	10
¹¹⁵ Sn	50	65	0.01265	10
¹²⁰ Te	52	68	0.0046	10
¹²⁴ Xe	54	70	0.00694	5
¹²⁶ Xe	54	72	0.00602	5
¹³⁰ Ba	56	74	0.00460	7
¹³² Ba	56	76	0.00440	7
¹³⁸ La	57	81	0.000397	15
¹³⁶ Ce	58	78	0.00217	5
¹³⁸ Ce	58	80	0.00293	5
¹⁴⁴ Sm	62	82	0.00781	10
¹⁵⁶ Dy	66	90	0.000216	10
¹⁵⁸ Dy	66	92	0.000371	10
¹⁶² Er	68	94	0.000350	7
¹⁶⁸ Yb	70	98	0.000323	7
¹⁷⁴ Hf	72	102	0.000275	10
¹⁸⁰ Ta	73	107	0.00000258	7
¹⁸⁰ W	74	106	0.000153	7
¹⁸⁴ Os	76	108	0.000133	7
¹⁹⁰ Pt	78	112	0.000185	7
¹⁹⁶ Ha	80	116	0.00063	50

Tab. 5.1 Nuclides that are mainly produced by the p-process; ¹⁸⁰Ta and ¹⁸⁰W may also be synthesized by the s-process; see, for example, Arlandini et al. (1999) for s-process contributions to the solar system abundances. (a) Solar system abundance (from Lodders 2003); the values are given relative to 10⁶ Si atoms. (b) Errors are given for elemental abundances only.

energy is required to remove a neutron from increasingly neutron-deficient nuclei. At the same time, the proton and α -particle separation energies decrease when moving along the isotopic chain from the stability valley toward the proton dripline. In other words, the proton-richer an isotope the less energy is required to remove a proton or α -particle and the larger the (γ ,p) and (γ , α) decay constants become (Fig. 5.77b). Clearly, at some even-*N* nucleus along the isotopic chain, either the (γ ,p) or (γ , α) reaction will dominate over the competing (γ ,n) reaction. When this first occurs (at ¹²⁰Te in Fig. 5.77), the abundance flow branches off to an isotope of a different element (here ¹¹⁶Sn) and the sequence of events repeats itself in the chain of Sn isotopes.



Fig. 5.77 (a) Section of the chart of the nuclides in the region of neutron-deficient Sn, Sb, and Te isotopes. Stable nuclides are shown as shaded squares. The letters s, r or p refer to their productions in the s-, r- or p-process, respectively. (b) Decay constants of neutron-deficient tellurium iso-

topes, calculated for a temperature of T =2.5 GK (from Rauscher and Thielemann 2000). At ¹²⁰Te, the (γ, α) reaction dominates over the competing (γ, n) reaction. As a consequence, the abundance flow branches off to an isotope of a different element (here ¹¹⁶Sn).

In each isotopic chain of proton number Z, the branch point is defined by the condition

$$\lambda_{\gamma p} + \lambda_{\gamma \alpha} > \lambda_{\gamma n} \tag{5.203}$$

From the arguments presented above it is also clear that in each isotopic chain the longest photodisintegration lifetimes on the flow path tend to occur near the branch point. These even-N nuclei become waiting points and material accumulates at their location. This applies especially to nuclei with closed neutron or proton shells since they have unusually large separation energies (Section 1.6.2). On the other hand, little accumulation of material is expected at odd-N nuclei since their neutron separation energies are relatively small and, consequently, their (γ, n) decay constants are large. In the above example, the branch (and waiting) point occurs at a stable nucleus (¹²⁰Te) which

becomes a p-nucleus (Table 5.1). Ultimately, the original seed nucleus (here ¹²²Te) is photodisintegrated into several lighter waiting point nuclei until the iron peak is reached where further photodisintegrations become energetically unfavorable. Of course, in any realistic situation there will be a distribution of s- and r-process seed nuclei extending up to Pb, all subject to the same hot photon environment. The abundance flow then reaches from lead down to iron and, along the way, is fed by the destruction of many seed nuclei. The mean lifetimes for photodisintegration reactions, $\tau_{\gamma i} \equiv 1/\lambda_{\gamma i}$, along the pprocess path are < 100 s and thus β -decays, which are typically much slower, are negligible for the nucleosynthesis as long as relatively high temperatures of *T* \approx 2–3 GK are maintained. In the above discussion, proton- or α -particleinduced reactions [for example, (p, γ) or (α, γ)] play no role. Because of the dominance of photodisintegrations, the above mechanism of the p-process is sometimes referred to as the γ -process (Woosley and Howard 1978). Some of the neutrons released during the p-process may also contribute to the nucleosynthesis. It was shown that these impede at higher temperatures the reverse (γ, n) reactions, especially in the region of the lighter p-nuclei (Rayet, Prantzos and Arnould 1990).

A few points must be clarified. First, if the hot photon environment is maintained for a too long period of time, then all seed nuclei would be photodissociated into iron peak nuclei, free protons, neutrons, and α -particles, as dictated by nuclear statistical equilibrium (Section 5.5.5). Thus, for any realistic site responsible for the synthesis of the p-nuclei, the values of temperature and time scales must guarantee the occurrence of some nuclear transformations, yet not so intense as to entirely reduce all nuclei to iron. These arguments support the conclusion that the p-process occurs during stellar explosions with an associated rapid expansion and cooling of material. Thus, the nucleosynthesis during the p-process will depend sensitively on the distribution of temperatures and expansion time scales, the abundances of seed nuclei, and the hydrodynamic conditions of the explosion. Second, in the above example (Fig. 5.77), the waiting point coincided with a p-nucleus. This is generally the case for lighter-mass nuclei. In the region of heavier-mass nuclei, however, the waiting points correspond to proton-rich progenitors which subsequently transmute to p-nuclei via β^+ -decays after cooling, expansion, and ejection of the material. For example, the stable p-nucleus ¹⁹⁶Hg is produced by the decay of the unstable waiting point nucleus ¹⁹⁶Pb, that is, via ¹⁹⁶Pb($\beta^+\nu$)¹⁹⁶Tl($\beta^+\nu$)¹⁹⁶Hg. Third, the abundance flow at the waiting point nucleus ¹²⁰Te (Fig. 5.77) continues via a (γ , α) reaction. This is the preferred path in the region of the heaviermass nuclei. On the other hand, most (but not all) decays of waiting point nuclei in the lighter mass range proceed via the (γ, p) reaction (Rauscher 2005). Fourth, since the photodisintegration rates are highly temperature dependent, the location of the branch point in a given isotopic chain depends on the value

of the temperature. A branch point has the tendency to shift toward more proton-rich nuclei for increasing temperatures (Problem 5.17). The question of why almost all p-nuclei exhibit an even number of protons is explored in Problem 5.18.

It is interesting to consider the total photodisintegration decay constant, $\Lambda = \lambda_{\gamma\alpha} + \lambda_{\gamma p} + \lambda_{\gamma n}$, of the p-nuclei or, if appropriate, of their proton-rich progenitors. The results are shown in Fig. 5.78 as a function of mass number for temperatures of *T* = 2.0, 2.5, and 3.0 GK. The decay constants $\lambda_{\gamma i}$ are obtained from Hauser-Feshbach reaction rates. The structure seen in the curves is influenced by nuclear shell effects but will not concern us here. The outstanding feature shown in Fig. 5.78 is the large variation of Λ at each temperature by several orders of magnitude over the displayed mass range. Suppose that all the p-nuclei were synthesized at the same single and constant value of temperature. If that would be the case, then any photon exposure sufficient to produce the lighter p-nuclei in the A = 70-100 range would completely destroy all the heavy p-nuclei in the A = 160-200 region. Thus the strong variation of Λ shown in Fig. 5.78 supports the conclusion that stellar regions of different temperature are responsible for the synthesis of the p-nuclei. The heavy p-nuclei are produced at relatively low temperature while the light pnuclei are created at relatively high values of *T*. Note also that Λ is an *increas*ing function of mass number. If the opposite were the case, then any photon exposure sufficient to destroy the heavy seed nuclei (for example, lead) would also destroy the photodisintegration products of lead, and so on, until the iron region is reached. The nucleosynthesis of intermediate-mass nuclei could not occur and the p-process model described above would be inappropriate.

Most investigations to date have assumed that the p-process occurs in type II supernovae when the shock wave passes through the O-Ne-rich layer of a massive star (Section 1.4.3). For a short period of time (≈ 1 s), the shock wave compresses and heats this stellar region. During the explosion, different zones in the O-Ne-rich layer will undergo different thermodynamic histories and thus will achieve different peak temperatures. Calculations show that during the p-process the range of peak temperatures in these zones amounts to $T_{\text{peak}} \approx 1.8$ –3.3 GK. The weak s-process component operating mainly during the preceding core helium burning stage in the pre-supernova star enhances strongly the p-process seed abundances in the $A \approx 60-90$ region (Section 5.6.1). It has been demonstrated that p-nuclei with masses of $A \leq 92$, $A \approx$ 92–144, and $A \ge$ 144 are mainly produced in stellar zones with peak temperatures of $T_{\text{peak}} \ge 3$ GK, $T_{\text{peak}} \approx 2.7$ –3.0 GK, and $T_{\text{peak}} \le 2.5$ GK, respectively. In fact, each p-nucleus is synthesized in a relatively narrow temperature range only (Rayet, Prantzos and Arnould 1990). The abundances obtained from such calculations have been weighted and averaged over a range of stars with different masses. As a result, about 60% of the p-nuclei are repro-



Fig. 5.78 Total photodisintegration rates Λ for the p-nuclei or their proton-rich progenitors at stellar temperatures of T = 2.0, 2.5, and 3.0 GK as a function of mass number. The photodisintegration rates are calculated by using the Hauser–Feshbach model (Rauscher and Thielemann 2000).

duced within a factor of 3 of their solar system values. This must be regarded as a remarkable success in view of the complexities of the nuclear physics input (see below) and of the stellar models. However, a number of discrepancies persist. Most notable among those is the underproduction of the light p-nuclei ⁹²Mo, ⁹⁴Mo, ⁹⁶Ru, and ⁹⁸Ru. The odd-*A* nuclides ¹¹³In, ¹¹⁵Sn and the odd-odd species ¹³⁸La are also underproduced in most calculations. On the other hand, the rarest species occurring naturally in the solar system, the odd-odd nuclide ¹⁸⁰Ta, seems to be a product of the p-process, although the s-process during thermal pulses of certain AGB stars may also contribute to its observed solar system abundance (Gallino et al. 1998).

Several other sites have also been considered for the production of p-nuclei, including supernovae of type Ia and Ib/Ic (Section 1.4.3). Interestingly, although the stellar models for all these scenarios are very different, similar p-abundance distributions are obtained in each case. Most of the p-nuclei are reproduced within a factor of \approx 3 of their solar system values, while certain species (92 Mo, 94 Mo, 96 Ru, 98 Ru, 113 In, 115 Sn, 138 La) are significantly underproduced. Hence, it appears likely that the p-process occurs in a number of different sites. The underproduction of some nuclides is perhaps caused by nuclear physics uncertainties or by an unreliable estimate of the s-isotope seed distribution for the p-process. Alternatively, some of the underproduced species may be predominantly synthesized in a different site, such as sub-

Chandrasekhar white dwarf explosions. For more information on sites and other issues related to the p-process, see Arnould and Goriely (2003).

We now move from a qualitative discussion to a numerical treatment. The O-Ne layer during a type II supernova (that is, explosive oxygen burning) is chosen as an example for a p-process site. It was already pointed out that the p-nuclei, depending on their mass number, are synthesized in different zones that achieve different peak temperatures. We will discuss below the results of a network calculation performed for the explosive evolution of a single zone in a O–Ne layer of a $25 M_{\odot}$ star. The temperature–density profile of the selected zone is shown in Fig. 5.79. The profile starts at point A (T =1.4 GK, $\rho = 1.4 \times 10^5$ g/cm³), evolves to point B at the peak of the explosion $(T = 3.0 \text{ GK}, \rho = 6 \times 10^5 \text{ g/cm}^3)$ and then settles at point C $(T = 1.4 \text{ GK}, \rho =$ 7.0×10^4 g/cm³). The entire evolution from A to C lasts for about t = 1.1 s. The network consists of about 1100 nuclei, stretching from ¹H to ²⁰⁹Bi, and includes \approx 11000 reactions induced by neutrons, protons, α -particles, and their reverse reactions. The 3α reaction and the ${}^{12}C + {}^{12}C$ and ${}^{16}O + {}^{16}O$ reactions are also included. Above calcium, all reaction rates are adopted from Hauser-Feshbach statistical model calculations. For (n, γ) reactions on stable target nuclei (and, more importantly, for the corresponding reverse (γ ,n) reactions), the statistical model results have been renormalized to experimental values (Bao et al. 2000). Beta-decays are also included, but are expected to have a negligible influence, as discussed above, except after the termination of the explosion when some radioactive progenitors decay to stable p-nuclei. The network described above is much larger compared to all other burning processes discussed so far. It must also be stressed that, contrary to the s-process or the r-process, the concepts of steady flows or reaction rate equilibria cannot be used here in order to simplify this complex situation. The p-process operates far from equilibrium and, as a result, the entire network must be followed by an explicit computation. Clearly, p-process studies are among the most complicated nucleosynthesis processes. The initial abundances are adopted from the pre-supernova evolution models of Rayet et al. (1995). The most abundant species are ${}^{16}O(X_i = 0.73)$, ${}^{20}Ne(X_i = 0.17)$, and ${}^{24}Mg(X_i = 0.17)$ 0.05), while the seed abundances in the mass A = 60-90 region are significantly increased compared to a solar system composition due to the operation of the weak s-process component during the preceding core helium burning stage (Section 5.6.1).

Abundance flows integrated over the duration of the network calculation are presented in Fig. 5.79. We are only interested here in the nucleosynthesis that takes place in the mass region above germanium. The obtained flow pattern reflects the qualitative arguments presented above. The p-process has the remarkable property that the abundance flow proceeds from heavy nuclei at the top of the network down toward lighter nuclei. In other words, a particular p-nucleus is synthesized exclusively from those seed isotopes which are heavier than the p-nucleus itself. The seed nuclei are converted via (γ ,n) reactions until, in each isotopic chain, a branching point nucleus is reached. For the chosen conditions, the branching point nuclei above europium are almost exclusively destroyed via (γ , α) reactions. Also, the p-process path in this region is located on average 2–4 mass units away from the neutron-deficient side of the stability valley and, therefore, the branching point nuclei are all radioactive. In the region below europium, the branching point nuclei are destroyed either via (γ , α) or (γ ,p) reactions and frequently coincide with p-nuclei. Apart from these three photodisintegration reactions and certain (n, γ) reactions, no other processes are important for the nucleosynthesis. As already mentioned above, for lower peak temperatures the branching point nuclei have the tendency to shift to a location closer to the stability valley.

The ratio of final abundances obtained at the end of the calculation (t = 1.1 s) and the initial (seed) abundances is shown in Fig. 5.80. It can be seen that, for a peak temperature of $T_{\text{peak}} = 3.0$ GK achieved in this particular zone, most pnuclei (solid dots) in the A = 96–144 region are strongly overproduced, while other nuclides (open circles) are underproduced. Clearly, the net effect of the nucleosynthesis is the conversion of s (and r)-process seeds to p-nuclei. At this high peak temperature all species beyond A = 150 (p-nuclei and others) are destroyed and converted via photodisintegrations to p-nuclei in the A = 96–144 region. A proper analysis of overproduction factors requires an averaging over all stellar zones (peak temperatures) in the O–Ne layer and a normalization to solar system abundances instead of the initial abundances in the pre-supernova star. Nevertheless, it is interesting that even this one-zone calculation hints at an unsolved problem of current p-process computations, that is, the relative underproduction of species such as 92 Mo, 94 Mo, 113 In and 115 Sn.

Finally, we will address issues related to the nuclear physics input required for p-process calculations. A number of charged-particle reactions in the mass $A \le 25$ range may play an important role. For example, the ${}^{12}C(\alpha,\gamma){}^{16}O$ rate (Section 5.3.1) influences the pre-supernova evolution of the massive star and hence the composition of the O–Ne layer prior to core collapse (Rayet et al. 1995). The ${}^{22}Ne(\alpha,n){}^{25}Mg$ reaction is crucial since it is responsible for the weak s-process component (Section 5.6.1) during core helium burning in massive stars. An increase in this rate will enhance the s-nuclide seed abundances for the p-process and may reduce the underproduction of Mo and Ru p-nuclei in current type II supernova models (Arnould and Goriely 2003).

With relatively few exceptions, almost all rates for a very large number of reactions (> 10000) in the region of the p-process (A > 60) have to be calculated by using the Hauser–Feshbach model. As we have seen, the most important interactions are (γ ,n), (γ , α), and (γ ,p) photodisintegrations. Their decay constants are usually calculated from Eq. (3.45) by using the rates of the cor-





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Fig. 5.79 Time-integrated net abundance flows in the region above yttrium for the p-process during explosive oxygen burning of a type II supernova. The calculation represents the results from a single zone of the O–Ne layer for which the *T*– ρ profile is shown in the inset (Rapp et al. 2005). The peak temperature achieved during the explosion in this particular zone amounts to *T*_{peak} = 3.0 GK. The reaction network calculation is terminated after t = 1.1 s (the time it takes for the zone to evolve from point A to B to C during the explosion). Abundance flows are represented by arrows of three thicknesses: thick, intermediate and thin arrows show flows of $F^{\max}10^{-9} \ge F_{ij} > F^{\max}10^{-10}$, $F^{\max}10^{-10} \ge F_{ij} > F^{\max}10^{-11}$ and $F^{\max}10^{-11} \ge F_{ij} > F^{\max}10^{-12}$, respectively, where F^{\max} corresponds to the maximum flow of a link in the mass A < 60 region which is not shown in the figure.

responding forward reactions. Note that the p-process path involves neutrondeficient nuclei that are located close to the stability valley. This is a fortunate circumstance since the reaction *Q*-values (and separation energies) in this region are experimentally well known. It has been demonstrated that different prescriptions of the Hauser–Feshbach model influence sensitively the final pnuclei abundances obtained from type II supernovae. The predicted abundances of the heavier p-nuclei are most sensitive to the α -nucleus optical potential, while the lighter species are mainly affected by uncertainties in nuclear level densities and nucleon–nucleus optical potentials (Arnould and Goriely 2003).

