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STERN-GERLACH

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Abstract

The experiment performed by Otto Stern and Walther Gerlach in 1922 provided very convincing evidence of two important consequences of Modern Quantum Mechanics

1. Space quantization can occur even in non-periodic systems.
2. That some particles have an intrinsic angular momentum, and therefore magnetic moment.

The experiment did not really produce any new ideas but dramatically confirmed ideas developed indirectly from Spectroscopy and the Zeeman Effect. The Stern Gerlach experiment was in principle simple and its results were clear. It removed many of the lingering doubts that Quantum Mechanics is true.

The Original Experiment

Stern and Gerlach generated a beam of neutral silver atoms by evaporating silver from an oven. The process was performed in a vacuum so that the silver atoms moved without scattering. The atoms were collimated by slits and sent through a region with a large non-uniform magnetic field.

A magnetic field non-uniformity $\frac{\partial B_z}{\partial z}$ produces a force $\frac{\partial B_z}{\partial z}\mu_z$ on a magnetic moment where μ_z is the component of the magnetic moment μ in the z direction.

The silver atoms were thus deflected and allowed to strike a cold metallic plate. After about 8-10 hours the number of condensed silver atoms was large enough to show a visible trace. The trace showed 2 marks showing that the silver atoms had 2 possible components of μ_z.

This would not have been expected with classical physics since this would have predicted that the z component of μ would have been

$$\mu_z = \mu \cos\theta$$

where $\cos\theta$ could have all values from -1.0 to $+1.0$.

Even the original Schrödinger theory predicts an odd number of possible states. This could explain μ_z having, for example, 3 values: $-\mu$, 0 and $+\mu$

The obvious 2 states shown by the experiment is evidence that something is missing in the original Schrödinger Theory. The missing idea is that electrons have an intrinsic spin (a spin which cannot be removed) and that the angular momentum can be written in the form:

$$\vec{S} = \frac{1}{2}\hbar\vec{\sigma}.$$

The component of $\vec{S}$ in any specified direction, say the "z" or third direction must then have eigenvalues $\pm\frac{1}{2}\hbar$.

This Experiment

We include two improvements to the original experiment that allow it to be done more easily.

1. Potassium is used instead of silver because:

   (a) it is easier to evaporate (63.6°C for K instead of 961.9°C for Ag)

(b) The low temperature means that the Potassium atoms are moving at a lower speed and are more easily deflected.

(c) Potassium has a single valence electron outside its closed shells and the magnetic moment is more obviously due to the single valence electron.

(d) Potassium is easier to ionize than silver and is thus easier to detect electrically.

2. A hot wire detector is used instead of a cold metallic plate. The hot wire ionizes a fraction of the potassium atoms which strike it. The positive recoiling ions are then collected on a nearby negative electrode and the small ionization current is measured by an electrometer. This allows modern electronics to be used with its high sensitivity for small currents and fast reaction time.

Theory

One result of Schrödinger's Quantum Theory is that a system existing in a region of space with a unique axis of symmetry (such as the z axis) will have a wavefunction which can be expressed in the form of a product

$$\psi(r,\theta,\phi) = \Xi(r,\theta)\Phi(\phi)$$

The angle ϕ is the azimuthal angle measured by rotating around the axis of symmetry. The r, θ and ϕ form a set of spherical coordinates. Read Eisberg and Resnick pages 256–259 for the solution of the equation.

The function $\Phi(\phi)$ must repeat after ϕ is increased by 2π and consequently $\Phi(\phi)$ has a solution:

$$\Phi = e^{im\phi}$$

where m is an integer. The state has an angular momentum J_z about the z axis which is:

$$J_z = m\hbar.$$

The number of such spatial states is always **odd**, since m is any integer from $-j$ to $+j$ where j depends upon the form $\Xi(r,\theta)$. In our case of potassium, $j = 0$ and so only one state ($m = 0$) is possible.

However the intrinsic spin (introduced by the Dirac improvement to Schrödinger's Theory) may also be included as an extra product term if the space has a unique axis of symmetry. The total angular momentum about the z axis then includes that of the intrinsic spin $\vec{S}$

$$|S_z| = \tfrac{1}{2}\hbar.$$

In our case of potassium, the intrinsic spin causes the single $m = 0$ angular momentum to be split into **two** states with angular momenta of $\pm\frac{1}{2}\hbar$. The regular Schrödinger Theory can only predict an **odd** number of states and so the number of states is an important test for the existence of intrinsic spin.

