11. Free Electron Lasers

Synchrotron radiation is emitted when electromagnetic fields exert a force on a charged particle. This opens the possibility to apply external fields with specific properties for the stimulation of electrons to emit even more radiation. Of course, not just any external electromagnetic field would be useful. Fields at some arbitrary frequency would not work because particles interacting with such fields would in general be periodically accelerated and decelerated without any net energy transfer. The external field must have a frequency and phase such that a particle may continuously lose energy into synchrotron radiation. Generally, it is most convenient to recycle and use spontaneous radiation emitted previously by the same emission process. In this part, we will discuss in some detail the process of stimulation as it applies to a free electron laser.

In a free electron laser (FEL) quasi-monochromatic, spontaneous radiation emitted from an undulator is recycled in an optical cavity to interact with the electron beam causing accelerations which are periodic with the frequency of the undulator radiation. In order to couple the particle motion to the strictly transverse electromagnetic radiation field, the path of the electrons is modulated by periodic deflections in a magnetic field to generate transverse velocity components. In a realistic setup, this magnetic field is provided in an undulator magnet serving both as the source of radiation and the means to couple to the electric field. The transverse motion of the particle results in a gain or loss of energy from/to the electromagnetic field depending on the location of the particle with respect to the phase of the external radiation field. The principle components of a FEL are shown in Fig. B.1.

An electron beam is guided by a bending magnet unto the axis of an undulator. Upon exiting the undulator, the beam is again deflected away from the axis by a second bending magnet, both deflections to protect the mirrors of the optical cavity. Radiation that is emitted by the electron beam while travelling through the undulator is reflected by a mirror, travels to the mirror on the opposite side of the undulator and is reflected there again. Just as this radiation pulse enters the undulator again, another electron bunch joins to establish the emission of stimulated radiation. The electron beam pulse consists of a long train of many bunches, much longer than the length of the optical cavity such that many beam-radiation interactions can be established.

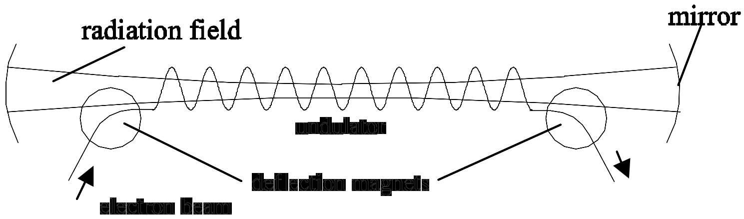

Fig. 11.1. Free electron laser setup (schematic)

We may follow this process in great detail observing an electron as it travels through say the positive half period of its oscillatory trajectory. During this phase, the electron experiences a negative acceleration from the undulator magnet field which is in phase with the oscillation amplitude. Acceleration causes a perturbation of the electric fields of the electron as was discussed in detail in Chap. 2. This perturbation travels away from the source at the speed of light, which is what we call electromagnetic radiation. For an electron, the electric radiation field points in the direction of the acceleration. As the electron travels through the positive half wave, it emits a radiation field made of half a wave. Simultaneously, this radiation field, being faster than the electron, travels ahead of the electron by precisely half a wavelength. This process tells us that the radiation wavelength is closely related to the electron motion and that it is quasi-monochromatic. Of course, for a strong undulator the sinusoidal motion becomes perturbed and higher harmonics appear, but the principle arguments made here are still true. Now, the electron starts performing the negative half of its oscillation and, experiencing a positive acceleration, emits the second halfwave of the radiation field matching perfectly the first halfwave. This happens in every period of the undulator and when the electron reaches the end of the last period a radiation wave composed of N_{p} oscillations exists ahead of the electron. This process describes the spontaneous radiation emission from an electron in an undulator magnet.

The radiation pulse just created is recycled in the optical cavity to reenter the undulator again at a later time. The length of the optical cavity must be adjusted very precisely to an integer multiple of both the radiation wavelength and the distance between electron bunches. Under these conditions, electron bunches and radiation pulses enter the undulator synchronously. A complication arises from the fact that the electrons are contained in a bunch which is much longer than the wavelength of the radiation. The electrons are distributed for all practical purposes uniformly over many wavelengths. For the moment, we ignore this complication and note that there is an electron available whenever needed.

We pick now an electron starting to travel through a positive half wave of its oscillation exactly at the same time and location as the radiation wave starts its positive field halfperiod. The electron, experiences then a downward acceleration from the radiation field. During its motion the electron is continuously accelerated until it has completed its travel through the positive half oscillation. At the same time, the full positive have wave of the radiation field has moved over the electron. At this moment the electron and the radiation field are about to start their negative half periods. Continuing its motion now through the negative half period, the electron still keeps loosing energy because it now faces a negative radiation field. The fact that the radiation field "slides" over the electron just one wavelength per undulator period ensures a continuous energy transfer from electron to the radiation field. The electron emits radiation which is now exactly in synchronism with the existing radiation field and the new radiation intensity is proportional to the acceleration or the external radiation field. Multiple recycling and interaction of radiation field with electron bunches results therefore in an exponential increase in radiation intensity.

At this point, we must consider all electrons, not just the one for which the stimulation works as just described. This process does not work that perfect for all particles. An electron just half a wavelength behind the one discussed above would continuously gain energy from the radiation field and any other electron would loose or gain energy depending on its phase with respect to the radiation. It is not difficult to convince oneself that on average there may not be any net energy transfer one way or another and therefore no stimulation or acceleration. To get actual stimulation, some kind of asymmetry must be introduced.

To see this, we recollect the electron motion in a storage ring in the presence of the rf-field in the accelerating cavity. In Sect. 7.1 we discussed the phase space motion of particles under the influence of a radiation field. The radiation field of a FEL acts exactly the same although at a much shorter wavelength. The electron beam extends over many buckets as shown in Fig.11.2 and it is obvious that in its interaction with the field half of the electrons gain and the other half loose energy from/to the radiation field. The effect of the asymmetry required to make the FEL work is demonstrated in Fig. 11.3. Choosing an electron beam energy to be off-resonance by a small amount, the energy gain and losses for all electrons within a bucket becomes unbalanced and we can choose a case where all electrons on average loose energy into (FEL) or gain energy (particle acceleration by a radiation field) from the radiation field. The arrows in the first bucket of Fig. 11.3 show clearly the imbalance of energy gain or loss. What it means to choose an electron beam energy off-resonance will be discussed in more detail in the next section, where we formulate quantitatively the processes discussed so far only qualitatively.

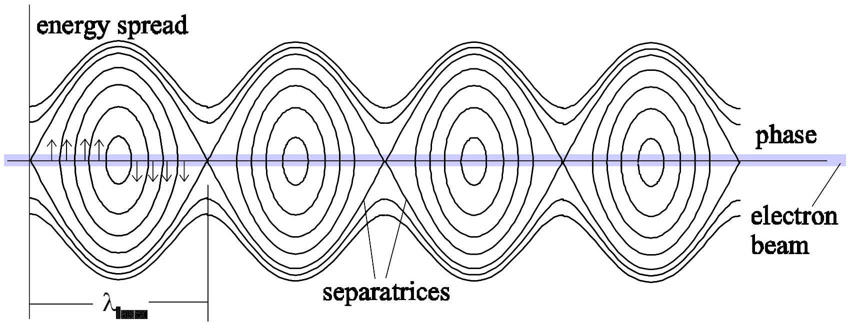

Fig. 11.2. Interaction of an electron beam (on-resosnace energy) with the radiation field of a FEL. The arrows in the first bucket indicate the direction of particle motion in its interaction with the electromagnetic field

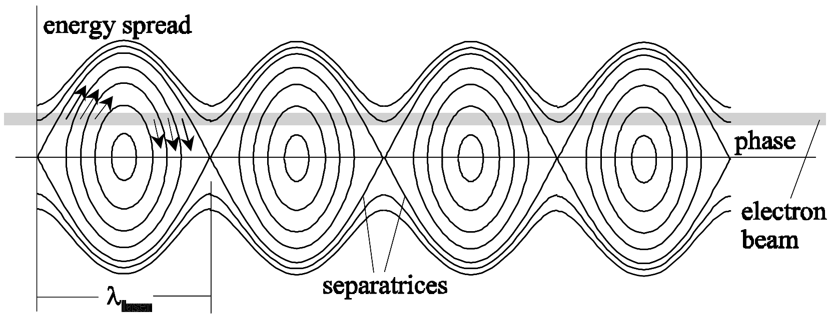

Fig. 11.3. Interaction of an electron beam (off-resonance energy) with the radiation field of a FEL

11.1 Small Gain FEL

We concentrate on the case where only a small fraction of the particle energy is extracted such that we can neglect effects on particle parameters. This regime is called the "small-gain" regime. Specifically, we ignore changes in the particle energy and redistribution in space as a consequence of the periodic energy modulation.

11.1.1 Energy Transfer

Transfer of energy between a charged particle and an electromagnetic wave is effected by the electric field term of the Lorentz force equation and the amount of transferred energy is

$$\Delta W = e \int \boldsymbol{E}_{\mathrm{L}} \, \mathrm{d}\boldsymbol{s} = e \int \boldsymbol{E}_{\mathrm{L}} \, \boldsymbol{v} \mathrm{d}t \,, \tag{11.1}$$

where $\boldsymbol{E}_\mathrm{L}$ is the external field or the Laser field in the optical cavity and $\boldsymbol{v}$ the particle velocity. In free space $\boldsymbol{v} \perp \boldsymbol{E}_\mathrm{L}$ and therefore there is no energy transfer possible, $\Delta W \equiv 0$. Generating some transverse velocity $\boldsymbol{v}_\perp$ through periodic deflection in an undulator, we get from (10.17a)

$$v_x = +\beta c \frac{K}{\gamma} \sin k_\mathrm{p} s \,, \tag{11.2}$$

where $k_\mathrm{p} = 2\pi/\lambda_\mathrm{p}$. The external radiation field can be expressed by

$$\boldsymbol{E}_\mathrm{L} = \boldsymbol{E}_{0\mathrm{L}} \cos\left(\omega_\mathrm{L} t - k_\mathrm{L} s + \varphi_0\right) \tag{11.3}$$

and the energy transfer is

$$\begin{aligned} \Delta W &= e \int \boldsymbol{v} \boldsymbol{E}_\mathrm{L} \mathrm{d}t = e \int v_x E_\mathrm{L} \mathrm{d}t \\ &= e\beta c \frac{K}{\gamma} E_{0\mathrm{L}} \int \cos\left(\omega_\mathrm{L} t - k_\mathrm{L} s + \varphi_0\right) \sin k_\mathrm{p} s \, \mathrm{d}t \\ &= \tfrac{1}{2} e\beta c \frac{K}{\gamma} E_{0\mathrm{L}} \int \left(\sin \Psi^+ - \sin \Psi^-\right) \mathrm{d}t \,, \end{aligned} \tag{11.4}$$

where

$$\Psi^\pm = \omega_\mathrm{L} t - \left(k_\mathrm{L} \pm k_\mathrm{p}\right) s + \varphi_0 \,. \tag{11.5}$$

The energy transfer appears to be oscillatory, but continuous energy transfer can be obtained if either $\Psi^+ =$ const. or $\Psi^- =$ const. In this case

$$\frac{\mathrm{d}\Psi^\pm}{\mathrm{d}t} = \omega_\mathrm{L} - \left(k_\mathrm{L} \pm k_\mathrm{p}\right) \dot{s} = 0 \tag{11.6}$$

and we must derive conditions for this to be true. The velocity $\dot{s}$ is from (10.17b)

$$\dot{s} = \bar{\beta} c + \beta c \frac{K^2}{4\gamma^2} \cos\left(2 k_\mathrm{p} z\right) \,, \tag{11.7}$$

where the average drift velocity $\bar{\beta} c$ is defined by

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}t} = \bar{\beta} c = \beta c \left(1 - \frac{K^2}{4\gamma^2}\right) \,. \tag{11.8}$$

We modify slightly the condition (11.6) and require that it be true only on average

$$\frac{\mathrm{d}\Psi^\pm}{\mathrm{d}t} = \omega_\mathrm{L} - \left(k_\mathrm{L} \pm k_\mathrm{p}\right) \frac{\mathrm{d}\bar{s}}{\mathrm{d}t} = 0 \,, \tag{11.9}$$

or

$$(k_{\mathrm{L}} \pm k_{\mathrm{p}})\,\beta\left(1-\frac{K^2}{4\gamma^2}\right) - k_{\mathrm{L}} = 0\,. \tag{11.10}$$

With $\beta \approx 1 - 1/2\gamma^2$ and $k_{\mathrm{p}} \ll k_{\mathrm{L}}$, (11.10) becomes

$$k_{\mathrm{L}}\left[\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{K^2}{4\gamma^2}\right) - 1\right] \pm k_{\mathrm{p}} \approx 0\,, \tag{11.11}$$

or for $\gamma \gg 1$

$$-\frac{k_{\mathrm{L}}}{2\gamma^2}\left(1+\tfrac{1}{2}K^2\right) \pm k_{\mathrm{p}} = 0\,. \tag{11.12}$$

Equation (11.12) can be met only for the +sign or for

$$k_{\mathrm{p}} = \frac{k_{\mathrm{L}}}{2\gamma^2}\left(1+\tfrac{1}{2}K^2\right)\,, \tag{11.13}$$

which is identical to the definition of the fundamental undulator radiation wavelength

$$\lambda_{\mathrm{L}} = \frac{\lambda_{\mathrm{p}}}{2\gamma^2}\left(1+\tfrac{1}{2}K^2\right)\,. \tag{11.14}$$

Radiation at the fundamental wavelength of undulator radiation guarantees a continuous energy transfer from the particles to the electromagnetic wave or stimulation of radiation emission by an external field. For this reason, it is most convenient to use spontaneous undulator radiation as the external field to start the build-up of the free electron laser.

11.1.2 Equation of Motion

The energy gain $\mathrm{d}W$ of the electromagnetic field is related to the energy change $\mathrm{d}\gamma$ of the electron by

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = -\frac{1}{mc^2}\frac{\mathrm{d}W}{\beta c \mathrm{d}t} \tag{11.15}$$

or with (11.4)

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = -\frac{eKE_{0\mathrm{L}}}{2\gamma mc^2}\left(\sin\Psi^+ - \sin\Psi^-\right)\,. \tag{11.16}$$

With the substitution $\sin x = -\operatorname{Re}\left(\mathrm{i}\,e^{\mathrm{i}x}\right)$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{eKE_{0\mathrm{L}}}{2\gamma mc^2}\operatorname{Re}\left(\mathrm{i}\,e^{\mathrm{i}\Psi^+} - \mathrm{i}\,e^{\mathrm{i}\Psi^-}\right)\,. \tag{11.17}$$

In $\Psi^\pm = \omega_{\mathrm{L}} t - (k_{\mathrm{L}} \pm k_{\mathrm{p}})\; s\,(t) + \varphi_0$, we replace the location function $s(t)$ by its expression (10.19b)

$$s(t) = \underbrace{\bar{\beta}ct}_{=\bar{s}} + \underbrace{\frac{K^2}{8\gamma^2 k_\mathrm{p}} \sin\left(2k_\mathrm{p}\bar{\beta}ct\right)}_{\ll \bar{\beta}ct}, \tag{11.18}$$

composed of an average position $\bar{s}$ and an oscillatory term. With $k_\mathrm{p} \ll k_\mathrm{L}$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{e\beta K E_{0\mathrm{L}}}{2\gamma mc^2} \operatorname{Re}\left\{\mathrm{i}\exp\left[\mathrm{i}\,\frac{k_\mathrm{L} K^2}{8\gamma^2 k_\mathrm{p}} \sin(2k_\mathrm{p}\bar{s})\right]\left[e^{\mathrm{i}\bar{\Psi}^+} - e^{\mathrm{i}\bar{\Psi}^-}\right]\right\} \tag{11.19}$$

and the the phase

$$\bar{\Psi}^{\pm} = \omega_\mathrm{L} t - (k_\mathrm{L} \pm k_\mathrm{p})\ \bar{s} + \varphi_0\,. \tag{11.20}$$

With the definition $\exp(\mathrm{i}x\sin\phi) = \sum_{n=-\infty}^{n=+\infty} J_n(x)\mathrm{e}^{\mathrm{i}n\phi}$ we get finally

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{e\beta K E_{0\mathrm{L}}}{2\gamma mc^2} \operatorname{Re}\left[\mathrm{i}\sum_{n=-\infty}^{n=+\infty} J_n\left(\frac{k_\mathrm{L} K^2}{8\gamma^2 k_\mathrm{p}}\right) e^{\mathrm{i}2nk_\mathrm{p}\bar{s}}\left(e^{\mathrm{i}\bar{\Psi}^+} - e^{\mathrm{i}\bar{\Psi}^-}\right)\right]. \tag{11.21}$$

The infinite sum reflects the fact that the condition for continuous energy transfer can be met not only at one wavenumber but also at all harmonics of that frequency. Combining the exponential terms and sorting for equal wavenumbers $h\,k_\mathrm{p}$, where h is an integer, we redefine the summation index by setting

$$2\,n\,k_\mathrm{p} + k_\mathrm{p} = h\,k_\mathrm{p} \quad \longrightarrow \quad n = \frac{h-1}{2} \tag{11.22a}$$

$$2\,n\,k_\mathrm{p} - k_\mathrm{p} = h\,k_\mathrm{p} \quad \longrightarrow \quad n = \frac{h+1}{2} \tag{11.22b}$$

and get

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{e\beta K E_{0\mathrm{L}}}{2\gamma mc^2} \sum_{h=1}^{\infty}\left[J_{\frac{h-1}{2}}(x) - J_{\frac{h+1}{2}}(x)\right] \underbrace{\operatorname{Re}\left\{\mathrm{i}\,e^{\mathrm{i}[(k_\mathrm{L}+h\,k_\mathrm{p})\,\bar{s}-\omega_\mathrm{L}\,t+\varphi_0]}\right\}}_{=-\sin[(k_\mathrm{L}+h\,k_\mathrm{p})\,\bar{s}-\omega_\mathrm{L}\,t+\varphi_0]}, \tag{11.23}$$

where $x = \frac{K^2}{4+2K^2}$. Introducing the abbreviation

$$[JJ] = J_{\frac{k-1}{2}}\left(\frac{K^2}{4+2K^2}\right) - J_{\frac{k+1}{2}}\left(\frac{K^2}{4+2K^2}\right), \tag{11.24}$$

the energy transfer is

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = -\frac{e\beta K E_{0\mathrm{L}}}{2\gamma mc^2} \sum_{h=1}^{\infty}[JJ]\sin\Psi\,. \tag{11.25}$$

For maximum continuous energy transfer $\sin\Psi = \text{const.}$ or

$$\begin{aligned}\frac{\mathrm{d}\Psi}{\mathrm{d}t} &= (k_\mathrm{L} + h\,k_\mathrm{p})\;\bar{\dot{s}} - \omega_\mathrm{L} \\ &= (k_\mathrm{L} + h\,k_\mathrm{p})\;\beta c\left(1 - \frac{K^2}{4\gamma^2}\right) - \omega_\mathrm{L} \\ &= (k_\mathrm{L} + h\,k_\mathrm{p})\left(1 - \frac{1}{2\gamma^2}\right) c\left(1 - \frac{K^2}{4\gamma^2}\right) - ck_\mathrm{L} \\ &= -\frac{ck_\mathrm{L}}{2\gamma^2}\left(1 + \tfrac{1}{2}K^2\right) + h\,k_\mathrm{p}c = 0\,,\end{aligned} \tag{11.26}$$

where we assumed that $k_\mathrm{L} \gg h\,k_\mathrm{p}$, which is true since $\lambda_\mathrm{p} \gg \lambda_\mathrm{L}$ and the harmonic number of interest is generally unity or a single digit number. This condition confirms our earlier finding (11.14) and extends the synchronicity condition to multiples h of the fundamental radiation frequency

$$\lambda_\mathrm{L} = \frac{\lambda_\mathrm{p}}{2\gamma^2 h}\left(1 + \tfrac{1}{2}K^2\right)\,. \tag{11.27}$$

The integer h therefore identifies the harmonic of the radiation frequency with respect to the fundamental radiation.