Experimental (n, γ) , (α, γ) , and (p, γ) rates on stable target nuclei in the mass A > 60 range (Bao et al. 2000, Arnould and Goriely 2003) play an important



Fig. 5.80 Ratio of final abundances obtained at the end of the pprocess calculation shown in Fig. 5.79 and the initial (seed) abundances. For a peak temperature of T = 3.0 GK, most p-nuclei (solid dots) in the A = 96-144 region are strongly overproduced, while other nuclides (open circles) are underproduced. The underproduction of the p-nuclei ⁹²Mo, ⁹⁴Mo, ¹¹³In, and ¹¹⁵Sn remains unexplained.
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role for p-process studies for two reasons. First, they are used for adjusting statistical model parameters and, as a result, Hauser-Feshbach rate predictions for a multitude of unmeasured reactions become more reliable. Second, the decay constants for the corresponding reverse photodisintegration reactions can be calculated from Eq. (3.45). A number of (γ ,n) reactions have also been measured directly by using real photons. We argued in Section 3.2.3 that the astrophysically most important energy range for a reaction $A(\gamma,n)B$ is located at a γ -ray energy of $E_{\gamma}^{\text{eff}} \approx S_{\text{n}} + kT/2$ (for $\ell = 0$ neutrons). The quantity S_n is the neutron separation energy of nucleus A [or the reaction Q-value of $B(n,\gamma)A$]. Consequently, direct (γ,n) measurements relevant to p-process studies (T < 3 GK or kT/2 < 0.15 MeV) have to be performed in a relatively narrow energy window close to the reaction threshold. This method has been applied, for example, in the study of the reaction 181 Ta (γ, n) 180 Ta (Utsunomiya et al. 2003). Note that Hauser-Feshbach calculations are also needed in order to estimate the contribution from thermally excited states, even if laboratory reaction rates are available. For reaction rate sensitivity studies of the p-process, see Arnould and Goriely (2003) and Rapp et al. (2005).

5.7

Origin of the Solar System Nuclides

We will close this chapter by briefly summarizing the origin of the nuclides in nature. It seems, in principle, possible to predict the main astrophysical source(s) for a given nuclide in the solar system by considering the fraction of the yield contributed to the interstellar medium by each of the sites mentioned in this book. Obviously, one would need to know: (i) how many stars of a given mass and metallicity there are (since stellar evolution depends on both parameters), (ii) the efficiency of various nucleosynthetic processes in each star, (iii) the fraction of the expelled matter (via explosion or stellar wind), and so on. As already mentioned, a large number of nuclides originate from massive stars. In this case, many of the predicted abundances depend strongly on the mass cut dividing the material that is ejected in the type II supernova explosion from material that falls back onto the remnant neutron star or black hole. There are major uncertainties associated with all of the issues mentioned above. Nevertheless, the overall picture regarding the origin of the nuclides in the solar system seems well established and this achievement certainly represents a triumph for the theory of nucleosynthesis.

The origin of the light nuclides with masses of $A \le 40$ is presented in Table 5.2. Only the dominant sources are listed for each species and somewhat uncertain assignments are placed in square parentheses. Hydrogen (¹H, ²H) and helium (³He, ⁴He) are made in the Big Bang (BB). Cosmic ray spallation

(CR) accounts for the abundances of ⁶Li, ⁹Be, ¹⁰B and perhaps ¹¹B, althouh the latter nuclide may also be made by the *v*-process during the core collapse of a massive star (Woosley et al. 1990). The species ^{12,13}C and ¹⁴N are synthesized during the asymptotic giant branch phase (AGB) of low-mass stars, while classical novae have been claimed to contribute substantially to the Galactic abundances of ¹³C, ¹⁵N, and ¹⁷O. About half of the ¹²C and all of ^{16,18}O is produced during hydrostatic helium burning in massive stars. All of the nuclides in the A= 20–40 region are made in massive stars during various phases of their evolution, either in hydrostatic carbon burning, explosive carbon, neon or oxygen burning, or during the weak s-process (hydrostatic helium burning). For a number of species listed in Table 5.2, notably ⁷Li, ¹⁹F, and ³⁶S, the dominant sources have not been identified yet with certainty. These may be produced in a number of sites, although the relative contributions are controversial at present.

All of the nuclides in the A = 40-60 mass region are most likely produced in supernovae. For example, the species ⁵⁶Fe is made chiefly as radioactive ⁵⁶Ni in both core-collapse and type Ia supernovae (Section 5.5.5). However, for many nuclides in this mass range the dominant production site (type II versus various type Ia scenarios, explosive silicon burning versus α -rich freeze-out, and so on) is not known with certainty. This can be seen from the differences in the results presented by Woosley, Heger and Weaver (2002) and Clayton

Tab. 5.2 Origin of the light nuclides. The labels denote: Big Bang (BB); cosmic ray spallation (CR); asymptotic giant branch stars (AGB); ν -process (ν); classical novae (CN); helium, carbon, neon, oxygen burning in massive stars (He, C, Ne, O), where an "x" in front of the symbol of a burning stage indicates explosive rather than hydrostatic burning; the weak s-process component is denoted by He(s). Information from Woosley, Heger and Weaver (2002), Clayton (2003), José, Lattanzio and Limongi (2006) (private communication). Uncertain or conflicting assignments are given in square parenthesis.

Nuclide	Origin	Nuclide	Origin	Nuclide	Origin
^{1}H	BB	¹⁷ O	CN	³⁰ Si	С
² H	BB	¹⁸ O	He	³¹ P	С
³ He	BB	¹⁹ F	[AGB, ν,]	³² S	хО
⁴ He	BB	²⁰ Ne	С	³³ S	xO, xNe
⁶ Li	CR	²¹ Ne	С	³⁴ S	хО
⁷ Li	[BB, AGB, CN]	²² Ne	He, AGB	³⁶ S	[He(s), xC,]
⁹ Be	CR	²³ Na	С	³⁵ CI	хО
¹⁰ B	CR	²⁴ Mg	С	³⁷ Cl	[xO, He(s),]
¹¹ B	[CR, <i>v</i>]	²⁵ Mg	C, AGB	³⁶ Ar	хО
¹² C	AGB, He	²⁶ Mg	C, AGB	³⁸ Ar	хО
¹³ C	AGB, CN	²⁶ AI	xC, xNe	40 Ar	[He(s), C,]
14 N	AGB	²⁷ AI	С	³⁹ K	хО
¹⁵ N	CN	²⁸ Si	xO	40 K	He(s)
¹⁶ O	He	²⁹ Si	С	40 Ca	xO

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(2003), and reflects our incomplete knowledge regarding the issues mentioned above.

Above mass A = 60 the situation is much clearer since for the overwhelming number of nuclides the relative contributions of the s-, r-, and p-process can be estimated in a straightforward manner (Section 5.6). These are shown in Fig. 5.76. For more information on the origin of the nuclides in nature, see Woosley, Heger and Weaver (2002) and Clayton (2003).

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5.1 Calculate the lifetime of: (i) a proton against destruction via the $p(p,e^+\nu)d$ reaction, and (ii) a deuteron against destruction via the $d(p,\gamma)^3$ He reaction for a temperature of T = 15 MK, a density of $\rho = 100$ g/cm³, and a hydrogen mass fraction of $X_{\rm H} = 0.5$. Use the following numerical values for the reaction rates: $N_A \langle \sigma v \rangle_{\rm pp} = 7.90 \times 10^{-20}$ cm³ mol⁻¹ s⁻¹, $N_A \langle \sigma v \rangle_{\rm dp} = 1.01 \times 10^{-2}$ cm³ mol⁻¹ s⁻¹ (Angulo et al. 1999).

5.2 Derive an expression for the temperature dependence of the decay constant for the 3α reaction (see Eq. (5.102)).

5.3 Consider the nucleus ³⁰S for the conditions T = 0.5 GK, $\rho = 10^4$ g/cm³ and $X_{\rm H} = 0.73$ (Fig. 5.81). Explain why, according to the top panel of Fig. 5.36, the net abundance flow prefers to follow the link ³⁰S($\beta^+ \nu$)³⁰P instead of the competing ³⁰S(p,γ)³¹Cl reaction. Use the values of $T_{1/2}$, Q, and $N_A \langle \sigma v \rangle$ given in the figure.



Fig. 5.81 (a) Section of the chart of the nuclides in the vicinity of ³⁰S. Nuclides that eventually reach equilibrium are shown as shaded squares. Values of $Q_{\rm P\gamma}$ (left-hand side) and $T_{1/2}$ (right-hand side) are adopted from Audi, Wapstra and Thibault (2003) and Audi et al. (2003), respectively. The values given for $N_A \langle \sigma v \rangle$ apply to a temperature of T = 0.5 GK (Iliadis et al. 2001). See Problem 5.3.

5.4 Consider the ⁶⁴Ge(p, γ)⁶⁵As reaction at T = 1.34 GK and $\rho = 5.9 \times 10^5$ g/cm³. Calculate the mean lifetime of ⁶⁴Ge versus destruction by sequential two-proton capture for a value of $Q_{64}_{Ge(p,\gamma)} = -0.38$ MeV. Furthermore, assume that the reaction rate for ⁶⁵As(p, γ)⁶⁶Se is $N_A \langle \sigma v \rangle = 1.0 \times 10^{-2}$ cm³ mol⁻¹ s⁻¹, that is, a factor of 10 smaller than the value predicted by a Hauser–Feshbach calculation (Goriely 1998). Assume a hydrogen mass fraction of $X_{\rm H} = 0.47$. The half-lives of ⁶⁵As and ⁶⁶Se are $T_{1/2} = 0.128$ s (Lopez Jimenez 2002) and $T_{1/2} = 0.033$ s (Audi et al. 2003), respectively. The reaction rate and decay constant for ⁶⁴Ge(p, γ)⁶⁵As and ⁶⁶Se(γ ,p)⁶⁵As are given by $N_A \langle \sigma v \rangle = 0.011$ cm³ mol⁻¹ s⁻¹ and $\lambda = 0.29$ s⁻¹, respectively (Goriely 1998). The spins and normalized partition functions are $j_{64}_{Ge} = 0$, $j_{65}_{\rm As} = 3/2$, $j_{\rm p} = 1/2$ and $G_{\rm form}^{\rm corm} = 1.005$, $G_{\rm 65}^{\rm norm} = 1.306$ and $G_{\rm p}^{\rm norm} = 1$ (Rauscher and Thielemann 2000).

5.5 Derive an approximate analytical expression for the energy generation rate during neon burning. Assume an ${}^{16}\text{O} + \alpha \leftrightarrow {}^{20}\text{Ne} + \gamma$ equilibrium and that the subsequent ${}^{20}\text{Ne}(\alpha,\gamma){}^{24}\text{Mg}$ reaction involves the equilibrium α -particle abundance. The spins of ${}^{4}\text{He}$, ${}^{16}\text{O}$, and ${}^{20}\text{Ne}$ are all $j_i = 0$. The normalized partition functions of these nuclei for typical neon burning temperatures are equal to unity (Rauscher and Thielemann 2000). Disregard all contributions to the energy generation rate from other (secondary) reactions.

5.6 According to Fig. 5.52, the mass fractions of ²⁸Si and ²⁴Mg at t = 100 s amount to $X_{28Si} = 0.45$ and $X_{24Mg} = 0.00011$, respectively. Calculate the equilibrium α -particle mass fraction at t = 100 s for the conditions T = 3.6 GK and $\rho = 3 \times 10^7$ g/cm³.

5.7 Derive the quasiequilibrium abundance ratio $N_{34_S}/N_{28_{Si}}$ explicitly by successive application of the Saha equation (see Fig. 5.54a). Compare your result to the one obtained directly from Eq. (5.149).

5.8 Prove the relationship for the light particle abundances during silicon burning (see Eq. (5.151)).

5.9 Derive the expression for the effective photodisintegration rate of 24 Mg, f_{an} , during silicon burning (see Eq. (5.156)).

5.10 Consider the reaction sequence ${}^{1}H\leftrightarrow{}^{2}H\leftrightarrow{}^{3}H\leftrightarrow{}^{4}He$ (Fig. 5.82). Calculate the number abundance of ${}^{4}He$ in nuclear statistical equilibrium by repeated application of the Saha equation. Compare your result to Eq. (5.151). Generalization of your result will yield directly Eq. (5.163).

5.11 Consider nuclear statistical equilibrium at $\eta = 0$. Assume that all the matter consist only of α -particles, protons, and neutrons. Find the temperature–density conditions at which the α -particle abundance (by mass) is equal to the total nucleon abundance, that is, $X_{\alpha} = 0.5$, $X_{p} = 0.25$, and $X_{n} = 0.25$ (see dotted line in Fig. 5.56).



Fig. 5.82 The reaction chain ${}^{1}H\leftrightarrow {}^{2}H\leftrightarrow {}^{3}H\leftrightarrow {}^{4}He$ in equilibrium. See Problem 5.10.

5.12 Estimate the r-process contribution to the solar system abundance of the s,r-isotope ¹²⁵Te. Use values of $N_{\odot}(124) = 0.2319$ and $N_{\odot}(125) = 0.3437$ for the number abundances of ¹²⁴Te and ¹²⁵Te per 10⁶ Si atoms, respectively (Lodders 2003). The Maxwellian-averaged neutron-capture cross sections at kT = 30 keV for ¹²⁴Te and ¹²⁵Te are $\langle \sigma \rangle_{124} = 155 \pm 2$ mb and $\langle \sigma \rangle_{125} = 431 \pm 4$ mb, respectively (Bao et al. 2000).

5.13 Solve the abundance evolution of 56 Fe in the s-process (see Eq. (5.178)) for an exponential distribution of neutron exposures (see Eq. (5.175)), that is, derive the solution given in Eq. (5.180).

5.14 Derive an expression (see Eq. (5.193)) for the number abundance of an isotope in the r-process by successive application of the Saha equation to an $(n,\gamma) \leftrightarrow (\gamma,n)$ equilibrium in an isotopic chain of a given element *Z*.

5.15 Find a quantitative criterion from Eq. (5.193) for predicting the location of the abundance maximum in an isotopic chain at $(n,\gamma) \leftrightarrow (\gamma,n)$ equilibrium in the r-process. Also, choose the conditions T = 1.25 GK and $N_n = 10^{22}$ cm⁻³ together with the $Q_{n\gamma}$ -values from Möller, Nix and Kratz (1997) in order to reproduce the abundance maxima shown in Fig. 5.70. Disregard partition functions and the spins of the heavy nuclides.

5.16 By using the waiting point and steady flow approximations of the r-process, calculate the half-life of ¹³⁰Cd from the measured half-lives (Audi et al. 2003) of ¹³¹In ($T_{1/2} = 280 \pm 30$ ms) and ¹³³In ($T_{1/2} = 165 \pm 3$ ms) and from the observed solar system r-abundances (Anders and Grevesse 1989, Ar-landini et al. 1999) of ¹³⁰Te (1.634), ¹³¹Xe (0.946), and ¹³²Xe (0.748). The latter values are given relative to Si ($N_{\text{Si}} \equiv 10^6$). Note that the measured branching ratio for the β -delayed neutron decay of ¹³³In amounts to $P_n = 85\%$ (Audi et al. 2003). Disregard all other β -delayed neutron decays (see Fig. 5.71).

5.17 Explain why, during the p-process, the branch point in a given isotopic chain has the tendency to shift to more proton-rich nuclei for increasing temperature.

5.18 The location of a branch point nuclide in a given isotopic chain is specified by the condition of Eq. (5.203). The decay constants for the (reverse)

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photodisintegrations can be calculated from Eq. (3.45) by using Hauser– Feshbach rates for the (forward) particle-induced reactions. Branch point nuclides for all elements between selenium (Z = 34) and lead (Z = 82) during p-process nucleosynthesis, calculated with this method, can be found in Rauscher (2005). Use these results together with a nuclidic chart to explain qualitatively why almost all p-nuclei exhibit an even number of protons.

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Appendix A Solutions of the Schrödinger Equation in Three Dimensions

The three-dimensional time-independent Schrödinger equation in cartesian coordinates is given by

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V(x,y,z)\psi = E\psi$$
(A.1)

with ψ the total wave function, V the potential, E the total energy, and m the particle mass. For many quantum mechanical problems, the potential V depends only on the distance but not on the direction, that is, $V(\vec{r}) = V(r)$. We call this a *central potential*. For such potentials, we can take advantage of the symmetry and replace the cartesian coordinates x, y, and z by the spherical coordinates r, θ , and ϕ . The wave function ψ for a central potential is separable into three different functions,

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \tag{A.2}$$

The Schrödinger equation is then separable as well and one obtains three different equations, one for each of the variables r, θ , and ϕ . The differential equation for Φ is

$$\frac{d^2\Phi}{d\phi^2} + m_\ell^2 \Phi = 0 \tag{A.3}$$

where m_{ℓ}^2 is the separation constant. The solution is

$$\Phi_{m_{\ell}}(\phi) = \frac{1}{\sqrt{2\pi}} e^{im_{\ell}\phi} \tag{A.4}$$

with $m_{\ell} = 0, \pm 1, \pm 2, ...,$ and so on. The quantity m_{ℓ} is called the *magnetic quantum number*. The equation for Θ is

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + \left[\ell(\ell+1) - \frac{m_{\ell}^2}{\sin^2\theta} \right] \Theta = 0$$
(A.5)

with $\ell = 0, 1, 2, ...,$ and so on, and $m_{\ell} = 0, \pm 1, ..., \pm \ell$. The quantity ℓ is referred to as the *orbital angular momentum quantum number*. The solutions can

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be expressed in terms of associated Legendre polynomials $P_{\ell}^{m_{\ell}}$,

$$\Theta_{\ell m_{\ell}}(\theta) = \sqrt{\frac{(2\ell+1)}{2} \frac{(\ell-m_{\ell})!}{(\ell+m_{\ell})!}} P_{\ell}^{m_{\ell}}(\theta)$$
(A.6)

The product of the two angle-dependent functions gives the spherical harmonics

$$Y_{\ell m_{\ell}}(\theta,\phi) = \Theta_{\ell m_{\ell}}(\theta)\Phi_{m_{\ell}}(\phi) \tag{A.7}$$

which describe the angular part of a wave function for any central potential. The parity π of a function describes the behavior under the coordinate transformation $\vec{r} \rightarrow -\vec{r}$ (space reflection), or in polar coordinates $r \rightarrow r, \theta \rightarrow \pi - \theta$, $\phi \rightarrow \pi + \phi$. Since two such transformations must yield again the original function ($\pi^2 = 1$), the parity can possess only the values $\pi = +1$ (positive or "even" parity) or $\pi = -1$ (negative or "odd" parity). The spherical harmonics have the important property

$$Y_{\ell m_{\ell}}(\pi - \theta, \pi + \phi) = (-1)^{\ell} Y_{\ell m_{\ell}}(\theta, \phi)$$
(A.8)

and hence the parity is even or odd for ℓ even or odd, respectively. In general, the functions $Y_{\ell m_{\ell}}$ are complex valued. For the special case of $m_{\ell} = 0$ the spherical harmonics are real valued and we obtain

$$Y_{\ell 0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}(\cos\theta) \tag{A.9}$$

where the functions $P_{\ell}(\cos \theta)$ are called *Legendre polynomials*. For the lowest values of ℓ they are given by

$$P_0(x) = 1$$
 (A.10)

$$P_1(x) = x \tag{A.11}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{A.12}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{A.13}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{A.14}$$

The equation for the radial function *R* is

$$-\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \left[V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right]R = ER$$
(A.15)

Note that only the radial equation depends on the central potential. The $\ell(\ell + 1)$ term is called the *centripetal potential*. It keeps the particle away from the

origin when $\ell > 0$. We can rewrite the radial equation by substituting u(r) = rR(r),

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2} \left[E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2mr^2} \right] u = 0$$
(A.16)

Frequently, one writes with $E = p^2/(2m) = \hbar^2 k^2/(2m)$

$$\frac{d^2u}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}V(r)\right]u = 0$$
(A.17)

where *k* is the wave number of the free particle. Applied to nuclear scattering, this equation is only correct for distances larger than the nuclear radius (r > R), since the motion inside the nucleus cannot be described by a wave function which depends on only one coordinate. The two general, linearly independent, solutions of Eq. (A.17) are denoted by $F_{\ell}(r)$ and $G_{\ell}(r)$. These satisfy the condition that the *Wronskian* combination is independent of *r*,

$$\left(\frac{dF_{\ell}}{dr}\right)G_{\ell} - F_{\ell}\left(\frac{dG_{\ell}}{dr}\right) = k \tag{A.18}$$

In the following we will discuss three special cases.