The spin is measured by an effect on the associated magnetic moment. The electron intrinsic magnetic moment is given by

$$\vec{\mu} = g\mu_B\vec{S}$$

where g is called the gyromagnetic factor and μ_B is the Bohr Magneton with value $\mu_B = \frac{e\hbar}{2m_e} = 9.27\times10^{-24}$ Joules/Tesla. The gyromagnetic factor for the electron is almost exactly 2. The two possible z components of the magnetic moment are given by: $\mu_z = g\mu_B m_s$ with $m_s = \pm\frac{1}{2}$, resulting in $\mu_z = \pm\mu_B$.

The suggestion of intrinsic spin by Goudsmit and Uhlenbeck and the original Stern-Gerlach experiment are discussed in Eisberg and Resnick, pages 296–302.

The Atoms

Potassium atoms have 19 protons and either 10 or 12 neutrons. The electrons are:

$n = 0$ shell		filled with 2
$n = 1$ shell		filled with 8
$n = 3$	$\ell = 0$ subshell	filled with 2
	$\ell = 1$ subshell	filled with 6
	$\ell = 2$ subshell	empty
$n = 4$	$\ell = 0$ subshell	- - one valence electron

thus 18 of the electrons form a closed core with the same configuration as in the Argon atom. The 19th electron sees a potential very similar to that of a single proton and thus:

1. The energy levels of the 19th electron (if excited above n = 4, $\ell = 0$) are somewhat similar to those of a Hydrogen atom.

2. The magnetic moments of the electrons in the core cancel to give no contribution to the total magnetic moment of the atom. The nuclear magnetic moments are 0.391 and 0.215 nuclear magnetons for ${}_{19}\mathrm{K}^{39}$ and ${}_{19}\mathrm{K}^{41}$ and since a nuclear magneton is about $\frac{1}{2000}$ of a Bohr magneton, the only significant magnetic moment is that of the valence electron.

3. The valence electron is in an $\ell = 0$ state and so it has no spatial **(orbital)** magnetic moment. The atom therefore has a magnetic moment approximately equal to the intrinsic magnetic moment of an electron.

The above was strongly suspected from spectroscopic studies. The idea of an ad-hoc "intrinsic magnetic moment" is however rather unsatisfactory. The Stern Gerlach experiment provided a direct method for finding the number of states and measuring the intrinsic magnetic moment. The observation of an even number of states then confirms the concept of "intrinsic" spin.

Magnetic Field

The apparatus sends a beam of neutral potassium atoms in the x direction through a strong magnetic field gradient transverse to the beam. The field exerts a deflecting force on the magnetic moments of the moving atoms, proportional to the magnetic field gradient given by:

$$\begin{aligned}\vec{F} &= -\vec{\nabla}(\text{potential of mag. mom. in field B}) \\ &= -\vec{\nabla}(\vec{\mu} \cdot \vec{B}).\end{aligned}$$

If the magnetic field gradient is taken to be in the z direction then the z component of the force is given by:

$$F_z = -\frac{\partial}{\partial z}(-\mu_z B) = \mu_z \frac{\partial B}{\partial z}.$$

Since μ_z is quantized and independent of x, y, and z, and since the magnetic poles are designed so that there is a region of constant $\frac{\partial B}{\partial z}$, the beam will split into two distinct beams, the force given by:

$$F_z = \pm \mu_B \frac{\partial B}{\partial z}.$$

Classically (pre-quantum theory), the spin of the atom may be at any angle to the z axis. The component of the magnetic moment along the z axis is then **any value** between $+\frac{1}{2}\mu_0$ and $-\frac{1}{2}\mu_0$ where μ_0 is the Bohr magneton.

The probability of the magnetic moment being between angles θ and $\theta + d\theta$ is equal to

$$\begin{aligned}
&= \frac{\text{area of section of sphere between } \theta \text{ and } (\theta + d\theta)}{\text{full area of sphere}} \\
&= \frac{(rd\theta)\ (2\pi r \sin\theta)}{4\pi r^2} \\
&= \frac{\sin\theta\, d\theta}{2} \quad .
\end{aligned}$$

Hence the number of atoms with the z component between μ and $\mu + d\mu$ is proportional to $\sin\theta\, d\theta$.