In a real particle beam with a finite energy spread we may not assume that all particles exactly meet the synchronicity condition. It is therefore useful to evaluate the tolerance for meeting this condition. To do this, we define a resonance energy

$$\gamma_\mathrm{r}^2 = \frac{k_\mathrm{L}}{2h\,k_\mathrm{p}}\left(1 + \tfrac{1}{2}K^2\right), \tag{11.28}$$

which is the energy at which the synchronicity condition is met exactly. For any other particle energy $\gamma = \gamma_\mathrm{r} + \delta\gamma$ we get from (11.26) and (11.28)

$$\frac{\mathrm{d}\Psi}{\mathrm{d}s} = 2\,h\,k_\mathrm{p}\frac{\delta\gamma}{\gamma_\mathrm{r}}\,. \tag{11.29}$$

With the variation of the energy deviation $\frac{\mathrm{d}}{\mathrm{d}s}\delta\gamma = \left.\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right|_{\gamma_\mathrm{r}} - \frac{\mathrm{d}\gamma_\mathrm{r}}{\mathrm{d}s} = \left.\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right|_{\gamma_\mathrm{r}}$ and (11.25) we get from (11.29) after differentiating with respect to s

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}s^2} = 2\,h\,k_\mathrm{p}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\delta\gamma}{\gamma_\mathrm{r}} = -\frac{eh\,k_\mathrm{p}K\,E_{0\mathrm{L}}}{\gamma_\mathrm{r}^2 mc^2}[JJ]\sin\Psi(s)\,, \tag{11.30}$$

where, for simplicity, we use only one harmonic h. This equation can be written in the form

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}s^2} + \Omega_\mathrm{L}^2\sin\Psi = 0 \tag{11.31}$$

exhibiting the dynamics of a harmonic oscillator. Equation (11.31) is known as the Pendulum equation [65] with the frequency

$$\Omega_{\mathrm{L}}^{2}=\frac{e h\, k_{\mathrm{p}} K\, E_{0\mathrm{L}}}{\gamma_{\mathrm{r}}^{2} m c^{2}}\,|\,[JJ]\,|\;. \tag{11.32}$$

While interacting with the external radiation field, the particles perform harmonic oscillations in a potential generated by this field. This situation is very similar to the synchrotron oscillation of particles in a storage ring interacting with the field of the rf-cavities as was discussed in section 7.1. In phase space, the electron perform synchrotron oscillations at the frequency Ω_{L} while exchanging energy with the radiation field.

11.1.3 FEL-Gain

Having established the possibility of energy transfer from an electron to a radiation field, we may evaluate the magnitude of this energy transfer or the gain in field energy per interaction process or per pass. One pass is defined by the interaction of an electron bunch with the radiation field while passing through the entire length of the undulator. The gain in the laser field $\Delta W_{\mathrm{L}} = -mc^2 n_{\mathrm{e}} \Delta\gamma$, where $\Delta\gamma$ is the energy loss per electron and pass to the radiation field and n_{e} the number of electrons per bunch. The energy in the laser field

$$W_{\mathrm{L}}=\frac{1}{8\pi}\frac{1}{2}E_{0\mathrm{L}}^{2}\,V\,, \tag{11.33}$$

where V is the volume of the radiation field. With this, we may define the FEL-gain for the k-th harmonic by

$$G_{k}=\frac{\Delta W_{\mathrm{L}}}{W_{\mathrm{L}}}=-\frac{mc^{2}n_{\mathrm{b}}\Delta\gamma}{\frac{1}{16\pi}E_{0\mathrm{L}}^{2}V}=-\frac{4\pi mc^{2}\gamma_{\mathrm{r}}n_{\mathrm{e}}}{hk_{\mathrm{p}}E_{0\mathrm{L}}^{2}V}\left\langle\Delta\Psi'\right\rangle_{n_{\mathrm{e}}}, \tag{11.34}$$

making use of (11.29). $\left\langle\Delta\Psi'\right\rangle_{n_{\mathrm{e}}}$ is the average value of $\Delta\Psi'=\Psi_{\mathrm{f}}'-\Psi_{0}'$ for all electrons per bunch, where Ψ_0' is defined at the beginning of the undulator and Ψ_{f}' at the end of the undulator. To further simplify this expression, we use (11.32), solve for the laser field

$$E_{0\mathrm{L}}=\frac{mc^{2}\gamma_{\mathrm{r}}^{2}\Omega_{\mathrm{L}}^{2}}{ehKk_{\mathrm{p}}V[JJ]}, \tag{11.35}$$

and define the electron density $n_{\mathrm{b}} = n_{\mathrm{e}}/V$. Here we have tacitly assumed that the volume of the radiation field perfectly overlaps the volume of the electron beam. This is not automatically the case and must be achieved by carefully matching the electron beam to the diffraction dominated radiation field. If this cannot be done, the volume V is the overlap volume, or the smaller of both. With this the FEL-gain becomes

$$G=-\frac{8\pi e^{2}n_{\mathrm{b}}hK^{2}k_{\mathrm{p}}[JJ]^{2}}{mc^{2}\gamma_{\mathrm{r}}^{3}\Omega_{\mathrm{L}}^{4}}\left\langle\Delta\Psi'\right\rangle_{n_{\mathrm{e}}} \tag{11.36}$$

Numerical evaluation of $\langle \Delta\Psi' \rangle_{n_e}$ can be performed with the pendulum equation. Multiplying the pendulum equation $2\Psi'$ and integrating we get

$$\Psi'^2 - 2\Omega_\mathrm{L}^2 \cos\Psi = \mathrm{const.} \tag{11.37}$$

Evaluating this at the beginning of the undulator

$$\Psi'^2 - \Psi_0'^2 = 2\Omega_\mathrm{L}^2 \left(\cos\Psi - \cos\Psi_0\right) , \tag{11.38}$$

which becomes with $\Psi_0' = 2N\, k_\mathrm{p} \frac{\gamma_0 - \gamma_\mathrm{r}}{\gamma_\mathrm{r}}$ and $\Psi_\mathrm{b}' = \Psi$ from (11.29)

$$\Psi'^2 = \left(2hk_\mathrm{p} \frac{\gamma_0 - \gamma_\mathrm{r}}{\gamma_\mathrm{r}}\right)^2 + 2\Omega_\mathrm{L}^2 \left(\cos\Psi - \cos\Psi_0\right) \tag{11.39}$$

Finally,

$$\Psi'(s) = 2\, h\, k_\mathrm{p} \frac{\gamma - \gamma_\mathrm{r}}{\gamma_\mathrm{r}} \sqrt{1 + \frac{\Omega_\mathrm{L}^2}{2k^2\, k_\mathrm{p}^2} \frac{\gamma_\mathrm{r}^2}{\left(\gamma - \gamma_\mathrm{r}\right)^2} \left[\cos\Psi(s) - \cos\Psi_0\right]}\, , \tag{11.40}$$

or with

$$w = h\, k_\mathrm{p}\, L_\mathrm{u} \frac{\gamma - \gamma_\mathrm{r}}{\gamma_\mathrm{r}} , \tag{11.41}$$

where $L_\mathrm{u} = N_\mathrm{p} \lambda_\mathrm{p}$ is the undulator length,

$$\Psi'(s) = \frac{2w}{L_\mathrm{u}} \sqrt{1 + \frac{L_\mathrm{u}^2 \Omega_\mathrm{L}^2}{2w^2} \left[\cos\Psi(s) - \cos\Psi_0\right]}\, . \tag{11.42}$$

We solve this by expansion and iteration. For a low gain FEL, the field $E_{0\mathrm{L}}$ is weak and does not influence the particle motion. Therefore $\Omega_\mathrm{L} \ll 1$ and (11.42) becomes

$$\begin{aligned} \Psi' \approx \frac{2w}{L} \Bigg[1 &+ \frac{1}{2} \frac{L^2 \Omega_\mathrm{L}^2}{2w^2} \left(\cos\Psi - \cos\Psi_0\right) \\ &- \frac{1}{8} \frac{L^4 \Omega_\mathrm{L}^4}{4w^4} \left(\cos\Psi - \cos\Psi_0\right)^2 + \ldots \Bigg] \, . \end{aligned} \tag{11.43}$$

In the lowest order of iteration $\Psi_0' = \frac{2w}{L}$ and $\Delta\Psi_{(0)}' = 0$ for all particles, which means there is no energy transfer. For first order approximation, we integrate $\Psi_0'(s) = \frac{2w}{L_\mathrm{u}}$ to get $\Psi_{(1)}(s) = \frac{2w}{L_\mathrm{u}} s + \Psi_0$ and

$$\Delta\Psi_{(1)}' = \Psi'(L_\mathrm{u}) - \Psi_1'(0) = \tfrac{L\, \Omega_\mathrm{L}^2}{2w} \left[\cos\left(2w + \Psi_0\right) - \cos\Psi_0\right] + \mathcal{O}(2) \tag{11.44}$$

from (11.43). Averaging over all initial phases occupied by electrons $0 \le \Psi_0 \le 2\pi$

$$\langle \Delta\Psi_1' \rangle = \frac{L\,\Omega_{\mathrm{L}}^2}{2w}\frac{1}{2\pi}\int\limits_0^{2\pi} \left[\cos\left(2w+\Psi_0\right)-\cos\Psi_0\right]\,\mathrm{d}\Psi_0 = 0\,. \tag{11.45}$$

No energy transfer to the laser field occurs in this approximation either. We need a still higher order approximation. The higher order correction to $\Psi_1'(s) = \Psi_0'(s) + \delta\Psi_1'(s)$ is from (11.43)

$$\delta\Psi_{(1)}' = \frac{L\Omega_{\mathrm{L}}^2}{2w}\left[\cos\Psi - \cos\Psi_0\right]\,, \tag{11.46}$$

and the correction to $\Psi_1(s)$ is

$$\begin{aligned}\delta\Psi_{(1)} &= \frac{L\Omega_{\mathrm{L}}^2}{2w}\int_0^L \left[\cos\left(\frac{2w}{L}s+\Psi_0\right)-\cos\Psi_0\right]\,\mathrm{d}s \\ &= \frac{L\Omega_{\mathrm{L}}^2}{4w^2}\left[\sin\left(2w+\Psi_0\right)-\sin\Psi_0 - 2w\cos\Psi_0\right]\,.\end{aligned} \tag{11.47}$$

The second order approximation to the phase is then $\Psi_1(s) = \frac{2w}{L_{\mathrm{u}}}s + \Psi_0 + \delta\Psi_{(1)}$ and using (11.43) in second order as well we get

$$\begin{aligned}\Delta\Psi_{(2)}' = &\tfrac{L\,\Omega_{\mathrm{L}}^2}{2w}\left[\cos\left(2w+\Psi_0+\delta\Psi_{(1)}\right)-\cos\Psi_0\right] \\ &-\tfrac{L^3\Omega_{\mathrm{L}}^4}{4w^2}\left[\cos\left(2w+\Psi_0\right)-\cos\Psi_0\right]^2+\dots\,,\end{aligned} \tag{11.48}$$

where in the second order term only the first order phase $\Psi_1(s) = \frac{2w}{L_{\mathrm{u}}}s + \Psi_0$ is used. The first term becomes with $\delta\Psi_{(1)} \ll \Psi_0 + 2w$

$$\begin{aligned}&\cos\left(2w+\Psi_0+\delta\Psi_1\right)-\cos\Psi_0 \\ &\qquad\approx \cos\left(2w+\Psi_0\right) - \delta\Psi_1\sin\left(2w+\Psi_0\right)-\cos\Psi_0\end{aligned}$$

and

$$\begin{aligned}\Delta\Psi_2' = \frac{L_{\mathrm{u}}^3\,\Omega_{\mathrm{L}}^4}{16w^3}\Bigg\{&\frac{8w^2}{L_{\mathrm{u}}^2\Omega^2}\left[\cos\left(2w+\Psi_0\right)-\cos\Psi_0\right] \\ &-2\sin\left(2w+\Psi_0\right)\left[\sin\left(2w+\Psi_0\right)-\sin\Psi_0-2w\cos\Psi_0\right] \\ &-\left[\cos\left(2w+\Psi_0\right)-\cos\Psi_0\right]^2+\dots\Bigg\}\,.\end{aligned} \tag{11.49}$$

Now, we average over all initial phases assuming a uniform distribution of particles in s or in phase. The individual terms become then

$$\begin{aligned}\langle\cos\left(2w+\Psi_0\right)-\cos\Psi_0\rangle &= 0 \\ \left\langle\sin^2\left(2w+\Psi_0\right)\right\rangle &= \tfrac{1}{2} \\ \langle\sin\left(2w+\Psi_0\right)\sin\Psi_0\rangle &= \tfrac{1}{2}\cos\left(2w\right) \\ \langle\sin\left(2w+\Psi_0\right)\cos\Psi_0\rangle &= \tfrac{1}{2}\sin\left(2w\right) \\ \langle\cos\left(2w+\Psi_0\right)\cos\Psi_0\rangle &= \tfrac{1}{2}\cos\left(2w\right)\,.\end{aligned} \tag{11.50a}$$

With this

$$\langle \Delta\Psi_2' \rangle = -\tfrac{L_\mathrm{u}^3\,\Omega_\mathrm{L}^4}{16w^3}\left[1-\cos\left(2w\right)-w\sin\left(2w\right)\right] \tag{11.51}$$

and finally with $\left[1-\cos\left(2w\right)-w\sin\left(2w\right)\right]/w^3 = -\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\sin w}{w}\right)^2$

$$\langle \Delta\Psi_2' \rangle = \frac{L_\mathrm{u}^3\,\Omega_\mathrm{L}^4}{8}\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\sin w}{w}\right)^2 . \tag{11.52}$$

The FEL-gain is finally from (11.36)

$$G_k = -\frac{\pi r_\mathrm{c} n_\mathrm{b} h\, K^2 L_\mathrm{u}^3 k_\mathrm{p}}{\gamma_\mathrm{r}^3}\left[JJ\right]^2\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\sin w}{w}\right)^2 , \tag{11.53}$$

where we may express the particle density n_b by beam parameters as obtained from the electron beam source

$$n_\mathrm{b} = \frac{n_\mathrm{e}}{V} = \frac{n_\mathrm{e}}{\pi^2\sigma^2\ell} , \tag{11.54}$$

where σ is the radius of a round beam. With these definitions, and $\hat{I} = cen_\mathrm{e}/\ell$ the electron peak current the gain per pass becomes

$$G_k = -\frac{2^{2/3}\pi r_\mathrm{c} h\,\lambda^{3/2} L_\mathrm{u}^3}{c\sigma^2\lambda_\mathrm{p}^{5/2}}\frac{\hat{I}}{\ell}\frac{K^2\left[JJ\right]^2}{\left(1+\frac{1}{2}K^2\right)^{3/2}}\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\sin w}{w}\right)^2 . \tag{11.55}$$

The gain depends very much on the choice of the electron beam energy through the function (11.41) , which is expressed by the gain curve as shown in Fig. 11.4.