A.1 Zero Orbital Angular Momentum and Constant Potential

For $\ell = 0$ and V = 0, the radial equation (see Eq. (A.17)) becomes

$$\frac{d^2u}{dr^2} + k^2 u = 0 (A.19)$$

Two independent solutions that satisfy this equation are the spherical wave functions e^{ikr} and e^{-ikr} . The general solution is given in terms of the linear combination

$$u = \alpha e^{ikr} + \beta e^{-ikr}, \qquad k^2 = \frac{2m}{\hbar^2}E$$
(A.20)

If $V(r) = \text{const} \neq 0$, then the general solution is given by

$$u = \alpha e^{i\hat{k}r} + \beta e^{-i\hat{k}r}, \qquad \hat{k}^2 = \frac{2m}{\hbar^2}(E - V)$$
(A.21)

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A.2 Arbitrary Orbital Angular Momentum and Zero Potential

For the special case of the free particle or for neutrons, the potential outside the nucleus is zero (V = 0). We write

$$\frac{d^2 u_\ell}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] u_\ell = 0$$
(A.22)

With the substitution $\rho = kr$ one finds

$$\frac{d^2 u_{\ell}}{d\rho^2} + \left[1 - \frac{\ell(\ell+1)}{\rho^2}\right] u_{\ell} = 0$$
(A.23)

The solutions to this radial equation depend on ρ . They are given by the *spher*ical Bessel functions $j_{\ell}(kr)$ and *spherical Neumann functions* $n_{\ell}(kr)$ (Abramowitz and Stegun 1965; note that other authors designate by n_{ℓ} the same function with the opposite sign)

$$F_{\ell} = (kr)j_{\ell}(kr) = (kr)\left(-\frac{r}{k}\right)^{\ell}\left(\frac{1}{r}\frac{d}{dr}\right)^{\ell}\frac{\sin(kr)}{kr}$$
(A.24)

$$G_{\ell} = (kr)n_{\ell}(kr) = (kr)\left(-\frac{r}{k}\right)^{\ell} \left(\frac{1}{r}\frac{d}{dr}\right)^{\ell} \frac{\cos(kr)}{kr}$$
(A.25)

Only the function j_{ℓ} is regular at the origin. For the special case of $\ell = 0$ (s-waves) we obtain

$$j_0(kr) = \frac{\sin(kr)}{kr}$$
 and $n_0(kr) = \frac{\cos(kr)}{kr}$ (A.26)

For the asymptotic values one finds

$$j_{\ell} \xrightarrow[kr \to \infty]{} \frac{1}{kr} \sin(kr - \ell\pi/2)$$
 and $n_{\ell} \xrightarrow[kr \to \infty]{} \frac{1}{kr} \cos(kr - \ell\pi/2)$ (A.27)

A.3

Arbitrary Orbital Angular Momentum and Coulomb Potential

For the Coulomb potential we have to consider the equation

$$\frac{d^2 u_\ell}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}V(r)\right]u_\ell = 0$$
(A.28)

where

$$V(r) = \frac{Z_p Z_t e^2}{r} \tag{A.29}$$

The quantities Z_p and Z_t are the charges of the projectile and target, respectively. With the substitutions $\eta = Z_p Z_t e^2 / (\hbar v) = m Z_p Z_t e^2 / (\hbar^2 k)$ and $\rho = kr$ we obtain

$$\frac{d^2 u_\ell}{d\rho^2} + \left[1 - \frac{\ell(\ell+1)}{\rho^2} - \frac{2\eta}{\rho}\right] u_\ell = 0$$
(A.30)

The solutions are called *regular* and *irregular Coulomb wave functions*, $F_{\ell}(\eta, \rho)$ and $G_{\ell}(\eta, \rho)$ (Abramowitz and Stegun 1965). These solutions cannot be written in terms of elementary functions. Tabulation of $F_{\ell}(\eta, \rho)$ and $G_{\ell}(\eta, \rho)$ is complicated by the fact that these functions depend both on energy (through k) and charge (through Z_pZ_t). The functions are best calculated by using available computer codes (see, for example, Barnett 1982). Numerically, we find for the arguments

$$\rho = 0.218735 \cdot r \sqrt{\frac{M_p M_t}{M_p + M_t}E} \tag{A.31}$$

$$\eta = 0.157489 \cdot Z_p Z_t \sqrt{\frac{M_p M_t}{M_p + M_t} \frac{1}{E}}$$
(A.32)

where M_i , E, and r are in units of u, MeV, and fm, respectively.

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Appendix B Quantum Mechanical Selection Rules

The quantum mechanical (selection) rules for the coupling of angular momenta or parities are explained in any quantum mechanics text (see, for example, Messiah 1999). Here we give, without proof, the most important results.

Consider a system composed of two parts with angular momentum vectors of \vec{j}_1 and \vec{j}_2 . The components have eigenfunctions $\phi_{j_1m_1}$ and $\phi_{j_2m_2}$ that are labeled according to their value of the total angular momentum quantum numbers j_1 and j_2 . The *z*-components of their total angular momenta are labeled by the magnetic quantum numbers m_1 and m_2 , where

$$m_i = -j_i, -j_i + 1, \dots, j_i - 1, j_i$$
 (B.1)

The composite system of angular momentum \overline{J} has an eigenfunction Φ_{JM} labeled according to the total angular momentum quantum number J and the magnetic quantum number M. The eigenfunction of the composite system can be expanded according to

$$\Phi_{JM}(j_1, j_2) = \sum_{m_1, m_2} (j_1 m_1 j_2 m_2 | JM) \phi_{j_1 m_1} \phi_{j_2 m_2}$$
(B.2)

The amplitudes $(j_1m_1j_2m_2|JM)$ are called *Clebsch–Gordan coefficients*. Their squares represent the probability of finding the coupled state $\Phi_{JM}(j_1, j_2)$ in the product state $\phi_{j_1m_1}\phi_{j_2m_2}$. The Clebsch–Gordan coefficients have important symmetry properties. They vanish unless the coupling of angular momentum vectors, $\vec{J} = \vec{j_1} + \vec{j_2}$, obeys the following rules:

$$|j_1 - j_2| \le J \le j_1 + j_2 \tag{B.3}$$

$$M = m_1 + m_2 = -J, -J + 1, \dots, J - 1, J$$
(B.4)

Clebsch–Gordan coefficients are widely tabulated (Rotenberg et al. 1959). They can also be calculated with readily available computer codes.

The total angular momentum \vec{J} and total parity Π are conserved in a nuclear reaction. While \vec{J} can be obtained from the above quantum mechanical rules of angular momentum coupling, the total parity of the composite system is given by the product of the parities for the individual parts (Appendix A). If a

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channel contains two nuclei 1 and 2 with spins $\vec{j_1}$, $\vec{j_2}$ and parities π_1 , π_2 , then \vec{J} and Π are given by

$$\vec{J} = \vec{\ell} + \vec{j}_1 + \vec{j}_2 = \vec{\ell} + \vec{s}$$
(B.5)

$$\Pi = \pi_1 \pi_2 (-1)^{\ell} \tag{B.6}$$

where $\ell = 0, 1, 2, 3, ...$, and so on, is the relative orbital angular momentum of the pair of nuclei and the vector sum $\vec{s} = \vec{j}_1 + \vec{j}_2$ is called the *channel spin*. If a channel contains only nucleus 1 plus a photon, then one has

$$\vec{J} = \vec{L} + \vec{j}_1 \tag{B.7}$$

$$\Pi = \pi_1 (-1)^L$$
 for electric (E) multipole radiation (B.8)

$$\Pi = \pi_1 (-1)^{L+1}$$
 for magnetic (M) multipole radiation (B.9)

where L = 1, 2, 3, ..., and so on, is the multipolarity of the electromagnetic radiation. Electric and magnetic radiation of the same multipolarity have opposite parities and hence cannot be emitted together in a transition connecting two given nuclear levels. Note also that γ -ray transitions to or from spin-0 states or those between spin- $\frac{1}{2}$ -states are pure, that is, they can only proceed via a single value of L and unique character (either electric or magnetic). A few examples will be given in the following in order to illustrate angular momentum and parity conservation in nuclear reactions and decays.

Example B.1

Suppose that excited ³²S levels are populated via resonances in the ²⁸Si + $\alpha \rightarrow {}^{32}$ S reaction. The spin and parity of both the α -particle and of ²⁸Si is 0⁺. The spins and parities of the populated levels (or, equivalently, of the resonances) are given by the quantum numbers j_r and π_r . Conservation of angular momentum and parity demands (see Eqs. (B.5) and (B.6))

$$\vec{j}_{\alpha} + \vec{j}_{28Si} + \vec{\ell}_{\alpha} = \vec{j}_r \qquad \text{and} \qquad \pi_{\alpha} \pi_{28Si} (-1)^{\ell_{\alpha}} = \pi_r \\ 0 \quad 0 \quad \ell_{\alpha} \to j_r \qquad (+1)(+1)(-1)^{\ell_{\alpha}} = \pi_r$$

The individual spins \vec{j}_{α} , \vec{j}_{2^8Si} can only couple to a unique value of the channel spin,

$$s = |j_{\alpha} - j_{28} |_{Si} |_{Si} = |0 - 0|, \dots, 0 + 0 = 0$$

In this case we simply find $j_r = \ell_{\alpha}$ and $\pi_r = (-1)^{\ell_{\alpha}}$. The allowed orbital angular momentum quantum numbers ℓ_{α} for particular values of j_r and π_r

are thus given by

α	+	²⁸ Si		\rightarrow	^{32}S
0^+		0^+	ℓ_{α}	\rightarrow	$j_r^{\pi_r}$
			0		0^+
			1		1^{-}
			2		2^{+}
			3		3-
			:		:

In other words, the resonance spin and parity are uniquely determined by the orbital angular momentum. For $\ell_{\alpha} = 0, 1, 2, ...$, and so on, the resonance quantum numbers are $j_r^{\pi_r} = 0^+, 1^-, 2^+, ...$, and so on. Levels with this combination of quantum numbers are referred to as *natural parity* states. In particular, levels of unnatural parity ($j_r^{\pi_r} = 0^-, 1^+, 2^-, ...$, and so on) cannot be populated in the ²⁸Si + $\alpha \rightarrow$ ³²S reaction (if the target and projectile are in their ground states). The spin and parities couple in exactly the same manner for the decay of excited ³²S levels into the channel ²⁸Si + α .

Example B.2

Suppose that excited ³³Cl levels are populated via resonances in the ³²S + p \rightarrow ³³Cl reaction. The spin and parity of the proton and of ³²S are 1/2⁺ and 0⁺, respectively. Conservation of angular momentum and parity demands

$\vec{j}_{\rm p} + \vec{j}_{32S} + \vec{\ell}_{\rm p} = \vec{j}_r$	and	$\pi_{\mathrm{p}}\pi_{\mathrm{32}_{\mathcal{S}}}(-1)^{\ell_{\mathrm{p}}}=\pi_{r}$
$\frac{1}{2}$ 0 $\ell_p \rightarrow j_r$		$(+1)(+1)(-1)^{\ell_p} = \pi_r$

The individual spins \vec{j}_{p} , \vec{j}_{32S} can only couple to the channel spin value of

$$s = |j_{\rm p} - j_{32_S}|, \dots, j_{\rm p} + j_{32_S} = \left|\frac{1}{2} - 0\right|, \dots, \frac{1}{2} + 0 = \frac{1}{2}$$

Thus we find in this case $\vec{j}_r = \vec{s} + \vec{l}_p$ and $\pi_r = (-1)^{\ell_p}$. The allowed orbital angular momentum quantum numbers ℓ_p for particular values of $j_r^{\pi_r}$ are, ac-

cording to $|j_r - s| \le \ell_p \le j_r + s$ (see Eq. (B.3)) and $\pi_r = (-1)^{\ell_p}$, given by

р	+	³² S		\rightarrow	³³ Cl
$\frac{1}{2}^+$		0^+	ℓ_{p}	\rightarrow	$j_r^{\pi_r}$
			0		$\frac{1}{2}^{+}$
			1		$\frac{1}{2}^{-}$
			2		$\frac{3}{2}^{+}$
			1		$\frac{3}{2}^{-}$
			÷		÷

As was the case in the previous example, a level (or resonance) of given spin and parity $(j_r^{\pi_r})$ can be populated only with a single value of the orbital angular momentum quantum number (ℓ_p) .

Example B.3

Suppose that excited ³²S levels are populated via resonances in the ³¹P + p \rightarrow ³²S reaction. The spin and parity of both the proton and of ³¹P is 1/2⁺. Conservation of angular momentum and parity demands

$$\vec{j}_{p} + \vec{j}_{31p} + \vec{\ell}_{p} = \vec{j}_{r} \qquad \text{and} \qquad \pi_{p} \pi_{31p} (-1)^{\ell_{p}} = \pi_{r}$$
$$\frac{1}{2} \quad \frac{1}{2} \qquad \ell_{p} \to j_{r} \qquad (+1)(+1)(-1)^{\ell_{p}} = \pi_{r}$$

The individual spins \vec{j}_p , \vec{j}_{31p} can couple to the channel spin values of

$$s = |j_p - j_{31_P}|, \dots, j_p + j_{31_P} = \left|\frac{1}{2} - \frac{1}{2}\right|, \dots, \frac{1}{2} + \frac{1}{2} = 0 \text{ or } 1$$

Thus we find in this case $\vec{j}_r = \vec{s} + \vec{\ell}_p$ and $\pi_r = (-1)^{\ell_p}$. The allowed orbital angular momentum quantum numbers ℓ_p for particular values of $j_r^{\pi_r}$ are, ac-

р	+	³¹ P		\rightarrow	³² S
$\frac{1}{2}^{+}$		$\frac{1}{2}^+$	$\ell_{\rm p}$	\rightarrow	$j_r^{\pi_r}$
			0 (s = 0)		0^+
			1(s = 1)		0^{-}
			0, 2 (<i>s</i> = 1)		1^+
			1(s = 0, 1)		1^{-}
			2(s = 0, 1)		2^+
			1,3 (<i>s</i> = 1)		2^{-}
			÷		÷

cording to $|j_r - s| \le \ell_p \le j_r + s$ (see Eq. (B.3)) and $\pi_r = (-1)^{\ell_p}$, given by

In this example, some ³²S levels are formed with unique values of ℓ_p and s ($j_r = 0$), while other levels can be formed with two different values of either ℓ_p or s ($j_r = 1, 2$). The relative contribution of the two components to the total cross section is described by parameters referred to as orbital angular momentum and channel spin *mixing ratios* (Appendix D).

Example B.4

We will next discuss the situation when a photon is present in a particular channel. Suppose that an excited level in ³²S has been populated by some means, for example, in an (α, γ) or (p, γ) reaction. The level has a spin and parity of $j_r^{\pi_r}$. The angular momentum and parity coupling in the incoming channel is described in Examples B.1 and B.3. We will now focus on the γ -ray decay of this level to lower-lying states in ³²S with spins and parities of j_1 and π_1 , respectively. Conservation of angular momentum and parity demands (see Eqs. (B.7)–(B.9))

 $\vec{j}_r = \vec{j}_1 + \vec{L}$ and $\pi_r = \pi_1(-1)^L$ for electric multipole radiation $\pi_r = \pi_1(-1)^{L+1}$ for magnetic multipole radiation

First suppose that the decaying ³²S level has a spin and parity of $j_r^{\pi_r} = 0^+$. The allowed values of the γ -ray multipolarity L for given values of $j_1^{\pi_1}$ for the lower lying states are, according to $|j_r - j_1| \le L \le j_r + j_1$ (see Eqs. (B.3) and

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(B.7)), given by

³² S*	\rightarrow	γ	+	³² S
0^+		L		$j_1^{\pi_1}$
		forbidden		0^+
		forbidden		0^{-}
		M1		1^{+}
		E1		1^{-}
		E2		2+
		M2		2^{-}
		÷		÷

The $0 \rightarrow 0$ transitions are forbidden since monopole radiation (L = 0) does not exist. In other words, photons must carry at least an angular momentum of \hbar . Such transitions may still proceed (Section 1.7.1) by internal conversion (deexcitation of the nucleus via emission of an atomic electron) or internal pair formation (de-excitation of the nucleus via emission of an electron-positron pair if the excitation energy exceeds an amount of $2m_ec^2$). All other γ -ray transitions proceed with a unique value of the multipolarity L.

If, on the other hand, the decaying level has a spin and parity of $j_r^{\pi_r} = 1^-$, then the following values of *L* are allowed for given values of $j_1^{\pi_1}$ for the lowerlying states:

${}^{32}S^*$	\rightarrow	γ	+	³² S
1^{-}		L		$j_1^{\pi_1}$
		E1		0^+
		M1		0^{-}
		E1, M2		1^{+}
		M1, E2		1^{-}
		E1, M2, E3		2^{+}
		M1, E2, M3		2-
		:		÷

The $1 \rightarrow 0$ transitions proceed either via electric or magnetic dipole radiation (L = 1). All other transitions can proceed via radiations of different multipolarities. The relative contribution of the individual components to the total transition probability is described by the γ -ray mixing ratio (see Eq. (1.31)).

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As already noted above, parity conservation implies that electric and magnetic radiation of the same multipolarity can never be emitted together in the same transition. The γ -ray transition probability decreases fast for increasing multipolarity and in practice one usually encounters the mixing of no more than the lowest two multipole radiations.