This fraction of the atoms will give magnetic moments between:

$$\frac{e\hbar}{4m}\cos\theta \text{ and } \frac{e\hbar}{4m}\cos(\theta + d\theta)$$

i.e., between:

$$\mu \text{ and } \mu - d\mu$$

where:

$$d\mu = -\frac{d}{d\theta}\left(\frac{e\hbar\cos\theta}{4m}\right) d\theta = \frac{e\hbar}{4m}\sin\theta\, d\theta.$$

Hence $d\mu$ is **also** proportional to $(\sin\theta\, d\theta)$. The number of atoms between μ and $\mu - d\mu$ is thus proportional to $d\mu$. The classical prediction is therefore that the z components of the magnetic moment will be uniformly distributed from $-\frac{e\hbar}{4m}$ to $+\frac{e\hbar}{4m}$.

z Displacement

An atom moving with velocity v in the x direction will be acted on by a force in the z direction $\mu_z\vec{\nabla}B$ for a time $t = d_1/v$, where d_1 is the distance travelled by the atom in the magnetic field. The acceleration along the z-axis (the direction of $\vec{\nabla}B$) will be

$$\vec{a}_z = \frac{\mu_z\vec{\nabla}B}{M}$$

where M is the atom's mass; the velocity and deflection of the atom in this direction as it leaves the magnetic field will be:

$$\begin{aligned}\vec{v}_z &= \vec{a}_z t \\ &= \frac{\mu_z \vec{\nabla} B\ d}{Mv}\end{aligned}$$

and

$$\vec{s}_z{}' = \vec{a}_z t^2/2 = \frac{\mu_z \vec{\nabla} B d_1^2}{2Mv}.$$

From that point to the detector, $\vec{v}_z$ remains constant, so that the deflection, $\vec{s}$, at the detector is:

$$\begin{aligned}\vec{s} &= \vec{v}_z d_z/v + \vec{s}_z{}' \\ &= \mu_z(\vec{\nabla} B)\frac{[d_1^2 + 2d_1 d_2]}{2Mv^2}\end{aligned}$$

where d_2 is the distance between the magnet exit and the detector. For fixed v, $\vec{s}$ is proportional to μ_z, and can be used to determine the distribution of μ_B. Quantum mechanically, μ_z can be only $+\mu_B$ or $-\mu_B$, so s will take on only two values, $+s_0$ and $-s_0$. Half of the atoms in the beam will arrive at each position. Classically the atoms would be distributed uniformly among all values of s between $+s_0$ and $-s_0$.

The Stern-Gerlach experiment, like many experiments utilizing molecular beam techniques, is limited to a certain extent by the fact that the atoms or molecules in a beam issuing from an oven at absolute temperature T do not have a unique velocity. The velocity distribution of the beam intensity is

$$I(v)dv = 2I_0(v/\alpha)^3 \ \ e^{-(v/\alpha)^2}\ d(v/\alpha),$$

where I_0 is the total beam intensity, and $\alpha = \sqrt{2kT/M}$ is the most probable velocity of atoms in the oven (but not the beam). k is the Boltzmann constant.

Now if s_α is the deflection for an atom of velocity α, then

$$s/s_\alpha = (\alpha/v)^2,$$

and changing variables in the equation above from v/α to s/s_α yields the following relation for the deflection pattern produced by a narrow beam of thermal atoms having moment μ_B:

