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International Centre
for Mechanical Sciences

Mechanics and Electrodynamics of Magneto- and Electro-Elastic Materials

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

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MECHANICS AND ELECTRODYNAMICS
OF MAGNETO- AND ELECTRO-ELASTIC
MATERIALS

EDITED BY

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PREFACE

This volume consists of Lecture Notes based on lectures delivered at the Advanced School on “Mechanics and Electrodynamics of Magneto- and Electro-elastic Materials” held at the International Centre for Mechanical Sciences (CISM) in Udine, Italy, in the period June 29 to July 3, 2009. The course was presented by 5 lecturers, from France, Germany, Italy, the UK and the USA.

The theory of electromagnetic continua has received considerable stimulus in the last few years because of the rapid development of elastomeric and polymeric materials that can respond dramatically to the application of an electric and/or magnetic field. Such materials, often referred to as ‘smart materials’, are being used in a variety of applications, ranging from high-speed actuators, sensors, and active car suspensions and vibration isolators, to artificial muscles, and other biomedical applications. The key point is that the mechanical properties of the materials can be changed rapidly and substantially by externally applied electromagnetic fields. Thus, the coupling between mechanics and electromagnetism is both strong and highly nonlinear. Mathematical and computational methods pervade research, development, testing, and evaluation problems encountered by researchers in the field of smart materials, and associated modelling issues have a fundamental role in the analysis of problems that arise in such complex materials.

Against this background the objective of this volume of lecture notes is to provide a state-of-the-art overview of the nonlinear continuum theory of both electro- and magneto-sensitive materials, theories that are applicable, in particular, to elastomers and polymers. This includes mathematical and computational aspects of the modelling of these materials from the point of view of material properties and the ‘smart-material’ control of their mechanical properties.

The first chapter includes a historical perspective of the general development of electromagnetic theory and its application to the mechanics of continua, including discussion of the notions of electromagnetic forces, internal stresses and configurational forces. This is followed by two chapters which deal separately with the fundamentals of electrostatics and magnetostatics, through to the coupling with continuum mechanics to produce general theories of nonlinear elec-

troelasticity and magnetoelasticity. In each case the necessary background from continuum mechanics is included, and constitutive laws describing electro-active and magneto-active deformable materials are developed and representative boundary-value problems are analyzed.

The last three chapters focus on different aspects of the interaction of mechanics and electric fields. There is a detailed analysis of the reduction from the three-dimensional equations of nonlinear electroelasticity to a two-dimensional model of an electroelastic membrane and the illustration of the theory for a particular membrane problem. Then follows a development of the necessary ingredients for the computational solution of boundary-value problems in nonlinear electroelasticity using a combination of the finite element method and the boundary element method, and example results are provided by way of illustration. The final chapter is concerned with both the static and dynamic response of a special class of materials, nematic elastomers, and summarizes recent developments in mathematical, computational and experimental aspects of their electro-mechanical coupling.

The combination of fundamental theory with application to precisely formulated, experimentally feasible problems, together with numerical simulation and the exploration of open questions remaining in the underlying framework of continuum electrodynamics results in a unique volume of lecture notes that is not available elsewhere.

We have pleasure in thanking our colleagues, Gérard Maugin, Luis Dorfmann, Paul Steinmann and Antonio DeSimone, for presenting their lectures and for preparing chapters for this volume. To the participants, who contributed to lively discussions we offer our grateful thanks. Special thanks are due to the Rector of CISM, Professor Giulio Maier, for his encouragement, enthusiasm and hospitality, and to Professor Paolo Serafini, Executive Editor of CISM, for his encouragement to publish these lecture notes. The assistance of the office staff at CISM was also much appreciated.

*Ray Ogden
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Electromagnetics in Deformable Solids

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Abstract. This series of lectures deals with the basics of electromagnetics in matter, first from point particles and then for a general continuum subjected to finite strains. The emphasis is placed on the notions of electromagnetic forces, momentum and stresses, on the general thermomechanical framework, and on applications to electroelasticity and magnetoelasticity at different scales, in particular with the introduction of the notions of internal stresses, electromagnetic internal variables of state, homogenization, polycrystals and configurational forces.

1 Introduction

1.1 Basic Aspects: Maxwell, Heaviside, Lorentz: Physics versus Electrical Engineering

In a time of rapid evolution of the conception of devices exploiting multiphysical couplings and of the required solution of problems posed by such devices, we must inevitably deal with the construction and analysis of physico-mathematical models where electro-magneto-mechanical interactions are at play. In spite of many works that elaborated on the matter in the past, there are still vivid discussions on how to couple the mechanics of continua and electromagnetic fields in a harmonious whole. This is presently marked by the publishing of papers that often ignore what was previously accomplished. Indeed, looking back upon previous achievements we must distinguish between different periods.

The first half of the 19th century is the time of the land-clearers such as Amp re, Faraday, Gauss, Poisson, and Oersted. The second half of the 19th is the time of unification in a grand scheme involving electricity and magnetism on equal footing, and culminating in the works of Kelvin, Weber, Helmholtz, and above all, Maxwell (1873) and Heaviside (1892) (to whom we owe the presently used form of Maxwell's equations). In parallel, coupled

effects of the electromechanical, magnetomechanical and galvano-magnetic types where discovered, among which electric conduction, piezoelectricity (Curie brothers) and magnetostriction (Joule) are still those that steer attention because of the many received applications. Then there followed a long period, early and first part of the 20th century, during which many relevant discussions were devoted to the relativistic framework, while electrical engineering took the front with applications to energy productive or transforming machines and to macroscopic electromechanical devices. It is only in the second part of the 20th century that we witness an indepth thinking about the continuum representation of multiphysical couplings with works of Truesdell and Toupin (1960), Brown (1966), Mindlin (1972), Nelson (1979), Eringen (1980), Tiersten (1990), to which we associate ourselves (Maugin and Eringen, 1977; Maugin, 1988; Eringen and Maugin, 1990) as we clearly agree with many of these developments, in particular with due consideration of interaction forces, and this in a pre fast-computer age. In parallel one must account for the constructive works of physicists such as Lorentz (1909) and De Groot and Suttrop (1972). Here we cite essentially books in which the provided material is rationally re-organised and not the many research papers which may have preceded these.

Much more recently new formal developments have appeared that often ignore, or are less physically rooted, than the just mentioned developments. Among these we note, e.g., the book of Kovetz (2000) and research papers by Dorfmann and Ogden (2003, 2004, 2005, 2006), Steigmann (2004), Ericksen (2007, 2008), Kankanala and Triantafyllidis (2008), Otténio et al. (2008), and probably others that we overlooked.

Here, in the light of above-mentioned references, we revisit the central notion of interactions between the mechanical system and the Maxwellian concepts which were *a priori* separated, the general agreement being that this cannot be just a linear superimposition.

The thermomechanics of solely deformable material continua and the electromagnetism of vacuum are two well established bodies of knowledge. The main question arises when material continua and electromagnetic fields co-exist spatially. It is then agreed upon that the relevant Maxwellian fields in matter, magnetic field \mathbf{H} and electric displacement \mathbf{D} differ from the characteristic electromagnetic fields of vacuum, the magnetic induction \mathbf{B} and the electric field \mathbf{E} , in such a way that with appropriate electromagnetic units (so-called Lorentz–Heaviside units) we have the equations

$$\mathbf{H} = \mathbf{B} - \mathbf{M}, \quad \mathbf{D} = \mathbf{E} + \mathbf{P}, \quad (1)$$

where \mathbf{M} and \mathbf{P} are the magnetization and electric polarization per unit volume, fields that differ from zero only in magnetized and electrically polarized

matter, respectively, i.e. when the celebrated set of Maxwell's equations in a fixed laboratory frame reads in full generality, according to Heaviside,

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \quad \nabla \cdot \mathbf{B} = 0, \quad (2)$$

and

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c} \mathbf{J}, \quad \nabla \cdot \mathbf{D} = q_f, \quad (3)$$

where c is the velocity of light in vacuum, \mathbf{J} is the electric current vector, and q_f is the density of free electric charges, the first set (2) being valid everywhere and yielding the notion of electromagnetic potentials. In general, to close the system of field equations (1)–(3), we are to be given electromagnetic constitutive equations, e.g., to give an idea to the reader, functional relations of the type

$$\mathbf{M} = \mathbf{M}(\mathbf{H}, \cdot), \quad \mathbf{P} = \mathbf{P}(\mathbf{E}, \cdot), \quad \mathbf{J} = \mathbf{J}(\mathbf{E}, \cdot), \quad (4)$$

where the dots stand for some other variables such as temperature or a strain in a deformable solid. While other possibilities exist, the selection (4) of dependent variables is not gratuitous. It pertains to the *characteristic* electromagnetic fields of *matter*. Several remarks are in order. First, by taking the divergence of (3)₁ and accounting for (3)₂, we obtain the law of *conservation of electric charge*:

$$\frac{\partial q_f}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (5)$$

a *strict* conservation law. Second, by a usual manipulation, one also deduces from (2)–(3) an *energy identity* called the “Poynting-Umov theorem”, such that

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = -\mathbf{J} \cdot \mathbf{E} - \nabla \cdot \mathbf{S}, \quad \mathbf{S} \equiv c \mathbf{E} \times \mathbf{H}, \quad (6)$$

without any hypothesis concerning the electromagnetic constitutive equations.

Third, *if we are in a vacuum* (for which the three quantities in (4) vanish identically), long before the proof of her “invariance” theorem by E. Noether, Maxwell proved the existence of the following vectorial strict conservation law:

$$\frac{\partial \mathbf{p}^{em.f}}{\partial t} - \operatorname{div} \mathbf{t}^{em.f} = \mathbf{0}, \quad (7)$$

wherein the electromagnetic momentum (in vacuum) and the so-called (symmetric) Maxwell stress tensor (stress tensor of *free* electromagnetic fields)

are defined by

$$\begin{aligned} \mathbf{p}^{em.f} &= \frac{1}{c} \mathbf{E} \times \mathbf{B}, & \mathbf{t}^{em.f} &= \mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B} - u^{em.f} \mathbf{1}, \\ u^{em.f} &= \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2), \end{aligned} \quad (8)$$

where the last quantity, $u^{em.f}$, is the electromagnetic energy of free fields per unit volume. This is a peculiar expression that holds here because of the inherent linearity of the electromagnetic constitutive equations ($\mathbf{H} = \mathbf{B}$, $\mathbf{D} = \mathbf{E}$, $\mathbf{J} = \mathbf{0}$) in vacuum. The latter serves as a (nonpolarized) *medium of comparison* for other electromagnetic media (an idea that will be successfully translated into mechanical behavior by J. R. Willis for studying effective properties of composites and deviations from a standard homogeneous electric model). Dealing with *energy* in a magnetized, electrically polarized, and conducting material in electromagnetism is a much more subtle matter as shown by the equation (6). The latter can be integrated in a usual conservation form for a global volume only if the electromagnetic constitutive equations are *linear* and the body is rigid. Indeed, with simple constitutive equations $\mathbf{B} = \mu \mathbf{H}$, $\mathbf{D} = \varepsilon \mathbf{E}$, for a rigid body occupying volume V bounded by regular boundary ∂V , of unit outward pointing normal \mathbf{n} , from (6) we would have the global balance of electromagnetic energy

$$\frac{d}{dt} \int_V u^{em.m} dV = - \int_V \mathbf{J} \cdot \mathbf{E} dV - \int_{\partial V} \mathbf{n} \cdot \mathbf{S} dA, \quad (9)$$

with

$$u^{em.m} = \frac{1}{2} (\varepsilon \mathbf{E}^2 + \mathbf{B}^2/\mu) = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}). \quad (10)$$

But this is **generally not true** as an expression for electromagnetic energy in an arbitrary deformable solid where (i) the electromagnetic constitutive equations may be strongly nonlinear and may even be dissipative (e.g., with relaxation, hysteresis); (ii) electromagnetic fields do **not** constitute an isolated thermodynamical system and they are in strong interaction with the deformation field. A consequence of this fact is that, if (6) is always true, it does not constitute a local statement of energy conservation for the *whole* mechanical-*plus*-electromagnetic system (sorry, the “plus” may be misleading with a connotation of simple “addition”). Similarly, equation (7) does not constitute an equation for conservation of so-called canonical momentum for the whole system. Much more work is required to reach this general result. What is remarkable is that, in spite of these words of caution, many authors have a natural tendency to think of expression such as (10)₂ as a starting point in any electromagnetic continuum. This is particularly true in

relativistically invariant theories where the *a priori* viewpoint of Minkowski concerning electromagnetic momentum and electromagnetic stress tensor (there the energy-momentum tensor) has been damaging. But Minkowski's reasoning is not based on a sophisticated physical model of field-matter interactions. The same must be said of Abraham's proposal (see all these in Maugin and Eringen, 1977; also Eringen and Maugin, 1990, pp. 62–64). This is where we fully agree with Nelson (1979), Tiersten (1990), and also Livens (1962). The same remark also applies concerning another energy quantity such as a Lagrangian density per unit volume. The density

$$\rho^{em.f} = \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) \quad (11)$$

strictly applies only to electromagnetic fields in a vacuum although it was proposed by authors such as Voigt, following Thomson and Maxwell, in analogy with a “mechanical” Lagrangian with kinetic and potential contributions. All this clearly means that part of the electromagnetic energy and of Lagrangian densities is stored also in the internal/free energy or “matter” Lagrangian for the combined mechanical-*plus*-electromagnetic medium that includes the missing interaction terms that should be expressed in terms of the essentially material fields (4).

One remark about the electric current. For all practical purposes, we note that the Joule term $\mathbf{J} \cdot \mathbf{E}$ can be interpreted as a power expended by an electric force. Indeed, we can write as an example

$$\mathbf{J} \cdot \mathbf{E} = (q \mathbf{v}) \cdot \mathbf{E} = (q \mathbf{E}) \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v}, \quad (12)$$

where $\mathbf{f} = q \mathbf{E}$ is seen in statics, according to Lorentz, as the elementary mechanical force acting on a point particle of electric charge q in an electric field \mathbf{E} . For a particle moving at velocity \mathbf{v} , we have the *Lorentz force*

$$\mathbf{f} = q \mathbf{E} + \frac{q}{c} \mathbf{v} \times \mathbf{B} = q \tilde{\mathbf{E}}, \quad \tilde{\mathbf{E}} = \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}, \quad (13)$$

where the electric field $\tilde{\mathbf{E}}$ is called the *electromotive intensity*.

Relationship with electric engineering

* Faraday's equation (2)₁ relates the circuit voltage that appears when the flux linkage varies in time, as in electrical generators. Indeed, by use of Stokes' theorem applied to a surface element S leaning on a circuit C , one shows that the difference of potential is given by

$$e.m.f = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}, \quad (14)$$

or

$$e = \frac{d\lambda}{dt} \quad (15)$$

in terms of the flux linkage λ .

* Ampère's law $(3)_1$ relates the magnetic field that curls around a current flux, corrected for unsteady values of electric fields (this last correction is due to Maxwell; cf. the notion of displacement current). By use of Stokes' theorem to a surface element S leaning on a circuit C , one finds, for a coil of n turns of length ℓ , the relation

$$\int_S \mathbf{H} \cdot d\ell = nI \quad \text{or} \quad H = nI/\ell, \quad (16)$$

where I is the current.

* Gauss–Poisson's equation $(3)_2$ tallies the field lines emanating (hence the divergence) from a distribution of charges.

* The last of Maxwell's equations $(2)_2$ reflects the circumstance that isolated magnetic poles do not exist. As a consequence a line of magnetic induction closes on itself. It does not “emanate” from a magnetic charge distribution as the latter does not exist.

Ampère's and Gauss–Poisson's equations are not used as such in circuitry, but the law of conservation of charges (5) yields, by integration, the circuitry equation

$$I = \frac{dq}{dt}, \quad (17)$$

where q is the electric charge. Then the system (14) – (17) is closed by the well known *constitutive equations* of passive circuit elements:

$$\begin{aligned} \lambda &= LI \quad (\text{inductor}), \\ q &= Ce \quad (\text{capacitor}), \\ e &= RI \quad (\text{resistor}), \end{aligned} \quad (18)$$

where L , C and R are an *inductance*, a *capacitance*, and a *resistance*, respectively. The last of these represents the celebrated Ohm law. There exist nonlinear generalizations of the constitutive equations (18) . Added to Kirchhoff's laws of currents at nodes, the above set equations (14) through (18) are all what one needs at the macroscale of *electricity*. Now we explore the other extreme, what happens at the microscopic scale.

2 Continuum Approach from Particle Approach: Ponderomotive Force, Couple and Energy

2.1 Information from a Microscopic Model

Rich information about the interactions between the mechanical system and electromagnetic fields in matter may be gained from a particle model due initially to Lorentz (1909). It was taken over by Dixon and Eringen (1964), Maugin and Eringen (1977) and Nelson (1979), to who we owe the present formulation. This analysis consists in evaluating the total force, couple and power acting on, or developed by, electromagnetic fields on the elementary electric charges contained in a stable cloud or volume element, and introducing the approximations of multipoles, a truncation of these at a certain order, and a volume or phase-space average. Lorentz's vision is essentially that of a free space containing point charged particles (Figure 1). To that purpose, each elementary electric charge δq_α , $\alpha = 1, 2, \dots$ contained

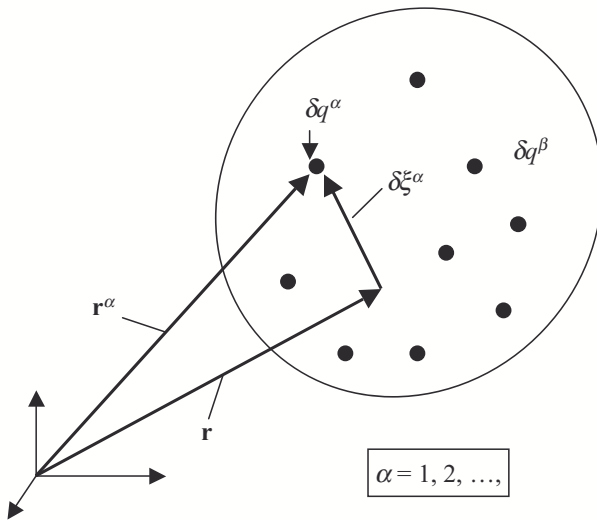


Figure 1.

in a volume element ΔV is acted upon by a *Lorentz force* (compare (13))

$$\delta \mathbf{f}_\alpha = \delta q_\alpha \left(\mathbf{e}(\mathbf{r}_\alpha) + \frac{1}{c} \dot{\mathbf{x}}_\alpha \times \mathbf{b}_\alpha(\mathbf{r}_\alpha) \right), \quad (19)$$

where \mathbf{e} and \mathbf{b} are the electric field and magnetic induction at the current placement \mathbf{r}_α of the charge δq_α . The computation consists then in evaluating the quantities (here, for the sake of simplicity, we adopt a simple volume average, while De Groot and Suttrop use a relativistically invariant phase average):

$$\sum_{\alpha \in \Delta V} \delta \mathbf{f}_\alpha, \quad \sum_{\alpha \in \Delta V} (\mathbf{r}_\alpha \times \delta \mathbf{f}_\alpha), \quad \sum_{\alpha \in \Delta V} \delta \mathbf{f}_\alpha \cdot \dot{\mathbf{x}}_\alpha, \quad (20)$$

and then dividing by ΔV . On neglecting quadrupole contributions and higher order multipoles, lengthy calculations (cf. Maugin and Eringen, 1977; Eringen and Maugin, 1990) lead to electromagnetic source terms of force, couple and energy per unit continuum volume:

$$\mathbf{f}^{em} = q_f \tilde{\mathbf{E}} + \frac{1}{c} (\tilde{\mathbf{J}} + \mathbf{P}^*) \times \mathbf{B} + (\mathbf{P} \cdot \nabla) \tilde{\mathbf{E}} + (\nabla \mathbf{B}) \cdot \tilde{\mathbf{M}}, \quad (21)$$

$$\mathbf{c}^{em} = \mathbf{r} \times \mathbf{f}^{em} + \tilde{\mathbf{c}}^{em}, \quad (22)$$

$$w^{em} = \mathbf{f}^{em} \cdot \mathbf{v} + \tilde{\mathbf{c}}^{em} \cdot \Omega + \rho h^{em}, \quad (23)$$

where \mathbf{r} refers to the center of charges of the volume element, ρ is the matter density, and \mathbf{v} is the physical velocity, Ω is the vorticity $\Omega = (\nabla \times \mathbf{v})/2$, and we have set

$$q_f(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \delta q_\alpha, \quad (24)$$

$$\mathbf{P}(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \delta q_\alpha \xi_\alpha(\mathbf{x}, t), \quad (25)$$

$$\mathbf{M}(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \frac{1}{2c} \delta q_\alpha \xi_\alpha \times \dot{\xi}_\alpha, \quad (26)$$

where $\xi_\alpha = \mathbf{x}_\alpha(t) - \mathbf{x}$ are internal coordinates vectors in ΔV . Note the lack of symmetry between polarization and magnetization effects. We have also defined the intrinsic electromagnetic sources of couple, energy and stress by (here $\text{tr} = \text{trace}$; subscript s stands for symmetrization)

$$\tilde{\mathbf{c}}^{em} = \mathbf{P} \times \tilde{\mathbf{E}} + \tilde{\mathbf{M}} \times \mathbf{B}, \quad (27)$$

$$\rho h^{em} = \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}} + \tilde{\mathbf{E}} \cdot \mathbf{P}^* - \tilde{\mathbf{M}} \cdot \mathbf{B}^* + \text{tr} (\tilde{\mathbf{t}}^{em} (\nabla \mathbf{v})_s), \quad (28)$$

and

$$\tilde{\mathbf{t}}^{em} = \mathbf{P} \otimes \tilde{\mathbf{E}} - \mathbf{B} \otimes \tilde{\mathbf{M}} + (\tilde{\mathbf{M}} \cdot \mathbf{B}) \mathbf{1}, \quad (29)$$

where the following fields are those in a co-moving frame (*Galilean approximation*; first of these is the conduction current *per se*):

$$\tilde{\mathbf{J}} = \mathbf{J} - q_f \mathbf{v}, \quad \tilde{\mathbf{E}} = \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}, \quad \tilde{\mathbf{M}} = \mathbf{M} + \frac{1}{c} \mathbf{v} \times \mathbf{P} \quad (30)$$

and \mathbf{E} and \mathbf{B} are simple volume averages of \mathbf{e} and \mathbf{b} . The first contribution in the r - h - s of (21) is none other than a ‘‘Lorentz force’’ (compare (19) and (13)) since

$$\mathbf{f}_L = q_f \mathbf{E} + \frac{1}{c} (q_f \mathbf{v}) \times \mathbf{B} = q_f \tilde{\mathbf{E}}. \quad (31)$$

Finally, a right asterisk denotes a so-called convected time derivative such that

$$\mathbf{P}^* = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{v}) + \mathbf{v} (\nabla \cdot \mathbf{P}) = \frac{d\mathbf{p}}{dt} - (\mathbf{P} \cdot \nabla) \mathbf{v} + \mathbf{P} (\nabla \cdot \mathbf{v}). \quad (32)$$

In principle, the above obtained *source terms*, once their origin forgotten, have to be carried into the classical balance laws of a continuum (with a possible *non symmetric* Cauchy stress), leaving however the internal/free energy of the medium to depend on the electromagnetic fields. A remarkable fact is that in spite of their farfetched outlook, some may be given a form that reminds us of some standard expression (such as in (5)). For instance, Maugin and Eringen (1977) have shown that (23) can also be written as

$$\begin{aligned} w^{em} &= \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v} (\mathbf{E} \cdot \mathbf{P})) \\ &= -\frac{\partial u^{em.f}}{\partial t} - \nabla \cdot (\mathbf{S} - \mathbf{v} (\mathbf{E} \cdot \mathbf{P})), \end{aligned} \quad (33)$$

in which we identify some of the terms in (6) or a possible direct combination with some of them.

Some identities. The electromagnetic volume force defined in (21) is sometimes called the electromagnetic *ponderomotive force*, $\tilde{\mathbf{c}}^{em}$ being then the *ponderomotive* couple. In 1974, Collet and Maugin (1974) proved the following remarkable identity at all regular material point:

$$\frac{\partial \mathbf{p}^{em}}{\partial t} - \operatorname{div} \mathbf{t}^{em} = -\mathbf{f}^{em}, \quad (34)$$

where

$$\mathbf{p}^{em} = \mathbf{p}^{em.f} = \frac{1}{c} \mathbf{E} \times \mathbf{B}, \quad (35)$$

$$\mathbf{t}^{em} = \mathbf{t}^{em.f} + \tilde{\mathbf{t}}^{em}. \quad (36)$$

Since we are dealing with nonsymmetric second-order tensors, we must specify that their divergence is taken on the first index. Simultaneously, the ponderomotive couple is the axial vector associated with the skew part of

$\tilde{\mathbf{t}}^{em}$. The latter vanishes together with the source terms in (21)–(23) outside matter, and (34) reverts to (7) in a vacuum. Because of the source term in its *r-h-s* equation (34) is *not* a conservation law for the whole physical system. But it allows one to rewrite the balance law of linear momentum for the whole continuum in a specific form (see Maugin, 1993, for these developments). We can also rewrite (21) emphasizing the occurrence of an *effective* Lorentz force \mathbf{f}_L^{eff} in the form

$$\mathbf{f}^{em} = \mathbf{f}_L^{eff} + \text{div } \tilde{\mathbf{t}}^{em}, \quad (37)$$

with (compare to (31))

$$\mathbf{f}_L^{eff} = q^{eff} \tilde{\mathbf{E}} + \frac{1}{c} \tilde{\mathbf{J}}^{eff} \times \mathbf{B}, \quad (38)$$

where

$$q^{eff} = q_f - \nabla \cdot \mathbf{P}, \quad \tilde{\mathbf{J}}^{eff} = \tilde{\mathbf{J}} + \mathbf{P}^* + c \nabla \times \tilde{\mathbf{M}}. \quad (39)$$

We easily check that there holds the identity

$$\frac{\partial \mathbf{p}^{em}}{\partial t} - \text{div } \mathbf{t}^{em.f} = -\mathbf{f}_L^{eff}. \quad (40)$$

Equations (34) and (40) are compatible, but they may suggest different ways to combine mechanics and electromagnetism in the balance of linear momentum as it may be tempting to many researchers to consider \mathbf{f}_L^{eff} as the primitive interaction force because effective charge and electric current appear also in Maxwell's equations (cf. Eringen and Maugin, 1990, p. 54) as natural perturbations of the vacuum equations, e.g., (3) also read

$$\nabla \cdot \mathbf{E} = q_f - \nabla \cdot \mathbf{P}, \quad \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c} \left(\mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + c \nabla \times \mathbf{M} \right). \quad (41)$$

These can be recast using convected field and time derivatives yielding source expressions such as in (39).

Note: While the above-given results are obtained, a similar treatment of Maxwell's equations in vacuum with source terms due to the individual electric charges, yield, after space average, the macroscopic equations (2) and (3) – this was the basic idea of Lorentz.

2.2 Postulate of Equations Accounting for Information from a Microscopic Model

This is the manner *à la* Newton–Cauchy. Global balances laws are written for linear and angular momenta along with the first and second laws of

thermodynamics, in which electromagnetic source terms as recalled above are introduced. This is the viewpoint expanded in Maugin (1988) and Eringen and Maugin (1990), and other authors, in great detail. Of course the result depends on the microscopic model used to obtain the sources or else, on an *a priori* and somewhat arbitrary choice for these sources (**not** our viewpoint). The full application of the method in Maugin (1988) and Eringen and Maugin (1990) shows its pertinence, albeit in spite of a complexity arising in the description of stresses. The latter are not symmetric *a priori* since there exists an applied couple (27), something that cannot be denied as otherwise there would not exist such an evident effect as the compass alignment with a magnetic field. But in the end the obtained thermomechanics proves to be satisfactory with an energy (internal or free-Helmholtz) containing part of the interactions, a part of constitutive origin. Among the results obtained in Maugin (1988) and Eringen and Maugin (1990) we note the formula for the stresses \mathbf{t} appearing in the local balance of linear momentum of a continuum (divergence of tensors taken on the first index; \mathbf{f} = body force such as gravity, ρ = actual matter density; $\dot{\mathbf{v}}$ = acceleration)

$$\rho \dot{\mathbf{v}} = \mathbf{f} + \mathbf{f}^{em} + \operatorname{div} \mathbf{t}, \quad (42)$$

with a *nonsymmetric* Cauchy stress

$$\mathbf{t} = \mathbf{t}^E + (\mathbf{t}^{em.f} - \mathbf{t}^{em}) = \mathbf{t}^E - \tilde{\mathbf{t}}^{em}, \quad (43)$$

or a total *symmetric* (*Cauchy*) stress $\boldsymbol{\tau}$ such that

$$\boldsymbol{\tau} = \mathbf{t} + \mathbf{t}^{em} = \mathbf{t}^E + \mathbf{t}^{em.f}, \quad (44)$$

where \mathbf{t}^E is a symmetric “elastic” stress such that, in components (here symmetric and skewsymmetric parts)

$$t_{(ij)}^E = t_{(ji)} + \tilde{t}_{(ji)}^{em}, \quad t_{[ji]}^E \equiv 0. \quad (45)$$

To the same degree of generality as (42), the local forms of the energy equation and inequality of entropy read (Eringen and Maugin, 1990)

$$\rho \dot{e} = \operatorname{tr} (\mathbf{t} (\nabla \mathbf{v})^T) - \mathbf{f}^{em} \cdot \mathbf{v} + w^{em} - \nabla \cdot \mathbf{q} + \rho h, \quad (46)$$

and

$$\rho \dot{\eta} \geq \rho h \theta^{-1} - \nabla \cdot (\mathbf{q} \theta^{-1}), \quad (47)$$

where e , η , θ , \mathbf{q} and h are the internal energy per unit actual mass, the entropy per unit actual mass, the thermodynamic temperature, the heat flux vector, and the external heat supply per unit actual mass, respectively.

The electromagnetic energy “source” w^{em} is given by (23) with expressions (27) through (29) valid. Equivalent forms were given in (33). An other equivalent expression is given by

$$w^{em} = \mathbf{f}^{em} \cdot \mathbf{v} + \rho \tilde{\mathbf{E}} \cdot \dot{\boldsymbol{\pi}} - \tilde{\mathbf{M}} \cdot \dot{\mathbf{B}} + \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}}, \quad (48)$$

where $\boldsymbol{\pi} = \mathbf{P}/\rho$ is the electric polarization per unit mass. On introducing the Helmholtz free energy function per unit mass

$$\psi = e - \eta \theta, \quad (49)$$

and substituting from (48), (49) and (46) in (47) we arrive at the so-called *Clausius–Duhem inequality*

$$-\rho(\dot{\psi} + \eta \dot{\theta}) + \text{tr}(\mathbf{t}(\nabla \mathbf{v})^T) + \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}} + \rho \tilde{\mathbf{E}} \cdot \dot{\boldsymbol{\pi}} - \tilde{\mathbf{M}} \cdot \dot{\mathbf{B}} - (\mathbf{q}/\theta) \cdot \nabla \theta \geq 0. \quad (50)$$

In a now well established tradition, this is conceived as a constraint on the formulation of constitutive equations for the fields $(\psi, \eta, \mathbf{t}, \tilde{\mathbf{J}}, \tilde{\mathbf{E}}, \tilde{\mathbf{M}}, \mathbf{q})$. The formulation (50) clearly emphasizes for electromagnetic processes the role of independent variables (causes) played by the pair $(\tilde{\mathbf{E}}, \nabla \theta)$, electric polarization and magnetic induction for galvanomagnetic couplings, electric polarization and magnetization effects respectively. Other sets of variables may be selected for the last two effects (see Maugin, 1988). What is more important here is that, in deformable solids, one often prefers to reformulate the theory in terms of so-called *material* fields. To that effect we set

$$\tilde{\mathbf{B}} = J_F \mathbf{F}^{-1} \cdot \mathbf{B}, \quad \tilde{\mathbf{D}} = J_F \mathbf{F}^{-1} \cdot \mathbf{D}, \quad \boldsymbol{\Pi} = J_F \mathbf{F}^{-1} \cdot \mathbf{P} = \rho_0 \mathbf{F}^{-1} \cdot \boldsymbol{\pi}, \quad (51)$$

$$\bar{\mathbf{E}} = \mathbf{E} \cdot \mathbf{F}, \quad \widetilde{M}_K = J_F F_{Kp}^{-1} \widetilde{M}_p, \quad \overline{M}_K = \widetilde{M}_i F_{iK}, \quad (52)$$

with

$$J_F = \det \mathbf{F}, \quad \rho_0 = \rho J_F, \quad \mathbf{F} = \{F_{iK} = x_{i,K}\}. \quad (53)$$

We then check that the relations (1) translate in material components to

$$\widetilde{D}_K = J_F C_{KL}^{-1} \bar{E}_L + \Pi_K, \quad \overline{H}_K = J_F^{-1} C_{KL} \widetilde{B}_L - \overline{M}_L, \quad (54)$$

with

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \{C_{KL} = x_{i,K} x_{i,L}\}, \quad \mathbf{C}^{-1} = (\mathbf{C})^{-1}. \quad (55)$$

First and second Piola–Kirchhoff stresses are defined by

$$\mathbf{T} = J_F \mathbf{F}^{-1} \cdot \mathbf{t}, \quad \mathbf{S} = \mathbf{T} \cdot \mathbf{F}^{-T}. \quad (56)$$

Similar definitions hold for Piola–Kirchhoff stresses associated with the stresses \mathbf{t}^E and $\mathbf{t}^{em.f}$. Thus we write

$$\mathbf{T}^E = J_F \mathbf{F}^{-1} \cdot \mathbf{t}^E, \quad \mathbf{S}^E = \mathbf{T}^E \cdot \mathbf{F}^{-T}, \quad \mathbf{T}^F = J_F \mathbf{F}^{-1} \cdot \mathbf{t}^{em.f}. \quad (57)$$

We let the reader prove by way of exercise that eqns. (42) and (50) can be rewritten as (here no body force)

$$\frac{\partial}{\partial t} \mathbf{P}_R^t \Big|_x - \operatorname{div}_R (\mathbf{T}^E + \mathbf{T}^F) = \mathbf{0}, \quad \mathbf{P}_R^t \equiv \rho_0 \left(\mathbf{v} + \frac{1}{\rho c} \mathbf{E} \times \mathbf{B} \right), \quad (58)$$

and

$$\begin{aligned} -(\dot{W} + N \dot{\theta}) + \frac{1}{2} S_{KL}^E \dot{C}_{KL} + \overline{E}_K \dot{\Pi}_K \\ - \overline{M}_K \dot{\overline{B}}_K + J_F (\tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}} - (\mathbf{q}/\theta) \cdot \nabla \theta) \geq 0, \end{aligned} \quad (59)$$

where we have set

$$W = \rho_0 \psi, \quad N = \rho_0 \eta. \quad (60)$$

Once we have established constitutive equations for S_{KL}^E , \overline{E}_K and \overline{M}_K , we can return to the original Eulerian fields, including the Cauchy stress \mathbf{t} . We shall go further in the exploitation of the inequality (59) in the next section.

2.3 The Principle of Virtual Power

The application of the “principle of virtual power” or d’Alembert’s principle consists in a *weak formulation* in mathematical terms, as it allows one to envisage virtual velocity fields as test functions (in the sense of mathematical analysis and generalized functions). We have advocated this formulation as the most powerful one in the construction of complex electromagnetomechanical behaviors (Maugin, 1976, 1980), when the set of virtual velocity fields is enlarged so as to include velocities of characteristic electromagnetic fields, magnetization and electric polarization, and this to the desired gradient order for weakly nonlocal theories. The principle essentially replaces (i.e. is strictly equivalent to) global statements of the balance laws of continuum mechanics and those additional equations that will govern the magnetization and electric polarization fields in addition to Maxwell’s equations. It provides a sure and safe way to do this by following simple obvious rules even though in presence of dissipation processes. Let us illustrate this with the cases already considered in previous paragraphs.

The general principle formally reads

$$P_{inertia}^* = P_{internal}^* + P_{bulk}^* + P_{Boundary}^*, \quad (61)$$

where a right asterisk means the value taken in virtual fields, and the four powers in (61) represent, respectively, the power of inertial forces, the power of internal forces, the power of data in the bulk of the body V , and the power of data at the boundary ∂V of the body. “Internal forces” are those quantities for which one needs to produce a constitutive equation. This includes stresses but also here additional fields representing the interactions between the Maxwellian electromagnetic fields and the deformable body. Principle (61) must hold for any element of volume and surface and any virtual velocity field. We shall formally introduce factors belonging to the dual set of the set $\{\mathbf{v}^*, (\dot{\boldsymbol{\pi}})^*, (\dot{\boldsymbol{\mu}})^*\}$, where $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ are electric polarization and magnetization per unit mass in the actual configuration at time t , i.e.

$$\boldsymbol{\pi} = \mathbf{P}/\rho, \quad \boldsymbol{\mu} = \widetilde{\mathbf{M}}/\rho. \quad (62)$$

To have an idea of what should be put in P_{bulk} we need to recall that up to the Joule term and a term of the form $\rho d(\boldsymbol{\mu} \cdot \mathbf{B})/dt$ that can be left to be integrated in the time derivative of the internal energy density, it can be shown that the source term (23) is equivalent to the expression (Maugin and Eringen, 1977, no electric conduction)

$$w^{em} = \mathbf{f}^{em} \cdot \mathbf{v} + \rho \widetilde{\mathbf{E}} \cdot \dot{\boldsymbol{\pi}} + \rho \mathbf{B} \cdot \dot{\boldsymbol{\mu}}, \quad (63)$$

where real velocity fields will be replaced by virtual ones. As to $P_{internal}$, it is endowed with the fundamental property that it should vanish if the generalized rigid-body motion of V is a rigid body motion *per se*, i.e. after Killing’s theorem, $sym(\nabla \mathbf{v}^*) = \mathbf{0}$, and the local interactions present in $P_{internal}^*$ are frozen in (keeping a constant modulus and rotating at the local rotational velocity of the rigid body motion). A consequence of this statement is that $P_{internal}^*$ should *a priori* be written as a *continuous linear form on a set of objective virtual velocities* (Maugin, 1980). Introducing factors $\{\sigma_{ji} = \sigma_{ij}, E_i^L, B_i^L\}$, an example of such a form is given by

$$P_{internal}^* = - \int_V (\sigma_{ji} (v_i^*)_{,j} - \rho E_i^L (D_J \pi)_i^* - \rho B_i^L (D_J \mu)_i^*) dV, \quad (64)$$

where, e.g.,

$$(D_J \pi)_i^* = (\dot{\boldsymbol{\pi}})_i^* - \Omega_{ij}^* \pi_j, \quad \Omega_{ij}^* = \frac{1}{2} ((v_i^*)_{,j} - (v_j^*)_{,i}), \quad (65)$$

relates to a virtual Jaumann time derivative, and to a virtual vorticity tensor.

The power $P_{inertial}$ here contains only acceleration forces, i.e.

$$P_{inertia}^* = \int_V (\rho \dot{\mathbf{v}}) \cdot \mathbf{v}^* dV. \quad (66)$$

On applying (61) to any virtual field and any element of volume and surface, we are led to the following field equations:

$$\rho \dot{\mathbf{v}} = \mathbf{f} + \mathbf{f}^{em} + \operatorname{div} \mathbf{t}, \quad (67)$$

$$\tilde{\mathbf{E}} + \mathbf{E}^L = \mathbf{0}, \quad \mathbf{B} + \mathbf{B}^L = \mathbf{0}, \quad (68)$$

with

$$t_{ji} = \sigma_{ji} + E_{[j}^L P_{i]} + B_{[j}^L \widetilde{M}_{i]}, \quad \sigma_{ji} \equiv t_{(ji)}, \quad (69)$$

at any regular point in V . The set (67)–(69) is equivalent to the set formed by (42) and (43) although with a different decomposition of the symmetric part of \mathbf{t} . Here,

$$t_{(ji)} = \sigma_{ji}, \quad t_{[ji]} = -(P_{[j} E_{i]} + \widetilde{M}_{[j} B_{i]}) \quad (70)$$

and

$$\sigma_{ji} = t_{ji}^E - (P_{(j} \widetilde{E}_{i)} + \widetilde{M}_{(j} B_{i)}). \quad (71)$$

We do not pursue further this simple example. However, we note the rich potentialities offered by equations of *balance* such as (68) between a Maxwellian field and an interaction field for which we need a constitutive equation. In particular, equations such as (68) will contain additional terms (inertia, divergence of a tensor) when a more complicated interaction scheme is considered in the expression (64) – i.e. higher order gradients of $\boldsymbol{\mu}$ and $\boldsymbol{\pi}$ as is the case in ordered dielectrics (ferroelectrics) or ferromagnets. The inertial contribution (66) will most probably contain additional terms involving the dynamics of $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$. Note also that the “local” fields \mathbf{E}^L and \mathbf{B}^L are not restricted to thermodynamically recoverable phenomena; they may present dissipative contributions (relaxation, hysteresis). All this is documented in great detail in 41;12;39;40 and hereafter. Nonetheless, one must recognize in equations such as (68), the pioneering proposals of Toupin (1956) and Tiersten (1964) that inaugurated the rich development period of the 1960s–1980s.

3 Continuum Thermomechanics of Electromagnetic Solids: Standard Formulation

Now we examine the consequences of (59) for various simple classes of materials.

3.1 Nondissipative Materials

- A.** For **hyperelastic dielectric solids** ($q_f = 0$, $\widetilde{\mathbf{M}} = \mathbf{0}$, $\widetilde{\mathbf{J}} = \mathbf{0}$), (59) reduces to the equality

$$-(\dot{W} + N\dot{\theta}) + \frac{1}{2} S_{KL}^E \dot{C}_{KL} + \overline{E}_K \dot{\Pi}_K = 0, \quad (72)$$

from which there follows the constitutive equations

$$S_{KL}^E = 2 \frac{\partial \widehat{W}}{\partial C_{KL}}, \quad \overline{E}_K = \frac{\partial \widehat{W}}{\partial \Pi_K}, \quad N = -\frac{\partial \widehat{W}}{\partial \theta}, \quad (73)$$

wherein

$$W = \widehat{W}(C_{KL}, \Pi_K, \theta). \quad (74)$$

Accordingly, the following constitutive equations are obtained (Maugin, 1988; Eringen and Maugin, 1990):

$$t_{ji}^E = 2 J_F^{-1} F_{jK} F_{iL} \frac{\partial \widehat{W}}{\partial C_{KL}}, \quad \overline{E}_K = \frac{\partial \widehat{W}}{\partial \Pi_K}, \quad (75)$$

Then, after (43),

$$\mathbf{t} = \mathbf{t}^E - \mathbf{P} \otimes \widetilde{\mathbf{E}} = \mathbf{t}^E - J_F^{-1} \mathbf{F} \cdot \mathbf{\Pi} \otimes \widetilde{\mathbf{E}}, \quad (76)$$

hence in components for the Cauchy stress

$$t_{ji} = J_F^{-1} F_{jK} \left(2 \frac{\partial \widehat{W}}{\partial C_{KL}} - \Pi_K \frac{\partial \widehat{W}}{\partial \Pi_L} \right) F_{iL}. \quad (77)$$

- B.** For **hyperelastic magnetized (insulating) bodies** ($q_f = 0$, $\mathbf{P} = \mathbf{0}$, $\widetilde{\mathbf{J}} = \mathbf{0}$), (59) reduces to the equality

$$-(\dot{W} + N\dot{\theta}) + \frac{1}{2} S_{KL}^E \dot{C}_{KL} - \overline{M}_K \dot{\overline{B}}_K = 0. \quad (78)$$

Following along the same path as in the case of dielectrics, we obtain instead of (75)

$$t_{ji}^E = 2 J_F^{-1} F_{jK} F_{iL} \frac{\partial \overline{W}}{\partial C_{KL}}, \quad \widetilde{M}_K = -\frac{\partial \overline{W}}{\partial \overline{B}_K} = J_F F_{Kp}^{-1} \widetilde{M}_p, \quad (79)$$

with

$$W = \overline{W}(\mathbf{C}, \overline{\mathbf{B}}, \theta), \quad \overline{\mathbf{B}} = \{\overline{B}_K = B_i F_{iK}\}, \quad (80)$$

so that after (43)

$$\mathbf{t} = \mathbf{t}^E + \mathbf{B} \otimes \widetilde{\mathbf{M}} - (\widetilde{\mathbf{M}} \cdot \mathbf{B}) \mathbf{1}, \quad (81)$$

and thus

$$t_{ji} = J_F^{-1} F_{jK} \left(2 \frac{\partial \overline{W}}{\partial C_{KL}} - \overline{B}_K \frac{\partial \overline{W}}{\partial \overline{B}_L} + \frac{\partial \overline{W}}{\partial \overline{B}_Q} \overline{B}_Q C_{KL}^{-1} \right) F_{iL}. \quad (82)$$

In these two examples the energy density is per unit reference volume but we give the mechanical constitutive equation in the Eulerian configuration. But the following remark is in order. Instead of \widehat{W} and \overline{W} above, we could have considered energies (up to the temperature dependence)

$$W = \widehat{\Phi}(\mathbf{F}, \boldsymbol{\Pi}) \quad \text{and} \quad W = \overline{\Phi}(\mathbf{F}, \overline{\mathbf{B}}). \quad (83)$$

In the first case we note that (explicit derivative here means at fixed electromagnetic fields)

$$\frac{\partial W}{\partial \mathbf{F}} = \left(\frac{\partial W}{\partial \mathbf{F}} \right)_{explicit} + \frac{\partial W}{\partial \boldsymbol{\Pi}} \cdot \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{F}}, \quad (84)$$

so that with an explicit dependence on \mathbf{F} through \mathbf{C} for (75) we immediately check that (77) is equivalent to

$$t_{ji} = J_F^{-1} F_{jK} \frac{\partial \widehat{\Phi}}{\partial F_{iK}} - P_j \widetilde{E}_i, \quad \tau_{ji} = J_F^{-1} F_{jK} \frac{\partial \widehat{\Phi}}{\partial F_{iK}} + t_{ji}^{em.f}. \quad (85)$$

A similar proof holds for (82) which can be shown to be equivalent to

$$\tau_{ji} = J_F^{-1} F_{jK} \frac{\partial \overline{\Phi}}{\partial F_{iK}} + t_{ji}^{em.f}, \quad (86)$$

where in (85) and (86) the corresponding reduced form of $t_{ji}^{em.f}$ has to be used.

This play around with derivatives with respect to \mathbf{F} was exploited in (Maugin, 1993, pp. 186–189), and further by Trimarco and Maugin (2001). Results (57) and (58) coincide with those of Dorfmann and Ogden (2003, 2004, 2005, 2006), who followed Kovetz (2000) in considering a dependency of W on \mathbf{F} . But there is more to this, as the $t_{ji}^{em.f}$ term in (85) or (86) can also be derived from a potential. Indeed, in the case of *electroelasticity*, it is readily shown that the electric energy of free fields per unit reference volume can be written as (Nelson, 1979)

$$\frac{1}{2} J_F \mathbf{E}^2 = \frac{1}{2} J_F \overline{\mathbf{E}} \cdot \mathbf{C}^{-1} \cdot \overline{\mathbf{E}}, \quad \overline{\mathbf{E}} = \mathbf{E} \cdot \mathbf{F}. \quad (87)$$

Introducing then the *augmented energy* density

$$\widehat{\Omega} = \widehat{\Phi} - \frac{1}{2} J_F \overline{\mathbf{E}} \cdot \mathbf{C}^{-1} \cdot \overline{\mathbf{E}}, \quad (88)$$

we let the reader show that

$$\tau_{ji} = J_F^{-1} F_{jK} \frac{\partial \widehat{\Omega}}{\partial F_{iK}}, \quad (89)$$

while we can introduce a material electric displacement $\overline{\mathbf{D}}$ by

$$\overline{\mathbf{D}} = J_F \mathbf{C}^{-1} \overline{\mathbf{E}} + \mathbf{\Pi} = -\frac{\partial \widehat{\Omega}}{\partial \overline{\mathbf{E}}}. \quad (90)$$

In the case of *magnetoelasticity*, the magnetic energy of free fields per unit reference volume can be written as (Nelson, 1979)

$$\frac{1}{2} J_F \mathbf{B}^2 = \frac{1}{2} J_F^{-1} \overline{\mathbf{B}} \cdot \mathbf{C} \cdot \overline{\mathbf{B}}. \quad (91)$$

The augmented energy density then reads

$$\overline{\Omega} = \overline{\Phi} - \frac{1}{2} J_F^{-1} \overline{\mathbf{B}} \cdot \mathbf{C} \cdot \overline{\mathbf{B}}, \quad (92)$$

and we let the reader show that

$$t_{ji} = J_F^{-1} F_{jK} \frac{\partial \overline{\Omega}}{\partial F_{iK}}, \quad \overline{\mathbf{H}} = \mathbf{H} \cdot \mathbf{F} = J_F^{-1} \mathbf{C} \cdot \overline{\mathbf{B}} - \overline{\mathbf{M}} = \frac{\partial \overline{\Omega}}{\partial \overline{\mathbf{B}}}, \quad (93)$$

equations in agreement with Dorfmann and Ogden (2004) and Otténio et al. (2008). The general case of electro-magneto-elasticity can be treated in the same manner, the corresponding energy and Lagrangian density of the free fields per unit reference volume being such that (compare to (8)₃ and (11))

$$\begin{aligned} U^{em.f} &= \frac{1}{2} (J_F \overline{\mathbf{E}} \cdot \mathbf{C}^{-1} \cdot \overline{\mathbf{E}} + J_F^{-1} \overline{\mathbf{B}} \cdot \mathbf{C} \cdot \overline{\mathbf{B}}), \\ L^{em.f} &= \frac{1}{2} (J_F \overline{\mathbf{E}} \cdot \mathbf{C}^{-1} \cdot \overline{\mathbf{E}} - J_F^{-1} \overline{\mathbf{B}} \cdot \mathbf{C} \cdot \overline{\mathbf{B}}). \end{aligned} \quad (94)$$

Then all electromagnetic interactions in matter and the effects of free electromagnetic fields will have been integrated into a single augmented potential. This naturally leads us to make a digression concerning variational formulations.

3.2 Variational Formulations

Variational formulations in the Hamiltonian-Lagrangian style are the most economic formulations as they require only the postulate of a Lagrangian density. A drawback is the required knowledge of the behavior of the material *a priori* (to specify the dependency of the density of energy contained in the Lagrangian) and, contrary to the general method of virtual power, they are limited to nondissipative effects. Still they are very attractive in that they often provide the root for numerical implementation. Many of the now currently applied theories of electromagnetomechanical interactions were first proposed under this form. This is the case of the theory of finitely deformable, materially inhomogeneous, dielectrics in quasi-electrostatics for which we can consider the Lagrangian density

$$\ell = \ell^{em.f} + \ell^{matter} = \frac{1}{2} J_F^{-1} \mathbf{E}^2 + \left(\frac{1}{2} \rho_0(\mathbf{X}) \mathbf{v}^2 - W(\mathbf{C}, \mathbf{\Pi}; \mathbf{X}) \right) \quad (95)$$

per unit volume of a reference configuration, while for the theory of finitely deformable, materially inhomogeneous, magnetized isolators in quasi-magnetostatics, we would consider the Lagrangian density

$$\begin{aligned} \ell = \ell^{matter} + \ell^{em.f} = & \left(\frac{1}{2} \rho_0(\mathbf{X}) \mathbf{v}^2 - W(\mathbf{C}, \overline{\mathbf{B}}; \mathbf{X}) \right) \\ & - \frac{1}{2} J_F^{-1} \mathbf{B}^2 + J_F^{-1} \widetilde{\mathbf{M}} \cdot \mathbf{B}, \end{aligned} \quad (96)$$

where $\rho_0 = \rho J_F$ is the matter density at the reference configuration. There is no problem in coupling the two models (95) and (96) into a single one for the full electrodynamics of nondissipative electro-magneto-deformable solids except for the bulk of the writing (cf. Nelson, 1979; Maugin, 1993, Chapter 8; Trimarco and Maugin, 2001), in which we took notice of the second of (10). Of course we avoid here the temptation to use a direct misleading rewriting of the expression (28). Specific dependencies are given to the potential energy W in both (95) and (96). We could have chosen dependencies on electromechanical variables $(\mathbf{F}, \overline{\mathbf{E}})$ and magnetomechanical variables $(\mathbf{F}, \overline{\mathbf{B}})$ respectively. We would then be led to introducing the *augmented* energy densities $\widehat{\Phi}$ and $\overline{\Phi}$, respectively. In the case of quasi-statics, this greatly facilitates the problem of studying electroelastic or magnetoelastic stability (from the second variation of the potential) – cf. Otténio et al. (2008).