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Appendix C Kinematics

In the following, expressions are presented that describe the kinematics of a binary interaction $a + A \rightarrow b + B$, where species a, A, and B are particles with rest mass. For more detailed discussions of the kinematics in nuclear physics, the reader is referred to Marmier and Sheldon (1969) and references therein. Kinematics calculations can be conveniently performed by using readily available computer codes.

The kinematics of a nuclear reaction or of elastic scattering is determined by the conservation of total energy and linear momentum. Consider Fig. C.1 (left panel), showing a collision between a projectile *a* and a stationary target nucleus *A* in the laboratory. After the collision, the recoil nucleus *B* moves into a direction specified by the laboratory angle ϕ , while species *b* moves into a direction given by laboratory angle θ . If species *b* is a photon, then the collision represents a radiative capture process. If species *a* is identical to *b*, and species *A* is identical to *B* (that is, their state of excition), then the collision represents elastic scattering. First, expressions are given that relate quantities appropriate for the laboratory coordinate system only. Afterward, formulas for the transformation of quantities between laboratory and centerof-mass coordinate systems are presented.

C.1

Relationship of Kinematic Quantities in the Laboratory Coordinate System

Consider first a collision involving only particles with rest mass. The target nucleus *A* is assumed to be stationary in the laboratory system. Conservation of energy and linear momentum yields the three equations

$$m_a c^2 + E_a + m_A c^2 = m_b c^2 + E_b + m_B c^2 + E_B$$
(C.1)

$$\sqrt{2m_a E_a} = \sqrt{2m_B E_B \cos\phi} + \sqrt{2m_b E_b \cos\theta} \tag{C.2}$$

$$0 = \sqrt{2m_B E_B \sin \phi} - \sqrt{2m_b E_b \sin \theta} \tag{C.3}$$

where *E* and *m* denote the kinetic energy and the rest mass, respectively. The linear momenta are given by $p = \sqrt{2mE}$. The second and third expression

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describes the total linear momentum parallel and perpendicular, respectively, to the incident beam direction. It is usually difficult to detect species *B* if it represents a heavy recoil nucleus. By eliminating E_B and ϕ and by using the definition of the reaction *Q*-value, $Q = (m_a + m_A - m_b - m_B)c^2$ (see Eq. (1.4)), one finds

$$Q = E_b \left(1 + \frac{m_b}{m_B} \right) - E_a \left(1 - \frac{m_a}{m_B} \right) - \frac{2}{m_B} \sqrt{m_a m_b E_a E_b} \cos \theta \tag{C.4}$$

This expression is sometimes used to determine an unknown *Q*-value by measuring E_a , E_b , and θ if the masses m_a , m_b , and m_B are known. Frequently, one is interested in the energy E_b of the emitted particle as a function of the bombarding energy E_a and the angle θ . From Eq. (C.4) one obtains

$$\sqrt{E_b} = r \pm \sqrt{r^2 + s} \tag{C.5}$$

where

$$r = \frac{\sqrt{m_a m_b E_a}}{m_b + m_B} \cos \theta \quad \text{and} \quad s = \frac{E_a (m_B - m_a) + m_B Q}{m_b + m_B} \tag{C.6}$$

We assumed above that in low-energy nuclear reactions the speeds of the particles are sufficiently small to disregard relativistic effects. For very accurate work, one can take the relativistic correction into account if each mass m in the above expressions is replaced by $m + E/(2c^2)$. Only real and positive solutions of E_b in Eqs. (C.5) and (C.6) are physically allowed. A number of different cases can be distinguished. If the reaction is exothermic (Q > 0) and if the projectile mass is smaller than the mass of the residual nucleus ($m_a < m_B$), then s > 0 and there will only be one positive solution for E_b . Because of the $\cos \theta$ dependence of r, E_b has a minimum at $\theta = 180^\circ$. For very small projectile energies, for example, in reactions involving thermal neutrons, we find $r \to 0$ and hence

$$E_b(E_a \approx 0) \approx s \approx Qm_B / (m_B + m_b) \tag{C.7}$$

This implies that the kinetic energy of the emitted particle *b* has the same value for all angles. The situation is more complex if the reaction is endothermic (Q < 0). For very small projectile energies, $E_a \approx 0$, one has again $r \rightarrow 0$, but *s* becomes negative so that no positive value of E_b exists. Hence, for each angle θ there will be a minimum energy below which the reaction cannot proceed. The value of this minimum energy is smallest at $\theta = 0^\circ$ and is referred to as the *threshold energy*, given by

$$E_a^{\min}(\theta = 0^\circ) = E_a^{\text{thresh}} = -Q \frac{m_b + m_B}{m_b + m_B - m_a}$$
 (C.8)

C.1 Relationship of Kinematic Quantities in the Laboratory Coordinate System 591



Fig. C.1 Kinematic properties of a reaction A(a, b)B in the laboratory coordinate system (left) and the center-of-mass coordinate system (right). The target nucleus A is assumed to be stationary in the laboratory ($v_A = 0$). Unprimed and primed quantities are used in the laboratory and center-of-mass frame, respectively. The location of the center of mass is labeled "c."

At the threshold energy, the particles are emitted only in the direction $\theta = 0^{\circ}$ with an energy of

$$E_b(E_a = E_a^{\text{thresh}}) = E_a^{\text{thresh}} \frac{m_a m_b}{(m_b + m_B)^2}$$
(C.9)

If one increases the bombarding energy beyond the threshold energy, then the particles *b* can be emitted at angles greater than $\theta = 0^{\circ}$. It is also interesting to note that for endothermic reactions Eqs. (C.5) and (C.6) yield two positive solutions for $\theta < 90^{\circ}$. In other words, two particle groups of different discrete energies are emitted in the forward direction. For bombarding energies exceeding

$$E_a = -Q \frac{m_B}{m_B - m_a} \tag{C.10}$$

there exists only a single positive solution for Eqs. (C.5) and (C.6).

Consider now a radiative capture process $a + A \rightarrow B + \gamma$. In this case we have to replace in Eqs. (C.1)–(C.3) the total energy, $m_b c^2 + E_b$, and linear momentum, $\sqrt{2m_b E_b}$, of species *b* by E_γ and E_γ/c , respectively. Eliminating again E_B and ϕ and solving for the energy of the emitted photon yields

$$E_{\gamma} = Q + \frac{m_A}{m_B} E_a + E_{\gamma} \frac{v_B}{c} \cos \theta - \frac{E_{\gamma}^2}{2m_B c^2} = Q + \frac{m_A}{m_B} E_a + \Delta E_{\text{Dopp}} - \Delta E_{\text{rec}}$$
(C.11)

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The photon energy is given by a sum of four terms: (i) the value of $Q = (m_a + m_A - m_B)c^2 = E_B + E_{\gamma} - E_a$; (ii) the bombarding energy in the centerof-mass system (see below); (iii) the *Doppler shift* since the photon is emitted by a recoil nucleus *B* moving at a speed of $v_B = v_a(m_a/m_B)$; and (iv) the *recoil shift* which is caused by the energy shift of the recoiling nucleus. The last two contributions represent relatively small corrections and are numerically given by

$$\Delta E_{\text{Dopp}} = 4.63367 \times 10^{-2} \frac{\sqrt{M_a E_a}}{M_B} E_\gamma \cos\theta \qquad (\text{MeV})$$
(C.12)

$$\Delta E_{\rm rec} = 5.36772 \times 10^{-4} \frac{E_{\gamma}^2}{M_B} \qquad ({\rm MeV})$$
(C.13)

where all energies are in units of MeV and the rest masses are in units of u. The calculation of the photon energy from Eq. (C.11) is not strictly valid since E_{γ} also occurs on the right-hand side. If an answer with a precision of a few keV or so is sufficient, then one may replace the masses with (integer) mass numbers and use the approximation $E_{\gamma} \approx Q + E_a(m_A/m_B)$ on the right-hand side of Eq. (C.11). For accurate work, however, the masses of *a*, *A*, and *B* in Eqs. (C.1)–(C.3) should be replaced by the factors $m_i + E_i/(2c^2)$. The exact relativistic expression for the photon energy is then given by

$$E_{\gamma} = \frac{Q(m_a c^2 + m_A c^2 + m_B c^2)/2 + m_A c^2 E_a}{m_a c^2 + m_A c^2 + E_a - \cos \theta \sqrt{E_a (2m_a c^2 + E_a)}}$$
(C.14)

The relationship between the photon emission angle θ and the recoil angle ϕ can be obtained from the ratio of Eqs. (C.2) and (C.3),

$$\phi = \arctan\left(\frac{\sin\theta}{E_{\gamma}^{-1}\sqrt{2m_ac^2E_a} - \cos\theta}\right)$$
(C.15)

The maximum angle of ϕ is obtained when the photon is emitted perpendicular to the incident beam direction ($\theta = 90^{\circ}$),

$$\phi_{\max} = \arctan\left(\frac{E_{\gamma}}{\sqrt{2m_a c^2 E_a}}\right) \tag{C.16}$$

Hence, the recoil nucleus *B* is emitted in the forward direction into a cone of half-angle ϕ_{max} .

A few comments are in order. If the reaction $A + a \rightarrow B + b$ or $A + a \rightarrow B + \gamma$ populates an excited state in nucleus *B*, then the *Q*-value in the above expressions must account for the energy of the excited state,

$$Q = Q_0 - E_x \tag{C.17}$$

where Q_0 is the *Q*-value for the ground state of *B*. Several excited levels may be populated in a given reaction. For a fixed angle θ , each of these states will give rise to a different value for the energy of the reaction products (for example, E_b or E_{γ}), where the largest observed energy corresponds to the population of the ground state. From a measurement of E_b or E_γ we may thus deduce an unknown excitation energy E_x by using Eqs. (C.5), (C.11), or (C.14). Note that for a radiative capture reaction the maximum emission angle ϕ_{max} of *B* is given by Eq. (C.16), with E_{γ} denoting the photon energy for the ground state transition, even if the primary decay proceeds to an excited level (since subsequent de-excitation photons may also be emitted at $\theta = 90^{\circ}$). The above expressions disregard the beam energy loss in the target and assume that the reaction is induced with a bombarding energy of E_a in the laboratory. If the reaction excites a narrow resonance, then the interaction is induced at the resonance energy E_r rather than at the actual incident beam energy. In this case, E_a in the above expressions represents E_r . Finally, for the case of radiative capture reactions it is assumed that the γ -ray emission occurs on a sufficiently short time scale for recoil energy losses in the target to be negligible, that is, the emitted photon experiences the full Doppler energy shift. If the photon is emitted after the recoil nucleus experienced an energy loss in the target, then the Doppler shift is attenuated. It is sometimes possible to deduce the mean lifetime of a nuclear level by measuring the attenuated Doppler shift (see, for example, Bertone et al. 2001).

C.2

Transformation Between Laboratory and Center-of-Mass Coordinate System

In experimental nuclear physics, all observations take place in a reference frame that is at rest in the laboratory. It is referred to as the *laboratory coordinate system*. From the theoretical point of view, however, the motion of the center of mass is of no consequence for the properties of a nuclear reaction. It is then often more convenient to use a moving coordinate frame in which the center of mass of the two colliding nuclei is at rest. It is called the *center-ofmass coordinate system*. Most kinematic quantities in Chapters 3 and 5 are given in the center-of-mass system. However, in Chapter 4 these quantities are frequently presented in the laboratory system, as is customary in the nuclear physics literature, since this is where the quantities are directly observed. We will only consider here the nonrelativistic transformation of kinematic quantities between these two reference frames. For the relativistic case, see Marmier and Sheldon (1969) and references therein.

The kinematic properties of a nuclear reaction A(a, b)B in the laboratory and center-of-mass frames are shown in Fig. C.1. Unprimed and primed quantities

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will be used in this section for the former and the latter coordinate system, respectively. It is assumed that the target nucleus is stationary in the laboratory ($v_A = 0$). In the center-of-mass frame, the total linear momentum is always equal to zero and, therefore, the nuclei *b* and *B* will recede in opposite directions. In other words, there is only one scattering angle θ' .

We will first consider the situation before the collision. The velocity \vec{v}_c of the center-of-mass is given by the relations

$$(m_a + m_A)\vec{v}_c = m_a\vec{v}_a + m_A \cdot 0$$
 or $\vec{v}_c = \frac{m_a}{m_a + m_A}\vec{v}_a$ (C.18)

and hence the projectile and target have velocities in the center-of-mass frame of

$$\vec{v}_a' = \vec{v}_a - \vec{v}_c = \left(1 - \frac{m_a}{m_a + m_A}\right)\vec{v}_a = \frac{m_A}{m_a + m_A}\vec{v}_a \tag{C.19}$$

$$\vec{v}_A' = \vec{v}_A - \vec{v}_c = -\vec{v}_c = -\frac{m_a}{m_a + m_A}\vec{v}_a$$
 (C.20)

Since the total linear momentum of a + A is zero in the center-of-mass frame, we find for the ratio of speeds

$$m_a \vec{v}'_a = m_A \vec{v}'_A$$
 or $\frac{v'_a}{v'_A} = \frac{m_A}{m_a}$ (C.21)

The kinetic energies of the two particles in the center-of-mass system are given by (see Eqs. (C.19) and (C.20))

$$E'_{a} = \frac{1}{2}m_{a}(v'_{a})^{2} = \frac{1}{2}m_{a}v_{a}^{2}\left(\frac{m_{A}}{m_{a}+m_{A}}\right)^{2} = E_{a}\frac{m_{A}^{2}}{(m_{a}+m_{A})^{2}}$$
(C.22)

$$E'_{A} = \frac{1}{2}m_{A}(v'_{A})^{2} = \frac{1}{2}m_{A}v_{a}^{2}\left(\frac{m_{a}}{m_{a}+m_{A}}\right)^{2} = E_{a}\frac{m_{A}m_{a}}{(m_{a}+m_{A})^{2}}$$
(C.23)

and the total kinetic energy in the center-of-mass system before the collision is related to the laboratory bombarding energy by

$$E'_{i} = E'_{a} + E'_{A} = E_{a} \frac{m_{A}^{2} + m_{A}m_{a}}{(m_{a} + m_{A})^{2}} = E_{a} \frac{m_{A}}{m_{a} + m_{A}}$$
(C.24)

The laboratory bombarding energy, E_a , can be expressed as the sum of total kinetic energy in the center-of-mass system before the collision, E'_i , and the kinetic energy of the center-of-mass motion, E_c , as can be seen from (see Eqs. (C.18) and (C.24))

$$E_{a} = \frac{1}{2}m_{a}v_{a}^{2} = \frac{1}{2}\frac{m_{A}m_{a}}{m_{a} + m_{A}}v_{a}^{2} + \frac{1}{2}\frac{m_{a}^{2}}{m_{a} + m_{A}}\frac{m_{a} + m_{A}}{m_{a} + m_{A}}v_{a}^{2}$$
$$= E_{a}\frac{m_{A}}{m_{a} + m_{A}} + \frac{1}{2}(m_{a} + m_{A})v_{c}^{2} = E_{i}' + E_{c}$$
(C.25)

Furthermore, we find from Eq. (C.24)

$$E'_{i} = \frac{1}{2} \frac{m_{a}m_{A}}{m_{a} + m_{A}} v_{a}^{2} = \frac{1}{2} m_{aA} v_{a}^{2}$$
(C.26)

and thus the total center-of-mass kinetic energy can be expressed in terms of the laboratory bombarding velocity, v_a , and the *reduced mass* of particles *a* and *A*, defined as $m_{aA} \equiv m_a m_A / (m_a + m_A)$. Obviously, the above expressions apply equally to a radiative capture reaction, $A(a, \gamma)B$, or to elastic scattering, A(a, a)A.

We will now consider the situation after the collision. The total linear momentum in the center-of-mass system remains zero. For a reaction A(a, b)B, the two residual particles *b* and *B* separate in opposite directions with equal but opposite linear momenta,

$$m_b v'_b = m_B v'_B \tag{C.27}$$

The kinetic energies in the center-of-mass system are given by

$$E'_b = \frac{1}{2}m_b(v'_b)^2 \tag{C.28}$$

$$E'_B = \frac{1}{2}m_B(v'_B)^2 = \frac{1}{2}m_b(v'_b)^2 m_B \frac{m_b}{m_B^2} = \frac{m_b}{m_B}E'_b$$
(C.29)

The total kinetic energy in the center-of-mass system after the collision is then

$$E'_{f} = E'_{b} + E'_{B} = E'_{b} + \frac{m_{b}}{m_{B}}E'_{b} = E'_{b}\left(1 + \frac{m_{b}}{m_{B}}\right)$$
(C.30)

The kinetic energies in the center-of-mass system after the collision can be expressed in terms of the laboratory bombarding energy by using $E'_i + Q = E'_f$ (see Eq. (1.5)). The total kinetic energy is given by (see Eq. (C.24))

$$E'_{f} = E'_{i} + Q = E_{a} \frac{m_{A}}{m_{a} + m_{A}} + Q = Q + E_{a} \left(1 - \frac{m_{a}}{m_{a} + m_{A}}\right)$$
(C.31)

After some algebra one obtains for the kinetic energies of the particles

$$E'_b = \frac{m_B}{m_b + m_B} \left[Q + E_a \left(1 - \frac{m_a}{m_b + m_B} \right) \right]$$
(C.32)

$$E'_B = \frac{m_b}{m_b + m_B} \left[Q + E_a \left(1 - \frac{m_a}{m_b + m_B} \right) \right] \tag{C.33}$$

Finally, we will present the transformation equations for the angles and solid angles in the laboratory and center-of-mass systems. After the collision, we have for a reaction A(a, b)B (see Eq. (C.19))

$$\vec{v}_b' = \vec{v}_b - \vec{v}_c \tag{C.34}$$

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or, in terms of the components parallel with and perpendicular to the beam direction

$$v_b' \cos \theta' = v_b \cos \theta - v_c \tag{C.35}$$

$$v_b'\sin\theta' = v_b\sin\theta - 0 \tag{C.36}$$

From these expressions one can derive either of the following two relationships:

$$\tan \theta = \frac{v'_b \sin \theta'}{v'_b \cos \theta' + v_c} = \frac{\sin \theta'}{\cos \theta' + v_c / v'_b} = \frac{\sin \theta'}{\cos \theta' + \gamma}$$
(C.37)

$$\cos\theta = \frac{\gamma + \cos\theta'}{\sqrt{1 + \gamma^2 + 2\gamma\cos\theta'}} \tag{C.38}$$

The parameter γ is defined by the ratio of velocities of the center of mass and of particle *b* in the center-of-mass system,

$$\gamma \equiv \frac{v_c}{v_b'} = \sqrt{\frac{m_a m_b E_a}{m_B (m_b + m_B)Q + m_B (m_B + m_b - m_a)E_a}}$$
$$\approx \sqrt{\frac{m_a m_b}{m_A m_B} \frac{E_a}{(1 + m_a/m_A)Q + E_a}}$$
(C.39)

where the approximation is obtained by setting $m_a + m_A \approx m_b + m_B$. For a very heavy target nucleus, one finds $\gamma \approx 0$ and hence the angle of the emitted particle *b* has about the same value in the laboratory and center-of-mass systems ($\theta \approx \theta'$). For elastic scattering, $m_a = m_b$, $m_A = m_B$, Q = 0, and thus one finds $\gamma = m_a/m_A$. For a radiative capture reaction, $A(a, \gamma)B$, the laboratory and center-of-mass angle of the emitted photon are related by (given here without proof)

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \tag{C.40}$$

where the relativistic parameter β is defined as

$$\beta \equiv \frac{\sqrt{E_a(E_a + 2m_a c^2)}}{m_A c^2 + m_a c^2 + E_a} \tag{C.41}$$

The definition of the differential cross section implies that the same number of reaction products are emitted into the solid angle element $d\Omega$ in the direction θ (laboratory system) as are emitted into $d\Omega'$ in the corresponding direction θ' (center-of-mass system). Thus

$$\left(\frac{d\sigma}{d\Omega}\right)_{\theta} d\Omega = \left(\frac{d\sigma}{d\Omega}\right)'_{\theta'} d\Omega' \tag{C.42}$$

We assume that the cross section depends on θ or θ' , but not on the azimuthal angle. Hence

$$\frac{(d\sigma/d\Omega)'_{\theta'}}{(d\sigma/d\Omega)_{\theta}} = \frac{d\Omega}{d\Omega'} = \frac{d(\cos\theta)}{d(\cos\theta')}$$
(C.43)

From Eq. (C.38) we find for a reaction A(a, b)B

$$\frac{d(\cos\theta)}{d(\cos\theta')} = \frac{1+\gamma\cos\theta'}{(1+\gamma^2+2\gamma\cos\theta')^{3/2}} = \frac{\sqrt{1-\gamma^2\sin^2\theta}}{\left(\gamma\cos\theta+\sqrt{1-\gamma^2\sin^2\theta}\right)^2} \quad (C.44)$$

For a radiative capture reaction, $A(a, \gamma)B$, one obtains from Eq. (C.40)

$$\frac{d(\cos\theta)}{d(\cos\theta')} = \frac{1-\beta^2}{(1+\beta\cos\theta')^2} \tag{C.45}$$

for the relationship of the solid angles of the emitted photon.