$$I(s)ds = aI_0\frac{s_\alpha^2}{s^3}e^{-(s_\alpha/s)}ds$$

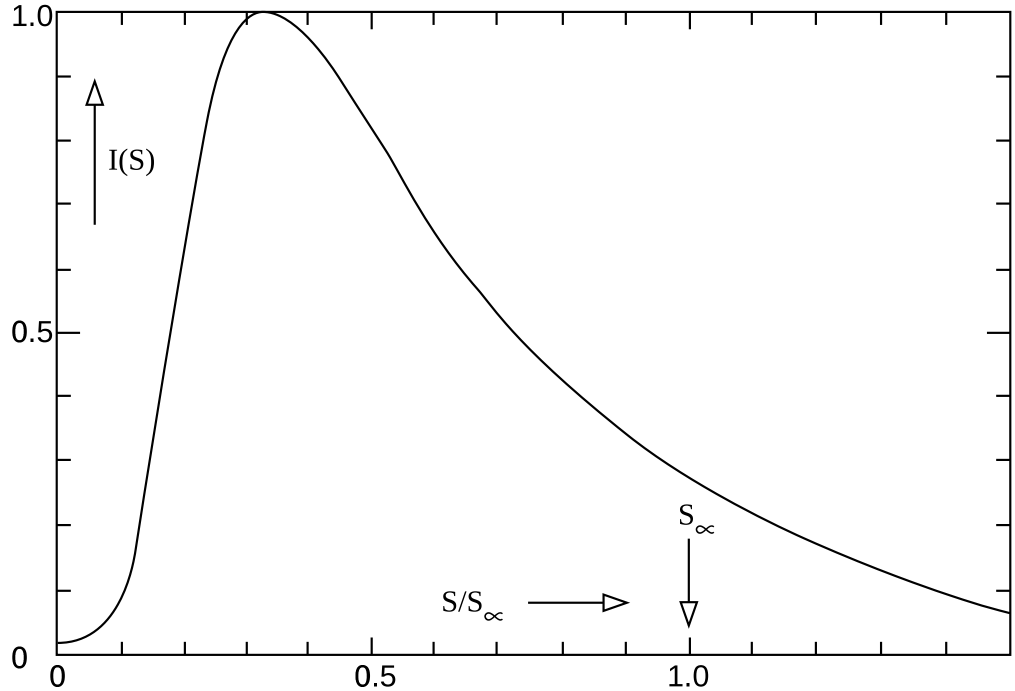

Figure 1: Deflection pattern for atoms having a unique value of μ_B. Undeflected beam width, a, is $s_\alpha/10$.

where the half width of the undeflected beam is a, $a \ll s_\alpha$, and its intensity is I_0. For non-zero a, we must integrate the above equation over the width of the undeflected beam. The resulting pattern, for $a = s_\alpha/10$, is shown in Fig. 1. A peak occurs at s_{max}, which equals $s_\alpha/3$.

Each value of μ_B gives rise to a pattern similar to Fig. 1, but with a different value for s_α and therefore for s_{max}. The two values of μ_B given by quantum theory produce two patterns, one as shown, and one its mirror image. Classically an infinite number of values of μ_B occur, and the resultant composite pattern could be a roughly bell-shaped curve centered at $s = 0$.

Summary of Predictions of z direction distributions

1. The classical (pre-quantum theory) predicts a uniform distribution between the limits. This is smeared by the thermal velocity distribution.

2. The Schrödinger Quantum Mechanics (no spin) predicts an **odd** number of groups. In the case of Potassium it predicts one group.

3. The Dirac Theory predicts intrinsic spin. In the case of Potassium it predicts **two** groups.

The Apparatus

Potassium Oven

A slug of potassium has been sealed into the oven by a steel plug and sealed with a copper gasket. The oven is thermally isolated on 3 mounts and is heated by a nichrome cartridge heater. The heater is powered by a variable AC source at about 3.5 Watts (40 V rms) producing an oven temperature of about 135° C. The oven operating temperature is above the 62° C melting point of potassium and under these conditions a hot potassium vapor beam exits the oven through a small hole covered by two steel precision slits. The heater has been carefully located with respect to the slits so that the slit area is kept hotter than the bulk storage chamber and is therefore kept clean.

Oven Heater

The oven heater is driven from an AC variable voltage source. The potassium atomic beam intensity depends upon the vapor pressure of the potassium and this depends critically upon the temperature. The stabilized heater thus ensures a steady beam after a warm-up period. The oven is typically run somewhere between 120-150° C depending on the condition of the potassium load.

Monitoring the Oven Temperature

The temperature of the oven is monitored by an iron-constantan thermocouple which passes through the oven mounting face. The thermocouple is connected to a voltmeter calibrated in °C. The approximate vapor pressure of potassium is 10^{-6} Torr at 63°C and 10^{-3} Torr at 161°C.

Vacuum System

A pressure of less than 5×10^{-6}mm Hg (Torr) is necessary for a good signal since the potassium atoms can be easily scattered. The system is pumped by a turbo pump which vents to a mechanical forepump. The vacuum is monitored by two triode vacuum gauges-one on either side of the magnet box. Migration of unwanted potassium through the vacuum system from the oven to the detector is reduced by additional slits and by running the oven at low output. The slits force any potassium atom travelling from the oven to the detector to strike at least one surface before reaching the detector unless the atom is travelling along the prescribed path of the potassium beam. Tempered pyrex glass pipe forms the major part of the vacuum system and allows visual monitoring of the position of the beam gate, the temperature and position of the detector, and possible accidental contamination of the oven end of the system by potassium or pump vapors.