There is no gain if the beam energy is equal to the resonance energy, $\gamma = \gamma_\mathrm{r}$. As has been discussed in the introduction to this chapter, we must introduce an asymmetry to gain stimulation of radiation or gain and this asymmetry is generated by a shift in energy. For a monochromatic electron beam maximum gain can be reached for $w \approx 1.2$. A realistic beam, however, is not monochromatic and the narrow gain curve indicates that a beam with too large an energy spread may not produce any gain. There is no precise upper limit for the allowable energy spread but from Fig. 11.4 we see that gain is all but gone when $|w| \gtrsim 5$. We use this with (11.41) and (11.28)to formulate a condition for the maximum allowable energy spread

$$\left|\frac{\delta\gamma}{\gamma}\right| \ll \frac{2\gamma_\mathrm{r}^2\,\lambda_\mathrm{L}}{1+\frac{1}{2}K^2} . \tag{11.56}$$

For efficient gain the geometric size of the electron beam and the radiation field must be matched. In (11.54) we have introduced a volume for the electron bunch. Actually, this volume is the overlap volume of radiation field

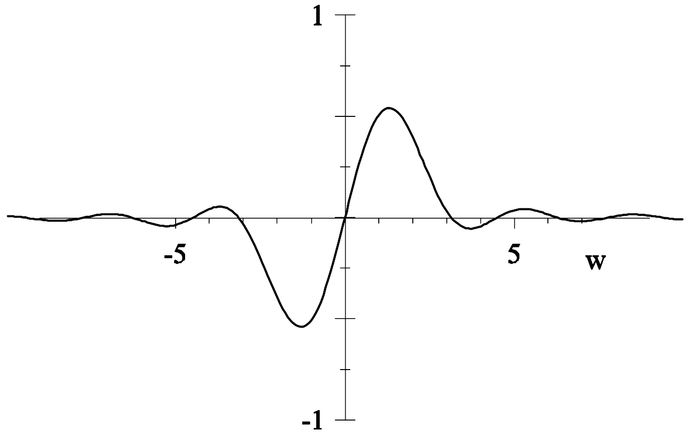

Fig. 11.4. Free electron laser gain curve $G \propto -\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\sin w}{w}\right)^2$

and electron bunch. Ideally, one would try to get a perfect overlap by forming both beams to be equal. This is in fact possible and we will discuss the conditions for this to happen. First, we assume that the electron beam size varies symmetrically about the center of the undulator. From our discussion in Section 6.6.2 the beam size develops like

$$\sigma^2(s) = \sigma_0^2 + \left(\frac{\epsilon}{\sigma_0}\right)^2 s^2 \tag{11.57}$$

with distance s from the beam waist. To maximize gain we look for the minimum average beam size within an undulator. This minimum demands a symmetric solution about the undulator center. Furthermore, we may select the optimum beam size at the center by looking for the minimum value of the maximum beam size within the undulator. From $\mathrm{d}\sigma^2/\mathrm{d}\sigma_0^2 = 0$, the optimum solution is obtained for $s = \frac{1}{2}L_\mathrm{u} = \sigma_0^2/\epsilon = \beta_0$. For $\beta_0 = \frac{1}{2}L_\mathrm{u}$ the beam cross section grows from a value of σ_0^2 in the middle of the undulator to a maximum value of $2\sigma_0^2$ at either end.

The radiation field is governed by diffraction. Starting at a beam waist the growth of the radiation field cross section due to diffraction is quantified by the Rayleigh length

$$s_\mathrm{R} = \pi\frac{w_0^2}{\lambda}, \tag{11.58}$$

where w_0 is the beam size at the waist and λ the wavelength. This length is defined as the distance from the radiation source (waist) to the point at which the cross section of the radiation beam has grown by a factor of two. For a Gaussian beam we have for the beam size at a distance s from the waist

$$w^2(s) = w_0^2 + \Theta^2 s^2 \,, \tag{11.59}$$

where $\Theta = \frac{\lambda}{\pi w_0}$ is the divergence angle of the radiation field. This is exactly the same condition as we have just discussed for the electron beam assuming the center of the undulator as the source of radiation.

Exercises *

Exercise 11.1 (S). Consider an electron travelling through an undulator producing radiation. Show, that the radiation front travels faster than the electron by one fundamental radiation wavelength per undulator period.

Exercise 11.2. Choose an undulator with $N_\mathrm{p} = 50$ and $K = 4.0$. Specify the undulator period length λ_p and electron beam energy from a linear accelerator such that the fundamental wavelength of the radiation is 1μm. What is the upper limit on the beam energy spread to make an FEL to function?

* The argument (S) indicates an exercise for which a solution is given in Appendix A.

A. Solutions to Exercises

Solution 1.1. Solve $E^2 = (cp)^2 + (mc^2)^2$ for $(cp)^2 = E^2 - (mc^2)^2$, extract E , and replace $E/mc^2 = \gamma$, and $E = E_{\text{kin}} + mc^2$ to get with $\beta = \sqrt{1-\gamma^{-2}}$ finally $cp = \beta(E_{\text{kin}} + mc^2)$. Replacing β we get after some manipulation: $cp = mc^2\sqrt{(E_{\text{kin}}^2/mc^2 + 1)^2 - 1}$, and finally $E_{\text{kin}} = mc^2(\gamma - 1)$.

Solution 1.2. For very large energies $\gamma \gg 1$ and $cp \approx E_{\text{kin}}$, and $E_{\text{kin}} \approx mc^2\gamma$. For nonrelativistic particles, set $\gamma \approx 1+\delta$, where $\delta = E_{\text{kin}}/mc^2 = \frac{1}{2}\beta^2$, therefore $\beta \approx \sqrt{2\delta}$. With $E_{\text{kin}} = \frac{1}{2}mv^2$ and keeping only terms linear in β, we get $cp = \sqrt{2\delta}(1+\delta)mc^2 \approx cmv$ or the classical definition of the momentum $p = mv$.

Solution 1.3. For a kinetic energy of $E_{\text{kin}} = 200$ MeV and $m_pc^2 = 938.27$ MeV; the total energy $E = 1138.27$ MeV. The velocity $\beta = \sqrt{1-\gamma^{-2}} = \sqrt{1-(\frac{938.27}{1138.27})^2} = 0.566$ and the momentum $cp = \beta E = 644.44$ MeV or $p = 644.44$ MeV/c.

Solution 1.4. A length ds of the linac in the laboratory system appears to the electron Lorentz contracted by the relativistic factor γ. Since the electron energy varies along the linac we must integrate the contraction along the full length of the linac. The linac length as seen by the electron is therefore $\int \frac{ds}{\gamma(s)}$, where $\gamma(s) = \gamma_0 + \alpha s$, and the acceleration $\alpha = 20/mc^2 = 39.14$ 1/m. From $\beta_0 = \frac{1}{2}$ we get $\gamma_0 = \frac{4}{3}$ and with $\gamma = \frac{4}{3} + 39.14 \cdot 3000 = 1.17 \times 10^5$ the integral is $\int \frac{\text{d}s}{\gamma(s)} = \frac{1}{\alpha}\ln\frac{\gamma}{\gamma_0}$ or numerically $\frac{1}{39.14}\ln\frac{1.17\times10^5}{4/3} = 0.291$ m. For an electron coasting with energy γ along a 3000 m long tube, this tube appears to be $3000/\gamma = 0.0256$ m or 2.56 cm long.

Solution 1.5. The revolution frequency is $f_{\text{rev}} = c/C = 1.0 \times 10^6$ 1/s and the total number of particles orbiting $n_{\text{e}} = I/e/f_{\text{rev}} = 1.5604 \times 10^{12}$ electrons or 3.1208×10^9electron/bunch. The photon pulses image exactly those of the electron bunches. Therefore, there is a 1 cm long photon pules every 0.6 m, or in time one 30 ps photon pulses every 2 ns.

Solution 1.6. The mass of the pion is $m_\pi c^2 = 139.6$ MeV and $\gamma = 1.716$. From this the velocity is $\beta = 0.813$ or $v = 2.44 \times 10^8$ m/s. The travel time along a 15 m long beam line is $t = 6.15 \times 10^{-8}$ and the survival probability

without time dilatation! is 2 = exp (−6d5τ2d6) = 0d094c0 Ch h
e.ar C h C h h h
C hh C h h h
2 = 0 − M e M8 M M and the available pion flux is greater by a factor of 2.7.

Solution 1.7. 0 c c c / c C
c tco = 2. s>n β magi = ... nr 1 = c0 y.I o)o n = 6 n 7 a − β^t ·
50M= x 00 β^0 / ./ o = 2.3eg . **Solut o** β β . . γ . .e
tt o t o E_{heC} = 797.6 ./

Solution 1.8. 0 he geometry of the field lines in the particle system of reference can be expressed by $\ell = z$ 2 α , 23 α
3 3 23 3 z 3
3 z b aγa γ γ
aγ $\ell = \frac{\text{tio } \alpha}{4}$ z e 0 9 ution of radial field lines is compressed in the z-direction by a factor In

Solution 1.9. 0 he circulating beam current is defined by i = 234 ran =
ab , b b irp oducpt s tip tdio T 34 irp tpsdiso
tpdpo b ≈ r the particle velocity and C the accelerator circumference. The number of particles representing a current of 1A are: $n = iC/ev$ = I.ns 1 yn02nf
f
6 = b /c = T. μs. 0 he synchrotron produces 100 pulses of 1μ duration and at a pulse current of 1A. The average beam current is therefore $i_{avg} = 100 \cdot 6 \cdot 1 = 33.3$ μA.

Solution 2.1. 0 he Cherenkov condition is β. air cos θ = 1. For electro0
β (10) = 0r99869 0 (50) = 0r9999478r 0 0
0 (/τ. θ ℓ t 0vt 3 t γ bWithmi,an dm ,, ,, m
,ta im WWmW ,m m,ta W i i WhmhWW m,ta
i 57 3 t0, t/ τ0
θ = 1 √ 0 / 1. √0. − − − = × × d57=

Solvi **ion 2.2.** 0 ki nkii i θ ha = 4% 0 / e.hh· w . = 53.9 d9
99 9 9 9 9 9 9
9 999 99 9 9 9
θS = γ . d. . **tan nta tn t ta at t**
a n t t a a t aat a n 3τ.
t t

b γ. ℒ a a0, ae, an, ba f ev
b b , be, bn, bf ev eth ehnumbr to a0 n0 + at vt + e ve vvv 20 818 ℒ*
8 a0* i0* + * i * + * i * − a+* w+* e e e
ee a0* S0* + a2* 2* + a3* 3* − a+* C+* = a0 z0 4 cw zw 4 och −

$\beta\gamma a_4)(\gamma b_3 - \beta\gamma b_4) - (-\beta\gamma a_3 + \gamma a_4)(-\beta\gamma b_3 + \gamma b_4) = a_1 b_1 + a_2 b_2 + \gamma^2 a_3 b_3 - \beta\gamma^2 a_3 b_4 - \beta\gamma^2 a_4 b_3 + \beta^2\gamma^2 a_4 b_4 - \beta^2\gamma^2 a_3 b_3 + \beta\gamma^2 a_3 b_4 + \beta\gamma^2 a_4 b_3 - \gamma^2 a_4 b_4 = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_4 b_4$ q.e.d.

Solution 2.4. The 4-momentum is $(c\hbar\boldsymbol{k}, \mathrm{i}\hbar\omega) = (c\hbar k\boldsymbol{n}, \mathrm{i}\hbar\omega)$, the 4-spacetime vector $(\boldsymbol{r}, \mathrm{i}ct)$ and with $ck = \omega$ their product is $c\hbar k n_x x + c\hbar k n_y y + c\hbar k n_z z - c\hbar\omega t = \hbar\omega\,(n_x x + n_y y + n_z z - ct)$ which is $\hbar c$-times the phase (2.18) of a plane wave.

Solution 2.5. The relativistic Doppler effect is $\omega^*\gamma\,(1 + \beta_z\, n_z^*) = \omega$ and for the classical case we set $\gamma = 1$, $n_z^* = \cos\vartheta$ and $\beta = v/v_0$, where v_0 is the velocity of the wave (light or acoustic). The relative Doppler shift is then $\frac{\Delta f}{f_s} = \frac{v}{v_0}\cos\vartheta$.

Solution 2.6. We use the uncertainty relation $\Delta x\,\Delta p = \Delta x\,\hbar k \geq \hbar$ or $\Delta x \geq 1/k$ and the "characteristic volume" of a photon is $V_{\mathrm{ph}} = \frac{\lambda^3}{8\pi^3}$. The average electric field within this volume is from $\varepsilon = \frac{1}{2\epsilon_0}E^2 V_{\mathrm{ph}} = \hbar\omega$ or $E = k^2\sqrt{2\epsilon_0\hbar c}$. For a 0.1238 eV photon (CO_2-laser) the wavelength is $\lambda = 10\ \mu$m and the average electric field is $E = 2.96 \times 10^{-7}$ V/m. In case of a 10 keV x-ray photon the field is $E = 1.93$ kV/m

Solution 2.7. We describe EM-waves by $\boldsymbol{E} = \boldsymbol{E}_0 \exp\left[\mathrm{i}\,(\omega t - \boldsymbol{kr})\right]$ and $\boldsymbol{B} = \boldsymbol{B}_0 \exp\left[\mathrm{i}\,(\omega t - \boldsymbol{kr})\right]$ where $\boldsymbol{kr}{=}\boldsymbol{nr}k$ and $\boldsymbol{n}$ is a unit vector parallel to $\mathbf{k}$. Inserted into Maxwell's equation $\nabla \times \boldsymbol{E} = -\frac{[c]}{c}\dot{\boldsymbol{B}}$ we get with $k = \omega/c$ for the l.h.s.: $\nabla \times (\mathrm{i}\,\boldsymbol{nr}k)\boldsymbol{E} = \nabla(\mathrm{i}\boldsymbol{nr}k) \times \boldsymbol{E}$ and for the r.h.s. $\dot{\boldsymbol{B}} = \frac{\mathrm{i}\omega}{c}[c]\boldsymbol{B} = \mathrm{i}\,[c]k\boldsymbol{B}$. With $\nabla(\boldsymbol{nr}) = \nabla(n_x x + n_y y + n_z z) = \boldsymbol{n}$ we get finally $[c]\boldsymbol{B} = \boldsymbol{n}\times\boldsymbol{E}$. This equation tells us that the electric and magnetic wavefields are orthogonal.

Solution 2.8. With $[c]\boldsymbol{B} = \boldsymbol{n}\times\boldsymbol{E}$ from Exercise 2.7 and $(B.10)$ we get $\boldsymbol{E}\times(\boldsymbol{n}\times\boldsymbol{E}) = \boldsymbol{E}^2\boldsymbol{n}$, what was to be demonstrated.

Solution 3.1. The energy loss per turn is from (3.18) $U_0 = 20.32$ keV and the total radiation power $P = 20.32$ kW. In case of muons, we have the mass ratio $m_\mu/m_{\mathrm{e}} = 206.8$ and the energy loss is reduced by the $4^{\mathrm{th}}-$ power of this ratio to become $U_{0\mu} = 11.1\ \mu$eV, which is completely negligible.

Solution 3.2. The maximum photon flux occurs at a photon energy of about $\varepsilon = 0.286\,\varepsilon_{\mathrm{c}}$ and $S(0.286) \approx 0.569$. To find the 1% photon energy we use (3.38) to scale the photon flux and have $0.777\sqrt{x}/\exp x = 0.00569$, which is solved by $x = 5.795$. Appreciable radiation exists up to almost six times the critical photon energy.

Solution 3.3. From (3.28) we have $E = [0.4508\,\epsilon_{\mathrm{c}}\,(\mathrm{keV})\,\rho\,(\mathrm{m})]^{1/3} = 8.0423^{1/3} = 2.003\,5$ GeV. The magnetic field necessary for a bending radius of $\rho = 1.784$ m would be $B = 3.75$ T, which is way beyond conventional magnet technology. Either superconducting magnets must be used to preserve the ring geometry or a new ring must be constructed with bending magnets which must be longer by at least a factor of 2.5 .

Solution 3.4. The bending radius is $\rho = 2887$ m, the energy loss is $U_0 = 88.5\frac{E^4}{\rho}\left(\frac{m_e}{m_p}\right)^4 = 399.4$ keV and the critical photon energy $\varepsilon_c = 2.2\frac{E^3}{\rho}\left(\frac{m_e}{m_p}\right)^3 = 929.3$ eV. The synchrotron radiation power is $P = 65.5$ kW.

Solution 3.5. The critical photon energy is $\varepsilon_c = 38.04$ keV and $\varepsilon/\varepsilon_c = 0.21$. The universal function is $S(0.21) = 0.5625$ and the photon flux $\frac{d\dot{N}_{ph}}{d\psi} = C_\psi\, E\, I\, \frac{\Delta\omega}{\omega}\, S\,(0.21) = 3.118\,5{\times}10^{12}$ photons/mrad. The vertical opening angle $\sqrt{2\pi}\sigma_\theta = 0.251$ mr resulting in an effective beam height at the experiment of $Y = 3.77$ mm. A beam size of 10 μm at 15 m corresponds to an angle of 0.667 μrad at the source. The total photon flux into the required sample cross section is then $\dot{N}_{ph} = 5.53 \times 10^6$ photons/s, which is more than required. For a still higher photon flux one might apply some photon focusing.

Solution 3.6. In the horizontal plane the radiation distribution is uniform and an angle of $\Delta\psi = 0.2$ mr will produce a photon beam width of 1 mm at a distance of 5 m. The critical photon energy is $\varepsilon_c = 563$ eV and $\varepsilon/\varepsilon_c = 0.124/563 = 0.00022$. For the IR radiation the vertical opening angle $\theta_{rad} = 11.3$ mr ($\gg 1/\gamma\,!$) and the source length $L = 0.045$ m. The total source height is $\sigma_{tot,y} = \sqrt{0.11^2 + 0.107^2} = 0.153$ mm and the vertical divergence $\sigma_{tot,y'} = 14.9$ mr. The photon flux for $\lambda = 10$ μm and $S(0.00022) = 0.0805$ is $d\dot{N}_{ph}/d\psi = 1.275\,10^{15}$ photons/s/mr/100%BW. The photon brightness is then $\mathcal{B} = \frac{(d\dot{N}_{ph}/d\psi)\Delta\psi}{2\pi\sigma_{tot,y}\sigma_{tot,y'}} = \frac{1.275\cdot 10^{15}\,0.2}{2\pi\cdot 0.153\cdot 14.9} = 1.780 \times 10^{13}\ \frac{\text{photons}}{\text{s}\cdot\text{mm}^2\cdot\text{mr}^2\cdot 100\%\text{BW}}$.

Solution 3.7. From Exercise 3.6 $L = 0.045$ m and the diffraction limited source size and divergence are $\sigma_r = \frac{1}{2\pi}\sqrt{\lambda L} = 0.107$ mm and $\sigma_{r'} = \sqrt{\frac{\lambda}{L}} = 14.9$ mr, respectively. This is to be compared with the electron beam parameters $(\sigma_{b,x}, \sigma_{b,y}) = (1.1, 0.11)$ mm and $\sigma\,(_{b,x'}, \sigma_{b,y'}) = (0.11, 0.011)$ mr. There is a considerable mismatch in the x-plane with $\sigma_r/\sqrt{2} = 0.076$ mm $\ll \sigma_{b,x}$ and $\sigma_{r'}/\sqrt{2} = 10.5$ mr $\gg \sigma_{b,x'}$. In the vertical plane the mismatch is small. In both planes the diffraction limited photon emittance is $\epsilon_{ph,x,y} = 797$ nm, which is much larger than the electron beam emittances in both planes. The 10 μm IR radiation is therefore spatially coherent.