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Appendix D Angular Correlations

Traditionally, angular correlation measurements have been used in nuclear physics as a powerful tool in order to determine the angular momenta of states participating in nuclear transitions. It also turns out that angular correlations are sensitive to the ratios of nuclear matrix elements (that is, mixing ratios; see later) that correspond to different possibilities of coupling angular momenta in a specific transition. We will not attempt here to summarize this vast field, but will focus on aspects that are of primary importance in low-energy nuclear astrophysics measurements.

Uncertainties in thermonuclear reaction rates are caused by contributions from resonances or nonresonant reaction processes that are as yet unobserved. The goal of the experimentalist is to measure such contributions. If the detection system covers the entire solid angle (4π sr), the measured intensities represent angle-integrated yields. These may then be converted to cross sections or resonance strengths (Sections 4.8 and 4.9). However, in most experimental setups the detector(s) will cover only a fraction of the full solid angle. What is measured in such cases are *differential* yields that may be influenced by angular correlation effects. It should be pointed out that the angular momenta for many levels participating in astrophysically important reactions are known or, at least, have been restricted to a certain range of values by previous nuclear structure studies. Hence, it becomes in principle possible to estimate angular correlation effects by making reasonable assumptions and, if necessary, to correct the measured differential yields appropriately.

A comprehensive theory of angular correlations is beyond the scope of the present work. The interested reader is referred to the specialized literature (see, for example, Devons and Goldfarb 1957). The focus of this section is on angular correlations in astrophysically important reactions, that is, processes such as A(a, b)B or $A(a, \gamma)B$, where *a* and *b* denote particles with rest mass. We will briefly explain the origin of angular correlations in such processes and examples of the application of angular correlations to specific cases will be given. In this section, all angles θ refer to the center-of-mass system.

Nuclear Physics of Stars. Christian Iliadis Copyright © 2007 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim ISBN: 978-3-527-40602-9

D Angular Correlations **D.1 General Aspects**

For the discussions in this section, we will make the following assumptions: (i) the beam is unpolarized and the target nuclei are randomly oriented; (ii) the nuclear levels involved in the transitions at each stage have unique spin and well-defined parity; (iii) the polarization of the detected radiations is not observed. These assumptions apply to most cases of interest here. The term *radiation* denotes bombarding (incident) particles or γ -rays as well as emitted (outgoing) particles or γ -rays. An angular correlation between two radiations (for example, between the incident beam and an outgoing radiation, or between two successive outgoing radiations) is the result of the alignment of a particular nuclear level. An aligned level of spin J is prepared by some process that populates its 2J + 1 magnetic substates *unequally* with the condition that the population of the +m substate will be equal to the population of the -msubstate (since we assume unpolarized beam and target nuclei). Particles or γ -rays that are emitted from a specific substate *m* of the aligned level and that populate a substate m_f of a final level will then have a characteristic radiation pattern, or angular correlation, with respect to some (z-)axis of quantization, depending on the value of $\Delta m = m - m_f$. The total radiation pattern will consist of the superposition of all allowed transitions $m \rightarrow m_f$ between substates. The alignment in reactions of type A(a, b)B or $A(a, \gamma)B$ is achieved by the fact that the orbital angular momentum carried by the incident radiation is perpendicular to its direction of motion. This simple circumstance, plus the additional fact that angular momentum is conserved, forms the foundation of the angular correlation theory for unpolarized radiations.

As a simple example, we will consider an excited level of spin and parity $J^{\pi} = 1^{-}$ that decays to a 0⁺ ground state via emission of electric dipole (E1; L = 1) radiation (Fig. D.1). The spatial distribution of the emitted photons will depend on the magnetic quantum numbers m and m_f of the decaying and the final level, respectively, where each allowed value of $\Delta m = m - m_f$ gives rise to a different radiation pattern. In our example, the decaying level consists of $(2 \cdot 1 + 1) = 3$ substates and the final level has only $(2 \cdot 0 + 1) = 1$ substate. The allowed transitions are then described by $m - m_f = 0 - 0 = 0$ and $m - m_f = \pm 1 - 0 = \pm 1$. The corresponding radiation patterns are given by $W_{\Delta m=0}(\theta) \sim \sin^2 \theta$ and $W_{\Delta m=\pm 1}(\theta) \sim (1 + \cos^2 \theta)/2$, respectively (Jackson 1975). These are plotted as polar intensity diagrams in Fig. D.1. Suppose first that the $J^{\pi} = 1^{-}$ level is populated by the β -decay of a parent state and that the β -particles are not detected. Under such conditions, the β -decay populates the magnetic substates *equally* that is, with a probability of p(m) = 1/(2J + 1) =

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1/3. The total photon radiation pattern is thus given by

$$W(\theta) = \sum_{m} p(m) W_{m \to m_{f}}(\theta)$$

$$\sim \frac{1}{3} \cdot \frac{1}{2} (1 + \cos^{2} \theta) + \frac{1}{3} \sin^{2} \theta + \frac{1}{3} \cdot \frac{1}{2} (1 + \cos^{2} \theta) = \text{const}$$
(D.1)

and hence becomes isotropic. Now suppose that the $J^{\pi} = 1^{-}$ level is instead populated as a resonance in a capture reaction $A(a, \gamma)B$ involving target and projectile spins and parities of $j_A = 0^+$ and $j_a = 0^+$. The resonance can only be formed by absorption of particles *a* with an orbital angular momentum of $\ell_a =$ 1 (Example B.1). Provided the incident particle beam is well collimated, the projection of the orbital angular momentum vector along the incident beam direction is zero (Fig. 2.4). The allowed range of magnetic substates of the resonance that can be populated in this type of capture reaction is then given by $m_{\text{res}} \leq j_A + j_a$ (see Eqs. (B.3) and (B.4)). It follows that, among the three different magnetic substates of the resonance, only the m = 0 substate can be populated in the reaction. In other words, we obtain p(0) = 1 and $p(\pm 1) = 0$, and the γ -ray decay must proceed from m = 0 to $m_f = 0$. Consequently, the total radiation pattern is given by the $\Delta m = m - m_f = 0$ transition only

$$W(\theta) = \sum_{m} p(m) W_{m \to m_f}(\theta) \sim \sin^2 \theta$$
(D.2)

The alignment in this example is exceptionally strong and thus the variation of the γ -ray counting rate with angle is relatively large. If the beam or target nuclei have nonzero spin, then the alignment will be weaker, but angular correlation effects are in general nevertheless observed.

In certain situations involving nuclear reactions, all magnetic substates are populated equally, independent of the mode of formation. For example, the capture of unpolarized protons by spin-zero target nuclei leading to a J = 1/2 resonance will always populate the $m = \pm 1/2$ magnetic substates of the resonance uniformly. As a result, the total radiation pattern will be isotropic. Similar arguments apply to a resonance of spin J = 0. In this case, only one magnetic substate exists and the transitions to the various substates in the final state proceed with equal probabilities. As a result, the total radiation pattern must necessarily be isotropic.

We considered so far only the angular correlation caused by the alignment of levels produced in nuclear reactions (also termed *angular distribution*). Another type of angular correlation occurs if an excited level de-excites to a final state through an intermediate level by emitting two successive radiations (for example, two photons). In this case, measurement of the direction of the first radiation will produce an aligned intermediate state. The result is again a nonuniform intensity distribution of the second radiation with respect to the 602 D Angular Correlations





Fig. D.1 (a) Level scheme for an excited state $(J^{\pi} = 1^{-})$ that can be populated either via β -decay from nucleus B' or via the capture reaction $A + a \rightarrow B + \gamma$. Both the target and the projectile have spins and parities of 0^+ . The state decays via E1 emission to the ground state $(J^{\pi} = 0^+)$. In the first case, the radiation pattern will be isotropic, while in the second case, the pattern is anisotropic because of a strong alignment. (b) Dipole radiation pattern for $\Delta m = 0$ (top) and $\Delta m = \pm 1$ (bottom).

measured direction of the first radiation. We encountered this situation in the discussion of angular correlation effects for γ -ray detector summing corrections (Section 4.5.2). As will be seen in the following, the angular correlation formalism is quite general and describes this situation as well.

The summation over magnetic quantum numbers is performed explicitly in Eq. (D.1). In more complicated situations involving a number of unobserved or coupled orientations, such a calculation becomes very tedious. Much more convenient, but equivalent, expressions have been developed where the magnetic substates are not explicitly introduced and where the sums over substates are automatically performed. A number of different formalisms and expressions can be found in the literature. Here, we will follow the work of Biedenharn (1960).

Any correlation where only two directions of motion are measured can be expressed as a Legendre polynomial series in the angle between those directions (see also Eqs. (A.9)–(A.14)). We write

$$W(\theta) = \frac{1}{b_0} \sum_{n=0}^{n_{\max}} b_n P_n(\cos \theta)$$

= $1 + \frac{b_2}{b_0} P_2(\cos \theta) + \frac{b_4}{b_0} P_4(\cos \theta) + \dots + \frac{b_{n_{\max}}}{b_0} P_{n_{\max}}(\cos \theta)$ (D.3)

If the process under consideration is a nuclear reaction, then $W(\theta)$ is related to the differential and total cross section by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\theta} = \frac{1}{4\pi} \,\sigma \,W(\theta) \tag{D.4}$$

An isotropic differential cross section implies $W(\theta) = 1$. The sum in Eq. (D.3) is restricted to even values of *n* because we are making the assumption that the reaction (or the successive decay) involves at each stage nuclear states of well-defined parity. The wave function describing the exit channel must then have the same parity as the resonance (or the intermediate state). The corresponding intensity of the emitted radiation (that is, the square of the wave function) has even parity and is unchanged by the inversion $\vec{r} \rightarrow -\vec{r}$, or more specifically, by the substitution $\theta \rightarrow \pi - \theta$ (since for unpolarized beams and randomly oriented target nuclei the intensity does not depend on the azimuthal angle ϕ). The condition $W(\theta) = W(\pi - \theta)$ implies that $W(\theta)$ is symmetric about $\theta = 90^{\circ}$ and, consequently, all odd Legendre polynomial terms in Eq. (D.3) must vanish.

The coefficients b_n in Eq. (D.3) depend on the angular momenta and nuclear matrix elements involved in the process. Theoretical expressions for b_n are given in the following. They can be factored into components referring separately to each transition. Each of these components, in turn, is expressed in terms of vector coupling (Clebsch–Gordan and Racah) coefficients. We will be using the coefficients F_n , defined by (Biedenharn 1960)

$$F_n(LL'jJ) \equiv (-)^{j-J-1} \sqrt{(2L+1)(2L'+1)(2J+1)(L1L'-1|n0)} W(JJLL';nj)$$
(D.5)

where *j* and *J* are angular momenta (spins) of nuclear states and *L* and *L'* are orbital angular momenta (for particles) or multipolarities (for photons) of radiations; (L1L' - 1|n0) and W(IJLL';nj) denotes a Clebsch–Gordan and a Racah coefficient, respectively. A tabulation of the functions $F_n(LjJ) \equiv F_n(LLjJ)$ is given in Biedenharn and Rose (1953). Numerical values of the mixed correlation coefficients $F_n(LL'jJ)$ for $L \neq L'$ can be found in Appel (1968). For n = 0, we obtain $F_0(LL'jJ) = 0$ and $F_0(LjJ) = 1$. In order to determine how many terms have to be taken into account in the sum of Eq. (D.3), it is useful to consider the symmetry properties of the functions $F_n(LL'jJ)$ which follow directly from those of the Clebsch–Gordan and Racah coefficients. For given values of L, L', and J, we obtain $F_n(LL'jJ) \neq 0$ only for $|L - L'| \leq n \leq \min(2J, L + L')$. It follows that $F_n(LjJ) \neq 0$ only for $0 \leq n \leq \min(2J, 2L)$.

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D.2 Pure Radiations in a Two-Step Process

We start by considering a two-step process, where each step proceeds via a pure transition. An intermediate state of spin *J* is formed from an initial state of spin j_1 via absorption or emission of some radiation of angular momentum L_1 . The intermediate state decays then to the final state of spin j_2 via emission of radiation with angular momentum L_2 . We write symbolically $j_1(L_1)J(L_2)j_2$. The angular correlation function between the directions of the two radiations is then given in terms of the coefficients $A_n(i)$ and the particle parameters $a_n(i)$ by

$$W(\theta) = \sum_{n=0,2,\dots} [a_n(1)A_n(1)][a_n(2)A_n(2)]P_n(\cos\theta)$$
(D.6)

for photons:

$$a_n(i) = 1; \qquad \qquad A_n(i) = F_n(L_i j_i J)$$

(D.7)

for
$$s = 0$$
 particles: $a_n(i) = \frac{2L_i(L_i+1)}{2L_i(L_i+1) - n(n+1)}; \quad A_n(i) = F_n(L_ij_iJ)$
(D.8)

for
$$s \neq 0$$
 particles: $a_n(i) = \frac{2L_i(L_i+1)}{2L_i(L_i+1) - n(n+1)}; \quad A_n(i) = F_n(L_i j_s J)$
(D.9)

For photons or particles, L_i denotes the γ -ray multipolarity or the orbital angular momentum, respectively. If a particle has a nonzero spin s, then the channel spin given by $\vec{j_s} = \vec{j_i} + \vec{s}$ and $|j_i - s| \le j_s \le j_i + s$ replaces the initial state spin j_i . The sum in Eq. (D.6) is restricted to $0 \le n \le \min(2L_1, 2L_2, 2J)$.

Example D.1

The β -decay of ⁶⁰Co populates a 4⁺ level in the ⁶⁰Ni daughter nucleus. This level decays to an intermediate state of spin 2⁺, which in turn decays to the ground state of spin 0⁺ (Fig. D.2a). Calculate the angular correlation between the two de-excitation γ -rays.

We encountered this case in Section 4.5.2 and Fig. 4.30. The β -decay electron is emitted into a random direction and is not observed. Thus, the initial 4⁺ level populated in the daughter nucleus is not aligned. The first γ -ray is also emitted into a random direction. If it is detected in a counter, then a line connecting the radioactive source with the detector represents a preferred direction relative to which the second γ -ray is emitted. Both transitions in this direction–direction correlation are γ -rays and θ represents the angle between

their correlated emission directions. Both the first and the second γ -ray decay can only proceed via an E2 transition (Example B.4). Thus, we have to consider the angular momentum sequence $j_1(L_1)J(L_2)j_2 \rightarrow 4(2)2(2)0$. From Eqs. (D.6) and (D.7) we obtain

$$W(\theta) = \sum_{n=0,2,\dots} F_n(L_1 j_1 J) F_n(L_2 j_2 J) P_n(\cos \theta) \quad \text{with} \quad 0 \le n \le \min(2L_1, 2L_2, 2J)$$

Hence

$$W(\theta) = \sum_{n=0,2,4} F_n(242)F_n(202)P_n(\cos\theta)$$

= 1 + F_2(242)F_2(202)P_2(\cos\theta) + F_4(242)F_4(202)P_4(\cos\theta)
= 1 + (-0.1707)(-0.5976)P_2(\cos\theta) + (-0.0085)(-1.069)P_4(\cos\theta)
= 1 + 0.1020P_2(\cos\theta) + 0.0091P_4(\cos\theta)

Example D.2

A resonance with spin and parity of $J^{\pi} = 2^+$ is populated in the ${}^{32}S(\alpha,\gamma){}^{36}Ar$ reaction. The resonance decays to a final state with $J^{\pi} = 0^+$ (Fig. D.2b). Calculate the expected angular correlation between the incident beam (α -particles) and the emitted γ -radiation.