3.3 Internal Strains and Stresses

These notions are of great interest in thermoelasticity, electro- and magneto-elasticity. By way of example, we consider the case of materially homogeneous isothermal magneto-elasticity (of soft ferromagnets) in which

the free energy per unit reference volume is given by the simple sufficiently regular function (in particular allowing for inversion)

$$W = \overline{W}(\mathbf{F}, \mathbf{M}) \quad (97)$$

where both deformation gradient \mathbf{F} and material magnetization per unit reference volume \mathbf{M} are evaluated at material point \mathbf{X} . Then we raise the following question. Can we bring the formula (97) in a form that would remind us of the response of a pure elastic body for which the energy depends on a single argument, a deformation gradient. This can be achieved locally at \mathbf{X} by applying a local change of configuration $\mathbf{K}(\mathbf{X})$ so that we can write, accounting for the concomitant change in volume

$$W = J_K^{-1} \widehat{W}(\mathbf{F}(\mathbf{K}(\mathbf{M}(\mathbf{X}))))). \quad (98)$$

This is an operation (thought experiment) introduced by 21 to deal with material inhomogeneities. In the present case, we shall set $\mathbf{F}^\mu = \mathbf{K}^{-1}$, so that (98) reads

$$W = J_{F^\mu} \widehat{W}(\mathbf{F}^e) = \widetilde{W}(\mathbf{F}, \mathbf{F}^\mu), \quad (99)$$

where we have introduced the “elastic” deformation “gradient”

$$\mathbf{F}^e = \mathbf{F}(\mathbf{F}^\mu)^{-1}, \quad (100)$$

so that we have the following multiplicative decomposition of \mathbf{F} :

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^\mu, \quad (101)$$

into two quantities of which none is a true gradient. Only \mathbf{F} is integrable in a displacement, while the elements in the right-hand side of (101) are called “gradients” by abuse of language. Thus, \mathbf{F}^μ is the magnetic “deformation gradient”. We can also verify the following:

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{F}} = \frac{\partial \widetilde{W}}{\partial \mathbf{F}}, \quad -\frac{\partial \widetilde{W}}{\partial \mathbf{K}} \mathbf{K}^T = W \mathbf{I}_R - \mathbf{T} \mathbf{F} = \mathbf{b}, \quad (102)$$

where the last quantity will be later on (Section 6) identified as an Eshelby material stress.

Internal strains are defined as those strains that exit in the absence of imposed forces, and **internal stresses** are those stresses that are associated with internal strains *via* the purely elastic response. Accordingly, internal strains due to magnetization are in principle defined by $\mathbf{T} = \mathbf{0}$, i.e.

$$\frac{\partial \overline{W}}{\partial \mathbf{F}} = 0, \quad \Rightarrow \overline{W}(\mathbf{F} = \mathbf{F}_{intern}^\mu, \mathbf{M}) \quad \Rightarrow \mathbf{F}_{intern}^\mu = \overline{\mathbf{F}}(\mathbf{M}), \quad (103)$$

and thus

$$\mathbf{T}_{intern}^\mu = \frac{\partial W}{\partial \mathbf{F}^e}, \quad (104)$$

computed at $\mathbf{F}^e = \mathbf{F}_{intern}^\mu$. But here some caution must be taken because “deformation gradients” involve *rotation* contributions (think of the polar decomposition). Indeed the above reasoning hints at the view that \mathbf{T} is the thermodynamical dual of \mathbf{F} , so that one is tempted to choose \mathbf{T} as the natural variable stresswise. This choice would even be additionally supported by the fact that \mathbf{T} is naturally related to the traction. But several reasons make that this would be an illegitimate choice. First, \mathbf{T} being a two-point tensor field, it is not frame invariant as a tensor. Second, \mathbf{T} is not positive (or negative) definite, so that no polar decomposition applies to it. As a result, basing on \mathbf{T} as the only variable, the complementary energy W_c could not be made depending on material quantities, as constitutive laws are requested. Finally, a lack of uniqueness – which is potentially contained in the last of (103) – for the inversion with strains could emerge even in the presence of fixed tractions at the boundary and unique solution in stresses. This lack of uniqueness is due to the indeterminacy of the finite rotation \mathbf{R} (see Ogden, 1984). For the appropriate choice of the stress tensor and for a detailed discussion of this problem, the reader is referred to this author. Thus the present reasoning should be done in terms of true strains and conjugated stress tensors. The introduction of the *complementary energy* density W_c through a Legendre transformation allows one to shift from strains to stresses as independent variables. We may consider the second Piola–Kirchhoff stress \mathbf{S} in such a way that the relevant Legendre transformation reads as follows for an elastic material in finite strains:

$$W(\mathbf{S}, \mathbf{E}) = \text{tr}(\mathbf{S} \cdot \mathbf{E}) - W_c(\mathbf{S}), \quad (105)$$

where \mathbf{E} is not the electric field but the Lagrangian finite strain $\mathbf{E} = (\mathbf{C} - \mathbf{1}_R)/2$. We have the reciprocal constitutive relations

$$\mathbf{E} = \frac{\partial W_c}{\partial \mathbf{S}}, \quad \mathbf{S} = \frac{\partial W}{\partial \mathbf{E}}. \quad (106)$$

In the magnetizable case W_c remains a function of the material magnetization. Then we can expect that

$$\frac{\partial W(\mathbf{E}, \mathbf{M})}{\partial \mathbf{E}} = \mathbf{0} \quad \Rightarrow \quad \mathbf{E}_{intern}^\mu = \bar{\mathbf{E}}(\mathbf{M}) = \left. \frac{\partial W_c}{\partial \mathbf{S}} \right|_{\mathbf{S}=\mathbf{0}}, \quad (107)$$

and

$$\mathbf{S}_{intern} = \frac{\partial W}{\partial \mathbf{E}}(\mathbf{E} = \mathbf{E}_{intern}^\mu). \quad (108)$$

In the case of small strains, the multiplicative decomposition reduces to an additive decomposition because of the initial equation $\mathbf{F} = \mathbf{1} + (\nabla \mathbf{u})^T$; (101) is replaced by the small-strain version

$$\boldsymbol{\varepsilon} = (\nabla \mathbf{u})_S = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^\mu, \quad (109)$$

where only $\boldsymbol{\varepsilon}$ is the symmetric part of the displacement gradient.

In homogeneous anisotropic bodies and small strains, the Cauchy stress is given by the simple expression

$$\boldsymbol{\sigma} = L[\boldsymbol{\varepsilon}] + \lambda[\mathbf{M} \otimes \mathbf{M}] \quad \text{or} \quad \sigma_{ij} = L_{ijkl} \varepsilon_{kl} + \lambda_{ijkl} M_k M_l, \quad (110)$$

where the operators L and λ have the following symmetries

$$L_{ijkl} = L_{(ij)(kl)} = L_{klij}, \quad \lambda_{ijkl} = \lambda_{(ij)(kl)}. \quad (111)$$

For isotropic bodies there are only two independent elasticity coefficients, the Lamé parameters, and only two magnetostriction coefficients. Magnetostriction is a magnetoelastic coupling that exists for all material symmetries, to a smaller or larger extent of course. Natural piezomagnetism – a linear magnetoelastic coupling – is a rare event, requiring the right combination of crystal and magnetic symmetries (cf. Maugin, 1988). Let L^{-1} the inverse operator of L in \mathbb{R}^6 (due to the symmetries exhibited in (111)₁, L is a symmetric linear operator of \mathbb{R}^6 onto itself). Then the magnetic internal strain is given by

$$\varepsilon_{ij}^\mu = -L_{ijkl}^{-1} \lambda_{klpq} M_p M_q, \quad (112)$$

and

$$\varepsilon_{ij} = L_{ijkl}^{-1} \sigma_{kl} + \varepsilon_{ij}^\mu = \varepsilon_{ij}^e + \varepsilon_{ij}^\mu. \quad (113)$$

Thus

$$W(\boldsymbol{\varepsilon}, \mathbf{M}) = \frac{1}{2} \varepsilon_{ij} L_{ijkl} \varepsilon_{kl} + \varepsilon_{ij} \lambda_{ijkl} M_k M_l, \quad (114)$$

and

$$W_c(\boldsymbol{\sigma}, \mathbf{M}) = \sigma_{ij} \varepsilon_{ij} - W(\boldsymbol{\varepsilon}, \mathbf{M}). \quad (115)$$

We can rewrite (110)₂ as

$$\sigma_{ij} = L_{ijkl} (\varepsilon_{kl} + L_{klpq}^{-1} \lambda_{pqmn} M_m M_n) = L_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^\mu), \quad (116)$$

so that we can also consider that

$$\boldsymbol{\sigma} = \frac{\partial \widehat{W}}{\partial \boldsymbol{\varepsilon}}, \quad \widehat{W} = \frac{1}{2} (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^\mu) \cdot L \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^\mu), \quad (117)$$

and

$$\varepsilon = \frac{\partial \widehat{W}_c}{\partial \sigma}, \quad \widehat{W}_c(\sigma, \mathbf{M}) = \frac{1}{2} \sigma : L^{-1} : \sigma + \sigma : \varepsilon^\mu, \quad (118)$$

$$\varepsilon^e = \frac{\partial \check{W}}{\partial \sigma}, \quad \check{W}(\sigma) = \frac{1}{2} \sigma : L^{-1} : \sigma. \quad (119)$$

This is but one example of the inclusion of internal strains in the continuum formalism. A similar formalism holds for internal strains due to electroelastic couplings (electrostriction, piezoelectricity) and to thermoelasticity. In the latter case, in the finite-strain-framework one is led to introducing a “thermal deformation gradient” \mathbf{F}^θ and in small strains a thermal strain ε^θ such that, including magnetoelastic effects, (101) and (109) are replaced by the equations

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^\mu \mathbf{F}^\theta \quad \text{and} \quad \varepsilon = \varepsilon^e + \varepsilon^\mu + \varepsilon^\theta. \quad (120)$$

In isotropic media, ε^θ reduces to a dilatation (cf. Maugin, 1988). As to ε^μ it is generally considered as isochoric. Hence *trace* $\varepsilon^\mu = 0$. It must be noted that ε^μ in classical magnetostrictive materials such as cubic *Fe* or *Ni* is expressed as a tensor in terms of two nondimensional coefficients λ_{100} and λ_{111} – measured along the crystallographic axes [100] and [111] – and the director cosines of the magnetization (cf. Du Trémolet de Lacheisserie, 1993). The order of magnitude is such as 10^{-5} , hence small. It is only in newly developed materials exhibiting so-called “giant magnetostriction” that two orders of magnitude can be gained, becoming then competitive with electroelastic internal strains.

Interesting as they are from some view point (e.g., in the theory of material incompatibilities of E. Kröner (1958); remember the elements of decomposition in (120) are not individually integrable into a displacement), the couplings epitomized in decompositions (120) are *not* related to thermodynamically irreversible effects.

4 Dissipative Processes: Relaxation, Hysteresis

4.1 Standard Relaxation

As an illustrative example we select the case of the relaxation of electric polarization. This is a standard approach in the sense that no further variables and/or microstructure need be introduced. Indeed, the Clausius–Duhem inequality (59) in a nonmagnetizable insulator (neither heat nor electric conduction). Thus

$$-(\dot{W} + N \dot{\theta}) + \frac{1}{2} S_{KL}^E \dot{C}_{KL} + \overline{E}_K \dot{\Pi}_K \geq 0. \quad (121)$$

On assuming that the dependent thermodynamic variable $(W, N, S_{KL}^E, \bar{E}_K)$ may depend on the set

$$(\theta, C_{KL}, \Pi_K; \dot{C}_{KL}, \dot{\Pi}_{KL}), \quad (122)$$

and considering a linear dependence for mechanical and electric dissipative processes we can have the following set of constitutive equations:

$$N = -\frac{\partial W}{\partial \theta}, \quad S_{KL}^\nu = S_{KL}^E - 2 \frac{\partial W}{\partial C_{KL}}, \quad \bar{E}_K^r = \bar{E}_K - \frac{\partial W}{\partial \Pi_K}, \quad (123)$$

$$\Phi = \frac{1}{2} S_{KL}^\nu \dot{C}_{KL} + \bar{E}_K^r \dot{\Pi}_K \geq 0, \quad (124)$$

$$S_{KL}^\nu = L_{KL} [\dot{C}], \quad \dot{\Pi}_K = L_K [\bar{E}^r]. \quad (125)$$

For instance, we may have the following special case for the last of these

$$\dot{\Pi}_K = \frac{\chi_P}{\tau_P} (\bar{E}_K - \chi_P^{-1} \Pi_K), \quad (126)$$

an equation that also reads

$$\tau_P \dot{\Pi}_K + \Pi_K = \chi_P \bar{E}_K. \quad (127)$$

The electric susceptibility χ_P and relaxation time τ_P could be function of temperature. As to the first of (125) it obviously refers to viscosity of the Kelvin–Voigt type. Another formulation of electric-polarization relaxation would have followed if we had first performed an electric Legendre transformation of the free energy, i.e. replaced (121) by

$$-(\dot{\bar{W}} + N \dot{\theta}) + \frac{1}{2} S_{KL}^E \dot{C}_{KL} - \Pi_K \dot{\bar{E}}_K \geq 0, \quad \bar{W} = W - \Pi_K \bar{E}_K. \quad (128)$$

Then a standard exploitation of this would yield a relaxation of the electric field rather than one of electric polarization. The precise physical truth may be mixture of this and of (126) or (127) yielding, in the isotropic case, an electric response with two relaxation times (cf. Maugin, 1995).

4.2 The Notion of Internal Variable of State

The mechanisms at work at a microscopic scale that are responsible for the macroscopically observable irreversibilities, are far too complicated to be accounted for as such. But it has been found expedient, in fact extremely efficient, to represent these phenomena with the help of a few variables, well identified by gifted observers and measurable by experimentalists, but not

directly controlled by direct means. Accordingly, they can be listed, characterized by a certain tensorial order, incorporated in the energy function, but they produce no direct source term in the work of external forces both in the bulk and at the boundary. Their only, but essential, virtue, is to dissipate and, therefore, to be governed by the second law of thermodynamics. From this discursive definition, such variables are called *internal variables of state*. The richness and intricacies, but relative simplicity, of the irreversible thermodynamics of internal variables of state (for short TIV) are exposed in a book (Maugin, 1999). If α designates the ordered arrow of the components of the relevant internal variables of state, then the corresponding dissipation is given by

$$\Phi = A \cdot \dot{\alpha}, \quad A = -\partial W(\cdot, \alpha) / \partial \alpha, \quad (129)$$

where A is the thermodynamic force associated with α , and the dot between A and $\dot{\alpha}$ stands for the appropriate inner product between the two spaces of “forces” and rates placed in duality. Application of the second law yields a thermodynamically admissible relationship between A and $\dot{\alpha}$, hence a more or less regular *evolution equation* for α . In a purely mechanical framework, such a thermodynamical formulation, TIV, offers an efficient phenomenological representation of many irreversible phenomena such as the viscosity of complex fluids, the plasticity and viscoplasticity of solids, damage, and phase-transformation phenomena. In the case of electromagnetic continua, relaxation and hysteresis of various types are obvious candidates for the exploitation of TIV.

4.3 First Example: Dielectric Relaxation in Ceramics

We consider the Clausius–Duhem inequality (121), but the functional dependence (122) is replaced by

$$W = W(\mathbf{C}, \mathbf{\Pi}, \theta; \mathbf{\Pi}^{int}), \quad (130)$$

where $\mathbf{\Pi}^{int}$ is a material vector whose components are akin to an electric polarization. On computing the time derivative of W and exploiting (121) we obtain the laws of state as

$$N = -\frac{\partial W}{\partial \theta}, \quad S_{KL}^E = 2 \frac{\partial W}{\partial C_{KL}}, \quad \bar{E}_K = \frac{\partial W}{\partial \Pi_K}, \quad E_K^{int} \equiv -\frac{\partial W}{\partial \Pi_K^{int}}, \quad (131)$$

along with the following residual dissipation inequality (compare (129)₁)

$$\Phi = E_K^{int} \dot{\Pi}_K^{int} \geq 0. \quad (132)$$

As an example we may consider an energy of the following type (isotropic body):

$$W = W_1(\mathbf{C}, \mathbf{\Pi}, \theta) + W_2(\mathbf{\Pi}, \mathbf{\Pi}^{int}), \quad (133)$$

with

$$W_2 = \frac{1}{2} a \Pi_K \Pi_K + \frac{1}{2} b (\Pi_K - \Pi_K^{int}) (\Pi_K - \Pi_K^{int}). \quad (134)$$

With proportionality between the two factors in (132), such a model yields an electric relaxation in the form

$$\tau_d \dot{\Pi}_K^{int} = \Pi_K - \Pi_K^{int}, \quad (135)$$

where $\tau_d \geq 0$ is the dielectric (polarization) relaxation time. This time is practically directly accessible to experiments in a study of *shock-wave propagation* in dielectrics (see Maugin et al., 1992).

4.4 Second Example: Electromechanical Hysteresis in Ferroelectrics

This thermodynamically admissible modelling is performed in analogy with plasticity with work hardening. In this case we introduce two vectorial internal variables (materials vectors), one of which, $\mathbf{\Pi}^R$, will essentially be the irreversible part of the electric polarization (analog of the plastic deformation in the additive decomposition of strain in small strains), and the other, $\mathbf{\Pi}^{int}$, will play a role analogous to that played by the work-hardening variables in plasticity. Accordingly, they produce a dissipation of the form

$$\Phi = \overline{E}_K \dot{\Pi}_K^R + E_K^{int} \dot{\Pi}_K^{int} \geq 0, \quad (136)$$

where

$$\mathbf{\Pi}^R = \mathbf{\Pi} - \mathbf{\Pi}^r, \quad E_K^{int} = -\frac{\partial W}{\partial \Pi_K^{int}}, \quad (137)$$

where $\mathbf{\Pi}^r$ is the reversible electric polarization (the analog of the elastic deformation in small strains). A typical hysteretic behavior is obtained when the dissipation (136) is assumed to be a non-negative (first-order) homogeneous function of the present time rates, the electric fields in (136) remain in a convex domain in the space of “forces” ($\overline{\mathbf{E}}, \mathbf{E}^{int}$), and we have saturation in the polarization field $\mathbf{\Pi}^R$. Such a model was extensively developed by Bassiouny et al. (1988) and improved by many authors. An analogous modelling of ferromagnetic hysteresis was developed before by Sabir and Maugin (1990). A discussion of various forms of magnetic hysteresis may be found in Maugin (1991, 1992, 1993). A finer modelling accounting for

the domain structure of ferromagnets was started by Motogi and Maugin (1993), and further developed by French authors (see Section 5 below).

5 Different Scales: Homogenization, Ferromagnetic Polycrystalline Bodies

5.1 General Problem

There are about nine orders of magnitude in size between the atomic structure and the macroscale of tools utilized in industry, e.g., between, the ordered array of atoms and magnetic spins in a perfect ferromagnetic crystal and the iron-nickel plates used in building transformers. Easily identifiable scales in this problem are those of the atomic structure (10^{-9} meters), of the magnetic domain, of the grain-monocrystal (10^{-6} meters), of the representative volume element (RVE) of a polycrystal (10^{-3} meters), and the industrial scale of the order of 10^{-1} – 10^0 meters. Basic physical mechanisms are really at the smallest scale having a quantum-physical justification (without which magnetism would not exist at all). Macroscopically observed magnetoelastic couplings and hysteresis are far from this. Very few researchers have endeavoured to bridge the gap between these two extreme scales but this becomes now a legitimate goal. More reasonably, it is conceivable to build a bridge between the last three higher scales by exploiting the now available tools of statistical physics and mathematical homogenisation. The LMT-Cachan (France) group around R. Billardon (L. Hirsinger, N. Buiron, L. Danied, O. Hubert) has been very active along this line and we shall borrow from them the essential steps.

5.2 The Magnetic-Domain Scale

A magnetic domain (labelled α) is a material region of supposedly uniform magnetization \mathbf{M}^α of constant magnitude (saturation) and uniform director cosines γ^α . All magnetic quantities being uniform within the domain, exchange energy vanishes inside the domain and the only energies left (cf. Maugin, 1988) in the domain are the elastic energy, the energy due to the applied magnetic field (itself uniform in the domain) and the magnetocrystalline energy (function of the director cosines and here written for cubic symmetry) easily integrated over the volume of the domain:

$$W^\alpha = \frac{1}{2} \sigma^\alpha \cdot L_\alpha^{-1} \cdot \sigma^\alpha - \mathbf{M}^\alpha \cdot \mathbf{H}^\alpha + K_1 (\gamma_1^2 \gamma_2^2 + \gamma_2^2 \gamma_3^2 + \gamma_3^2 \gamma_1^2) + K_2 (\gamma_1^2 \gamma_2^2 \gamma_3^2), \quad (138)$$

where L_α^{-1} is the tensor of elastic compliances within domain α , and K_1 and K_2 are magnetic anisotropy constants (cf. Maugin, 1988). The orientation

of the domain (i.e. of the uniform magnetization \mathbf{M}^α) is defined by two angles θ_α and δ_α in a crystallographic frame.

5.3 The Grain-Monocrystal Scale

A grain or monocrystal presents homogeneous elastic properties. But it contains a large, finite number of magnetic domains. This is where some statistical physics may come into the picture. For each family of domains one can introduce a volume fraction f_α that is a function of the potential energy of the considered family. A Boltzmann function can be introduced such that the volume fraction f_α is given by

$$f_\alpha = \frac{\exp(-A_s W^\alpha)}{\sum_\alpha \exp(-A_s W^\alpha)}, \quad (139)$$

such that

$$W^\alpha(\theta_\alpha, \phi_\alpha) = \min(W^\alpha), \quad \theta_\alpha \in [0, \pi], \quad \phi_\alpha \in [0, 2\pi]. \quad (140)$$

In other words, the orientation of domains inside the grain is such as to minimize the energy (138). Here A_s is an adjustable parameter estimated to be given by $A_s = 3\chi_0/M_s^2$, where χ_0 is the initial susceptibility of the material (cf. Daniel, 2003). With ε_g^μ the magnetostrictive strain in the grain, the total free energy of the magnetic domain will be given by (compare to (138))

$$W^\alpha = -\mathbf{M}^\alpha \cdot \mathbf{H}_g - \sigma_g : \varepsilon_g^\mu + K_1 (\gamma_1^2 \gamma_2^2 + \gamma_2^2 \gamma_3^2 + \gamma_3^2 \gamma_1^2) + K_2 (\gamma_1^2 \gamma_2^2 \gamma_3^2), \quad (141)$$

where \mathbf{H}_g pertains to the grain. Herein above, averages for a monocrystal (with Latin subscript label I) are such that

$$\mathbf{M}_I = \langle \mathbf{M}^\alpha \rangle = \sum_\alpha f_\alpha \mathbf{M}^\alpha, \quad \varepsilon_I^\mu = \langle \varepsilon_\alpha^\mu \rangle = \sum_\alpha f_\alpha \varepsilon_\alpha^\mu \neq 0, \quad \mathbf{H}_g = \mathbf{H}_\alpha. \quad (142)$$

Of course, this modelling is rather crude as one could have thought that the energy of the monocrystal would be the total sum of the energies of the individual domains to which should be added the energy of domain walls. The evaluation of the latter is rather intricate as none of the fields is uniform within a wall (in particular, magnetization itself changes its orientation), and strain incompatibilities are concentrated there in order to allow for the assembly of various domains. To achieve such a description we should return to the theory of *micromagnetism* (cf. Maugin, 1988). All this is avoided here in the approach due initially to Buiron (2000) – and then Hubert (2008); Daniel et al. (2008) – with the help of the introduction of the adjustable scalar parameter A_s .

5.4 The Polycrystal Scale

Passing from the monocrystal to the polycrystal constitutes now a typical problem of *homogenisation*, a technique that owes much to Hill, Kröner and Eshelby in the mechanical case (Sanchez-Palencia and Zaoui, 1987). It has become somewhat standard in elasticity, but here it must also be applied to magnetic properties that are already nonhomogeneous at the monocrystal scale.

First we remind the reader of the main step in the purely mechanical homogenisation of elastic polycrystals. Assume we know the properties at the scale of the monocrystal. Let Σ and E the macro stress and strain. We have

$$\Sigma = \langle \sigma_I \rangle = L_{eff} : E, \quad \sigma_I = B_I : \Sigma, \quad \varepsilon_I = A_I : E, \quad (143)$$

where L_{eff} is the looked for effective tensor of elasticity coefficients, and B_I and A_I are fourth-order tensor linear operators on the set of symmetric tensors, called stress-concentration and strain-localization tensors, respectively. They have for expressions

$$B_I = L_I : A_I : L_{eff}^{-1}, \quad A_I = (L_I + L_*)^{-1} : (L_{eff} + L_*), \quad (144)$$

where the influence tensor of Hill, L_* , is given by

$$L_* = L_0 : (S_E^{-1} - \mathbf{I}). \quad (145)$$

Here \mathbf{I} is the unit in fourth-order tensors and S_E is Eshelby's inclusion problem tensor, which depends on the elastic moduli of the matrix and the shape of the inclusion, in the celebrated inclusion problem of J. D. Eshelby (1957), while L_0 is the tensor of elasticities of the matrix, or comparison elasticities. The self-consistent model consists in taking $L_0 = L_{eff}$ (that we do not know) so that we can obtain

$$L_{eff} = \langle L_I : (L_I + L_*)^{-1} : (L_{eff} + L_*) \rangle. \quad (146)$$

But this is an implicit equation that will require an iterative procedure for its solution. This can be treated analytically in the case of isotropy.

In the magneto-elastic case under study we must also treat the magnetic problem and the coupled one. The magnetic problem follows along the line of the mechanical one while dealing with vectors instead of tensors. Considering the case of a spherical magnetic inclusion (the Eshelby inclusion problem for magnetostatics), and for an isotropic matrix, we will take

$$\mathbf{M}_0 = \chi_0 \mathbf{H}_{ext} \quad (147)$$

as the comparison magnetic behavior and show that the macroscopic magnetic constitutive relation reads

$$\mathbf{M} = \langle \mathbf{M}_I \rangle = \chi_{eff} \cdot \mathbf{H}_{ext} \quad (148)$$

with an effective electric susceptibility tensor given by (compare to (146))

$$\chi_{eff} = 3(1 + \chi_0) \langle \chi_I ((3 + 2\chi_0) \mathbf{1} + \chi_I)^{-1} \rangle. \quad (149)$$

It remains to select χ_0 . Again, the self-consistent model consists in taking $\chi_0 = \chi_{eff}$ and the resulting equation (149) to be solved iteratively.

Finally the magneto-elastic coupling is taken care of as follows. We can write

$$\sigma_I = B_I : \Sigma + L_I : (S_E - \mathbf{I}) : \varepsilon_I^\mu, \quad (150)$$

and

$$\varepsilon_I = L_I^{-1} : (B_I : \Sigma) + S_E : \varepsilon_I^\mu. \quad (151)$$

Then it is directly shown that

$$\mathbf{M} = \langle \mathbf{M}_I \rangle, \quad \mathbf{E} = \langle \varepsilon_I \rangle, \quad (152)$$

of which the former yields (148) and the latter provides the relation

$$\mathbf{E} = L_{eff}^{-1} : \Sigma + \langle B_I^T : \varepsilon_I^\mu \rangle. \quad (153)$$

But the latter has to be solved iteratively since the magnetostrictive strain state in the monocrystal depends on the mechanical stress. The reader will find in Daniel (2003) and Daniel et al. (2008) an attempt at determining all parameters of the macroscopic behavior for polycrystalline ferromagnets.

Note that the present study is conducted for pure reversible behaviors involving neither plasticity nor magnetic hysteresis. There is still a long way to justify through a multiscale approach of the present type the phenomenological macroscopic model of elastoplastic materials exhibiting magnetic hysteresis as constructed by Sabir and Maugin (1990) with the aid of mixed magneto-mechanical internal variables of state. Magnetic hysteresis is related to plasticity in the sense that magnetic-domain walls are anchored on structural defects.

6 Electromagnetic Configurational Mechanics

6.1 Material Momentum and Eshelby Stress

Before considering the case of full matter, let us consider the case where the Lagrangian is none other than the one usually considered in vacuum, but

now written per unit volume of matter, i.e. a simple Lagrangian expressed on the basis of (11), that we write first as

$$\ell^{em.f}(\mathbf{E}, \mathbf{B}) = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \quad (154)$$

per unit of actual volume in K_t . Per unit of undeformed volume this yields

$$\ell_R^{em.f}(\mathbf{E}, \mathbf{B}; \mathbf{F}) = J_F \ell^{em.f}. \quad (155)$$

Recalling the “material” fields

$$\bar{\mathbf{E}} := \mathbf{E} \cdot \mathbf{F}, \quad \tilde{\mathbf{B}} \equiv J_F \mathbf{F}^{-1} \cdot \mathbf{B}, \quad (156)$$

this yields

$$\ell_R^{em.f} = \frac{1}{2} J_F \bar{\mathbf{E}} \cdot \mathbf{C}^{-1} \cdot \bar{\mathbf{E}} - \frac{1}{2} J_F^{-1} \tilde{\mathbf{B}} \cdot \mathbf{C} \cdot \tilde{\mathbf{B}}, \quad (157)$$

an expression obtained by Nelson (1979) up to the notation. This expression tells us how $\ell_R^{em.f}$ depends on the deformation gradient. Of course this function cannot depend explicitly on \mathbf{X} . In particular, for *quasi-electrostatics*, there remains only the first contribution in the right-hand side of (157), and Maugin and Epstein (1991) have proved that computing the material gradient of $\ell_R^{em.f}$, one obtains the following identity:

$$\nabla_R \ell_R^{em.f} - \text{div}_R \left(\frac{\partial \ell_R^{em.f}}{\partial \mathbf{F}} \cdot \mathbf{F} \right) = \text{div}_R \left(\ell_R^{em.f} \mathbf{1}_R - \frac{\partial \ell_R^{em.f}}{\partial \mathbf{F}} \mathbf{F} \right) \equiv \mathbf{0}. \quad (158)$$

This means that the material divergence of the Eshelby stress tensor of *free* electromagnetic fields is not balanced by any material force (inhomogeneity force). This is checked directly by computing the following two quantities,

$$\frac{\partial}{\partial X^K} \left(\frac{1}{2} J_F \mathbf{E}^2 \right), \quad \frac{\partial}{\partial X^I} \left(\frac{\partial (J_F \ell^{em.f})}{\partial F^i} F^i_{.K} \right),$$

and subtracting the second result from the first. The following more general result can be checked (full dynamic electromagnetic case):

$$\frac{\partial}{\partial t} \mathbf{P}^{emf} \Big|_x - \text{div}_R \mathbf{b}^{emf} \equiv \mathbf{0}, \quad (159)$$

where

$$\mathbf{P}^{emf} := \frac{\partial \ell_R^{em.f}}{\partial \mathbf{V}}, \quad \mathbf{b}^{emf} = -(\ell_R^{em.f} \mathbf{1}_R + \mathbf{T}^F \mathbf{F}), \quad \mathbf{T}^F := -\frac{\partial \ell_R^{em.f}}{\partial \mathbf{F}}. \quad (160)$$

In plain words, *free* electromagnetic fields, do *not*, by themselves, develop any inhomogeneity force. This property encapsulates the essential difference of nature between the pervasive *pure field* contributions and those which pertain to true *material* fields (e.g., magnetization and electric polarization). Accordingly, the usual Maxwell stress tensor introduced in (8)₂ cannot contribute to the balance of material momentum. Along the same line, the electromagnetic material momentum has to be different from the pull back of the electromagnetic momentum in vacuum.

Another prerequisite concerns the fact that the first group of Maxwell's equations – say (2) – is automatically taken care of by the introduction of the electromagnetic potentials. Thus a variational formulation will necessarily involve independent variations of $\hat{\varphi}$ and $\hat{\mathbf{A}}$ and either material or Eulerian variations of the motion. Accordingly, for the whole system consisting of matter *plus* electromagnetic fields, we may have to consider Lagrangian densities such as

$$\ell = \ell_R^{em.f}(\bar{\mathbf{E}}, \tilde{\mathbf{B}}, \mathbf{F}, \mathbf{V}) + \ell^{md}(\mathbf{v}, \mathbf{F}, \bar{\mathbf{E}}, \tilde{\mathbf{B}}; \mathbf{X}) \quad (161)$$

for a Lagrangian per unit volume in K_R and a direct-motion description, and

$$\check{\ell} = J_F^{-1} \ell = J_F^{-1} \left[\ell_R^{em.f} + \ell^{mi}(\mathbf{V}, \mathbf{F}^{-1}, \bar{\mathbf{E}}, \tilde{\mathbf{B}}; \mathbf{X}) \right], \quad \mathbf{V} = -\mathbf{F}^{-1} \cdot \mathbf{v}, \quad (162)$$

for a Lagrangian per unit volume of K_t and an inverse-motion description, where

$$\ell^{md} = \frac{1}{2} \rho_0(\mathbf{X}) \mathbf{v}^2 - W(\mathbf{F}, \bar{\mathbf{E}}, \tilde{\mathbf{B}}; \mathbf{X}) \quad (163)$$

and

$$\ell^{mi} = \frac{1}{2} \rho_0(\mathbf{X}) \mathbf{V} \cdot \mathbf{C} \cdot \mathbf{V} - \bar{W}(\mathbf{F}^{-1}, \bar{\mathbf{E}}, \tilde{\mathbf{B}}; \mathbf{X}). \quad (164)$$

It is understood that the local interactions of matter and electromagnetic fields which give rise to magnetization and electric polarization, are contained in W or \bar{W} , from which we shall derive these notions.

The variational principle *per se*:

The Hamiltonian action considered reads

$$A \{ \chi; \mathbf{E}, \mathbf{B} \} = \int_t dt \int_{B_R} (\ell_R^{em.f} + \ell^{md}) dV. \quad (165)$$

Then we have the following fundamental results (Maugin, 1990; Maugin et al., 1992a,b):

Theorem 6.1. *From a material variation accompanied by proper variations of the electromagnetic potentials of the action (165) there follows the second group of Maxwell's equations (3) – in material form – in the absence of electricity conduction, the equation of motion (58) in the absence of body force, and the general constitutive equations*

$$\mathbf{T}^E = \frac{\partial W}{\partial \mathbf{F}}, \quad \mathbf{\Pi} = -\frac{\partial W}{\partial \bar{\mathbf{E}}}, \quad \bar{\mathbf{M}} = -\frac{\partial W}{\partial \tilde{\mathbf{B}}}, \quad (166)$$

with the

Corollary 6.2. *By applying Noether's theorem to (165) for material space translations \mathbf{X} , we obtain the balance of material momentum for the system matter plus field in matter as*

$$\left. \frac{\partial \mathbf{P}^{tot}}{\partial t} \right|_x - \text{div}_R \mathbf{b}^{tot} = \mathbf{f}^{inh}, \quad (167)$$

where we have defined the following entities

$$\mathbf{P}^{tot} = \mathbf{P}^{mech} + \mathbf{P}^{emm}, \quad \mathbf{P}^{mech} = \rho_0 \mathbf{C} \cdot \mathbf{V}, \quad \mathbf{P}^{emm} = \frac{1}{c} \mathbf{\Pi} \times \tilde{\mathbf{B}}, \quad (168)$$

$$\mathbf{b}^{tot} = -(\ell^{md} \mathbf{1}_R + \mathbf{S} \cdot \mathbf{C}), \quad (169)$$

where the second Piola–Kirchhoff stress is given by

$$\mathbf{S} = \mathbf{S}^E - (\mathbf{C}^{-1} \cdot \bar{\mathbf{E}}) \otimes \mathbf{\Pi} + (\mathbf{C}^{-1} \cdot \bar{\mathbf{M}}) \otimes \tilde{\mathbf{B}} - (\bar{\mathbf{M}} \cdot \tilde{\mathbf{B}}) \mathbf{1}_R, \quad (170)$$

and the constitutive equations read

$$\mathbf{S}^E = 2 \frac{\partial W}{\partial \mathbf{C}}, \quad \mathbf{\Pi} = -\frac{\partial W}{\partial \bar{\mathbf{E}}}, \quad \bar{\mathbf{M}} = -\frac{\partial W}{\partial \tilde{\mathbf{B}}}, \quad W = W(\mathbf{C}, \bar{\mathbf{E}}, \tilde{\mathbf{B}}; \mathbf{X}), \quad (171)$$

while there exists a material force of inhomogeneity such as

$$\mathbf{f}^{inh} = \left. \frac{\partial \ell^{md}}{\partial \mathbf{X}} \right|_{expl}. \quad (172)$$

The last of (171) provides an *objective* (materially indifferent) form of the energy W . The *material* electromagnetic momentum defined by the last of (168) exists only in electrodynamics, but it in fact exists only if the material is electrically polarized *and* placed in a magnetic field. It is not the pull

back, changed of sign, of the electromagnetic momentum defined by (58). Concerning the stress involved in the total Eshelby stress, the general form of the effective second Piola–Kirchhoff stress (170) is of great interest. Its electromagnetic contribution is essentially a second Piola–Kirchhoff stress built from the interaction “Cauchy-like” stress $\tilde{\mathbf{t}}^{em}$ introduced in (29). This again means, like in a previous remark, that free fields are *filtered out* by the material manifold, the latter retaining only those terms which contain a true material quantity, such as magnetization or electric polarization (of course in the presence of electromagnetic fields, \mathbf{E} and \mathbf{B}).

Because of its length and technical aspect, the proof of Theorem 6.1 and Corollary 6.2 is not reported here. It is to be found elsewhere (Maugin, 1993).

6.2 Global Balance of Material Momentum

Although the general framework is not so relevant from the applicative viewpoint, some general feeling can be gathered from some results in foregoing sections. This is the case of the material momentum equation (167). Integrated over a *regular* material volume V , this yields

$$\frac{d}{dt}\mathbf{P}(V) = \mathbf{B}_E(V) + \mathbf{F}^{inh}(V), \quad (173)$$

with

$$\begin{aligned} \mathbf{P}(V) &= \int_V \mathbf{P}^{tot} dV, & \mathbf{B}_E(V) &= \int_{\partial V} \mathbf{N} \cdot \mathbf{b}^{tot} dA, \\ \mathbf{F}^{inh}(V) &= \int_V \mathbf{f}^{inh} dV. \end{aligned} \quad (174)$$

Of particular interest here is the surface contribution, which can be rewritten as

$$\mathbf{B}_E(V) = \int_{\partial V} \left\{ P \mathbf{N} - \tilde{\mathbf{T}}^E \cdot \mathbf{C} + Q_P \tilde{\mathbf{E}} - \widehat{\mathbf{M}}(\tilde{\mathbf{B}} \cdot \mathbf{N}) \right\} dA, \quad (175)$$

where we have defined a pressure-like term P , a surface traction $\tilde{\mathbf{T}}^E$, and a surface electric charge density due to electric polarization, Q_P , by

$$P := -\ell^{md} + \widehat{\mathbf{M}} \cdot \tilde{\mathbf{B}}, \quad \tilde{\mathbf{T}}^E := \mathbf{N} \cdot \mathbf{S}^E, \quad Q_P := \mathbf{\Pi} \cdot \mathbf{N}. \quad (176)$$

Several important cases are as follows. For a *nonmagnetizable* material (vanishing magnetization in a co-moving frame), we obviously have $\widehat{\mathbf{M}} = \mathbf{0}$, so that (176) reduces to a form relevant to the case of *electroelastic* materials

$$\mathbf{B}_E(V) = \int_{\partial V} \left\{ -L \mathbf{N} - \tilde{\mathbf{T}}^E \cdot \mathbf{C} + Q_P \tilde{\mathbf{E}} \right\} dA. \quad (177)$$

In quasi-statics, but keeping both magnetization and electric polarization, we obtain an approximation valid for both *electroelasticity* and *magnetoelasticity*:

$$\mathbf{B}_E(V) = \int_{\partial V} \left\{ W \mathbf{N} - \widetilde{\mathbf{T}}^E \cdot (\mathbf{1}_R + 2 \mathbf{E}) - Q_P \nabla_R \hat{\varphi} - \widehat{\mathbf{M}} (\nabla_R \times \widehat{\mathbf{A}}) \cdot \mathbf{N} \right\} dA, \quad (178)$$

where \mathbf{E} is the finite strain expressible in terms of the displacement gradient, and $\hat{\varphi}$ and $\widehat{\mathbf{A}}$ are the material electromagnetic potentials. The expression (178) gives in advance an idea of what will be the J -integral generalized to electroelasticity or magnetoelasticity. This shows an essential difference between electric and magnetic processes because there is no magnetic equivalent here to the notion of polarization surface charge.

For a homogeneous material and for homogeneous boundary conditions or vanishing fields at infinity (in the case of an integration over the whole material space), (173) reduces to a pure conservation of the total material (canonical) momentum. This is what happens in certain problems of propagation dealing with solitons. More on the canonical Hamilton formalism associated with the general electrodynamic case in (Maugin, 1993, pp. 192–193). Instead of dealing with this mundane subject, we prefer to revisit the problem posed by electroelastic solids in finite strains (and its analog in magnetoelastic bodies) because of its many applications in electromechanical devices and in the industry of electronic components.

6.3 Electroelastic Bodies

For many applications it is sufficient to consider a quasi-static approximation to the general equations presented in Section 6.2. In particular, acceleration terms are discarded in the basic equation of motion, magnetization is ignored as well as couplings between electric and magnetic phenomena (although there exist magneto-electric materials of great interest) and most of the time the material is assumed to be a *dielectric*, i.e. it does not conduct electricity and is free of charges. In these conditions the basic field equations at any regular material point \mathbf{X} are reduced to

- Balance of linear (physical) momentum in its equilibrium form:

$$\operatorname{div}_R (\mathbf{T}^E + \mathbf{T}^F) = \mathbf{0}; \quad (179)$$

- Reduced Faraday equation:

$$\nabla_R \times \widehat{\mathbf{E}} = \mathbf{0} \quad \Rightarrow \quad \widehat{\mathbf{E}} = -\nabla_R \hat{\varphi}; \quad (180)$$

- Reduced Gauss equation:

$$\nabla_R \cdot \widehat{\mathbf{D}} = 0, \quad (181)$$

where (179) introduces the first Piola–Kirchhoff stresses associated with elasticity and the “free” electromagnetic fields, while $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{D}}$ are material electric fields. Because we are in quasi-electrostatics, we have the relations:

$$\begin{aligned} \widehat{\mathbf{E}} &= \mathbf{E} \cdot \mathbf{F}, & \widehat{\mathbf{D}} &= J_F \mathbf{F}^{-1} \cdot \mathbf{D}, \\ \mathbf{\Pi} &= J_F \mathbf{F}^{-1} \cdot \mathbf{P}, & \widehat{\mathbf{D}} &= J_F \mathbf{C}^{-1} \cdot \widehat{\mathbf{E}} + \mathbf{\Pi}, \end{aligned} \quad (182)$$

where \mathbf{E} , \mathbf{D} , and \mathbf{P} are the standard fields in a laboratory frame.

Because of our special interest in fracture and the evaluation of energy-release rates, the equation of energy associated with equations (179)–(181) is most relevant. To that purpose we note that

$$\mathbf{T}^F = \overline{\mathbf{E}} \otimes \mathbf{E} - \frac{1}{2} \mathbf{F}^{-1} (\widehat{\mathbf{E}} \cdot \overline{\mathbf{E}}), \quad \overline{\mathbf{E}} := J_F \mathbf{F}^{-1} \cdot \mathbf{E}. \quad (183)$$

It is checked that

$$\operatorname{div}_R \mathbf{T}^F = -(\nabla_R \cdot \mathbf{\Pi}) \mathbf{E}, \quad (184)$$

where we recognize in the quantity within parentheses a so-called polarization charge density.

With an objective energy density for a homogeneous material, per unit reference volume,

$$W = \overline{W}(\mathbf{C}, \widehat{\mathbf{E}}),$$

we have the mechanical and electric constitutive equations

$$\mathbf{S} = 2 \frac{\partial \overline{W}}{\partial \mathbf{C}}, \quad \mathbf{\Pi} = -\frac{\partial \overline{W}}{\partial \widehat{\mathbf{E}}}, \quad (185)$$

corresponding to the energy equation (no dissipation of any kind)

$$\dot{W} = \frac{1}{2} \operatorname{tr}(\mathbf{S} \cdot \dot{\mathbf{C}}) - \mathbf{\Pi} \cdot \dot{\widehat{\mathbf{E}}}. \quad (186)$$

We can as well consider the so-called *electric enthalpy*

$$\widetilde{W} = \overline{W} - \frac{1}{2} \widehat{\mathbf{E}} \cdot \overline{\mathbf{E}}, \quad (187)$$

so that, on account of the last of (186), instead of (185) we have the following material constitutive equations

$$\mathbf{S} = 2 \frac{\partial \widetilde{W}}{\partial \mathbf{C}}, \quad \widehat{\mathbf{D}} = -\frac{\partial \widetilde{W}}{\partial \widehat{\mathbf{E}}}. \quad (188)$$

It is the multiplicity of possible electromechanical energies which causes some problems in the sequel. Indeed, starting from the field equations (179) through (181) and the energy equation (186), we can deduce some identities which will be useful for the evaluation of energy-release rates. All these are obtained at regular material points. For instance, (186) can first be rewritten as

$$\dot{W} = \text{tr}(\mathbf{T}^E \cdot \dot{\mathbf{F}}) - \mathbf{\Pi} \cdot \dot{\hat{\mathbf{E}}}. \quad (189)$$

But this can be transformed to

$$\frac{d}{dt} \widetilde{W} + \widehat{\mathbf{D}} \cdot \dot{\hat{\mathbf{E}}} = \text{tr}(\mathbf{T}^E \cdot (\nabla_R \mathbf{v})^T) + \frac{1}{2} (\overline{\mathbf{E}} \cdot \dot{\hat{\mathbf{E}}} - \dot{\hat{\mathbf{E}}} \cdot \widehat{\mathbf{E}}). \quad (190)$$

But we can write

$$\text{tr}(\mathbf{T}^E \cdot (\nabla_R \mathbf{v})^T) = \nabla_R \cdot ((\mathbf{T}^E + \mathbf{T}^F) \cdot \mathbf{v}) - \text{tr}(\mathbf{T}^F \cdot (\nabla_R \mathbf{v})^T), \quad (191)$$

and also prove that

$$\text{tr}(\mathbf{T}^F \cdot (\nabla_R \mathbf{v})^T) = \frac{1}{2} (\overline{\mathbf{E}} \cdot \dot{\hat{\mathbf{E}}} - \dot{\hat{\mathbf{E}}} \cdot \widehat{\mathbf{E}}). \quad (192)$$

The nontrivial proof of this is given in the appendix to Dascalu and Maugin (1994). On combining (190) through (192), we obtain that after introduction of the electric potential and enthalpy,

$$\frac{d}{dt} \widetilde{W} = \nabla_R \cdot ((\mathbf{T}^E + \mathbf{T}^F) \cdot \mathbf{v} + \widehat{\mathbf{D}} \dot{\hat{\varphi}}). \quad (193)$$

This is a remarkable form of the energy equation because it is written as *a strict conservation law*. The dual material contravector to $\widehat{\mathbf{D}} \dot{\hat{\varphi}}$, $-\dot{\hat{\varphi}} \partial \widehat{\mathbf{D}} / \partial t$, is the *Poynting vector* of quasi-electrostatics (see Maugin, 1988, p. 238). In contrast, we can write another form, also totally admissible, by considering (190) and noting that

$$\frac{d}{dt} \left(\frac{1}{2} \widehat{\mathbf{E}} \cdot \overline{\mathbf{E}} \right) = \nabla_R \cdot (\mathbf{T}^F \cdot \mathbf{v} + \overline{\mathbf{E}} \dot{\hat{\varphi}}) + h^{elec}, \quad (194)$$

where we have set

$$h^{elec} = (\text{div}_R \mathbf{T}^F) \cdot \mathbf{v} + (\nabla_R \cdot \overline{\mathbf{E}}) \dot{\hat{\varphi}}. \quad (195)$$

Whence (193) takes on the form

$$\frac{d}{dt} \overline{W} = \nabla_R \cdot (\mathbf{T}^E \cdot \mathbf{v} + \mathbf{\Pi} \dot{\hat{\varphi}}) + h^{elec}, \quad (196)$$

which is not in the form of a strict conservation law, but it emphasizes the consideration of the electric polarization as compared to that of the electric displacement. These two forms will necessarily have different consequences in so far as the evaluation of the corresponding energy-release rate of fracture is concerned.

6.4 Evaluation of the Energy-Release Rate in Electroelastic Fracture

The proofs will follow exactly those of the pure elastic case and, therefore, details are not repeated, the reader being referred to the original research papers (here essentially Dascalu and Maugin, 1994). For instance, in the first case where we start from the local energy equation (193), with a classical notation, we shall obtain the following global energy balance in the presence of the straight through crack

$$\frac{d}{dt} \int_B \widetilde{W} \, dA + G^{crack} = \int_S \left\{ \mathbf{N} \cdot (\mathbf{T}^E + \mathbf{T}^F) \cdot \mathbf{v} + \dot{\phi} (\widehat{\mathbf{D}} \cdot \mathbf{N}) \right\} \, dS, \quad (197)$$

where S is the boundary of $B-C$, where C is the crack, and we have defined the energy-release rate by

$$G^{crack} = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \left\{ \widetilde{W} (\overline{\mathbf{V}} \cdot \mathbf{N}) + \mathbf{N} \cdot (\mathbf{T}^E + \mathbf{T}^F) \cdot \mathbf{v} + \dot{\phi} (\widehat{\mathbf{D}} \cdot \mathbf{N}) \right\} \, d\Gamma \quad (198)$$

where $\overline{\mathbf{V}}$ is the material velocity of the irreversible progress of the tip of the crack in the body. Here (197) results from the application of Reynolds' theorem for the evolving body. Clearly, the expression (198) involves not only the flux of electric enthalpy, but also the full contributions of coupled and free electric fields. We can say that this is a "natural" formulation. If instead we start with the identity (196) we shall obtain the global energy balance as

$$\begin{aligned} \frac{d}{dt} \int_B \overline{W} \, dA + G_*^{crack} &= \int_S \left\{ \mathbf{N} \cdot \mathbf{T}^E \cdot \mathbf{v} + \dot{\phi} (\boldsymbol{\Pi} \cdot \mathbf{N}) \right\} \, dS \\ &+ \int_B h^{elec} \, dA, \end{aligned} \quad (199)$$

with an energy release rate given by

$$G_*^{crack} = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \left\{ \overline{W} (\overline{\mathbf{V}} \cdot \mathbf{N}) + \mathbf{N} \cdot \mathbf{T}^E \cdot \mathbf{v} + \dot{\phi} (\boldsymbol{\Pi} \cdot \mathbf{N}) \right\} \, dS \quad (200)$$

Notice that the stress contribution related to the free electric field is not involved in this formula. In order to obtain (200) we assumed that the integrals in this relation are convergent. For some linear piezoelectric materials (Pak, 1990; Sosa and Pak, 1990; or in the case of linearized electrostriction as show by Dascalu and Maugin, 1995), the behavior of the solution at the tip shows that the term h^{elec} is of order r^{-2} , so that the last integral in (199) generally diverges. However, if $h^{elec} = 0$, then the above computation holds good. If we examine the definition (195) we can show that this is nothing but

$$h^{elec} = -(\nabla_R \cdot \mathbf{\Pi}) \frac{\partial \varphi}{\partial t}, \quad (201)$$

where φ is the electrostatic potential in the actual configuration. Thus, as $\partial \varphi / \partial t$ cannot be forced to vanish, the condition of vanishing h^{elec} can be realized only if we impose the constraint

$$\nabla_R \cdot \mathbf{\Pi} = 0, \quad (202)$$

as a sufficient condition. This can be achieved in some concrete electroelastic problem (cf. Dascalu and Maugin, 1995).