Solution 4.1. The magnetic field of 2 T would limit the proton energy to $E \leq 0.3B\rho = 90000$ GeV or 90 TeV. The energy loss at 90 TeV would be from (3.18) with (D.1) and (3.15) $\Delta E = 3.41$ keV and the total radiation power is 34.1 W. The total radiation power is less than the available rf-power and therefore the energy is limited by the magnetic field. The critical photon energy is from (3.26) and (3.27) corrected for protons $\epsilon_c = \hbar C_c \frac{E^3}{\rho} = 1.74 \times 10^{-6}$ GeV $= 1.74$ keV and the radiation is therefore mainly concentrated in the soft x-ray regime.

Solution 4.2. To keep the 66/99 keV contamination below 1% we expect to work on the high energy end of the synchrotron radiation spectrum. In this regime the spectral intensity scales like $I(x) \propto 0.777\sqrt{x}/\mathrm{e}^{x}$, where $x = \epsilon_{\mathrm{ph}}/\epsilon_{\mathrm{c}}$. The task requires that $I_{66}/I_{33} \leq 0.01$ or $\sqrt{2}\mathrm{e}^{x_{33}-x_{66}} = \sqrt{2}\mathrm{e}^{-x_{33}} \leq 0.01$. Solving, we get $x_{33} = \ln\left(100/\sqrt{2}\right) = 4.258\,6$ or $\epsilon_{\mathrm{c}} \leq 33/4.2586 = 7.749$ keV. The spectral photon flux at 33 keV is then from (3.38) with $x = 4.2586$ given by $\dot{N}_{\mathrm{ph}} \approx 2.25 \times 10^{13} n_p E I$ photons/s/0.1%BW, where we added a factor n_{p} to account for the number of wiggler poles. The magnetic field and beam energy are related by the critical photon energy and we get from (3.26) $E^2\left(\mathrm{GeV}^2\right) B\,(\mathrm{T}) \leq 19.256$. A reasonable field level for a superconducting wiggler magnet is $B = 4-6$ T; we take $B = 6$ T. For economy, we would like to keep the beam energy low and the ring size small. Here we try $E = 1.5$ GeV. In this case $\epsilon_{\mathrm{c}} = 8.9775$ keV and $x_{33} = 3.676$. For this decreased value of x_{33} the flux is increased and we have now $\dot{N}_{\mathrm{ph}} \approx 5.61 \times 10^{13} n_p I$ photons/s/0.1%BW. With a beam current of say 0.5 A and a wiggler magnet with $n_{\mathrm{p}} = 6$ poles we have finally a flux of $\dot{N}_{\mathrm{ph}}(33\ \mathrm{keV}) \approx 1.68 \times 10^{14}$ photons/s/0.1%BW, which is not quite what we want. To get more flux, one would have to increase either the beam energy, the current or the number of wiggler poles. The final choice of parameters is now determined by technical limitations or economic considerations.

Solution 4.3. In first approximation, we assume that all the fields are contained within the two rows of poles and no field leaks out. Separating the poles by $\mathrm{d}g$ requires to generate the additional field energy $\mathrm{d}\varepsilon = F\mathrm{d}g$, where F is the force between poles. Since $\mathrm{d}\varepsilon > 0$ for $\mathrm{d}g > 0$, the force is attractive, meaning that the poles are attracted. The force is then $F = \frac{\mathrm{d}\varepsilon_{\mathrm{m}}}{\mathrm{d}g} = \frac{w}{2\mu_0}\int_0^{15\lambda_{\mathrm{p}}} B^2(z)\,\mathrm{d}z = 20889$ N= 2.13 tons.

Solution 4.4. The instantaneous radiation power is given by (3.9) $P_\gamma\,(\mathrm{GeV/s}) = 379.35\,B^2\,E^2$. The total energy loss of an electron due to wiggler radiation power can be obtained by integrating through the wiggler field for $\Delta E\,(\mathrm{GeV}) = 189.67\,B_0^2\,E^2\frac{L_{\mathrm{u}}}{\beta c}$ and the total radiation power for a beam current I is $P_{\mathrm{u}}\,(\mathrm{W}) = 632.67 B_0^2 E^2 L_{\mathrm{u}} I$.

Solution 4.5. In the electron rest frame energy conservation requires $\hbar\omega + mc^2 = \hbar\omega' + \sqrt{c^2p^2 + (mc^2)^2}$, where $\hbar\omega$ and $\hbar\omega'$ are the incoming and outgoing photon energies, respectively and cp the electron momentum after the scattering process. Solving for cp we get $c^2p^2 = \hbar^2\left(\omega-\omega'\right)^2 + 2\hbar mc^2\left(\omega-\omega'\right)$. For momentum conservation, we require that $\hbar\boldsymbol{k} = \hbar\boldsymbol{k'} + \boldsymbol{p}$ with the angle ϑ between $\boldsymbol{k}$ and $\boldsymbol{p}$. From this we get $c^2p^2 = \left(\hbar\omega'\right)^2 + \left(\hbar\omega\right)^2 - 2\hbar^2\omega\,\omega'\cos\vartheta$. Comparing booth expressions for cp we get $-2\hbar\omega\omega' + 2\hbar mc^2\left(\omega-\omega'\right) = -2\hbar^2\omega\,\omega'\cos\vartheta$ or $\frac{\hbar}{mc^2}\left(1-\cos\vartheta\right) = \frac{1}{\omega'} - \frac{1}{\omega} = \frac{\lambda'-\lambda}{2\pi c}$. We look for radiation emitted in the forward direction or for $\vartheta = 180^o$ and get for the scattered wavelength $\lambda' = \lambda$, because $\frac{4\pi\hbar c}{mc^2} \approx 4.8\,10^{-12} \ll \lambda$. Note, that all quantities are still defined in the electron rest frame. The wavelength of the undulator field in the electron system is $\lambda = \frac{\lambda_p^*}{\gamma}$, where now $\mathcal{L}^*$ is the laboratory system of reference and the scattered radiation in the laboratory system due to the Doppler effect is $\lambda^* = \frac{\lambda_p^*}{2\gamma^2}\left(1 + \frac{1}{2}K^2\right)$, which is the expression for the fundamental wavelength of undulator radiation.

Solution 4.6. The fundamental wave length for a very weak undulator ($K \ll 1$, e.g. wide open gap) is $\lambda\,(800\text{ MeV}) = 102$ Å and $\lambda\,(7\text{ GeV}) = 1.33$ Å which are the shortest achievable wavelength. For a 10 mm gap the field is from (4.5) $B = 1.198$ T and the maximum value of the strength parameter is $K = 5.595$. With this the longest wavelength in the fundamental is $\lambda = 1698.5$ Å for the 800 MeV ring and $\lambda = 22.147$ Å for the 7 GeV ring.

Solution 4.7. The short wavelength limits are given for a wide open undulator, $K \ll 1$, and are $\lambda = 3.13$ Å for $\lambda_p = 15$ mm and $\lambda = 15.7$ Å for $\lambda_p = 75$ mm. The long wavelength limits are determined by the magnetic fields when the undulator gaps are closed to 10 mm. The fields are from (4.5) $B_0\,(\lambda_p = 15\text{ mm}) = 0.19$ T and $B_0\,(\lambda_p = 75\text{ mm}) = 1.66$ T, respectively. The undulator strengths are $K\,(\lambda_p = 15\text{ mm}) = 0.270$ and $K\,(\lambda_p = 75\text{ mm}) = 1.35$ and the wavelengths $\lambda\,(\lambda_p = 15\text{ mm}) = 3.24$ Å and $\lambda\,(\lambda_p = 75\text{ mm}) = 30.0$ Å. The tuning range is very small for the 15 mm undulator and about a factor of two for the long period undulator. The ranges are so different because the K-value can be varied much more for longer period undulators.

Solution 5.1. In each bunch there are $n_e = \tau\, I_b/e = 1.87\times10^7$ electrons and the circulating beam current per bunch is $i_b = en_e f_{rev} = 3.0\ \mu$A. To reach a circulating beam current of 200 mA 66667 injection pulses are required. To deliver that many pulses an injector operating at 10 Hz would require 1.85 hours. To reduce the injection time to less than 5 min each injection pulse must deliver at least 23 bunches. Additional time and bunches are required if the injection process is less than 100% effective.

Solution 6.1. A uniform field B_y in the laboratory system transforms into a field $\left(E_x^*, E_y^*, E_z^*\right) = ([c]\,\beta\gamma B_y,\, 0,\, 0)$ and $\left(B_x^*, B_y^*, B_z^*\right) = (0,\, \gamma B_y,\, 0)$. The particle velocity is zero in it's own system and therefore the magnetic field B_y^* is ineffective. There is a nonzero electric field $E_x^* = [c]\,\beta\gamma B_y$ which deflects an electron in the negative x-direction just like the magnetic field in the laboratory system does. The gain in transverse momentum is $\Delta p_x^* = \int e\, E_x^*\,\mathrm{d}t^* = \int \frac{eE_x^*}{\beta c\gamma}\,\mathrm{d}s = \frac{eE_x^*}{\beta c\gamma}\ell_b$ where we set $\mathrm{d}t^* = \frac{\mathrm{d}t}{\gamma} = \frac{\mathrm{d}s}{\beta c\gamma}$, and ℓ_b is the length of the bending magnet. The deflection angle is then $\psi = \frac{\Delta p_x^*}{p_0^*} = [c]\frac{eB_y}{\beta E}\ell_b$, which is the same as (6.11) with (6.7) .

Solution 6.2. From the Lorentz force equation we get $\boldsymbol{E} = [c]\beta\boldsymbol{B}$. Solving for β, we get for our case $\beta = \frac{E}{[c]B} = 3.336\times10^{-3}$ or with $\gamma \approx 1 + \frac{1}{2}\beta^2$ the kinetic energy for force equivalence is $E_{\text{kin}} = 852.35$ eV. Magnetic fields are more effective for electrons with a kinetic energy of more than 852 eV. That equivalence point is much higher for protons and ions.

Solution 6.3. For the total ring we have $0.3\frac{B}{E}n_{\text{b}}\ell = 2\pi$ or $n_{\text{b}} = \frac{2\pi E}{0.3B\ell} = 26.18$. The number of magnets must be an integer and we set $n_{\text{b}} = 28$ (it is customary to use an even number of magnets, although not necessary), the field $B = 1.122$ T, the bending radius $\rho = 8.913$ m, and the deflection angle is $\psi = 360/28 = 12.86$ degrees per magnet.

Solution 6.4. To generate a focusing magnet, we require a magnetic field $B_y(y=0) = gx$, where g is a constant. This field can be derived from a potential $V = -gxy$. The surface of ferromagnetic pole is an equipotential surface and its cross section in the xy-plane is given by $xy = $ const. The shape of a pole is that of a hyperbola. The pole tip at the intersection of the 45^o-line from the magnet axis and the pole profile is at the coordinates $\left(R/\sqrt{2},\, R/\sqrt{2}\right)$. With this the equation for the pole shape becomes $xy = \frac{1}{2}R^2$ which is (6.16).

Solution 6.5. The transformation matrix for quadrupole and 5 m drift space is $M = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{5}{f} & 5 \\ -\frac{1}{f} & 1 \end{pmatrix}$. To focus a parallel ray to a focal point at 5 m, we require $x_{\text{F}} = \left(1-\frac{5}{f}\right)x_0 = 0$ or $f = \frac{1}{kl} = 5$ m. From this $k = 1.0\ \text{m}^{-2}$ and $g = \frac{1.5\cdot 1}{0.3} = 5.0$ T/m. The relation between excitation current and field gradient in a quadrupole can be derived in a similar way as done for a bending magnet and is given by $\mu_0 I_{\text{coil}} = \frac{1}{2}gR^2$. For the case on hand, the excitation current is $I_{\text{coil}} = 4973.6$ A·turns.

Solution 6.6. Starting from the waist in the center of the drift space the betatron function is $\beta(s) = \beta_{\text{w}} + \frac{s^2}{\beta_{\text{w}}}$. If we choose a very small value for $\beta_{\text{w}} = \beta(0)$ we get a very large value at $s = \frac{1}{2}L$. Similarly, if β_{w} is very large $\beta\left(\frac{1}{2}L\right)$ is large too. There must be a minimum for $\beta\left(\frac{1}{2}L\right)$. We calculate $\partial\beta/\partial\beta_{\text{w}} = 0$, solve for β_{w} and get for the optimum value $\beta_{\text{w}} = \frac{1}{2}L$. For this value at the waist we have the minimum variation of the betatron function or of the beam size along the drift space L. If $\sigma_{\text{w}} = \sqrt{\epsilon\beta_{\text{w}}}$ is the beam size at the waist then the beam size at the ends of the drift space is $\sigma = \sqrt{2}\sigma_{\text{w}}$. This is the same result as is known for a photon beam defining the Rayleigh length.

Solution 6.7. We form FODO cells which are each $2L = 8$ m long. That leaves between bending magnets 2 m of space for quadrupoles and drift spaces. For $\kappa = \sqrt{2}$ we get the minimum beam sizes along the FODO-cell, which is define as the minimum beam radius, or $R_{\text{min}} = \left.\sqrt{\sigma_x^2 + \sigma_y^2}\right|_{\text{min}}$. The horizontal and vertical betatron functions in the center of the QD-quadrupole are $\beta_x = 2.343$ m and $\beta_y = 13.657$ m. The focal length of the half-quadrupole is $f_{\mathrm{hq}} = \kappa L = 5.657$ m. Note, in FODO theory we use half quadrupoles. That means the focal length of the full quadrupole is $f_{\mathrm{q}} = \frac{1}{2} f_{\mathrm{hq}} = 2.8285$ m. Each half FODO-cell now has the following structure: $\frac{1}{2}$QF(0.5 m)–Drift(0.5 m)–Bend(2 m)–Drift(0.5 m)–$\frac{1}{2}$QD(0.5 m).

Solution 6.8. In the center of say a QF the optimum betatron function is given by $\beta_0 = L \frac{\kappa(\kappa+1)}{\sqrt{\kappa^2-1}}$, and $\alpha_0 = 0$. The transformation matrix from the center to the exit of the thin quadrupole is $\begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix}$ and the betatron functions at the quadrupole exit are therefore: $\beta = \beta_0$, $\alpha = -\frac{1}{f}\beta_0$ and $\gamma = \frac{1+\alpha^2}{\beta_0}$. These are the starting values for the drift space which spans the space between quadrupoles since the bending magnets are not included in the focusing scheme in this approximation. The expression for the betatron function between a QF and QD quadrupole is therefore $\beta(s) = \beta_0 - 2\alpha s + \gamma s^2$ and the phase advance $\Psi = \int_0^L \frac{\mathrm{d}s}{\beta(s)} = \sqrt{\kappa^2-1} \int_0^1 \frac{\mathrm{d}s}{\kappa(\kappa+1) - 2(\kappa+1)s + 2s^2} = -\arctan \frac{\kappa-1}{\sqrt{\kappa^2-1}} + \arctan \frac{\kappa+1}{\sqrt{\kappa^2-1}}$. For the optimum FODO-cell with $\kappa = \sqrt{2}$, the phase advance per half cell is $\Psi_{\frac{1}{2}} = 45^o$, and because of symmetry for the full cell $\Psi_{\mathrm{FODO}} = 90^o$.

Solution 7.1. In this case we do not expand the rf-voltage and keep $V_{\mathrm{rf}}(t) = V_{\mathrm{rf}} \sin \omega_{\mathrm{rf}} t$. We also ignore damping, because it is a very small effect. Equation (7.2) becomes then with (7.3)
$\ddot{\tau} + \frac{\eta_{\mathrm{c}} e V_{\mathrm{rf}}}{T_0 E_0} (\sin \psi_{\mathrm{s}} \cos \omega_{\mathrm{rf}} \tau + \cos \psi_{\mathrm{s}} \sin \omega_{\mathrm{rf}} \tau - \sin \psi_{\mathrm{s}}) = 0$, where $\eta_{\mathrm{c}} = \frac{1}{\gamma^2} - \alpha_{\mathrm{c}}$.
We multiply this equation with $\dot{\tau}$ and integrate to get
$\frac{1}{2}\dot{\tau}^2 + \frac{\eta_{\mathrm{c}} e V_{\mathrm{rf}}}{\omega_{\mathrm{rf}} T_0 E_0} (\sin \psi_{\mathrm{s}} \sin \omega_{\mathrm{rf}} \tau - \cos \omega_{\mathrm{rf}} \tau \cos \psi_{\mathrm{s}} - \tau \sin \psi_{\mathrm{s}})$ =const. For simplicity, we assume that $\psi_{\mathrm{s}} \to 0$, which is not true for storage rings but describes very well the situation in FELs, and get $\dot{\tau}^2 - \frac{2\eta_{\mathrm{c}} e V_{\mathrm{rf}}}{\omega_{\mathrm{rf}} T_0 E_0} \cos \omega_{\mathrm{rf}} \tau$ =const. Plotting $\dot{\tau}$ as a function of τ results in the phase space diagram as shown in Fig. 11.2.

Solution 7.2. We use the storage ring of Exercise 6.3 with an energy of 3 GeV and a bending radius of $\rho = 8.913$ m. This ring has 14 FODO cells, each 8 m long. To make it more realistic we assume that all insertion straight sections will occupy the same length such that the ring circumference is $C = 224.0$ m. The synchrotron radiation power is from (3.14) $\langle P_\gamma \rangle = 1075.9$ GeV/s and the horizontal and vertical damping times are then $\tau_{x,y} = 2.788$ ms. The synchrotron damping time is half that or $\tau_s = 1.394$ ms. The beam energy spread is from (7.32) $\left(\frac{\sigma_\varepsilon}{E}\right) = 0.0609$ %.

Solution 7.3. The probability to emit a photon of energy ε in a unit time is $\dot{n}(\varepsilon_{\mathrm{ph}}) = \frac{P_\gamma}{\varepsilon_{\mathrm{c}}^2} \frac{S(x)}{x}$. We are looking for the case $\varepsilon = \sigma_\varepsilon = \frac{E^2}{mc^2} \sqrt{\frac{55\hbar c}{64\sqrt{3} mc^2 J_s \rho}} = 10.9$ MeV. For $\varepsilon_{\mathrm{c}} = \frac{3}{2}\hbar c \frac{\gamma^3}{\rho} = 19166$ eV, the ratio $x = \frac{1}{\gamma}\sqrt{\frac{55 mc^2 \rho}{144\sqrt{3} J_s \hbar c}} = 227.54 \gg 1$ and $\frac{P_\gamma}{\varepsilon_{\mathrm{c}}^2} = 23826$ 1/eVs. The probability becomes with this $\dot{n}(\varepsilon_{\mathrm{ph}}) \approx$

1.86×10^{-96}! We may, without calculation conclude that no second photon of this energy will be emitted within a damping time. Energy is emitted in very small fractions of the electron energy.