Beam Gate

The beam gate is attached to the long vertical baffle seen in Fig. 2 in the cross tube. It is able to pivot and stay in position, either open or closed, since its center of gravity is higher than the pivot point. The gate is moved from one position to another by a hand-held magnet outside the vacuum system, and rests against the glass in both positions, either fully open or fully closed. When the gate is closed, the oven end of the system is well separated from the detector end of the system.

The overall layout of the potassium beam path is shown in Fig. 3.

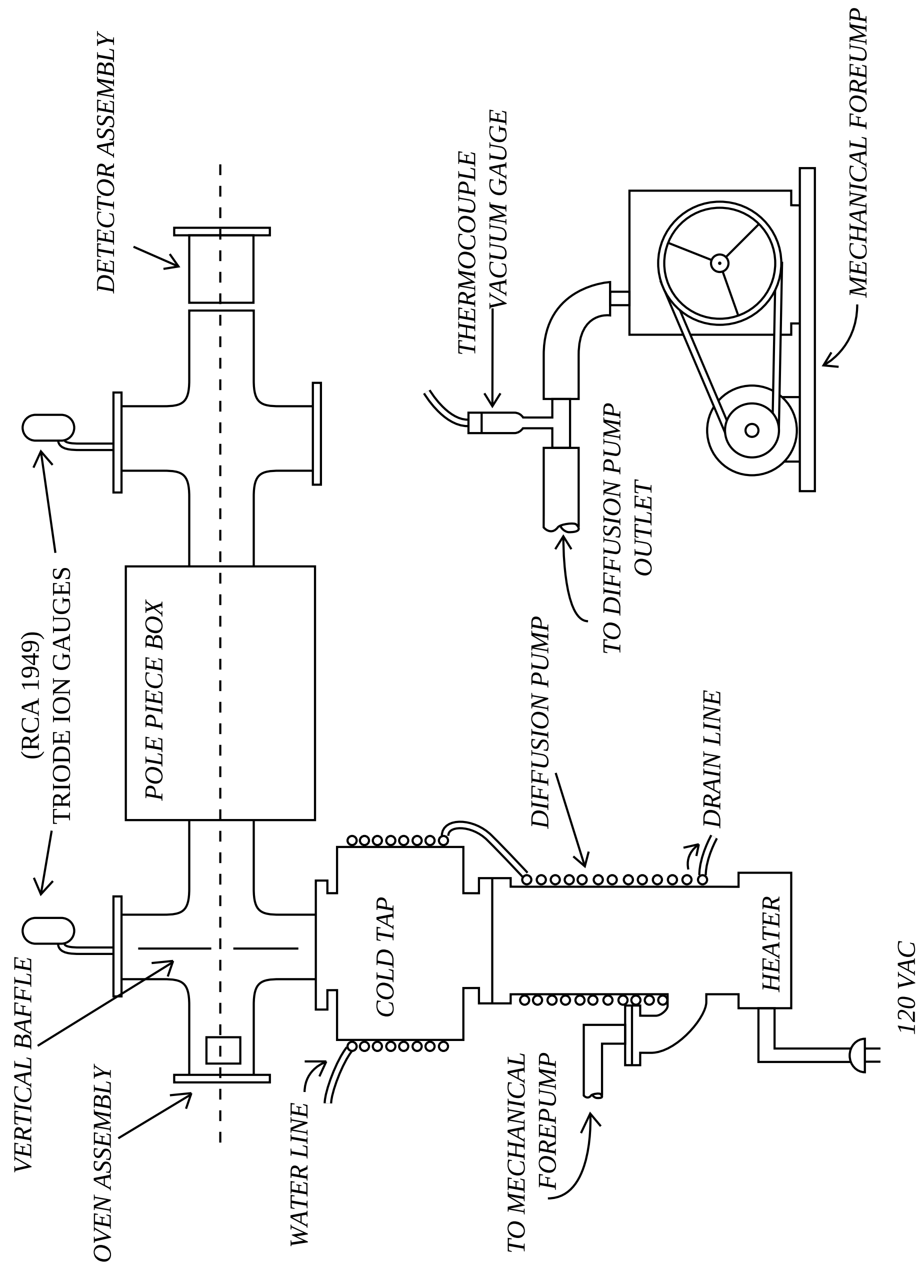
DETECTOR ASSEMBLY
(RCA 1949)
TRIODE ION GAUGES
POLE PIECE BOX
VERTICAL BAFFLE
OVEN ASSEMBLY
WATER LINE
COLD TAP
TO MECHANICAL
FOREPUMP
DIFFUSION PUMP
DRAIN LINE
HEATER
120 VAC
THERMOCOUPLE
VACUUM GAUGE
TO DIFFUSION PUMP
OUTLET
MECHANICAL FOREUMP