6.5 Electroelastic Path-Independent Integrals

Now we express the above-obtained energy-release rates in terms of contour integrals that do not depend on the integration path, an essential property for easy computation. These integrals were obtained first by Pak and Herrmann (1986) and Maugin and Epstein (1991). To do this we need an estimate of the degree of singularity of electro-mechanical fields in the neighborhood of the crack tip. Suppose that both the displacement \mathbf{u} and the electric potential $\hat{\varphi}$ have a regular time behavior as observed from the crack tip. So, just the same as in pure elasticity (Gurtin, 1979; Nguyen Quoc Son, 1980), this assumption allows us to write

$$\dot{\mathbf{u}} = -\bar{\mathbf{V}} \cdot \nabla_R \mathbf{u} + \mathbf{w}, \quad \dot{\hat{\varphi}} = -\bar{\mathbf{V}} \cdot \nabla_R \hat{\varphi} + \psi, \quad (203)$$

where \mathbf{w} and ψ have no singular behavior at the crack tip. Then the terms in G^{crack} containing these fields will vanish for $\Gamma \rightarrow 0$. This allows us to show that G^{crack} takes on the form

$$G^{crack} = \lim_{\Gamma \rightarrow 0} \bar{\mathbf{V}} \cdot \int_{\Gamma} \left\{ \widetilde{\mathbf{W}} \mathbf{N} - \nabla_R \mathbf{u} \cdot (\mathbf{T}^E + \mathbf{T}^F)^T \cdot \mathbf{N} - \nabla_R \hat{\varphi} (\widehat{\mathbf{D}} \cdot \mathbf{N}) \right\} dS \quad \text{as } \Gamma \rightarrow 0. \quad (204)$$

The same technique applied to G_*^{crack} yields

$$G_*^{crack} = \lim_{\Gamma \rightarrow 0} \bar{\mathbf{V}} \cdot \int_{\Gamma} \left\{ \widetilde{W} \mathbf{N} - \nabla_R \mathbf{u} \cdot (\mathbf{T}^E)^T \cdot \mathbf{N} - \nabla_R \varphi (\Pi \cdot \mathbf{N}) \right\} dS \quad (205)$$

In this relation we can replace $\nabla_R \mathbf{u}$ by \mathbf{F} making the assumption that

$$\lim_{\Gamma \rightarrow 0} \int_{\Gamma} \mathbf{N} \cdot \mathbf{T}^E dS = 0 \quad \text{as } \Gamma \rightarrow 0, \quad (206)$$

an assumption that is verified when \mathbf{T}^E behaves like $r^{-1/2}$ near the tip, just like in classical elasticity (cf. Pak, 1990; Sosa and Pak, 1990; Dascalu and Maugin, 1995). Then (204) reads

$$G^{crack} = \lim_{\Gamma \rightarrow 0} \bar{\mathbf{V}} \cdot \int_{\Gamma} \left\{ \widetilde{W} \mathbf{N} - \mathbf{F}^T \cdot (\mathbf{T}^E + \mathbf{T}^F)^T \cdot \mathbf{N} - \nabla_R \hat{\varphi} (\widehat{\mathbf{D}} \cdot \mathbf{N}) \right\} dS \quad \text{as } \Gamma \rightarrow 0. \quad (207)$$

Now we restrict the analysis to the case of a straight through crack along the X_1 -axis and thus

$$\bar{\mathbf{V}} = \dot{\ell}(t) \mathbf{E}_1. \quad (208)$$

With this we formulate electroelastic J -integrals. For a piecewise smooth non intersecting path Γ which begins and ends on the crack and surrounds the tip of the crack, we define

$$J(\Gamma) = \int_{\Gamma} \left\{ \widetilde{W} N_1 - \mathbf{N} \cdot (\mathbf{T}^E + \mathbf{T}^F) \cdot \frac{\partial \mathbf{u}}{\partial X_1} - (\widehat{\mathbf{D}} \cdot \mathbf{N}) \frac{\partial \hat{\varphi}}{\partial X_1} \right\} dS. \quad (209)$$

If J does not depend on Γ , then (207) yields

$$G^{crack} = J \dot{\ell}, \quad (210)$$

the familiar dissipation form of the product of a “force” and a “velocity”. The J -integral (209) was obtained by Pak and Herrmann (1986) using the Eshelby theory of inhomogeneities. Their argument for the path independence stems from the relation

$$\operatorname{div}_R \mathbf{b} = \mathbf{0}, \quad (211)$$

with an electromechanical Eshelby stress given by

$$\mathbf{b} = \widetilde{W} \mathbf{1}_R - (\mathbf{T}^E + \mathbf{T}^F) \cdot (\nabla_R \mathbf{u})^T - \widehat{\mathbf{D}} \otimes \nabla_R \hat{\varphi}. \quad (212)$$

Here also, the path independence requires that some conditions hold true along the faces of the crack. These conditions are

$$\mathbf{N} \cdot \widehat{\mathbf{D}}^\pm = 0, \quad \mathbf{N} \cdot (\mathbf{T}^E + \mathbf{T}^F)^\pm = \mathbf{0}, \quad (213)$$

i.e. neither electric charges nor tractions along the crack. If, however, we deal with paths which start and end at the same point along the crack, then only the jumps in (213) are required to vanish.

To deal with G_*^{crack} , we define another J -integral, J^* , by

$$J^* = \int_{\Gamma} \left\{ \overline{W} N_1 - \mathbf{N} \cdot \mathbf{S}^E \cdot \mathbf{C} \cdot \mathbf{E}_1 - (\boldsymbol{\Pi} \cdot \mathbf{N}) \frac{\partial \hat{\varphi}}{\partial X_1} \right\} dS, \quad (214)$$

which was obtained by Maugin and Epstein (1991) using the same method of Eshelby but remarking that an identity verified by the free electric fields permits us to work with the following electroelastic Eshelby stress tensor (cf. Section 6.1)

$$\mathbf{b}^* = \overline{W} \mathbf{1}_R - \mathbf{S}^E \cdot \mathbf{C} - \boldsymbol{\Pi} \otimes \nabla_R \hat{\varphi}, \quad (215)$$

for which

$$\operatorname{div}_R \mathbf{b}^* = \mathbf{0}, \quad (216)$$

at all regular material points \mathbf{X} . For the path independence of J^* we must have

$$\mathbf{N} \cdot (\mathbf{T}^E)^\pm = \mathbf{0}, \quad \mathbf{N} \cdot \boldsymbol{\Pi}^\pm = 0, \quad (217)$$

on the faces of the crack. The same conditions, but for the jumps, are valid when the contour of J^* starts and ends at the same point.

6.6 Electroelastic Phase-Transition Fronts

Here we consider a more general framework than in the preceding section by allowing the presence of thermal effects and of a density of electric charges. The last ingredient may have some importance in electroelastic bodies that may contain these charges such as in piezoelectric semiconductors. Furthermore, they allow for interesting developments in the case of transition zones such as phase-transition fronts.

A. General equations

Still in the frame work of electrostatics we have at any regular material point:

- Maxwell's electrostatic equations:

$$\nabla_R \times \widehat{\mathbf{E}} = \mathbf{0}, \quad \nabla_R \cdot \widehat{\mathbf{D}} = \widehat{Q}_f; \quad (218)$$

- Conservation of mass:

$$\left. \frac{\partial}{\partial t} \rho_0 \right|_x = 0; \quad (219)$$

- Balance of linear (physical) momentum:

$$\left. \frac{\partial}{\partial t} \mathbf{P} \right|_x - \operatorname{div}_R \mathbf{T} = \mathbf{f}^{em}; \quad (220)$$

- Balance of energy:

$$\left. \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 \mathbf{v}^2 + E - \widehat{Q}_f \widehat{\varphi} \right) \right|_x - \nabla_R \cdot (\mathbf{T} \cdot \mathbf{v} + \mathbf{S}^e - \mathbf{Q}) = h^e; \quad (221)$$

- Balance of entropy:

$$\left. \theta \frac{\partial S}{\partial t} \right|_x + \nabla_R \cdot \mathbf{Q} = 0. \quad (222)$$

This writing (Maugin, 1988) isolates the electric force \mathbf{f}^{em} , incorporates the action of the free charge in the total energy, and emphasizes the notions of electric Poynting vector \mathbf{S}^e and electric energy source h_e with

$$\mathbf{f}^{em} = (\widehat{Q}_f + \mathbf{\Pi} \cdot \nabla_R) \mathbf{E}, \quad \mathbf{S}^e = -\widehat{\varphi} \left. \frac{\partial \widehat{\mathbf{D}}}{\partial t} \right|_x, \quad h^e = -\widehat{Q}_f \left. \frac{\partial \widehat{\varphi}}{\partial t} \right|_x, \quad (223)$$

together with

$$\begin{aligned} \widehat{\mathbf{D}} &= \overline{\mathbf{E}} + \mathbf{\Pi}, & \widehat{\mathbf{E}} &= \mathbf{E} \cdot \mathbf{F}, \\ \overline{\mathbf{E}} &= J_F \mathbf{F}^{-1} \cdot \mathbf{E} = J_F \mathbf{C}^{-1} \cdot \widehat{\mathbf{E}}, & \widehat{\mathbf{E}} &= -\nabla_R \widehat{\varphi}. \end{aligned} \quad (224)$$

The following identities can be proved (Maugin, 1988; Dascalu and Maugin, 1995):

$$\mathbf{f}^{em} = \operatorname{div}_R \mathbf{T}^{em}, \quad \operatorname{div}_R \mathbf{T}^F = (\widehat{Q}_f - \nabla_R \cdot \mathbf{\Pi}) \mathbf{E}, \quad (225)$$

$$\mathbf{T}^{em} = \widehat{\mathbf{D}} \otimes \mathbf{E} - \frac{1}{2} (\widehat{\mathbf{E}} \cdot \overline{\mathbf{E}}) \mathbf{F}^{-1}, \quad \mathbf{T}^F = \overline{\mathbf{E}} \otimes \mathbf{E} - \frac{1}{2} (\widehat{\mathbf{E}} \cdot \overline{\mathbf{E}}) \mathbf{F}^{-1}, \quad (226)$$

and

$$\operatorname{tr}(\mathbf{T}^F \cdot (\nabla_R \mathbf{v})^F) = \frac{1}{2} \left(\overline{\mathbf{E}} \cdot \left. \frac{\partial \widehat{\mathbf{E}}}{\partial t} \right|_x - \left. \frac{\partial \overline{\mathbf{E}}}{\partial t} \right|_x \cdot \widehat{\mathbf{E}} \right), \quad (227)$$

$$\nabla_R \cdot \overline{\mathbf{E}} = \widehat{Q}_f - \nabla_R \cdot \mathbf{\Pi}, \quad (228)$$

$$\frac{d}{dt} \left(\frac{1}{2} \widehat{\mathbf{E}} \cdot \overline{\mathbf{E}} \right) = -\nabla_R \cdot \left(\mathbf{T}^F \cdot \mathbf{v} + \overline{\mathbf{E}} \left. \frac{\partial \widehat{\varphi}}{\partial t} \right|_x \right) + H, \quad (229)$$

$$H = \mathbf{v} \cdot \operatorname{div}_R \mathbf{T}^F + (\nabla_R \cdot \overline{\mathbf{E}}) \left. \frac{\partial \widehat{\varphi}}{\partial t} \right|_x = (\widehat{Q}_f - \nabla_R \cdot \mathbf{\Pi}) \left. \frac{\partial \varphi}{\partial t} \right|_x. \quad (230)$$

Here $\varphi = \hat{\varphi}(\mathbf{X}(\mathbf{x}, t), t) = \varphi(\mathbf{x}, t)$ is the Eulerian electrostatic potential such that

$$\left. \frac{\partial \varphi}{\partial t} \right|_x = \left. \frac{\partial \hat{\varphi}}{\partial t} \right|_x + \mathbf{V} \cdot \nabla_R \cdot \hat{\varphi}. \quad (231)$$

Equation (220) can also be written as a strict conservation law in the form

$$\left. \frac{\partial}{\partial t} \mathbf{P} \right|_x - \operatorname{div}_R \mathbf{T}^{tot} = \mathbf{0}, \quad (232)$$

wherein

$$\mathbf{T}^{tot} = \mathbf{T} + \mathbf{T}^{em} = \mathbf{T}^E + \mathbf{T}^F. \quad (233)$$

Constitutive relations

We recall that \mathbf{S}^E is the second (material and symmetric) Piola–Kirchhoff stress associated with \mathbf{T}^E , and W denotes the free energy per unit reference volume, $W = E - S\theta$, so that the Clausius–Duhem inequality reads

$$-(\dot{W} + S\dot{\theta}) + \frac{1}{2} \operatorname{tr}(\mathbf{S}^E \cdot \dot{\mathbf{C}}) + \hat{\mathbf{E}} \cdot \dot{\mathbf{\Pi}} - \theta^{-1} \mathbf{Q} \cdot \nabla_R \theta \geq 0, \quad (234)$$

or

$$-(\dot{\bar{W}} + S\dot{\theta}) + \frac{1}{2} \operatorname{tr}(\bar{\mathbf{S}}^E \cdot \dot{\mathbf{C}}) - \mathbf{\Pi} \cdot \dot{\hat{\mathbf{E}}} - \theta^{-1} \mathbf{Q} \cdot \nabla_R \theta \geq 0, \quad (235)$$

with

$$\bar{W} = W - \hat{\mathbf{E}} \cdot \mathbf{\Pi}, \quad \bar{\mathbf{S}}^E = \mathbf{S}^E - (\hat{\mathbf{E}} \cdot \mathbf{\Pi}) \mathbf{C}^{-1}. \quad (236)$$

Still another possibility is provided with (Maugin and Trimarco, 1997)

$$-(\dot{\hat{W}} + S\dot{\theta}) + \frac{1}{2} \operatorname{tr}(\hat{\mathbf{S}}^E \cdot \dot{\mathbf{C}}) - \hat{\mathbf{D}} \cdot \dot{\hat{\mathbf{E}}} - \theta^{-1} \mathbf{Q} \cdot \nabla_R \theta \geq 0, \quad (237)$$

wherein

$$\hat{W} = \bar{W} + \frac{1}{2} \hat{\mathbf{E}} \cdot \bar{\mathbf{E}}, \quad \hat{\mathbf{S}}^E = \bar{\mathbf{S}}^E + \frac{1}{2} (\hat{\mathbf{E}} \cdot \bar{\mathbf{E}}) \mathbf{C}^{-1} - J_F^{-1} \bar{\mathbf{E}} \otimes \bar{\mathbf{E}}. \quad (238)$$

With the choice of free energy

$$\hat{W} = \hat{W}(\mathbf{C}, \hat{\mathbf{E}}, \theta; \mathbf{X}), \quad (239)$$

the usual argument of thermodynamic admissibility yields the constitutive equations

$$\hat{\mathbf{S}}^E = 2 \frac{\partial \hat{W}}{\partial \mathbf{C}}, \quad \hat{\mathbf{D}} = -\frac{\partial \hat{W}}{\partial \hat{\mathbf{E}}}, \quad S = -\frac{\partial \hat{W}}{\partial \theta}, \quad (240)$$

while there remains the residual dissipation inequality

$$\mathbf{Q}(\mathbf{C}, \hat{\mathbf{E}}, \theta, \nabla_R \theta; \mathbf{X}) \cdot \nabla_R \theta \leq 0. \quad (241)$$

B. Canonical balance laws

These are the balance laws of energy and momentum associated with the space-time parametrization (\mathbf{X}, t) . Since there already are several equivalent forms of the local balance of energy, the most sensible one is that obtained *via* the equation of mechanical energy, obtained by inner product of (220) with \mathbf{v} , and then combining with (221) to yield

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 \mathbf{v}^2 + \overline{E} \right) \Big|_x - \nabla_R \cdot \left\{ (\mathbf{T}^E + \mathbf{T}^F) \cdot \mathbf{v} + \widehat{\mathbf{D}} \frac{\partial \hat{\varphi}}{\partial t} \Big|_x - \mathbf{Q} \right\} \\ = -\widehat{Q}_f \frac{\partial \hat{\varphi}}{\partial t} \Big|_x, \end{aligned} \quad (242)$$

where

$$\overline{E} = W + S\theta - \widehat{\mathbf{D}} \cdot \widehat{\mathbf{E}}. \quad (243)$$

Equation (242) has the advantage that, although not in a strict conservative form, it emphasizes the analogy between the roles played by the “velocities” $\mathbf{v} = \partial \chi / \partial t|_x$ and $\partial \hat{\varphi} / \partial t|_x$, hence between the elastic displacement and the electrostatic potential. It obviously takes the form of a strict conservation law in the absence of free charges. Equation (221) is immediately recovered from (242) by noting that

$$\nabla_R \cdot \mathbf{S}^e = \frac{\partial}{\partial t} (\widehat{\mathbf{E}} \cdot \widehat{\mathbf{D}} - \widehat{Q}_f \hat{\varphi}) + \nabla_R \cdot \left(\widehat{\mathbf{D}} \frac{\partial \hat{\varphi}}{\partial t} \Big|_x \right), \quad E = \overline{E} + \widehat{\mathbf{E}} \cdot \widehat{\mathbf{D}}. \quad (244)$$

The second canonical balance law, that of material momentum, is obtained in strict parallel with (242) following now a routine procedure. The result is

$$\frac{\partial \mathbf{P}}{\partial t} \Big|_x - \operatorname{div}_R \mathbf{b} = \mathbf{f}^{inh} + \mathbf{f}^{th} + \mathbf{f}^e, \quad (245)$$

where we have set

$$\mathbf{P} = -\mathbf{p} \cdot \mathbf{F} = \rho_0 \mathbf{C} \cdot \mathbf{V}, \quad (246)$$

$$\mathbf{b} = -(\ell^{th} \mathbf{1}_R + (\mathbf{T}^E + \mathbf{T}^F) \cdot \mathbf{F} - \widehat{\mathbf{D}} \otimes \widehat{\mathbf{E}}),$$

$$\ell^{th} = \frac{1}{2} \rho_0(\mathbf{X}) \mathbf{v}^2 - W(\mathbf{C}, \widehat{\mathbf{E}}, \theta; \mathbf{X}), \quad \mathbf{f}^{inh} = \left(\frac{\partial L^{th}}{\partial \mathbf{X}} \right)_{expl}, \quad (247)$$

$$\mathbf{f}^{th} = S \nabla_R \theta, \quad \mathbf{f}^e = -\widehat{Q}_f \widehat{\mathbf{E}}. \quad (248)$$

The really new quantity here is the last material force \mathbf{f}^e , which is none other than the pull back of the original volume force due to free charges, changed of sign. It is of interest to evaluate its power in a material motion. That is,

the quantity $\mathbf{f}^e \cdot \mathbf{V}$. We have the following remarkable result (Maugin and Trimarco, 1997):

$$P_B^e := \mathbf{f}^e \cdot \mathbf{V} = \widehat{Q}_f \left(\left. \frac{\partial \varphi}{\partial t} \right|_x - \left. \frac{\partial \hat{\varphi}}{\partial t} \right|_x \right). \quad (249)$$

This really exemplifies the fictitious nature of some material forces such as \mathbf{f}^e , as their power vanishes identically when the distinction between actual and reference configurations is lost.

C. Jump relations at a front

We consider from the start a homothermal singular surface with no dislocations so that we have the following two conditions of continuity:

$$[\theta] = 0, \quad [\mathbf{V}] = \mathbf{0} \quad \text{at } \Sigma. \quad (250)$$

Now, without further ado we can apply the thumb rule to replace the partial differential operators ∇_R and $\partial/\partial t|_x$ applied to functions $f(\mathbf{X}, t)$ by the jump operators $\mathbf{N} \cdot [\cdot]$ and $-\overline{\mathbf{V}}_N [\cdot]$ and to introduce unknown surface source terms for those equations which are not strict conservation laws at regular material points. Thus we have the following roster of jump equations in the present case ($\overline{\mathbf{V}}$ is the material velocity of the front)

$$\mathbf{N} \times [\widehat{\mathbf{E}}] = \mathbf{0}, \quad \mathbf{N} \cdot [\widehat{\mathbf{D}}] = Q_\Sigma^e, \quad (251)$$

$$\overline{\mathbf{V}}_N [\rho_0] = 0, \quad (252)$$

$$\mathbf{N} \cdot [\overline{\mathbf{V}} \otimes \mathbf{p} + \mathbf{T}^E + \mathbf{T}^F] = \mathbf{0}, \quad (253)$$

$$\mathbf{N} \cdot \left[\overline{\mathbf{V}} \left(\frac{1}{2} \mathbf{v}^2 + \overline{E} \right) + (\mathbf{T}^E + \mathbf{T}^F) \cdot \mathbf{v} + \widehat{\mathbf{D}} \left. \frac{\partial \hat{\varphi}}{\partial t} \right|_x - \mathbf{Q} \right] = h_\Sigma^e, \quad (254)$$

$$\mathbf{N} \cdot [\overline{\mathbf{V}} \otimes \mathbf{P} + \mathbf{b}] + \mathbf{f}_\Sigma = \mathbf{0}, \quad (255)$$

$$\mathbf{N} \cdot [\overline{\mathbf{V}} S \theta - \mathbf{Q}] - q_\Sigma = 0, \quad (256)$$

and

$$\mathbf{N} \cdot [\overline{\mathbf{V}} S - (\mathbf{Q}/\theta)] = \sigma_\Sigma \geq 0, \quad (257)$$

where a set of unknown surface sources is present in (254) through (257). These must be all consistent in order to respect the second law of thermodynamics expressed by the inequality in (257). Already, the consistency between (256) and (257) requires that

$$q_\Sigma \geq 0 \quad \text{at } \Sigma(t). \quad (258)$$

We shall not give the details of the derivation of the results that follows. On one hand we compute

$$P_\Sigma := \mathbf{f}_\Sigma \cdot \bar{\mathbf{V}} = -\bar{V}_N [\mathbf{P} \cdot \bar{\mathbf{V}}] - [\mathbf{N} \cdot \mathbf{b} \cdot \bar{\mathbf{V}}], \quad (259)$$

from which there follows that

$$q_\Sigma = -\bar{V}_N \text{Hugo}_{PT} + Q_\Sigma^e \left\langle \frac{\partial \varphi}{\partial t} \Big|_x - \frac{\partial \hat{\varphi}}{\partial t} \Big|_x \right\rangle - \left(h_\Sigma^e + [\mathbf{N} \cdot \hat{\mathbf{D}}] \frac{\partial \varphi}{\partial t} \Big|_x \right) \geq 0 \quad (260)$$

or

$$q_\Sigma = -\bar{V}_N \text{Hugo}_{PT} - Q_\Sigma^e \left\langle \frac{\partial \hat{\varphi}}{\partial t} \Big|_x \right\rangle - \left(h_\Sigma^e + \langle \mathbf{N} \cdot \hat{\mathbf{D}} \rangle \left[\frac{\partial \varphi}{\partial t} \Big|_x \right] \right) \geq 0, \quad (261)$$

where we have introduced the Hugoniot–Gibbs functional (that depends on the value of fields on both “sides” of Σ):

$$\text{Hugo}_{PT} := \left[W - \langle \mathbf{N} \cdot (\mathbf{T}^E + \mathbf{T}^F) \rangle \cdot \frac{\partial \chi}{\partial N} - \langle \mathbf{N} \cdot \hat{\mathbf{D}} \rangle \frac{\partial \varphi}{\partial N} \right], \quad (262)$$

where $\partial/\partial N$ denotes the normal derivative. The expression (261) deserves the following comments. First, only the normal component of $\bar{\mathbf{V}}$ is involved, thus emphasizing the local normal growth of one phase with respect to the other. The other two contributions in (261) are peculiar. We notice that application of a gauge condition at $\Sigma(t)$ for quasi-electrostatics processes requires that

$$\left[\frac{\partial \varphi}{\partial t} \Big|_x \right] = 0. \quad (263)$$

This condition is formally analogous to the coherency condition (250)₂. Therefore, it could be referred to as the *electric coherency condition*. Then from (261) there remains

$$q_\Sigma = -\bar{V}_N \text{Hugo}_{PT} - Q_\Sigma^e \left\langle \frac{\partial \hat{\varphi}}{\partial t} \Big|_x \right\rangle - h_\Sigma^e \geq 0. \quad (264)$$

The formally introduced surface heat h_Σ^e may be viewed as some kind of *latent heat* characteristic of the examined electroelastic crystal.

Had we considered a *dielectric* material to start with, we would have introduced neither h_Σ^e nor Q_Σ^e , and so (261) would reduce to the “simple” expression

$$\theta_\Sigma \sigma_\Sigma = q_\Sigma = f_\Sigma \bar{V}_N \geq 0, \quad (265)$$

where the scalar *driving* force f_Σ is introduced through the following *surface balance* of scalar material forces:

$$f_\Sigma + Hugo_{pT} = 0 \quad \text{at } \Sigma(t). \quad (266)$$

This emphasizes the different ontological nature of the two scalar material forces as $Hugo_{pT}$ is a (functional) field quantity, while f_Σ is a thermodynamical force determined by the application of the theory of irreversible thermodynamics to the pair of conjugate variables (f_Σ, \bar{V}_N) , yielding eventually a *kinetic relation* of the type

$$\bar{V}_N = \tilde{V}(f_\Sigma; \theta_\Sigma) \quad (267)$$

respecting the inequality (265).

Returning to the more general case (264), we note that the expression

$$P_\Sigma^e = -Q_\Sigma^e \left\langle \frac{\partial \hat{\varphi}}{\partial t} \Big|_x \right\rangle \quad (268)$$

has formally the same structure as the bulk power (249) because, if a gauge condition of the type $\partial\varphi/\partial t|_x$ holds at all material points, the latter reduces to

$$P_B^e = -Q_f \frac{\partial \hat{\varphi}}{\partial t} \Big|_x, \quad (269)$$

of which the similarity with (249) is obvious. Quantities such as (269) will naturally appear if, at Σ , we have possible re-combination of charges as it occurs at junctions in electroelastic semiconductors (classical jump conditions for these are given in Daher and Maugin, 1986, 1988). Concerning works in the line of the presentation in this section, we note Jiang (1994), but this is erroneous.

6.7 Case of Magnetized Materials

Developments devoted to magnetizable deformable solids, and parodying the electroelastic case have been given by Sabir and Maugin (1996) and Fomethé and Maugin (1996), in so far as magnetoelastic fracture and the propagation of magnetoelastic phase-transition fronts and magnetic domain walls are concerned. We refer the reader to these authors. Here we just quote a few exemplary results in which the reader can identify the analogies and differences with the electroelastic case, keeping in mind the general electro-magnetic expressions given in previous lectures. In particular, we note the following reduction of Maxwell's equations in quasi-magnetostatics in insulators (in the Laboratory frame)

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{0}, \quad \mathbf{H} = \mathbf{B} - \mathbf{M}. \quad (270)$$

This translates into the material form as

$$\nabla_R \cdot \widehat{\mathbf{B}} = 0, \quad \nabla_R \times \widehat{\mathbf{H}} = \mathbf{0}, \quad (271)$$

with

$$\widehat{\mathbf{B}} = J_F \mathbf{F}^{-1} \cdot \mathbf{B}, \quad \widehat{\mathbf{H}} = \mathbf{H} \cdot \mathbf{F}, \quad \widehat{\mathbf{M}} = \mathbf{M} \cdot \mathbf{F}, \quad \widehat{\mathbf{H}} = J_F^{-1} \mathbf{C} \cdot \widehat{\mathbf{B}} - \widehat{\mathbf{M}}. \quad (272)$$

The coupling between the crystal lattice and the magnetization field is represented by a local bulk balance equation, comparable to (68) – case of soft ferromagnets –, i.e.

$$\mathbf{B} + \mathbf{B}^L = \mathbf{0}, \quad (273)$$

where \mathbf{B} is the Maxwellian field appearing in the first of (270) and \mathbf{B}^L is the local magnetic induction for which one needs a constitutive equation in terms of magneto-mechanical fields. Because of (270), there exists a quasi-static magnetic potential φ in the actual configuration or $\hat{\varphi}$ in the material framework, so that

$$\mathbf{H} = -\nabla\varphi, \quad \widehat{\mathbf{H}} = -\nabla_R\hat{\varphi}. \quad (274)$$

Then the perspicacious reader has already noticed that we can practically translate all what we did in the quasi-electrostatic case to this magnetic case. This was indeed achieved by Sabir and Maugin (1996) who gave the canonical equations of energy and material momentum for this case and corresponding J -integrals for *magnetoelastic fracture*, noting that, just like in the electric case, there is a plurality of formulations of the energy conservation, and, therefore, the possibility to construct different J -integrals. This applies in particular to materials with high *magnetostrictive coupling* (piezomagnetism being a rare event) such as TERFENOL-D. We refer the reader to these authors for such developments.

Much more interesting from the conceptual viewpoint is the case of elastic (hard) ferromagnets because such materials exemplify the problem of formulating canonical balance laws in media equipped with a microstructure, here a magnetic one, which is equivalent to considering additional internal degrees of freedom in a continuum. Here, this is materialized by the fact that equation (273) is replaced by a true dynamical equation containing a flux. In addition, the new internal degree of freedom, represented by the precession of a *spin* is peculiar in the sense that it is of gyroscopic nature having no closed form for its kinetic energy in classical physics (the phenomenon is inherently quantum mechanical). It is a so-called *d'Alembertian inertia couple* that does not expend power (see Maugin, 1988, Chapter 6). This is an interesting challenge for the formulation of canonical balance

laws in the material framework. This was achieved by Fomethé and Maugin (1996) with applications to the propagation of ferromagnetic phase-transition fronts and ferromagnetic domain walls (Fomethé and Maugin, 1997; Maugin and Fomethé, 1997).

For the sake of illustration we just report the expression of the driving force on a phase-transformation front in a soft-ferromagnet. Considering small strains, instead of (262) we have the reduced form

$$Hugo_{PT}^{s,fer} = [W - \langle \mathbf{B} \rangle \cdot \mathbf{M} - \langle \mathbf{N} \cdot \mathbf{T} \rangle \cdot \mathbf{F} \cdot \mathbf{N}], \quad (275)$$

with constitutive equations

$$\mathbf{B} = \frac{\partial W}{\partial \mathbf{M}}, \quad \mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}, \quad S = -\frac{\partial W}{\partial \theta}, \quad W = W(\mathbf{F}, \mathbf{M}, \theta). \quad (276)$$

If we set

$$\begin{aligned} \overline{W}(\mathbf{F}, \mathbf{H}, \theta) &= W + \frac{1}{2} (\mathbf{H}^2 - \mathbf{B}^2) = W - \mathbf{M} \cdot \mathbf{B} + \frac{1}{2} \mathbf{M}^2 \\ &= W - \left(\frac{1}{2} \mathbf{H}^2 + \mathbf{M} \cdot \mathbf{H} \right), \end{aligned} \quad (277)$$

and introducing the magnetic scalar potential φ and the strain ε and the elastic displacement \mathbf{u} we immediately show that (275) takes on the following form:

$$Hugo_{PT}^{s,fer} = [\overline{W}(\varepsilon, \mathbf{H}, \theta) - \langle \mathbf{M} \rangle \cdot \nabla \varphi - \langle \mathbf{N} \cdot \mathbf{T} \rangle \cdot (\nabla \mathbf{u})^T \cdot \mathbf{N}], \quad (278)$$

with the constitutive equations

$$\mathbf{M} = \frac{\partial \overline{W}}{\partial \mathbf{H}}, \quad \mathbf{T} = \frac{\partial \overline{W}}{\partial \varepsilon}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}. \quad (279)$$

An equivalent formulation reads

$$Hugo_{PT}^{s,fer} = [\widetilde{W}(\varepsilon, \mathbf{H}, \theta) - \langle \mathbf{B} \rangle \cdot \nabla \varphi - \langle \mathbf{N} \cdot \mathbf{T} \rangle \cdot (\nabla \mathbf{u})^T \cdot \mathbf{N}], \quad (280)$$

with

$$\mathbf{B} = \frac{\partial \widetilde{W}}{\partial \mathbf{H}}, \quad \mathbf{T} = \frac{\partial \widetilde{W}}{\partial \varepsilon}, \quad S = -\frac{\partial \widetilde{W}}{\partial \theta}. \quad (281)$$

We recognize in the formulas (280) and (278) the structure of the two possible Eshelby stresses and the two J -integrals introduced previously in the study of the fracture of elastic paramagnets and soft ferromagnets by Sabir and Maugin (1996). The transformation (277) was originally introduced in (Maugin, 1971, p. 85c) – also in Abd-Alla and Maugin (1987).

7 Comments and Conclusions

In these lectures we have exhibited, contrasted, or shown complementarity between, different approaches to the construction of models of electromagneto-mechanical interactions in solids, in a nonrelativistic framework, but with possible finite strains. Information gathered directly from a microscopic model (here the Lorentz one, Section 2) keeps us close to physical reality and avoids too much arbitrariness. The method of virtual power (accounting for the first point) is certainly the method most pregnant of generalizations to complex models. This is fully illustrated by the cases of electroelastic semiconductors and media with interfaces (Daher and Maugin, 1986, 1988) where there are necessarily present dissipative effects. It would also provide a safe way to build a rational model when the deformation field itself is specialized to those found in essentially two-dimensional (plates, shells) or one-dimensional (rods) structures. Variational formulations are most convenient with possible direct application to numerical schemes and the study of stability (see, e.g., Kankanala and Triantafyllidis, 2008). Furthermore, with direct application of the celebrated Noether's theorem, they allow for the direct production of conservation laws of energy and canonical momentum (cf. (25) in vacuum), the latter playing an essential role in the theory of material inhomogeneities and defects (see Maugin, 1993 for some applications to electromagnetic solids). The present chapter concerns solids. The case of fluids can also be considered, especially when dealing with electro- and magneto-rheology (see, for instance, Eringen and Maugin, 1990, Chapter 5; Rajagopal and Růžička, 2001; Drouot and Racineux, 2005). The second law of thermodynamics can be used as a constraint to formulate coupled constitutive equations, especially when there exist dissipative processes such as electric or magnetic relaxation and hysteresis (Section 4). But the thermodynamics with internal variables of states remains, for the time being, the most fruitful method to formulate such effects. Of course, a bridge between the microscopic phenomena and those exhibited at the exploitation scale of the relevant materials is desirable; an example of such as effort at bridging this gap was presented in Section 5 but in the linear theory. Finally, Eshelbian mechanics and the theory of material inhomogeneities as viewed by the author provide an efficient means to formulate the relevant problems of fracture and propagation of field discontinuities (Section 6).

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Modeling Nonlinear Electroelastic Materials

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Abstract. In this chapter we provide an overview of the basic equations governing the mechanical behavior of electroelastic materials capable of finite deformations. To describe the nonlinear electromechanical interactions in a deformable material, we first review the fundamentals of the theory of electrostatics and then provide a summary of the required equations of nonlinear continuum mechanics. Electromagnetic field variables and the corresponding boundary conditions, which, in general, are defined with respect to the current configuration, are re-cast in Lagrangian form and the Lagrangian forms of the field equations are derived. An overview of different constitutive formulations involving different ‘stress tensors’ and ‘electric body forces’ and the associated equations of mechanical equilibrium in the presence of electromechanical interactions is included. In particular, we consider an isotropic electroelastic material for which the constitutive equations can be expressed in terms of six invariants involving the deformation and an electric vector field, which reduce to five for an incompressible material. To illustrate the theory, we determine the influence of a radial electric field on the axial shear response of a thick-walled circular cylindrical tube. The last section focuses on the governing equations describing the linearized response of electroelastic solids superimposed on a state of finite deformation in the presence of an electric field.

1 Introduction

The nonlinear theory describing electromechanical coupling has received considerable attention in the last few years because of the rapid development of elastomeric and polymeric materials that in response to the application of an electric field undergo large deformations (Bar-Cohen, 2002). Such materials, often referred to as ‘smart materials’, are being used in a variety of applications, ranging from high-speed actuators, and sensors to artificial muscles and other biomedical applications (Pelrine et al., 2000).

The key point is that the mechanical properties of the materials can be changed rapidly and reversibly by externally applied electric fields. Thus, the coupling between mechanics and electromagnetism is both strong and highly nonlinear. The purpose of this chapter is to present an overview of fundamental concepts of the electromagnetic theory and solid mechanics and the use of these theories to develop a consistent framework to describe the behavior of electro-sensitive materials.

The nonlinear theory of electroelasticity was originally developed by Toupin (1956) for the static situation and extended to include thermal effects by Tiersten (1971). Relevant background information is provided in the books by Truesdell and Toupin (1960) and Landau and Lifshitz (1960). The theory of nonlinear electroelasticity has seen a renewed interest recently due to the development of highly deformable and polarizable materials that offer exciting possibilities for many new devices (Bar-Cohen, 2002; Pelrine et al., 2000). This interest is evidenced by a range of recent books dealing with the nonlinear interaction between mechanical and electromagnetic fields, for example, Hutter and van de Ven (1978), Nelson (1979), Maugin (1988), Eringen and Maugin (1990), Kovetz (2000) and Hutter et al. (2006). More recent theoretical developments of the nonlinear theory of electromechanical interactions are included in articles by Rinaldi and Brenner (2002), Erickson (2002, 2007), McMeeking and Landis (2005), McMeeking et al. (2007), Fosdick and Tang (2007), Suo et al. (2008), Bustamante et al. (2009a,b) and Dorfmann and Ogden (2005, 2006, 2010a,b).

In this chapter we provide an overview of the basic equations governing the *electromechanical time-independent response* of electroelastic materials capable of finite deformations. It begins in Section 2 with a review of *basic concepts in electrostatics*, such as point and distributed charges and associated electromagnetic interactions. In particular, this idealization is used to introduce the Lorentz force, Coulomb's Law, time-independent electric fields, charge conservation, electrostatic potential, field of a dipole and the Gauss's theorem. We then review some fundamental concepts from the theory of magnetostatics as a prerequisite to collecting the four Maxwell's equations governing the electric and magnetic fields when the charge density and current density are known. Starting from the connection between moving charged particles and magnetic induction, we define the magnetic vector potential and summarize the first and second equations of magnetostatics. *Maxwell's equations*, for time dependent fields, are first defined in terms of the electric field vector and the magnetic induction and the connection is made to the homogeneous and inhomogeneous wave equations for electromagnetic fields. For a homogeneous isotropic and nonconducting source-free region, using the electric permittivity and the magnetic perme-

ability in free space, we show how the electric field is connected to the electric displacement and how the magnetic field vector can be expressed in terms of the magnetic induction. These enable Maxwell's equations to be written in alternative forms.

For electromagnetic continua, the electric and magnetic properties of a material are defined by introducing the notions of *electric polarization* and *magnetization*. We show how the charge density can be divided into free charges and bound charges and how the latter are related to the electric polarization. Similarly, we define the free current density, the polarization current density and the bound current density and establish the connection between the latter and the magnetization in the material. Very briefly, constitutive equations for the electric and magnetic fields are shown as they apply to *linear isotropic media*. In order to solve boundary-value problems involving electromagnetic fields, we need to define continuity conditions at the bounding surfaces of the material. This is achieved by writing Maxwell's equations in integral form together with the application of the divergence and Stokes' theorems, as appropriate.

In Section 3, the development next takes account of the deformability of material media. To describe nonlinear electroelastic interactions in a deformable material, a review of continuum kinematics is provided. We recall some important kinematic identities, which are valuable for converting formulas between Eulerian and Lagrangian descriptions. Electromagnetic field variables, the associated field equations and boundary conditions, which, in general, are defined with respect to the current configuration, are re-cast in Lagrangian form. A complete treatment of mechanics of continua and nonlinear elasticity can be found in the texts by Holzapfel (2000) and Ogden (1997).

At the beginning of Section 4, for convenience, we summarize the equations of electrostatics as they are critically important for further developments in this chapter. The first part of this section focuses on the many different ways in which the equations of *mechanical equilibrium* can be written in the presence of electromechanical interactions. We provide an overview of alternative energy formulations, which differ by the independent electric variable used, and derive the corresponding equilibrium equations. These require different definitions of 'stress tensor' and associated 'electric body force' terms that are included in tabular form for six different energy functions; see the recent publication by Bustamante et al. (2009a). In the second part of this section a simple, but *general constitutive law* is derived governing the finite deformation of electroelastic materials in the presence of an electric field. The formulation is based on a 'total' energy function with either the electric field or the electric displacement as the independent

electric variable, together with an appropriate measure of deformation. Attention is then focused on isotropic unconstrained electroelastic materials for which the constitutive equation can be expressed in terms of six invariants involving the deformation and the electric field variable, which reduce to five for an incompressible material. To illustrate the theory, we specialize the field equations to problems for which the cylindrical symmetry is maintained during deformation. Specifically, we determine the influence of a radial electric field on the axial shear response of a thick-walled circular cylindrical tube. For the solution of additional boundary-value problems we refer to Dorfmann and Ogden (2005, 2006).

The last section of this chapter focuses on the governing equations describing the linearized response of electroelastic solids superimposed on a state of finite deformation in the presence of an electric field for independent incremental changes in the electric displacement and the deformation within the material. We derive the *incremental forms* of the *total stress tensor* and the *electric field vector* within the material and its surrounding space for unconstrained and incompressible materials. The incremental equations require fourth-, third- and second-order *electroelastic moduli tensors*, which are derived explicitly for an isotropic material. The *incremental boundary conditions* to accompany the governing equations are derived as well. This theory can be applied to the analysis of stability requirements and wave propagation problems for electroelastic materials. However, due to space limitations no examples on the use of the incremental theory are given, but the interested reader is referred to Dorfmann and Ogden (2010a,b) for more details.

2 Electrostatics

2.1 Lorentz Force

Historically, electromagnetism has been defined as a macroscopic phenomenon, the concepts of distributed charges and corresponding interactions being idealizations that allow a mathematical description of the experimentally observed phenomena. Following this development, we consider a time-independent distribution of charges and quantify the corresponding electromagnetic interactions. The charged particles assume positions of equilibrium and interact with one another by generating electrostatic forces. By placing a test particle with point charge e at an arbitrary location \mathbf{x} we measure the resultant interaction force \mathbf{f} of all particles. The magnitude of the test charge e must be small enough not to alter the original arrangement of the particles. As the magnitude of e approaches zero it is obvious that the measured force goes to zero as well. However, in the limit the ratio of

force \mathbf{f} over charge e remains finite and identifies the *electric field vector* \mathbf{E} at location \mathbf{x} by

$$\mathbf{E} = \lim_{e \rightarrow 0} \frac{\mathbf{f}}{e}. \quad (1)$$

For a given electric field \mathbf{E} and for time-independent phenomena a particle carrying charge e which is at rest at location \mathbf{x} is subject to a force \mathbf{f} given by

$$\mathbf{f} = e\mathbf{E}. \quad (2)$$

When an electric field \mathbf{E} and a magnetic induction \mathbf{B} are both present at \mathbf{x} and the particle moves with constant velocity \mathbf{v} , it experiences an additional force perpendicular to its direction of motion and proportional to the magnitude of \mathbf{v} . The total force is then given by

$$\mathbf{f} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (3)$$

which is known as the *Lorentz force* and which identifies the *magnetic induction vector* \mathbf{B} and the *electric field* \mathbf{E} in terms of the electromagnetic force on a charged particle e .

2.2 Coulomb's Law

Coulomb, based on experimental data, showed that the electric field \mathbf{E} due to an isolated and stationary particle is proportional to its charge e and varies inversely with the square of the distance from the particle. The electric field at the point \mathbf{x} due to a point charge e located at the origin therefore has the form

$$\mathbf{E}(\mathbf{x}) = k e \frac{\mathbf{x}}{r^3} = k e \frac{\hat{\mathbf{x}}}{r^2}, \quad (4)$$

where $r = |\mathbf{x}|$, $\hat{\mathbf{x}} = \mathbf{x}/r$ is a unit vector and k is a constant of proportionality that depends on the units used. If the particle is located at the fixed point \mathbf{x}' instead of the origin then equation (4) is replaced by

$$\mathbf{E}(\mathbf{x}) = k e \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \quad (5)$$

In addition, Coulomb was able to quantify the force of interaction between two charged particles at rest. If the two particles have charges e_1 and e_2 and are placed at locations \mathbf{x}_1 and \mathbf{x}_2 , respectively, the interaction force is given by *Coulomb's Law*

$$\mathbf{f} = k e_1 e_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}, \quad (6)$$

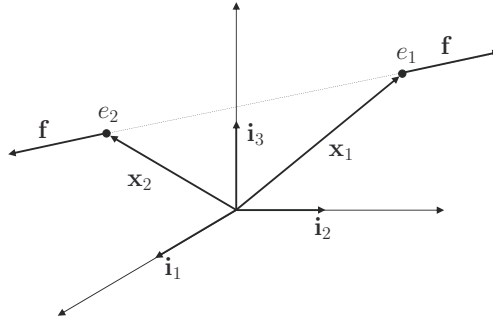


Figure 1. Repulsive interaction force between two charged particles e_1 and e_2 .

which is attractive if the charges are of the opposite type, repulsive otherwise, as depicted in Figure 1. If N particles are interacting, the resultant net force acting on e_1 due to all other charged particles is obtained using the principle of superposition. It has the form

$$\mathbf{f} = \sum_{i=2}^N k e_1 e_i \frac{\mathbf{x}_1 - \mathbf{x}_i}{|\mathbf{x}_1 - \mathbf{x}_i|^3}. \quad (7)$$

In a similar way, if we consider a point charge e moving with uniform velocity \mathbf{v} , the resulting magnetic induction \mathbf{B} at position \mathbf{x} relative to the point charge is proportional to

$$\mathbf{B} = k' e \frac{\mathbf{v} \times \hat{\mathbf{x}}}{r^2}, \quad (8)$$

where the constant k' also depends on the system of units used. Unlike Coulomb's law this is an approximation in the sense that it is only valid in the non-relativistic situation (when $|\mathbf{v}|$ is much smaller than the speed of light and the acceleration is negligible). Relativistic effects are not considered in the present work.

2.3 Units

In the SI system, the unit of electric charge is the Coulomb (C), the electric current is given in Ampères (A), the force in Newtons (N) and the length in meters (m). The electric charge of an electron, for example, is $e = -1.602 \times 10^{-19}$ C, and the unit of the electric field \mathbf{E} is Volt per meter

(V m^{-1}); the magnetic induction \mathbf{B} has units in Tesla ($\text{N A}^{-1} \text{m}^{-1}$). The constants of proportionality k and k' introduced in (6) and (8) are chosen such that the electric field and magnetic induction are given respectively by

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0} \frac{\hat{\mathbf{x}}}{r^2}, \quad \mathbf{B} = \frac{\mu_0 e}{4\pi} \frac{\mathbf{v} \times \hat{\mathbf{x}}}{r^2}, \quad (9)$$

where $\epsilon_0 \approx 8.854 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$ is the *permittivity of free space* and μ_0 , which is equal to $4\pi \times 10^{-7} \text{ N A}^{-2}$, is the *magnetic permeability of free space*. It turns out that for SI units, the speed c of propagation of electromagnetic effects (the speed of light) in free space is given by $c^2 = 1/(\mu_0\epsilon_0)$.

2.4 Charge Conservation

The definition of the electric field up to this point assumes the existence of a set of discrete point charges. We now expand this concept to include a charge distributed over a certain region in space. Consider an infinitesimal element of volume dV and let $\rho_e dV$ be the total charge within this element. Then ρ_e is the *charge density*, which may be positive or negative and depends, in general, on the position \mathbf{x} and time t , i.e. $\rho_e = \rho_e(\mathbf{x}, t)$.

If \mathbf{v} is the mean velocity of the individual charges in dV , then

$$\mathbf{J} = \rho_e \mathbf{v}, \quad (10)$$

defines the *current density*. The Lorentz force for a discrete point charge subject to electromagnetic fields \mathbf{E} and \mathbf{B} has been defined in equation (3). For a distribution with charge density ρ_e and current density \mathbf{J} , the Lorentz force per unit volume is given by

$$\mathbf{f} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}. \quad (11)$$

Consider a *fixed* volume in space V bounded by a surface S with unit outward normal \mathbf{n} . The charge density per unit volume within V is ρ_e and the rate at which charge flows across S is given by $\mathbf{J} \cdot \mathbf{n}$ per unit area. The rate of increase of charge within V must arise from the influx. Thus,

$$\frac{d}{dt} \int_V \rho_e dV = - \int_S \mathbf{J} \cdot \mathbf{n} dS = - \int_V \text{div } \mathbf{J} dV, \quad (12)$$

where the divergence theorem has been used to convert the surface integral to an integral over the volume V . It follows that

$$\int_V \left(\frac{\partial \rho_e}{\partial t} + \text{div } \mathbf{J} \right) dV = 0, \quad (13)$$

which must hold for arbitrary V . Provided the integrand in (13) is continuous we may deduce the local form of the *charge conservation equation* as

$$\frac{\partial \rho_e}{\partial t} + \operatorname{div} \mathbf{J} = 0, \quad (14)$$

where the partial derivative indicates that the charge density ρ_e may also depend on the location \mathbf{x} in V . In a steady state situation (no time dependence) we have $\partial \rho_e / \partial t = 0$ and equation (14) reduces to

$$\operatorname{div} \mathbf{J} = 0. \quad (15)$$

The corresponding integral form is

$$\int_S \mathbf{J} \cdot d\mathbf{S} = 0 \quad (16)$$

for arbitrary closed surfaces S .

2.5 The Field of a Static Charge Distribution

As we have seen, the electric field at a location \mathbf{x} due to an isolated point charge e located at the origin is given by equation (9)₁. Equivalently, this can be written as

$$\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi\epsilon_0} \frac{\hat{\mathbf{x}}}{r^2} = -\frac{e}{4\pi\epsilon_0} \operatorname{grad} \left(\frac{1}{r} \right). \quad (17)$$

When the point charge is placed at the position \mathbf{x}' , the electric field is given by equation (5) or, alternatively, by

$$\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi\epsilon_0} \frac{\mathbf{R}}{R^3} = -\frac{e}{4\pi\epsilon_0} \operatorname{grad} \left(\frac{1}{R} \right), \quad (18)$$

where $R = |\mathbf{R}|$ and $\mathbf{R} = \mathbf{x} - \mathbf{x}'$. Note that the grad operator is with respect to the location \mathbf{x} where the electric field \mathbf{E} is determined.

If we consider the charge within the volume V to be continuously distributed, the point charge e can be replaced by the charge $\rho_e dV$ in the volume element dV . If $\rho_e = 0$ outside the specified volume V , then the electric field at \mathbf{x} is the sum of all contributions $\rho_e dV$ within the volume V . It is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho_e(\mathbf{x}') \frac{\mathbf{R}}{R^3} dV(\mathbf{x}') = -\frac{1}{4\pi\epsilon_0} \int_V \rho_e(\mathbf{x}') \operatorname{grad} \left(\frac{1}{R} \right) dV(\mathbf{x}'), \quad (19)$$

where the integration is with respect to the \mathbf{x}' variable. Since the grad operator is with respect to \mathbf{x} , it can be taken outside the integral. Thus,

$$\mathbf{E}(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0} \text{grad} \int_V \frac{\rho_e(\mathbf{x}')}{R} dV(\mathbf{x}'). \quad (20)$$

The gradient operator in the above equation acts on a scalar function. It is therefore convenient to formalize this process by explicitly introducing a scalar potential function ϕ , known as the *electrostatic potential*. Equation (20) is then written compactly as

$$\mathbf{E}(\mathbf{x}) = -\text{grad}\phi(\mathbf{x}), \quad (21)$$

where the scalar potential ϕ depends on the charge density function ρ_e and is given by

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_e(\mathbf{x}')}{R} dV(\mathbf{x}'). \quad (22)$$

Since $\text{curl}(\text{grad}\phi) \equiv \mathbf{0}$ for any scalar function ϕ , we obtain the *first equation of electrostatics*

$$\text{curl}\mathbf{E} = \mathbf{0}. \quad (23)$$

Far from the charge distribution the field is approximately that of a point charge situated at the origin with a charge equal to the total charge within the distribution. In this case we have $1/R \approx 1/r$ and the electrostatic potential (22) can be approximated by

$$\phi(\mathbf{x}) \approx \frac{e}{4\pi\epsilon_0 r}, \quad (24)$$

where

$$e = \int_V \rho_e(\mathbf{x}') dV(\mathbf{x}') \quad (25)$$

is now the total charge in V .

2.6 The Field of a Dipole

Consider now a distribution of charge with density $\rho_e(\mathbf{x}')$ confined to a finite volume V , where \mathbf{x}' is the position vector of a typical point in V relative to an origin O located within V and $\rho_e = 0$ outside V . Let \mathbf{x} be the position vector of a point P far from V at which the electrostatic field is to be calculated (see Figure 2).

Then $|\mathbf{x}'| \ll |\mathbf{x}|$ for all \mathbf{x}' in V , and we may use the Taylor expansion to obtain the approximation

$$\frac{1}{R} \equiv \frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \frac{1}{r} - \mathbf{x}' \cdot \text{grad} \left(\frac{1}{r} \right), \quad (26)$$

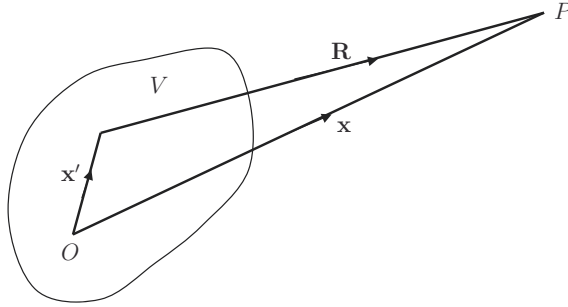


Figure 2. Volume V containing a charge distribution with density $\rho_e(\mathbf{x}')$ such that $\rho_e = 0$ outside V , showing field point P having position vector \mathbf{x} relative to origin O in V and $\mathbf{R} = \mathbf{x} - \mathbf{x}'$.

recalling that $r = |\mathbf{x}|$. Hence, from (22), the electrostatic potential at \mathbf{x} is approximated as

$$\phi(\mathbf{x}) \approx \frac{e}{4\pi\epsilon_0 r} - \frac{1}{4\pi\epsilon_0} \boldsymbol{\mu} \cdot \text{grad} \left(\frac{1}{r} \right), \quad (27)$$

where e is the total charge in V given by the formula (25) and $\boldsymbol{\mu}$ is defined by

$$\boldsymbol{\mu} = \int_V \rho_e(\mathbf{x}') \mathbf{x}' dV(\mathbf{x}'). \quad (28)$$

If $e \neq 0$ then the origin can be translated to the center of charge (analogous to the center of mass in mechanics) so that $\boldsymbol{\mu} = \mathbf{0}$, in which case

$$\phi(\mathbf{x}) \approx \frac{e}{4\pi\epsilon_0 r}, \quad (29)$$

which is the field of a point charge e located at the origin. Thus, the field of a charge distribution at a large distance is indistinguishable from that of a point charge. On the other hand, if $e = 0$ and $\boldsymbol{\mu} \neq \mathbf{0}$ we have

$$\phi(\mathbf{x}) \approx -\frac{1}{4\pi\epsilon_0} \boldsymbol{\mu} \cdot \text{grad} \left(\frac{1}{r} \right) = \frac{\boldsymbol{\mu} \cdot \mathbf{x}}{4\pi\epsilon_0 r^3}. \quad (30)$$

This is the potential due to an *electric dipole* of strength $\boldsymbol{\mu}$ situated at the origin. This is equivalent to having two charges of magnitude e and of equal and opposite signs very close together, say at distances $\pm \mathbf{d}/2$ from the origin. Using equation (28) the electric dipole is then given by $\boldsymbol{\mu} = e\mathbf{d}$.