Solution 7.4. From (9.167) we get the number of photons emitted per unit time $\dot{N}_{\mathrm{ph}} = \frac{15\sqrt{3}}{8}\frac{P_\gamma}{\varepsilon_{\mathrm{c}}} = 3.158 \times 10^6 \frac{\gamma}{\rho}$ and per radian $\dot{n} = 0.01063\gamma \approx \frac{\gamma}{100}$.

Solution 9.1. Integration of (9.76) over φ results in factors 2π and π for the two terms in the nominator, respectively and we have the integrals $2\pi \int_0^\pi \frac{\sin\theta}{(1-\beta\cos\theta)^3}\mathrm{d}\theta - \pi\left(1-\beta^2\right)\int_0^\pi \frac{\sin^3\theta}{(1-\beta\cos\theta)^5}\mathrm{d}\theta = \frac{4\pi}{(1-\beta^2)^2} - \frac{\frac{4}{3}\pi}{(1-\beta^2)^2}$ $= 4\pi\gamma^4\left(1-\frac{1}{3}\right) = 4\pi\gamma^4\frac{2}{3}$. With this, the radiation power is $P_{\mathrm{tot}} = \frac{2}{3}r_{\mathrm{c}}mc^2c\frac{\beta^4\gamma^4}{\rho^2}$, which is (9.59).

Solution 9.2. The vertical opening angle is $1/\gamma = 0.085$ mr and therefore all radiation will be accepted. The spectral photon flux into an opening angle of $\Delta\psi = 10$ mr is therefore given from (9.156) by $\dot{N}_{\mathrm{ph}} = C_\psi EI\frac{\Delta\omega}{\omega}S\left(\frac{\omega}{\omega_{\mathrm{c}}}\right)\Delta\psi$. With the critical photon energy is $\varepsilon_{\mathrm{c}} = 23.94$ keV the spectral photon flux from an ESRF bending magnet is $\dot{N}_{\mathrm{ph}} = 4.75 \times 10^{14}\, S\left(\frac{\varepsilon_{\mathrm{ph}}(\mathrm{keV})}{23.94}\right)$.

Solution 9.3. We use (9.106) and get with $\xi = \frac{1}{2}\frac{\omega}{\omega_{\mathrm{c}}}(1+\gamma^2\theta^2)^{3/2}$ for the $p\%$ point $\frac{\mathrm{d}^2W(10\%)}{\mathrm{d}\Omega\mathrm{d}\omega}/\frac{\mathrm{d}^2W}{\mathrm{d}\Omega\mathrm{d}\omega} = (1+\gamma^2\theta^2)^2\left[\frac{K_{2/3}^2(\xi)}{K_{2/3}^2(0)} + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\frac{K_{1/3}^2(\xi)}{K_{2/3}^2(0)}\right] = 0.1$. Solving for θ gives the angle at which the intensity has dropped to 10%. For low frequencies $\frac{\mathrm{d}^2W(10\%)}{\mathrm{d}\Omega\mathrm{d}\omega}/\frac{\mathrm{d}^2W}{\mathrm{d}\Omega\mathrm{d}\omega} \underset{\xi\to 0}{\longrightarrow} 1 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\frac{\Gamma^4(1/3)}{2^{8/3}\Gamma^4(2/3)}\left(\frac{\omega}{\omega_{\mathrm{c}}}\right)^{4/3} = p$, and for large arguments $\frac{\mathrm{d}^2W(10\%)}{\mathrm{d}\Omega\mathrm{d}\omega}/\frac{\mathrm{d}^2W}{\mathrm{d}\Omega\mathrm{d}\omega} \underset{\xi\to\infty}{\longrightarrow} \frac{\exp\left(\frac{\omega}{\omega_{\mathrm{c}}}\right)}{\exp\left[\frac{\omega}{\omega_{\mathrm{c}}}(1+\gamma^2\theta^2)^{3/2}\right]}\frac{1+2\gamma^2\theta^2}{\sqrt{1+\gamma^2\theta^2}} = p$. All expressions have to be evaluated numerically. The angle at which the total radiation intensity has dropped to 10% is from (9.115) given by $\frac{\mathrm{d}W(10\%)}{\mathrm{d}\Omega}/\frac{\mathrm{d}W}{\mathrm{d}\Omega} = \frac{1}{(1+\gamma^2\theta^2)^{5/2}}\left(1+\frac{5}{7}\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\right) = p$, which can be solved by $\gamma\theta = 1.390$ for $p = 10\%$.

Solution 10.1. From (10.20) the amplitude of the oscillatory motion in an undulator is $a_\perp = \frac{\lambda_{\mathrm{p}}K}{2\pi\gamma} = \frac{0.05\cdot 1}{2\pi(7/0.000511)} = 0.581$ μm. The longitudinal oscillation amplitude is from (10.19*b*) $a_\parallel = \frac{K^2}{8\gamma^2 k_{\mathrm{p}}} = 0.053$ Å. Both amplitudes are very small, yet are responsible for the high intensities of radiation.

Solution 10.2. The focal length for a single pole end is given by (10.23) $\frac{1}{f_{1,y}} = \frac{\pi^2}{2\gamma^2}\frac{K^2}{\lambda_{\mathrm{p}}} = 2.58 \times 10^{-5}$ m^{-1} and for the whole undulator $\frac{1}{f_y} = \frac{\pi^2}{2\gamma^2}\frac{K^2}{\lambda_{\mathrm{p}}}2N = 0.00258$ m^{-1} or $f_y = 387.60$ m. This focal length is very long compared to the focal lengths of the ring quadrupole, which are of the order of the distance between quadrupoles. Typically, the focal length of any insertion should be more than about 50 m to be negligible. The wiggler magnet with $K = 5$, on the other hand, produces a focal length of $f_y = 15.50$ m which is too strong to be ignored and must be compensated. The difference comes from the fact that its the deflection angle which is responsible for focusing and $\frac{1}{f_y} \propto \theta^2$. Focusing occurs only in the nondeflecting plane and $\frac{1}{f_x} = 0$.

Solution 10.3. This result appears nonphysical, yet it is correct, but requires some interpretation. The number of photons emitted into the forward cone is constant. Note, that the forward cone angle decreases with increasing number of periods. The constant number of photons is emitted into a smaller and smaller cone. Outside this forward cone there is still much radiation and integration of all radiation would give the more intuitive result that the total radiation power increases with number of undulator periods.

Solution 10.4. To solve this problem , we do not rely on exact calculations, but are satisfied with the precision of reading graphs in Chapter D. We also use iterations to get the solution we want. The fundamental flux drops below 10% for $K < 0.25$, and we use this value to get 15 keV radiation. From the definition of the fundamental photon energy we get the periodlength $\lambda_\mathrm{p} = 3.0$ cm. To generate 4 keV radiation we need to change K enough to raise the factor $\left(1 + \frac{1}{2}K^2\right)$ from a low value of 1.031 by a factor of 15/4 to a value of 3.87 or to a high of $K = 2.4$, which corresponds to a field of $B = 0.857$ T. Unfortunately, that field requires a gap of $g = 8.1$ mm which is less than allowed. We have to increase the periodlength to say $\lambda_\mathrm{p} = 3.5$ cm, which gives a maximum photon energy for $K = 0.25$ of $\varepsilon_\mathrm{ph} = 12.9$ keV. We plan to use the 3$^\mathrm{rd}$ harmonic to reach 15 keV. To reach $\varepsilon_\mathrm{ph} = 4$ keV, we need $K = 2.16$, a field of $B = 0.661$ T, which requires an allowable gap of $g = 11.7$ mm. We use the 3$^\mathrm{rd}$ harmonic to reach $\varepsilon_\mathrm{ph} = 15$ keV at $K = 1.82$. With this result we may even extend the spectral range on both ends.

Solution 10.5. In the electron system the wavelength of the laser beam is Lorentz contracted by a factor of $\frac{1}{2\gamma}$, where the factor of two is due to the fact that the relative velocity between both beams is $2c$. The wavelength in the laboratory system is therefore $\lambda = \frac{\lambda_\mathrm{L}}{4\gamma^2}$, since $K \ll 1$ for the laser field.

Solution 10.6. The maximum transverse oscillation amplitude is 4.57 μm and the transverse velocity in units of c is just equal to the maximum deflection angle $\beta_\perp = \theta = K/\gamma = 0.38$ mr. The transverse relativistic factor $\gamma_\perp \approx 1 + 7.22 \times 10^{-8}$, indeed very small, yet enough to start generating relativistic perturbations in the transverse particle motion.

Solution 10.7. The fundamental wavelength is given by the expression $\lambda = \frac{\lambda_\mathrm{p}}{2\gamma^2}\left(1 + \frac{1}{2}K^2 + \gamma^2\vartheta^2\right)$ and for $\vartheta = 0$, we have the fundamental wavelength $\lambda = 10.88$ Å. The natural bandwidth is $1/N_\mathrm{p} = 1\%$ and we look therefore for an angle $\hat{\vartheta}$ such that the wavelength has increased by no more than 9%, or $\frac{\gamma^2\hat{\vartheta}^2}{1+\frac{1}{2}K^2} = 0.09$ and solving for $\hat{\vartheta}$, we get $\hat{\vartheta} = \pm 62.6$ μrad.

Solution 11.1. We may solve this problem two ways. First, we use the average drift velocity $\bar{\beta} = \beta\left(1 - \frac{K^2}{4\gamma^2}\right)$ and calculate the time it takes the electron to travel along one period, $t_\mathrm{e} = \frac{\lambda_\mathrm{p}}{c\bar{\beta}} \approx \frac{\lambda_\mathrm{p}}{c\beta}\left(1 + \frac{K^2}{4\gamma^2}\right)$. During that same time the photon travels a distance $s_\gamma = ct_\mathrm{e} = \frac{\lambda_\mathrm{p}}{\beta}\left(1 + \frac{K^2}{4\gamma^2}\right)$ and the difference is $\delta s = s_\gamma - \lambda_\mathrm{p} = \frac{\lambda_\mathrm{p}}{\beta}\left(1 + \frac{K^2}{4\gamma^2}\right) - \lambda_\mathrm{p} = \lambda_\mathrm{p}\left(\frac{1}{\beta} - 1\right) + \frac{\lambda_\mathrm{p}}{\beta}\frac{K^2}{4\gamma^2} \approx \frac{\lambda_\mathrm{p}}{2\gamma^2}\left(1 + \frac{1}{2}K^2\right)$ which is just equal to the fundamental radiation wave length. We may also integrate the path length along the sinusoidal trajectory and get for one quarter period $s_\mathrm{e} = \frac{\lambda_\mathrm{p}}{2\pi}\int_0^{\pi/2}\sqrt{1 + \theta^2\cos^2 x}\,dx = \frac{\lambda_\mathrm{p}}{2\pi}\,\mathrm{EllipticE}\left(\sqrt{-\theta^2}\right)$ an elliptical function. Since the argument will always be very small we may expand $\mathrm{EllipticE}\left(\sqrt{-\theta^2}\right) \approx \frac{\pi}{2} + 0.393\,\theta^2$. The electron travel time for one period is then $t_\mathrm{e} = 4\frac{\lambda_\mathrm{p}}{2\pi}\frac{1}{c\beta}\left(\frac{\pi}{2} + 0.393\,\theta^2\right)$ and the path length difference is $\delta s = ct_\mathrm{e} - \lambda_\mathrm{p} = \lambda_\mathrm{p}\frac{1}{\beta}\left(1 + \frac{0.393\,\theta^2}{\pi/2}\right) - \lambda_\mathrm{p} \approx \frac{\lambda_\mathrm{p}}{2\gamma^2}\left(1 + \underbrace{\frac{8\cdot 0.393}{\pi}}_{\approx 1}\frac{1}{2}K^2\right)$ which is again the wavelength of the fundamental radiation.

B. Mathematical Constants and Formulas

B.1 Constants

$$\pi = 3.141592653589793238 \tag{B.1}$$
$$\mathrm{e} = 2.718281828459045235 \tag{B.2}$$

$$\Gamma(1/3) = 2.6789385 \tag{B.3}$$
$$\Gamma(2/3) = 1.351179 \tag{B.4}$$

B.2 Series Expansions

For $x \ll 1$

$$\mathrm{e}^x \approx 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... \tag{B.5}$$
$$\sin x \approx \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \tag{B.6}$$
$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \tag{B.7}$$
$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4 - ... \tag{B.8}$$
$$\sqrt{1+x} \approx 1 + \tfrac{1}{2}x - \tfrac{1\cdot 1}{2\cdot 4}x^2 + \tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 + \tfrac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + ... \tag{B.9}$$

B.3 Multiple Vector Products

In a vector product $\boldsymbol{a}\times\boldsymbol{b}{=}\boldsymbol{c}$, vectors $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$ form a right handed orthogonal system.

$\boldsymbol{a} = (a_x, a_y, a_z)$;

$\boldsymbol{a}\times\boldsymbol{b} = (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x)$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\boldsymbol{b}) \tag{B.10}$$

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a}\boldsymbol{c}) - \boldsymbol{a}(\boldsymbol{b}\boldsymbol{c}) \tag{B.11}$$

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = \boldsymbol{c}\,[(\boldsymbol{a}\times\boldsymbol{b})\,\boldsymbol{d}] - \boldsymbol{d}\,[(\boldsymbol{a}\times\boldsymbol{b})\,\boldsymbol{c}] \tag{B.12}$$

$$\boldsymbol{a}(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}(\boldsymbol{a}\times\boldsymbol{b}) \tag{B.13}$$

$$(\boldsymbol{a}\times\boldsymbol{b})(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\boldsymbol{c})(\boldsymbol{b}\boldsymbol{d}) - (\boldsymbol{b}\boldsymbol{c})(\boldsymbol{a}\boldsymbol{d}) \tag{B.14}$$

B.4 Differential Vector Expressions

$\boldsymbol{a}, \boldsymbol{b}$ vectors; ψ scalar;

$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$;

$\Delta = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)$

$$\boldsymbol{\nabla}(\boldsymbol{a}\psi) = \psi\,\boldsymbol{\nabla}\boldsymbol{a} + \boldsymbol{a}\boldsymbol{\nabla}\psi \tag{B.15}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{a}\psi) = \psi\,(\boldsymbol{\nabla}\times\boldsymbol{a}) - (\boldsymbol{a}\times\boldsymbol{\nabla}\psi) \tag{B.16}$$

$$\boldsymbol{\nabla}(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\,(\boldsymbol{\nabla}\times\boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{\nabla}\times\boldsymbol{b}) \tag{B.17}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{b}\boldsymbol{\nabla})\,\boldsymbol{a} - (\boldsymbol{a}\boldsymbol{\nabla})\,\boldsymbol{b} + \boldsymbol{a}(\nabla\boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{\nabla}\boldsymbol{a}) \tag{B.18}$$

$$\boldsymbol{\nabla}\,(\boldsymbol{a}\boldsymbol{b}) = (\boldsymbol{b}\boldsymbol{\nabla})\,\boldsymbol{a} + (\boldsymbol{a}\boldsymbol{\nabla})\,\boldsymbol{b} + \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) \tag{B.19}$$

$$\boldsymbol{\nabla}\,(\boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}^2\psi = \Delta\psi \tag{B.20}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) = 0 \tag{B.21}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0 \tag{B.22}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\times\boldsymbol{a}) = 0 \tag{B.23}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) = \boldsymbol{\nabla}\,(\boldsymbol{\nabla}\boldsymbol{a}) - \Delta\boldsymbol{a} \tag{B.24}$$

if $\boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{0}$ a scalar function ψ exists with $\boldsymbol{a} = \boldsymbol{\nabla}\psi$ (B.25)

if $\boldsymbol{\nabla}\boldsymbol{a} = \boldsymbol{0}$ a vector function $\boldsymbol{b}$ exists with $\boldsymbol{a} = \boldsymbol{\nabla}\times\boldsymbol{b}$ (B.26)

B.5 Theorems

$$\int_V \boldsymbol{\nabla}\psi \, \mathrm{d}V = \oint (\psi \boldsymbol{n}) \, \mathrm{d}a \,. \tag{B.27}$$

Gauss's theorem

$$\int_V \boldsymbol{\nabla} \boldsymbol{a} \mathrm{d}V = \oint (\boldsymbol{a}\boldsymbol{n}) \, \mathrm{d}a \,. \tag{B.28}$$

Stokes' theorem

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{a})_{\mathrm{n}} \, \mathrm{d}\boldsymbol{a} = \oint \boldsymbol{a} \, \mathrm{d}\boldsymbol{s} \,. \tag{B.29}$$

Fourier transform

$$f(\omega) = \int_{-\infty}^{\infty} f(t) \, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \,, \tag{B.30}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) \, \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega \,. \tag{B.31}$$

Parseval's theorem

$$\int_{-\infty}^{\infty} f^2(t) \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^2(\omega) \, \mathrm{d}\omega \,. \tag{B.32}$$

B.6 Coordinate Systems

Cartesian coordinates (x, y, z)

$$\boldsymbol{\nabla}\phi = \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right] \tag{B.33}$$

$$\boldsymbol{\nabla} \boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \tag{B.34}$$

$$\boldsymbol{\nabla} \times \boldsymbol{a} = \left[\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}, \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right] \tag{B.35}$$

$$\Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{B.36}$$

Cylindrical coordinates (ρ, φ, ζ)

$$\boldsymbol{\nabla}\phi = \left[\frac{\partial\phi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}, \frac{\partial\phi}{\partial\zeta}\right] \tag{B.37}$$

$$\boldsymbol{\nabla}\boldsymbol{a} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho a_\rho\right) + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi} + \frac{\partial a_\zeta}{\partial\zeta} \tag{B.38}$$

$$\boldsymbol{\nabla}\times\boldsymbol{a} = \left[\frac{1}{\rho}\frac{\partial a_\zeta}{\partial\varphi} - \frac{\partial a_\varphi}{\partial\zeta}, \frac{\partial a_\rho}{\partial\zeta} - \frac{\partial a_\zeta}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho a_\varphi\right) - \frac{1}{\rho}\frac{\partial a_\rho}{\partial\varphi}\right] \tag{B.39}$$

$$\Delta\phi = \frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial\zeta^2} \tag{B.40}$$