Figure 2

POTASSIUM BEAM PATH

What does the Stern-Gerlach apparatus look like from the inside?

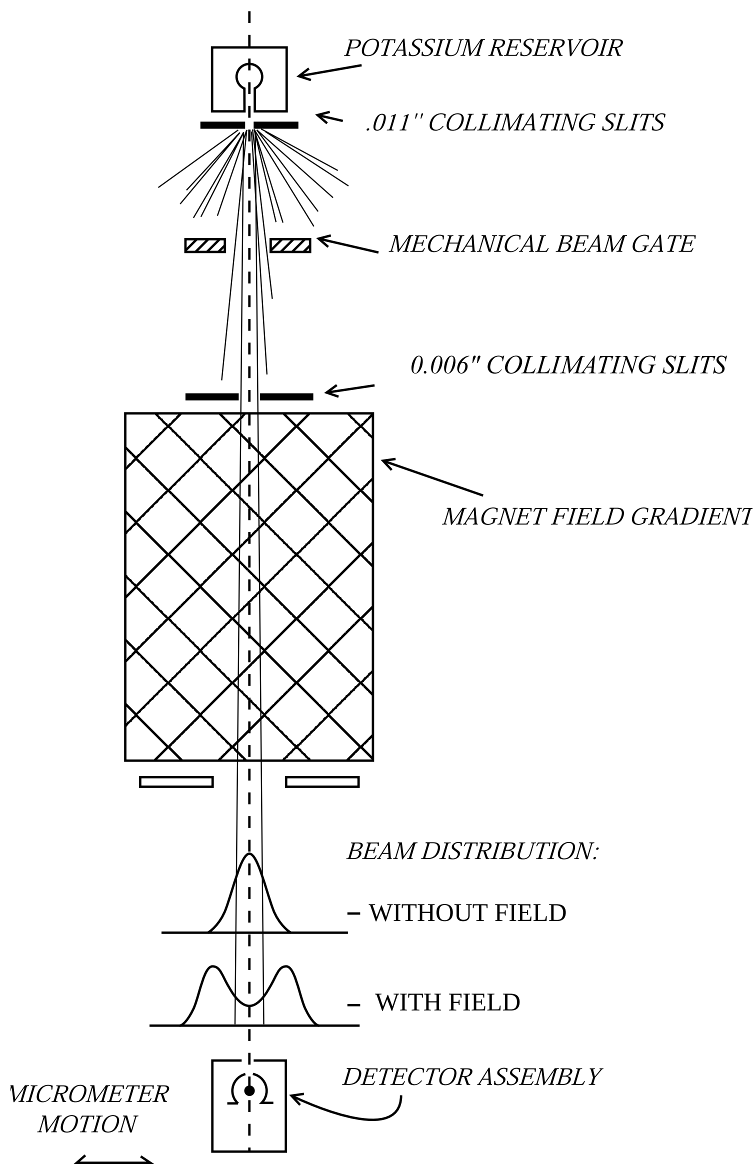

Figure 3

Vacuum System Maintenance

The maximum operating pressure of the Stern-Gerlach apparatus is approximately 5.0 $\times 10^{-6}$mm of Hg. At higher pressures the potassium beam is scattered by the residual gases and no beam is detected.

The present system, when uncontaminated will maintain a vacuum better than 1.0×10^{-6}mm of Hg with the oven off.

Due to the finite pumping speed through the pole piece box, the pressure at the detector end will be slightly higher than the pressure in the oven end.