2.7 Gauss's Theorem

To derive Gauss's theorem consider first a point charge e contained within a volume V bounded by a closed surface S . Equation (18) gives the electric field \mathbf{E} at \mathbf{x} due to a charge e placed at position \mathbf{x}' . The vector \mathbf{R} again denotes the vector $\mathbf{x} - \mathbf{x}'$. Now suppose \mathbf{x} lies on the surface S and that \mathbf{n} is the unit normal vector pointing outward at that point. Let $d\mathbf{S}$ ($= \mathbf{n}dS$) be an infinitesimal area element on the surface S located at \mathbf{x} . Then, the flux of \mathbf{E} across $d\mathbf{S}$ is given by

$$\mathbf{E} \cdot d\mathbf{S} = \frac{e}{4\pi\epsilon_0} \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3}, \quad (31)$$

and the total flux of \mathbf{E} across the closed surface S is

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{e}{4\pi\epsilon_0} \int_S \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3}. \quad (32)$$

The integrand on the right-hand side defines the solid angle, denoted $d\Omega$, subtended by $d\mathbf{S}$ at \mathbf{x}' , i.e.

$$d\Omega = \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3}, \quad (33)$$

and equation (32) may therefore be written

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{e}{4\pi\epsilon_0} \int_S d\Omega. \quad (34)$$

If \mathbf{x}' lies within the volume V , then the solid angle is equal to 4π ; see Figure 3. On the other hand, if \mathbf{x}' lies outside the bounding surface, then for every positive quantity $\mathbf{R} \cdot d\mathbf{S}/R^3$, there is an equal and opposite quantity which cancels the first. Thus,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \begin{cases} e/\epsilon_0 & \text{if } e \text{ is within } V \\ 0 & \text{if } e \text{ is outside } V. \end{cases} \quad (35)$$

Now consider a continuous charge distribution ρ_e within a volume V' , so that

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_e(\mathbf{x}') \mathbf{R}}{R^3} dV(\mathbf{x}'), \quad (36)$$

and the flux of \mathbf{E} across a closed surface S , the boundary of V (not necessary coinciding with V'), is

$$\int_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{S}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \rho_e(\mathbf{x}') dV(\mathbf{x}') \int_S \frac{\mathbf{R} \cdot d\mathbf{S}(\mathbf{x})}{R^3}. \quad (37)$$

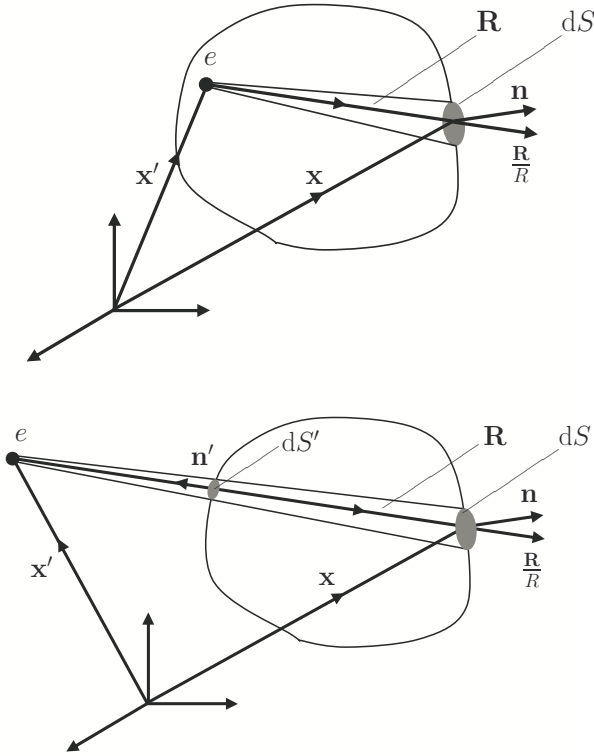


Figure 3. Point charge e at location \mathbf{x}' inside the volume V (top) and outside the volume (below). The solid angle for \mathbf{x}' subtended by the whole surface S at \mathbf{x}' is 4π when \mathbf{x}' is inside, zero if \mathbf{x}' is outside V .

Using again the properties of solid angle we have

$$\int_S \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3} = \begin{cases} 4\pi & \text{if } \mathbf{x}' \text{ is within } V \\ 0 & \text{if } \mathbf{x}' \text{ is outside } V, \end{cases} \quad (38)$$

and hence

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{V'(S)} \rho_e(\mathbf{x}') dV(\mathbf{x}'), \quad (39)$$

where $V'(S)$ is that part of V' contained within V . This leads to *Gauss's*

theorem

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{e}{\varepsilon_0}, \quad (40)$$

which states that the flux of \mathbf{E} across a closed surface S is proportional to the total charge e contained within S .

To obtain the local form of Gauss's theorem we first rewrite the integral form as

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho_e dV. \quad (41)$$

Then, using the divergence theorem, equation (41) becomes

$$\int_V \left(\operatorname{div} \mathbf{E} - \frac{\rho_e}{\varepsilon_0} \right) dV = 0, \quad (42)$$

which must hold for arbitrary V . Therefore, provided the integrand in (42) is continuous, we deduce that

$$\operatorname{div} \mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \quad (43)$$

which is the local form of Gauss's theorem and the *second equation of electrostatics*.

The equations

$$\operatorname{curl} \mathbf{E} = \mathbf{0}, \quad \operatorname{div} \mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \quad (44)$$

govern the electrostatic field \mathbf{E} . Since equation (44)₁ is equivalent to $\mathbf{E} = -\operatorname{grad} \phi$, we obtain Poisson's equation

$$\nabla^2 \phi = -\frac{\rho_e}{\varepsilon_0} \quad (45)$$

for the scalar potential ϕ for a given charge distribution with density ρ_e . For regions where $\rho_e = 0$, equation (45) reduces to Laplace's equation

$$\nabla^2 \phi = 0. \quad (46)$$

2.8 Maxwell's Equations

We begin this subsection by first introducing some fundamental concepts from the theory of magnetostatics. These will be combined with the equations of electrostatics extended to the time-dependent situation to summarize the four Maxwell equations. Note that a detailed presentation of the theory of magnetostatics is given in the following chapter.

Recall that equation (9)₁ gives the electric field \mathbf{E} at a location \mathbf{x} due to an isolated point charge e at rest at the origin. If the charge is moving with uniform velocity \mathbf{v} then, in addition to an electric field, a magnetic induction \mathbf{B} will be produced; see equation (9)₂. This equation can easily be generalized by replacing the point charge e moving with velocity \mathbf{v} by the current density $\mathbf{J} = \rho_e \mathbf{v}$ at \mathbf{x}' within a volume V such that the magnetic induction at location \mathbf{x} has the form

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{R}}{R^3} dV(\mathbf{x}'), \quad (47)$$

which is the magnetic analog to the electric field equation (19). Using standard vector identities, it can be shown that the above equation can be written in the alternative form

$$\mathbf{B}(\mathbf{x}) = \text{curl } \mathbf{A}, \quad (48)$$

where \mathbf{A} is the magnetic vector potential given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{R} dV(\mathbf{x}'). \quad (49)$$

Note that the current density \mathbf{J} in equation (49) is defined with respect to the \mathbf{x}' variable, but the curl operator in (48) with respect to \mathbf{x} . Since the divergence of a curl is always zero, it follows from (48) that the magnetic induction \mathbf{B} satisfies the important equation

$$\text{div } \mathbf{B} = 0, \quad (50)$$

which is the *first equation of magnetostatics*. The *second equation of magnetostatics* (see the following chapter) is

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J}. \quad (51)$$

The two equations (44) describe the electrostatic phenomena and (50) and (51) are the field equations governing magnetostatic fields. When there is time dependence, the charge conservation equation $\text{div } \mathbf{J} = 0$, which appeared in (15), is no longer valid and is replaced by

$$\text{div } \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0. \quad (52)$$

Equations (44)₂ and (50) remain unchanged, while equation (44)₁ is replaced by

$$\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (53)$$

which shows that a changing magnetic field induces an electric field. This leaves equation (51), which no longer holds since it implies $\operatorname{div} \mathbf{J} = 0$ and not (52). To compensate for this difference we write, instead of (51),

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} + \mathbf{G}, \quad (54)$$

where \mathbf{G} is unknown and must be determined. On taking the divergence of this equation and using (44)₂ and (52) we obtain

$$\operatorname{div} \mathbf{G} = -\mu_0 \operatorname{div} \mathbf{J} = \mu_0 \frac{\partial \rho_e}{\partial t} = \mu_0 \varepsilon_0 \operatorname{div} \left(\frac{\partial \mathbf{E}}{\partial t} \right). \quad (55)$$

The equations are now self-consistent if we set

$$\mathbf{G} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (56)$$

so that

$$\mu_0^{-1} \operatorname{curl} \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (57)$$

The additional term $\varepsilon_0 \partial \mathbf{E} / \partial t$ in (57) is called the *displacement current*.

We now collect together the four fundamental differential equations derived above as

$$\operatorname{div} \mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \quad \operatorname{div} \mathbf{B} = 0, \quad (58)$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (59)$$

These are the four *Maxwell's equations*, which govern the fields \mathbf{E} and \mathbf{B} everywhere when the charge density ρ_e and current density \mathbf{J} are known. When coupled with the Lorentz force law they constitute an exact and complete description of classical (non-relativistic) electromagnetic phenomena.

On taking the curl of (59)₂ and making use of (59)₁ we obtain

$$\operatorname{curl}(\operatorname{curl} \mathbf{E}) = -\frac{\partial}{\partial t}(\operatorname{curl} \mathbf{B}) = -\frac{\partial}{\partial t} \left(\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \quad (60)$$

Combining this with the identity $\operatorname{curl}(\operatorname{curl} \mathbf{E}) = \operatorname{grad}(\operatorname{div} \mathbf{E}) - \nabla^2 \mathbf{E}$ and (58)₁, we arrive at the equation

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \operatorname{grad} \left(\frac{\rho_e}{\varepsilon_0} \right) + \mu_0 \frac{\partial \mathbf{J}}{\partial t}. \quad (61)$$

This is the *inhomogeneous wave equation* for \mathbf{E} , where $c = (\mu_0 \varepsilon_0)^{-1/2}$ is the wave speed (the speed of electromagnetic effects in free space). The right-hand side of (61) is the *source* term. Similarly, taking the curl of equation

(59)₁ and using (58)₂ and (59)₂, the corresponding wave equation for the magnetic induction \mathbf{B} is obtained as

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \text{curl} \mathbf{J}. \quad (62)$$

In free space, where $\rho_e = 0$ and $\mathbf{J} = \mathbf{0}$, we obtain the *homogeneous wave equations*

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}. \quad (63)$$

We now define two further electromagnetic field vectors, denoted \mathbf{D} and \mathbf{H} , which are designated as the *electric displacement vector* and the *magnetic field vector*, respectively. In free space these are simply related to \mathbf{E} and \mathbf{B} , respectively, by a constant factor in each case. Thus,

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad (64)$$

where ε_0 is the electric permittivity of free space and μ_0 the magnetic permeability of free space, which appeared in (9). These enable Maxwell's equations (58) and (59) to be written in the forms

$$\text{div} \mathbf{D} = \rho_e, \quad \text{div} \mathbf{B} = 0, \quad (65)$$

$$\text{curl} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad \text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (66)$$

For a detailed treatment of Maxwell's equations, see, for example, the classic text by Jackson (1999) and, for an interesting historical overview, we refer to the preceding chapter by Maugin.

2.9 Polarization and Magnetization in Materials

In material media the relations (64) do not hold in general and they must be replaced by *constitutive laws*, which describe the electric and magnetic properties of the material in question. When an electromagnetic field is applied to material media certain kinds of charges and currents are generated. These are conveniently described by two additional vectors, known as the *electric polarization*, denoted \mathbf{P} and the *magnetization*, denoted \mathbf{M} , which are defined in terms of the other field vectors by the standard formulas

$$\mathbf{P} = \mathbf{D} - \varepsilon_0 \mathbf{E}, \quad \mathbf{M} = \mu_0^{-1} \mathbf{B} - \mathbf{H}. \quad (67)$$

To be more specific, the charge density ρ_e may be decomposed in the form $\rho_e = \rho_f + \rho_b$, where ρ_f is the density of *free charges* and ρ_b the density

of *bound charges*. The polarization arises from the accumulation and re-arrangement of bound charges when an external electric field is applied to the material, and it is related to ρ_b by $\rho_b = -\operatorname{div}\mathbf{P}$. It follows from (65)₁ and (67)₁ that $\operatorname{div}\mathbf{D} = \rho_f$. The magnetization arises from the response of the material to an external magnetic field and corresponds to the magnetic dipole moment density. The effect of the magnetization is to induce a *bound current* density, denote here by \mathbf{J}_b , which is given by $\mathbf{J}_b = \operatorname{curl}\mathbf{M}$. Moreover, when the polarization changes in time it generates an additional current, characterized by the *polarization current* density, which is denoted \mathbf{J}_p and given by $\mathbf{J}_p = \partial\mathbf{P}/\partial t$. The difference

$$\mathbf{J} - \mathbf{J}_b - \mathbf{J}_p = \mathbf{J} - \operatorname{curl}\mathbf{M} - \frac{\partial\mathbf{P}}{\partial t} \quad (68)$$

is the *free current* density, denoted \mathbf{J}_f . It follows from (59)₁ and (67) that $\operatorname{curl}\mathbf{H} = \mathbf{J}_f + \partial\mathbf{D}/\partial t$.

Notice that the conservation of charge holds separately for free and bound charges/currents since $\operatorname{div}\mathbf{J}_f + \partial\rho_f/\partial t = 0$ and $\operatorname{div}\mathbf{J}_p + \partial\rho_b/\partial t = 0$ follow from the above relations.

To summarize, the four Maxwell's equations in material matter may be written as

$$\operatorname{div}\mathbf{D} = \rho_f, \quad \operatorname{div}\mathbf{B} = 0, \quad (69)$$

$$\operatorname{curl}\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}, \quad \operatorname{curl}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad (70)$$

which are equivalent to (65) and (66).

Equation (67)₁ gives an explicit expression for \mathbf{P} in terms of either \mathbf{D} or \mathbf{E} as the independent electric variable when \mathbf{E} (respectively \mathbf{D}) is given in terms of \mathbf{D} (respectively \mathbf{E}) by a constitutive law. Similarly, the magnetization \mathbf{M} is given in terms of either \mathbf{H} or \mathbf{B} as the independent variable when \mathbf{B} (respectively \mathbf{H}) is given in terms of \mathbf{H} (respectively \mathbf{B}) by a second constitutive equation.

Basic examples of constitutive laws include those for *linear isotropic media*, for which equations (64) are replaced by

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}, \quad \mathbf{B} = \mu\mu_0\mathbf{H}, \quad (71)$$

where ε and μ are the *relative dielectric permittivity* and *relative magnetic permeability*, respectively. From equations (67) and (71), the polarization and magnetization are given by

$$\mathbf{P} = \frac{\varepsilon - 1}{\varepsilon}\mathbf{D}, \quad \mathbf{M} = \frac{\mu - 1}{\mu_0\mu}\mathbf{B}, \quad (72)$$

so that \mathbf{P} and \mathbf{M} are parallel to the corresponding field vectors. In vacuo or in non-polarizable material $\varepsilon = 1$, while in vacuo or in non-magnetizable media $\mu = 1$. In polarizable materials $\varepsilon > 1$ and \mathbf{P} is in the same direction as \mathbf{D} . For most materials $\mu > 1$; however, there are some magnetizable materials for which $\mu < 1$ and \mathbf{M} is therefore opposite in direction to \mathbf{B} .

2.10 Boundary Conditions

Maxwell's equations (65) and (66) are valid for any material medium provided \mathbf{D} and \mathbf{H} are given by appropriate constitutive laws. To these equations we need to append boundary conditions in order to formulate and solve boundary-value problems. In general the field vectors $\mathbf{E}, \mathbf{D}, \mathbf{B}$ and \mathbf{H} are discontinuous across a surface between different media or across a surface bounding the material. In this section we derive, using equations (65) and (66) in integral form together with the divergence and Stokes' theorems, as appropriate, the equations satisfied by the discontinuities.

2.11 Boundary Conditions for \mathbf{E} and \mathbf{D}

Let S be a *stationary* surface which carries free surface charge σ_f per unit area. The two sides of S are distinguished as side 1 and side 2 and field vectors on the two sides of S are identified with subscripts 1 and 2. Let \mathbf{n} be the unit normal to S pointing from side 1 to side 2. The 'jump' in a vector on S is the difference between its values on side 2 and side 1, evaluated on S . Thus \mathbf{E} , for example, has jump $\mathbf{E}_2 - \mathbf{E}_1$, which is denoted $[[\mathbf{E}]]$, and similarly for the other vectors. The jump conditions satisfied by \mathbf{E} and \mathbf{D} are summarized as

$$\mathbf{n} \times [[\mathbf{E}]] = \mathbf{0}, \quad \mathbf{n} \cdot [[\mathbf{D}]] = \sigma_f. \quad (73)$$

We now prove these results.

Consider the Maxwell equation (66)₂ integrated over an open surface Σ with bounding curve Γ . After application of Stokes' theorem it becomes

$$\oint_{\Gamma} \mathbf{E} \cdot d\mathbf{x} = - \int_{\Sigma} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \quad (74)$$

Let Σ be an infinitesimal plane rectangular surface with Γ identified by its corner points $ABCD$ lying in the plane of the unit normal \mathbf{n} to a surface S and a unit tangent vector \mathbf{t} to the surface and intersecting S , as shown in Figure 4. The sides AB and CD of Γ are parallel to \mathbf{t} and have lengths δs . The sides BC and DA are parallel to \mathbf{n} and have lengths δh . Then,

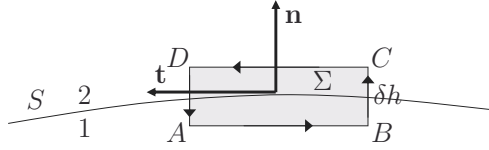


Figure 4. A small plane area Σ intersecting the surface S in the plane of the unit normal \mathbf{n} to the surface and a unit tangent vector \mathbf{t} . The unit normal points from side 1 to side 2 of the surface. The bounding curve of Σ is traversed in the direction of the arrows along the path $ABCD A$.

application of (74) to Σ and Γ yields the approximate result

$$-\int_{AB} \mathbf{E} \cdot \mathbf{t} ds + \int_{BC} \mathbf{E} \cdot \mathbf{n} dh + \int_{CD} \mathbf{E} \cdot \mathbf{t} ds - \int_{DA} \mathbf{E} \cdot \mathbf{n} dh \approx -\frac{\partial \mathbf{B}}{\partial t} \cdot (\mathbf{n} \times \mathbf{t}) \delta h \delta s. \tag{75}$$

Taking the limit as $\delta h \rightarrow 0$ and then dividing by δs and letting $\delta s \rightarrow 0$ we obtain $\mathbf{E}_2 \cdot \mathbf{t} - \mathbf{E}_1 \cdot \mathbf{t} = 0$, i.e. $\mathbf{t} \cdot \llbracket \mathbf{E} \rrbracket = 0$. This holds for an arbitrary \mathbf{t} normal to \mathbf{n} , and hence the result (73)₁ follows.

Now consider a cylinder (or ‘pill box’) of infinitesimal height δh and cross-sectional area $\delta S = \mathbf{n} \delta S$ straddling the surface S , as depicted in Figure 5. Equation (65)₁, integrated over the volume V of the cylinder and application of the divergence theorem, gives

$$\int_{\Sigma} \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_e dV, \tag{76}$$

where Σ is the bounding surface of the cylinder. Since δh is infinitesimal and

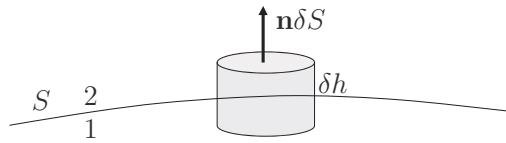


Figure 5. A ‘pill-box’ of height δh and cross-sectional area δS intersecting the surface S with unit normal \mathbf{n} pointing from side 1 to side 2 of S .

the flux of \mathbf{D} across the lateral surface of the cylinder becomes negligible as $\delta h \rightarrow 0$, the only contribution to the surface integral comes from the top and bottom surfaces of the cylinder. The right-hand side of (76) is the total free charge in V , which consists of the surface charge $\sigma_f \delta S$. Equation (76)

is therefore approximated simply as $\mathbf{D}_2 \cdot \mathbf{n} \delta S - \mathbf{D}_1 \cdot \mathbf{n} \delta S \approx \sigma_f \delta S$, which, after dividing by δS and taking the limit $\delta S \rightarrow 0$, yields $\mathbf{n} \cdot \llbracket \mathbf{D} \rrbracket = \sigma_f$, and hence (73)₂ is established. Clearly, if the surface S is free of distributed charge σ_f , then the normal component of \mathbf{D} is continuous.

2.12 Boundary Conditions for \mathbf{B} and \mathbf{H}

The counterparts of the boundary conditions (73) for the magnetic vectors are

$$\mathbf{n} \times \llbracket \mathbf{H} \rrbracket = \mathbf{K}_f, \quad \mathbf{n} \cdot \llbracket \mathbf{B} \rrbracket = 0, \quad (77)$$

where \mathbf{K}_f is the free current surface density on the surface S per unit area. The proof of (77) follows the same pattern as for (73) and we refer to the following chapter of this volume for a detailed derivation.

3 Deformable Electromagnetic Materials

3.1 Continuum Kinematics

Consider a deformable electromagnetically sensitive body that is initially in an unstressed configuration. Let the region in three-dimensional Euclidean space occupied by the body in this configuration be denoted \mathcal{B}_0 , with boundary $\partial\mathcal{B}_0$, and let \mathbf{X} be the position vector of a generic material particle. Suppose that the body deforms under the combined action of mechanical loads and an electromagnetic field, so that the point \mathbf{X} occupies the new position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. The resulting deformed configuration is denoted by \mathcal{B} and its boundary by $\partial\mathcal{B}$. The vector field $\boldsymbol{\chi}$, which is a one-to-one, orientation-preserving mapping with suitable regularity properties, describes the motion of the body and is defined for $\mathbf{X} \in \mathcal{B}_0 \cup \partial\mathcal{B}_0$.

The deformation gradient tensor \mathbf{F} relative to \mathcal{B}_0 is defined by

$$\mathbf{F} = \text{Grad} \boldsymbol{\chi}, \quad \mathbf{X} \in \mathcal{B}_0, \quad (78)$$

where Grad denotes the gradient operator with respect to \mathbf{X} . We also adopt the notation

$$J = \det \mathbf{F}, \quad (79)$$

which by standard convention is positive.

Associated with \mathbf{F} are the symmetric Cauchy–Green tensors. To avoid a conflict of standard notations we use here the lower case characters \mathbf{c} and \mathbf{b} to represent, respectively, the right and left Cauchy–Green tensors. These are defined in terms of the deformation gradient by

$$\mathbf{c} = \mathbf{F}^T \mathbf{F}, \quad \mathbf{b} = \mathbf{F} \mathbf{F}^T, \quad (80)$$

where T denotes the transpose of a second-order tensor. Suggested references for additional details of the kinematics of solid continua are, for example, the texts by Ogden (1997) and Holzapfel (2000).

In what follows, the notations grad , div and curl are used for the standard differential operators with respect to \mathbf{x} , while Grad , Div and Curl are the corresponding operators with respect to \mathbf{X} . We also use the convention that the divergence operator, when applied to tensors, acts on the first index of the tensor that follows. For example, $\text{div} \mathbf{F} \equiv \partial F_{j\alpha} / \partial x_j$ and $\text{Div}(\mathbf{F}^T) \equiv \partial F_{j\alpha} / \partial X_\alpha$.

We note the important kinematic identities

$$\text{Div}(J\mathbf{F}^{-1}) = \mathbf{0}, \quad \text{div}(J^{-1}\mathbf{F}) = \mathbf{0}, \quad \text{Curl}(\mathbf{F}^T) = \mathbf{0}, \quad \text{curl}(\mathbf{F}^{-T}) = \mathbf{0}, \quad (81)$$

where $\mathbf{0}$ denotes the zero vector and \mathbf{O} the zero second-order tensor. Equations (81) are valuable for converting formulas between Eulerian and Lagrangian descriptions.

Suppose that $\mathbf{a} = \mathbf{a}(\mathbf{x})$ is an Eulerian vector defined in the deformed configuration \mathcal{B} . Using equations (81)_{1,3}, we have

$$\text{Div}(J\mathbf{F}^{-1}\mathbf{a}) = J\text{div} \mathbf{a}, \quad \text{Curl}(\mathbf{F}^T\mathbf{a}) = J\mathbf{F}^{-1}\text{curl} \mathbf{a}. \quad (82)$$

Similarly, let $\mathbf{A} = \mathbf{A}(\mathbf{X})$ be a Lagrangian vector defined in the reference configuration \mathcal{B}_0 . Then, by using (81)_{2,4}, we obtain

$$\text{div}(J^{-1}\mathbf{F}\mathbf{A}) = J^{-1}\text{Div} \mathbf{A}, \quad \text{curl}(\mathbf{F}^{-T}\mathbf{A}) = J^{-1}\mathbf{F}\text{Curl} \mathbf{A}. \quad (83)$$

By using the connection $\mathbf{a} = J^{-1}\mathbf{F}\mathbf{A}$ we see that the divergence identities in equations (82)₁ and (83)₁ are equivalent. Equally, the equations involving the curl operator coincide if, instead, we set $\mathbf{a} = \mathbf{F}^{-T}\mathbf{A}$.

3.2 Lagrangian Formulation

The electromagnetic field equations in the preceding section are expressed in Eulerian form involving the operators div and curl . In this section we re-cast the equations in Lagrangian form using the operators Div and Curl along with the boundary conditions, with the independent spatial variable \mathbf{x} replaced by \mathbf{X} .

Applying the divergence theorem to the integral form of equation (65)₁, with $\rho_e = 0$, we obtain

$$\int_{\mathcal{B}} \text{div} \mathbf{D} dv = \int_{\partial \mathcal{B}} \mathbf{D} \cdot \mathbf{n} da = 0, \quad (84)$$

where \mathbf{n} is the unit outward normal to $\partial\mathcal{B}$. For time-independent phenomena the integral form of equation (66)₂ gives

$$\int_{\mathcal{S}} \operatorname{curl} \mathbf{E} \cdot \mathbf{n} da = \oint_{\partial\mathcal{S}} \mathbf{E} \cdot d\mathbf{x} = 0, \quad (85)$$

where \mathcal{S} is an open surface in the deformed configuration and $\partial\mathcal{S}$ is a closed curve bounding \mathcal{S} defined in the usual sense relative to the unit normal \mathbf{n} to \mathcal{S} .

In equation (84), we now make use of the standard Nanson formula $\mathbf{n} da = J\mathbf{F}^{-T}\mathbf{N} dA$ connecting referential and current area elements (respectively dA and da), where the unit normal \mathbf{N} is the referential counterpart of \mathbf{n} . We also use, in (85), the rule $d\mathbf{x} = \mathbf{F} d\mathbf{X}$ connecting line elements of material. Equations (84) and (85) can then be written in the referential forms

$$\int_{\partial\mathcal{B}_0} J(\mathbf{F}^{-1}\mathbf{D}) \cdot \mathbf{N} dA = \int_{\mathcal{B}_0} \operatorname{Div}(J\mathbf{F}^{-1}\mathbf{D}) dV = 0, \quad (86)$$

$$\oint_{\partial\mathcal{S}_0} (\mathbf{F}^T\mathbf{E}) \cdot d\mathbf{X} = \int_{\mathcal{S}_0} \operatorname{Curl}(\mathbf{F}^T\mathbf{E}) \cdot \mathbf{N} dA = 0, \quad (87)$$

where \mathcal{S}_0 is the surface in the reference configuration that deforms into \mathcal{S} , and the closed curve $\partial\mathcal{S}_0$ is its boundary.

This leads to the introduction of Lagrangian counterparts of \mathbf{D} and \mathbf{E} , denoted \mathbf{D}_l and \mathbf{E}_l and defined by

$$\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}, \quad \mathbf{E}_l = \mathbf{F}^T\mathbf{E}. \quad (88)$$

Similarly, using equations (65)₂ and (66)₁ with $\mathbf{J} = \mathbf{0}$, for time-independent phenomena, the Lagrangian counterparts of \mathbf{B} and \mathbf{H} can be derived. Applying a process identical to the one used for the electric field variables, gives

$$\int_{\mathcal{B}} \operatorname{div} \mathbf{B} dv = \int_{\partial\mathcal{B}} \mathbf{B} \cdot \mathbf{n} da = 0, \quad (89)$$

$$\int_{\mathcal{S}} \operatorname{curl} \mathbf{H} \cdot \mathbf{n} da = \oint_{\partial\mathcal{S}} \mathbf{H} \cdot d\mathbf{x} = 0. \quad (90)$$

Again, using Nanson's formula and the rule connecting line elements of material, the identities in (89) and (90) can be written in the referential forms

$$\int_{\partial\mathcal{B}_0} J(\mathbf{F}^{-1}\mathbf{B}) \cdot \mathbf{N} dA = \int_{\mathcal{B}_0} \operatorname{Div}(J\mathbf{F}^{-1}\mathbf{B}) dV = 0, \quad (91)$$

$$\oint_{\partial\mathcal{S}_0} (\mathbf{F}^T\mathbf{H}) \cdot d\mathbf{X} = \int_{\mathcal{S}_0} \operatorname{Curl}(\mathbf{F}^T\mathbf{H}) \cdot \mathbf{N} dA = 0. \quad (92)$$

This prompts introduction of the notations

$$\mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}, \quad \mathbf{H}_l = \mathbf{F}^T\mathbf{H}, \quad (93)$$

with $J = 1$ for an incompressible material, which are the Lagrangian counterparts of \mathbf{B} and \mathbf{H} respectively.

From (82)₁ it follows that

$$\text{Div}\mathbf{D}_l = J\text{div}\mathbf{D}, \quad \text{Div}\mathbf{B}_l = J\text{div}\mathbf{B}, \quad (94)$$

and using (82)₂ gives

$$\text{Curl}\mathbf{E}_l = J\mathbf{F}^{-1}\text{curl}\mathbf{E}, \quad \text{Curl}\mathbf{H}_l = J\mathbf{F}^{-1}\text{curl}\mathbf{H}. \quad (95)$$

Maxwell's equations (69) may therefore be written in Lagrangian form as

$$\text{Div}\mathbf{D}_l = \rho_F, \quad \text{Div}\mathbf{B}_l = 0, \quad (96)$$

where $\rho_F = J\rho_f$ is the free charge density per unit reference volume. For time-independent phenomena, equations (70) become

$$\text{Curl}\mathbf{E}_l = 0, \quad \text{Curl}\mathbf{H}_l = \mathbf{J}_l, \quad (97)$$

where $\mathbf{J}_l = J\mathbf{F}^{-1}\mathbf{J}_f$ is the Lagrangian counterpart of the free current density.

The Lagrangian forms of the boundary conditions are entirely analogous to their Eulerian counterparts in (73) and (77). The boundary conditions associated with equations (96) are

$$\mathbf{N} \cdot \llbracket \mathbf{D}_l \rrbracket = \sigma_F, \quad \mathbf{N} \cdot \llbracket \mathbf{B}_l \rrbracket = 0, \quad (98)$$

where \mathbf{N} is the unit normal to the reference boundary $\partial\mathcal{B}_0$ associated with \mathbf{n} through Nanson's formula $\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA$ and σ_F is the surface charge per unit reference area. Similarly, the boundary conditions associated with equations (97) are

$$\mathbf{N} \times \llbracket \mathbf{E}_l \rrbracket = \mathbf{0}, \quad \mathbf{N} \times \llbracket \mathbf{H}_l \rrbracket = \mathbf{K}_l, \quad (99)$$

where $\mathbf{K}_l = \mathbf{F}^{-1}\mathbf{K}_f da/dA$ is the Lagrangian free surface current, defined per unit reference area.

In the remainder of this article we illustrate the use of the equations summarized above by focusing attention on the electrostatic situation. In particular, we develop the constitutive theory that describes the nonlinear interaction between elastic deformation and electric fields.

4 Nonlinear Electroelastic Materials

4.1 The Equations of Electrostatics

Recall that \mathbf{E} , \mathbf{D} and \mathbf{P} denote, respectively, the electric field, the electric displacement and the polarization in the deformed configuration \mathcal{B} and are regarded as functions of \mathbf{x} .

The fields \mathbf{E} and \mathbf{D} must satisfy the field equations

$$\operatorname{curl} \mathbf{E} = \mathbf{0}, \quad \operatorname{div} \mathbf{D} = 0, \quad (100)$$

which are the appropriate specializations of Maxwell's equations in the absence of magnetic interactions, distributed charges and time dependence. The polarization vector \mathbf{P} can be considered as a derived quantity and is defined in terms of \mathbf{E} and \mathbf{D} by the standard equation

$$\mathbf{P} = \mathbf{D} - \varepsilon_0 \mathbf{E}, \quad (101)$$

where we recall that the constant ε_0 is the permittivity of free space. Equation (101) gives an explicit expression for \mathbf{P} in terms of either \mathbf{E} or \mathbf{D} as the independent variable when \mathbf{D} (respectively \mathbf{E}) is given in terms of \mathbf{E} (respectively \mathbf{D}) by a constitutive law. In vacuum or in non-polarizable material, $\mathbf{P} = \mathbf{0}$ and (101) reduces to the simple relation

$$\mathbf{D} = \varepsilon_0 \mathbf{E}. \quad (102)$$

Across a surface of discontinuity within the body or across the boundary $\partial\mathcal{B}$, the fields \mathbf{E} and \mathbf{D} have to satisfy certain continuity conditions. Here we do not consider internal surfaces, and therefore the continuity conditions refer only to $\partial\mathcal{B}$. It is a simple matter to allow for internal surfaces as and when needed. Let open square brackets signify the jump in the enclosed quantity in passing from the inside to the outside of the body. For example, $[[\mathbf{E}]] = \mathbf{E}^\circ - \mathbf{E}^i$, where $^\circ$ and i designate 'outside' and 'inside', respectively. Then, in the absence of surface charges, the continuity conditions satisfied by the fields are

$$\mathbf{n} \times [[\mathbf{E}]] = \mathbf{0}, \quad \mathbf{n} \cdot [[\mathbf{D}]] = 0 \quad \text{on } \partial\mathcal{B}, \quad (103)$$

where \mathbf{n} is the unit outward normal to $\partial\mathcal{B}$. Then, using (101), it is easy to show that

$$\varepsilon_0 [[\mathbf{E}]] = (\mathbf{n} \cdot \mathbf{P})\mathbf{n}, \quad [[\mathbf{D}]] = (\mathbf{n} \cdot \mathbf{P})\mathbf{n} - \mathbf{P} \quad \text{on } \partial\mathcal{B}. \quad (104)$$

We remark that in some circumstances the surface of a dielectric may retain a surface charge distribution, but we do not include this in our main discussion.

Equations (100) can be written in the alternative Lagrangian forms. These are

$$\text{Curl } \mathbf{E}_l = \mathbf{0}, \quad \text{Div } \mathbf{D}_l = 0, \quad (105)$$

where we recall that Curl and Div operate with respect to \mathcal{B}_0 . The boundary conditions for the electric field and displacement, again in the absence of magnetic interactions, distributed charges and time dependence, assume the Lagrangian forms

$$\mathbf{N} \times \llbracket \mathbf{E}_l \rrbracket = \mathbf{0}, \quad \mathbf{N} \cdot \llbracket \mathbf{D}_l \rrbracket = 0, \quad (106)$$

which apply on $\partial\mathcal{B}_0$.

4.2 Equilibrium, Stress and Constitutive Laws.

There are many different ways in which the equations of mechanical equilibrium can be written in the presence of electromechanical interactions, and there are many possible definitions of ‘stress tensor’ that can be included in the equilibrium equations. Perhaps the simplest structure is provided by the so-called *total stress tensor*, here denoted $\boldsymbol{\tau}$, which is *symmetric* and is the analogue of the Cauchy stress tensor arising in elasticity theory (Dorfmann and Ogden, 2005). In terms of $\boldsymbol{\tau}$ the equilibrium equation has the form

$$\text{div } \boldsymbol{\tau} + \rho \mathbf{f} = \mathbf{0}, \quad (107)$$

where \mathbf{f} is the *mechanical* body force per unit mass and ρ is the mass density of the material in the configuration \mathcal{B} . We emphasize that $\boldsymbol{\tau}$ incorporates terms that may be considered as electric body forces rather than stresses, the diversity of which is discussed in the following subsections.

If \mathbf{t}_a is the applied mechanical traction per unit area of $\partial\mathcal{B}$, then the stress $\boldsymbol{\tau}$ calculated inside the material must satisfy

$$\boldsymbol{\tau} \mathbf{n} = \mathbf{t}_a + \boldsymbol{\tau}_m \mathbf{n} \quad \text{on } \partial\mathcal{B}, \quad (108)$$

where $\boldsymbol{\tau}_m$ is the Maxwell stress due to the electric field outside the material. It is given by

$$\boldsymbol{\tau}_m = \mathbf{D} \otimes \mathbf{E} - \frac{1}{2}(\mathbf{D} \cdot \mathbf{E})\mathbf{I}, \quad (109)$$

where \mathbf{I} is the identity tensor and with $\mathbf{D} = \varepsilon_0 \mathbf{E}$ the expression (109) is symmetric. In general, \mathbf{t}_a may be prescribed on only part of $\partial\mathcal{B}$ or not at all. Here we do not consider alternative mechanical boundary conditions such as prescription of \mathbf{x} on part of $\partial\mathcal{B}$. The formulation can be adapted to accommodate such boundary conditions when required.

In what follows we summarize some alternative formulations of the equilibrium equation (107). We assume throughout that the material is not

subject to any internal mechanical constraint such as incompressibility, although extension of the theory presented here to mechanically internally constrained materials is straightforward. Furthermore, we consider only conservative materials and, in particular, we assume that there is no hysteresis.

Formulations based on the polarization. We now examine different forms of the equilibrium equation and the associated stress measures. We first consider the formulation introduced by Toupin (1956). This is based on use of $\bar{\mathbf{P}} \equiv \mathbf{P}/\rho$ as the independent electric variable and an energy function per unit mass, here denoted $\bar{\chi}(\mathbf{F}, \bar{\mathbf{P}})$, which does not include the electric self energy. From this are calculated a Cauchy-like stress, denoted $\bar{\boldsymbol{\sigma}}$, and \mathbf{E} via

$$\bar{\boldsymbol{\sigma}} = \rho \mathbf{F} \frac{\partial \bar{\chi}}{\partial \mathbf{F}}, \quad \mathbf{E} = \frac{\partial \bar{\chi}}{\partial \bar{\mathbf{P}}}. \quad (110)$$

Although this is not immediately apparent, we note that in general $\bar{\boldsymbol{\sigma}}$ is not symmetric.

The equilibrium equation (107) can be expressed in the alternative form

$$\operatorname{div} \bar{\boldsymbol{\sigma}} + (\operatorname{grad} \mathbf{E})^T \mathbf{P} + \rho \mathbf{f} = \mathbf{0}, \quad (111)$$

the term $(\operatorname{grad} \mathbf{E})^T \mathbf{P}$ having the role of an electric body force relative to the stress tensor $\bar{\boldsymbol{\sigma}}$. Here we use the convention (in Cartesian components) that $(\operatorname{grad} \mathbf{E})_{ij} = \partial E_i / \partial x_j$. The stress $\bar{\boldsymbol{\sigma}}$ is easily seen to be related to $\boldsymbol{\tau}$ by

$$\boldsymbol{\tau} = \bar{\boldsymbol{\sigma}} + \mathbf{D} \otimes \mathbf{E} - \frac{1}{2} \varepsilon_0 (\mathbf{E} \cdot \mathbf{E}) \mathbf{I}, \quad (112)$$

which reduces to the Maxwell stress (109) outside the material (where $\mathbf{P} = \mathbf{0}$). Note that, since $\operatorname{curl} \mathbf{E} = \mathbf{0}$, $(\operatorname{grad} \mathbf{E})^T \mathbf{P} = (\operatorname{grad} \mathbf{E}) \mathbf{P} \equiv (\mathbf{P} \cdot \operatorname{grad}) \mathbf{E}$. In general, $\mathbf{D} \otimes \mathbf{E}$ is not symmetric, and since $\boldsymbol{\tau}$ is symmetric it follows that $\bar{\boldsymbol{\sigma}}$ is not symmetric.

If, instead of \mathbf{F} and $\bar{\mathbf{P}}$, we use \mathbf{F} and \mathbf{P} as the independent variables and write $\chi(\mathbf{F}, \mathbf{P})$ as the energy function then equations (110) are replaced by

$$\boldsymbol{\sigma}_\chi \equiv \rho \mathbf{F} \frac{\partial \chi}{\partial \mathbf{F}} = \bar{\boldsymbol{\sigma}} + (\mathbf{E} \cdot \mathbf{P}) \mathbf{I}, \quad \mathbf{E} = \rho \frac{\partial \chi}{\partial \mathbf{P}}. \quad (113)$$

This requires use of the formula $\partial J / \partial \mathbf{F} = J \mathbf{F}^{-1}$ and the connection $\rho_0 = \rho J$, where ρ_0 is the mass density in \mathcal{B}_0 and ρ the density in the deformed configuration \mathcal{B} .

In terms of $\boldsymbol{\sigma}_\chi$ the equilibrium equation has the form

$$\operatorname{div} \boldsymbol{\sigma}_\chi - (\operatorname{grad} \mathbf{P})^T \mathbf{E} + \rho \mathbf{f} = \mathbf{0}. \quad (114)$$

Next, we define $\chi^*(\mathbf{F}, \mathbf{P})$ by

$$\rho\chi^*(\mathbf{F}, \mathbf{P}) = \rho\chi(\mathbf{F}, \mathbf{P}) + \frac{1}{2}\varepsilon_0^{-1}\mathbf{P} \cdot \mathbf{P}, \quad (115)$$

and the associated derivatives are

$$\boldsymbol{\sigma}_\chi^* \equiv \rho\mathbf{F} \frac{\partial\chi^*}{\partial\mathbf{F}} = \boldsymbol{\sigma}_\chi + \frac{1}{2}\varepsilon_0^{-1}(\mathbf{P} \cdot \mathbf{P})\mathbf{I}, \quad \varepsilon_0^{-1}\mathbf{D} = \rho \frac{\partial\chi^*}{\partial\mathbf{P}}. \quad (116)$$

The equilibrium equation can then be written

$$\operatorname{div} \boldsymbol{\sigma}_\chi^* - \varepsilon_0^{-1}(\operatorname{grad} \mathbf{P})^T \mathbf{D} + \rho \mathbf{f} = \mathbf{0}, \quad (117)$$

where $-\varepsilon_0^{-1}(\operatorname{grad} \mathbf{P})^T \mathbf{D}$ is the electric body force relative to $\boldsymbol{\sigma}_\chi^*$. Neither $\boldsymbol{\sigma}_\chi$ nor $\boldsymbol{\sigma}_\chi^*$ is symmetric in general. Yet another alternative can be obtained by using $\overline{\mathbf{P}}$ instead of \mathbf{P} in the above, but we omit the details of this case.

Formulations based on the electric field. Formulations of the equations based on use of \mathbf{E} as the independent electric variable are considered next. First, the energy function $\psi(\mathbf{F}, \mathbf{E})$ is obtained by the partial Legendre transform

$$\rho\psi(\mathbf{F}, \mathbf{E}) = \rho\chi(\mathbf{F}, \mathbf{P}) - \mathbf{P} \cdot \mathbf{E}, \quad (118)$$

from which we obtain

$$\boldsymbol{\sigma}_\psi \equiv \rho\mathbf{F} \frac{\partial\psi}{\partial\mathbf{F}} = \overline{\boldsymbol{\sigma}}, \quad \mathbf{P} = -\rho \frac{\partial\psi}{\partial\mathbf{E}}. \quad (119)$$

Note, in particular, that this yields the same stress tensor $\overline{\boldsymbol{\sigma}}$ as does $\overline{\chi}$.

If next we define $\psi^*(\mathbf{F}, \mathbf{E})$ by

$$\rho\psi^*(\mathbf{F}, \mathbf{E}) = \rho\psi(\mathbf{F}, \mathbf{E}) - \frac{1}{2}\varepsilon_0\mathbf{E} \cdot \mathbf{E}, \quad (120)$$

then we obtain

$$\boldsymbol{\sigma}_\psi^* \equiv \rho\mathbf{F} \frac{\partial\psi^*}{\partial\mathbf{F}} = \boldsymbol{\sigma}_\psi - \frac{1}{2}\varepsilon_0(\mathbf{E} \cdot \mathbf{E})\mathbf{I}, \quad \mathbf{D} = -\rho \frac{\partial\psi^*}{\partial\mathbf{E}}, \quad (121)$$

and the equilibrium equation can be translated to

$$\operatorname{div} \boldsymbol{\sigma}_\psi^* + (\operatorname{grad} \mathbf{E})^T \mathbf{D} + \rho \mathbf{f} = \mathbf{0}, \quad (122)$$

$(\operatorname{grad} \mathbf{E})^T \mathbf{D}$ being the electric body force associated with the stress $\boldsymbol{\sigma}_\psi^*$ (which is not in general symmetric).

Formulations based on the electric displacement. A third option is to base the formulation on the electric displacement vector \mathbf{D} and to define an energy function $\phi(\mathbf{F}, \mathbf{D})$ via the transformation

$$\rho\phi(\mathbf{F}, \mathbf{D}) = \rho\psi(\mathbf{F}, \mathbf{E}) - \frac{1}{2}\varepsilon_0^{-1}\mathbf{P} \cdot \mathbf{P}. \quad (123)$$

This yields a stress tensor, denoted $\boldsymbol{\sigma}_\phi$, and the polarization \mathbf{P} in the forms

$$\boldsymbol{\sigma}_\phi \equiv \rho\mathbf{F}\frac{\partial\phi}{\partial\mathbf{F}} = \boldsymbol{\sigma}_\psi - \frac{1}{2}\varepsilon_0^{-1}(\mathbf{P} \cdot \mathbf{P})\mathbf{I}, \quad \mathbf{P} = -\varepsilon_0\rho\frac{\partial\phi}{\partial\mathbf{D}}, \quad (124)$$

while the equilibrium equation becomes

$$\operatorname{div}\boldsymbol{\sigma}_\phi + \varepsilon_0^{-1}(\operatorname{grad}\mathbf{D})^T\mathbf{P} + \rho\mathbf{f} = \mathbf{0}. \quad (125)$$

In this case it is the term $\varepsilon_0^{-1}(\operatorname{grad}\mathbf{D})^T\mathbf{P}$ that has the role of an electric body force (in respect of the stress $\boldsymbol{\sigma}_\phi$). In general, $\boldsymbol{\sigma}_\phi$ is not symmetric.

The final option we consider here is based on the potential $\phi^*(\mathbf{F}, \mathbf{D})$ given by

$$\rho\phi^*(\mathbf{F}, \mathbf{D}) = \rho\phi(\mathbf{F}, \mathbf{D}) + \frac{1}{2}\varepsilon_0^{-1}\mathbf{D} \cdot \mathbf{D}, \quad (126)$$

or, equivalently, through the partial Legendre transform,

$$\rho\phi^*(\mathbf{F}, \mathbf{D}) = \rho\psi^*(\mathbf{F}, \mathbf{E}) + \mathbf{D} \cdot \mathbf{E}. \quad (127)$$

The stress tensor associated with ϕ^* , denoted $\boldsymbol{\sigma}_\phi^*$, and the electric field \mathbf{E} are given by

$$\boldsymbol{\sigma}_\phi^* \equiv \rho\mathbf{F}\frac{\partial\phi^*}{\partial\mathbf{F}} = \boldsymbol{\sigma}_\phi + \frac{1}{2}\varepsilon_0^{-1}(\mathbf{D} \cdot \mathbf{D})\mathbf{I}, \quad \mathbf{E} = \rho\frac{\partial\phi^*}{\partial\mathbf{D}}, \quad (128)$$

and the equilibrium equation can then be re-cast as

$$\operatorname{div}\boldsymbol{\sigma}_\phi^* - (\operatorname{grad}\mathbf{D})^T\mathbf{E} + \rho\mathbf{f} = \mathbf{0}, \quad (129)$$

now with electric body force $-(\operatorname{grad}\mathbf{D})^T\mathbf{E}$ and (in general unsymmetric) stress $\boldsymbol{\sigma}_\phi^*$.

As in the magnetostatic situation discussed by Bustamante et al. (2008), the formulations listed in the above sections are all equivalent, but they are not the only possible ones. Generically, the equilibrium equation can be written in the form

$$\operatorname{div}\hat{\boldsymbol{\sigma}} + \hat{\mathbf{f}}_e + \rho\mathbf{f} = \mathbf{0}, \quad (130)$$

where $\hat{\boldsymbol{\sigma}}$ is a Cauchy-like stress tensor and $\hat{\mathbf{f}}_e$ is an electric body force (defined per unit volume in \mathcal{B}). The latter is always expressible in the

form $\hat{\mathbf{f}}_e = \text{div } \hat{\boldsymbol{\tau}}_m$, where $\hat{\boldsymbol{\tau}}_m$ is a ‘Maxwell stress’ within the material and $\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\tau}}_m = \boldsymbol{\tau}$. Clearly, the concepts of ‘stress’, ‘electric body force’ and ‘Maxwell stress’ inside the material are not uniquely defined, as is well known, and, in particular, the electric ‘body force’ and ‘Maxwell stress’ terms are different for each choice of ‘stress tensor’. Moreover, the boundary conditions for each stress tensor are different since, from (108), $\hat{\boldsymbol{\sigma}}$ must satisfy

$$\hat{\boldsymbol{\sigma}}^T \mathbf{n} = \mathbf{t}_a + (\boldsymbol{\tau}_m - \hat{\boldsymbol{\tau}}_m)^T \mathbf{n} \quad \text{on } \partial\mathcal{B}, \quad (131)$$

where, it should be emphasized, $\boldsymbol{\tau}_m$ is the Maxwell stress calculated on the boundary from the exterior fields, while $\hat{\boldsymbol{\tau}}_m$ is the Maxwell stress calculated on the boundary from the fields inside the material.

To be more specific we now consider the stress tensor $\bar{\boldsymbol{\sigma}}$ since the associated body force $(\mathbf{P} \cdot \text{grad})\mathbf{E}$ is the direct continuum analogue of the force on a single electric dipole in an electric field. On reference to equation (112) we may write this as $\text{div } \bar{\boldsymbol{\tau}}_m$, where $\bar{\boldsymbol{\tau}}_m$ is the associated Maxwell stress (inside the material) defined by

$$\bar{\boldsymbol{\tau}}_m = \mathbf{D} \otimes \mathbf{E} - \frac{1}{2}\varepsilon_0(\mathbf{E} \cdot \mathbf{E})\mathbf{I}, \quad (132)$$

which, in general, is not symmetric. We recall that the Maxwell stress exterior to the body is given by (109) and, since $\mathbf{D} = \varepsilon_0\mathbf{E}$ there, this is symmetric. By using equations (104) we then see that there is a discontinuity across $\partial\mathcal{B}$ in the traction associated with the Maxwell stress, namely

$$(\boldsymbol{\tau}_m - \bar{\boldsymbol{\tau}}_m^T)\mathbf{n} = \frac{1}{2}\varepsilon_0^{-1}(\mathbf{P} \cdot \mathbf{n})^2\mathbf{n}. \quad (133)$$

From (108) and (112) it follows that $\bar{\boldsymbol{\sigma}}$ must satisfy the boundary condition

$$\bar{\boldsymbol{\sigma}}^T \mathbf{n} = \mathbf{t}_a + \frac{1}{2}\varepsilon_0^{-1}(\mathbf{P} \cdot \mathbf{n})^2\mathbf{n} \quad \text{on } \partial\mathcal{B}. \quad (134)$$

The distinctions between the relative contributions of the many different body forces or, equivalently, Maxwell stresses are therefore somewhat artificial. In any case, it is unlikely that experiment will be able to distinguish between different choices of Maxwell stress by direct measurement, notwithstanding the fact that for a given body force, the associated Maxwell stress is undetermined to within an additive divergence-free stress. For a detailed discussion of different dipole-type models in electromagnetism associated with different body forces we refer to Hutter and van de Ven (1978); Hutter et al. (2006).

From the mathematical point of view, the formulation based on the ‘total stress’ is the cleanest and avoids the need for defining either an electric

Energy	Stress $\hat{\boldsymbol{\sigma}}$	Body force \mathbf{f}_e	Electric vector
$\chi(\mathbf{F}, \mathbf{P})$	$\boldsymbol{\sigma}_\chi = \rho \mathbf{F} \frac{\partial \chi}{\partial \mathbf{F}}$	$-(\text{grad } \mathbf{P})^T \mathbf{E}$	$\mathbf{E} = \rho \frac{\partial \chi}{\partial \mathbf{P}}$
$\chi^*(\mathbf{F}, \mathbf{P})$	$\boldsymbol{\sigma}_\chi^* = \rho \mathbf{F} \frac{\partial \chi^*}{\partial \mathbf{F}}$	$-\varepsilon_0^{-1} (\text{grad } \mathbf{P})^T \mathbf{D}$	$\varepsilon_0^{-1} \mathbf{D} = \rho \frac{\partial \chi^*}{\partial \mathbf{P}}$
$\psi(\mathbf{F}, \mathbf{E})$	$\boldsymbol{\sigma}_\psi = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}}$	$(\text{grad } \mathbf{E})^T \mathbf{P}$	$\mathbf{P} = -\rho \frac{\partial \psi}{\partial \mathbf{E}}$
$\psi^*(\mathbf{F}, \mathbf{E})$	$\boldsymbol{\sigma}_\psi^* = \rho \mathbf{F} \frac{\partial \psi^*}{\partial \mathbf{F}}$	$(\text{grad } \mathbf{E})^T \mathbf{D}$	$\mathbf{D} = -\rho \frac{\partial \psi^*}{\partial \mathbf{E}}$
$\phi(\mathbf{F}, \mathbf{D})$	$\boldsymbol{\sigma}_\phi = \rho \mathbf{F} \frac{\partial \phi}{\partial \mathbf{F}}$	$\varepsilon_0^{-1} (\text{grad } \mathbf{D})^T \mathbf{P}$	$\mathbf{P} = -\varepsilon_0 \rho \frac{\partial \phi}{\partial \mathbf{D}}$
$\phi^*(\mathbf{F}, \mathbf{D})$	$\boldsymbol{\sigma}_\phi^* = \rho \mathbf{F} \frac{\partial \phi^*}{\partial \mathbf{F}}$	$-(\text{grad } \mathbf{D})^T \mathbf{E}$	$\mathbf{E} = \rho \frac{\partial \phi^*}{\partial \mathbf{D}}$

Table 1. Listing of energy functions based on \mathbf{P} , \mathbf{E} or \mathbf{D} and the associated stress $\hat{\boldsymbol{\sigma}}$ and electric body force $\hat{\mathbf{f}}_e$ in equation (130), together with the derived field vectors.

body force or a Maxwell stress *within* a polarizable material. Indeed, as shown by Dorfmann and Ogden (2005), the stress $\boldsymbol{\tau}$ has a very simple expression in terms of a modified form of the energy function. This will be recalled and highlighted in the following subsection. For ease of reference the main features of the above formulations are collected together in Table 1. However, for the remainder of this chapter we shall not make use of electric body forces or Maxwell stresses within the material. The formulations discussed above are all Eulerian in character. In the following subsection, we discuss the Lagrangian counterparts of some of these formulations.