Transformation to cylindrical coordinates (ρ, φ, ζ)

$$\begin{aligned}(x, y, z) &= (\rho\cos\varphi, \rho\sin\varphi, \zeta)\\ \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}\zeta^2\\ \mathrm{d}V &= \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\zeta\end{aligned} \tag{B.41}$$

Polar coordinates (r, φ, θ)

$$\boldsymbol{\nabla}\phi = \left[\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\varphi}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\theta},\right] \tag{B.42}$$

$$\nabla\boldsymbol{a} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\sin\varphi\, a_\varphi\right) + \frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial\theta} \tag{B.43}$$

$$\boldsymbol{\nabla}\times\boldsymbol{a} = \left[\begin{array}{c}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\, a_\zeta)}{\partial\varphi} - \frac{\partial a_\varphi}{\partial\theta}\right),\\ \frac{1}{r\sin\theta}\left(\frac{\partial a_r}{\partial\theta} - \sin\theta\,\frac{\partial(r a_\theta)}{\partial r}\right),\\ \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r a_\varphi\right) - \frac{\partial a_r}{\partial\varphi}\right).\end{array}\right] \tag{B.44}$$

$$\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) \tag{B.45}$$

Transformation to polar coordinates (r, φ, θ)

$$\begin{aligned}(x, y, z) &= (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)\\ \mathrm{d}s^2 &= \mathrm{d}r^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 + r^2\mathrm{d}\theta^2\\ \mathrm{d}V &= r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta\end{aligned} \tag{B.46}$$

B.7 Gaussian Distribution

1-dim Gaussian distribution (σ: standard deviation) (Fig. 13.1)

$$\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}\frac{x^2}{\sigma^2}} \tag{B.47}$$

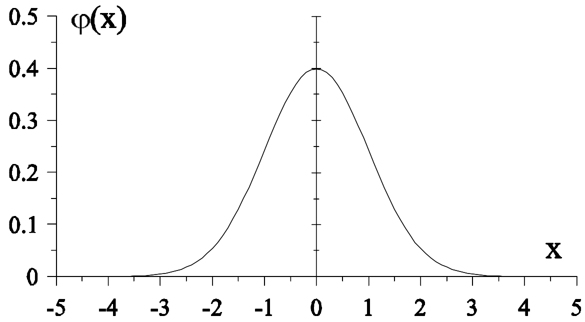

Fig. B.1. Gaussian error function

Table B.1. Gaussian error functions $\varphi(x)$ and $\Phi(x)$

x	$\varphi(x)$	$\Phi(x)$	x	$\varphi(x)$	$\Phi(x)$	x	$\varphi(x)$	$\Phi(x)$
0.0	0.3989	0.0000	1.0	0.2420	0.6827	2.0	0.0540	0.9545
0.1	0.3970	0.0797	1.1	0.2179	0.7287	2.25	0.0317	0.9756
0.2	0.3910	0.1585	1.2	0.1942	0.7699	2.5	0.0175	0.9876
0.3	0.3814	0.2358	1.3	0.1714	0.8064	2.75	0.0091	0.9940
0.4	0.3683	0.3108	1.4	0.1497	0.8385	3.0	0.0044	0.9973
0.5	0.3521	0.3830	1.5	0.1295	0.8664	3.5	8.73e-4	0.9995
0.6	0.3332	0.4514	1.6	0.1109	0.8904	4.0	1.34e-4	0.9999
0.7	0.3123	0.5161	1.7	0.0941	0.9109	5.0	1.49e-6	1-6e-7
0.8	0.2897	0.5762	1.8	0.0790	0.9281	7.5	2.4e-13	≈ 1
0.9	0.2661	0.6319	1.9	0.0656	0.9426	10.0	7.7e-23	≈ 1

Integral of Gaussian error function (Fig. 13.2)

$$\Phi(x/\sigma) = 2\int_0^{x/\sigma} \varphi(\bar{x}/\sigma)\, \mathrm{d}\frac{\bar{x}}{\sigma} \tag{B.48}$$

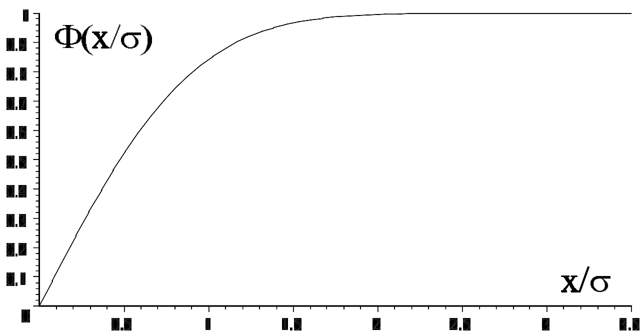

Fig. B.2. Integral of Gaussian error function

2-dim Gaussian distribution

$$\varphi(r) = \frac{1}{2\pi\,\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right] \tag{B.49}$$

2-dim Gaussian distribution (round) $(\sigma_x = \sigma_y = \sigma_r)$

$$\varphi(r) = \frac{1}{2\pi\sigma_r^2} \mathrm{e}^{-\frac{1}{2}\frac{r^2}{\sigma_r^2}} \tag{B.50}$$

B.8 Miscelaneous Mathematical Formulas

Element of solid angle (θ polar angle, ψ azimuthal angle)

$$\mathrm{d}\,\Omega = \sin\theta\,\mathrm{d}\psi\,\mathrm{d}\,\theta \tag{B.51}$$

Integrated solid angle within polar angle θ

$$\Delta\Omega = 2\pi\,(1 - \cos\theta) \tag{B.52}$$

Modified Bessel's functions (Fig. 13.3)

$$K_{1/3}(\xi) = \sqrt{3}\int_0^\infty \cos\left[\frac{1}{2}\,\xi(3x + x^3)\right]\,\mathrm{d}x\,, \tag{B.53}$$

$$K_{2/3}(\xi) = \sqrt{3}\int_0^\infty \sin\left[\frac{1}{2}\,\xi(3x + x^3)\right]\,\mathrm{d}x\,. \tag{B.54}$$

for small arguments $\xi \to 0$

$$K_{1/3}(\xi \to 0) \approx \frac{\Gamma^2\,(1/3)}{2^{2/3}}\left(\frac{\omega}{\omega_c}\right)^{-2/3}\frac{1}{1+\gamma^2\theta^2}\,, \tag{B.55}$$

$$K_{2/3}(\xi \to 0) \approx 2^{2/3}\Gamma^2\,(2/3)\left(\frac{\omega}{\omega_c}\right)^{-4/3}\frac{1}{\left(1+\gamma^2\theta^2\right)^2}\,, \tag{B.56}$$

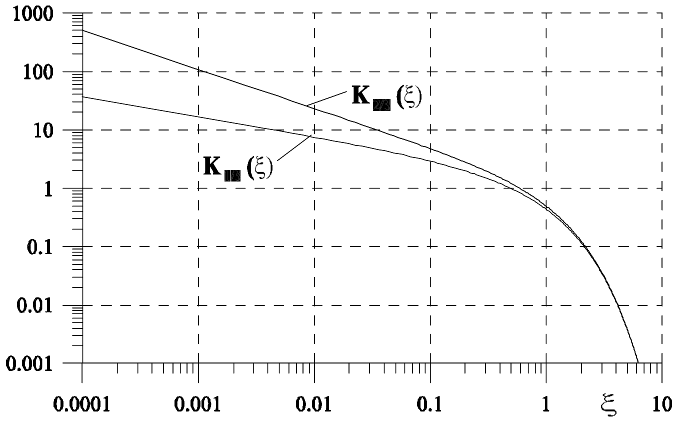

Fig. B.3. Modified Bessel's functions $\mathrm{K}_{1/3}(\xi)$ and $\mathrm{K}_{1/3}(\xi)$

and for large arguments $\xi \to \infty$

$$K_{1/3}^2(\xi \to \infty) \approx \frac{\pi}{2\xi\,\mathrm{e}^{2\xi}}\,, \tag{B.57}$$

$$K_{2/3}^2(\xi \to \infty) \approx \frac{\pi}{2\xi\,\mathrm{e}^{2\xi}}\,. \tag{B.58}$$

For $\nu = \frac{\omega}{\omega_{\mathrm{L}}} \gg 1$, where the Larmor frequency $\omega_{\mathrm{L}} = c/\rho$,

$$K_{1/3}(\xi) = \frac{\sqrt{3}\pi}{\sqrt{1-\beta^2\cos^2\theta}} J_\nu(\nu\beta\cos\theta)\,, \tag{B.59}$$

$$K_{2/3}(\xi) = \frac{\sqrt{3}\pi}{\sqrt{\left(1-\beta^2\cos^2\theta\right)}} J\,'_\nu(\nu\beta\cos\theta)\,. \tag{B.60}$$

Airy's **functions**

$$\mathcal{A}\mathrm{i}\,(z) = \frac{\sqrt{z}}{\sqrt{3}\pi} K_{1/3}(\xi)\,, \tag{B.61}$$

$$\mathcal{A}'\mathrm{i}\,(z) = -\frac{z}{\sqrt{3}\pi} K_{2/3}(\xi)\,. \tag{B.62}$$

C. Physical Formulas and Parameters

C.1 Constants

velocity of light in vacuum	c	$= 2.99792458 \times 10^{8}$	m / s
electric charge unit	e	$= 1.60217733 \times 10^{-19}$	C
	e^2	$= 14.399652$	eV Å
electron rest energy	$m_e c^2$	$= 0.5110034$	MeV
fine structure constant	α	$= 7.29735308 \times 10^{-3}$	
		$= 1/137.04$	
Avogadro's number	A	$= 6.0221367 \times 10^{23}$	1/ mol
molar volume at STP		22.41410×10^{-3}	m^3/mol
atomic mass unit	amu	$= 931.49432$	MeV
classical electron radius	r_c	$= 2.81794092 \times 10^{-15}$	m
proton/electron mass ratio	m_p/m_e	$= 1836.2$	
Planck's constant:	h	$= 6.6260755 \times 10^{-34}$	J s
		$= 4.1356692 \times 10^{-15}$	eV s
Planck's constant	$\hbar$	$= 1.05457266 \times 10^{-34}$	J s
		$= 6.5821220 \times 10^{-16}$	eV s
	$\hbar c$	$= 197.327053$	MeV s
electron Compton wavelength	λ_C	$= 2.42631058 \times 10^{-12}$	m
wavelength for 1eV	$\hbar c/e$	$= 12398.424$	Å
el.cyclotron frequency/field	ω_{Cy}/B	$= e/m_e$	
		$= 1.75881962 \times 10^{11}$	rad/(s T)
Thomson cross section	σ_T	$= 0.66524616 \times 10^{-28}$	m^2
Boltzmann constant	k	$= 1.3806568 \times 10^{-23}$	J / K
Stephan-Boltzmann constant	σ	$= 5.67051 \times 10^{-8}$	W /(m^2 K^4)
Permittivity of vacuum	ϵ_0	$= 8.854187817 \times 10^{-12}$	C/(V m)
Permeability of vacuum	μ_0	$= 1.2566370614 \times 10^{-6}$	Vs/(A m)

C.2 Unit Conversion

Numerical conversion factors:

Table C.1. Numerical conversion factors

quantity	label	replace cgs units	by SI units
voltage	U	1 esu	300 V
electric field	E	1 esu	$3\ 10^4$ V/cm
current	I	1 esu	$10\ c = 2.9979\ 10^9$ A
charge	q	1 esu	$(10c)^{-1} = 3.3356\ 10^{-10}$ C
resistance	R	1 s/cm	$8.9876\ 10^{11}\ \Omega$
capacitance	C	1 cm	$(1/8.9876)\ 10^{-11}$ F
inductance	L	1 cm	$1\ 10^9$ Hy
magnetic induction	B	1 Gauss	$3\ 10^{-4}$ Tesla
magnetic field	H	1 Oersted	$1000/4\pi = 79.577$ A/m
force	f	1 dyn	10^{-5} N
energy	E	1 erg	10^{-7} J

Equation conversion factors:

Table C.2. Equation conversion factors

variable	replace cgs variable	by SI variable
potential,voltage	$V_{\rm cgs}$	$\sqrt{4\pi\epsilon_0}\, V_{\rm MKS}$
electric field	$E_{\rm cgs}$	$\sqrt{4\pi\epsilon_0}\, E_{\rm MKS}$
current, current density	$I_{cgs}, j_{\rm cgs}$	$1/\sqrt{4\pi\epsilon_0}\, I_{\rm MKS}, j_{\rm MKS}$
charge, charge density	q, ρ	$1/\sqrt{4\pi\epsilon_0}\, q_{\rm MKS}, \rho_{\rm MKS}$
resistance	$R_{\rm cgs}$	$\sqrt{4\pi\epsilon_0}\, R_{\rm MKS}$
capacitance	$C_{\rm cgs}$	$1/\sqrt{4\pi\epsilon_0}\, C_{\rm MKS}$
inductance	$L_{\rm cgs}$	$\sqrt{4\pi\epsilon_0}\, L_{\rm MKS}$
magnetic induction	$B_{\rm cgs}$	$\sqrt{4\pi/\mu_0}\, B_{\rm MKS}$

Formulas are written for use of either unit. Include factors in square brackets [...] for MKS-units and omit those factors using cgs-units:

C.3 Relations of Fundamental Parameters

fine structure constant	α	$= \frac{e^2}{[4\pi\epsilon_0]\,\hbar c}$
classical electron radius	r_c	$= \frac{e^2}{[4\pi\epsilon_0]\,m_e c^2}$
electron Compton wavelength	λ_C	$= \frac{2\pi\hbar c}{m_e c^2}$

C.4 Energy Conversion

Table C.3. Energy conversion table

	calories [cal]	Joule [J]	eVolt [eV]	wavenumber [1/cm]	degKelvin [°K]
1 cal	1	4.186	$2.6127\ 10^{19}$	$2.1073\ 10^{23}$	$3.0319\ 10^{23}$
1 J	0.23889	1	$6.2415\ 10^{18}$	$5.0342\ 10^{22}$	$7.2429\ 10^{22}$
1 eV	$3.8274\ 10^{-20}$	$1.6022\ 10^{-19}$	1	8065.8	11604
1/cm	$4.7453\ 10^{-24}$	$1.9864\ 10^{-23}$	$1.2398\ 10^{-4}$	1	1.4387
1 °K	$3.2984\ 10^{-24}$	$1.3807\ 10^{-23}$	$8.6176\ 10^{-5}$	0.69507	1

C.5 Maxwell's Equations

$$\nabla \boldsymbol{E} = \frac{4\pi}{[4\pi\epsilon_0]\,\epsilon_r}\rho\,, \tag{C.1}$$

$$\nabla \boldsymbol{B} = 0\,, \tag{C.2}$$

$$\nabla \times \boldsymbol{E} = -\frac{[c]}{c}\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{C.3}$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\left[\frac{c}{4\pi}\right][\mu_0]\mu_r\,\rho\boldsymbol{v} + \frac{[c]}{c}[\epsilon_0\mu_0]\epsilon_r\,\mu_r\frac{\partial \boldsymbol{E}}{\partial t}\,. \tag{C.4}$$

C.5.1 Lorentz Force

$$\boldsymbol{F} = q\boldsymbol{E} + [c]\,\frac{q}{c}\,[\boldsymbol{v} \times \boldsymbol{B}] \tag{C.5}$$

C.6 Wave and Field Equations

Definition of potentials

$$\text{vector potential } \boldsymbol{A}: \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{C.6}$$

$$\text{scalar potential } \varphi: \quad \boldsymbol{E} = -\frac{[c]}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\varphi, \tag{C.7}$$

Wave equations in vacuum

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \frac{4\pi}{[4\pi\epsilon_0]}\rho\,\boldsymbol{\beta} \tag{C.8}$$

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} = -\frac{4\pi}{[4\pi\epsilon_0]}\,\rho \tag{C.9}$$

Vector and scalar potential in vacuum

$$\boldsymbol{A}(t) = \frac{1}{[4\pi c\epsilon_0]}\frac{1}{c}\int \left.\frac{\boldsymbol{v}\rho(x,y,z)}{R}\right|_{t_{\mathrm{ret}}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{C.10}$$

$$\varphi(t) = \frac{1}{[4\pi c\epsilon_0]}\frac{1}{c}\int \left.\frac{\rho(x,y,z)}{R}\right|_{t_{\mathrm{ret}}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{C.11}$$

Vector and scalar potential for a point charge q in vacuum

$$\boldsymbol{A}(P,t) = \frac{1}{[4\pi c\epsilon_0]}\left.\frac{q}{R}\frac{\boldsymbol{\beta}}{1+\boldsymbol{n}\boldsymbol{\beta}}\right|_{t_{\mathrm{ret}}} \tag{C.12}$$

$$\boldsymbol{\varphi}(P,t) = \frac{1}{[4\pi\epsilon_0]}\left.\frac{q}{R}\frac{1}{1+\boldsymbol{n}\boldsymbol{\beta}}\right|_{t_{\mathrm{ret}}} \tag{C.13}$$

Radiation field in vacuum

$$\boldsymbol{E}(t) = \frac{1}{[4\pi\epsilon_0]}\frac{q}{cr^3}\left.\left\{\boldsymbol{R}\times\left[(\boldsymbol{R}+\beta\boldsymbol{R})\times\dot{\boldsymbol{\beta}}\right]\right\}\right|_{t_{\mathrm{ret}}} \tag{C.14}$$

$$\boldsymbol{B}(t) = \frac{1}{c}\left[\boldsymbol{E}\times\boldsymbol{n}\right]_{t_{\mathrm{ret}}} \tag{C.15}$$

C.7 Relativistic Relations

Quantities x^* etc. are taken in the particle system $\mathcal{L}^*$,while quantities x etc. refer to the laboratory system $\mathcal{L}$. The particle system $\mathcal{L}^*$ is assumed to move at the velocity β along the z-axis with respect to the laboratory system $\mathcal{L}$.