It is important to remember that:

a. the ion gauges are never to be operated at pressures above 1.0×10^{-4}mm of Hg.

b. the grid current can be reduced by a factor of [10] to lengthen filament life; (in this case the pressure reads lower by factor of [10] also).

c. The turbo pump must always be on. The pressure will rise rapidly if the turbo pump is off.

d. the mechanical forepump must operate **continuously** to vent the tubo pump. The rubber hose from the forepump to the turbo pump must be free from crimps or sharp bends that would restrict the flow of gas inside.

Triode Ion Gauges

The 1949 ion vacuum gauge contains a plate, a grid, and two tungsten filaments, one of which is a spare. The filament is operated at a voltage such that all of the emitted electrons are drawn to the grid. The grid is operated at a positive potential (V_p). Electrons from the filament are accelerated to the positive grid. They bombard and ionize some of the molecules of any gas present in the tube. The resulting positive ions are attracted to the negative plate and constitute the plate current (I_p). The ratio of this current to the grid current (I_c) is proportional to the gas pressure for pressures below about 0.0001mm Hg (10^{-4} Torr). The pressure gauge meter reads correctly for 8 mA grid current.

Nonuniform Magnetic Field

A cross section of the pole tips is shown in Fig. 4. The iron pole tips consist of a .218 inch diameter convex half cylinder spaced approximately 0.155 inch (3.97mm) from a .500 inch (1.27cm) diameter concave half cylindrical groove.

The concave circle, if extended, intersects the convex circle at the ends of its vertical diameter. These two curves are magnetic equipotentials of the magnetic field that would be present if the pole tips were replaced by two current carrying wires centered on the intersection points of the circles. The magnet is designed to approximate this field, which has a reasonably homogeneous gradient in the region the beam is allowed to pass through.

Field strength. A magnet capable of producing 0.4 Tesla (4000 gauss) across a .750 inch (1.91 cm) gap with a face 4 inches (10.16 cm) in diameter will produce the best results (this corresponds to about 7000 ampere turns), although respectable Stern-Gerlach patterns can be obtained with half that strength. A 7000 ampere turn magnet produces a gradient $\vec{\nabla} B$ of a little over 100 Tesla/meter.

Hot Wire Detector

The hot wire detector is an important advance over the techniques used in the original Stern-Gerlach experiment. With it, the beam can be continuously sampled and experimental runs can be completed in a short time. The detector is a hot pure tungsten filament wire, .005 inch in diameter,

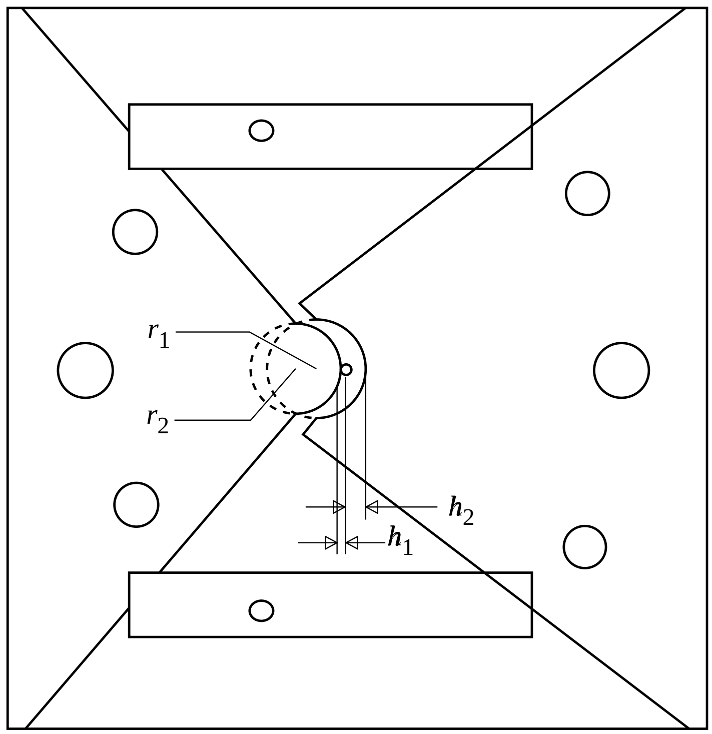

Figure 4: View of the pole unit looking “down the beam path” with one end piece removed. The extension of the concave circle (r_1 = .25 inches) intersects the ends of the vertical diameter of the convex circle (r_2 = .218 inches). The dimensions of the spacings are h_1 = .055 inches and h_2 = .100 inches.

surrounded by a collecting cylinder kept at a voltage of about 15 volts below the wire. The collecting cylinder has a narrow slit aperture to admit the potassium atoms. The wire is heated by a variable stabilized voltage supply at about 1.4 A which "floats" at the bias voltage of 15 volts

The detector can be moved laterally, by means of a micrometer, from outside the system; its position is indicated by the scale on the micrometer drive. The micrometer shaft passes through an O-ring into the vacuum chamber.