Lagrangian variables and governing equations. A suitable translation of the total stress tensor $\boldsymbol{\tau}$ to the Lagrangian context is the *total nominal stress* (the counterpart of the nominal stress in elasticity theory), here denoted \mathbf{T} , which is a two-point tensor related to $\boldsymbol{\tau}$ by

$$\mathbf{T} = J \mathbf{F}^{-1} \boldsymbol{\tau}. \quad (135)$$

Then, the equilibrium equation (107) may be written in the alternative form

$$\text{Div } \mathbf{T} + \rho_0 \mathbf{f} = \mathbf{0}, \quad (136)$$

where we recall that $\rho_0 = \rho J$ is the mass density of the material in \mathcal{B}_0 . In the absence of mechanical body forces, the equilibrium equation simplifies

to

$$\text{Div} \mathbf{T} = \mathbf{0}. \quad (137)$$

The traction boundary condition associated with (136) and analogous (108) can be re-cast as

$$\mathbf{T}^T \mathbf{N} = \mathbf{t}_A + \mathbf{t}_E, \quad (138)$$

where $\mathbf{t}_E = \mathbf{T}_E^T \mathbf{N}$ and $\mathbf{T}_E = J \mathbf{F}^{-1} \boldsymbol{\tau}_m$ and we have used Nanson's formula $\mathbf{n} da = J \mathbf{F}^{-T} \mathbf{N} dA$ relating area elements to define \mathbf{t}_A by $\mathbf{t}_A dA = \mathbf{t}_a da$ as the traction per unit reference area.

4.3 Constitutive Equations

The energy function $\psi(\mathbf{F}, \mathbf{E})$ was introduced in the previous subsection. From the connection (88)₂ between \mathbf{E} and \mathbf{E}_l we may regard $\psi(\mathbf{F}, \mathbf{E})$, equivalently, as a function of \mathbf{F} and \mathbf{E}_l , and we write

$$\Psi(\mathbf{F}, \mathbf{E}_l) \equiv \psi(\mathbf{F}, \mathbf{F}^{-T} \mathbf{E}_l). \quad (139)$$

Note that since \mathbf{E}_l is a Lagrangian vector it is indifferent to observer transformations in the deformed configuration, while, as is standard in nonlinear elasticity, the dependence on \mathbf{F} is via the right Cauchy–Green tensor \mathbf{c} defined in (80)₁, which ensures objectivity of Ψ .

The stress tensor $\boldsymbol{\sigma}_\psi = \bar{\boldsymbol{\sigma}}$ discussed in Section 4.2 is now given by

$$\bar{\boldsymbol{\sigma}} = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}} = \rho \mathbf{F} \frac{\partial \Psi}{\partial \mathbf{F}} - \mathbf{P} \otimes \mathbf{E}, \quad (140)$$

the total stress $\boldsymbol{\tau}$ by

$$\boldsymbol{\tau} = \rho \mathbf{F} \frac{\partial \Psi}{\partial \mathbf{F}} + \varepsilon_0 [\mathbf{E} \otimes \mathbf{E} - \frac{1}{2} (\mathbf{E} \cdot \mathbf{E}) \mathbf{I}], \quad (141)$$

and the corresponding total nominal stress \mathbf{T} by (135). Note that the first term on the right-hand side of (141) provides yet another example of a stress tensor, in this case symmetric.

We also have

$$\mathbf{P}_l = -\rho_0 \frac{\partial \Psi}{\partial \mathbf{E}_l}, \quad \mathbf{P} = -\rho \mathbf{F} \frac{\partial \Psi}{\partial \mathbf{E}_l}, \quad (142)$$

where we have defined a Lagrangian form \mathbf{P}_l of the polarization \mathbf{P} by

$$\mathbf{P}_l = J \mathbf{F}^{-1} \mathbf{P}. \quad (143)$$

Following Dorfmann and Ogden (2005) we now introduce the ‘total’ energy (density) function, denoted $\Omega = \Omega(\mathbf{F}, \mathbf{E}_l)$ and defined by

$$\Omega = \rho_0 \Psi - \frac{1}{2} \varepsilon_0 J \mathbf{E}_l \cdot (\mathbf{c}^{-1} \mathbf{E}_l). \quad (144)$$

This enables \mathbf{T} and $\boldsymbol{\tau}$ to be given in the simple forms

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \boldsymbol{\tau} = J^{-1} \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}}, \quad (145)$$

and the Lagrangian and Eulerian forms \mathbf{D}_l and \mathbf{D} of the electric displacement field simplify to

$$\mathbf{D}_l = -\frac{\partial \Omega}{\partial \mathbf{E}_l}, \quad \mathbf{D} = -J^{-1} \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{E}_l}. \quad (146)$$

Given that \mathbf{E}_l can be written $\mathbf{E}_l = -\text{Grad} \varphi$ and that $\mathbf{F} = \text{Grad} \mathbf{x}$, and given the mechanical body force and the form of Ω , the relevant equations to be solved for \mathbf{x} and φ are the coupled nonlinear equations

$$\text{Div} \mathbf{T} + \rho_0 \mathbf{f} = \mathbf{0}, \quad \text{Div} \mathbf{D}_l = 0, \quad (147)$$

in conjunction with the boundary conditions $(98)_1$, $(99)_1$ and (138) .

As an alternative to using \mathbf{E}_l , the Lagrangian displacement vector \mathbf{D}_l can be used as the independent electric variable through the function $\Omega^*(\mathbf{F}, \mathbf{D}_l)$, which is defined by the partial Legendre transformation

$$\Omega^*(\mathbf{F}, \mathbf{D}_l) = \Omega(\mathbf{F}, \mathbf{E}_l) + \mathbf{E}_l \cdot \mathbf{D}_l. \quad (148)$$

We recall that we are considering conservative materials with no hysteresis so that it is appropriate to assume that \mathbf{D}_l is a monotonic function of \mathbf{E}_l . The transformation (148) is then well defined and it follows that

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{E}_l = \frac{\partial \Omega^*}{\partial \mathbf{D}_l}, \quad (149)$$

which have Eulerian counterparts

$$\boldsymbol{\tau} = J^{-1} \mathbf{F} \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{E} = \mathbf{F}^{-\text{T}} \frac{\partial \Omega^*}{\partial \mathbf{D}_l}. \quad (150)$$

Incompressible materials. The expressions for the various stress tensors in the foregoing apply for a material that is not subject to any internal mechanical constraint. For an important class of materials, including electro-sensitive elastomers, it is appropriate to adopt the constraint of incompressibility, in which case the expressions for the stresses require modification.

For an incompressible material we have the constraint

$$\det \mathbf{F} \equiv 1. \quad (151)$$

The amended free energy function (144) then simplify to

$$\Omega = \rho_0 \Psi - \frac{1}{2} \varepsilon_0 \mathbf{E}_l \cdot (\mathbf{c}^{-1} \mathbf{E}_l), \quad (152)$$

and total nominal stress \mathbf{T} and Cauchy stress $\boldsymbol{\tau}$ given by (145) in terms of Ω are then amended in the forms

$$\boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}} - p \mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p \mathbf{F}^{-1}, \quad (153)$$

respectively, where p is a Lagrange multiplier associated with the constraint (151). The expressions (146) are unchanged except that (151) is in force. In terms of Ω^* we have, instead of (153),

$$\boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^* \mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^* \mathbf{F}^{-1}, \quad (154)$$

where, in general, the p in (153) is not the same as the p^* in (154).

4.4 Material Symmetry Considerations

Thus far no restrictions have been placed on the forms of the free energy functions other than those required by objectivity, so that considerable generality remains. Other restrictions may be physically or mathematically based. For example, physical restrictions arise from the nature of the material itself, such as its inherent symmetry. Electro-sensitive elastomers are typically isotropic in their response in the absence of an electric field, but application of an electric field endows the material with a preferred direction. Thus, the electric field vector \mathbf{E} generates a preferred direction in the deformed configuration \mathcal{B} . However, from the point of view of constitutive law development, it is advantageous to make use of the Lagrangian field \mathbf{E}_l instead of \mathbf{E} , and to consider the free energy functions Ω .

For simplicity we now restrict attention to so-called *isotropic magneto-sensitive materials*, for which the material symmetry considerations are similar to those that arise for a transversely isotropic elastic material, which possesses a preferred direction in the reference configuration. This is appropriate for fiber-reinforced materials, for which the preferred direction is the fiber direction in the reference configuration. The vector field \mathbf{E}_l has an analogous role in the present context.

The electroelastic material considered here is said to be *isotropic* if Ω is an isotropic function of the two tensors \mathbf{c} and $\mathbf{E}_l \otimes \mathbf{E}_l$. Note that the latter expression is unaffected by reversal of the sign of \mathbf{E}_l . Then, the form of Ω is reduced to dependence on the principal invariants I_1, I_2, I_3 of \mathbf{c} , defined

by

$$I_1 = \operatorname{tr} \mathbf{c}, \quad I_2 = \frac{1}{2} [(\operatorname{tr} \mathbf{c})^2 - \operatorname{tr}(\mathbf{c}^2)], \quad I_3 = \det \mathbf{c} = J^2, \quad (155)$$

together with three invariants that depend on \mathbf{E}_l . A convenient choice of the latter is

$$I_4 = |\mathbf{E}_l|^2, \quad I_5 = (\mathbf{c}\mathbf{E}_l) \cdot \mathbf{E}_l, \quad I_6 = (\mathbf{c}^2\mathbf{E}_l) \cdot \mathbf{E}_l. \quad (156)$$

Note that for a transversely isotropic elastic material the counterpart of the invariant I_4 would be absent since in that case the preferred direction is a unit vector.

In the following the subscripts 1, 2, \dots , 6 on Ω signify differentiation with respect to I_1, I_2, \dots, I_6 , respectively. A direct calculation based on (145)₂ leads to

$$\begin{aligned} \boldsymbol{\tau} = & J^{-1}[2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) + 2I_3\Omega_3\mathbf{I} + 2\Omega_5\mathbf{b}\mathbf{E} \otimes \mathbf{b}\mathbf{E} \\ & + 2\Omega_6(\mathbf{b}\mathbf{E} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{b}\mathbf{E})], \end{aligned} \quad (157)$$

which is symmetric, and (146)₂ gives

$$\mathbf{D} = -2J^{-1}(\Omega_4\mathbf{b}\mathbf{E} + \Omega_5\mathbf{b}^2\mathbf{E} + \Omega_6\mathbf{b}^3\mathbf{E}), \quad (158)$$

where we recall that $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy–Green deformation tensor. The corresponding Lagrangian forms may be obtained from the connections $\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}$ and $\mathbf{E}_l = \mathbf{F}^T\mathbf{E}$.

For an incompressible material $I_3 \equiv 1$ and (157) is replaced by

$$\boldsymbol{\tau} = 2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5\mathbf{b}\mathbf{E} \otimes \mathbf{b}\mathbf{E} + 2\Omega_6(\mathbf{b}\mathbf{E} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{b}\mathbf{E}), \quad (159)$$

while (158) is unchanged in form, but with $J = 1$ and I_3 absent from Ω .

If we work with Ω^* instead of Ω then the invariants based on \mathbf{E}_l have to be changed to invariants based on \mathbf{D}_l . These are denoted here by K_4, K_5, K_6 and may be defined, analogously to (156), by

$$K_4 = \mathbf{D}_l \cdot \mathbf{D}_l, \quad K_5 = (\mathbf{c}\mathbf{D}_l) \cdot \mathbf{D}_l, \quad K_6 = (\mathbf{c}^2\mathbf{D}_l) \cdot \mathbf{D}_l. \quad (160)$$

The associated formulas for $\boldsymbol{\tau}$ are similar to those based on Ω . For an incompressible material, for example, we have

$$\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5^*\mathbf{D} \otimes \mathbf{D} + 2\Omega_6^*(\mathbf{D} \otimes \mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D} \otimes \mathbf{D}). \quad (161)$$

The electric field is given by (150)₂ and has the form

$$\mathbf{E} = 2(\Omega_4^*\mathbf{b}^{-1}\mathbf{D} + \Omega_5^*\mathbf{D} + \Omega_6^*\mathbf{b}\mathbf{D}), \quad (162)$$

and its Lagrangian counterpart is obtained via $\mathbf{E}_l = \mathbf{F}^T \mathbf{E}$. In the above equations, for which $\Omega^* = \Omega^*(I_1, I_2, K_4, K_5, K_6)$, Ω_i^* is defined as $\partial\Omega^*/\partial I_i$ for $i = 1, 2$ and $\partial\Omega^*/\partial K_i$ for $i = 4, 5, 6$.

There are important differences between the formulations based on Ω and Ω^* in respect of their application to particular boundary-value problems. If \mathbf{D}_l is taken as the independent variable then it has to satisfy $\text{Div} \mathbf{D}_l = 0$. The resulting \mathbf{E}_l , calculated from (149)₂, then has to satisfy the vector equation $\text{Curl} \mathbf{E}_l = \mathbf{0}$, which, for some problems, puts severe restrictions on the class of constitutive laws that admit the deformation in question for the considered electric displacement field \mathbf{D}_l . On the other hand, if we start with \mathbf{E}_l as the independent variable it has to satisfy $\text{Curl} \mathbf{E}_l = \mathbf{0}$ and then the resulting \mathbf{D}_l , calculated from (146)₁, has to satisfy the scalar equation $\text{Div} \mathbf{D}_l = 0$. This also may, in some situations, put restrictions on the admissible class of constitutive laws, but they are different from and generally less severe than for the other formulation.

4.5 Exterior Fields

In vacuum or in a non-polarizable material the electric displacement is related to the electric field by

$$\mathbf{D}^* = \varepsilon_0 \mathbf{E}^* \quad (163)$$

where the star is now introduced to denote a quantity exterior to the material. The expression of the total stress tensor $\boldsymbol{\tau}$, given by (141), now simplifies and reduces to the Maxwell stress (109). The latter can be written in the equivalent form

$$\boldsymbol{\tau}^* = \varepsilon_0 \left[\mathbf{E}^* \otimes \mathbf{E}^* - \frac{1}{2} (\mathbf{E}^* \cdot \mathbf{E}^*) \mathbf{I} \right], \quad (164)$$

which satisfy the corresponding Maxwell's equations (100) and the equilibrium equation

$$\text{div} \boldsymbol{\tau}^* = \mathbf{0}. \quad (165)$$

4.6 Representative Example

We now consider a circular cylindrical tube subject to *axial shear* and apply the relevant equations derived earlier in this chapter to solve for the unknown stress components and the corresponding deformation. In particular, we consider an incompressible electro-sensitive material in the absence of mechanical body forces subject to an electrostatic field. The analysis is based on the use of the energy function $\Omega^*(\mathbf{F}, \mathbf{D}_l)$, which for

isotropic materials can equivalently be expressed in terms of the five invariants I_1, I_2, K_4, K_5, K_6 ; see equations (148), (155) and (160). Due to the incompressibility condition, the third invariant $I_3 \equiv 1$ and is therefore not included. For the solution of a circular cylindrical tube subject to axial shear, using the energy function Ω , we refer to Dorfmann and Ogden (2005).

The equations needed for the solution of a typical nonlinear electroelastic boundary-value problem are, for completeness, summarized below. We begin with the field equations in Eulerian form,

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \quad \operatorname{curl} \mathbf{E} = \mathbf{0}, \quad \operatorname{div} \mathbf{D} = 0, \quad (166)$$

where we assume that the charge density $\rho_e = 0$. For the formulation based on $\Omega^*(\mathbf{F}, \mathbf{D}_l)$, the total stress tensor $\boldsymbol{\tau}$ is given by equation (161) and the electric field vector \mathbf{E} , in terms of the electric displacement \mathbf{D} , is shown in equation (162). The Lagrangian counterparts of equations (166) are

$$\operatorname{Div} \mathbf{T} = \mathbf{0}, \quad \operatorname{Curl} \mathbf{E}_l = \mathbf{0}, \quad \operatorname{Div} \mathbf{D}_l = 0, \quad (167)$$

where we used the Lagrangian-Eulerian connections

$$\mathbf{T} = \mathbf{F}^{-1} \boldsymbol{\tau}, \quad \mathbf{E}_l = \mathbf{F}^T \mathbf{E}, \quad \mathbf{D}_l = \mathbf{F}^{-1} \mathbf{D}. \quad (168)$$

We also recall the relations between the electric displacement and the electric field

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{D}^* = \varepsilon_0 \mathbf{E}^*, \quad (169)$$

where \mathbf{P} is the polarization in the material. The equations are valid, respectively, within the material and in the surrounding space. The Lagrangian forms of (169) are not needed here; we refer to Dorfmann and Ogden (2005) for the corresponding transformations.

In order to solve boundary-value problems, we need to add boundary conditions for the electric fields and mechanical traction forces. The electric fields \mathbf{E} and \mathbf{D} are discontinuous across a surface bounding the material. In Eulerian form, the jump conditions are

$$\mathbf{n} \times \llbracket \mathbf{E} \rrbracket = \mathbf{0}, \quad \mathbf{n} \cdot \llbracket \mathbf{D} \rrbracket = 0, \quad (170)$$

where we assumed that the bounding surface does not carry any free surface charge. The boundary conditions in Lagrangian form are, analogously,

$$\mathbf{N} \times \llbracket \mathbf{E}_l \rrbracket = \mathbf{0}, \quad \mathbf{N} \cdot \llbracket \mathbf{D}_l \rrbracket = 0, \quad (171)$$

where \mathbf{N} is the unit normal to the bounding surface in the reference configuration.

The total stress tensor $\boldsymbol{\tau}$, on the boundary, must satisfy the condition

$$\boldsymbol{\tau} \mathbf{n} = \mathbf{t}_a + \boldsymbol{\tau}_m \mathbf{n}, \quad (172)$$

where $\boldsymbol{\tau}_m$ is the Maxwell stress outside the material and \mathbf{t}_a is the prescribed mechanical traction force. This equation can be recast in Lagrangian form as

$$\mathbf{T}^T \mathbf{N} = \mathbf{t}_A + \mathbf{t}_E, \quad (173)$$

where the connection between \mathbf{t}_E and the Maxwell stress $\boldsymbol{\tau}_m$ is given following equation (138).

Note that exact solutions are available only for a very limited number of boundary-value problems, some of which are discussed in detail by Dorfmann and Ogden (2005, 2006). The main problem arises, when an electric field is present as compared to the purely elastic case, in the difficulty of meeting the boundary conditions for \mathbf{E} and \mathbf{D} for bodies with finite geometry. In particular, the boundary conditions on the ends of a tube of finite length are not in general compatible with those of the lateral surfaces; see Bustamante et al. (2007) for discussion of this in the magnetoelastic context. For this reason, we consider here an infinitely long tube, for which the boundary conditions on the cylindrical surfaces can be satisfied exactly.

We now specialize the field equations to problems for which the cylindrical symmetry is maintained during deformation. It is convenient to work in terms of cylindrical polar coordinates, which in the reference configuration are denoted by (R, Θ, Z) and in the deformed configuration by (r, θ, z) . The components of equations (166)_{2,3} are

$$\frac{1}{R} \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = 0, \quad \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = 0, \quad \frac{1}{r} \frac{\partial(rE_\theta)}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \theta} = 0, \quad (174)$$

and

$$\frac{\partial D_r}{\partial r} + \frac{D_r}{r} + \frac{1}{r} \frac{\partial D_\theta}{\partial \theta} + \frac{\partial D_z}{\partial z} = 0, \quad (175)$$

where (E_r, E_θ, E_z) are the components of the electric field vector \mathbf{E} and similarly (D_r, D_θ, D_z) are those of the electric displacement \mathbf{D} . We select to work in terms of $\Omega^*(\mathbf{F}, \mathbf{D}_l)$ with \mathbf{D}_l as the independent electric variable that satisfies $\text{Div} \mathbf{D}_l = 0$. It follows that the electric displacement $\mathbf{D} = \mathbf{F} \mathbf{D}_l$ in the deformed configuration automatically satisfies the corresponding equation $\text{div} \mathbf{D} = 0$; see equation (94)₁. The electric field, given by the constitutive formulation (162), must satisfy the field equation $\text{curl} \mathbf{E} = \mathbf{0}$, which may put restriction on the class of constitutive laws; details are given in Dorfmann and Ogden (2005).

With reference to a circular cylindrical tube of infinite length, the components of the electric field and electric displacement are independent of θ and z . The above equations simplify and, after integration, become

$$E_z = \text{constant}, \quad rE_\theta = \text{constant}, \quad rD_r = \text{constant}, \quad (176)$$

which show that no restrictions are imposed on the components E_r , D_θ and D_z . Similarly, the components of the equilibrium equation (166)₁ simplify, and those not automatically satisfied are

$$r \frac{d\tau_{rr}}{dr} = \tau_{\theta\theta} - \tau_{rr}, \quad \frac{d}{dr}(r^2\tau_{r\theta}) = 0, \quad \frac{d}{dr}(r\tau_{rz}) = 0. \quad (177)$$

Axial shear. Consider a circular cylindrical tube whose reference geometry is defined by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad -\infty < Z < \infty, \quad (178)$$

where A, B are positive constants defining the inner and outer radii, respectively. Let the inner surface of the cylinder be bonded to a rigid cylinder and apply a uniform axial shear to the outer surface of the tube. The resulting deformation is described by the equations

$$r = R, \quad \theta = \Theta, \quad z = Z + u(R), \quad (179)$$

where $u(R)$ is a function of R that has to be determined by the solution of the governing equations and application of the boundary conditions. These are

$$u(a) = 0, \quad u(b) = d, \quad (180)$$

which corresponds to the inner boundary held fixed and the outer displaced by an amount d . The components of the deformation gradient \mathbf{F} , referred to cylindrical polar coordinate axes and represented by the matrix \mathbf{F} , are given by

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u'(r) & 0 & 1 \end{pmatrix}, \quad (181)$$

where we now regard u as a function of $r (= R)$; correspondingly, we write $a = A, b = B$. We also use the notation $\gamma = u'(r)$ and note that the deformation is locally a *simple shear* with amount of shear γ . Note, however, that γ is not constant but depends on r : $\gamma = \gamma(r)$. Then, the left and right Cauchy–Green tensors $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{c} = \mathbf{F}^T\mathbf{F}$ have the matrix

representations

$$\mathbf{b} = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ \gamma & 0 & 1 + \gamma^2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 + \gamma^2 & 0 & \gamma \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix}. \quad (182)$$

The forms of the principal invariants I_1, I_2, I_3 associated with \mathbf{c} (equivalently \mathbf{b}) are defined in (155). For an incompressible material and for the deformation here considered, these become

$$I_1 = I_2 = 3 + \gamma^2, \quad I_3 \equiv 1. \quad (183)$$

Radial electric displacement field. Consider an electric displacement vector with components in Lagrangian form given by $(D_{lR}, 0, 0)$. The invariants K_4, K_5, K_6 , shown in (160), become

$$K_4 = D_{lR}^2, \quad K_5 = (1 + \gamma^2)K_4, \quad K_6 = (1 + 3\gamma^2 + \gamma^4)K_4. \quad (184)$$

In the deformed configuration, the components of the electric displacement vector \mathbf{D} are calculated from $\mathbf{D} = \mathbf{F}\mathbf{D}_l$ and are

$$D_r = D_{lR}, \quad D_\theta = 0, \quad D_z = \gamma D_{lR}, \quad (185)$$

and from equation (176)₃ we find

$$D_r = D_{lR} = \frac{D_b b}{r}, \quad (186)$$

where D_b is the value of D_r at the boundary $r = b$. The components of the electric field \mathbf{E} are given by equation (162) as

$$\begin{aligned} E_r &= 2 [\Omega_4^* + \Omega_5^* + (1 + \gamma^2)\Omega_6^*] D_{lR}, \\ E_z &= 2 [\gamma\Omega_5^* + \gamma(2 + \gamma^2)\Omega_6^*] D_{lR}, \end{aligned} \quad (187)$$

with $E_\theta = 0$. The polarization \mathbf{P} is given by equation (101) and has non-zero components

$$\begin{aligned} P_r &= [1 - 2\varepsilon_0 (\Omega_4^* + \Omega_5^* + (1 + \gamma^2)\Omega_6^*)] D_{lR}, \\ P_z &= [1 - 2\varepsilon_0 (\Omega_5^* + (2 + \gamma^2)\Omega_6^*)] \gamma D_{lR}. \end{aligned} \quad (188)$$

Stress components and equilibrium. In the space surrounding the material the Maxwell stress is given by (164), and its non-zero components are

$$\tau_{rr}^* = \frac{1}{2}\varepsilon_0[(E_r^*)^2 - (E_z^*)^2], \quad \tau_{rz}^* = \varepsilon_0 E_r^* E_z^*, \quad \tau_{zz}^* = -\frac{1}{2}\varepsilon_0[(E_r^*)^2 - (E_z^*)^2], \quad (189)$$

where the superscript $*$ is again used to indicate that the fields are calculated outside the material body. The total stress tensor $\boldsymbol{\tau}$ inside an incompressible material is given by the constitutive equation (161). The non-zero components have the form

$$\begin{aligned}\tau_{rr} &= -p + 2\Omega_1^* + 4\Omega_2^* + 2K_4 [\Omega_5^* + 2(1 + \gamma^2)\Omega_6^*], \\ \tau_{\theta\theta} &= -p + 2\Omega_1^* + 2(2 + \gamma^2)\Omega_2^*, \\ \tau_{zz} &= -p + 2(1 + \gamma^2)\Omega_1^* + 2(2 + \gamma^2)\Omega_2^* + 2K_4 [\gamma^2\Omega_5^* + 2(2\gamma^2 + \gamma^4)\Omega_6^*], \\ \tau_{rz} &= 2\gamma(\Omega_1^* + \Omega_2^*) + 2\gamma K_4 [\Omega_5^* + (3 + 2\gamma^2)\Omega_6^*],\end{aligned}\tag{190}$$

and the remaining two shear components $\tau_{\theta z}$ and $\tau_{r\theta}$ vanish. The first component of the equilibrium equation (177)₁, using the expressions for τ_{rr} and $\tau_{\theta\theta}$, becomes

$$r \frac{d\tau_{rr}}{dr} = 2\gamma^2\Omega_2^* - 2K_4 [\Omega_5^* + 2\Omega_6^*(1 + \gamma^2)],\tag{191}$$

and can be used to determine the Lagrange multiplier p for given γ and K_4 and specified boundary conditions. The second component of the equilibrium equations, involving $\tau_{r\theta}$, is satisfied automatically.

Equation (183) shows that the invariants I_1, I_2 depend on γ only, and from (184) we find that the invariants K_5, K_6 are given in terms of K_4 and γ . It is therefore convenient to define a reduced energy function, denoted ω^* , and given by

$$\omega^*(\gamma, K_4) = \Omega^*(I_1, I_2, K_4, K_5, K_6).\tag{192}$$

The expression for the stress τ_{rz} can now be written in the simplified form as $\tau_{rz} = \omega_\gamma^*$, where $\omega_\gamma^* = \partial\omega^*/\partial\gamma$. Integration of the third equilibrium equation (177)₃ and using the simplified expression for τ_{rz} gives

$$\tau_{rz} = \frac{\partial\omega^*}{\partial\gamma} = \frac{\tau_z b}{r},\tag{193}$$

where τ_z denotes the value of τ_{rz} on the boundary $r = b$. This equation, for a known function ω^* , can be integrated to give γ and used to determine the unknown displacement function $u(r)$. Recall that the invariant K_4 depends on D_{IR} , which is given by equation (186).

Finally, we recall that equation (176)₁ requires E_z to be constant. Therefore, from (187)₂ we have the condition

$$\gamma [\Omega_5^* + (2 + \gamma^2)\Omega_6^*] = cr\tag{194}$$

where c is a constant. This restriction can be satisfied, for example, by selecting an energy function independent of K_5 and K_6 , in which case $c = 0$.

Illustration. To illustrate the effect of an electric field on the response of an electro-sensitive material, we consider the energy function

$$\Omega^*(I_1, K_4) = \frac{\mu(K_4)}{k} \left[\frac{(I_1 - 1)^k}{2^k} - 1 \right] + \nu(K_4), \quad (195)$$

where μ and ν are functions of K_4 and k is a constant such that $k \geq 1/2$. The function ν describes the energy in the material associated with an electric field when the deformation $\gamma = 0$. It must vanish in the absence of an applied electric field, i.e. $\nu(0) = 0$. The form of (195) is independent of K_5 and K_6 and therefore satisfies the restriction imposed by equation (194).

Using equation (193), we calculate the associate shear stress

$$\mu(K_4)\gamma \left(\frac{2 + \gamma^2}{2} \right)^{k-1} = \frac{\tau_z b}{r}, \quad K_4 = \frac{D_b^2 b^2}{r^2}, \quad (196)$$

where we recall that $\gamma = u'(r)$. For any suitable choice of the function μ and for given k this equation can in principle be solved for γ and hence by integration for the displacement function $u(r)$ by integration of $u'(r) = \gamma$. In order to permit an explicit solution we restrict attention to the case $k = 1$ here, so that (196) simplifies to

$$\gamma = \frac{\tau_z b}{r\mu(K_4)}. \quad (197)$$

Next we specialize the function μ to be linear and write

$$\mu(K_4) = \mu(0) + \mu_1 K_4, \quad (198)$$

where $\mu(0)$ is the elastic shear modulus (in the absence of the electric field) and μ_1 is a positive material constant.

If we now introduce the dimensionless variables $\bar{\tau} = \tau_z/\mu(0)$ and $\bar{\mu}_1 = \mu_1/\mu(0)$ then equation (197) may be solved explicitly, on use of the boundary condition (180)₁, to give the displacement function

$$u(r) = \frac{1}{2}\bar{\tau}b \log \left(\frac{r^2 + \bar{\mu}_1 D_b^2 b^2}{a^2 + \bar{\mu}_1 D_b^2 b^2} \right). \quad (199)$$

Using the displacement boundary condition $u(b) = d$ prescribed on the outer surface of the tube, see (180)₂, we obtain the (shear) stress-displacement relation

$$d = \frac{1}{2}\bar{\tau}b \log \left(\frac{b^2 + \bar{\mu}_1 D_b^2 b^2}{a^2 + \bar{\mu}_1 D_b^2 b^2} \right). \quad (200)$$

In the absence of the electric field this reduces to the standard formula

$$d = \bar{\tau}b \log(b/a), \quad (201)$$

which is a special case of the more general solution derived in the purely elastic context by Jiang and Ogden (2000). For given values of a, b and $\bar{\mu}_1$ equation (200) shows how this relation is changed by the application of an electric field. In particular, for the chosen form of μ , the stress response of the material stiffens as the magnitude of the electric displacement $|D_b|$ increases.

5 Incremental Equations

5.1 Increments within the Material

We now examine the effect of an incremental deformation combined with an increment in the electric displacement superimposed on the configuration \mathcal{B} and the corresponding increment in the latter exterior to the material. Let increments be signified by superposed dots. Then, for example, $\dot{\mathbf{T}}$, $\dot{\mathbf{D}}_l$ and $\dot{\mathbf{E}}_l$ are increments in \mathbf{T} , \mathbf{D}_l and \mathbf{E}_l , respectively. The incremental forms of the equations (137) and (105) are

$$\text{Div } \dot{\mathbf{T}} = \mathbf{0}, \quad \text{Div } \dot{\mathbf{D}}_l = 0, \quad \text{Curl } \dot{\mathbf{E}}_l = \mathbf{0}, \quad (202)$$

in the first and third of which the (linearized) incremental forms of the constitutive equations (149) are required. For an unconstrained material these are given by

$$\dot{\mathbf{T}} = \mathcal{A}\dot{\mathbf{F}} + \Gamma\dot{\mathbf{D}}_l, \quad \dot{\mathbf{E}}_l = \Gamma^T\dot{\mathbf{F}} + \mathcal{K}\dot{\mathbf{D}}_l, \quad (203)$$

where $\dot{\mathbf{F}}$ is the increment in \mathbf{F} and \mathcal{A} , Γ and \mathcal{K} are, respectively, fourth-, third- and second-order tensors, which we refer to as *electroelastic moduli tensors*. Their components are defined by

$$\mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 \Omega^*}{\partial F_{i\alpha} \partial F_{j\beta}}, \quad \Gamma_{\alpha i \beta} = \frac{\partial^2 \Omega^*}{\partial F_{i\alpha} \partial D_{l\beta}}, \quad \mathcal{K}_{\alpha \beta} = \frac{\partial^2 \Omega^*}{\partial D_{l\alpha} \partial D_{l\beta}}, \quad (204)$$

which have the symmetries

$$\mathcal{A}_{\alpha i \beta j} = \mathcal{A}_{\beta j \alpha i}, \quad \mathcal{K}_{\alpha \beta} = \mathcal{K}_{\beta \alpha}. \quad (205)$$

Note that Γ has no corresponding indicial symmetry, although the order of the mixed derivatives in (204)₂ may be reversed. The products in (203) are defined, in component form, by

$$\dot{T}_{\alpha i} = \mathcal{A}_{\alpha i \beta j} \dot{F}_{j\beta} + \Gamma_{\alpha i \beta} \dot{D}_{l\beta}, \quad \dot{E}_{l\alpha} = \Gamma_{\beta i \alpha} \dot{F}_{i\beta} + \mathcal{K}_{\alpha \beta} \dot{D}_{l\beta}. \quad (206)$$

For an isotropic electroelastic material with no mechanical constraint, Ω^* is a function of the six invariants I_1, I_2, I_3 and K_4, K_5, K_6 that have been defined in (155) and (160), respectively. For convenience of notation, we now denote K_4, K_5, K_6 by I_4, I_5, I_6 such that

$$I_4 = \mathbf{D}_l \cdot \mathbf{D}_l, \quad I_5 = (\mathbf{cD}_l) \cdot \mathbf{D}_l, \quad I_6 = (\mathbf{c}^2\mathbf{D}_l) \cdot \mathbf{D}_l. \quad (207)$$

The expressions (204) can then be expanded in the forms

$$\begin{aligned} \mathcal{A}_{\alpha i \beta j} &= \sum_{m=1, m \neq 4}^6 \sum_{n=1, n \neq 4}^6 \Omega_{mn}^* \frac{\partial I_m}{\partial F_{i\alpha}} \frac{\partial I_n}{\partial F_{j\beta}} + \sum_{n=1, n \neq 4}^6 \Omega_n^* \frac{\partial^2 I_n}{\partial F_{i\alpha} \partial F_{j\beta}}, \\ \Gamma_{\alpha i \beta} &= \sum_{m=4}^6 \sum_{n=1, n \neq 4}^6 \Omega_{mn}^* \frac{\partial I_m}{\partial D_{l\beta}} \frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^6 \Omega_n^* \frac{\partial^2 I_n}{\partial F_{i\alpha} \partial D_{l\beta}}, \\ \mathcal{K}_{\alpha \beta} &= \sum_{m=4}^6 \sum_{n=4}^6 \Omega_{mn}^* \frac{\partial I_m}{\partial D_{l\alpha}} \frac{\partial I_n}{\partial D_{l\beta}} + \sum_{n=4}^6 \Omega_n^* \frac{\partial^2 I_n}{\partial D_{l\alpha} \partial D_{l\beta}}, \end{aligned} \quad (208)$$

where $\Omega_n^* = \partial \Omega^* / \partial I_n$, $\Omega_{mn}^* = \partial^2 \Omega^* / \partial I_m \partial I_n$. Expressions, in component form, for the first and second derivatives of $I_n, n = 1, \dots, 6$ with respect to \mathbf{F} and \mathbf{D}_l are given in Dorfmann and Ogden (2010a).

For an *incompressible* material, \mathbf{T} is given by (154)₂ and its increment is then

$$\dot{\mathbf{T}} = \mathcal{A}\dot{\mathbf{F}} + \Gamma\dot{\mathbf{D}}_l - p\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \quad (209)$$

which replaces (203)₁ in this case, while the expression (203)₂ for $\dot{\mathbf{E}}_l$ remains in force.

Let $\dot{\mathbf{T}}_0, \dot{\mathbf{D}}_{l0}, \dot{\mathbf{E}}_{l0}$ denote the ‘push forward’ versions of $\dot{\mathbf{T}}, \dot{\mathbf{D}}_l, \dot{\mathbf{E}}_l$, respectively. These are given by

$$\dot{\mathbf{T}}_0 = J^{-1}\mathbf{F}\dot{\mathbf{T}}, \quad \dot{\mathbf{D}}_{l0} = J^{-1}\mathbf{F}\dot{\mathbf{D}}_l, \quad \dot{\mathbf{E}}_{l0} = \mathbf{F}^{-T}\dot{\mathbf{E}}_l. \quad (210)$$

Then, equations (202) can be transformed into their Eulerian counterparts as

$$\operatorname{div} \dot{\mathbf{T}}_0 = \mathbf{0}, \quad \operatorname{div} \dot{\mathbf{D}}_{l0} = 0, \quad \operatorname{curl} \dot{\mathbf{E}}_{l0} = \mathbf{0}. \quad (211)$$

It is convenient now to use the notation \mathbf{u} for the incremental displacement $\dot{\mathbf{x}}$, with \mathbf{u} treated as a function of \mathbf{x} , so that $\dot{\mathbf{F}} = (\operatorname{grad} \mathbf{u})\mathbf{F}$. Let $\mathbf{d} = \operatorname{grad} \mathbf{u}$, with components defined by $d_{ij} = \partial u_i / \partial x_j$. Then, the incremental constitutive equations (203) can be re-cast in the forms

$$\dot{\mathbf{T}}_0 = \mathcal{A}_0 \mathbf{d} + \Gamma_0 \dot{\mathbf{D}}_{l0}, \quad \dot{\mathbf{E}}_{l0} = \Gamma_0^T \mathbf{d} + \mathcal{K}_0 \dot{\mathbf{D}}_{l0}, \quad (212)$$

where, in index notation, the tensors \mathcal{A}_0 , Γ_0 , and \mathcal{K}_0 are defined by

$$\begin{aligned}\mathcal{A}_{0jisk} &= J^{-1}F_{j\alpha}F_{s\beta}\mathcal{A}_{\alpha i\beta k}, \\ \Gamma_{0jik} &= F_{j\alpha}F_{\beta k}^{-1}\Gamma_{\alpha i\beta}, \\ \mathcal{K}_{0ij} &= JF_{\alpha i}^{-1}F_{\beta j}^{-1}\mathcal{K}_{\alpha\beta}\end{aligned}\quad (213)$$

for an unconstrained material. For an incompressible material $J = 1$ in (213), and (212) is replaced by

$$\dot{\mathbf{T}}_0 = \mathcal{A}_0\mathbf{d} + \Gamma_0\dot{\mathbf{D}}_{l0} + p\mathbf{d} - \dot{p}\mathbf{I}, \quad \dot{\mathbf{E}}_{l0} = \Gamma_0^T\mathbf{d} + \mathcal{K}_0\dot{\mathbf{D}}_{l0}, \quad (214)$$

and \mathbf{u} satisfies the incremental incompressibility condition

$$\operatorname{div} \mathbf{u} = 0. \quad (215)$$

We note that the symmetries of \mathcal{A} and \mathcal{K} carry over to \mathcal{A}_0 and \mathcal{K}_0 in the form

$$\mathcal{A}_{0jisk} = \mathcal{A}_{0skji}, \quad \mathcal{K}_{0ij} = \mathcal{K}_{0ji}, \quad (216)$$

while Γ_0 has the symmetry

$$\Gamma_{0ijk} = \Gamma_{0jik}, \quad (217)$$

which can be established by using the incremental form of the symmetry condition $\mathbf{FT} = (\mathbf{FT})^T$, as can the connections

$$\mathcal{A}_{0jisk} - \mathcal{A}_{0ijsk} = \tau_{js}\delta_{ik} - \tau_{is}\delta_{jk} \quad (218)$$

between the components of the tensors \mathcal{A}_0 and $\boldsymbol{\tau}$ for an unconstrained material. The corresponding connections for an incompressible material are

$$\mathcal{A}_{0jisk} - \mathcal{A}_{0ijsk} = (\tau_{js} + p\delta_{js})\delta_{ik} - (\tau_{is} + p\delta_{is})\delta_{jk}. \quad (219)$$

The latter two equations generalize to the electroelastic situation results that hold for a purely elastic material as shown by Chadwick and Ogden (1971) and Chadwick (1997), and are identical to corresponding formulas for a magnetoelastic material (Otténio et al., 2008).

We next decompose \mathcal{A}_0 as the sum

$$\mathcal{A}_0 = \mathcal{A}_0^{(0)} + \mathcal{A}_0^{(5)} + \mathcal{A}_0^{(6)}. \quad (220)$$

None of these terms involves any derivatives of Ω^* with respect to I_4 , while $\mathcal{A}_0^{(0)}$ does not involve derivatives with respect to I_5 or I_6 and is given in

component form by

$$\begin{aligned}
J\mathcal{A}_{0jisk}^{(0)} &= 2\Omega_1^*\delta_{ik}b_{js} + 2\Omega_2^*(2b_{ij}b_{ks} + \delta_{ik}\mathcal{N}_{js} - b_{jk}b_{is} - b_{ik}b_{js}) \\
&+ 2J^2\Omega_3^*(2\delta_{ij}\delta_{ks} - \delta_{is}\delta_{jk}) + 4\Omega_{11}^*b_{ij}b_{ks} + 4\Omega_{22}^*\mathcal{N}_{ij}\mathcal{N}_{ks} \\
&+ 4\Omega_{12}^*(b_{ks}\mathcal{N}_{ij} + b_{ij}\mathcal{N}_{ks}) + 4J^2\Omega_{13}^*(b_{ks}\delta_{ij} + b_{ij}\delta_{ks}) \\
&+ 4J^2\Omega_{23}^*(\mathcal{N}_{ks}\delta_{ij} + \mathcal{N}_{ij}\delta_{ks}) + 4J^4\Omega_{33}^*\delta_{ij}\delta_{ks}, \tag{221}
\end{aligned}$$

where

$$\mathcal{N}_{ij} = b_{kk}b_{ij} - b_{ik}b_{kj} \tag{222}$$

and b_{ij} are the components of \mathbf{b} .

The terms $\mathcal{A}_{0jisk}^{(5)}$ and $\mathcal{A}_{0jisk}^{(6)}$ do involve derivatives with respect to I_5 and I_6 and have the component forms

$$\begin{aligned}
\mathcal{A}_{0jisk}^{(5)} &= \mathcal{A}_{0jisk}^{0(5)}\Omega_5^* + \sum_{m=1, m \neq 4}^6 \mathcal{A}_{0jisk}^{m(5)}\Omega_{m5}^*, \\
\mathcal{A}_{0\alpha i \beta j}^{(6)} &= \mathcal{A}_{0jisk}^{0(6)}\Omega_6^* + \sum_{m=1, m \neq 4}^6 \mathcal{A}_{0jisk}^{m(6)}\Omega_{m6}^*, \tag{223}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_{0jisk}^{0(5)} &= 2J^{-1}a_ja_s\delta_{ik}, \quad \mathcal{A}_{0jisk}^{1(5)} = 4J^{-1}(a_ia_jb_{ks} + a_ka_s b_{ij}), \\
\mathcal{A}_{0jisk}^{2(5)} &= 4J^{-1}(a_ia_j\mathcal{N}_{ks} + a_ka_s\mathcal{N}_{ij}), \quad \mathcal{A}_{0jisk}^{3(5)} = 4J(a_ia_j\delta_{ks} + a_ka_s\delta_{ij}), \\
\mathcal{A}_{0jisk}^{5(5)} &= 4J^{-1}a_ia_ja_s a_k, \quad \mathcal{A}_{0jisk}^{6(5)} = 4J^{-1}(a_ia_j\mathcal{H}_{ks} + a_ka_s\mathcal{H}_{ij}), \tag{224}
\end{aligned}$$

with \mathcal{H}_{ij} and a_i defined by

$$\mathcal{H}_{ij} = b_{im}a_m a_j + b_{jm}a_m a_i, \quad a_i = F_{i\alpha}D_{l\alpha}. \tag{225}$$

Similarly, we have

$$\begin{aligned}
\mathcal{A}_{0jisk}^{0(6)} &= 2J^{-1}(\delta_{ik}\mathcal{H}_{js} + a_ia_k b_{js} + a_ia_s b_{jk} + a_ja_k b_{is} + a_ja_s b_{ik}), \\
\mathcal{A}_{0jisk}^{1(6)} &= 4J^{-1}(b_{ks}\mathcal{H}_{ij} + b_{ij}\mathcal{H}_{ks}), \quad \mathcal{A}_{0jisk}^{2(6)} = 4J^{-1}(\mathcal{H}_{ij}\mathcal{N}_{ks} + \mathcal{H}_{ks}\mathcal{N}_{ij}), \\
\mathcal{A}_{0jisk}^{3(6)} &= 4J(\mathcal{H}_{ks}\delta_{ij} + \mathcal{H}_{ij}\delta_{ks}), \quad \mathcal{A}_{0jisk}^{5(6)} = 4J^{-1}(a_ia_j\mathcal{H}_{ks} + a_ka_s\mathcal{H}_{ij}), \\
\mathcal{A}_{0jisk}^{6(6)} &= 4J^{-1}\mathcal{H}_{ij}\mathcal{H}_{ks}. \tag{226}
\end{aligned}$$

We decompose $\mathbf{\Gamma}_0$ in the form

$$\mathbf{\Gamma}_0 = \mathbf{\Gamma}_0^{(1)} + \mathbf{\Gamma}_0^{(2)} + \mathbf{\Gamma}_0^{(3)} + \mathbf{\Gamma}_0^{(5)} + \mathbf{\Gamma}_0^{(6)}, \tag{227}$$

the constituents of which have components

$$\begin{aligned}\Gamma_{0jik}^{(1)} &= 4b_{ij}\mathcal{M}_{1k}, & \Gamma_{0jik}^{(2)} &= 4\mathcal{N}_{ij}\mathcal{M}_{2k}, & \Gamma_{0jik}^{(3)} &= 4J^2\delta_{ij}\mathcal{M}_{3k}, \\ \Gamma_{0jik}^{(5)} &= 4a_i a_j \mathcal{M}_{5k} + 2\Omega_5^*(a_j\delta_{ik} + a_i\delta_{jk}), \\ \Gamma_{0jik}^{(6)} &= 4\mathcal{H}_{ij}\mathcal{M}_{6k} + 2\Omega_6^*(\delta_{ik}a_m b_{jm} + a_i b_{jk} + \delta_{jk}a_m b_{im} + a_j b_{ik}),\end{aligned}\tag{228}$$

where

$$\mathcal{M}_{ik} = (\Omega_{i4}^* F_{\beta k}^{-1} D_{l\beta} + \Omega_{i5}^* a_k + \Omega_{i6}^* b_{km} a_m).\tag{229}$$

Finally, we decompose \mathcal{K}_0 as

$$\mathcal{K}_0 = \mathcal{K}_0^{(4)} + \mathcal{K}_0^{(5)} + \mathcal{K}_0^{(6)},\tag{230}$$

where the components are given by

$$\begin{aligned}\mathcal{K}_{0ij}^{(4)} &= 2J(2F_{\alpha i}^{-1} D_{l\alpha} \mathcal{M}_{4j} + b_{ij}^{-1} \Omega_4^*), & \mathcal{K}_{0ij}^{(5)} &= 2J(2a_i \mathcal{M}_{5j} + \delta_{ij} \Omega_5^*), \\ \mathcal{K}_{0ij}^{(6)} &= 2J(2b_{in} a_n \mathcal{M}_{6j} + b_{ij} \Omega_6^*).\end{aligned}\tag{231}$$

For an incompressible material, we have $J = 1$ in all the above expressions and the terms Ω_3^* and Ω_{n3}^* , $n = 1, \dots, 6$, are dropped.

5.2 Exterior Incremental Fields

The relation $\mathbf{D}^* = \varepsilon_0 \mathbf{E}^*$, which is valid in vacuum and non-polarizable materials, has the incremental form

$$\dot{\mathbf{D}}^* = \varepsilon_0 \dot{\mathbf{E}}^*,\tag{232}$$

where $\dot{\mathbf{D}}^*$ and $\dot{\mathbf{E}}^*$ are the increments of \mathbf{D}^* and \mathbf{E}^* , respectively. The associated incremental Maxwell equations are

$$\operatorname{div} \dot{\mathbf{D}}^* = 0, \quad \operatorname{curl} \dot{\mathbf{E}}^* = \mathbf{0},\tag{233}$$

and the increment of the Maxwell stress (164) has the form

$$\dot{\boldsymbol{\tau}}^* = \varepsilon_0 [\dot{\mathbf{E}}^* \otimes \mathbf{E}^* + \mathbf{E}^* \otimes \dot{\mathbf{E}}^* - (\mathbf{E}^* \cdot \dot{\mathbf{E}}^*) \mathbf{I}],\tag{234}$$

which satisfies the equilibrium equation $\operatorname{div} \dot{\boldsymbol{\tau}}^* = \mathbf{0}$.

5.3 Incremental Boundary Conditions

On the boundary \mathcal{B} of the material, in addition to any applied traction \mathbf{t}_a (defined per unit area of $\partial\mathcal{B}$), there will in general be a contribution from

the electric Maxwell stress; see equation (108). Explicitly, this is a force $\boldsymbol{\tau}^* \mathbf{n}$ per unit current area, where $\boldsymbol{\tau}^*$ is given by (164). Recall, that on use of Nanson's formula this can be written as $\mathbf{t}_E = J \boldsymbol{\tau}^* \mathbf{F}^{-T} \mathbf{N}$ per unit reference area. Then, the boundary condition (138) has the explicit form

$$\mathbf{T}^T \mathbf{N} = \mathbf{t}_A + J \boldsymbol{\tau}^* \mathbf{F}^{-T} \mathbf{N} \quad (235)$$

on $\partial \mathcal{B}_0$. On taking the increment of this equation, we obtain

$$\dot{\mathbf{T}}^T \mathbf{N} = \dot{\mathbf{t}}_A + J \dot{\boldsymbol{\tau}}^* \mathbf{F}^{-T} \mathbf{N} - J \boldsymbol{\tau}^* \mathbf{F}^{-T} \dot{\mathbf{F}}^T \mathbf{F}^{-T} \mathbf{N} + \dot{J} \boldsymbol{\tau}^* \mathbf{F}^{-T} \mathbf{N} \quad (236)$$

on $\partial \mathcal{B}_0$, or, in Eulerian form,

$$\dot{\mathbf{T}}_0^T \mathbf{n} = \dot{\mathbf{t}}_A + \dot{\boldsymbol{\tau}}^* \mathbf{n} - \boldsymbol{\tau}^* \mathbf{d}^T \mathbf{n} + (\operatorname{div} \mathbf{u}) \boldsymbol{\tau}^* \mathbf{n} \quad (237)$$

on $\partial \mathcal{B}$, wherein we have used the standard formula $\dot{J} = J \operatorname{div} \mathbf{u}$.

Using the notation introduced in this section, we rewrite the boundary conditions for the electric field variables (106) in the equivalent form

$$(\mathbf{D}_l - J \mathbf{F}^{-1} \mathbf{D}^*) \cdot \mathbf{N} = 0 \quad (\mathbf{E}_l - \mathbf{F}^T \mathbf{E}^*) \times \mathbf{N} = \mathbf{0}. \quad (238)$$

On incrementing these conditions, we obtain

$$[\dot{\mathbf{D}}_{l0} - \dot{\mathbf{D}}^* + \mathbf{d} \mathbf{D}^* - (\operatorname{div} \mathbf{u}) \mathbf{D}^*] \cdot \mathbf{n} = 0 \quad (239)$$

and

$$(\dot{\mathbf{E}}_{l0} - \dot{\mathbf{E}}^* - \mathbf{d}^T \mathbf{E}^*) \times \mathbf{n} = \mathbf{0}, \quad (240)$$

both of which hold on $\partial \mathcal{B}$.