Lorentz transformation of coordinates

$$\begin{pmatrix} x^* \\ y^* \\ z^* \\ ct^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta\gamma \\ 0 & 0 & -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}. \tag{C.16}$$

Lorentz transformation of frequencies (relativistic Doppler effect)

$$\omega = \omega^* \gamma \left(1 + \beta n_z^*\right) \tag{C.17}$$

Lorentz transformation of angles (collimation)

$$\theta \approx \frac{\sin\theta^*}{\gamma(1 + \beta\cos\theta^*)} \,. \tag{C.18}$$

C.8 Four-Vectors

Properties of 4-vectors are used in this text to transform physical phenomena from one inertial system to another.

Space-time 4-vector

$$\tilde{s} = (x, y, z, \mathrm{i}ct) \,, \tag{C.19}$$

World time

$$\tau = \sqrt{-\tilde{s}^2}. \tag{C.20}$$

length of a 4-vector is Lorentz invariant
any product of two 4-vectors is Lorentz invariant

Lorentz transformation of time. From $(C.20)$

$$\begin{aligned} c\mathrm{d}\tau &= \sqrt{c^2\left(\mathrm{d}t\right)^2 - \left(\mathrm{d}x\right)^2 - \left(\mathrm{d}y\right)^2 - \left(\mathrm{d}z\right)^2} \\ &= \sqrt{c^2 - \left(v_x^2 + v_y^2 + v_z^2\right)}\mathrm{d}t \\ &= \sqrt{c^2 - v^2}\mathrm{d}t = \sqrt{1 - \beta^2}c\mathrm{d}t \end{aligned}$$

or

$$\mathrm{d}\tau = \frac{1}{\gamma}\,\mathrm{d}t \,. \tag{C.21}$$

Velocity 4-vector

$$\tilde{v} = \frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}\tilde{s}}{\mathrm{d}t} = \gamma\left(\dot{x}, \dot{y}, \dot{z}, \mathrm{i}c\right) \,. \tag{C.22}$$

4-acceleration

$$\tilde{a} = \frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\frac{\mathrm{d}\tilde{s}}{\mathrm{d}t}\right) \tag{C.23}$$

and with $(C.21, C.22)$

$$\tilde{a} = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma \frac{\mathrm{d}\tilde{s}}{\mathrm{d}t}\right). \tag{C.24}$$

4-acceleration $\tilde{a} = (\tilde{a}_x, \tilde{a}_y, \tilde{a}_z, \mathrm{i}\,\tilde{a}_t)$ in component form

$$\tilde{a}_x = \gamma^2 a_x + \gamma^4 \beta_x \left(\boldsymbol{\beta} \cdot \boldsymbol{a}\right) , \tag{C.25}$$

where $\boldsymbol{a}$ is the ordinary acceleration.

Square of the 4-acceleration

$$\tilde{a}^2 = \gamma^6 \left\{\boldsymbol{a}^2 - [\boldsymbol{\beta} \times \boldsymbol{a}]^2\right\} = \tilde{a}^{*2}. \tag{C.26}$$

in particle system $\beta = 0, \gamma = 1$ and therefore

$$\tilde{a}^{*2} = a^{*2}. \tag{C.27}$$

D. Electromagnetic Radiation

Notation (All variables are in SI units unless otherwise noted)

$E(GeV)$	particle energy	B	magnetic field
U_0	energy loss/turn	α	fine structure constant
e	unit of el. charge	ρ	bending radius
P_γ	synchrotron rad. power	$f_{\rm rev}$	revolution frequency
I	beam current	$n_{\rm b}$	number of bunches
N	number of circ. electrons	$N_{\rm b}$	electrons/bunch
$\dot{N}_{\rm ph}$	photon flux		
ω	photon frequency	ε	photon energy
$\omega_{\rm c}$	crit. photon frequency	$\varepsilon_{\rm c}$	crit.photon energy
$\Delta\omega/\omega$	band width		

D.1 Radiation Constants

$$C_\gamma = \frac{4\pi}{3}\frac{r_{\rm c}}{(mc^2)^3} = 8.8460\ 10^{-5}\ \frac{\rm m}{\rm GeV^3}\,, \tag{D.1}$$

$$C_P = \frac{2}{3}\frac{r_{\rm c}\,c^3}{(mc^2)^3} = 379.35\ \frac{1}{\rm s\ T^2\ GeV}\,, \tag{D.2}$$

$$C_\omega = \frac{2}{3}\frac{\hbar c}{(mc^2)^3} = 2.2182\,\frac{\rm m\,GeV}{\rm s} \tag{D.3}$$

$$C_{\rm u} = \frac{4\pi^2\,r_{\rm c}}{3\,mc^2} = 7.2567\times 10^{-20}\ \frac{\rm m}{\rm eV} \tag{D.4}$$

$$C_\Omega = \frac{3\alpha}{4\pi^2 e(mc^2)^2} = 1.3273\ 10^{22}\ \frac{\rm photons}{\rm s\ rad^2GeV^2A}, \tag{D.5}$$

$$C_\psi = \frac{4\alpha}{9e\,mc^2} = 3.967\ 10^{19}\ \frac{\rm photons}{\rm s\ rad\ A\ GeV}, \tag{D.6}$$

$$C_K = \frac{[c]\,e}{2\,\pi\,m\,c^2} = 0.93373\,\frac{1}{\rm T\,cm}, \tag{D.7}$$

$$C_{\rm q} = \frac{55}{32\sqrt{3}}\frac{\hbar c}{mc^2} = 3.84\ 10^{-13}\ {\rm m}\,, \tag{D.8}$$

$$C_{\mathrm{Q}} = \frac{55}{24\sqrt{3}} \frac{r_{\mathrm{e}} \hbar c}{(mc^2)^6} = 2.06\, 10^{-11} \frac{\mathrm{m}^2}{\mathrm{GeV}^5}, \tag{D.9}$$

$$C_{\mathrm{d}} = \frac{c}{3} \frac{r_{\mathrm{e}}}{(mc^2)^3} = 2110 \frac{\mathrm{m}^2}{\mathrm{GeV}^3 \mathrm{s}}, \tag{D.10}$$

$$C_{\rho} = [c]\, e = 0.299792 \frac{\mathrm{GeV}}{\mathrm{m\ T}}. \tag{D.11}$$

D.2 Bending Magnet Radiation

Notation for synchrotron radiation formulas

θ	angle of observation orthogonal to deflecting plane
ψ	angle of observation in the plane of deflection

For isomagnetic ring: all bending fields are equal, ρ =const.

Total radiation power

$$\boxed{P_{\gamma}(\mathrm{kW}) = 14.0788\ E^4\, \mathrm{I} \oint \frac{\mathrm{d}s}{\rho^2} \longrightarrow \underbrace{88.460 \frac{E^4}{\rho} I}_{\text{isomagnetic ring}}} \tag{D.12}$$

Energy loss per turn to synchrotron radiation

$$U_0(keV) = 14.0788\ E^4 \oint \frac{\mathrm{d}s}{\rho^2} \longrightarrow \underbrace{88.460 \frac{E^4}{\rho}}_{\text{isomagnetic ring}} \tag{D.13}$$

Fundamental photon energy of synchrotron radiation

$$\boxed{\epsilon_{\mathrm{c}}(\mathrm{keV}) = 2.2181 \frac{E^3}{\rho} = 0.665\, E^2\, B} \tag{D.14}$$

Radiation power into a beam line with acceptance angle $\Delta\psi$

$$\boxed{\Delta \mathrm{P}_{\gamma}(\mathrm{kW}) = 14.079 \frac{\Delta\psi}{\rho} E^4\, I} \tag{D.15}$$

Spatial and spectral photon flux

$$\boxed{\frac{\mathrm{d}^2 \dot{N}_{\mathrm{ph}}}{\mathrm{d}\theta \mathrm{d}\psi} \left[\frac{\text{photons}}{\text{s mrad}^2}\right] = 1.3273\ 10^{16}\, E^2 I \frac{\Delta\omega}{\omega} \left(\frac{\omega}{\omega_{\mathrm{c}}}\right)^2 K^{\,2}_{2/3}(\xi)\, F(\xi,\theta)\,,} \tag{D.16}$$

with

$$\xi = \tfrac{1}{2} \frac{\omega}{\omega_{\mathrm{c}}} \left(1 + \gamma^2 \theta^2\right)^{3/2} \tag{D.17}$$

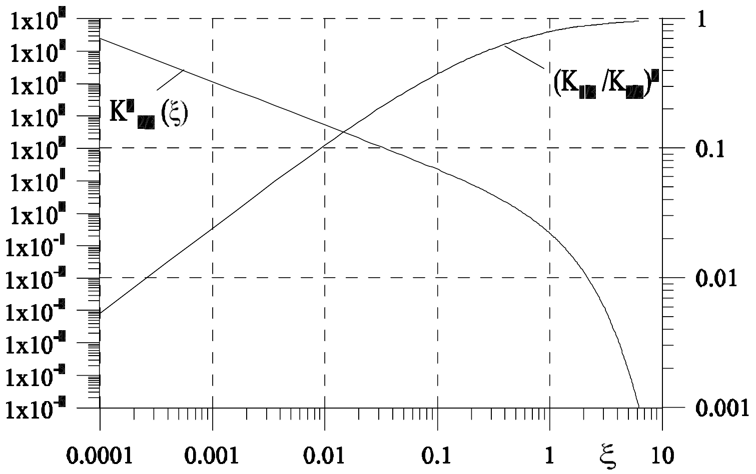

Fig. D.1. Functions $K_{2/3}^2(\xi)$ and $\left(K_{1/3}/K_{2/3}\right)^2$

and

$$F(\xi,\theta) = (1+\gamma^2\theta^2)^2 \left[1 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\frac{K_{1/3}^{\,2}(\xi)}{K_{2/3}^{\,2}(\xi)}\right]. \tag{D.18}$$

Spatial and spectral photon flux on axis ($\theta = 0$)

$$\boxed{\frac{\mathrm{d}^2\dot{N}_{\mathrm{ph}}}{\mathrm{d}\theta\mathrm{d}\psi}\left[\frac{\mathrm{photons}}{\mathrm{s\ mrad}^2}\right]\Bigg|_{\theta=0} = 1.3273\ 10^{16}\, E^2 I\,\frac{\Delta\omega}{\omega}\left(\frac{\omega}{\omega_{\mathrm{c}}}\right)^2 K_{2/3}^{\,2}\left(\frac{\omega}{2\omega_{\mathrm{c}}}\right)} \tag{D.19}$$

Photon flux per unit deflection angle

$$\boxed{\frac{\mathrm{d}\dot{N}_{\mathrm{ph}}}{\mathrm{d}\psi}\left[\frac{\mathrm{photons}}{\mathrm{s\ mrad}}\right] = 3.967\times 10^{16}\ E\,I\,\frac{\Delta\omega}{\omega}\,S\left(\frac{\omega}{\omega_{\mathrm{c}}}\right)} \tag{D.20}$$

Long and short wavelength approximations are

$$\begin{aligned} S\left(\frac{\omega}{\omega_{\mathrm{c}}}\right) &= \frac{9\sqrt{3}}{8\pi}\frac{\omega}{\omega_{\mathrm{c}}}\int_{\omega/\omega_{\mathrm{c}}}^{\infty} K_{5/3}(x)\mathrm{d}x \\ &= \begin{cases} 1.333\left(\frac{\omega}{\omega_{\mathrm{c}}}\right)^{1/3} & \text{for } \omega \ll \omega_{\mathrm{c}} \\ 0.777\sqrt{\frac{\omega}{\omega_{\mathrm{c}}}}\mathrm{e}^{-\omega/\omega_{\mathrm{c}}} & \text{for } \omega \gg \omega_{\mathrm{c}} \end{cases}. \end{aligned} \tag{D.21}$$

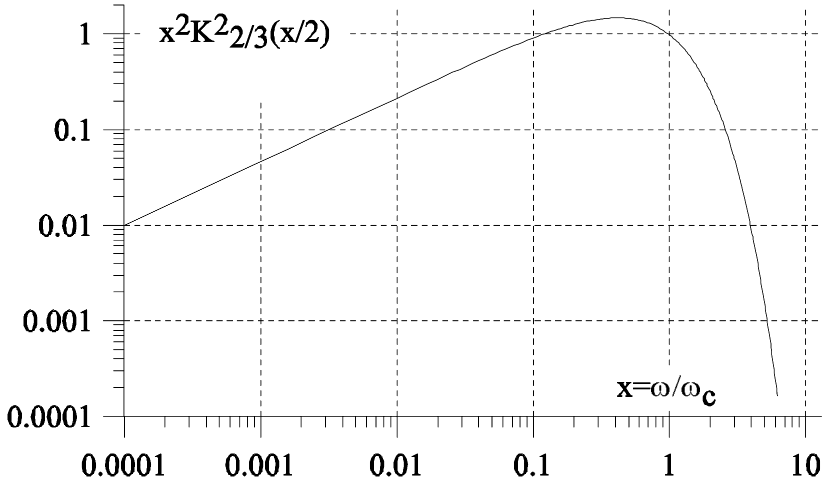

Fig. D.2. Function $\left(\frac{\omega}{\omega_c}\right)^2 K_{2/3}^2\left(\frac{1}{2}\frac{\omega}{\omega_c}\right)$ defining the forward photon flux

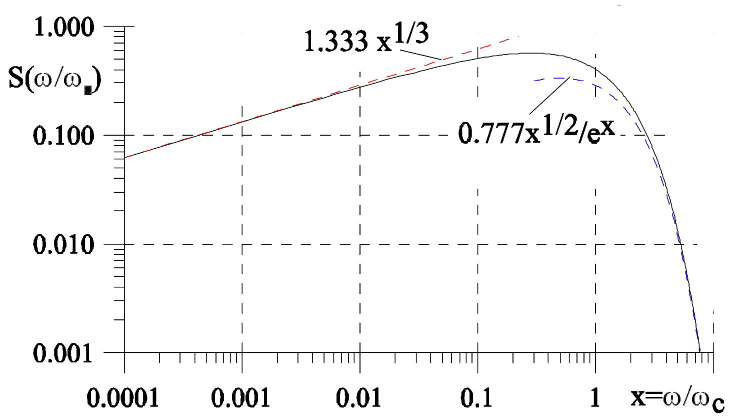

Fig. D.3. Universal Function $S\left(\frac{\omega}{\omega_c}\right)$

Vertical radiation cone angle defined by $\sqrt{2\pi}\sigma_\theta = \left(\frac{\mathrm{d}N_{\mathrm{ph}}}{\mathrm{d}\psi}\right) / \left(\frac{\mathrm{d}^2\dot{N}_{\mathrm{ph}}}{\mathrm{d}\theta\mathrm{d}\psi}\right)$

$$\boxed{\sigma_\theta\,(\mathrm{mrad}) = \frac{C_\psi}{\sqrt{2\pi}C_\Omega}\frac{1}{E}\frac{S(x)}{x^2K_{2/3}^2\left(\frac{1}{2}x\right)} = \frac{f(x)}{E(\mathrm{GeV})}}. \tag{D.22}$$

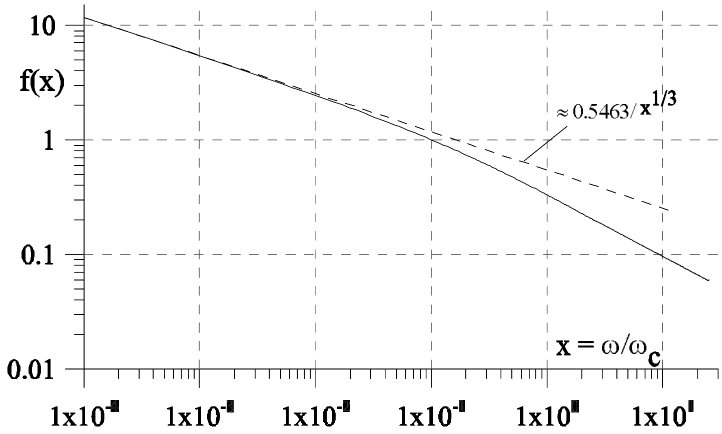

Fig. D.4. Scaling function $f(x) = \sigma_\theta(\mathrm{mrad})\ E(\mathrm{GeV})$ for the photon beam divergence

For long wavelengths ($x \ll 1$)

$$\sigma_\theta\,(\mathrm{mrad}) \approx \frac{0.5463\,x^{1/3}}{E(\mathrm{GeV})}. \tag{D.23}$$

D.3 Periodic Insertion Devices

Notation for insertion devices

ϑ	angle between observation and undulator axis
ψ	azimuthal angle of observation about undulator axis
B_0	maximum undulator field
θ	maximum deflection angle
λ_p	period length (m)
N_p	number of periods
L_u	undulator length (m) , $L_\mathrm{u} = N_\mathrm{p}\lambda_\mathrm{p}$

"horizontal" reflects the deflecting plane of the undulator

D.3.1 Insertion Device Parameter

Undulator field, on-axis

$$\boxed{B(z) = B_0\,\sin\frac{2\pi z}{\lambda_\mathrm{p}}} \tag{D.24}$$

Undulator strength parameter

$$\boxed{K\ = 93.4\,B_0\lambda_\mathrm{p} \approx B_0\lambda_\mathrm{p}(\mathrm{cm})} \tag{D.25}$$

D.3.2 Field Scaling for Hybrid Wiggler Magnets

The maximum on-axis field in a hybrid wiggler magnet depends on the gap aperture g and period length λ_p and is for $0.1\,\lambda_\mathrm{p} \lessapprox g \lessapprox 10\,\lambda_\mathrm{p}$ given by [34]

$$B_y(\mathrm{T}) \approx 3.33 \exp\left[-\frac{g}{\lambda_\mathrm{p}}\left(5.47 - 1.8\frac{g}{\lambda_\mathrm{p}}\right)\right] \tag{D.26}$$

D.3.3 Particle Beam Parameter

Beam width

$$\sigma_{\mathrm{b},x} = \sqrt{\epsilon_x \beta_x + \left(\eta_x \tfrac{\sigma_E}{E}\right)^2} \tag{D.27}$$

Horizontal beam divergence

$$\sigma_{\mathrm{b},x'} = \sqrt{\tfrac{\epsilon_x}{\beta_x} + \left(\eta'_x \tfrac{\sigma_E}{E}\right)^2} \tag{D.28}$$