Principle of Operation

The potassium atoms ionize when heated at the surface of wire. A voltage placed across the collector and the hot wire will collect the positive ions if the collector is negative with respect to the hot wire; the small current is detected by a picoammeter or electrometer.

The collector and the hot wire together are similar to a vacuum diode with a directly heated cathode. In the detector, the "plate" is always negative with respect to the "cathode." In fact, if the bias voltage were reversed, the detector would act like a conducting diode; the collector would collect the electrons from the potassium and, in addition, it would collect electrons from the electron gas surrounding the hot wire.

The picoammeter current obtained is dependent on the filament temperature. A low filament current causes a low temperature and hence a low ionization rate and hence sluggish picoammeter readings. A high filament current will shorten the lifetime of the filament. DO NOT EXCEED 2.0 A AT ANY TIME. A current of 1.4 A is normally used.

The ionization rate increases temporarily whenever the filament temperature is increased slightly as absorbed potassium atoms are boiled off. For this reason you should wait for about 5 minutes after changing the filament current before trusting the picoammeter readings. The setting of the bias voltage is not critical. A voltage of 15 volts is recommended.

A micrometer is used to move the detector across the beam. DO NOT EXCEED a micrometer setting of 325 mils. The micrometer screw rod forms part of the seal in maintaining a good vacuum. If the rod is pulled out farther than 350 mils the rod could slip out of the O-ring and this could destroy the vacuum and may damage the turbo pump.

Operating Conditions

1. Heat the oven at 20 W (100 V rms)to an operating temperature of about 145°C. Back off the heater voltage to $\sim$ 40 V when close to 140° so you don't overshoot. Stabilize the temperature at the normal operating temperature of about 145°. It is very important to make the actual measurements with a stable oven temperature.

2. The recommended hot wire current is 1.4 A. Sometimes the detector wire has to be baked at $\sim$1.7 A for 10 min. to clean the wire. Consult with the instructor.

3. This experiment requires a good vacuum. Good signals will be obtained when the starting pressures before the oven bake are $\leq 2 \times 10^{-6}$ Torr.

4. The signal current at the undeflected beam peak with a good vacuum can be as high as 10 pA. Depending on the conditions of the system it may be less.

Procedure

1. Estimate the expected width of the beam at the detector using the slit information in Fig. 3. You need to know the expected width of the beam since you are going to scan with the hot wire detector and you need to know what step size to use or you may miss the signal altogether.

2. Set up according to the above conditions and scan for the signal with the magnet off. Once you know where to look you should do a first pass to locate the peak.

3. At the peak, see if you can reduce the detector wire current without losing appreciable signal. This will result in a longer lifetime of the detector before it becomes contaminated with potassium. The bias voltage is not critical.

4. Do a fine scan over the peak. Each point may require both closed and open shutter readings. However if the background is stable you may only have to take a closed shutter reading at the beginning and at the end of a scan. Your data should be reproduceable.

5. Do the same with the magnet on at $\sim$1.0 A. You should immediately notice that the signal at the previous peak position is significantly smaller. Do a complete scan to map out the new distribution.

6. Possible additional studies include a different oven temperature and/or a different magnet setting.

Physics Analysis and Questions

1. Derive an expression for the most probable velocity of the potassium atoms in the beam and compare to the most probable value of the velocity in the oven. Why are they different? Evaluate both quantities at the oven operating temperature.

2. Calculate the magnetic field gradient from your data assuming the predicted magnetic moment for Potassium. This will involve making some geometry measurements and deciding on a procedure for analyzing the data. We can either compute the electron magnetic moment having a value for the magnetic field gradient or compute the magnetic field gradient having a value for the electron magnetic moment. We do the latter since we have no way to measure the magnetic field gradient. The field region is in vacuum and the pole piece gap is very small.