The governing equations describing the linearized response of electroelastic solids superimposed on a state of finite deformation in the presence of an electric field have been specialized in Dorfmann and Ogden (2010a) to evaluate the surface stability of an electrostatic half-space. The forms of the incremental equations and the expressions of the electrostatic moduli tensors are also applied to the analysis of waves in isotropic electroelastic materials. In particular, the analysis of plane waves propagating in a homogeneously deformed material with an underlying uniform electric field is given in Dorfmann and Ogden (2010b). In addition, the incremental equations are used to evaluate surface waves in a homogeneously deformed half-space of incompressible isotropic material in the presence of an electric field. In particular, the dependence of the wave speed on the deformation, the electric field and the electromechanical coupling parameters is evaluated numerically for a neo-Hookean electrostatic material. Due to space limitations a detailed discussion on the use of the incremental equations

will not be given here. The interested reader is referred to the references cited above.

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Magnetostatics: from Basic Principles to Nonlinear Interactions in Deformable Media

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Abstract. In these notes we provide a development of the basic principles of the classical theory of magnetostatics, from the fundamental notions of magnetic dipoles through to distributions of current in a non-deformable continuum, the equations governing the magnetic field and magnetic induction vectors in free space and in a magnetizable material, and then to the modifications of the theory required to account for the deformability of material media. A review of the relevant continuum mechanics is included as a prelude to the description of large magnetoelastic deformations. The constitutive equations for a nonlinear magnetoelastic material are presented first in Eulerian form and then an alternative formulation of the equations based on a Lagrangian approach is adopted, which leads to an elegant and relatively simple structure for the constitutive equations and the governing differential equations. The theory is specialized further to the case of an isotropic magnetoelastic material and representative prototype boundary-value problems are formulated and then solved using a simple model constitutive law in order to illustrate the nonlinear magnetoelastic coupling.

1 Introduction

Anyone who has ever played with a permanent magnet has been intrigued by how metal objects are attracted to it. The force of attraction acts not only on the object as a whole, but on each bit of material, inducing a change in shape and/or size of the object commonly known as *magnetostriction*. For typical metals this change is very small and the associated variations in the magnetic and mechanical properties of the material can be neglected. Only recently, however, have researchers come to appreciate the profound potential of multi-functional compliant magneto-sensitive materials as new polymer-based materials have been synthesized. These mechanically soft

materials possess high magneto-mechanical compliance and, unlike conventional magnetic metals, are capable of large elastic deformations under the influence of an external magnetic field, much larger than in conventional magnetostriction. The new materials are highly deformable and magnetizable polymers, typically elastomers composed of a rubber-like base matrix embedded with micron-sized magneto-active particles. Like a typical rubber, they have low mechanical stiffness and are very compliant, especially in low-dimensional structures such as membranes and rods, while demonstrating good magnetic susceptibility. The small particle size ensures that the materials are effectively homogeneous, and the material processing has already been advanced to the point where robust material characteristics can be achieved.

The nonlinearity in the response and the magneto-mechanical coupling of these materials opens the door for many new devices, offering a range of applications that could not be addressed with previously available materials. The nonlinearity is key. In his classic textbook on electrodynamics (Jackson, 1999) J. D. Jackson states that *“In substances other than ferromagnets, for weak enough fields the presence of an applied magnetic field induces a magnetization proportional to the magnitude of the applied field. We then say that the response of the medium is linear”*. In other words, the linear theory of magnetoelasticity, applicable to infinitesimal deformations and weak fields, neglects the magneto-mechanical coupling in the sense that there is no change in mechanical properties due to the applied magnetic field and no change in the magnetic properties due to mechanical deformations, i.e. there no change in the material constants. The availability of materials that can operate in a highly nonlinear magneto-mechanical regime offers very exciting possibilities and challenges from the perspectives of device design, materials science, constitutive modelling and magneto-mechanical theory.

At present the influence of magnetic fields on the behaviour of magneto-sensitive materials in the highly nonlinear regime is not well understood and the development of an appropriate theoretical framework is essential to further that understanding. While the extension of the theory of the magnetism of continuous media to highly deformable systems seems natural from an academic point of view, it has languished undeveloped because there has been no practical motivation hitherto. The materials did not exist! Recently, progress has been made in constructing a theoretical framework for the analysis of these materials, some aspects of which are described in this chapter. The theory requires further development so that it can describe accurately the nonlinear magneto-mechanical coupling when large deformations are involved. The theory of large deformations is of fundamental

interest, both in terms of the unique properties offered by magneto-sensitive elastomers and in terms of potential applications to, for example, sensors and controllable devices.

This chapter provides a basic framework for the analysis of large magnetoelastic deformations. It begins, in Section 2, with an overview of the fundamental principles of the classical theory of magnetostatics. For a more complete account we refer to Jackson (1999), for example. In electrostatics the fundamental unit from which the theory is built is the charged particle. There is no counterpart of this in the case of magnetostatics, i.e. magnetic ‘particles’ do not exist (or at least have not as yet been isolated), and the fundamental unit is the magnetic dipole, which is equivalent to a small current carrying circuit. Starting from the basic physics that gives the magnetic field due to a *moving* charged particle we construct the magnetostatic field generated by a continuous distribution of moving charge, which can be identified as a current density, leading to an explicit formula for the magnetic induction in terms of the current density known as the *Biot-Savart Law*, a variant of which enables a connection to be made between a magnetic dipole and the aforementioned small circuit of current. These results are then used to obtain the two differential equations that govern the magnetic induction vector in continuous media.

The magnetic field vector is introduced and the notion of magnetization in material media is discussed with particular reference to a linear (non-deformable) magnetic material. The continuity conditions across a material boundary for the magnetic field vectors are also derived.

The development next takes account of the deformability of material media. To describe the nonlinear magnetoelastic interactions in a deformable material, a review of continuum kinematics is necessary and this is provided in Section 3. More background on continuum mechanics and elasticity theory can be found in the texts by Ogden (1997) and Holzapfel (2001), for example. Magnetic field variables and boundary conditions, which, in general, are defined with respect to the current configuration, are re-cast in Lagrangian form and the Lagrangian forms of the governing equations are derived. The specialization of magnetostatics is then used in order to illustrate the application of the theory. In Section 4, we summarize in a simple form the equilibrium equations for a highly deformable magnetoelastic material whose mechanical properties can be changed significantly by the application of a magnetic field. We consider the nonlinear purely magnetoelastic coupling that does not involve dissipation. An overview of different ways in which the equations of mechanical equilibrium can be written in the presence of magneto-mechanical interactions is provided. In addition, we list some of the many possible definitions of ‘stress tensor’ that can be

included in the equilibrium equations along with the associated magnetic ‘body force’ terms. A valuable source of reference on magnetoelastic interactions is the classic text of Brown (1966).

The general constitutive law for a nonlinear magnetoelastic material is derived and expressed in a compact form, with either the magnetic field or the magnetic induction as the independent magnetic variable. Here we consider an isotropic magnetoelastic material for which the constitutive equations can be expressed in terms of six invariants involving the deformation and a magnetic vector, which reduce to five for an incompressible material, as is appropriate for elastomers. These equations are used in Section 5, for an incompressible material, in the solution of two representative boundary-value problems involving circular cylindrical geometry, specifically the helical shear of a circular cylindrical tube with an axial magnetic field and the extension and inflation of a circular cylindrical tube with a circumferential magnetic field. For each problem a general formulation is developed without specialization of the (isotropic) constitutive law, and then specific results are discussed briefly for a special choice of such a law. It is noted, in particular, that certain restrictions may be placed on the class of constitutive laws for a considered combination of deformation and magnetic field to be admitted. Many problems of this kind are analyzed in the volume by Eringen and Maugin (1990) and in the papers by Dorfmann and others (Dorfmann and Ogden, 2004, 2005; Ogden and Dorfmann, 2005; Bustamante et al., 2007), for example. For some discussion of stability analysis we refer to Eringen and Maugin (1990) and Otténio et al. (2008).

Necessarily, because of space limitations, only partial coverage of the vast subject of magnetic effects in deformable media can be provided in this chapter, and many interesting phenomena are not included. For example, the analysis developed here is purely static, but there are many applications that involve dynamic couplings that are not treated herein. For pointers to the extensive literature and for broader perspectives on both the mathematical and physical modelling of complex electro-magneto-mechanical couplings the reader is referred to the monographs by Maugin (1988), Eringen and Maugin (1990) and Maugin et al. (1992).

The preceding chapter in this volume by Dorfmann deals with the parallel development of nonlinear electroelastostatics and, to avoid repetition, we refer to that chapter for basic background material relating, in particular, to electric charges and dipoles and the Lorentz force.

2 Magnetostatics

In electrostatics the fundamental unit is the point charge e . When situated at the origin it generates, at a point with position vector \mathbf{x} , an electric field $\mathbf{E}(\mathbf{x})$ given by the inverse square law

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0} \frac{\mathbf{x}}{r^3},$$

where $r = |\mathbf{x}|$ and the constant ϵ_0 is the electric permittivity of free space, which has the approximate value $8.854 \times 10^{-12} \text{ C}^2\text{N}^{-1}\text{m}^{-2}$ in SI units, where C stands for Coulombs, N for Newtons and m for metres. While an electric field is generated by static charge a magnetic field is generated by the motion of charges. If a point charge e is instantaneously situated at the origin and moving with velocity \mathbf{v} it produces a magnetic field at the point \mathbf{x} , which may be expressed in terms of the *magnetic induction vector* \mathbf{B} by

$$\mathbf{B} = \frac{\mu_0 e}{4\pi} \frac{\mathbf{v} \times \hat{\mathbf{x}}}{r^2}, \quad (1)$$

where μ_0 is the *magnetic permeability of free space*, having value $4\pi \times 10^{-7} \text{ NA}^{-2}$, A standing for Ampères. The formula (1) is valid only in the non-relativistic approximation, for which $|\mathbf{v}| \ll c$, where c is the speed of light: note the connection $\mu_0\epsilon_0 = c^{-2}$.

Instead of a single charged particle suppose we now consider an infinitesimal element of volume dV and let $\rho_e dV$ be the total charge within this element. Then ρ_e is the *charge density*, which may be positive or negative and depends, in general, on position and time. If \mathbf{v} is the mean velocity of the individual charges in dV , then

$$\mathbf{J} = \rho_e \mathbf{v}, \quad (2)$$

defines the *current density* at the point at which dV is located, and in general \mathbf{J} depends on position and time.

A fundamental difference between electrostatics and magnetostatics is that there is no counterpart of the charged particle in magnetostatics. The basic building block in this context is the *magnetic dipole*, which we denote by \mathbf{m} . This idealization enables a relation to be established between the electric current and the magnetic induction vector \mathbf{B} . The flow of an electric current is associated with moving charges and is conveniently described by the current density \mathbf{J} defined by equation (2). When the current flow is steady (independent of time) then, according to the charge conservation equation (see preceding chapter), we have $\text{div } \mathbf{J} = 0$. Hence, by the divergence theorem, for a closed surface S we have $\int_S \mathbf{J} \cdot \mathbf{n} dS = 0$, where \mathbf{n} is the

unit outward normal to S , i.e. the net flux of current into (or out of) the enclosed volume vanishes. Geometrically, we can think of the *lines of current flow* as having tangent in the direction of \mathbf{J} at each point. A *tube of current flow* is then defined as the surface formed by all such lines that intersect a given closed curve at any instant (these are analogous to lines and tubes of flow in fluid dynamics). It follows that the flux of \mathbf{J} across a cross-section of the tube is the same for all cross-sections (by construction, there is no flow across the lateral surface of the tube). Steady current therefore consists of closed tubes of current flow. The total current I passing across an open surface S is just the flux of \mathbf{J} across S , and is given by

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}. \quad (3)$$

In practice, a thin conducting wire is a tube of flow of small cross-section $d\mathbf{S}$ and current $I \approx \mathbf{J} \cdot d\mathbf{S}$.

In electrostatics the force exerted on a charged particle at rest determines the magnitude and direction of the electric field. By contrast, the existence of a *magnetic field* is demonstrated by placing a small coil of wire carrying a current in a magnetic field. The coil experiences a force and a couple that can be used to quantify the magnitude and direction of the magnetic field. We will show that a small plane coil of area $d\mathbf{S}$ carrying a current I around its perimeter can be regarded as equivalent to a magnetic dipole with magnetic moment $\mathbf{m} = Id\mathbf{S}$.

2.1 The Biot-Savart Law and the Vector Potential

Recall now the formula (1), which gives the magnetic field at \mathbf{x} due to a point charge e at the origin moving with velocity \mathbf{v} . We now generalize this formula by considering a current distribution of density $\mathbf{J}(\mathbf{x}')$ at \mathbf{x}' within a volume V , which gives the magnetic field at \mathbf{x} as

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{R}}{R^3} dV(\mathbf{x}'), \quad (4)$$

where $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $R = |\mathbf{R}|$. This important basic formula is known as the *Biot-Savart Law*. There are some subtleties in the theory associated with singularities when the point \mathbf{x} is within V , but we do not discuss them here (see, for example, Jackson, 1999, for details).

Since

$$\text{curl} \left(\frac{\mathbf{J}(\mathbf{x}')}{R} \right) = \text{grad} \left(\frac{1}{R} \right) \times \mathbf{J}(\mathbf{x}') = \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{R}}{R^3}, \quad (5)$$

where the differentiations are with respect to \mathbf{x} (not \mathbf{x}'), equation (4) can be rewritten as

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \text{curl} \left(\frac{\mathbf{J}(\mathbf{x}')}{R} \right) dV(\mathbf{x}') = \frac{\mu_0}{4\pi} \text{curl} \left[\int_V \frac{\mathbf{J}(\mathbf{x}')}{R} dV(\mathbf{x}') \right]. \quad (6)$$

This suggests introducing the *magnetostatic vector potential*, denoted \mathbf{A} and defined as

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{R} dV(\mathbf{x}'), \quad (7)$$

leading to the important equation

$$\mathbf{B} = \text{curl} \mathbf{A} \quad (8)$$

for the magnetic induction vector, from which it follows that \mathbf{B} satisfies the equation

$$\text{div} \mathbf{B} = 0. \quad (9)$$

This is a fundamental equation of magnetostatics and expresses the fact that magnetic poles cannot be isolated, i.e. there is no counterpart in magnetostatics of the electrostatic point charge. In fact, equation (9) is general and holds even when there is time dependence and electromagnetic coupling, both in free space and in material media. It is one of the four Maxwell equations of conventional electromagnetic theory.

2.2 Scalar Magnetic Potential

Suppose that the volume V in (4) is a thin closed wire circuit C carrying current I . Then $\mathbf{J}(\mathbf{x}')dV(\mathbf{x}')$ can be replaced by $I d\mathbf{x}'$, and the volume integral by a line integral around C , where $d\mathbf{x}'$ is a line element in the direction of the current, i.e. along C . Thus, (4) becomes

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{x}' \times \mathbf{R}}{R^3} = \frac{\mu_0 I}{4\pi} \int_C \text{grad} \left(\frac{1}{R} \right) \times d\mathbf{x}', \quad (10)$$

and since the gradient is with respect to \mathbf{x} this may also be written

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \text{curl} \int_C \frac{d\mathbf{x}'}{R}. \quad (11)$$

For points distant from C for which $|\mathbf{x}'| \ll |\mathbf{x}|$ for all \mathbf{x}' in V we may use the Taylor expansion

$$\frac{1}{R} \equiv \frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \frac{1}{r} - \mathbf{x}' \cdot \text{grad} \left(\frac{1}{r} \right),$$

recalling that $r = |\mathbf{x}|$, and since C is a closed curve we obtain

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \operatorname{curl} \left[\mathcal{M} \operatorname{grad} \left(\frac{1}{r} \right) \right], \quad (12)$$

where

$$\mathcal{M} = I \int_C d\mathbf{x}' \otimes \mathbf{x}' \quad (13)$$

is a second-order tensor and \otimes signifies the tensor product of two vectors, so that for vectors \mathbf{a} and \mathbf{b} , for example, we have $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. Also, $(\mathcal{M}\mathbf{a})_i = \mathcal{M}_{ij} a_j$, $i, j \in \{1, 2, 3\}$, with summation over j . Now,

$$\mathcal{M} + \mathcal{M}^T = I \int_C d(\mathbf{x}' \otimes \mathbf{x}') = \mathbf{O}, \quad (14)$$

the zero tensor (again because C is closed), where T signifies the transpose of a second-order tensor. Hence \mathcal{M} is a skew-symmetric tensor, which we refer to as the *magnetic moment tensor*. Let \mathbf{m} denote the associated axial vector, defined by $\mathbf{m} = -\frac{1}{2}\epsilon\mathcal{M}$, where ϵ is the alternating tensor (in components, $m_i = -\frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk}$, with summation over indices j and k from 1 to 3). Then $\mathcal{M}\mathbf{a} = \mathbf{m} \times \mathbf{a}$, and (12) becomes

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \operatorname{curl} \left[\mathbf{m} \times \operatorname{grad} \left(\frac{1}{r} \right) \right] = \frac{\mu_0}{4\pi} \operatorname{curl} \operatorname{curl} \left(\frac{\mathbf{m}}{r} \right). \quad (15)$$

In view of the standard identity $\operatorname{curl} \operatorname{curl} = \operatorname{grad} \operatorname{div} - \nabla^2$ and the fact that $1/r$ satisfies Laplace's equation (provided $r \neq 0$), the above becomes

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \operatorname{grad} \operatorname{div} \left(\frac{\mathbf{m}}{r} \right) = -\frac{\mu_0}{4\pi} \operatorname{grad} \left(\frac{\mathbf{m} \cdot \mathbf{x}}{r^3} \right). \quad (16)$$

Thus, we may write

$$\mathbf{B}(\mathbf{x}) = -\operatorname{grad} \psi, \quad (17)$$

where ψ is a potential function given by

$$\psi(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \cdot \mathbf{x}}{r^3}. \quad (18)$$

The potential (18) has the same structure as the potential associated with an electric dipole (see preceding chapter), and thus (18) is interpreted as the magnetostatic potential of a *magnetic dipole* of strength \mathbf{m} situated at the origin. Moreover, since

$$\mathbf{m} = -\frac{1}{2}\epsilon\mathcal{M} = \frac{1}{2}I \int_C \mathbf{x}' \times d\mathbf{x}', \quad (19)$$

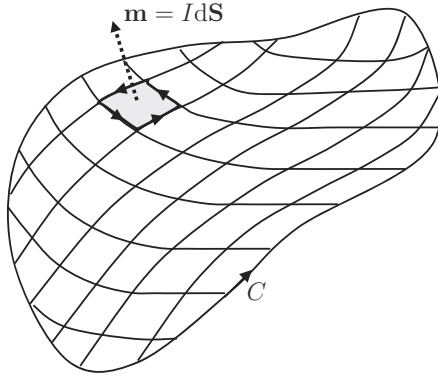


Figure 1. Circuit C carrying current I showing a network of curves made up of small current loops with current I corresponding to magnetic dipoles with magnetic moment $\mathbf{m} = I d\mathbf{S}$, where $d\mathbf{S}$ is the directed area element on the open surface whose edge is C .

the potential due to a magnetic dipole is equivalent to that due to a small current loop. More particularly, if C is a planar loop then

$$\mathbf{m} = I d\mathbf{S} = I n dS, \tag{20}$$

where dS is the plane area enclosed by the loop and \mathbf{n} is the unit normal to the plane of the loop, directed in the positive sense.

For a dipole situated at the point \mathbf{x}' , the potential in equation (18) is replaced by

$$\psi(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \cdot \mathbf{R}}{R^3}. \tag{21}$$

Now consider a circuit C of finite dimensions, carrying current I , as depicted in Figure 1. Let S be any regular surface that is bounded by C . Imagine that a fine network of curves is constructed on S such that each mesh is infinitesimal, effectively plane and with vector area element $d\mathbf{S}$. We may regard the current I as flowing in each curve of the mesh because it cancels out on adjoining meshes. In effect, we have a surface S consisting of a distribution of magnetic dipoles $I d\mathbf{S}$. The potential at \mathbf{x} is due to contributions from all such dipoles. Inserting $\mathbf{m} = I d\mathbf{S}$ into (21) and integrating we obtain the potential

$$\psi(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_S \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3}, \tag{22}$$

and we note that

$$\int_S \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3} = \Omega(\mathbf{x}) \quad (23)$$

is a purely geometrical quantity – the *solid angle* subtended by S at \mathbf{x} . Thus,

$$\psi(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \Omega(\mathbf{x}). \quad (24)$$

The solid angle $\Omega(\mathbf{x})$ has the property that its value changes by 4π as the point \mathbf{x} crosses the surface S . This means the potential function ψ is multi-valued and changes in value by $\mu_0 I$ each time \mathbf{x} traverses a curve which cuts S once. Otherwise ψ is continuous.

2.3 Ampère's Circuital Law

Consider a closed curve Γ which encircles the circuit C just once, and therefore cuts any open surface S that is bounded by C . The direction of Γ is related to that of I by the right-hand screw rule (see Figure 2).

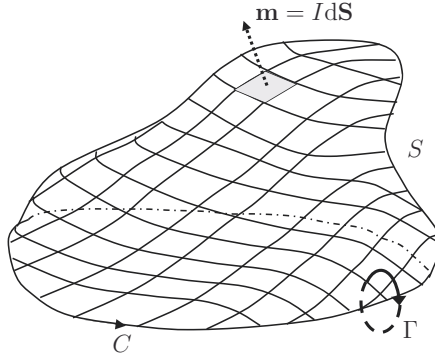


Figure 2. An open surface S whose bounding edge is the circuit C carrying current I . The closed curve Γ encircles C once and hence cuts S .

Since $\mathbf{B} = -\text{grad } \psi$, we obtain

$$\int_{\Gamma} \mathbf{B} \cdot d\mathbf{x} = - \int_{\Gamma} \text{grad } \psi \cdot d\mathbf{x} = -[\psi]_{\Gamma}, \quad (25)$$

where $[\psi]_{\Gamma}$ is the change in ψ as Γ is traversed once. This is non-zero since ψ is multi-valued, and since Γ cuts S just once in the sense described above, Ω increases by -4π , and hence ψ by $-\mu_0 I$, for a single traversal of

Γ . Therefore,

$$\int_{\Gamma} \mathbf{B} \cdot d\mathbf{x} = \mu_0 I. \quad (26)$$

The same argument can be applied to a thin wire or tube with a steady flow of current I , where the total current passing across an open surface is given by (3). Therefore,

$$\int_{\Gamma} \mathbf{B} \cdot d\mathbf{x} = \mu_0 \int_{\Sigma} \mathbf{J} \cdot d\mathbf{S}, \quad (27)$$

where Σ is an arbitrary regular open surface whose edge is Γ . Equation (27) is the mathematical statement of *Ampère's Circuital Law*. By applying Stokes' theorem to (27) we then obtain

$$\int_{\Sigma} (\text{curl} \mathbf{B} - \mu_0 \mathbf{J}) \cdot d\mathbf{S} = 0, \quad (28)$$

which holds for any open surface Σ . Provided the integrand in (28) is continuous we obtain the local form of one of the fundamental equations of magnetostatics as

$$\text{curl} \mathbf{B} = \mu_0 \mathbf{J}. \quad (29)$$

We recall that in deriving this equation it has been assumed that \mathbf{J} is time independent.

Returning to equation (8) we note that it is not affected by the addition of the gradient of an arbitrary scalar function (say φ) to the magnetic vector potential, i.e.

$$\mathbf{A} \rightarrow \mathbf{A} + \text{grad} \varphi, \quad (30)$$

which is known as a *gauge transformation*. This flexibility enables a restriction to be imposed on \mathbf{A} , which is usually taken in the form

$$\text{div} \mathbf{A} = 0. \quad (31)$$

Using equations (29) and (8) we have

$$\text{curl}(\text{curl} \mathbf{A}) = \mu_0 \mathbf{J}, \quad (32)$$

and, by using a standard vector identity, equation (32) can be written in the equivalent form

$$\text{grad}(\text{div} \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \quad (33)$$

Equation (31) is then used to reduce (33) to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \quad (34)$$

which, for given \mathbf{J} , is Poisson's equation for the magnetostatic vector potential. For an unbounded space, the solution of (34) for \mathbf{A} is given by (7).

2.4 Force and Couple on a Dipole in a Magnetic Field

The considerations thus far have not involved mechanical interactions with magnetic fields, but when such interactions are incorporated into the theory it will be necessary to account for the mechanical force exerted by the magnetic effects. We now derive expressions for the (mechanical) force and couple on a magnetic dipole placed in a magnetic field. For this purpose we recall (see the preceding chapter) that the Lorentz force acting on a charged particle e moving with velocity \mathbf{v} in an electromagnetic field with electric field \mathbf{E} and magnetic induction \mathbf{B} is $e\mathbf{E} + e\mathbf{v} \times \mathbf{B}$. In the case of a continuous distribution of charge with density ρ_e and current with density \mathbf{J} the Lorentz force density (per unit volume) is $\rho_e\mathbf{E} + \mathbf{J} \times \mathbf{B}$. Here we are only concerned with the magnetic contribution $\mathbf{J} \times \mathbf{B}$ to the Lorentz force.

Consider a material volume V in which there is a current distribution with density \mathbf{J} and let \mathbf{B} be the magnetic induction field permeating the material. Then, the magnetic contribution to the Lorentz force acting on V , which we denote by \mathbf{F}_m , is

$$\mathbf{F}_m = \int_V \mathbf{J} \times \mathbf{B} \, dV, \quad (35)$$

where the subscript m signifies ‘magnetic’. Now suppose that V consists simply of a single current loop C carrying current I . Then, we may replace the volume integral by a line integral around C and (35) becomes

$$\mathbf{F}_m = I \int_C d\mathbf{x} \times \mathbf{B} = I \int_S (d\mathbf{S} \times \text{grad}) \times \mathbf{B}, \quad (36)$$

where S is a regular open surface bounded by C and the latter integral has been obtained by an application of Stokes’ theorem.

Next, we take C and S to be infinitesimal so that the derivatives of \mathbf{B} are approximately uniform over S . Then (36) is approximated as $\mathbf{F}_m \approx I(d\mathbf{S} \times \text{grad}) \times \mathbf{B}$, and by setting $I d\mathbf{S} = \mathbf{m}$ to be the equivalent magnetic dipole and taking the limit $I \rightarrow \infty$ as $dS \rightarrow 0$ while keeping \mathbf{m} finite we obtain the exact result $\mathbf{F}_m = (\mathbf{m} \times \text{grad}) \times \mathbf{B}$, which is evaluated at the location of the dipole. By standard vector identities and the fact that $\text{div} \mathbf{B} = 0$ this force on a dipole \mathbf{m} in a magnetic induction field \mathbf{B} may be written as

$$\mathbf{F}_m = (\text{grad} \mathbf{B})^T \mathbf{m}. \quad (37)$$

In (37) and henceforth we adopt the following conventions: for two vector fields \mathbf{u} and \mathbf{v} we define the products $(\text{grad} \mathbf{u})^T \mathbf{v}$ and $(\text{grad} \mathbf{u}) \mathbf{v} \equiv (\mathbf{v} \cdot \text{grad}) \mathbf{u}$ via their index notation representations $u_{j,i} v_j$ and $u_{i,j} v_j$, respectively, where $_{,j} = \partial/\partial x_j$ and $(\text{grad} \mathbf{u})_{ij} = u_{i,j}$.

The (magnetic) couple on V , denoted \mathbf{G}_m , about a fixed origin due to the magnetic Lorentz force is given by

$$\mathbf{G}_m = \int_V \mathbf{x} \times (\mathbf{J} \times \mathbf{B}) dV, \quad (38)$$

where \mathbf{x} is the position vector relative to the chosen origin. When V consists of just a current loop C this becomes

$$\mathbf{G}_m = I \int_C \mathbf{x} \times (d\mathbf{x} \times \mathbf{B}) = I \int_C (d\mathbf{x} \otimes \mathbf{x}) \mathbf{B} - I \int_C (\mathbf{x} \cdot d\mathbf{x}) \mathbf{B}. \quad (39)$$

Once more we take C to be infinitesimal, but now it suffices as a first approximation to take \mathbf{B} to be uniform over C so that it can be taken outside the integrals. Then, since C is a closed circuit, the final integral in (39) vanishes, and on use of (13) \mathbf{G}_m can be written compactly as

$$\mathbf{G}_m = \mathcal{M}\mathbf{B} = \mathbf{m} \times \mathbf{B}, \quad (40)$$

again with \mathbf{B} evaluated at the location of the dipole, and this is exact in the limit described above. This is the *couple on a dipole \mathbf{m} in a magnetic induction field \mathbf{B}* .

Thus far the development has been based entirely on use of the *magnetic induction* vector \mathbf{B} , but at this point it is necessary to introduce the so-called *magnetic field* vector, which is denoted by \mathbf{H} . For the field due to an isolated dipole placed in a vacuum, for example, \mathbf{B} and \mathbf{H} are simply related by $\mathbf{B} = \mu_0 \mathbf{H}$, where the constant μ_0 is again the permeability of free space. This relationship applies at any point in free space or in non-magnetizable materials, whatever the source of the magnetic field, in which case \mathbf{B} and \mathbf{H} satisfy the same equations. In particular, $\text{curl} \mathbf{H} = \mathbf{0}$, or equivalently $(\text{grad} \mathbf{H})^T = \text{grad} \mathbf{H}$, and equations (37) and (40) can be written in the alternative forms

$$\mathbf{F}_m = \mu_0 (\mathbf{m} \cdot \text{grad}) \mathbf{H}, \quad \mathbf{G}_m = \mu_0 \mathbf{m} \times \mathbf{H}. \quad (41)$$

We emphasize that while the two expressions for \mathbf{F}_m and those for \mathbf{G}_m are equivalent in the present context, their counterparts are not equivalent in magnetizable media and the distinction will be recognized as important, in particular, when dealing with deformable media.

2.5 Magnetization in Material Media

In material media the relation $\mathbf{B} = \mu_0 \mathbf{H}$ does not hold in general and it must be replaced by a *constitutive law*, which describes the magnetic

properties of the material in question. When a magnetic field is applied to material media currents are generated. This effect is conveniently described by an additional vector, known as the *magnetization*, denoted \mathbf{M} , which is defined in terms of the other field vectors by the standard formula

$$\mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H}. \quad (42)$$

To be more specific, the magnetization arises from the response of the material to an external magnetic field and corresponds to the magnetic dipole density. The effect of the magnetization is to induce a *bound current* density, denote here by \mathbf{J}_b , which is given by $\mathbf{J}_b = \text{curl}\mathbf{M}$. The difference

$$\mathbf{J} - \mathbf{J}_b = \mathbf{J} - \text{curl}\mathbf{M} \quad (43)$$

is the *free current* density, denoted \mathbf{J}_f . It follows from (29) and (42) that $\text{curl}\mathbf{H} = \mathbf{J}_f$.

The governing equations of magnetostatics in material media may now be summarized as

$$\text{div}\mathbf{B} = 0, \quad \text{curl}\mathbf{H} = \mathbf{J}_f. \quad (44)$$

Equation (42) gives an expression for the magnetization \mathbf{M} in terms of either \mathbf{H} or \mathbf{B} as the independent variable when \mathbf{B} (respectively \mathbf{H}) is given in terms of \mathbf{H} (respectively \mathbf{B}) by an appropriate constitutive equation.

A basic example of a constitutive law is that for a *linear isotropic material*, for which the equation $\mathbf{B} = \mu_0\mathbf{H}$ is replaced by

$$\mathbf{B} = \mu\mu_0\mathbf{H}, \quad (45)$$

where μ is the *relative magnetic permeability*. From equation (45), the magnetization is given by

$$\mathbf{M} = \frac{\mu - 1}{\mu_0\mu}\mathbf{B}, \quad (46)$$

so that \mathbf{M} is parallel to \mathbf{B} and \mathbf{H} . In vacuo or in non-magnetizable media $\mu = 1$. For most materials $\mu > 1$. However, there are some magnetizable materials for which $\mu < 1$ and \mathbf{M} is therefore opposite in direction to \mathbf{B} .

2.6 Boundary Conditions

The equations (44) are valid for any material medium and must be coupled with a constitutive equation for either \mathbf{B} or \mathbf{H} . To these equations we need to append boundary conditions in order to formulate and solve boundary-value problems. In general the field vectors \mathbf{B} and \mathbf{H} are discontinuous across a surface between different media or across a surface bounding the material. In this section we derive, using equations (44) in integral

form together with the divergence and Stokes' theorems, as appropriate, the equations satisfied by the discontinuities.

Let S be a *stationary* surface. The two sides of S are distinguished as side 1 and side 2 and field vectors on the two sides of S are identified by subscripts 1 and 2. Let \mathbf{n} be the unit normal to S pointing from side 1 to side 2. The 'jump' in a vector on S is the difference between its values on side 2 and side 1, evaluated on S . Thus \mathbf{H} , for example, has jump $\mathbf{H}_2 - \mathbf{H}_1$, which is denoted $[[\mathbf{H}]]$, and similarly for \mathbf{B} . The jump conditions satisfied by \mathbf{B} and \mathbf{H} are summarized as

$$\mathbf{n} \cdot [[\mathbf{B}]] = 0, \quad \mathbf{n} \times [[\mathbf{H}]] = \mathbf{K}_f, \tag{47}$$

where \mathbf{K}_f is the free current surface density on the surface S per unit area. We now establish these formulas.

Consider a cylinder (or 'pill box') of infinitesimal height δh and cross-sectional area $\delta S = \mathbf{n} \delta S$ straddling the surface S , as depicted in Figure 3. Equation (44)₁, when integrated over the volume V of the cylinder followed by an application of the divergence theorem, yields

$$\int_{\Sigma} \mathbf{B} \cdot d\mathbf{S} = 0, \tag{48}$$

where Σ is the bounding surface of the cylinder.

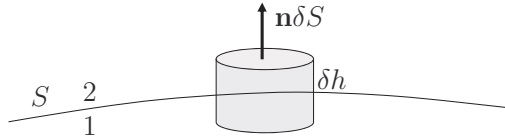


Figure 3. A 'pill-box' of height δh and cross-sectional area δS intersecting the surface S with unit normal \mathbf{n} pointing from side 1 to side 2 of S .

Since δh is infinitesimal and the flux of \mathbf{B} across the lateral surface of the cylinder becomes negligible as $\delta h \rightarrow 0$, the only contribution to the surface integral comes from the top and bottom surfaces of the cylinder. Equation (48) is therefore approximated simply as $\mathbf{B}_2 \cdot \mathbf{n} \delta S - \mathbf{B}_1 \cdot \mathbf{n} \delta S \approx 0$, which, after dividing by δS and taking the limit $\delta S \rightarrow 0$, yields $\mathbf{n} \cdot [[\mathbf{B}]] = 0$, and hence (47)₁ is established.

Next, consider equation (44)₂ integrated over an open surface Σ with bounding curve Γ . After application of Stokes' theorem it becomes

$$\oint_{\Gamma} \mathbf{H} \cdot d\mathbf{x} = \int_{\Sigma} \mathbf{J}_f \cdot d\mathbf{S}. \tag{49}$$

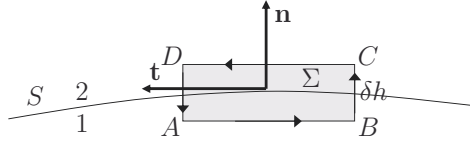


Figure 4. A small plane area Σ intersecting the surface S in the plane of the unit normal \mathbf{n} to the surface and a unit tangent vector \mathbf{t} . The unit normal points from side 1 to side 2 of the surface. The bounding curve of Σ is traversed in the direction of the arrows along the path $ABCD A$.

Now let Σ be an infinitesimal plane rectangular surface with Γ identified by its corner points $ABCD$ lying in the plane of the unit normal \mathbf{n} to the surface S and a unit tangent vector \mathbf{t} to the surface and intersecting S , as shown in Figure 4. The sides AB and CD of Γ are parallel to \mathbf{t} and have lengths δs . The sides BC and DA are parallel to \mathbf{n} and have lengths δh . Then, application of (49) to Σ and Γ yields the approximate result

$$\begin{aligned}
 - \int_{AB} \mathbf{H} \cdot \mathbf{t} \, ds + \int_{BC} \mathbf{H} \cdot \mathbf{n} \, dh + \int_{CD} \mathbf{H} \cdot \mathbf{t} \, ds - \int_{DA} \mathbf{H} \cdot \mathbf{n} \, dh \\
 \approx \left[\left(\int_{BC} \mathbf{J}_f \, dh \right) \times \mathbf{n} \right] \cdot \mathbf{t} \, \delta s. \quad (50)
 \end{aligned}$$

In the limit as $\delta h \rightarrow 0$ the integral on the right-hand side becomes the surface current density \mathbf{K}_f , such that $\mathbf{n} \cdot \mathbf{K}_f = 0$, and then dividing by δs and letting $\delta s \rightarrow 0$ we obtain $\mathbf{H}_2 \cdot \mathbf{t} - \mathbf{H}_1 \cdot \mathbf{t} = (\mathbf{K}_f \times \mathbf{n}) \cdot \mathbf{t}$. Setting $\mathbf{n} \times \mathbf{t} = \mathbf{k}$ and noting that $\mathbf{k} \times \mathbf{n} = \mathbf{t}$ it follows that $\{\mathbf{n} \times [\mathbf{H}]\} \cdot \mathbf{k} = \mathbf{K}_f \cdot \mathbf{k}$. Since \mathbf{t} is an arbitrary tangent then so is \mathbf{k} . This holds for arbitrary \mathbf{k} normal to \mathbf{n} , and hence the result (47)₂ follows.

3 Deformable Magnetic Materials

3.1 Continuum Kinematics

Consider a deformable magnetically sensitive body that is initially in an unstressed configuration. Let the region in three-dimensional Euclidean space occupied by the body in this configuration be denoted \mathcal{B}_0 , with boundary $\partial\mathcal{B}_0$, and let \mathbf{X} be the position vector of a generic material particle. Suppose that the body is deformed quasi-statically under the combined action of mechanical loads and a magnetic field, so that the point \mathbf{X} occupies the new position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in the resulting deformed configuration. We denote the deformed configuration by \mathcal{B} and its boundary by $\partial\mathcal{B}$. The vector

field $\boldsymbol{\chi}$, which is a one-to-one, orientation-preserving mapping with suitable regularity properties, describes the deformation of the body and is defined for $\mathbf{X} \in \mathcal{B}_0 \cup \partial\mathcal{B}_0$.

The deformation gradient tensor \mathbf{F} relative to \mathcal{B}_0 is defined by

$$\mathbf{F} = \text{Grad } \boldsymbol{\chi}, \quad \mathbf{X} \in \mathcal{B}_0, \tag{51}$$

where Grad denotes the gradient operator with respect to \mathbf{X} . We also adopt the notation

$$J = \det \mathbf{F}, \tag{52}$$

which by standard convention is positive.

Associated with \mathbf{F} are the symmetric Cauchy–Green tensors. To avoid a conflict of standard notations we use here the lower case characters \mathbf{c} and \mathbf{b} to represent, respectively, the right and left Cauchy–Green tensors. These are defined in terms of the deformation gradient by

$$\mathbf{c} = \mathbf{F}^T \mathbf{F}, \quad \mathbf{b} = \mathbf{F} \mathbf{F}^T, \tag{53}$$

and we recall that T denotes the transpose of a second-order tensor.

In what follows, the notations grad, div and curl are used for the standard differential operators with respect to \mathbf{x} , while Grad, Div and Curl are the corresponding operators with respect to \mathbf{X} . We use the convention that the divergence and curl operators, when applied to tensors, act on the first index of the tensor that follows them. For example, $\text{div } \mathbf{F} \equiv \partial F_{j\alpha} / \partial x_j$ and $\text{Div}(\mathbf{F}^T) \equiv \partial F_{j\alpha} / \partial X_\alpha$. We also recall that by convention we define the components of the gradient of a vector according to $(\text{grad } \mathbf{v})_{ij} = \partial v_i / \partial x_j$, so that for vectors \mathbf{u} and \mathbf{v} , $(\mathbf{v} \cdot \text{grad})\mathbf{u} = (\text{grad } \mathbf{u})\mathbf{v}$.

The kinematic identities

$$\text{Div}(J\mathbf{F}^{-1}) = \mathbf{0}, \quad \text{div}(J^{-1}\mathbf{F}) = \mathbf{0}, \quad \text{Curl}(\mathbf{F}^T) = \mathbf{O}, \quad \text{curl}(\mathbf{F}^{-T}) = \mathbf{O}, \tag{54}$$

where $\mathbf{0}$ denotes the zero vector and \mathbf{O} the zero second-order tensor, are valuable for converting formulas between Eulerian and Lagrangian descriptions.

Suppose that $\mathbf{a} = \mathbf{a}(\mathbf{x})$ is an Eulerian vector defined in the deformed configuration \mathcal{B} . Using equations (54)_{1,3}, we have

$$\text{Div}(J\mathbf{F}^{-1}\mathbf{a}) = J\text{div } \mathbf{a}, \quad \text{Curl}(\mathbf{F}^T\mathbf{a}) = J\mathbf{F}^{-1}\text{curl } \mathbf{a}. \tag{55}$$

Similarly, let $\mathbf{A} = \mathbf{A}(\mathbf{X})$ be a Lagrangian vector defined in the reference configuration \mathcal{B}_0 . Then, by using (54)_{2,4}, we obtain

$$\text{div}(J^{-1}\mathbf{F}\mathbf{A}) = J^{-1}\text{Div } \mathbf{A}, \quad \text{curl}(\mathbf{F}^{-T}\mathbf{A}) = J^{-1}\mathbf{F}\text{Curl } \mathbf{A}. \tag{56}$$

By using the connection $\mathbf{a} = J^{-1}\mathbf{F}\mathbf{A}$ we see that the divergence identities in equations (55)₁ and (56)₁ are equivalent. Equally, the equations involving the curl operator coincide if, instead, we set $\mathbf{a} = \mathbf{F}^{-T}\mathbf{A}$.

3.2 Eulerian and Lagrangian Formulations

For convenience of reference we recall that the field variables \mathbf{B} and \mathbf{H} satisfy the equations

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{H} = \mathbf{J}_f \quad \text{in } \mathcal{B}, \quad (57)$$

and the boundary conditions

$$\mathbf{n} \cdot \llbracket \mathbf{B} \rrbracket = 0, \quad \mathbf{n} \times \llbracket \mathbf{H} \rrbracket = \mathbf{K}_f \quad \text{on } \partial\mathcal{B}. \quad (58)$$

These equations apply for both non-deformable and deformable media. They are expressed in Eulerian form and, in particular, the differential equations involve the operators div and curl .

We now re-cast the equations and boundary conditions in Lagrangian form. For this purpose the operators Div and Curl are used, and the independent spatial variable is \mathbf{X} instead of \mathbf{x} . Bearing in mind the formula (55)₂, we introduce the Lagrangian counterpart of \mathbf{H} , denoted \mathbf{H}_l and defined by

$$\mathbf{H}_l = \mathbf{F}^T \mathbf{H}, \quad (59)$$

so that

$$\operatorname{curl}\mathbf{H} = J^{-1}\mathbf{F}\operatorname{Curl}\mathbf{H}_l. \quad (60)$$

Similarly, we introduce Lagrangian counterpart of \mathbf{B} , denoted \mathbf{B}_l . This is defined by

$$\mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}, \quad (61)$$

and from (55)₁ it follows that

$$\operatorname{Div}\mathbf{B}_l = J\operatorname{div}\mathbf{B}. \quad (62)$$

The equations (57) may therefore be written in Lagrangian form as

$$\operatorname{Div}\mathbf{B}_l = 0, \quad \operatorname{Curl}\mathbf{H}_l = \mathbf{J}_l \quad \text{in } \mathcal{B}_0, \quad (63)$$

where $\mathbf{J}_l = J\mathbf{F}^{-1}\mathbf{J}_f$ is the Lagrangian counterpart of the free current density \mathbf{J}_f . The Lagrangian form of the charge conservation equation is simply $\operatorname{Div}\mathbf{J}_l = 0$.

The Lagrangian forms of the boundary conditions are entirely analogous to their Eulerian counterparts in (58). The boundary conditions associated with (63) are

$$\mathbf{N} \cdot \llbracket \mathbf{B}_l \rrbracket = 0, \quad \mathbf{N} \times \llbracket \mathbf{H}_l \rrbracket = \mathbf{K}_l \quad \text{on } \partial\mathcal{B}_0, \quad (64)$$

where \mathbf{N} is the unit normal to the reference boundary $\partial\mathcal{B}_0$ corresponding to \mathbf{n} through Nanson's formula $\mathbf{n}dS = J\mathbf{F}^{-T}\mathbf{N}dS_0$, dS and dS_0 are the area elements on $\partial\mathcal{B}$ and $\partial\mathcal{B}_0$, respectively, and $\mathbf{K}_l = \mathbf{F}^{-1}\mathbf{K}_f dS/dS_0$ is the Lagrangian free surface current, defined per unit reference area.

In the remainder of this chapter we illustrate the use of the equations summarized above by focusing attention on the magnetoelastic interactions. In particular, we develop the constitutive theory that describes the nonlinear coupling between elastic deformations and magnetic fields.

4 Nonlinear Magnetoelastic Interactions

In this section we consider a highly deformable elastic material in which the mechanical and magnetic effects are fully coupled. The relevant equations and boundary conditions are then (57) with (58) or, equivalently, (63) with (64).

Note that since $\mathbf{M} = \mathbf{0}$ outside the material then by combining the two boundary conditions in (58) and using the connection $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ we obtain

$$\llbracket \mathbf{H} \rrbracket = (\mathbf{n} \cdot \mathbf{M})\mathbf{n} - \mathbf{n} \times \mathbf{K}_f, \quad \llbracket \mathbf{B} \rrbracket = \mu_0 \mathbf{n} \times (\mathbf{n} \times \mathbf{M}) - \mu_0 \mathbf{n} \times \mathbf{K}_f \quad \text{on } \partial\mathcal{B}. \quad (65)$$

A corresponding equation can be obtained for the Lagrangian fields, but is omitted here and left as an exercise for the reader. The above equations and boundary conditions are conjoined with appropriate forms of the mechanical equilibrium equation and mechanical boundary conditions. We now provide an overview of different ways in which the equations of mechanical equilibrium and the accompanying traction boundary conditions can be written in the presence of magneto-mechanical interactions, which requires consideration of several different stress tensors.

4.1 Equilibrium, Stress and Constitutive Laws

Magnetic forces and couples. We recall that the Lorentz force on a material volume V containing a distribution of current with density \mathbf{J} is given by (35), which can be expanded in the form

$$\mathbf{F}_m = \int_V \mathbf{J} \times \mathbf{B} dV = \mu_0^{-1} \int_V (\text{curl} \mathbf{B}) \times \mathbf{B} dV = \int_V \text{div} \boldsymbol{\tau}_B dV, \quad (66)$$

where $\mathbf{J} = \mu_0^{-1} \text{curl} \mathbf{B} = \text{curl} \mathbf{H} + \text{curl} \mathbf{M}$ is the total current density, $\boldsymbol{\tau}_B$ is defined by

$$\boldsymbol{\tau}_B = \mu_0^{-1} [\mathbf{B} \otimes \mathbf{B} - \frac{1}{2} (\mathbf{B} \cdot \mathbf{B}) \mathbf{I}], \quad (67)$$

which is symmetric, and the identity $(\text{curl} \mathbf{B}) \times \mathbf{B} = (\text{grad} \mathbf{B}) \mathbf{B} - (\text{grad} \mathbf{B})^T \mathbf{B}$ has been used. Note that the latter can be written as $\mu_0 \text{div} \boldsymbol{\tau}_B$.

We now suppose that the volume V , with boundary S , corresponds to the material volume \mathcal{B} in the deformed configuration, with boundary $\partial \mathcal{B}$ on which the discontinuity in \mathbf{B} given by (65)₂ holds. In the following analysis we use the notation V for the volume and S for the boundary. The discontinuity in \mathbf{B} carries over to a discontinuity in $\boldsymbol{\tau}_B$, which must be accounted for in the evaluation of the final integral in (66).

Consider a surface S^+ adjacent to S and entirely enclosing S , and let V^+ be the volume within S^+ . Then, by applying the divergence theorem to V^+ , we obtain

$$\int_{V^+} \text{div} \boldsymbol{\tau}_B \, dV = \int_{S^+} \boldsymbol{\tau}_B \mathbf{n} \, dS, \quad (68)$$

where \mathbf{n} is the unit outward normal on S^+ . Note that the integral over V^+ is well defined even when $\text{div} \boldsymbol{\tau}_B$ is discontinuous across an internal surface provided its discontinuity is finite, which we assume to be the case. In the limit $V^+ \rightarrow V$, $S^+ \rightarrow S$, we then have

$$\mathbf{F}_m = \int_S \boldsymbol{\tau}_B \mathbf{n} \, dS, \quad (69)$$

where $\boldsymbol{\tau}_B$ is evaluated on the *outside* of S . Outside V we have $\mathbf{M} = \mathbf{0}$ and $\mathbf{B} = \mu_0 \mathbf{H}$, and $\boldsymbol{\tau}_B$ may be expressed in terms of either \mathbf{B} or \mathbf{H} . It turns out to be convenient to use a combination of \mathbf{B} and \mathbf{H} , and we adopt the notation defined by

$$\boldsymbol{\tau}_m = \mathbf{B} \otimes \mathbf{H} - \frac{1}{2} \mu_0 (\mathbf{H} \cdot \mathbf{H}) \mathbf{I}. \quad (70)$$

Outside V , $\boldsymbol{\tau}_m$ is symmetric and equal to $\boldsymbol{\tau}_B$. We shall also use the definition (70) within V , where $\boldsymbol{\tau}_m$ is not in general symmetric and not equal to $\boldsymbol{\tau}_B$. Then, with the help of (65), we obtain the discontinuity

$$\llbracket \boldsymbol{\tau}_m^T \rrbracket \mathbf{n} = \frac{1}{2} \mu_0 (\mathbf{M} \cdot \mathbf{n})^2 \mathbf{n} + \mathbf{K}_f \times \mathbf{B} + \mu_0 [(\mathbf{n} \times \mathbf{M}) \cdot \mathbf{K}_f] \mathbf{n} - \frac{1}{2} \mu_0 (\mathbf{K}_f \cdot \mathbf{K}_f) \mathbf{n}, \quad (71)$$

where \mathbf{B} and \mathbf{M} are evaluated on S from the *inside*. For convenience we denote the vector on the right-hand side of (71) by $\bar{\mathbf{t}}_m$, which represents a mechanical traction on the boundary S due to magnetic effects. Then, since $\boldsymbol{\tau}_B = \boldsymbol{\tau}_m$ on the outside of S we may use (71) and the divergence theorem

on V to re-write (69) as

$$\mathbf{F}_m = \int_V \operatorname{div} \boldsymbol{\tau}_m \, dV + \int_S \bar{\mathbf{t}}_m \, dS. \quad (72)$$

Now consider the expression for $\operatorname{div} \boldsymbol{\tau}_m$ within V . There are various useful ways to write it in terms of two or all three of \mathbf{B} , \mathbf{H} and \mathbf{M} , starting from the definition (70) and using $\operatorname{div} \mathbf{B} = 0$ and the connection $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. However, for simplicity, we assume now and henceforth that there is no free current and no free surface current, so that $\operatorname{curl} \mathbf{H} = \mathbf{J}_f = \mathbf{0}$ and $\mathbf{K}_f = \mathbf{0}$, and the resulting expressions for $\operatorname{div} \boldsymbol{\tau}_m$ reduce to the two main alternatives

$$\operatorname{div} \boldsymbol{\tau}_m = \mu_0(\operatorname{grad} \mathbf{H})\mathbf{M} = (\operatorname{grad} \mathbf{B})^T \mathbf{M} - \frac{1}{2} \mu_0 \operatorname{grad}(\mathbf{M} \cdot \mathbf{M}), \quad (73)$$

while the expression for $\bar{\mathbf{t}}_m$ simplifies to

$$\bar{\mathbf{t}}_m = \frac{1}{2} \mu_0 (\mathbf{M} \cdot \mathbf{n})^2 \mathbf{n}. \quad (74)$$

The magnetic force \mathbf{F}_m in (72) can now be written in two alternative ways, as either

$$\mathbf{F}_m = \mu_0 \int_V (\mathbf{M} \cdot \operatorname{grad}) \mathbf{H} \, dV + \int_S \bar{\mathbf{t}}_m \, dS, \quad (75)$$

or

$$\mathbf{F}_m = \int_V (\operatorname{grad} \mathbf{B})^T \mathbf{M} \, dV + \int_S \bar{\bar{\mathbf{t}}}_m \, dS, \quad (76)$$

where $\bar{\bar{\mathbf{t}}}_m$ is defined by

$$\bar{\bar{\mathbf{t}}}_m = -\frac{1}{2} \mu_0 [(\mathbf{n} \times \mathbf{M}) \cdot (\mathbf{n} \times \mathbf{M})] \mathbf{n} = \bar{\mathbf{t}}_m - \frac{1}{2} \mu_0 \mathbf{M} \cdot \mathbf{M}. \quad (77)$$

Notice that the associated volumetric force densities $\mu_0(\mathbf{M} \cdot \operatorname{grad}) \mathbf{H}$ and $(\operatorname{grad} \mathbf{B})^T \mathbf{M}$ are analogous to the expressions for the force on a single dipole given by (41)₁ and (37), respectively, but, unlike the latter, they are not the same. In the present context there is a contribution to the force from the boundary term which does not arise for a single dipole. The expressions (75) and (76) are entirely equivalent and can be generalized, if required, to accommodate non-zero current distributions, albeit with some loss of simplicity.