Beam height

$$\sigma_{\mathrm{b},y} = \sqrt{\epsilon_y \beta_y + \left(\eta_y \tfrac{\sigma_E}{E}\right)^2} \longrightarrow \underbrace{\sqrt{\epsilon_y \beta_y}}_{\text{flat ring}} \tag{D.29}$$

Vertical beam divergence

$$\sigma_{\mathrm{b},y'} = \sqrt{\tfrac{\epsilon_y}{\beta_y} + \left(\eta'_y \tfrac{\sigma_E}{E}\right)^2} \longrightarrow \underbrace{\sqrt{\tfrac{\epsilon_y}{\beta_y}}}_{\text{flat ring}} \tag{D.30}$$

Average drift velocity

$$\bar{\beta} = \beta\left(1 - \tfrac{K^2}{4\gamma^2}\right) \tag{D.31}$$

Transverse particle coordinate

$$x(t) = \frac{K}{\gamma k_\mathrm{p}} \cos\left(k_\mathrm{p}\,\bar{\beta}\,ct\right) \tag{D.32}$$

Maximum oscillation amplitude

$$a = \frac{K}{\gamma k_\mathrm{p}} = \frac{\lambda_\mathrm{p} K}{2\pi\gamma} \tag{D.33}$$

Maximum deflection angle

$$\theta = \pm\frac{K}{\gamma} \tag{D.34}$$

Longitudinal coordinate

$$z(t) = \bar{\beta}ct + \tfrac{K^2}{8\gamma^2 k_\mathrm{p}} \sin^2\left(2k_\mathrm{p}\,\bar{\beta}\,ct\right) \tag{D.35}$$

D.4 Undulator Radiation

Total radiation powerfrom an undulator magnet

$$P(\mathrm{kW}) = 0.6336\, E^2\, B_0^2\; I\; L_\mathrm{u} \tag{D.36}$$

Energy loss in an undulator

$$\Delta E(\mathrm{keV}) \;=\; 0.725\, \frac{E^2\, K^2}{\lambda_\mathrm{p}^2(\mathrm{cm})}\, L_\mathrm{u}, \tag{D.37}$$

Wavelength, frequency and photon energy for k-th harmonic

$$\boxed{\lambda_k(\text{Å}) = 13.056\, \tfrac{\lambda_\mathrm{p}(\mathrm{cm})}{k\, E^2}\left(1 + \tfrac{1}{2}K^2 + \gamma^2\vartheta^2\right)} \tag{D.38}$$

$$\boxed{\omega_k = 1.4427 \times 10^{18}\, \frac{k\, E^2}{\lambda_\mathrm{p}(\mathrm{cm})\left(1+\frac{1}{2}K^2+\gamma^2\vartheta^2\right)}} \tag{D.39}$$

$$\boxed{\varepsilon_k(\mathrm{keV}) = 0.9496\, \frac{k\, E^2}{\lambda_\mathrm{p}(\mathrm{cm})\left(1+\frac{1}{2}K^2+\gamma^2\vartheta^2\right)}} \tag{D.40}$$

Undulator on-axis differential photon flux for k-th harmonic

$$\boxed{\left.\frac{\mathrm{d}\dot{N}_\mathrm{ph}(k\omega_k)}{\mathrm{d}\Omega}\right|_{\vartheta=0} = 1.7443\, 10^{23}\, E^2 N_\mathrm{p}{}^2 \frac{\Delta\omega}{\omega} I\; A_k(K),} \tag{D.41}$$

Harmonic amplitude functions

$$A_k(K) = \frac{k^2\, K^2}{\left(1 + \frac{1}{2}K^2\right)^2}\left[J_{\frac{1}{2}(k-1)}\left(\tfrac{kK^2}{4+2K^2}\right) + J_{\frac{1}{2}(k+1)}\left(\tfrac{k\,K^2}{4+2K^2}\right)\right]^2 \tag{D.42}$$

Opening angle (diffraction limit) for on-axis photon flux

$$\sigma_\vartheta \approx \frac{1}{\gamma}\sqrt{\frac{1+\frac{1}{2}\,K^2}{2\,k\,N_\mathrm{p}}} \tag{D.43}$$

Pinhole solid angle

$$\mathrm{d}\Omega = 2\pi\,\sigma_\vartheta^2 \tag{D.44}$$

Pinhole photon flux

$$\boxed{\dot{N}_\mathrm{ph}(k\omega_1)|_{\vartheta=0} = 1.4309\, 10^{17}\, N_\mathrm{p}\, \tfrac{\Delta\omega}{\omega}\, G_k(K)} \tag{D.45}$$

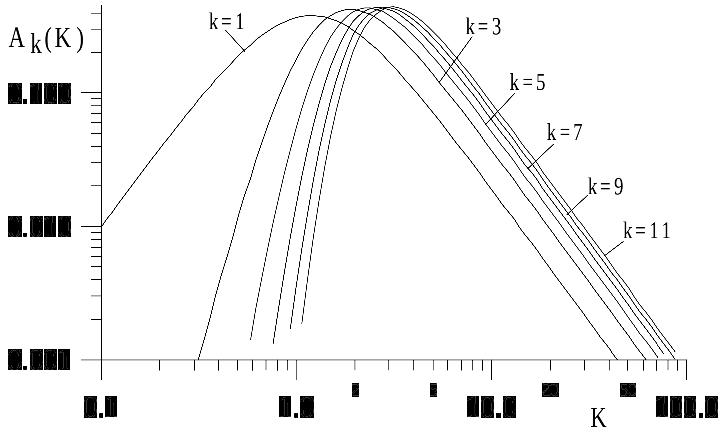

Fig. D.5. Functions $A_k(K)$

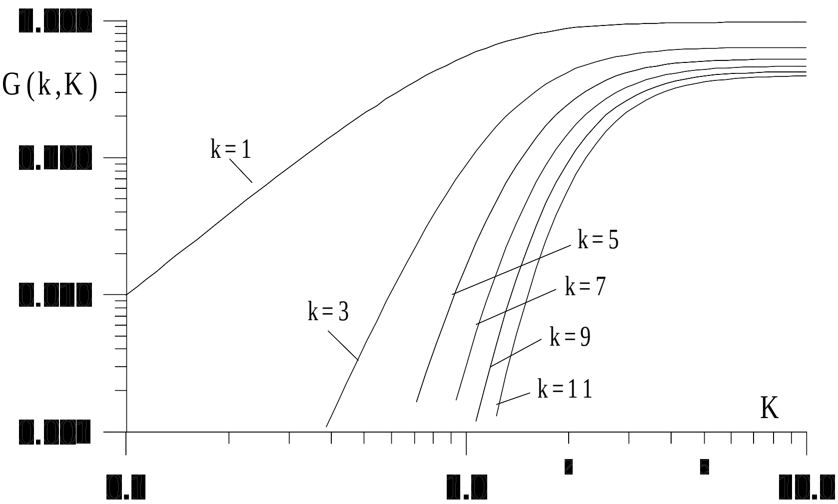

Fig. D.6. Function $G_k(K)$

$$G_k(K) = \frac{1+\frac{1}{2}K^2}{k} A_k\left(K\right) \tag{D.46}$$
$$= \frac{k\,K^2}{\left(1+\frac{1}{2}K^2\right)} \left[J_{\frac{1}{2}(k-1)}\left(\frac{kK^2}{4+2K^2}\right) + J_{\frac{1}{2}(k+1)}\left(\frac{k\,K^2}{4+2K^2}\right)\right]^2$$

Band width of radiation

$$\frac{\Delta\omega}{\omega_k} = \frac{1}{kN_{\mathrm{p}}} \tag{D.47}$$

D.5 Photon Beam Brightness

Spectral brightness $\equiv$ 6−dim photon phase space density

$$\boxed{\mathcal{B}(\omega)=\frac{\dot{N}_{\mathrm{ph}}(\omega)}{4\pi^2\,\sigma_x\sigma_{x'}\sigma_y\sigma_{y'}(\mathrm{d}\omega/\omega)}} \tag{D.48}$$

Diffraction limited source size

$$\boxed{\begin{array}{rl}\text{radial source size:} & \sigma_r = \frac{1}{2\pi}\sqrt{\lambda L_\mathrm{s}}\\ \text{radial source divergence:} & \sigma_{r'} = \sqrt{\frac{\lambda}{L_\mathrm{s}}}\end{array}} \tag{D.49}$$

Source length L_s

$$L_\mathrm{s} = L_\mathrm{u} \qquad \text{for undulator,} \tag{D.50}$$

$$L_\mathrm{s} = 2\rho/\gamma \qquad \text{for wiggler and bending magnets} \tag{D.51}$$

Diffraction limited brightness

$$\boxed{\mathcal{B}_{\max}=\dot{N}_{\mathrm{ph}}\,\frac{(4\,/\,\lambda^2)}{\mathrm{d}\omega/\omega}} \tag{D.52}$$

D.5.1 Effective Source Parameter

The beam parameters σ_{b0} are to be taken at the beginning of the source, e.g. at the entrance of the undulator and beam is assumed to be symmetric within undulator.

Horizontal source size

$$\boxed{\sigma^2_{\mathrm{t},x}=\tfrac{1}{2}\sigma^2_r+\sigma^2_{\mathrm{b0},x}+\left(\tfrac{\lambda_\mathrm{p}K}{2\pi\gamma}\right)^2+\tfrac{1}{12}\sigma^2_{\mathrm{b0},x'}\mathrm{L}^2_\mathrm{s}+\tfrac{1}{36}\theta^2\mathrm{L}^2_\mathrm{s}} \tag{D.53}$$

with increased source width due to electron path oscillations $\frac{\lambda_\mathrm{p}K}{2\pi\gamma}$, due to finite beam divergence $\frac{1}{12}\sigma^2_{\mathrm{b0},x'}L^2_\mathrm{s}$, and due to oblique horizontal observation angle $\frac{1}{36}\theta^2L^2_\mathrm{s}$.

Horizontal divergence

$$\sigma^2_{\mathrm{t},x'} = \tfrac{1}{2}\sigma^2_{r'} + \sigma^2_{\mathrm{b0},x'}\,, \tag{D.54}$$

Vertical source size

$$\sigma^2_{\mathrm{t},y} = \tfrac{1}{2}\sigma^2_r + \sigma^2_{\mathrm{b0},y} + \tfrac{1}{12}\sigma^2_{\mathrm{b0},y'}L^2_\mathrm{s} + \tfrac{1}{36}\psi^2L^2_\mathrm{s}\,, \tag{D.55}$$

with increased source height due to finite beam divergence $\frac{1}{12}\sigma^2_{\mathrm{b0},y'}L^2_\mathrm{s}$, and due to oblique vertical observation angle $\frac{1}{36}\psi^2L^2_\mathrm{s}$.

Vertical divergence

$$\sigma_{\mathrm{t},y'}^2 = \tfrac{1}{2}\sigma_{r'}^2 + \sigma_{\mathrm{b0},y'}^2 \,. \tag{D.56}$$

Effective spectral brightness

$$\boxed{\mathcal{B}(\omega) = \frac{\dot{N}_{\mathrm{ph}}(\omega)}{4\pi^2\, \sigma_{\mathrm{t},x}\, \sigma_{\mathrm{t},x'}\, \sigma_{\mathrm{t},y}\, \sigma_{\mathrm{t},y'} (\mathrm{d}\omega/\omega)}} \,. \tag{D.57}$$

References

1. A. Liènard, L'Eclairage Electrique **16**, 5 (1898).
2. E. Wiechert, Archives Neerlandaises 546 (1900).
3. G. Schott, Annalen der Physik **24**, 635 (1907).
4. G. Schott, Phil. Mag.[6] **13**, 194 (1907).
5. G. Schott, *Electromagnetic Radiation* (Cambridge Univ. Press, New York, 1912).
6. A. S. N.A. Vinokurov, Preprint INP 77-59, Institute of Nuclear Physics, Novosibirsk (unpublished).
7. The author would like to thank Prof. M. Eriksson, Lund, Sweden, for introducing him to this approach into the theory of synchrotron radiation.
8. D.W. Kerst and R. Serber, Phys. Rev. **60**, 53 (1941).
9. D. Ivanenko and I.Ya. Pomeranchouk, Phys. Rev. **65**, 343 (1944).
10. J. Blewett, Phys. Rev. **69**, 87 (1946).
11. describing work of C. Sutis, Sci. News Lett. **51**, 339 (1947).
12. describing work of F. Haber, Electronics **20**, 136 (1947).
13. F. Elder, A. Gurewitsch, R. Langmuir, and H. Pollock, Phys. Rev. **71**, 829 (1947).
14. M. Sands, in *Physics with Intersecting Storage Rings*, edited by B. Touschek (Academic, New York, 1971), p. 257.
15. R. Coisson, Opt.Com. **22**, 135 (1977).
16. R. Bossart, J. Bosser, L. Burnod, R. Coisson, E. D'Amico, A. Hofmann, and J. Mann, Nucl. Instrum. Methods **164**, 275 (1979).
17. R. Bossart, J. Boser, L. Burnod, E. D'Amico, G. Ferioli, J. Mann, and F. Meot, Nucl. Instrum. Methods **184**, 349 (1981).
18. *The Large Hadron Collider in the LEP Tunnel*, edited by G. Brianti and K. Hübner (CERN, Geneva, 1985).
19. J. Jackson, *Classical Electrodynamics*, 2nd. ed. (Wiley, New York, 1975).
20. D. Ivanenko and A.A. Sokolov, DAN (USSR) **59**, 1551 (1972).
21. J. Schwinger, Phys. Rev. **75**, 1912 (1949).
22. D.H. Tomboulian and P.L. Hartman, Phys. Rev. **102**, 102 (1956).
23. G. Bathow, E. Freytag, and R. Haensel, J. Appl. Phys. **37**, 3449 (1966).
24. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).
25. M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1975).
26. G. Airy, Trans. Cambr. Phil. Soc. **5**, 283 (1835).
27. L. Schiff, Rev. Sci. Instrum. **17**, 6 (1946).
28. T. Nakazato, M. Oyamada, N. Niimura, S. Urasawa, O. Konno, A. Kagaya, R. Kato, T. Kamiyama, Y. Torizuka, T. Nanba, Y. Kondo, Y. Shibata, K. Ishi, T. Oshaka, and M. Ikezawa, Phys. Rev. Lett. **63**, 1245 (1989).
29. E.B.Blum, U.Happek, and A.J. Sievers, Nucl. Instrum. Methods 568 (1992).
30. H. Wiedemann, P. Kung, and H.C. Lihn, Nucl. Instrum. Methods **A**, 1 (1992).

31. F. Michel, Phys. Rev. Lett. **48**, 580 (1982).
32. M. Berndt, W. Brunk, R. Cronin, D. Jensen, R. Johnson, A. King, J. Spencer, T. Taylor, and H. Winick, IEEE Trans. Nucl. Sci. 3812 (1979).
33. The author thanks T. Rabedau, SSRL, for providing this picture.
34. K. Halbach, J. Physique (1983).
35. W. Heitler, *The Quantum Theory of Radiation* (Clarendon, Oxford, 1954).
36. B. Kincaid, J. Appl. Phys. **48**, 2684 (1977).
37. W. Lavender, Ph.D. thesis, Stanford University, 1988.
38. R. Milburn, Phys. Rev. Lett. **4**, 75 (1963).
39. F.A. Arutyunian and V.A. Tumanian, Phys. Rev. Lett. **4**, 176 (1963).
40. F.A. Arutyunian, I.I. Goldman, and V.A Tumanian, ZHETF(USSR) **45**, 312 (1963).
41. I.F. Ginzburg, G.L. Kotin, V.G. Serbo, and V.I. Telnov, Preprint 81-102, Inst. of Nucl.Physics, Novosibirsk, USSR, (unpublished).
42. E.D. Cournat and H.S. Snyder, ap **3**, 1 (1958).
43. E. McMillan, Phys. Rev. **68**, 143 (1945).
44. V. Veksler, DAN(USSR) **44**, 393 (1944).
45. H. Wiedemann, *Particle Accelerator Physics I*, 2nd ed. (Springer, Heidelberg, 1999).
46. H. Wiedemann, *Particle Accelerator Physics II*, 2nd ed. (Springer, Heidelberg, 1999).
47. J.M. Paterson, J.R. Rees, and H. Wiedemann, Technical report, Stanford Linear Accelerator Center (unpublished).
48. W.K.H. Panofsky and W.A. Wenzel, Rev. Sci. Instrum. **27**, 967 (1956).
49. J. Larmor, Philos. Mag. **44**, 503 (1897).
50. J. Schwinger, Proc. Nat. Acad. of Sci. USA **40**, 132 (1954).
51. A.A. Sokolov and I.M. Ternov, *Synchrotron Radiation* (Pergamon, Oxford, 1968).
52. G. Watson, *Bessel Functions* (The Macmillan Company, New York, 1945).
53. V. Kostroun, Nucl. Instrum. Methods **172**, 371 (1980).
54. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed. (Academic, New York, 1965), prepared by Yu.V. Geronimus and M.Yu. Tseytlin, translation edited by A. Jeffrey.
55. V. Baier, in *Physics with Intersecting Storage Rings*, edited by B. Touschek (Academic, New York, 1971), p. 1.
56. H. Motz, J. Appl. Phys. **22**, 527 (1951).
57. L.R. Elias, W.M. Fairbanks, J.M.J. Madey, H.A. Schwettmann, and T.J. Smith, Phys. Rev. Lett. **36**, 717 (1976).
58. W. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, 1950).
59. D.F. Alferov, Y.A. Bashmakov, and E.G. Bessonov, Sov. Phys.-Tech. Phys. **18**, 1336 (1974).
60. S. Krinsky, IEEE Trans. Nucl. Sci. 307 (1983).
61. A. Bienenstock, G. Brown, H. Wiedemann, and H. Winick, Rev. Sci. Instrum. **60**, 7 (1989).
62. R. Carr, NIM **306**, 391 (1991).
63. S. Sasaki, K. Kakuno, T. Takada, T. Shimada, K. Yanagida, and Y. Miyahara, Nucl. Instrum. Methods **331**, 763 (1993).
64. R. Carr and S. Lidia, in *Proc. of the SPIE* (SPIE, Bellingham, WA (USA), 1993), Vol. 2013.
65. W. Colson, pla **64**, (1977).

Index