The ³²S target nuclei and the α -particles have both a spin and parity of 0⁺. Therefore, the $J^{\pi} = 2^+$ resonance can only be formed from an α -particle orbital angular momentum of $\ell_{\alpha} = 2$ (Example B.1). Furthermore, the γ -ray transition can only be of E2 character (Example B.4). The angular momentum sequence is therefore given by $j_1(L_1)J(L_2)j_2 \rightarrow j_{32} \int (\ell_{\alpha})J(L_{\gamma})j_{36} A_r \rightarrow 0(2)2(2)0$. We obtain from Eqs. (D.6) and (D.8)

$$W(\theta) = \sum_{n=0,2,\dots} \frac{2L_1(L_1+1)}{2L_1(L_1+1) - n(n+1)} F_n(L_1j_1J) F_n(L_2j_2J) P_n(\cos\theta)$$

From $0 \le n \le \min(2L_1, 2L_2, 2J)$ we find n = 0, 2, and 4. Hence

$$W(\theta) = \sum_{n=0,2,4} \frac{2 \cdot 2(2+1)}{2 \cdot 2(2+1) - n(n+1)} F_n(202) F_n(202) P_n(\cos \theta)$$

= $1 + \frac{12}{12 - 6} F_2(202) F_2(202) P_2(\cos \theta) + \frac{12}{12 - 20} F_4(202) F_4(202) P_4(\cos \theta)$
= $1 + 2(-0.5976)(-0.5976) P_2(\cos \theta) + (-1.5)(-1.069)(-1.069) P_4(\cos \theta)$
= $1 + 0.7143 P_2(\cos \theta) - 1.7143 P_4(\cos \theta)$

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D.3 Mixed Radiations in a Two-Step Process

Sometimes the angular momentum coupling in a sequential nuclear decay or in a nuclear reaction allows for different possibilities, each involving a unique combination of angular momenta. In general, these pure transitions will interfere, that is, their contributions to the total angular correlation add either incoherently or coherently. In either case, new parameters have to be introduced that describe quantitatively the degree of mixing. These mixing ratios are usually determined experimentally by fitting the data and are eventually interpreted in terms of some nuclear model.

Incoherent interference applies, for example, to the channel spin j_s . Since we assumed that the beam and target nuclei are unpolarized, the channel spin is *randomly oriented*. As a consequence, the total angular correlation is given by the sum of the individual (pure) correlations, each weighted according to the probability for a particular channel spin value to occur. We write $W(\theta) =$ $W_{i_s}(\theta) + \delta_c^2 W_{i'_s}(\theta)$, where the *channel spin mixing ratio* $\delta_c^2 \equiv P_{i'_s} / P_{i_s}$ is defined as the ratio of probabilities for forming (or of decay from) the intermediate state via the channel spins j'_s and j_s , where $j'_s > j_s$.

Coherent interference occurs when definite phase relationships are important. This is the case if several possible values of multipolarites are allowed for a specific γ -ray transition, or if the intermediate state can be formed (or decay) by several possible values of orbital angular momenta. In practice, only the smallest two allowed values of γ -ray multipolarities (L_i and $L_i + 1$) or orbital angular momenta (ℓ_i and ℓ_i + 2) need to be considered (Example B.4). In such cases we have to use in Eq. (D.6) the expression

$$a_{n}(i)A_{n}(i) = a_{n}(L_{i}L_{i})F_{n}(L_{i}j_{i}J) + 2\delta_{i}a_{n}(L_{i}L_{i}')F_{n}(L_{i}L_{i}'j_{i}J) + \delta_{i}^{2}a_{n}(L_{i}'L_{i}')F_{n}(L_{i}'j_{i}J)$$
(D.10)

 $a_n(i)=1$ for photons: (D.11) for particles: $a_n(L_iL'_i) = \cos(\xi_{L_i} - \xi_{L'_i}) \frac{(L_i0L'_i0|n0)}{(L_i1L'_i - 1|n0)}$ $= \cos(\xi_{L_i} - \xi_{L'_i}) \frac{2\sqrt{[L_i(L_i+1)][L'_i(L'_i+1)]}}{L_i(L_i+1) + L'_i(L'_i+1) - n(n+1)}$ (D.12)

where the primed quantities refer to the higher value of angular momentum (particle orbital angular momentum or γ -ray multipolarity). For particles with spin, the channel spin j_s replaces again the initial state spin j_i in Eq. (D.10).

The γ -ray multipolarity mixing ratio δ_{γ} is defined by the relation $\delta_{\gamma}^2 \equiv$ $\Gamma_{\gamma L+1}/\Gamma_{\gamma L}$, with $\Gamma_{\gamma L}$ the γ -ray partial width for the transition with multi-

polarity *L* (see Eq. (1.31)). The total angular correlation not only depends on the value but also on the phase (plus or minus) of δ_{γ} . Hence, the sign convention (that is, the definition of δ_{γ} in terms of the nuclear matrix elements) becomes important when interpreting the data. We will adopt here the convention used by Biedenharn (1960). See Ferguson (1965) for a different sign convention.

For the mixing of particle orbital angular momenta, one introduces the *orbital angular momentum mixing ratio*, defined by $\delta_a^2 \equiv \Gamma_{L+2}/\Gamma_L$, with Γ_L the particle partial width for orbital angular momentum *L*. For charged particles, the phase shifts ξ_L are given by (Ferguson 1965)

$$\xi_L = -\arctan\left(\frac{F_L}{G_L}\right) + \sum_{n=1}^{L}\arctan\left(\frac{\eta}{n}\right)$$
(D.13)

where F_L and G_L are the regular and irregular Coulomb wave functions, respectively, and η is the Sommerfeld parameter (see Section 2.4.3 and Appendix A.3). The first term in the above expression is the hardsphere phase shift and the second term is the Coulomb phase shift which is absent for neutral particles. It is obvious that the phase shift ξ_L is energy dependent.

Note that if a transition is mixed with a mixing parameter of δ_i^2 , then the total angular correlation is normalized to $(1 + \delta_i^2)$ instead of unity. If two or more different mixing processes are present with mixing parameters of δ_i^2 , $\delta_{i+1}^2, \delta_{i+2}^2, \ldots$, and so on, then $W(\theta)$ is normalized to the product $(1 + \delta_i^2)(1 + \delta_{i+1}^2)(1 + \delta_{i+2}^2)\ldots$, and so on.

Example D.3

A resonance with spin and parity of $J^{\pi} = 1^{-}$ is formed in the ³¹P(p, α)²⁸Si reaction. The α -particle emission populates the ground state in the final ²⁸Si nucleus (Fig. D.2c). Calculate the angular correlation between the incident proton beam and the emitted α -particles.

Both the ³¹P target nucleus and the proton have a spin and parity of $J^{\pi} = 1/2^+$. Thus, the angular momentum coupling of the target and projectile can produce either one of two channel spin possibilities: $|1/2 - 1/2| \le j_s \le 1/2 + 1/2$, hence $j_s = 0$ or 1. The value of the orbital angular momentum is unique for the incoming and outgoing reaction channel ($\ell_p = 1$ and $\ell_{\alpha} = 1$). First, the angular correlations for the pure transitions will be calculated, that is, each channel spin case will be treated separately. We have to consider the angular momentum sequences $j_1(L_1)J(L_2)j_2 \rightarrow j_s(\ell_p)J(\ell_{\alpha})j_{2^8}j_i \rightarrow 0(1)1(1)0$ ($j_s = 0$) and $\rightarrow 1(1)1(1)0$ ($j_s = 1$). For either channel spin the sum in Eq. (D.6) is
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restricted to $0 \le n \le \min(2 \cdot 1, 2 \cdot 1)$, that is, n = 0 and 2. We obtain

$$W_{j_s=0}(\theta) = \sum_{n=0,2} \frac{2L_1(L_1+1)}{2L_1(L_1+1) - n(n+1)} F_n(L_1 j_s J) \frac{2L_2(L_2+1)}{2L_2(L_2+1) - n(n+1)} \\ \times F_n(L_2 j_2 J) P_n(\cos \theta) \\ = 1 + \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(101) \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(101) P_2(\cos \theta) \\ = 1 + (-2)(0.7071)(-2)(0.7071) P_2(\cos \theta) = 1 + 2P_2(\cos \theta)$$

Similarly

$$W_{j_s=1}(\theta) = 1 + \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(111) \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(101) P_2(\cos \theta)$$

= 1 + (-2)(-0.3536)(-2)(0.7071) P_2(\cos \theta) = 1 - P_2(\cos \theta)

The total angular correlation is given by the sum of the correlations for the individual channel spins, each weighted according to the probability of the particular j_s value. Thus

$$W(\theta) = W_{j_s=0}(\theta) + \delta_c^2 W_{j_s=1}(\theta) = [1 + 2P_2(\cos\theta)] + \delta_c^2 [1 - P_2(\cos\theta)]$$

= 1 + $\delta_c^2 + [2 - \delta_c^2] P_2(\cos\theta)$

with $\delta_c^2 = P_{j_s=1}/P_{j_s=0}$ the ratio of the probabilities, or the ratio of the squares of the matrix elements, of forming the resonance via $j_s = 1$ relative to $j_s = 0$.

Example D.4

A resonance of spin and parity of $J^{\pi} = 1^{-}$ is populated in the ²⁹Si(p, γ)³⁰P reaction. The resonance decays via γ -ray emission to a final state in ³⁰P with spin and parity of $J^{\pi} = 1^{-}$ (Fig. D.2b). Calculate the angular correlation of the emitted γ -rays with respect to the incident proton beam direction.

The spin and parity of both the ²⁹Si target nucleus and the proton is $1/2^+$. Thus, two values for the channel spin are allowed, $j_s = 0$ and 1. The only allowed value for the orbital angular momentum of the proton is $\ell_p = 1$. The γ -ray decay may proceed either via a M1 or E2 transition. Hence, the angular correlation expression will contain two additional parameters, the channel spin mixing ratio δ_c and the γ -ray multipolarity mixing ratio δ_{γ} . We will first consider the two channel spins separately and write symbolically

$$j_1(L_1)J(L_2)j_2 \to j_s(\ell_p)J(L_\gamma)j_{30p} \to 0$$
(1)1 $\begin{pmatrix} 1\\2 \end{pmatrix}$ 1 and $\to 1$ (1)1 $\begin{pmatrix} 1\\2 \end{pmatrix}$ 1

For either channel spin the sum in Eq. (D.6) is restricted to $0 \le n \le 2J$, that is, n = 0 and 2. We obtain

$$\begin{split} W_{j_s=0}(\theta) &= \sum_{n=0,2} \left[\frac{2L_1(L_1+1)}{2L_1(L_1+1) - n(n+1)} F_n(L_1 j_s J) \right] \\ &\times \left[F_n(L_2 j_2 J) + 2\delta_\gamma F_n(L_2 L_2' j_2 J) + \delta_\gamma^2 F_n(L_2' j_2 J) \right] P_n(\cos \theta) \\ &= (1 + \delta_\gamma^2) + \left[\frac{2 \cdot 1(1+1)}{2 \cdot 1(1+1) - 2(2+1)} F_2(101) \right] \\ &\times \left[F_2(111) + 2\delta_\gamma F_2(1211) + \delta_\gamma^2 F_2(211) \right] P_2(\cos \theta) \\ &= (1 + \delta_\gamma^2) + \left[(-2)0.7071 \right] \\ &\times \left[(-0.3536) + 2\delta_\gamma (-1.0607) + \delta_\gamma^2 (-0.3535) \right] P_2(\cos \theta) \\ &= (1 + \delta_\gamma^2) + (0.5 + 3\delta_\gamma + 0.5\delta_\gamma^2) P_2(\cos \theta) \end{split}$$

Similarly

$$\begin{split} W_{j_s=1}(\theta) &= (1+\delta_{\gamma}^2) + \left[\frac{2 \cdot 1(1+1)}{2 \cdot 1(1+1) - 2(2+1)} F_2(111) \right] \\ &\times [F_2(111) + 2\delta_{\gamma}F_2(1211) + \delta_{\gamma}^2F_2(211)] P_2(\cos\theta) \\ &= (1+\delta_{\gamma}^2) + [(-2)(-0.3536)] \\ &\times [(-0.3536) + 2\delta_{\gamma}(-1.0607) + \delta_{\gamma}^2(-0.3535)] P_2(\cos\theta) \\ &= (1+\delta_{\gamma}^2) + (-0.25 - 1.5\delta_{\gamma} - 0.25\delta_{\gamma}^2) P_2(\cos\theta) \end{split}$$

The total angular correlation is given by the incoherent sum of the expressions for the individual channel spins,

$$\begin{split} W(\theta) &= W_{j_s=0}(\theta) + \delta_c^2 W_{j_s=1}(\theta) \\ &= (1 + \delta_\gamma^2) + (0.5 + 3\delta_\gamma + 0.5\delta_\gamma^2) P_2(\cos\theta) \\ &+ \delta_c^2 [(1 + \delta_\gamma^2) + (-0.25 - 1.5\delta_\gamma - 0.25\delta_\gamma^2) P_2(\cos\theta)] \\ &= (1 + \delta_\gamma^2) + \delta_c^2 (1 + \delta_\gamma^2) \\ &+ (0.5 + 3\delta_\gamma + 0.5\delta_\gamma^2 - 0.25\delta_c^2 - \delta_c^2 1.5\delta_\gamma - \delta_c^2 0.25\delta_\gamma^2) P_2(\cos\theta) \\ &= (1 + \delta_\gamma^2) (1 + \delta_c^2) + 0.5(1 + 6\delta_\gamma + \delta_\gamma^2) (1 - 0.5\delta_c^2) P_2(\cos\theta) \end{split}$$

The channel spin and γ -ray multipolarity mixing ratios are given by $\delta_c^2 = P_{j_s=1}/P_{j_s=0}$ and $\delta_{\gamma}^2 = \Gamma_{\gamma E2}/\Gamma_{\gamma M1}$, respectively.

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Example D.5

Consider the ¹⁹F(p, γ)²⁰Ne reaction, populating a resonance with a spin and parity of $J^{\pi} = 2^{-}$. The resonance decays to a lower lying state in ²⁰Ne with a spin and parity of $J^{\pi} = 1^{+}$ (Fig. D.2b). Calculate the angular correlation of the emitted γ -rays with respect to the incident proton beam direction.

Both the ¹⁹F target nucleus and the proton have a spin and parity of $1/2^+$. The channel spin has two allowed values, $j_s = 0$ and 1. However, the 2^- resonance cannot be formed from $j_s = 0$ since total angular momentum and parity must be conserved simultaneously. Hence, only the channel spin $j_s = 1$ plays a role in this process. The resonance can be formed via orbital angular momenta of $\ell_p = 1$ and 3 and thus this transition is mixed. For the sake of simplicity, we will assume that the γ -ray decay proceeds via an E1 transition only. We write symbolically

$$j_1(L_1)J(L_2)j_2 \rightarrow j_s(\ell_p)J(L_\gamma)j_{20Ne} \rightarrow 1 \begin{pmatrix} 1\\ 3 \end{pmatrix} 2(1)1$$

The sum in Eq. (D.6) is restricted to $n \le 2$ since we assumed $L_{\gamma} = 1$. It follows

$$\begin{split} W(\theta) &= \sum_{n=0,2} \left[\cos(\xi_{L_1} - \xi_{L_1}) \frac{2L_1(L_1 + 1)}{2L_1(L_1 + 1) - n(n+1)} F_n(L_1 j_s J) \right. \\ &+ 2\delta_a \cos(\xi_{L_1} - \xi_{L_1'}) \frac{2\sqrt{[L_1(L_1 + 1)][L_1'(L_1' + 1)]}}{L_1(L_1 + 1) + L_1'(L_1' + 1) - n(n+1)} F_n(L_1 L_1' j_s J) \right. \\ &+ \delta_a^2 \cos(\xi_{L_1'} - \xi_{L_1'}) \frac{2L_1'(L_1' + 1)}{2L_1'(L_1' + 1) - n(n+1)} F_n(L_1' j_s J) \right] \\ &\times F_n(L_2 j_2 J) P_n(\cos \theta) \\ &= \left[1 \cdot 1 \cdot 1 + \delta_a^2 \cdot 1 \cdot 1 \cdot 1 \right] \cdot 1 + \left[1 \cdot \frac{2 \cdot 1 \cdot (1+1)}{2 \cdot 1 \cdot (1+1) - 2(2+1)} F_2(112) \right. \\ &+ 2\delta_a \cos(\xi_{\ell=1} - \xi_{\ell=3}) \frac{2\sqrt{[1(1+1)][3(3+1)]}}{1(1+1) + 3(3+1) - 2(2+1)} F_2(1312) \right. \\ &+ \delta_a^2 \cdot 1 \cdot \frac{2 \cdot 3(3+1)}{2 \cdot 3(3+1) - 2(2+1)} F_2(312) \right] F_2(112) P_2(\cos \theta) \\ &= 1 + \delta_a^2 + \left[1 \cdot (-2)(0.4183) + 2\delta_a \cos(\xi_{\ell=1} - \xi_{\ell=3})(1.2247)(0.2390) \right. \\ &+ \delta_a^2 \cdot 1(1.333)(-0.7171) \right] (0.4183) P_2(\cos \theta) \\ &= 1 + \delta_a^2 + \left[-0.35 + 0.25\delta_a \cos(\xi_{\ell=1} - \xi_{\ell=3}) - 0.4\delta_a^2 \right] P_2(\cos \theta) \\ \end{split}$$

D.4 Three-Step Process with Unobserved Intermediate Radiation

It is sometimes of interest in a particle capture reaction to determine the angular correlation of secondary γ -rays with respect to the incident beam direction. In this case, we have a three-step process, involving: (i) the formation of a resonance with spin J through the capture of an incident particle with orbital angular momentum L_1 , (ii) the first (primary) γ -ray decay of multipolarity Lto an intermediate level of spin \overline{J} , and (iii) finally the subsequent secondary γ ray decay of multipolarity L_2 to the final state of spin j_2 (see Fig. D.2d). Only the incident beam and the secondary γ -ray transition are observed, while the primary γ -ray transition is unobserved. We write symbolically

$$j_1 \begin{pmatrix} L_1 \\ L'_1 \end{pmatrix} J \begin{pmatrix} L \\ L' \end{pmatrix} \overline{J} \begin{pmatrix} L_2 \\ L'_2 \end{pmatrix} j_2$$
(D.14)

The angular correlation expression is then given by

$$W(\theta) = \sum_{n=0,2,\dots} [a_n(1)A_n(1)]C_n[a_n(2)A_n(2)]P_n(\cos\theta)$$
(D.15)

$$C_n = \sqrt{(2J+1)(2\overline{J}+1)}W(JnL\overline{J};J\overline{J})$$
(D.16)

The first link $(j_1 \rightarrow J)$ and last link $(\overline{J} \rightarrow j_2)$ are described by the terms $a_n(1)A_n(1)$ and $a_n(2)A_n(2)$, respectively, and are handled as before. The term C_n describes the unobserved primary radiation. Unobserved γ -rays of multipolarities L and L' mix incoherently, that is, the total correlation is given by $W(\theta) = W_L(\theta) + \delta_{\gamma LL'}^2 W_{L'}(\theta)$. Furthermore, the sum over n is restricted by the condition $0 \le n \le \min(2L_1, 2L_2, 2J, 2\overline{J})$. In particular, the angular correlation becomes isotropic for either J or \overline{J} equal to 0 or 1/2. Note that the multipolarity L of the unobserved primary radiation does not limit the sum over n.