The analogue of the couple $\mathbf{m} \times \mathbf{B}$ on a single magnetic dipole, given by (40), is the couple per unit volume $\mathbf{M} \times \mathbf{B}$, which may also be written

as $\mu_0 \mathbf{M} \times \mathbf{H}$ or $\mathbf{B} \times \mathbf{H}$. The total magnetic couple \mathbf{G}_m about the origin corresponding to the representation (75) based on use of \mathbf{H} is then

$$\mathbf{G}_m = \mu_0 \int_V \{ \mathbf{x} \times [(\mathbf{M} \cdot \text{grad})\mathbf{H}] + \mathbf{M} \times \mathbf{H} \} dV + \int_S \mathbf{x} \times \bar{\mathbf{t}}_m dS, \quad (78)$$

while that based on use of \mathbf{B} is

$$\mathbf{G}_m = \int_V \{ \mathbf{x} \times [(\text{grad}\mathbf{B})^T \mathbf{M}] + \mathbf{M} \times \mathbf{B} \} dV + \int_S \mathbf{x} \times \bar{\bar{\mathbf{t}}}_m dS, \quad (79)$$

where we have assumed that there are no intrinsic mechanical couples.

We emphasize that the above derivations, starting with the Lorentz force (66), have been applied to the *whole* body. However, it can be shown that the formulas (75), (76), (78) and (79) actually apply to an arbitrary material volume V with boundary S (Brown, 1966). The derivation of (75), for example, is in this case more delicate and (66) requires modification to include boundary terms since the identification $\boldsymbol{\tau}_B = \boldsymbol{\tau}_m$ cannot be made *within a magnetized material*.

We now embrace these two alternatives by introducing a generic magnetic body force density and surface force density, denoted $\hat{\mathbf{f}}_m$ and $\hat{\mathbf{t}}_m$, respectively, and a corresponding intrinsic magnetic couple $\hat{\mathbf{g}}_m$. Then,

$$\mathbf{F}_m = \int_V \hat{\mathbf{f}}_m dV + \int_S \hat{\mathbf{t}}_m dS, \quad (80)$$

and

$$\mathbf{G}_m = \int_V (\mathbf{x} \times \hat{\mathbf{f}}_m + \hat{\mathbf{g}}_m) dV + \int_S \mathbf{x} \times \hat{\mathbf{t}}_m dS. \quad (81)$$

Thus, we have either $\hat{\mathbf{f}}_m = \mu_0(\mathbf{M} \cdot \text{grad})\mathbf{H}$ with $\hat{\mathbf{t}}_m = \bar{\mathbf{t}}_m$ or $\hat{\mathbf{f}}_m = (\text{grad}\mathbf{B})^T \mathbf{M}$ with $\hat{\mathbf{t}}_m = \bar{\bar{\mathbf{t}}}_m$, and in each case $\hat{\mathbf{g}}_m = \mu_0 \mathbf{M} \times \mathbf{H} = \mathbf{B} \times \mathbf{H} = \mathbf{M} \times \mathbf{B}$. We also introduce a generic counterpart of $\boldsymbol{\tau}_m$, denoted $\hat{\boldsymbol{\tau}}_m$, so that $\hat{\mathbf{f}}_m = \text{div} \hat{\boldsymbol{\tau}}_m$, and it follows from (73) that $\hat{\boldsymbol{\tau}}_m = \boldsymbol{\tau}_m$ and $\hat{\boldsymbol{\tau}}_m = \boldsymbol{\tau}_m + \frac{1}{2} \mu_0 (\mathbf{M} \cdot \mathbf{M}) \mathbf{I}$, respectively, for the two specialization above. Moreover, we note that $\boldsymbol{\epsilon} \boldsymbol{\tau}_m = \mathbf{B} \times \mathbf{H} = \hat{\mathbf{g}}_m$ in each case, where we recall that $\boldsymbol{\epsilon}$ is the alternating tensor defined in Section 2.2.

The generic definitions above admit the possibility of expressions for the magnetic body and surface force densities other than the two introduced here. Such expressions may or may not have direct physical interpretations but they may be useful from the point of view of the mathematical formulation of the governing equations. In the following we incorporate the magnetic force and couple into the mechanical balance equations.

Mechanical equilibrium. In equilibrium the total force and total couple acting on a body in its deformed configuration \mathcal{B} must each vanish. Let \mathbf{f} be the *mechanical body force* per unit mass, ρ the mass density of the material and \mathbf{t}_a the *mechanical traction* per unit area of the boundary $\partial\mathcal{B}$. Then, on taking account of the magnetic force and couple given by (80) and (81) (with V replaced by \mathcal{B}), we have

$$\int_{\mathcal{B}} (\rho\mathbf{f} + \hat{\mathbf{f}}_m) dV + \int_{\partial\mathcal{B}} (\mathbf{t}_a + \hat{\mathbf{t}}_m) dS = \mathbf{0}, \quad (82)$$

and

$$\int_{\mathcal{B}} [\mathbf{x} \times (\rho\mathbf{f} + \hat{\mathbf{f}}_m) + \hat{\mathbf{g}}_m] dV + \int_{\partial\mathcal{B}} [\mathbf{x} \times (\mathbf{t}_a + \hat{\mathbf{t}}_m)] dS = \mathbf{0}. \quad (83)$$

These balance equations apply not only to the whole body but also to any sub-volume of \mathcal{B} and its boundary. Then, by a standard tetrahedron argument from continuum mechanics applied to (82) we deduce that there exists a second-order (Cauchy-like) stress tensor, which we denote by $\hat{\boldsymbol{\sigma}}$, defined in \mathcal{B} , such that

$$\hat{\boldsymbol{\sigma}}^T \mathbf{n} = \mathbf{t}_a + \hat{\mathbf{t}}_m \quad \text{on } \partial\mathcal{B} \quad (84)$$

and $\hat{\boldsymbol{\sigma}}$ is independent of \mathbf{n} . Substitution of this into (82) and application of the divergence theorem yields

$$\int_{\mathcal{B}} (\rho\mathbf{f} + \hat{\mathbf{f}}_m + \text{div } \hat{\boldsymbol{\sigma}}) dV = \mathbf{0}. \quad (85)$$

This applies also to an arbitrary sub-volume of \mathcal{B} and provided the integrand is continuous we may then deduce the local form of the equilibrium equation, namely

$$\text{div } \hat{\boldsymbol{\sigma}} + \rho\mathbf{f} + \hat{\mathbf{f}}_m = \mathbf{0} \quad \text{in } \mathcal{B}. \quad (86)$$

Substitution of (84) into (83) followed by another application of the divergence theorem then leads to

$$\int_{\mathcal{B}} (\boldsymbol{\epsilon}\hat{\boldsymbol{\sigma}} + \hat{\mathbf{g}}_m) dV = \mathbf{0}, \quad (87)$$

which has local form

$$\boldsymbol{\epsilon}\hat{\boldsymbol{\sigma}} + \hat{\mathbf{g}}_m = \mathbf{0} \quad \text{in } \mathcal{B}. \quad (88)$$

The latter shows that in general $\boldsymbol{\epsilon}\hat{\boldsymbol{\sigma}} \neq \mathbf{0}$, i.e. $\hat{\boldsymbol{\sigma}}$ is not symmetric.

Now, for the two examples considered above we have $\hat{\mathbf{g}}_m = \boldsymbol{\epsilon}\hat{\boldsymbol{\tau}}_m$, and hence $\boldsymbol{\epsilon}(\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\tau}}_m) = \mathbf{0}$. Let us introduce the second-order tensor $\boldsymbol{\tau}$ defined by

$$\boldsymbol{\tau} = \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\tau}}_m, \quad (89)$$

so that $\epsilon\boldsymbol{\tau} = \mathbf{0}$, i.e. $\boldsymbol{\tau}$ is *symmetric*. Since $\hat{\mathbf{f}}_m = \text{div}\hat{\boldsymbol{\tau}}_m$ it follows from (86) that $\boldsymbol{\tau}$ satisfies the equilibrium equation

$$\text{div}\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0}. \quad (90)$$

On application of the divergence theorem to the global form of this equation, in conjunction with (82), in which $\hat{\mathbf{f}}_m$ is replaced by $\text{div}\hat{\boldsymbol{\tau}}_m$, followed by another application of the divergence theorem, we obtain

$$\int_{\partial\mathcal{B}} \boldsymbol{\tau}\mathbf{n} \, dS = - \int_{\mathcal{B}} \rho\mathbf{f} \, dV = \int_{\partial\mathcal{B}} (\mathbf{t}_a + \hat{\mathbf{t}}_m + \hat{\boldsymbol{\tau}}_m^T\mathbf{n}) \, dS. \quad (91)$$

This suggests that we should identify $\boldsymbol{\tau}\mathbf{n}$ with $\mathbf{t}_a + \hat{\mathbf{t}}_m + \hat{\boldsymbol{\tau}}_m^T\mathbf{n}$ on $\partial\mathcal{B}$. But, by the discontinuity conditions on $\partial\mathcal{B}$, we have $\hat{\mathbf{t}}_m + \hat{\boldsymbol{\tau}}_m^T\mathbf{n} = \boldsymbol{\tau}_m\mathbf{n}$, with $\boldsymbol{\tau}_m$ evaluated on the exterior of $\partial\mathcal{B}$. Thus, the boundary condition for $\boldsymbol{\tau}$ may be written in the form

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_a + \mathbf{t}_m \quad \text{on } \partial\mathcal{B}, \quad (92)$$

where \mathbf{t}_m is defined by

$$\mathbf{t}_m = \boldsymbol{\tau}_m\mathbf{n}. \quad (93)$$

We refer to $\boldsymbol{\tau}$ as the ‘total stress tensor’ since it enables the magnetic body forces to be treated as stresses. An advantage of the total stress tensor is that it is *symmetric*. Moreover, it is interesting to note that the intrinsic magnetic couple is absorbed by use of this stress tensor, and the rotational balance equation is satisfied automatically. Its global form is simply

$$\int_{\mathcal{B}} \rho\mathbf{x} \times \mathbf{f} \, dV + \int_{\partial\mathcal{B}} \mathbf{x} \times (\mathbf{t}_a + \mathbf{t}_m) \, dS = \mathbf{0}. \quad (94)$$

Constitutive equations – Eulerian formulations. At our disposal we have the three magnetic field vectors $\mathbf{B}, \mathbf{H}, \mathbf{M}$, with the connection $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ from (42). Any one of these can be used as the independent magnetic variable in the formulation of a constitutive law for a deformable magnetizable material along with the deformation gradient tensor \mathbf{F} . Such a constitutive law involves a scalar potential function or ‘energy’ function. We now examine several examples of such potential functions and these will then be considered within an energy balance framework via a virtual work formulation.

As a first example we consider a formulation based on use of the magnetic induction vector \mathbf{B} and we introduce the energy density function $\phi(\mathbf{F}, \mathbf{B})$, defined per unit mass. Based on standard thermodynamic arguments involving a free energy function (see, for example, the chapter by Maugin in

this volume for a general discussion), this yields the stress tensor, denoted $\boldsymbol{\sigma}$, and the magnetization \mathbf{M} in the forms

$$\boldsymbol{\sigma} = \rho \mathbf{F} \frac{\partial \phi}{\partial \mathbf{F}}, \quad \mathbf{M} = -\rho \frac{\partial \phi}{\partial \mathbf{B}}, \quad (95)$$

and it can be shown that the equilibrium equation takes the form

$$\operatorname{div} \boldsymbol{\sigma} + (\operatorname{grad} \mathbf{B})^T \mathbf{M} + \rho \mathbf{f} = \mathbf{0}. \quad (96)$$

Thus, in this case we have

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}, \quad \hat{\boldsymbol{\tau}}_m = \boldsymbol{\tau}_m + \frac{1}{2} \mu_0^{-1} (\mathbf{M} \cdot \mathbf{M}) \mathbf{I}, \quad \hat{\mathbf{f}}_m = (\operatorname{grad} \mathbf{B})^T \mathbf{M}. \quad (97)$$

A second example also involves \mathbf{B} as the independent magnetic variable. This makes use of the energy density function $\phi^*(\mathbf{F}, \mathbf{B})$, which is related to ϕ by

$$\rho \phi^*(\mathbf{F}, \mathbf{B}) = \rho \phi(\mathbf{F}, \mathbf{B}) + \frac{1}{2} \mu_0^{-1} \mathbf{B} \cdot \mathbf{B}. \quad (98)$$

This yields a stress tensor, denoted $\boldsymbol{\sigma}^*$, and the magnetic field:

$$\boldsymbol{\sigma}^* = \rho \mathbf{F} \frac{\partial \phi^*}{\partial \mathbf{F}}, \quad \mathbf{H} = \rho \frac{\partial \phi^*}{\partial \mathbf{B}}. \quad (99)$$

In this case we have

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^*, \quad \hat{\boldsymbol{\tau}}_m = \mathbf{B} \otimes \mathbf{H} - (\mathbf{H} \cdot \mathbf{B}) \mathbf{I}, \quad \hat{\mathbf{f}}_m = -(\operatorname{grad} \mathbf{B})^T \mathbf{H}. \quad (100)$$

An alternative starting point is to consider the magnetic field \mathbf{H} as the independent magnetic variable and to work in terms of the potential function $\psi(\mathbf{F}, \mathbf{H})$. This yields a stress tensor, which we denote by $\bar{\boldsymbol{\sigma}}$, and the magnetization in the forms

$$\bar{\boldsymbol{\sigma}} = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}}, \quad \mathbf{M} = -\mu_0^{-1} \rho \frac{\partial \psi}{\partial \mathbf{H}}, \quad (101)$$

and now we have

$$\hat{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}, \quad \hat{\boldsymbol{\tau}}_m = \boldsymbol{\tau}_m, \quad \hat{\mathbf{f}}_m = \mu_0 (\operatorname{grad} \mathbf{H}) \mathbf{M}. \quad (102)$$

The final example introduces the potential function $\psi^*(\mathbf{F}, \mathbf{H})$, related to ψ by

$$\rho \psi^*(\mathbf{F}, \mathbf{H}) = \rho \psi(\mathbf{F}, \mathbf{H}) - \frac{1}{2} \mu_0 \mathbf{H} \cdot \mathbf{H}, \quad (103)$$

and the associated stress, denoted $\bar{\boldsymbol{\sigma}}^*$, and the magnetic induction are derived as

$$\bar{\boldsymbol{\sigma}}^* = \rho \mathbf{F} \frac{\partial \psi^*}{\partial \mathbf{F}}, \quad \mathbf{B} = -\rho \frac{\partial \psi^*}{\partial \mathbf{H}}. \quad (104)$$

In this case we have

$$\hat{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}^*, \quad \hat{\boldsymbol{\tau}}_m = \mathbf{B} \otimes \mathbf{H}, \quad \hat{\mathbf{f}}_m = (\text{grad } \mathbf{H})\mathbf{B}. \quad (105)$$

The following connections between the potential functions are also noted for completeness:

$$\rho\psi(\mathbf{F}, \mathbf{H}) = \rho\phi(\mathbf{F}, \mathbf{B}) + \frac{1}{2}\mu_0\mathbf{M} \cdot \mathbf{M}, \quad (106)$$

$$\bar{\rho}\phi^*(\mathbf{F}, \mathbf{B}) = \rho\psi^*(\mathbf{F}, \mathbf{H}) + \mathbf{B} \cdot \mathbf{H}. \quad (107)$$

The latter is a Legendre-type transformation.

The above formulations are just a selection of the possible alternatives. For convenience the relevant expressions are collected together in Table 1, together with corresponding expressions for two possibilities for which the magnetization is the independent variable.

Potential	Stress $\hat{\boldsymbol{\sigma}}$	Magnetic Vector	Body Force $\hat{\mathbf{f}}_m$
$\phi(\mathbf{F}, \mathbf{B})$	$\rho\mathbf{F} \frac{\partial\phi}{\partial\mathbf{F}}$	$\mathbf{M} = -\rho \frac{\partial\phi}{\partial\mathbf{B}}$	$(\text{grad } \mathbf{B})^T \mathbf{M}$
$\phi^*(\mathbf{F}, \mathbf{B})$	$\rho\mathbf{F} \frac{\partial\phi^*}{\partial\mathbf{F}}$	$\mathbf{H} = \rho \frac{\partial\phi^*}{\partial\mathbf{B}}$	$-(\text{grad } \mathbf{B})^T \mathbf{H}$
$\psi(\mathbf{F}, \mathbf{H})$	$\rho\mathbf{F} \frac{\partial\psi}{\partial\mathbf{F}}$	$\mathbf{M} = -\mu_0^{-1}\rho \frac{\partial\psi}{\partial\mathbf{H}}$	$\mu_0(\mathbf{M} \cdot \text{grad})\mathbf{H}$
$\psi^*(\mathbf{F}, \mathbf{H})$	$\rho\mathbf{F} \frac{\partial\psi^*}{\partial\mathbf{F}}$	$\mathbf{B} = -\rho \frac{\partial\psi^*}{\partial\mathbf{H}}$	$(\mathbf{B} \cdot \text{grad})\mathbf{H}$
$\chi(\mathbf{F}, \mathbf{M})$	$\rho\mathbf{F} \frac{\partial\chi}{\partial\mathbf{F}}$	$\mathbf{H} = \mu_0^{-1}\rho \frac{\partial\chi}{\partial\mathbf{M}}$	$-\mu_0(\text{grad } \mathbf{M})^T \mathbf{H}$
$\chi^*(\mathbf{F}, \mathbf{M})$	$\rho\mathbf{F} \frac{\partial\chi^*}{\partial\mathbf{F}}$	$\mathbf{B} = \rho \frac{\partial\chi^*}{\partial\mathbf{M}}$	$-(\text{grad } \mathbf{M})^T \mathbf{B}$

Table 1: Energy (potential) functions based on \mathbf{B} , \mathbf{H} , \mathbf{M} , and in each case the associated stress $\hat{\boldsymbol{\sigma}}$, the derived magnetic field vector, and the magnetic body force $\hat{\mathbf{f}}_m$.

None of these options, however, allows the total stress tensor to be given directly in the form $\rho\mathbf{F}\partial(\text{potential function})/\partial\mathbf{F}$, although the equilibrium equation has its simplest mathematical statement in terms of the total stress and it avoids the need to define either a Maxwell stress or a magnetic body force within the material. We shall return to this point shortly in deriving a formulation that allows for a potential of the desired kind.

The notions of ‘stress’, ‘Maxwell stress’ and ‘magnetic body force’ *inside a magnetizable material* are clearly not uniquely defined. Outside the material the situation is clearer. We suppose that the deformable and magnetizable material is confined to the domain \mathcal{B}_0 in the reference configuration so that the deformation gradient \mathbf{F} is defined only for points \mathbf{X} within \mathcal{B}_0 . However, the magnetic field is not so restricted and can permeate the whole space. Outside the material we have $\mathbf{M} = \mathbf{0}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ and $\phi = \psi = 0$ and the associated stresses $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}$ vanish, leaving the total stress as

$$\boldsymbol{\tau} = \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{H})\mathbf{I}. \quad (108)$$

Note that ϕ^* and ψ^* do not vanish, nor do $\boldsymbol{\sigma}^*$ and $\bar{\boldsymbol{\sigma}}^*$. In fact, we have $\rho\phi^* = \frac{1}{2}\mathbf{B} \cdot \mathbf{H}$, which represents the magnetostatic energy density (per unit volume) outside the material (although the factor ρ has no meaning there).

Let us now denote the magnetic and magnetic induction field vectors outside the material as $\mathbf{H}^{(o)}$ and $\mathbf{B}^{(o)} = \mu_0 \mathbf{H}^{(o)}$, respectively, and the corresponding Maxwell stress as $\boldsymbol{\tau}_m^{(o)}$, which we write as

$$\boldsymbol{\tau}_m^{(o)} = \mathbf{B}^{(o)} \otimes \mathbf{H}^{(o)} - \frac{1}{2}(\mathbf{B}^{(o)} \cdot \mathbf{H}^{(o)})\mathbf{I}. \quad (109)$$

This induces a ‘traction’ $\boldsymbol{\tau}_m^{(o)} \mathbf{n}$ on the boundary $\partial\mathcal{B}$ of the material that is equivalent to the effect of a body force. If this is combined with an applied *mechanical* traction, \mathbf{t}_a say, per unit area of $\partial\mathcal{B}$ then the total traction is $\mathbf{t}_a + \boldsymbol{\tau}_m^{(o)} \mathbf{n}$ per unit area of the exterior of $\partial\mathcal{B}$. For equilibrium this must be matched by the corresponding total traction calculated on the interior of $\partial\mathcal{B}$, i.e. we must have

$$\boldsymbol{\tau} \mathbf{n} = \mathbf{t}_a + \boldsymbol{\tau}_m^{(o)} \mathbf{n} \quad \text{on } \partial\mathcal{B}, \quad (110)$$

where $\boldsymbol{\tau}$ is the (symmetric) total stress in \mathcal{B} .

The boundary condition (110) may also be expressed in terms of other stress tensors by making use of the connection (89), bearing in mind that in general $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\tau}}_m$ are not symmetric. This yields the boundary condition for $\hat{\boldsymbol{\sigma}}$ as

$$\hat{\boldsymbol{\sigma}}^T \mathbf{n} = \mathbf{t}_a + (\boldsymbol{\tau}_m^{(o)} - \hat{\boldsymbol{\tau}}_m^T) \mathbf{n}. \quad (111)$$

The Maxwell traction jump in (111) can be evaluated by making use of the jump conditions (65). In the case of $\bar{\boldsymbol{\sigma}}$, for example, this gives, after some manipulations, the standard result

$$\bar{\boldsymbol{\sigma}}^T \mathbf{n} = \mathbf{t}_a + \bar{\mathbf{t}}_m \equiv \mathbf{t}_a + \frac{1}{2}\mu_0(\mathbf{M} \cdot \mathbf{n})^2 \mathbf{n} \quad \text{on } \partial\mathcal{B}. \quad (112)$$

It is worth pointing out here that in general there is no stress tensor $\hat{\boldsymbol{\sigma}}$ for which the jump term in (111) vanishes and which therefore matches the mechanical traction alone. This is because the final term in (112) is not linear in \mathbf{n} . However, exceptions to this are when either $\mathbf{M} \cdot \mathbf{n} = 0$ or $\mathbf{n} \times \mathbf{M} = \mathbf{0}$ *everywhere* on $\partial\mathcal{B}$.

Virtual work formulation. With reference to the generic formulation based on the equilibrium equation (86) and the traction boundary condition (84), we now consider the total virtual work consisting of the virtual mechanical work of the body and surface forces on a virtual displacement, denoted $\dot{\mathbf{x}}$, and the virtual magnetic work, the form of which depends on the choice of variables. The total virtual work is then written in the form

$$\int_V (\rho \mathbf{f} + \hat{\mathbf{f}}_m) \cdot \dot{\mathbf{x}} dV + \int_S (\mathbf{t}_a + \hat{\mathbf{t}}_m) \cdot \dot{\mathbf{x}} dS + \int_V \dot{w}_m dV, \quad (113)$$

where the first two integrals represent the virtual mechanical work due to all the (mechanical and magnetic) forces, while the third integral is written in terms of the virtual magnetic work density \dot{w}_m (per unit volume), where the superimposed dot signifies a virtual increment.

By using (84) in the second integral and applying the divergence theorem and using (86) the expression (113) reduces to

$$\int_V [\text{tr}(\hat{\boldsymbol{\sigma}} \text{grad} \dot{\mathbf{x}}) + \dot{w}_m] dV. \quad (114)$$

In pure hyperelasticity theory, i.e. without the magnetic term, this would represent the virtual increase in stored elastic energy. By analogy, we can consider (114) to represent the virtual increase in stored magnetoelastic energy. This suggests the introduction of an energy density (per unit mass), which we denote by $\hat{\phi}$, such that

$$\rho \dot{\hat{\phi}} = \text{tr}(\hat{\boldsymbol{\sigma}} \text{grad} \dot{\mathbf{x}}) + \dot{w}_m. \quad (115)$$

Now suppose that $\hat{\phi}$ depends on the deformation gradient \mathbf{F} and some magnetic vector, say $\boldsymbol{\mu}$: $\hat{\phi}(\mathbf{F}, \boldsymbol{\mu})$. Then it follows that

$$\dot{\hat{\phi}} = \text{tr} \left(\frac{\partial \hat{\phi}}{\partial \mathbf{F}} \dot{\mathbf{F}} \right) + \frac{\partial \hat{\phi}}{\partial \boldsymbol{\mu}} \cdot \dot{\boldsymbol{\mu}}, \quad (116)$$

where again a superimposed dot signifies a virtual increment, and we note that $\dot{\mathbf{F}} = (\text{grad} \dot{\mathbf{x}}) \mathbf{F}$.

Comparison of (116) with (115), which must hold for all virtual increments, shows that

$$\hat{\boldsymbol{\sigma}} = \rho \mathbf{F} \frac{\partial \hat{\phi}}{\partial \mathbf{F}}, \quad \dot{w}_m = \rho \frac{\partial \hat{\phi}}{\partial \boldsymbol{\mu}} \cdot \dot{\boldsymbol{\mu}}. \quad (117)$$

We illustrate this in relation to two of the examples considered above. First, with $\boldsymbol{\mu} = \mathbf{B}$ we have $\hat{\phi} = \phi(\mathbf{F}, \mathbf{B})$, and hence, via (95)

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} = \rho \mathbf{F} \frac{\partial \phi}{\partial \mathbf{F}}, \quad \mathbf{M} = -\rho \frac{\partial \phi}{\partial \mathbf{B}}, \quad \dot{w}_m = -\mathbf{M} \cdot \dot{\mathbf{B}}. \quad (118)$$

Second, with $\boldsymbol{\mu} = \mathbf{H}$ we have $\hat{\phi} = \psi(\mathbf{F}, \mathbf{H})$, and hence, from (101),

$$\hat{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}} = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}}, \quad \mu_0 \mathbf{M} = -\rho \frac{\partial \psi}{\partial \mathbf{H}}, \quad \dot{w}_m = -\mu_0 \mathbf{M} \cdot \dot{\mathbf{H}}. \quad (119)$$

With reference to the connection (106) between ψ and ϕ it can be seen that in moving from the virtual work balance based on ϕ to that based on ψ the term $\mu_0 \mathbf{M} \cdot \dot{\mathbf{M}}$ is added to \dot{w}_m . Clearly, the different formulations are equivalent in terms of energy balance, but the choice of magnetic variable influences the description of magnetic energy within the material.

Constitutive equations – Lagrangian formulations. The equilibrium equation (90) in terms of the total Cauchy stress tensor $\boldsymbol{\tau}$ may be converted to Lagrangian form by defining, as is done in the context of nonlinear elasticity theory (see, for example, Ogden, 1997), an associated total *nominal* stress tensor, which is here denoted \mathbf{T} and defined by

$$\mathbf{T} = J \mathbf{F}^{-1} \boldsymbol{\tau}. \quad (120)$$

Then, by using (55), equation (90) may be written in the alternative form

$$\text{Div } \mathbf{T} + \rho_0 \mathbf{f} = \mathbf{0}, \quad (121)$$

where $\rho_0 = \rho J$ is the mass density of the material in the reference configuration \mathcal{B}_0 . This change to Lagrangian form is now coupled with a corresponding change in the representation of the potential functions, which leads to an elegant formulation of the constitutive law for a nonlinear magnetoelastic material with an accompanying simple structure of the governing equations.

We base the following development on the potential function ϕ^* , which depends on \mathbf{F} and \mathbf{B} . Thus, we write $\phi^* = \phi^*(\mathbf{F}, \mathbf{B})$. In view of the connection (61) between \mathbf{B} and \mathbf{B}_l we may regard $\phi^*(\mathbf{F}, \mathbf{B})$, equivalently, as a function of \mathbf{F} and \mathbf{B}_l , and we introduce the notation Φ^* for this purpose. This is defined by

$$\Phi^*(\mathbf{F}, \mathbf{B}_l) \equiv \phi^*(\mathbf{F}, J^{-1} \mathbf{F} \mathbf{B}_l). \quad (122)$$

Note that since \mathbf{B}_l is a Lagrangian vector it is indifferent to observer transformations in the deformed configuration, i.e. in the present context it is unaffected by a superimposed rotation defined by the proper orthogonal tensor \mathbf{Q} in the deformed configuration, while the deformation gradient \mathbf{F} changes to \mathbf{QF} . For Φ^* to be frame indifferent (objective) we must have

$$\Phi^*(\mathbf{QF}, \mathbf{B}_l) = \Phi^*(\mathbf{F}, \mathbf{B}_l) \quad (123)$$

for all proper orthogonal \mathbf{Q} . This requirement is guaranteed if Φ^* is regarded as a function of the right Cauchy–Green tensor $\mathbf{c} = \mathbf{F}^T \mathbf{F}$, which we assume implicitly to be the case. Thus, Φ^* is a function of \mathbf{c} and \mathbf{B}_l .

We have shown above that in the Eulerian formulation we have

$$\boldsymbol{\sigma}^* = \rho \mathbf{F} \frac{\partial \phi^*}{\partial \mathbf{F}}, \quad \mathbf{H} = \rho \frac{\partial \phi^*}{\partial \mathbf{B}}. \quad (124)$$

When re-cast in terms of Φ^* these equations become

$$\boldsymbol{\sigma}^* = \rho \mathbf{F} \frac{\partial \Phi^*}{\partial \mathbf{F}} - \mathbf{B} \otimes \mathbf{H} + (\mathbf{B} \cdot \mathbf{H}) \mathbf{I}, \quad \mathbf{H} = \rho J \mathbf{F}^{-T} \frac{\partial \Phi^*}{\partial \mathbf{B}_l}, \quad (125)$$

and hence, by (89) and (100)_{1,2}, we obtain the simple formula

$$\boldsymbol{\tau} = \rho \mathbf{F} \frac{\partial \Phi^*}{\partial \mathbf{F}}. \quad (126)$$

By (120) and (59) the corresponding Lagrangian expressions are

$$\mathbf{T} = \rho_0 \frac{\partial \Phi^*}{\partial \mathbf{F}}, \quad \mathbf{H}_l = \rho_0 \frac{\partial \Phi^*}{\partial \mathbf{B}_l}, \quad (127)$$

wherein we have used the connection $\rho_0 = \rho J$.

For convenience we now define the potential function $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l)$, per unit reference *volume*, by

$$\Omega(\mathbf{F}, \mathbf{B}_l) = \rho_0 \Phi^*(\mathbf{F}, \mathbf{B}_l), \quad (128)$$

so that the formulas (127) become simply

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}. \quad (129)$$

The corresponding formulas for $\boldsymbol{\tau}$ and \mathbf{H} are

$$\boldsymbol{\tau} = J^{-1} \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-T} \frac{\partial \Omega}{\partial \mathbf{B}_l}. \quad (130)$$

When \mathbf{B}_l is used as the independent magnetic variable, equations (129)₁ and (129)₂ are inserted into the equilibrium equation (121) and the equation (63)₂, while \mathbf{B}_l itself satisfies (63)₁. These coupled equations, when combined with appropriate boundary conditions, provide the equations governing the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, with $\mathbf{F} = \text{Grad } \mathbf{x}$, and a vector potential \mathbf{A}_l , with $\mathbf{B}_l = \text{Curl } \mathbf{A}_l$.

If, instead of \mathbf{B}_l , we wish to use \mathbf{H}_l as the independent magnetic variable then we can adopt the following approach. Let us now define, analogously to the definition (122), the potential function Ψ^* by

$$\Psi^*(\mathbf{F}, \mathbf{H}_l) = \psi^*(\mathbf{F}, \mathbf{F}^{-\text{T}} \mathbf{H}_l). \quad (131)$$

Then, by using (107), the connections (59) and (61), and $\rho_0 = \rho J$, we obtain

$$\rho_0 \Psi^* = \rho_0 \Phi^* - \mathbf{B}_l \cdot \mathbf{H}_l. \quad (132)$$

Introduction of the notation Ω^* , defined by

$$\Omega^*(\mathbf{F}, \mathbf{H}_l) = \rho_0 \Psi^*(\mathbf{F}, \mathbf{H}_l), \quad (133)$$

leads to the Legendre transformation

$$\Omega^* = \Omega - \mathbf{B}_l \cdot \mathbf{H}_l, \quad (134)$$

and in terms of Ω^* we then obtain the counterparts of equations (129) as

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l}. \quad (135)$$

For the validity of the Legendre transform one would require that \mathbf{B}_l be a monotonic function of \mathbf{H}_l . However, one could avoid this by starting with Ω^* instead of deriving it via (134). In this case equation (63)₂, with $\mathbf{J}_l = \mathbf{0}$, is satisfied by taking the independent variable \mathbf{H}_l in the form $-\text{Grad } \varphi_l$ for some scalar function φ_l , and the remaining equations are then coupled as equations for $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ and $\varphi_l(\mathbf{X})$.

In terms of the virtual work energy balance, here we set $\rho_0 \hat{\phi}$ as Ω , respectively Ω^* . The corresponding virtual magnetic work is then given by $\hat{w}_m = J^{-1} \mathbf{H}_l \cdot \dot{\mathbf{B}}_l$, respectively $\hat{w}_m = -J^{-1} \mathbf{B}_l \cdot \dot{\mathbf{H}}_l$. For a review of related variational approaches to the formulation of the equations of magnetoelasticity, which are not discussed here, we refer to the paper by Bustamante et al. (2008).

Incompressible materials. The expressions for the various stress tensors in the foregoing apply for a material that is not subject to any internal mechanical constraint. For an important class of materials, including magneto-sensitive elastomers, it is appropriate to adopt the constraint of incompressibility, in which case the expressions for the stresses require modification.

For an incompressible material we have the constraint

$$\det \mathbf{F} \equiv 1. \quad (136)$$

The total nominal and Cauchy stresses given by (129)₁ and (130)₁ in terms of Ω are then amended in the forms

$$\boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \quad (137)$$

respectively, where p is a Lagrange multiplier associated with the constraint (136). The expressions (129)₂ and (130)₂ are unchanged except that (136) is in force. In terms of Ω^* we have, instead of (137),

$$\boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega^*}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}} - p\mathbf{F}^{-1}. \quad (138)$$

In general the p in (138) need not be the same as in (137).

4.2 Material Symmetry Considerations

Thus far no restrictions have been placed on the forms of the potential functions other than those required by objectivity, so that considerable generality remains. Other restrictions may be physically or mathematically based. For example, physical restrictions arise from the nature of the material itself, such as its inherent symmetry. Magneto-sensitive elastomers are typically isotropic in their response in the absence of a magnetic field, but application of a magnetic field endows the material with a preferred direction. Thus, the magnetic induction vector \mathbf{B} generates a preferred direction in the deformed configuration \mathcal{B} . However, from the point of view of constitutive law development, it is advantageous to make use of the Lagrangian field \mathbf{B}_l instead of \mathbf{B} , and to consider the potential function Ω .

For simplicity we restrict attention to so-called *isotropic magnetoelastic materials*, for which the material symmetry considerations are similar to those that arise for a transversely isotropic elastic material, which possesses a preferred direction in the reference configuration (see, for example, Merodio and Ogden, 2005). This is appropriate for fiber-reinforced materials, for which the preferred direction is the fiber direction in the reference configuration. The vector field \mathbf{B}_l has an analogous role in the present context.

The magnetoelastic material considered here is said to be *isotropic* if Ω is an isotropic function of the two tensors \mathbf{c} and $\mathbf{B}_l \otimes \mathbf{B}_l$. Note that the latter expression is unaffected by reversal of the sign of \mathbf{B}_l . Then, the form of Ω is reduced to dependence on the principal invariants I_1, I_2, I_3 of \mathbf{c} , defined by

$$I_1 = \text{tr } \mathbf{c}, \quad I_2 = \frac{1}{2} [(\text{tr } \mathbf{c})^2 - \text{tr}(\mathbf{c}^2)], \quad I_3 = \det \mathbf{c} = J^2, \quad (139)$$

together with three invariants that depend on \mathbf{B}_l . A convenient choice of the latter, but by no means the only option, is

$$I_4 = |\mathbf{B}_l|^2, \quad I_5 = (\mathbf{c}\mathbf{B}_l) \cdot \mathbf{B}_l, \quad I_6 = (\mathbf{c}^2\mathbf{B}_l) \cdot \mathbf{B}_l. \quad (140)$$

Note that for a transversely isotropic elastic material the counterpart of the invariant I_4 would be absent since in that case the preferred direction is a unit vector.

In the following the subscripts $1, 2, \dots, 6$ on Ω signify differentiation with respect to I_1, I_2, \dots, I_6 , respectively. A direct calculation based on (130)₁ leads to

$$\boldsymbol{\tau} = J^{-1} [2\Omega_1 \mathbf{b} + 2\Omega_2 (I_1 \mathbf{b} - \mathbf{b}^2) + 2I_3 \Omega_3 \mathbf{I} + 2\Omega_5 \mathbf{B} \otimes \mathbf{B} + 2\Omega_6 (\mathbf{B} \otimes \mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B} \otimes \mathbf{B})], \quad (141)$$

which is clearly symmetric, and

$$\mathbf{H} = 2(\Omega_4 \mathbf{b}^{-1} \mathbf{B} + \Omega_5 \mathbf{B} + \Omega_6 \mathbf{b}\mathbf{B}), \quad (142)$$

and we recall that $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy–Green deformation tensor. The corresponding Lagrangian forms may be obtained from the connections $\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}$ and $\mathbf{H}_l = \mathbf{F}^T \mathbf{H}$.

For an incompressible material $I_3 \equiv 1$ and (141) is replaced by

$$\boldsymbol{\tau} = 2\Omega_1 \mathbf{b} + 2\Omega_2 (I_1 \mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5 \mathbf{B} \otimes \mathbf{B} + 2\Omega_6 (\mathbf{B} \otimes \mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B} \otimes \mathbf{B}), \quad (143)$$

while (142) is unchanged in form, but with I_3 absent from Ω .

If we work with Ω^* instead of Ω then the invariants based on \mathbf{B}_l have to be changed to invariants based on \mathbf{H}_l . These are denoted here by K_4, K_5, K_6 and may be defined, analogously to (140), by

$$K_4 = |\mathbf{H}_l|^2, \quad K_5 = (\mathbf{c}\mathbf{H}_l) \cdot \mathbf{H}_l, \quad K_6 = (\mathbf{c}^2\mathbf{H}_l) \cdot \mathbf{H}_l. \quad (144)$$

The associated formulas for $\boldsymbol{\tau}$ are similar to those based on Ω . For an incompressible material, for example, we have

$$\boldsymbol{\tau} = 2\Omega_1^* \mathbf{b} + 2\Omega_2^* (I_1 \mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5^* \mathbf{b}\mathbf{H} \otimes \mathbf{b}\mathbf{H} + 2\Omega_6^* (\mathbf{b}\mathbf{H} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{b}\mathbf{H}). \quad (145)$$

The magnetic induction is

$$\mathbf{B} = -2(\Omega_4^* \mathbf{b}\mathbf{H} + \Omega_5^* \mathbf{b}^2 \mathbf{H} + \Omega_6^* \mathbf{b}^3 \mathbf{H}). \quad (146)$$

The latter may be rearranged, if required, by using the Cayley-Hamilton theorem in the form

$$\mathbf{b}^3 = I_1 \mathbf{b}^2 - I_2 \mathbf{b} + \mathbf{I}, \quad (147)$$

for an incompressible material, for which $\Omega^* = \Omega^*(I_1, I_2, K_4, K_5, K_6)$. In the above equations Ω_i^* is defined as $\partial\Omega^*/\partial I_i$ for $i = 1, 2$ and $\partial\Omega^*/\partial K_i$ for $i = 4, 5, 6$.

There are important differences between the formulations based on Ω and Ω^* in respect of their application to particular boundary-value problems. If \mathbf{B}_l is taken as the independent variable then it has to satisfy $\text{Div} \mathbf{B}_l = 0$. The resulting \mathbf{H}_l , calculated from (129)₂, then has to satisfy the vector equation $\text{Curl} \mathbf{H}_l = \mathbf{0}$, which, for some problems, puts severe restrictions on the class of constitutive laws that admit the deformation in question for the considered magnetic induction field \mathbf{B}_l . On the other hand, if we start with \mathbf{H}_l as the independent variable it has to satisfy $\text{Curl} \mathbf{H}_l = \mathbf{0}$ and then the resulting \mathbf{B}_l , calculated from (135)₂, must satisfy the scalar equation $\text{Div} \mathbf{B}_l = 0$. This also may, in some situations, put restrictions on the admissible class of constitutive laws, but they are different from and generally less severe than for the other formulation.

A more general model than the isotropic model considered here has been developed by Bustamante (2010). This is a transversely isotropic model for which, in addition to the preferred direction due to the applied field, there is a second preferred direction, defined in the reference configuration, which is associated with alignment of magnetic particles during the curing process and is ‘frozen in’ to the material by the cure (see, for example, Bellan and Bossis, 2002; Varga et al., 2005, 2006). For details of this model, which involves a total of 10 invariants (9 for an incompressible material), we refer to Bustamante (2010).

5 Representative Boundary-Value Problems

We now apply the equations described in the preceding sections to two representative boundary-value problems in order to illustrate the theory. Consider an incompressible isotropic magnetoelastic body in the absence of mechanical body forces. The equations to be solved are summarized conveniently here as

$$\text{div} \mathbf{B} = 0, \quad \text{curl} \mathbf{H} = \mathbf{0}, \quad \text{div} \boldsymbol{\tau} = \mathbf{0} \quad \text{in } \mathcal{B}, \quad (148)$$

in Eulerian form, and

$$\text{Div} \mathbf{B}_l = 0, \quad \text{Curl} \mathbf{H}_l = \mathbf{0}, \quad \text{Div} \mathbf{T} = \mathbf{0} \quad \text{in } \mathcal{B}_0, \quad (149)$$

in Lagrangian form, with the Lagrangian–Eulerian interconnections

$$\mathbf{B}_l = \mathbf{F}^{-1} \mathbf{B}, \quad \mathbf{H}_l = \mathbf{F}^T \mathbf{H}, \quad \mathbf{T} = \mathbf{F}^{-1} \boldsymbol{\tau}. \quad (150)$$

We also recall the definition of the magnetization. In Eulerian form this is

$$\mathbf{M} = \mu_0^{-1} \mathbf{B} - \mathbf{H}. \quad (151)$$

It is possible to define a Lagrangian form of the magnetization, but we shall not need this here.

The associated Eulerian and Lagrangian boundary conditions are collected here from (58) and (64) as

$$\mathbf{n} \cdot \llbracket \mathbf{B} \rrbracket = 0, \quad \mathbf{n} \times \llbracket \mathbf{H} \rrbracket = \mathbf{0} \quad \text{on } \partial \mathcal{B} \quad (152)$$

and

$$\mathbf{N} \cdot \llbracket \mathbf{B}_l \rrbracket = 0, \quad \mathbf{N} \times \llbracket \mathbf{H}_l \rrbracket = \mathbf{0} \quad \text{on } \partial \mathcal{B}_0, \quad (153)$$

respectively.

On any part of the boundary, say $\partial \mathcal{B}_t \subset \partial \mathcal{B}$, where the mechanical traction \mathbf{t}_a is prescribed the traction boundary condition is given by

$$\boldsymbol{\tau} \mathbf{n} = \mathbf{t}_a + \boldsymbol{\tau}_m^{(o)} \mathbf{n} \quad \text{on } \partial \mathcal{B}_t, \quad (154)$$

where $\boldsymbol{\tau}_m^{(o)} \mathbf{n}$ is the effective traction due to the Maxwell stress calculated on the exterior of $\partial \mathcal{B}_t$. The boundary condition (154) may be expressed in Lagrangian form by making use of Nanson's formula $\mathbf{n} dS = \mathbf{J} \mathbf{F}^{-T} \mathbf{N} dS_0$ to give

$$\mathbf{T}^T \mathbf{N} = \mathbf{t}_A + \boldsymbol{\tau}_m^{(o)} \mathbf{F}^{-T} \mathbf{N} \quad \text{on } \partial \mathcal{B}_{t0}, \quad (155)$$

where \mathbf{t}_A is the applied mechanical traction per unit reference area and $\partial \mathcal{B}_{t0}$ is the reference counterpart of $\partial \mathcal{B}_t$.

5.1 Application to Circular Cylindrical Geometry

We now specialize to problems with circular cylindrical geometry for which it is convenient to express the variables and equations in terms of cylindrical polar coordinates. In the reference configuration these are denoted by (R, Θ, Z) and in the deformed configuration by (r, θ, z) . The component forms of the equations (149)_{1,2} are

$$\frac{\partial B_{lR}}{\partial R} + \frac{1}{R} B_{lR} + \frac{1}{R} \frac{\partial B_{l\Theta}}{\partial \Theta} + \frac{\partial B_{lZ}}{\partial Z} = 0, \quad (156)$$

and

$$\frac{1}{R} \frac{\partial H_{lZ}}{\partial \Theta} - \frac{\partial H_{l\Theta}}{\partial Z} = 0, \quad \frac{\partial H_{lR}}{\partial Z} - \frac{\partial H_{lZ}}{\partial R} = 0, \quad \frac{1}{R} \frac{\partial (RH_{l\Theta})}{\partial R} - \frac{1}{R} \frac{\partial H_{lR}}{\partial \Theta} = 0, \quad (157)$$

where $(B_{lR}, B_{l\Theta}, B_{lZ})$ are the components of \mathbf{B}_l and $(H_{lR}, H_{l\Theta}, H_{lZ})$ those of \mathbf{H}_l . Note that if \mathbf{H}_l is chosen as the independent variable, the corresponding magnetic induction \mathbf{B}_l is given by the constitutive equation (146) with $\mathbf{B}_l = \mathbf{F}^{-1}\mathbf{B}$, and its components must satisfy the scalar equation (156). On the other hand, if the vector \mathbf{B}_l is adopted as the independent variable, the corresponding magnetic field is given by the constitutive law (142) with $\mathbf{H}_l = \mathbf{F}^T\mathbf{H}$, the components of which must satisfy the equations (157). It is somewhat easier to work with \mathbf{H}_l as the independent variable since this leaves only a single scalar equation, namely (156), to be satisfied by the components of \mathbf{B}_l .

We apply these equations to a circular cylindrical tube and specialize by considering the components of \mathbf{B}_l and \mathbf{H}_l to be independent of Θ and Z , so that circular cylindrical symmetry is preserved. We consider the tube to have infinite length in order to avoid difficulties associated with compatibility of the magnetic boundary conditions on the ends of the tube and on its lateral surfaces. The above equations can now be integrated to give

$$B_{lR} = \frac{C_1}{R}, \quad H_{l\Theta} = \frac{C_2}{R}, \quad H_{lZ} = C_3, \quad (158)$$

where C_1, C_2, C_3 are constants. Corresponding solutions can be given for B_r, H_θ and H_z . However, since the normal component B_{lR} must be continuous across the inner boundary of the tube we must take $C_1 = 0$ so that there is no singularity at $R = 0$. Thus, we must have $B_{lR} = 0$.

The corresponding component forms of the equilibrium equation (149)₃ may also be written down. However, here we use their Eulerian counterparts. When dependence on θ and z is omitted the components of equation (148)₃ are expressed compactly as

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0, \quad \frac{d}{dr}(r^2\tau_{r\theta}) = 0, \quad \frac{d}{dr}(r\tau_{rz}) = 0. \quad (159)$$

5.2 Helical Shear

Let the reference geometry of the tube be defined by the inequalities

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad -\infty < Z < \infty, \quad (160)$$

so it has inner and outer radii A and B , respectively. The tube is subject to the helical shear deformation described by the equations

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z + w(R), \quad (161)$$

where $g(R)$ and $w(R)$ are functions of R to be determined by solving the governing equations and applying the boundary conditions. The deformation defined by (161) is a combination of a pure axial shear deformation and a pure azimuthal shear deformation. Since $r = R$, we use r as the variable in the functions g and w , and we also set $a = A$ and $b = B$. We assume that the inner boundary of the tube is fixed and let the outer boundary be subject to an azimuthal rotation through an angle β and an axial displacement d , so that

$$g(a) = 0, \quad g(b) = \beta, \quad w(a) = 0, \quad w(b) = d. \quad (162)$$

Referred to the two sets of cylindrical polar coordinate axes, the components of the deformation gradient \mathbf{F} are represented by the matrix \mathbf{F} , which is given by

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ rg'(r) & 1 & 0 \\ w'(r) & 0 & 1 \end{pmatrix}, \quad (163)$$

where the prime indicates differentiation with respect to r . For convenience, we use the notations

$$\gamma_\theta = rg'(r), \quad \gamma_z = w'(r), \quad \gamma^2 = \gamma_\theta^2 + \gamma_z^2. \quad (164)$$

The combined deformation is locally a simple shear with amount of shear γ (see, for example, Ogden et al., 1973). The corresponding matrices of the left and right Cauchy–Green tensors $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{c} = \mathbf{F}^T\mathbf{F}$, written \mathbf{b} and \mathbf{c} , respectively, are given by

$$\mathbf{b} = \begin{pmatrix} 1 & \gamma_\theta & \gamma_z \\ \gamma_\theta & 1 + \gamma_\theta^2 & \gamma_\theta\gamma_z \\ \gamma_z & \gamma_\theta\gamma_z & 1 + \gamma_z^2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 + \gamma^2 & \gamma_\theta & \gamma_z \\ \gamma_\theta & 1 & 0 \\ \gamma_z & 0 & 1 \end{pmatrix}. \quad (165)$$

The matrix corresponding to \mathbf{b}^2 is

$$\mathbf{b}^2 = \begin{pmatrix} 1 + \gamma^2 & (2 + \gamma^2)\gamma_\theta & (2 + \gamma^2)\gamma_z \\ (2 + \gamma^2)\gamma_\theta & 1 + (3 + \gamma^2)\gamma_\theta^2 & (3 + \gamma^2)\gamma_\theta\gamma_z \\ (2 + \gamma^2)\gamma_z & (3 + \gamma^2)\gamma_\theta\gamma_z & 1 + (3 + \gamma^2)\gamma_z^2 \end{pmatrix}, \quad (166)$$

and a similar expression for \mathbf{c}^2 can be written down if required.

From equation (139), the principal invariants I_1, I_2 are obtained as

$$I_1 = I_2 = 3 + \gamma^2, \quad (167)$$

while $I_3 \equiv 1$. The invariants K_4, K_5, K_6 in (144) depend on the choice of magnetic field and will be quantified in what follows.

Axial magnetic field. We work in terms of the energy function Ω^* and assume that the Lagrangian field \mathbf{H}_l has components $(0, 0, H_{lZ})$. The invariants K_4, K_5, K_6 are then calculated from equation (144), using $(165)_2$, as

$$K_4 = H_{lZ}^2, \quad K_5 = K_4, \quad K_6 = K_4(1 + \gamma^2). \quad (168)$$

Thus, the only remaining kinematic and magnetic variables are γ_θ, γ_z and K_4 . One can think of H_{lZ} as being applied in the reference configuration along with the stresses required to maintain the reference geometry, followed by application of the helical shear deformation and the necessary additional accompanying stresses.

In the deformed configuration, the magnetic field vector is given by $\mathbf{H} = \mathbf{F}^{-T}\mathbf{H}_l$ and has components

$$H_r = -\gamma_z H_{lZ}, \quad H_\theta = 0, \quad H_z = H_{lZ}. \quad (169)$$

The constitutive equation (146) is then used to determine the corresponding components of the magnetic induction vector \mathbf{B} as

$$B_r = -2[\Omega_5^* + \Omega_6^*(2 + \gamma^2)]\gamma_z H_z, \quad (170)$$

$$B_\theta = -2[\Omega_5^* + \Omega_6^*(3 + \gamma^2)]\gamma_\theta \gamma_z H_z, \quad (171)$$

$$B_z = -2[\Omega_4^* + \Omega_5^* + \Omega_6^* + \{\Omega_5^* + \Omega_6^*(3 + \gamma^2)\}\gamma_z^2]H_z. \quad (172)$$

Equation (170) shows that the radial component $B_r \neq 0$. However, because B_r depends only on r the equation $\text{div } \mathbf{B} = 0$ reduces to $d(rB_r)/dr = 0$, and hence $B_r = c/r$ for some constant c . As already indicated in respect of B_{lR} the normal component must be continuous across the inner boundary of the tube and, to avoid a singularity at $r = 0$, we must have $c = 0$. Thus, $B_r = 0$ everywhere. This implies that for a circular cylindrical tube subject to helical shear with $\gamma_z \neq 0$ and an applied axial field the constitutive restriction

$$\Omega_5^* + \Omega_6^*(2 + \gamma^2) = 0 \quad (173)$$

must be satisfied. There are many ways to satisfy this requirement. An immediate example is by using an energy function that does not depend

on K_5 and K_6 . Another possibility is an energy function depending on the variables I_1, K_4 and the combination $(I_1 - 1)K_5 - K_6 = K_4(1 + \gamma_\theta^2)$.

Since $B_r = 0$ everywhere, then $H_r = 0$ outside the material because of the connection $H_r = \mu_0^{-1}B_r$, while H_r is given by (169)₁ inside the material. Equation (152)₂ requires that the tangential component of \mathbf{H} be continuous across the inner and outer cylindrical boundaries of the tube. Thus, the component H_z of the magnetic field, which is tangential to the boundaries $r = a$ and $r = b$ of the tube and is constant, must be continuous across those boundaries.

The magnetization \mathbf{M} is obtained from equation (151), which in component form and with $B_r = 0$ gives

$$M_r = -H_r, \quad M_\theta = \mu_0^{-1}B_\theta, \quad M_z = \mu_0^{-1}B_z - H_z, \quad (174)$$

where the components B_θ and B_z are given by (171) and (172).