Example D.6

Consider the ¹¹B(p, γ)¹²C reaction leading to the formation of a resonance with spin and parity of $J^{\pi} = 2^+$ (Fig. D.2d). The resonance γ -ray decays to an intermediate state ($J^{\pi} = 2^+$) which, in turn, decays to the ¹²C ground state ($J^{\pi} = 0^+$). Calculate the angular correlation of the second γ -ray transition with respect to the incident beam direction.

The spin and parity of the ¹¹B ground state is $J^{\pi} = 3/2^{-}$. The two possible channel spins are $j_s = 1$ and 2. Of the two allowed proton orbital angular momenta ($\ell_p = 1$ and 3), we will consider only the lower ℓ_p value. Similarly, of the two γ -ray multipolarities for the unobserved primary transition (M1 and E2) we will only consider the M1 case. Only one possibility is allowed

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for the multipolarity of the secondary $\gamma\text{-ray}$ transition (E2). Symbolically we write

$$j_s \begin{pmatrix} L_1 \\ L'_1 \end{pmatrix} J \begin{pmatrix} L \\ L' \end{pmatrix} \overline{J} \begin{pmatrix} L_2 \\ L'_2 \end{pmatrix} j_2 \to 1(1)2(1)2(2)0 \text{ and } \to 2(1)2(1)2(2)0$$

For either channel spin, the summation is restricted to $n\leq 2$ (because of $\ell_{\rm p}$ = 1). The angular correlation is given by

$$\begin{split} W_{j_s=1}(\theta) &= \sum_{n=0,2,\dots} [a_n(1)A_n(1)]C_n[a_n(2)A_n(2)]P_n(\cos\theta) \\ &= \sum_{n=0,2} \frac{2L_1(L_1+1)}{2L_1(L_1+1) - n(n+1)}F_n(L_1j_sJ)\sqrt{(2J+1)(2\overline{J}+1)} \\ &\times W(JnL\overline{J};J\overline{J})F_n(L_2j_2J)P_n(\cos\theta) \\ &= 1\cdot 1\cdot \sqrt{(2\cdot 2+1)(2\cdot 2+1)}W(2012;22)\cdot 1 \\ &+ \frac{2\cdot 1\cdot 2}{2\cdot 1\cdot 2 - 2\cdot 3}F_2(112)\sqrt{(2\cdot 2+1)(2\cdot 2+1)} \\ &\times W(2212;22)F_2(202)P_2(\cos\theta) \\ &= 1\cdot 5\cdot 0.2\cdot 1 + (-2)(0.4183)\cdot 5\cdot 0.1\cdot (-0.5976)P_2(\cos\theta) \\ &= 1+0.25P_2(\cos\theta) \end{split}$$

Similarly

$$W_{j_s=2}(\theta) = 1 \cdot 1 \cdot \sqrt{(2 \cdot 2 + 1)(2 \cdot 2 + 1)W(2012;22) \cdot 1} + \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(122) \sqrt{(2 \cdot 2 + 1)(2 \cdot 2 + 1)} \times W(2212;22) F_2(202) P_2(\cos \theta) = 1 \cdot 5 \cdot 0.2 \cdot 1 + (-2)(-0.4183) \cdot 5 \cdot 0.1 \cdot (-0.5976) P_2(\cos \theta) = 1 - 0.25 P_2(\cos \theta)$$

The total angular correlation is given by the incoherent sum of the expressions for the individual channel spins,

$$W(\theta) = W_{j_s=1}(\theta) + \delta_c^2 W_{j_s=2}(\theta)$$

= $[1 + 0.25P_2(\cos \theta)] + \delta_c^2 [1 - 0.25P_2(\cos \theta)]$
= $1 + \delta_c^2 + 0.25(1 - \delta_c^2)P_2(\cos \theta)$



Fig. D.2 Schematic level diagrams indicating the quantum numbers involved in various angular correlation schemes. See the text.

D.5 Experimental Considerations

Experimental angular correlations and differential yields measured in the laboratory system must have both their intensities and angles converted to the center-of-mass system (Appendix C) before they can be compared to the theoretical expressions given above. Another important correction has to be performed since, strictly speaking, the theoretical angular correlation of Eq. (D.3) applies only to an ideal detector of negligible size. In an actual experiment, the measured intensities are obtained by integrating the theoretical angular correlation over the finite solid angle subtended by the detector. Hence, the effect of the finite solid angle is to reduce the anisotropy. For a detector of axial symmetry and for its symmetry axis pointing toward the source of the emitted radiation (Fig. 4.32), it can be shown that the form of the angular correlation function remains unchanged, but each term in the series of Eq. (D.3) becomes multiplied by a correction factor. For example, if radiation originating from a nuclear reaction is detected, then the experimental angular correlation mea614 D Angular Correlations

sured by a specific detector is given by

$$W_{\exp}(\theta) = \frac{1}{b_0} \sum_{n=0}^{n_{\max}} b_n Q_n P_n(\cos \theta)$$
(D.17)

Similarly, the experimental angular correlation between two emitted radiations a and b measured with two different detectors (or with the same detector, as was the case for coincidence summing in Section 4.5.2) can be written as

$$W_{\exp}(\theta) = \frac{1}{b_0} \sum_{n=0}^{n_{\max}} b_n Q_n^{(a)} Q_n^{(b)} P_n(\cos \theta)$$
(D.18)

The *attenuation factors* Q_n are given by (Rose 1953)

$$Q_n = \frac{\int_0^{\beta_{\max}} P_n(\cos\beta)\eta(\beta, E)\sin\beta\,d\beta}{\int_0^{\beta_{\max}}\eta(\beta, E)\sin\beta\,d\beta} \tag{D.19}$$

with β the angle between the radiation incident on the detector and the detector symmetry axis, β_{max} the maximum angle subtended by the detector, and $\eta(\beta, E)$ the detector efficiency for the radiation of energy *E* at angle β . It is apparent that the factors Q_n depend on the detector geometry, the energy of the radiation, and the kind of event that takes place in the detection process (for example, total versus partial energy deposition for γ -rays; see Section 4.5.2).

If the intrinsic detector efficiency is unity, as is generally the case for charged particle detectors, then the attenuation factor reduces to (Rose 1953)

$$Q_n = \frac{P_{n-1}(\cos\beta_{\max}) - \cos\beta_{\max}P_n(\cos\beta_{\max})}{(n+1)(1 - \cos\beta_{\max})}$$
(D.20)

Attenuation factors calculated from this expression are displayed in Fig. D.3a for values of n = 1, 2, 3, and 4.

In the case of γ -ray detectors, where the efficiency for detecting an incident photon is smaller than unity, the attenuation factors will be larger than given by Eq. (D.20), that is, they will be closer to unity and, consequently, the difference between measured and theoretical angular correlation will be smaller. The *total efficiency* attenuation factors can be estimated with the same method used for calculating total efficiencies (Section 4.5.2). One simply substitutes $\eta^T(\beta, E) = 1 - e^{-\mu(E)x(\beta)}$ for the total efficiency in Eq. (D.19) and solves the integrals numerically. Similarly, the *peak efficiency* attenuation factors can be estimated if the peak efficiency attenuation factors estimated in this way for a HPGe detector are displayed in Fig. D.3b. The curves show values of Q_n versus γ -ray energy for a fixed source-detector distance of 1.6 cm. As expected,



Fig. D.3 (a) Attenuation factors for an intrinsic detector efficiency of unity (for example, a silicon charged-particle counter). The horizontal axis displays the ratio r/h, with r and h the radius of the detector aperture and the source-detector distance, respectively. Note that $\tan \beta_{\max} = r/h$. The curves represent different values of n. After Gove (1959).

(b) Attenuation factors for a HPGe detector versus γ -ray energy. The detector volume and source-detector distance amount to 582 cm³ and 1.6 cm, respectively. The curves represent different values of *n* and are obtained by calculating peak efficiencies in Eq. (D.19) with the Monte Carlo code GEANT4. Courtesy of Richard Longland.

for decreasing photon energy the peak efficiency $\eta^{P}(\beta, E)$ increases and hence Q_{n} becomes smaller.

D.6 Concluding Remarks

We conclude this section with a few useful remarks. Since the angular momenta in low-energy nuclear reactions are rather small, the symmetry properties of the functions F_n restrict the series in Eq. (D.3) to a small number of terms. In practice, terms beyond n = 4 are rarely encountered. If for some reason the n = 4 term is zero or negligible, then we obtain $W(\theta = 55^\circ) \approx 1$ or $W(\theta = 125^\circ) \approx 1$, since the $P_2(\cos \theta)$ term is equal to zero at these angles (see Eq. (A.12)). Hence, the angle-integrated yield can be measured with a

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single detector located at a center-of-mass angle of $\theta = 55^{\circ}$ or $\theta = 125^{\circ}$. This circumstance has major practical advantages if very small yields need to be measured with a single detector in very close geometry to the target.

It is sometimes possible to simplify the theoretical angular correlation by making reasonable assumptions about the nuclear transition matrix elements. Mixtures of M1/E2 γ -ray multipolarites occur frequently, but E1/M2 mixtures are rarely important. In the latter case, it is often safe to assume that the E1 multipolarity dominates the γ -ray transition strength, hence $\delta_{\gamma M2/E1} \approx 0$. Similar arguments apply to the mixing of orbital angular momenta. Because of parity conservation (Appendix B), interfering orbital angular momenta must differ by two units, that is, ℓ_i and ℓ_{i+2} . The penetration factors decrease strongly for increasing values of orbital angular momentum, as can be seen from Fig. 2.21. Therefore, unless the reduced width (or spectroscopic factor) of the ℓ_{i+2} component is much larger than that of the ℓ_i component, the degree of orbital angular momentum mixing will be small, that is, $\delta_{a\ell_{i+2}/\ell_i} \approx 0$. Both of these simplifying assumptions should be treated with caution if the purpose of an angular correlation measurement is the determination of unknown nuclear spins. However, they are quite useful in nuclear astrophysics measurements if the level spins are known and if one is mostly interested in estimating angular correlation corrections for measured differential yields.

Sometimes a single γ -ray detector is placed at $\theta = 0^{\circ}$ in very close geometry to the target in order to maximize counting efficiency. Angular correlation effects may be significant for a specific primary γ -ray transition, but it may prove difficult to calculate the angular correlation if, for example, certain mixing ratios are unknown. In such cases, it could be of advantage to analyze instead the intensity of a corresponding secondary γ -ray decay for the calculation of the total yield. This is especially useful if the secondary γ -ray transition proceeds from a level with a spin of 0 or 1/2 since then its angular correlation is isotropic.

We pointed out that the series of Eq. (D.3) will contain only terms with n = even if the correlation involves an intermediate state of well-defined parity. However, if a reaction proceeds through two or more overlapping resonances of opposite parity, then the resulting angular correlation will not be symmetric about 90° anymore and terms with n = odd will appear in the series of Eq. (D.3). We will not consider here the more involved angular correlation resulting from the interference of two overlapping resonances. The interested reader is referred to Biedenharn (1960). Expressions for the angular correlation in direct radiative capture, and for the interference between resonant and direct contributions, are given in Rolfs (1973).

Example D.7

A resonance at $E_r^{\text{lab}} = 519$ keV is excited in the ${}^{17}\text{O}(\text{p},\gamma){}^{18}\text{F}$ reaction. The strongest primary transition occurs to the ${}^{18}\text{F}$ level at $E_x = 1121$ keV ($E_\gamma \approx 5 \text{ MeV}, B_\gamma = 0.55 \pm 0.03$). The theoretical angular correlation is given by

 $W_{\gamma}(\theta) = 1 - 0.10P_2(\cos\theta)$

The γ -ray counter is located at $\theta = 0^{\circ}$ with respect to the proton beam direction in very close geometry to the target. A (peak) attenuation factor of $Q_2 = 0.62 \pm 0.05$ is estimated from Eq. (D.19) for this geometry. The measured peak intensity is $N_{\gamma} = 1530 \pm 47$ for a certain total number of incident protons. The peak efficiency at $E_{\gamma} \approx 5$ MeV amounts to $\eta_{\gamma}^{P} = 0.015$ ($\pm 5\%$). Calculate the total number of reactions that took place. Ignore coincidence summing effects.

The measured angular correlation is given by

 $W_{\exp,\gamma}(\theta) = 1 - 0.10Q_2P_2(\cos\theta) = 1 - 0.10(0.62 \pm 0.05)P_2(\cos\theta)$

At $\theta = 0^{\circ}$ we obtain $P_2(\cos \theta) = 1$ and hence

$$W_{\exp,\gamma}(0) = 1 - 0.10(0.62 \pm 0.05) \cdot 1 = 0.94(\pm 5\%)$$

From Eq. (4.69) we find

$$\mathcal{N}_{R} = \frac{\mathcal{N}_{\gamma}}{B_{\gamma}\eta_{\gamma}^{P}W_{\gamma}} = \frac{1530(\pm 3\%)}{[0.55(\pm 5\%)][0.015(\pm 5\%)][0.94(\pm 5\%)]} = 197292(\pm 9\%)$$

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Appendix E Constants, Data, Units, and Notation

The physical constants used in this book are adopted from Mohr and Taylor (2005) and are listed in Appendix E.1. The mathematical symbols, units, and prefixes follow common usage and are given in Appendix E.2. Symbols for physical quantities are summarized in Appendix E.4. In several cases, the use of the same symbol for different physical quantities was unavoidable. Further help to the reader is provided by numbers given in parenthesis after a description of a symbol with multiple meanings. These refer to the chapter in which the specific meaning is used. For example, the symbol *N* denotes a normalization factor (in Chapters 2 or 3), the number *density* of particles or photons (in Chapters 3, 4, or 5), and the neutron number (in Chapters 1 or 5). The symbol \mathcal{N} denotes the number (without units) of particles, photons, disintegrations, or reactions throughout the text.

E.1

Physical Constants and Data

- *a*₀ Bohr radius; $a_0 = 0.5291772108 \times 10^{-10}$ m
- *c* speed of light in a vacuum; c = 299792458 m/s
- *e* elementary charge; $e = 1.60217653 \times 10^{-19} \text{ C}$
- $$\begin{split} h & \quad \text{Planck constant; } h = 4.13566743 \times 10^{-15} \text{ eV s} = 6.6260693 \times 10^{-34} \text{ J s}, \\ \hbar \equiv h/(2\pi) = 6.58211915 \times 10^{-16} \text{ eV s} = 1.05457168 \times 10^{-34} \text{ J s}, \\ \hbar c = 197.327 \text{ MeV fm} \end{split}$$
- *k* Boltzmann constant; $k = 8.617343 \times 10^{-5} \text{ eV/K}$
- L Loschmidt constant; $L = 2.6867773 \times 10^{25} \text{ m}^{-3}$
- L_{\odot} luminosity of the Sun (bolometric); $L_{\odot} = 3.826 \times 10^{26}$ W
- $m_{\rm e}$ electron mass; $m_{\rm e} = 9.1093826 \times 10^{-31}$ kg
- m_u atomic mass constant; $1 m_u \equiv \frac{1}{12}m(^{12}\text{C}) = 1.66053886 \times 10^{-27} \text{ kg}$

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$$m_{\rm n}$$
 neutron mass; $m_{\rm n} = 1.00866491560$ u

proton mass; $m_{\rm p} = 1.00727646688$ u $m_{\rm p}$

$$m_u c^2$$
 energy equivalent of m_u ; $m_u c^2 = 931.494043$ MeV

- m_ec^2 electron rest energy; $m_ec^2 = 0.510998918$ MeV
- $m_{\rm n}c^2$ neutron rest energy; $m_{\rm n}c^2 = 939.565360$ MeV
- $m_{\rm p}c^2$ proton rest energy; $m_{\rm p}c^2 = 938.272029$ MeV
- M_{\odot} mass of the Sun; $M_{\odot} = 1.989 \times 10^{30}$ kg
- Avogadro constant; $N_A = 6.0221415 \times 10^{23} \text{ mol}^{-1}$ N_A

E.2

Mathematical Expressions

- equal to =
- proportional to \sim
- defined as \equiv
- approximately equal to \approx
- greater than >
- <less than
- much greater than \gg
- \ll much less than
- limit toward \rightarrow
- infinity ∞
- Laplace operator; in Cartesian coordinates $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ ∇^2
- complex conjugate; $z^* = \operatorname{Re} z \operatorname{Im} z$
- $|z|^{2}$ absolute magnitude of z; $|z|^2 = z^* z$
- expectation value of *x*; $\langle x \rangle = \int_x x P(x) dx$, where P(x) is a normalized $\langle x \rangle$ probability distribution of *x*
- Kronecker delta; $\delta_{ij} = 1$ if i = j and 0 otherwise δ_{ij}

- *e* base of natural logarithm; e = 2.71828...
- $\exp(x)$ exponential function; $\exp(x) \equiv e^x$
- *i* imaginary unit; $i \equiv \sqrt{-1}$
- Im z imaginary part of z
- $j_{\ell}(kr)$ spherical Bessel function
- $\ln(x)$ natural logarithm; $\ln(x) = \log_{e}(x)$
- log(x) common (base 10) logarithm; $log(x) = log_{10}(x)$
- $n_{\ell}(kr)$ spherical Neumann function
- π ratio of a circle's circumference to its diameter; $\pi = 3.14159...$
- $P_{\ell}(x)$ Legendre polynomial
- Re z real part of z

 $Y_{\ell m_{\ell}}(\theta, \phi)$ spherical harmonic

 Δa difference; $\Delta a \equiv a_2 - a_1$

 Ω solid angle

E.3 Prefixes and Units

Prefixes

- f- femto-; 10^{-15}
- **p** pico-; 10⁻¹²
- n- nano-; 10⁻⁹
- µ- micro-; 10⁻⁶
- m- milli-; 10^{-3}
- **c-** centi-; 10⁻²
- k- kilo-; 10³
- M- mega-; 10⁶
- **G-** giga-; 10⁹

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Units

0	degree of arc; $1^\circ = \frac{\pi}{180}$ rad
А	ampere; $1 \text{ A} = 1 \text{ C/s}$
b	barn; 1 b = 10^{-24} cm ² = 10^{-28} m ²
Bq	becquerel; 1 Bq = 1 s^{-1}
°C	degree Celsius
С	coulomb
Ci	curie; 1 Ci = $3.7 \times 10^{10} \text{ s}^{-1}$
erg	cgs unit of energy; 1 erg = 10^{-7} J
eV	electron volt; 1 eV = $1.60217653 \times 10^{-19}$ J
g	gram
Hz	hertz; 1 Hz = $1 s^{-1}$
J	joule
K	kelvin
m	meter
min	minute
m w.e.	meter water equivalent
rad	radian; 1 rad = 57.29578°
S	second
sr	steradian; solid angle over entire sphere amounts to 4π sr; 1 sr = 3282.80635 deg ²
u	(unified) atomic mass unit; $1 u = m_u = \frac{1}{12}m(^{12}C)$
V	volt
W.u.	Weisskopf unit
у	year; 1 sidereal year = 3.1558149984×10^7 s

E.4 **Physical Quantities** Α activity (4); area (2, 3); mass number (1, 2, 3, 4, 5) hardsphere potential scattering amplitude Apot $A_{\rm res}$ resonance scattering amplitude diffuseness of Woods-Saxon potential а (*a*,*b*) nuclear reaction involving incoming particle *a* and emitted particle *b* В branching ratio (1, 2, 3, 4, 5); binding energy (1, 5); magnetic field strength (4) $B(\overline{\omega}L)$ reduced γ -ray transition probability С isospin Clebsch-Gordan coefficient (2); net number of counts (4); peak centroid in pulse height spectrum (4) reaction channel С number density of deuterium or ²H (5); number of disintegra-D tions (4) d deuteron target or absorber thickness (4), distance of point source to detector d front face (4) Ε energy energy location of Gamow peak maximum E_0 E_r observed resonance energy electron, also ee e^+ positron $e^{-2\pi\eta}$ Gamow factor $(e^+\nu)$ nuclear emission of positron (e^{-}, v) nuclear electron capture F_{ii} time-integrated net abundance flow between species *i* and *j* fraction of ⁴He nuclei produced in the pp*i* chain F_{ppi} F(Z, p)Fermi function

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f	fraction of the number of ⁵⁶ Fe seed nuclei that have been subjected to an exponential distribution of neutron exposures	
fan	effective rate of ²⁸ Si consumption	
f_ℓ	logarithmic derivative at the boundary for orbital angular momentum ℓ	
f_{ij}	net abundance flow between species i and j	
<i>f</i> _{ppi}	fraction of total energy retained in the star if ⁴ He nucleus is produced in ppi chain	
f_s	screening factor	
$f(\theta)$	scattering amplitude	
$f(E_i, E, E)$	') probability that particle incident at energy E_i has energy E at a depth inside the target corresponding to E'	
$f(Z, E_{e}^{\max})$ Fermi integral		
FWHM	full width at half maximum	
G	partition function	
G_A	axial-vector coupling constant	
G_V	vector coupling constant	
G ^{norm}	normalized partition function	
8µ	statistical weight of nuclear state μ	
$g(E_0, E_i)$	probability that particle in incident beam of mean energy E_0 has energy of E_i	
Н	Hamiltonian (2); number density of 1 H (5); pulse height (4)	
H_{fi}	weak interaction matrix element	
Ι	current (4); particle spin (2)	
J	nuclear spin (1); resonance spin (3); total particle spin (2)	
j	current density (2); total particle spin (2, 3, 5)	
Κ	kinetic energy (4); recoil energy of Compton electron (4); wave num- ber (2)	
k	wave number, also κ , \hat{k} , K	

l	orbital angular momentum quantum number
L	γ -ray multipolarity (1, 2, 3); length of flight path in time-of-flight experiment (4)
Ĺ	angular momentum vector
М	relative atomic mass in units of u
M_F	Fermi matrix element
M_{GT}	Gamow-Teller matrix element
M.E.	atomic mass excess
M_W^2	γ -ray transition strength in Weisskopf units
т	atomic mass, nuclear mass (1, 2, 3, 4); magnetic quantum number (1, 2)
m _{ij}	reduced mass of particles <i>i</i> and <i>j</i>
Ν	harmonic oscillator quantum number (1); neutron number (1, 5); normalization factor (2, 3); number <i>density</i> of particles or photons (3, 4, 5)
\mathcal{N}	number (without units) of particles, photons, disintegrations, or reactions
$N_A \langle \sigma v \rangle$	reaction rate per particle pair in units of $\mathrm{cm}^3 \mathrm{mol}^{-1} \mathrm{s}^{-1}$
n	neutron
п	exponent in temperature dependence of reaction rate (3, 5); number of nodes in radial wave function (2); number of target or sample nuclei per unit area (4); radial quantum number (1)
n_c	average number of neutrons captured per 56 Fe seed nucleus
n_{e^-}	electron density
Р	beam power (4); gas pressure (4); Maxwell–Boltzmann distribu- tion (3); particle density (2); penetration factor (2, 3); population probability of excited levels (1, 3); probability (4); production rate of radioactive nuclei (4)
р	proton
р	linear momentum

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- $p(\tau)$ exponential probability distribution of neutron exposures
- *Q* angular correlation attenuation factor (4); *Q*-value (1, 2, 3, 4, 5); total accumulated charge (4)
- *q* ion charge state (4); neutrino linear momentum (1); parameter describing absorption in the nuclear interior (2)
- *R* nuclear radius (2, 3); radius of cylindrical detector (4)
- R_0 radius of square-well potential (2, 3); radius of Woods–Saxon potential (1)
- *R*₁ radius of outer boundary of square-barrier potential
- *R_c* classical turning point
- *R_D* Debye–Hückel radius
- *R*_{tt} stellar enhancement factor
- RUL recommended upper limit
- \Re *R*-function or *R*-matrix
- *r* reaction rate in units of number of reactions per time per volume
- \vec{r} radius vector
- r_0 radius parameter
- *S* astrophysical *S*-factor (2, 3, 4, 5); shift factor (2); spectroscopic factor (1, 2, 3); stopping power (4)
- S_n , S_p , S_α neutron, proton, and α -particle separation energy
- s channel spin
- *T* neutron transmission (4); temperature (3, 4, 5)
- \hat{T} transmission coefficient
- $T_{1/2}$ half-life
- T_9 temperature in units of GK, $T_9 \equiv T/10^9$ K
- t length of detector crystal (4); time (1, 2, 3, 4, 5)
- *U* electric potential
- *U_s* perturbing potential caused by electron shielding charge density

u(r)	radial wave function, $u(r) \equiv rR(r)$
V	potential (2); volume (3, 4)
V_C	Coulomb barrier
V_s	screening potential
υ	velocity
v_T	location of the maximum of the Maxwell–Boltzmann velocity distri- bution
W	angular correlation
w	parameter $w = (Q_{\beta} + m_{\rm e}c^2)/m_{\rm e}c^2$
Χ	mass fraction
x	parameter for electron screening, $x(E) \equiv R_c/R_D$
Y	mole fraction (1, 3, 5); yield (4)
Ζ	atomic number (1, 4, 5); charge (2, 3, 4, 5)
α	alpha-particle
$\alpha(I_1I_2)$	specific pair of nuclei 1 and 2 with spins of I_1 and I_2
β	nuclear emission or capture of electron or positron
eta^+	nuclear emission of positron or electron capture
β^{-}	nuclear emission of electron
$(\beta \nu a)$	β -delayed emission of particle <i>a</i>
Г	total width of resonance or compound nucleus level
Γ _a	partial width for emission or absorption of particle <i>a</i>
Γ_{γ}	partial width for emission or absorption of γ -ray
Γ_i^o	"observed" total or partial width
γ	γ -ray or photon
γ^2	reduced width
Δ	level shift (2); parameter $\Delta \equiv R_1 - R_0$ (2); systematic difference be- tween tabulated and experimental stopping power (4); 1/e width of

be-n of Gaussian approximation to Gamow peak (3, 5)

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δ	mixing ratio (1, 3); scattering phase shift (2); δ electrons (4)
$\delta_{\alpha}, \delta_{p}, \delta_{n}$	number of α -particles, protons, and neutrons of nucleus ${}^A_Z \Upsilon_N$ in excess of their number in ${}^{28}Si$
ε	nuclear energy generation per unit time and per volume (3, 5); stopping power (4)
ϵ	dimensionless parameter, $\epsilon \equiv E/E_0$
ζ	parameter for electron screening
η	detector efficiency (4); neutron excess parameter (1, 5); Sommerfeld parameter (2, 3)
θ	angle (2, 4); parameter $\theta \equiv (2\pi m_u kT/h^2)^{3/2}$ (5)
θ^2	dimensionless reduced width
θ_{pc}^2	dimensionless single-particle reduced width
θ_e	electron degeneracy factor
Λ	total photodisintegration decay constant
λ	de Broglie wavelength (2, 4); decay constant (1, 3, 4, 5); mean free path of photons or neutrons (4)
μ	linear absorption coefficient for photons (4); muon (4)
ν	frequency (3); neutrino (1, 3, 5); neutron number (5)
$\bar{\nu}$	antineutrino
π	parity (1, 2, 5); proton number (5)
ρ	mass density (1, 3, 4, 5); product $ ho \equiv kr$ (2)
σ	cross section (2, 3, 4, 5); experimental stopping power error (4)
$\hat{\sigma}$	effective reaction cross section
$\overline{\sigma}$	average reaction cross section
$\langle \sigma \rangle_T$	Maxwellian-averaged cross section
σ_ℓ	Coulomb phase shift
$\langle \sigma v \rangle$	reaction rate per particle pair
τ	duration of r-process (5); mean lifetime (1, 2, 3, 5); neutron exposure in units of neutrons per area (5); parameter $\tau \equiv 3E_0/(kT)$ (3, 5)

$ au_{cycle}$	fission cycling time
$\tau_{\rm NSE}$	time to reach nuclear statistical equilibrium
ω	angular frequency (2); spin factor $\omega \equiv (2J+1)(1+\delta_{01})/[(2j_0+1)(2j_1+1)]$ (3, 4, 5)
$\omega\gamma$	resonance strength
Φ	time-integrated neutron flux in units of particles per area
φ	angle (2, 4); incident particle flux in units of particles per area and per time (4, 5); wave function (1)
ϕ_{ij}	parameter $\phi_{ij} \equiv r_{i \to j} - r_{j \to i} / \max(r_{i \to j}, r_{j \to i})$

 Ψ , ψ wave function

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Color Plates



Fig. 1 The bright globular cluster M 10. It is located at a distance of \approx 14 000 light years from Earth and has an approximate diameter of \approx 80 light years. The bright reddish-orange stars are red giants that fuse hydrogen to helium via the CNO-cycles in a shell surrounding a helium core. The bright blue stars are horizontal branch stars that fuse helium to carbon and oxygen in the core and hydrogen to helium in a shell. Only the faint, grey-looking stars (that is, those with the lowest mass) are most likely main sequence stars that fuse hydrogen to helium via the pp-chains in the core. The image is a two color composite. Reprinted with permission. Credit and copyright: T. Credner and S. Kohle, Observatorium Hoher List, Sternwarte Bonn.

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Fig. 2 An average, but very special, main sequence star of spectral class G2 that fuses hydrogen to helium in its core via the pp chains. Its mean distance from the center of our Galaxy, which hosts more than 100 million similar stars, is 27 000 light years. The surface temperature amounts to \approx 5800 K and its diameter is about 1.4 million kilometer. The Sun goes through an 11-year activity cycle, which is caused by variations of its magnetic field. The above image was taken in 1997 in the ultraviolet

light emitted by a specific type of ionized helium. Particularly hot areas appear in white, while cooler areas are displayed in red. The material in the eruptive prominence visible on the lower left side is at temperatures of \approx 70 000 K and is much cooler than the surrounding corona which has a temperature typically in excess of one million kelvin. Courtesy of SOHO/EIT consortium. SOHO (SOlar and Heliospheric Observatory) is a project of international cooperation between ESA and NASA.



Fig. 3 The Dumbbell Nebula (M 27). It was the first planetary nebula ever discovered. Its distance from Earth is about 1200 light years. The red and green colors originate from the emission of hydrogen and oxygen, respectively. The gas is heated and excited by the ultraviolet radiation from a star that is located in the center of the nebula (visible at the middle of the image). The cen-

tral star has a high surface temperature of \approx 85 000 K. Planetary nebulae are the result of a natural evolutionary stage of low mass stars (see text). The Sun is expected to become the central star of a planetary nebula in several billion years. The image is a three color composite. Reprinted with permission. Credit and copyright: European Southern Observatory.

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Fig. 4 The planetary nebula NGC 6543, known as the Cat's Eye. It is located in the constellation Draco, at a distance of \approx 3000 light years from Earth. The image is a false-color composite of an X-ray part (shown in purple; obtained with the Chandra X-Ray Observatory) and an optical part (shown in red and green; obtained with the Hubble Space Telescope), and reveals where the hot, X-ray emitting gas appears in relation to the cooler material seen at optical wavelengths. The central star has a surface temperature of \approx 50 000 K and is expected to collapse to a white dwarf in a few million years. The fast stellar wind emitted from

the central star shock-heats the gas that was previously expelled and gives rise to the X-ray emitting bubble (shown in purple). Pockets of hot gas seem to border on cooler gas emitting strongly at optical wavelengths, which may indicate that the expanding hot gas is sculpting the visible filaments and structures. The mechanisms that produced the complicated morphology of the planetary nebula are still not well understood. Credits: (X-ray) NASA/UIUC/Y. Chu et al.; (optical) NASA/J. P. Harrington, K. J Borkowski (UMD); (composite) Z. Levay (STScl).



Fig. 5 The nebula M1-67 surrounding the Wolf-Rayet star WR 124, located in the constellation Sagittarius. The distance of the nebula from Earth is about 15 000 light years. The central Wolf-Rayet star is very hot (\approx 50 000 K). It is also massive and hence short-lived. Wolf-Rayet stars go through a phase of enormous mass loss via a strong stellar wind. The image reveals hot blobs of ejected gas, indicating that the stellar wind does not flow smoothly into space

but has instabilities which make the nebula appear clumpy. The age of the nebula is less than 10 000 years. The false color image was obtained with the Wide Field Planetary Camera 2 of the Hubble Space Telescope. Credit: Yves Grosdidier (University of Montreal and Observatoire de Strasbourg), Anthony Moffat (Universitie de Montreal), Gilles Joncas (Universite Laval), Agnes Acker (Observatoire de Strasbourg), and NASA.

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Fig. 6 Supernova 1987A in the Large Magellanic Cloud (a nearby small galaxy that is a satellite of our Galaxy) was the brightest exploding star seen in 400 years. Its distance from Earth is \approx 160 000 light years. The supernova was of type II and its progenitor was a massive star (a blue supergiant). The shock wave from the supernova has been moving toward a ring of matter, about two light years across, that was probably ejected by the central star about 20 000 years before the explosion. The image

shows many hot spots that are created by the supernova shock compressing and heating the gas of the ring. (The brightest spot on the lower right side of the ring is a star that happens to lie along the Hubble Space Telescope's line of sight). The elongated and expanding object in the middle of the ring is the debris from the explosion. Credit: NASA, P. Challis, R. Kirshner (Harvard-Smithsonian Center for Astrophysics) and B. Sugerman (STScl).



Fig. 7 The Crab Nebula (M 1). The nebula consists of matter ejected in a supernova explosion. The material is spread over a volume of 10 light years in diameter and is still expanding at velocities of \approx 1800 km/s. Its distance from Earth is about 6000 light years. The supernova explosion was detected on July 4, 1054, by Chinese astronomers. It is one of the very few historically observed supernovae in our Galaxy. The remnant of the supernova, located in the middle of the nebula, is a neutron star that spins with a period of \approx 30 ms (pulsar).

The presence of a remnant neutron star and of hydrogen in the ejecta supports the association of the Crab Nebula with a type II supernova. The image is a three color composite. The green light is predominantly produced by hydrogen emission from material that was ejected by the exploding star. The blue light arises mainly from relativistic electrons that spiral in a large-scale magnetic field (synchrotron radiation) and that are continuously ejected from the rapidly spinning neutron star. Credit: NASA, ESA, and J. Hester (Arizona State University).



Fig. 8 The spiral galaxy NGC 4526 in the constellation Virgo, about 100 million light years away from Earth. The bright spot at the lower left is Supernova 1994D. (The designation means that it was the fourth supernova discovered in 1994). The light emitted during the weeks after the stellar explosion showed that the supernova was of type Ia. Credit: NASA, ESA, The Hubble Key Project Team, and The High-Z Supernova Search Team.



Fig. 9 Tycho's supernova remnant in the constellation Cassiopeia, located at a distance of 7500 light years from Earth. The supernova was recorded by the Danish astronomer Tycho Brahe on November 11, 1572. The false color X-ray image was obtained with the Chandra X-ray Observatory. The colors represent different X-ray energies (red: 0.95–1.26 keV; green: 1.63–2.26 keV; blue: 4.1–6.1 keV). The remnant glows at X-ray energies because of the strong interaction between the high-

velocity expanding matter and the interstellar gas that was swept up by the explosion. No hot compact object has been found in the remnant, supporting the theory that the supernova was of type Ia. The cloud is nearly spherical with a diameter of about 20 light years, indicating both a spherical ejection of matter and a rather homogeneous environment in the explosion. Credit: NASA/CXC/Rutgers/J. Warren and J. Hughes et al.

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Fig. 10 Nova V1974 Cygni 1992 erupted on February 19, 1992 and was one of the brightest classical novae in 20 years, reaching naked-eye visibility for a brief period of time. The distance of Nova Cygni from Earth is about 10000 light years. The image reveals a nearly spherical and slightly lumpy ring-like structure, which represents the edge of a bubble of hot gas ejected into space by the outburst. The shell contains elements such as nitrogen, oxygen, neon, silicon and sulfur, which are overabundant relative to their solar system values. This implies that the explosion occured on the surface of an oxygen-neon white dwarf. These outbursts are also referred to as "neon novae". The white dwarf and the companion star in the center of the image are so close that they revolve around each other in about two hours. The image was taken by the Hubble Space Telescope in ultraviolet light 467 days after the explosion. Credit: F. Paresce, R. Jedrzejewski (STScI), NASA/ESA.

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dle of the image. Gamma-ray intensity is represented by a false color map - green (low) to yellow (high). It has been estimated that the Galaxy produces ²⁶Al at a rate of about two solar masses per million years. Reprinted with permission from S. Plüschke, R. Diehl, V. Schönfelder, et al., ESA SP 459, 55 (2001).



Fig. 12 The nuclear astrophysics facilities of ISAC at the TRIUMF laboratory in Vancouver, Canada. The radioactive ion beam exits the accelerator in the top lefthand corner of the photo and enters either the DRAGON recoil spectrometer or the TUDA beamline. For radiative proton- and α capture reactions, the 21-m long DRAGON facility (photo, center) is used. The drawing shows the major components of DRAGON, including the windowless gas target surrounded by a 30-element γ -ray detection array, the two independent stages of electromagnetic separators that suppress the beam by a factor of $< 10^{13}$, and the appropriately configured recoil detector. For other nonradiative-capture measurements, such as direct reactions and elastic or inelastic scattering, TUDA is employed. The TUDA chamber and shielded electronics room (photo, right-hand side) allow for multiple arrangements of various detectors specifically chosen for the reaction of interest. Credit: Dave Hutcheon, Chris Ruiz, Götz Ruprecht, Mike Trinczek, Christof Vockenhuber and TRIUMF.

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