Outside the material the Maxwell stress components are obtained from (108) as

$$\tau_{rr} = -\frac{1}{2}\mu_0^{-1}(B_\theta^2 + B_z^2), \quad \tau_{\theta\theta} = \frac{1}{2}\mu_0^{-1}(B_\theta^2 - B_z^2), \quad (175)$$

$$\tau_{zz} = -\frac{1}{2}\mu_0^{-1}(B_\theta^2 - B_z^2), \quad \tau_{\theta z} = \mu_0^{-1}B_\theta B_z, \quad (176)$$

with $\tau_{r\theta} = 0$ and $\tau_{rz} = 0$.

The total stress tensor $\boldsymbol{\tau}$ inside the material is given by equation (145), and its components are

$$\tau_{rr} = -p + 2(\Omega_1^* + 2\Omega_2^*), \quad (177)$$

$$\tau_{\theta\theta} = -p + 2\Omega_1^*(1 + \gamma_\theta^2) + 2\Omega_2^*(2 + \gamma^2), \quad (178)$$

$$\tau_{zz} = -p + 2\Omega_1^*(1 + \gamma_z^2) + 2\Omega_2^*(2 + \gamma^2) + 2K_4[\Omega_5^* + 2\Omega_6^*(1 + \gamma_z^2)], \quad (179)$$

$$\tau_{r\theta} = 2(\Omega_1^* + \Omega_2^*)\gamma_\theta, \quad (180)$$

$$\tau_{rz} = 2(\Omega_1^* + \Omega_2^*)\gamma_z + 2\Omega_6^*K_4\gamma_z, \quad (181)$$

$$\tau_{\theta z} = 2\Omega_1^*\gamma_\theta\gamma_z + 2\Omega_6^*K_4\gamma_\theta\gamma_z. \quad (182)$$

Clearly, the original five invariants now depend on just three independent quantities, namely γ_θ, γ_z and K_4 . This allows us, for this problem, to define a *reduced energy function*, denoted ω^* and given by

$$\omega^*(\gamma_\theta, \gamma_z, K_4) = \Omega^*(I_1, I_2, K_4, K_5, K_6). \quad (183)$$

Using the explicit expressions for I_1 and I_2 in (167) and K_6 in (168) and applying the chain rule then enables equations (180) and (181) to be reduced to the simple forms

$$\tau_{r\theta} = \frac{\partial\omega^*}{\partial\gamma_\theta}, \quad \tau_{rz} = \frac{\partial\omega^*}{\partial\gamma_z}. \quad (184)$$

Equations (159)_{2,3} may now be integrated and combined with (184) to give

$$\tau_{r\theta} = \frac{\partial\omega^*}{\partial\gamma_\theta} = \frac{\tau_\theta b^2}{r^2}, \quad \tau_{rz} = \frac{\partial\omega^*}{\partial\gamma_z} = \frac{\tau_z b}{r}, \quad (185)$$

where τ_θ is the value of $\tau_{r\theta}$ on the boundary $r = b$ and τ_z that of τ_{rz} .

Since the components of the exterior Maxwell stress corresponding to $\tau_{r\theta}$ and τ_{rz} are zero, τ_θ and τ_z should match the externally applied mechanical traction components in the azimuthal and axial directions, respectively. Thus, we can consider these as appropriate boundary conditions as alternatives to specifying β and d in (162). By definition of the deformation there is no change in the radius so a radial traction boundary condition is not required, and equation (159)₁ may be used simply to calculate the radial mechanical traction required to maintain the deformation. We do not need this calculation here. Given τ_θ and τ_z , equations (185) can in principle be used to determine γ_θ and γ_z and then by integration the displacement functions $g(r)$ and $w(r)$, subject to (162)_{1,3}.

Illustration. We now apply the equations in the above section to a particular material model, with Ω^* depending only on the invariants I_1 and K_4 . Specifically, we set

$$\Omega^*(I_1, K_4) = \frac{\mu(K_4)}{k} \left[\left(\frac{I_1 - 1}{2} \right)^k - 1 \right] + \nu(K_4), \quad (186)$$

where μ and ν are functions of K_4 and k is a constant such that $k \geq 1/2$. In particular, μ is such that $\mu(0) (> 0)$ is the shear modulus of the material in the undeformed configuration in the absence of a magnetic field. The term $\nu(K_4)$ represents the magnetic energy in the material in the absence of deformation and requires that $\nu(0) = 0$. The form (186) of Ω^* is based on a strain-energy function introduced by Jiang and Ogden (1998) in the purely elastic context.

Using (185) and the function (186), we obtain the shear stresses in the forms

$$\tau_{r\theta} = \mu(K_4)\gamma_\theta \left(\frac{2 + \gamma^2}{2} \right)^{k-1} = \frac{\tau_\theta b^2}{r^2}, \quad (187)$$

$$\tau_{rz} = \mu(K_4)\gamma_z \left(\frac{2 + \gamma^2}{2} \right)^{k-1} = \frac{\tau_z b}{r}. \quad (188)$$

The dependence of the shear stresses τ_θ and τ_z on the rotation angle β and the axial displacement d can in principle be determined by integration of

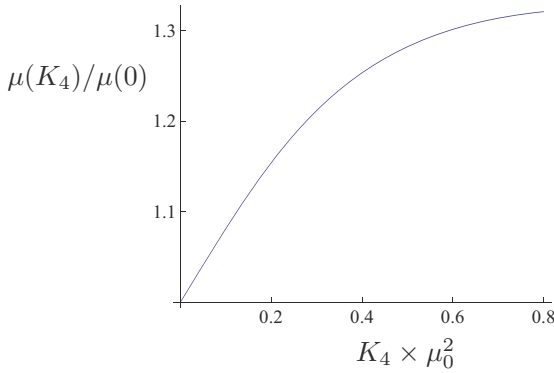


Figure 5. Representative dependence of the shear modulus μ as a function of the magnetic field strength, as measured by the invariant K_4 . The vertical axis corresponds to the dimensionless shear modulus $\mu(K_4)/\mu(0)$. The horizontal axis is scaled by μ_0^2 . Note the distinction between $\mu(0)$ and μ_0 .

(187) and (188) via $\gamma_\theta = rg'(r)$ and $\gamma_z = w'(r)$ together with (162). In the special case $k = 1$ explicit results can be obtained for $g(r)$ and $w(r)$, which are

$$g(r) = \frac{\tau_\theta b^2}{2\mu(K_4)} \left(\frac{1}{a^2} - \frac{1}{r^2} \right), \quad w(r) = \frac{\tau_\theta b}{\mu(K_4)} \log \left(\frac{r}{a} \right). \quad (189)$$

Hence

$$\beta = \frac{\tau_\theta b^2}{2\mu(K_4)} \left(\frac{1}{a^2} - \frac{1}{b^2} \right), \quad d = \frac{\tau_\theta b}{\mu(K_4)} \log \left(\frac{b}{a} \right), \quad (190)$$

and it can be seen that the stiffness of the mechanical response of the tube to either azimuthal or axial shearing increases with $\mu(K_4)$, and hence with the magnitude of the applied magnetic field if μ is an increasing function of K_4 . Data presented by Jolly et al. (1996) for magnetorheological elastomers suggest that the behaviour of the shear modulus $\mu(K_4)$ is as depicted schematically in Figure 5. The higher the concentration of particles the greater the stiffness of the material response up to a limiting value of the volume fraction, which is about 27%. For lower volume fractions the general character of the behaviour of $\mu(K_4)$ is typically as illustrated in Figure 5, but as the limiting volume fraction is approached there is evidence that $\mu(K_4)$ reaches a maximum at a certain value of K_4 and then decreases with further increases in K_4 .

The components of the magnetization vector are given by equation (174). In the reference configuration, where $\gamma_\theta = \gamma_z = 0$, the only non-zero com-

ponent of the magnetic field is H_z , and, by (172), $B_z = -2\Omega_4^* H_z$ in respect of the model (186). The corresponding component of the magnetization is then given by (174) with (186) as

$$M_z = -[1 + 2\mu_0^{-1}\nu'(K_4)]H_z. \quad (191)$$

Thus, while the function $\mu(K_4)$ characterizes the dependence of the shear response of the material on the magnetic field, this shows that $\nu(K_4)$, through its derivative, characterizes the magnetization in the undeformed configuration. In the deformed configuration, the magnetization depends on both functions.

5.3 Extension and Inflation of a Tube

We now consider a second problem for the tube with reference geometry given by (160). The tube is deformed by combining axial extension and radial expansion according to the equations

$$r = [a^2 + \lambda_z^{-1}(R^2 - A^2)]^{1/2}, \quad \theta = \Theta, \quad z = \lambda_z Z, \quad (192)$$

where λ_z is the constant axial stretch. The deformation gradient is diagonal with respect to the cylindrical coordinate axes, with principal stretches in the radial, azimuthal and axial directions given by

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \lambda = \frac{r}{R}, \quad \lambda_3 = \lambda_z, \quad (193)$$

wherein the notation λ is defined and use has been made of the incompressibility constraint $\lambda_1\lambda_2\lambda_3 \equiv 1$ to give λ_1 in terms of the independent stretches λ and λ_z .

Circumferential magnetic field. In this problem, by contrast with that of helical shear discussed above, we consider a circumferential magnetic field, with azimuthal component $H_{l\Theta}$ in the reference configuration. From equation (158)₂ we then have

$$H_{l\Theta} = \frac{C_2}{R}, \quad (194)$$

where C_2 is a constant, and, from (150)₂,

$$H_\theta = \lambda^{-1}H_{l\Theta} = \frac{C_2}{r}. \quad (195)$$

This magnetic field can be generated by a current flowing along the core of the tube or a surface current along its inner boundary, so there is no

difficulty associated with a possible singularity on $r = 0$ in this example. The boundary conditions require that either H_θ is continuous across the inner cylindrical surface $r = a$ or that H_θ on $r = a$ (calculated from $r > a$) is matched by an axial surface current on $r = a$ via the specialization of (47)₂, for example, and H_θ is also continuous across the outer cylindrical surface located at $r = b$.

The invariants K_4, K_5, K_6 defined in (144) assume now the values

$$K_4 = H_{l\Theta}^2, \quad K_5 = \lambda^2 K_4, \quad K_6 = \lambda^4 K_4, \quad (196)$$

while, on use of the incompressibility condition, I_1 and I_2 can be expressed in terms of λ and λ_z :

$$I_1 = \lambda^2 + \lambda_z^2 + \lambda^{-2} \lambda_z^{-2}, \quad I_2 = \lambda^{-2} + \lambda_z^{-2} + \lambda^2 \lambda_z^2. \quad (197)$$

Similarly to (183), this allows a reduced formulation of the energy function Ω^* to be defined. Again we use the notation ω^* , this time defined by

$$\omega^*(\lambda, \lambda_z, K_4) = \Omega(I_1, I_2, K_4, K_5, K_6), \quad (198)$$

so that only three independent deformation/magnetic variables are required in this specialization.

In the deformed configuration both the magnetic induction and the magnetization are circumferential, with

$$B_\theta = -2\lambda^2 \frac{\partial \omega^*}{\partial K_4} H_\theta, \quad M_\theta = -2\mu_0^{-1} \lambda^2 \frac{\partial \omega^*}{\partial K_4} H_\theta - H_\theta. \quad (199)$$

For this particular field the equation $\operatorname{div} \mathbf{B} = 0$ is automatically satisfied and therefore places no restriction on the form of the constitutive equation. We note in passing, however, that in the formulation based on Ω with \mathbf{B} as the independent variable restrictions are placed on the class of material models for the considered deformation and circumferential field to be admissible (Dorfmann and Ogden, 2005).

In the space surrounding the tube, where $\mathbf{B} = \mu_0 \mathbf{H}$, the components of the Maxwell stress tensor may again be obtained by specializing (108). The only non-zero components are

$$\tau_{rr} = \tau_{zz} = -\frac{1}{2} \mu_0 H_\theta^2, \quad \tau_{\theta\theta} = \frac{1}{2} \mu_0 H_\theta^2, \quad (200)$$

which depend on the radius r . For later use, we define the notation

$$\tau_m(r) = \frac{1}{2} \mu_0 H_\theta^2, \quad (201)$$

where H_θ is given by (195) as a function of r .

The components of $\boldsymbol{\tau}$ within the material are obtained from the specialization of (145). The only non-zero components are $\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}$ and it is straightforward to show, by using (196) and (198), that

$$\tau_{\theta\theta} - \tau_{rr} = \lambda \frac{\partial \omega^*}{\partial \lambda}, \quad \tau_{zz} - \tau_{rr} = \lambda_z \frac{\partial \omega^*}{\partial \lambda_z}. \quad (202)$$

For the considered deformation, the equilibrium equations (159)_{2,3} in the circumferential and axial directions are satisfied trivially, while the radial equation

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0 \quad (203)$$

remains. On substituting from (202)₁ we may integrate this equation in the form

$$\tau_{rr} = \int_a^r \lambda \frac{\partial \omega^*}{\partial \lambda} \frac{dr}{r} + \tau_{rr}(a), \quad (204)$$

where $\tau_{rr}(a)$ is the value of τ_{rr} on $r = a$. For the traction boundary conditions we suppose that the inner boundary $r = a$ is subjected to a pressure P while the outer boundary $r = b$ is free of mechanical load. Thus, we must have

$$\tau_{rr}(a) = -P - \tau_m(a), \quad \tau_{rr}(b) = -\tau_m(b), \quad (205)$$

and by using these in (204) we obtain

$$P = \int_a^b \lambda \frac{\partial \omega^*}{\partial \lambda} \frac{dr}{r} + \tau_m(b) - \tau_m(a), \quad (206)$$

which involves the Maxwell stress $\tau_m(r)$ at the inner and outer surfaces. Since

$$\tau_m(b) - \tau_m(a) = \frac{1}{2} \mu_0 C_2^2 \left(\frac{1}{b^2} - \frac{1}{a^2} \right) \quad (207)$$

is negative this means that the resultant effect of the magnetic field is similar to that of the pressure in that it can cause inflation of the tube at fixed axial extension. In particular, the magnetic field can induce inflation in the absence of internal pressure.

Although we are considering a tube of infinite length it is interesting to obtain an expression for the resultant axial force, denoted N , on any cross-section of the tube. This is

$$N = 2\pi \int_a^b \tau_{zz} r \, dr = 2\pi \left[\int_a^b (\tau_{zz} - \tau_{rr}) r \, dr + \int_a^b \tau_{rr} r \, dr \right]. \quad (208)$$

By integrating the second integral on the right-hand side by parts, using (203) and the fact that $a^2\tau_m(a) = b^2\tau_m(b)$, we obtain the formula

$$N = \pi \int_a^b (2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta})r \, dr + \pi a^2 P. \quad (209)$$

Use of equations (202) allows N to be rewritten in terms of the reduced energy function ω^* as

$$N = \pi \int_a^b \left(2\lambda_z \frac{\partial \omega^*}{\partial \lambda_z} - \lambda \frac{\partial \omega^*}{\partial \lambda} \right) r \, dr + \pi a^2 P. \quad (210)$$

Thus, when Ω^* , and hence ω^* , is given both the pressure and the axial load can be calculated. The reader is invited to evaluate the expressions for P and N explicitly for the case of the model energy function (186) with $k = 1$ by choosing a linear form $\mu(K_4) = \mu(0) + \mu_0\eta K_4$ for the shear modulus, where η is a dimensionless constant.

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Analysis of Nonlinear Electrostatic Membranes

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Abstract. We give a concise treatment of the interaction of a nonlinear elastic membrane with an electrostatic field. The focus is on the generation of reduced dimension model equations for continuum electrodynamics based on the methodology used by Steigmann (2007) in the purely mechanical context. We formulate the background theory for reduced dimension models pertaining to continuum electrostatics and implement the resulting equations numerically for an example problem of practical interest.

1 Introduction

In this chapter we describe a systematic approach to the generation of reduced dimension model equations which govern equilibrium for a nonlinear elastic membrane acted on by an electrostatic field. The primary purpose of this work is two-fold – to contribute to the strong foundation laid by Kovetz (2000) for work in continuum electrodynamics by adopting conventions used there, and to apply the technique of Steigmann (2007) for generating reduced dimension equations pertaining to equilibrium electro-elastostatics problems.

To implement the theory we present, the system we study in some detail is that of an elastomeric dielectric membrane with deformable electrodes fixed to opposing lateral surfaces. The similarly treated magnetostatic case has been previously considered by Steigmann (2004). For the academic, the combination of finite deformation and electric field interaction presents a challenging system to analyze, one for which many fundamental ideas from continuum mechanics are required. From a practical standpoint, the combination of an elastomeric dielectric material along with a coexisting system of charged electrodes is an active area of research and development,

with applications in the field of artificial muscle actuator technologies (Bar-Cohen, 2000).

The benefit of developing reduced dimension models for problems of electro-elastostatics is apparent upon examination of the complexity of the fully coupled mathematical system. We will formulate a two-dimensional approximate theory for continuum electrostatics problems that is more tractable than the exact three-dimensional theory. This is also of practical benefit due to the high aspect ratio geometry commonly employed for electrostatic actuators (Carpi, 2005, 2007).

There are several contributions of note in this paper. First, generation of the two-dimensional reduced forms of three-dimensional balance equations in a method adapted from Steigmann (2007) is applied to Maxwell equations for electrostatics, which has not been previously presented to our knowledge. Secondly, we describe the electromechanical coupling in a way that is consistent with accepted thermodynamic considerations, following Kovetz (2000). This includes utilizing the general constitutive framework for elastic materials guided by nonlinear continuum thermodynamics – by which response functions are expressed in terms of derivatives of a stored energy function. The result is a proposed stored energy function which reduces to the Mooney–Rivlin stored energy function for incompressible elastomers if the effect of the electric field is ignored. This stored energy function also produces a polarization response which agrees with the familiar linear dielectric constitutive relation.

We present this topic in four sections. First the general background of continuum electrostatics will be reviewed, primarily stating the three-dimensional balance laws and boundary conditions along with considerations of electrostatic-to-mechanical coupling. Secondly, we present the reduced dimension model equations generated from the governing three-dimensional mechanical and electromagnetic balance laws using a systematic procedure introduced by Steigmann (2007). Third, we discuss constitutive equations for electroelastic materials which are consistent with nonlinear continuum thermodynamic principles and propose a stored energy function. Finally we numerically implement the proposed reduced dimension model for a chosen material system and geometry, that of an axisymmetric membrane. Qualitative aspects of the numerical results are discussed, our goal being simply to use the posed problem as an illustrative platform in order to exercise the model we are presenting.

Explicit examples of boundary value problems of the sort we present are somewhat lacking in the literature, so we feel this work will be a useful contribution both for those interested in examples of applications of continuum electrodynamics theory and to those interested in reduced dimension

continuum mechanics models.

2 Theoretical Background

In this section the background theory needed for the study of continuum electrostatics is briefly reviewed. A more in-depth discussion of the general three-dimensional balance equations can be found in (Steigmann, 2009), or (Kovetz, 2000). We will only consider the static (equilibrium) case, i.e. $d(\cdot)/dt = 0$, where (\cdot) is any quantity of interest. We also will neglect the effect of magnetic fields.

2.1 Conservation Laws, Maxwell's Equations

The governing equations which describe the physical behavior of deforming materials in an electromagnetic field come from the balance laws for linear momentum and Maxwell's equations. First we consider these two sets of equations separately, without any coupling between linear momentum and Maxwell's equations.

Linear momentum. The conservation of linear momentum can be written as

$$\frac{d}{dt} \int_{\Omega} \rho v \, dV = \int_{\partial\Omega} t(F; n) + \int_{\Omega} \rho f_{mech} \, dV, \quad (1)$$

where v is the material velocity vector, ρ is the mass density, $t(F; n)$ is the traction vector, a material response function depending only on elastic deformation gradient $F \equiv \text{Grad } y$ and surface normal n , and f_{mech} is a force vector term arising from agencies external to the body, e.g., gravity. By Cauchy's Theorem, $t(F; n)$ is a linear transformation on the set of unit vectors, and there exists a second-order tensor field, the Cauchy stress tensor T , such that $Tn = t$ (Liu, 2002). For the equilibrium case ($v = 0$), with no body forces, the global integral equation for balance of linear momentum, equation (1), along with use of Stokes' theorem gives the local form of the balance of linear momentum as

$$\text{div } T = 0, \quad (2)$$

where div is the divergence operator. The jump condition on the stress tensor is

$$[[T]]n = 0, \quad (3)$$

where n is the outward pointing unit normal vector to the surface of interest, and $[[\cdot]] \equiv (\cdot)^o - (\cdot)^i$ is the jump in (\cdot) across the same surface. The superscripts o and i refer to 'outside' and 'inside', respectively.

Maxwell's equations. The governing equations for the electromagnetic fields are described by Maxwell's equations

$$\operatorname{div} d = q_f, \quad \operatorname{curl} e + \frac{\partial b}{\partial t} = 0, \quad (4)$$

$$\operatorname{div} b = 0, \quad \operatorname{curl} h - \frac{\partial d}{\partial t} = j, \quad (5)$$

where d is the electric displacement vector, b is the magnetic induction, e is the electric field, h is the magnetic field, q_f is the free electric charge density per unit volume, and j is the free electric current density.

For our study we will only be concerned with the time independent, electrostatic case, so that the relevant Maxwell equations simplify to

$$\operatorname{div} d = q_f, \quad \operatorname{curl} e = 0. \quad (6)$$

We can immediately satisfy equation (6)₂ by introducing the electrostatic potential V related to the electric field by

$$e = -\operatorname{grad} V,$$

so that $\operatorname{curl} e \equiv 0$.

The electric displacement d is expressed in terms of the electric field and polarization vector as

$$d = \epsilon_0 e + p(F, e), \quad (7)$$

where it is emphasized here that the polarization field $p(F, e)$ is a material response function, i.e. it is given by a constitutive equation for the material. For our purposes the polarization will be assumed to depend predominantly on the elastic deformation and the electric field, as shown by the indicated dependence in equation (7).

The jump conditions for the Maxwell equations are

$$[[d]] \cdot n = \sigma_f, \quad n \times [[e]] = 0, \quad (8)$$

where σ_f is the surface free charge density and \times denotes the cross product.

2.2 Continuum Electrodynamics

Linear momentum, Lorentz force. We now consider continuum electrodynamic theory wherein the governing balance equations are fully coupled. The link between linear momentum balance and electrostatics comes from the addition of the Lorentz force term f_L into the linear momentum balance as a body force term, i.e.

$$\frac{d}{dt} \int_{\Omega} \rho v \, dV = \int_{\partial\Omega} t(F, e; n) + \int_{\Omega} \rho f_{mech} \, dV + \int_{\Omega} f_L \, dV, \quad (9)$$

where $f_L = qe + j \times b$, q is the total electric charge density, and the material traction response $t(F, e; n)$ is now, in general, allowed to depend on the electric field in addition to the deformation gradient and surface normal.

The balance of linear momentum equation with the effect of the electric field added through the Lorentz force term, equation (9), can be reformulated in a multitude of ways, which has led to some confusion in the subject due to their apparent non-agreement. However, these different formulations have been shown to be equivalent by Steigmann (2009). We follow the approach of Kovetz (2000) to rearrange equation (9) into a more useful form. After many manipulations of equation (9) using Maxwell's equations and Stokes' theorem, the Lorentz force gives rise to the Maxwell stress tensor, which for the electrostatic case is

$$T_M = \epsilon_0 \left(e \otimes e - \frac{1}{2} |e|^2 I \right), \quad (10)$$

where I is the identity tensor and $|e| \equiv \sqrt{e \cdot e}$ is the magnitude of the electric field.

As an important aside, in many works on problems of applied continuum electrodynamics, researchers disregard the constitutive dependence of the traction t on the electromagnetic field and use a version of the Maxwell stress tensor to incorporate the electromechanical coupling – replacing the permittivity of free space ϵ_0 in equation (10) with the material permittivity ϵ . We will later present a form for the stress tensor which does not use this approach and is more consistent with accepted ideas of continuum thermodynamics.

With the effect of the electric field added to the balance of linear momentum through the Maxwell stress tensor T_M the equilibrium equation (2) becomes (Steigmann, 2009)

$$\operatorname{div} \hat{T} = 0, \quad (11)$$

where $\hat{T} \equiv T(F, e) + T_M$ and, as indicated, the Cauchy stress T depends on the electric field in addition to the deformation gradient.

2.3 Stored Energy Function, Constitutive Relations

Elastic materials, by definition, entail the specification of a stored energy function which encompasses all constitutive behavior through its derivatives. Accordingly we assume the existence of a stored energy function for the material ψ which depends on the deformation and electric field, i.e.

$$\psi = \hat{\psi}(F, e). \quad (12)$$

The constitutive forms for the response functions T , p can be obtained in terms of this stored energy function using the Clausius–Duhem inequality (Kovetz, 2000). One obtains the relations for Cauchy stress, T , and polarization, p , as

$$T = \rho \frac{\partial \psi}{\partial F} F^T + e \otimes p \quad (13)$$

and

$$p = -\rho \frac{\partial \psi}{\partial e}. \quad (14)$$

By the principle of material frame invariance, we must have a functional representation for the stored energy function ψ which satisfies the equality

$$\psi(F, e) = \Phi(C, E) \quad (15)$$

where $\Phi(C, E)$ is the stored energy function ψ , expressed in terms of $C = F^T F$ and $E = F^T e$, the right Cauchy–Green stretch and the electric field in the reference frame respectively. Equating variations in $\psi(F, e)$ and $\Phi(C, E)$, i.e.

$$\frac{\partial \psi}{\partial F} \cdot \dot{F} + \frac{\partial \psi}{\partial e} \cdot \dot{e} = \frac{\partial \Phi}{\partial C} \cdot \dot{C} + \frac{\partial \Phi}{\partial E} \cdot \dot{E}, \quad (16)$$

allows one to eventually read off the constitutive expressions for T, p in terms of derivatives of Φ using equations (13) and (14). One obtains

$$T = 2\rho F \text{sym} \left(\frac{\partial \Phi}{\partial C} \right) F^T, \quad (17)$$

where $\text{sym}(\cdot)$ denotes the symmetrization operation: $\text{sym}(A) = \frac{1}{2}(A + A^T)$. Note that the term $e \otimes p$ in equation (13) ends up canceling in the course of obtaining equation (17). The constitutive expression (14) for the polarization, using the energy function $\Phi(C, E)$, gives the result

$$p = -\rho F \frac{\partial \Phi}{\partial E}. \quad (18)$$

Reference configuration. The Piola stress tensor $P = TF^*$, where $F^* \equiv JF^{-T}$ and $J \equiv \det F$, is convenient to employ in order to perform calculations in a coordinate system fixed to material points. Use of equation (17) expressed in terms of P gives

$$P = 2\rho_0 F \text{sym} \left(\frac{\partial \Phi}{\partial C} \right), \quad (19)$$

where $\rho_0 \equiv \rho J$ is the mass density in material coordinates.

The analogous reference quantity for the electric displacement is given by the transformation law for d (Kovetz, 2000) as

$$D = JF^{-1}d = JF^{-1}(\epsilon_0 e + p) = \epsilon_0 J C^{-1} E + \Pi \quad (20)$$

where

$$\Pi = JF^{-1}p = -\rho_0 \frac{\partial \Phi}{\partial E}$$

is the polarization response function in the material reference configuration.

The balance equations (4) and (11) can be expressed in the reference configuration as

$$\text{Div } \hat{P} = 0, \quad \text{Div } D = 0, \quad (21)$$

where $\hat{P} \equiv \hat{T}F^*$ and Div is the divergence operator in material coordinates.

The PDEs given in equations (21) along with appropriate boundary conditions and constitutive behavior given by specification of the energy potential function $\Phi(C, E)$ enables solution of general three-dimensional problems in finite deformation continuum electrostatics. In the next section we consider applying these equations to a material body which has a dimension h of much less extent than any other dimension L , i.e. $h/L \ll 1$. This will lead to a set of two-dimensional approximate versions of the PDEs (21), and, with additional algebraic equations, will together form the system of equations to solve for the membrane material.

3 Membrane Approximation

In this section, we apply the systematic procedure adapted from Steigmann (2007) to derive the leading order membrane approximations to the governing balance equations (21). The technique integrates the balance equations (21) over the whole body then expands the integration along the smallest dimension in a Taylor series expansion. Cutting the expansion off at the leading order gives the membrane approximation.

3.1 Membrane Kinematics

We begin with a general discussion of our kinematic conventions for the membrane problem. We use a convected coordinate system, so that a material point in the current configuration possesses the same coordinate identification throughout deformation, and this material coordinate system is the coordinate system to which the differential operations will be referred.

Consider a material coordinate system where the coordinate axes are aligned with the body Ω , as shown in Figure 1. The body has boundary $\partial\Omega$ and outward pointing surface normal N . The coordinate ζ describes

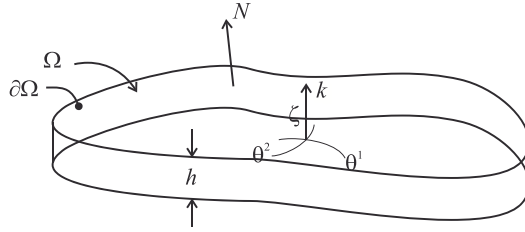


Figure 1. Reference geometry for material amenable to membrane approximation. The body Ω has boundary $\partial\Omega$, outward pointing normal N , through thickness coordinate ζ , and in-plane coordinates $\theta^{1,2}$. The dimension h is much smaller than any other dimension for the material.

the extent along the thinnest dimension, which has maximum extent h . The other material coordinates are general curvilinear coordinate functions, denoted θ^α , $\alpha = 1, 2$. The through thickness dimension h is much less than the other dimensions in the material.

The position vector in the material configuration is described by decomposition into the sum of a component vector to the midsurface at $\zeta = 0$, denoted x_0 in Figure 2, and a component along the k direction with coordinate ζ , i.e. $x = x_0 + \zeta k$.

In the physical configuration the position vector is denoted y . The position of the deformed midsurface, with coordinates $\zeta = 0$, is denoted by y_0 . Material point positions of the body are described by a Taylor series expansion about the deformed midsurface, e.g.,

$$y(\theta^\alpha, \zeta) = y_0(\theta^\alpha) + \zeta y'_0(\theta^\alpha) + \frac{1}{2} \zeta^2 y''_0(\theta^\alpha) + O(\zeta^3), \quad (22)$$

where

$$(\cdot)' \equiv \frac{\partial}{\partial \zeta}, \quad (\cdot)'' \equiv \frac{\partial^2}{\partial \zeta^2}.$$

The deformation gradient F is found by applying the referential gradient operator to the material point in the physical configuration (22). Thus,

$$F = \text{Grad } y = \nabla y_0 + y'_0 \otimes k + \zeta \nabla y'_0 + \zeta y''_0 \otimes k + O(\zeta^2), \quad (23)$$

where ∇ is the two-dimensional gradient operator in the θ^α coordinates.

We will see later that imposing the membrane approximation to the balance equations will only use the leading order term in these expansions, so that for future clarity we use the first order expansion for F ,

$$F \approx F_0 = \nabla y_0 + y'_0 \otimes k, \quad (24)$$

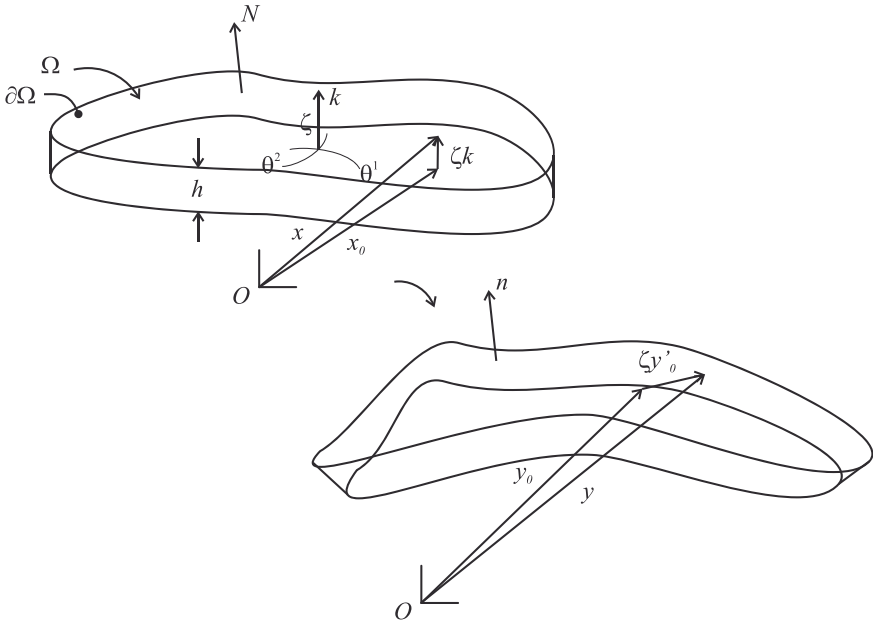


Figure 2. Reference geometry for the physical configuration.

where y'_0 is the director field.

Having established the kinematics, in the next sections we use the approach of Steigmann (2007) for generating the two-dimensional membrane approximations to the governing equations (21). We first cast the local equations in weak form as an integration over the body, then apply the thin body approximation to the integrals by computing the leading order Taylor series expansion of the through thickness integration for small h . The resulting model gives a two-dimensional system of differential–algebraic equations which are more easily solved than the fully three-dimensional system.

We consider the reductions for the electrostatic case first, followed by those for the linear momentum balance.

3.2 Membrane Reduction of Maxwell’s Equations

The weak form for Maxwell’s equation is obtained by multiplying (21) by the variation \dot{V} in the electrostatic potential. We then have by the product rule

$$0 = \text{Div } D \cdot \dot{V} = \text{Div } (D\dot{V}) - D \cdot \text{Grad } \dot{V}. \tag{25}$$

Integration of equation (25) over the body gives the weak form

$$\int_{\Omega} D \cdot \text{Grad } \dot{V} \, dV = \int_{\partial\Omega} DN \cdot \dot{V} \, dA. \tag{26}$$

We first consider the right-hand side of equation (26). The integration over the surface, $\partial\Omega$ can be decomposed into top and bottom surface integrations over Γ^+ and Γ^- along with the product of the boundary of the midsurface plane with the through thickness coordinate, $\partial\Gamma \times \{-h/2, h/2\}$, i.e.

$$\int_{\partial\Omega} (\cdot) \, dA = \int_{\Gamma^+} (\cdot)^+ \, dA + \int_{\Gamma^-} (\cdot)^- \, dA + \int_{\partial\Gamma \times \{-h/2, h/2\}} (\cdot) \, dA; \tag{27}$$

see Figure 3.

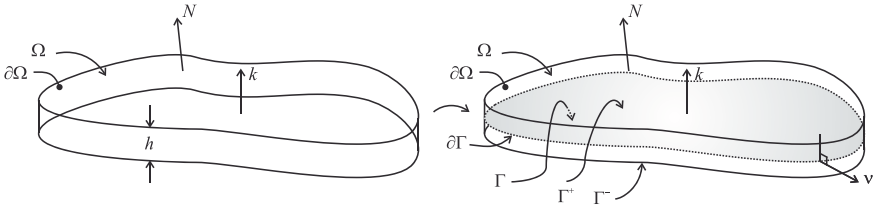


Figure 3. The three-dimensional material body Ω , with boundary $\partial\Omega$ and outward pointing normal N is divided into a two-dimensional area Γ with boundary $\partial\Gamma$ and outward pointing normal ν . The top and bottom surfaces are denoted Γ^+ and Γ^- , respectively.

Using this, the surface integral term in equation (26) can be rewritten as

$$\int_{\partial\Omega} D \cdot N \dot{V} \, dA = \int_{\partial\Gamma} \left(\int_{-h/2}^{h/2} 1D \cdot \nu \dot{V} \, d\zeta \right) ds + \int_{\Gamma^+} D^+ \cdot N^+ \dot{V}^+ \, dA + \int_{\Gamma^-} D^- \cdot N^- \dot{V}^- \, dA, \tag{28}$$

where $1 = I - k \otimes k$ is the projection operator, I is the identity, and ν is the outward pointing normal from the midsurface, lying in the plane of Γ . In obtaining equation (28) we have used the decomposition

$$D = ID = 1D + (k \cdot D)k = 1D + D_k k, \tag{29}$$

with $D_k = D \cdot k$, and it is noted that $N|_{\zeta=0} = k$. We have also used $\nu \cdot k = 0$.

The key point in generating the membrane theory is that since h is small, a Taylor series expansion about $h = 0$ is applicable to the ζ integration. For arbitrary integrand $g(s)$, using the Leibniz rule gives

$$\int_{-h/2}^{h/2} g(\zeta) \, d\zeta = h \frac{\partial}{\partial h} \left(\int_{-h/2}^{h/2} g(\zeta) \, d\zeta \right)_{h=0} + o(h) = hg(0) + o(h). \quad (30)$$

Use of this expansion in the first term on the right-hand side of equation (28) gives

$$\begin{aligned} \int_{\partial\Omega} D \cdot N \dot{V} \, dA &= h \int_{\partial\Gamma} 1 D_0 \cdot \nu \dot{V}_0 \, ds \\ &+ \int_{\Gamma^+} D^+ \cdot N^+ \dot{V}^+ \, dA + \int_{\Gamma^-} D^- \cdot N^- \dot{V}^- \, dA. \end{aligned} \quad (31)$$

To progress further, lateral boundary information must be imposed to simplify the terms involving $(D, N)^\pm$.

For our implementation of the membrane theory, we consider placing flexible electrodes on the top and bottom lateral surfaces of the membrane. This is a configuration of practical importance (Pelrine, 1998; Shahinpoor, 1998; Mockensturm and Goulbourne, 2004; Carpi, 2005, 2007) We can consider this electrode system as painting a free surface charge of density σ on the material in the physical configuration; see Figure 4. This electrode arrangement also simplifies the electrostatic problem, to the degree of approximation we are willing to accept. If the membrane were infinite in extent and non-deforming, the imposition of equal and opposite electric charge on either side of the membrane would result in the field configuration illustrated in Figure 5. In that situation, the field is zero outside the electrodes. We will use this approximation even when the membrane is highly deformed in order to avoid complicating the problem solution. This is tantamount to assuming the flexible electrodes to be perfect conductors.

The boundary condition $(8)_1$ with $d^o = \epsilon_0 e^o = 0$, where $(\cdot)^o$ denotes a quantity evaluated outside the membrane, then gives at the top surface Γ^+ ,

$$\begin{aligned} \int_{\Gamma^+} D^+ \cdot N^+ \dot{V}^+ \, dA &= \int_{\Gamma^+} J^+(F^{-1})^+ d^+ \cdot N^+ \dot{V}^+ \, dA \\ &= \int_{\Gamma^+} -\sigma^+ \alpha^+ \dot{V}^+ \, da, \end{aligned} \quad (32)$$

where we have used the jump condition $[[d]] \cdot n = -d^i \cdot n = -d^+ \cdot n^+ = \sigma$ and the kinematic condition $JF^{-T}N = \alpha n$, where $\alpha \equiv da/dA$ is the area ratio between material and physical coordinates. By equating the total charge in

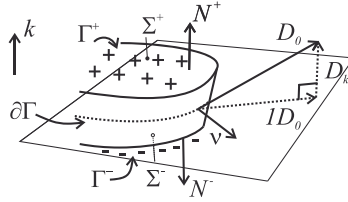


Figure 4. Electrostatic boundary conditions. Surface charge densities Σ^\pm are painted on the top and bottom of the membrane. The projection of electric displacement D onto the midsurface and normal direction is also shown.

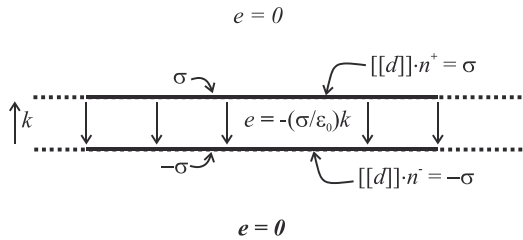


Figure 5. Illustration of infinite parallel plane electrodes with equal and opposite charge densities $\pm\sigma$. The solution to Maxwell’s equations shows that the field is zero except within the membrane.

both material and physical coordinates, we have

$$\int_{\partial\Omega} \sigma \, da = \int_{\partial\Omega} \sigma \alpha \, dA \equiv \int_{\partial\Omega} \Sigma \, dA,$$

so that $\Sigma \equiv \alpha\sigma$ is the surface charge density in material coordinates.

After similarly handling the bottom surface integral at Γ^- we obtain the intermediate result

$$\int_{\partial\Omega} D \cdot N \dot{V} \, dA = h \int_{\partial\Gamma} 1D_0 \cdot \nu \dot{V}_0 \, ds - \int_{\Gamma^+} \Sigma^+ \dot{V}^+ \, dA - \int_{\Gamma^-} \Sigma^- \dot{V}^- \, dA,$$

where

$$\dot{V}^\pm = \dot{V}_0 \pm \frac{h}{2} \dot{V}'_o + o(h).$$

After applying the expansion for \dot{V}^\pm we get

$$\begin{aligned} \int_{\partial\Omega} D \cdot N\dot{V} \, dA &= h \int_{\partial\Gamma} 1D_0 \cdot \nu \dot{V}_0 \, ds - \int_{\Gamma} (\Sigma^+ + \Sigma^-) \dot{V}_0 \, dA \\ &\quad - \frac{h}{2} \int_{\Gamma} (\Sigma^+ + \Sigma^-) \dot{V}'_0 \, dA + o(h). \end{aligned} \quad (33)$$

We now consider imposing equal and opposite charged electrodes, $\Sigma^- = -\Sigma^+$, $\Sigma^+ \equiv \Sigma$. Then

$$\int_{\partial\Omega} D \cdot N\dot{V} \, dA = h \left(\int_{\partial\Gamma} 1D_0 \cdot \nu \dot{V}_0 \, ds - \int_{\Gamma} \Sigma \dot{V}'_0 \, dA \right) + o(h). \quad (34)$$

Now consider the left-hand side of equation (26). By decomposing onto the midsurface plane and the midsurface normal direction k we can write, for the gradient of the variation in the electrostatic potential

$$\text{Grad } \dot{V} = I(\text{Grad } \dot{V}) = 1(\text{Grad } \dot{V}) + (k \cdot \text{Grad } \dot{V})k = \nabla \dot{V} + \dot{V}'k, \quad (35)$$

where ∇ is the two-dimensional gradient operator on the reference plane. For the displacement, we again use the decomposition (29). Thus we obtain the result

$$D \cdot \text{Grad } \dot{V} = 1D \cdot \nabla \dot{V} + D_k \dot{V}', \quad (36)$$

where we have used the result $(1a) \cdot k = 0$ for any vector a . Using (36) we rewrite the left-hand side of equation (26), after using another h -integral expansion and the Leibniz rule as

$$\begin{aligned} \int_{\Omega} D \cdot \text{Grad } \dot{V} \, dV &= \int_{\Gamma} \left(\int_{-h/2}^{h/2} 1D \cdot \nabla \dot{V} + D_k \dot{V}' \, d\zeta \right) dA \\ &= h \int_{\Gamma} \left(1D_0 \cdot \nabla \dot{V}_0 + D_k|_0 \dot{V}'_0 \right) dA + o(h). \end{aligned} \quad (37)$$

Substitution of equations (34), (37) into equation (26), dividing by h and letting h go to zero gives

$$\int_{\Gamma} 1D_0 \cdot \nabla \dot{V}_0 + (D_k|_0 + \Sigma) \dot{V}'_0 \, dA = \int_{\partial\Gamma} (1D_0 \cdot \nu) \dot{V}_0 \, ds. \quad (38)$$

Use of the divergence theorem expands the boundary integral term over $\partial\Gamma$ on the right-hand side of equation (38) as

$$\begin{aligned} \int_{\partial\Gamma} (1D_0 \cdot \nu \dot{V}_0) \, ds &= \int_{\Gamma} \nabla \cdot (1D_0 \dot{V}_0) \, dA \\ &= \int_{\Gamma} (\nabla \cdot (1D_0) \dot{V}_0 + 1D_0 \cdot \nabla \dot{V}_0) \, dA. \end{aligned} \quad (39)$$

Use of equation (39) in equation (38) then gives

$$\int_{\Gamma} (D_k|_0 + \Sigma) \dot{V}_0' dA = \int_{\Gamma} \nabla \cdot (1D_0 \dot{V}_0) dA. \quad (40)$$

Since \dot{V}_0 and \dot{V}_0' are independent, considering $\dot{V}_0 = 0, \dot{V}_0' \neq 0$, and $\dot{V}_0 \neq 0, \dot{V}_0' = 0$ in succession gives the local equations

$$\nabla \cdot (1D_0) = 0, \quad D_k = -\Sigma. \quad (41)$$

These are the membrane approximations for electrostatics, specific to the lateral boundary condition considered.

Next, we consider the membrane approximation procedure applied to the linear momentum balance equations.

3.3 Membrane Reduction of Linear Momentum Balance

As before, we first cast the local equation (21)₁ into the form

$$0 = \text{Div } \hat{P} \cdot \dot{y} = \text{Div} (\hat{P}^T \dot{y}) - \hat{P} \cdot \text{Grad } \dot{y}. \quad (42)$$

Integration of equation (42) with the divergence theorem gives the weak form

$$\int_{\Omega} \hat{P} \cdot \dot{F} dV = \int_{\partial\Omega} \hat{P} N \cdot \dot{y} dA, \quad (43)$$

where $\dot{F} \equiv \text{Grad } \dot{y}$ and N is the normal vector in the material coordinate system. Decompositions into the midsurface plane and normal direction are facilitated by use of the projection $1 \equiv I - k \otimes k$. We can decompose the Piola stress tensor as

$$\hat{P} = \hat{P}I = \hat{P}1 + \hat{P}k \otimes k, \quad (44)$$

and the variation in the deformation gradient (24) decomposes as

$$\dot{F} = \dot{F}_0 = \dot{F}_0 I = \dot{F}_0 1 + \dot{F}_0 k \otimes k \approx \nabla y_0 + \dot{y}'_0 \otimes k, \quad (45)$$

where we have used the membrane approximation for F given in equation (24). Then

$$\hat{P} \cdot \dot{F} = \hat{P}1 \cdot \nabla y_0 + \hat{P}k \cdot \dot{y}'_0, \quad (46)$$

and, using the Leibniz rule as before,

$$\int_{\Omega} \hat{P} \cdot \dot{F} dV = h \int_{\Gamma} (\hat{P}_0 1 \cdot \nabla y_0 + \hat{P}_0 k \cdot \dot{y}'_0) dA + o(h). \quad (47)$$

We now have

$$\int_{\partial\Omega} \hat{P}N \cdot \dot{y} \, dA = h \int_{\partial\Gamma} \hat{P}_0\nu \cdot \dot{y}_0 \, ds + \int_{\Gamma^+} \hat{P}^+N^+ \cdot \dot{y}^+ \, dA + \int_{\Gamma} \hat{P}^-N^- \cdot \dot{y}^- \, dA + o(h). \quad (48)$$

As with the electrostatic case, to progress we must consider a specific lateral boundary loading situation. In our implementation we will consider the problem of a pressurized membrane, with boundary/jump condition on the stress given by equation (3), and shown in Figure 6.

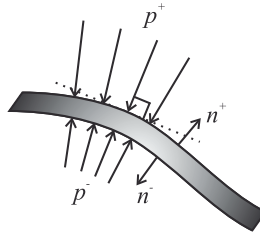


Figure 6. Pressure boundary condition. The applied traction t_a is given by $t_a^\pm = -p^\pm n^\pm$.

In terms of the Piola stress this becomes

$$\hat{P}^\pm N^\pm = \alpha^\pm t_a^\pm = -\alpha^\pm p^\pm n^\pm, \quad (49)$$

where p^\pm is the applied pressure in the physical configuration to the top and bottom surfaces respectively and n^\pm is the surface normal at the top and bottom surfaces in the physical configuration. Here we have also used the fact that the Maxwell stress T_M is zero outside the membrane since the electric field is zero. In obtaining equation (49), we have used the kinematic result $F^*N = \alpha n$ along with the identity

$$\int_{\partial B} Tn \, da = \int_{\partial\Omega} PN \, dA,$$

where B denotes the material body Ω in the current configuration.

We then use an expansion for the physical position at the top and bottom surfaces $\zeta = \pm h/2$ as

$$\dot{y}^\pm = \dot{y}_0 \pm \frac{h}{2} \dot{y}'_0 + o(h). \quad (50)$$

Similar expansions for the surface normal n , and the area ratio α give

$$n^\pm = \pm n|_0 \pm \frac{h}{2} n'_0 + o(h), \quad \alpha^\pm = \alpha|_0 \pm \frac{h}{2} \alpha'_0 + o(h), \quad (51)$$

where $n|_0$ is the outward pointing normal at the midsurface plane, in this convention pointing from $\zeta = -h/2$ to $\zeta = h/2$.

Applying equations (50) and (51) to equation (48) gives

$$\int_{\partial\Omega} \hat{P}N \cdot \dot{y} \, dA = h \int_{\partial\Gamma} \hat{P}_0\nu \cdot \dot{y}_0 \, ds + \int_{\Gamma} \delta p [\alpha n \cdot \dot{y}_0 + O(h)] \, dA + o(h), \quad (52)$$

where $\delta p \equiv p_- - p_+$ is the net pressure applied across the membrane. Next we expand the pressure differential as a function of the membrane thickness, i.e.

$$\delta p = h\delta\bar{p} + o(h), \quad (53)$$

where $\delta\bar{p} = O(1)$. Dividing by h and letting $h \rightarrow 0$ we obtain the leading order model:

$$\int_{\Gamma} (\hat{P}_0 1 \cdot \nabla \dot{y}_0 + \hat{P}_0 k \cdot \dot{y}'_0) \, dA = \int_{\partial\Gamma} (\hat{P}_0 1) \nu \cdot \dot{y}_0 \, ds + \int_{\Gamma} \delta\bar{p} \alpha n \cdot \dot{y}_0 \, dA. \quad (54)$$

The boundary integral term over $\partial\Gamma$ can be expanded using Stokes' theorem as

$$\int_{\partial\Gamma} (\hat{P}_0 1) \nu \cdot \dot{y}_0 \, ds = \int_{\Gamma} (\nabla \cdot \hat{P}_0 1) \cdot \dot{y}_0 + \hat{P}_0 1 \cdot \nabla \dot{y}_0 \, dA. \quad (55)$$

Use of equation (55) in equation (54) gives

$$\int_{\Gamma} \hat{P}_0 k \cdot \dot{y}'_0 \, dA = \int_{\Gamma} (\nabla \cdot \hat{P}_0 1 + \delta\bar{p} \alpha n) \cdot \dot{y}_0 \, dA. \quad (56)$$

Since the variations \dot{y}_0, \dot{y}'_0 are independent, taking $\dot{y}_0 \neq 0, \dot{y}'_0 = 0$ and $\dot{y}_0 = 0, \dot{y}'_0 \neq 0$ in turn in (56) gives the local form of the membrane equations:

$$\nabla \cdot \hat{P}_0 1 = -\delta\bar{p} \alpha n, \quad \hat{P}_0 k = 0. \quad (57)$$

Note that (57)₂ is the familiar plane stress condition from introductory solid mechanics.

3.4 Summary of Membrane Equations

Altogether, the results for the membrane reduction of the general balance PDEs becomes

$$\nabla \cdot (\hat{P}_0 1) = -\delta\bar{p} \alpha n, \quad \hat{P}_0 k = 0, \quad (58)$$

$$\nabla \cdot (1D_0) = 0, \quad D_k|_0 = -\Sigma. \quad (59)$$

An illustration of the lateral boundary conditions we have considered is depicted in Figure 7. It is important to note that different lateral boundary conditions would influence the model and change the form of the source terms Σ and $\delta\bar{\rho}\alpha n$.

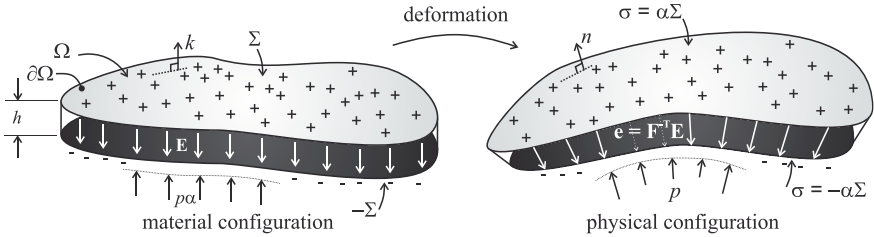


Figure 7. The membrane body with pressure loading and applied surface charge density is illustrated. The equations describing this system are given in (41), (57).

4 Proposed Constitutive Relation

Constitutive relations are required to obtain a solution to the algebraic-differential membrane equations, equations (58), (59). We use a constitutive formulation consistent with nonlinear continuum thermodynamics, by specifying a stored energy function whose derivatives give the appropriate response function, stress or polarization. In this section we derive an expression for this stored energy function $\Phi(C, E)$ which predicts constitutive behavior consistent with linear dielectric polarization and the mechanical response of a Mooney–Rivlin material.

4.1 Dielectric Constitutive Formulation

The constitutive relation used for the linear dielectric material model is $d = \epsilon e$. The linear dielectric relation can be expressed in terms of polarization p to obtain

$$p = d - \epsilon_0 e = \epsilon e - \epsilon_0 e = (\epsilon - \epsilon_0)e. \tag{60}$$

Recall the thermodynamic constitutive formulation in Section 2.3, which yields

$$p = -\rho \frac{\partial \psi}{\partial e}. \tag{61}$$

In this section we seek to find energy functions ψ that are consistent with the polarization predicted for linear dielectric behavior given in equation

(60). To this end, it is convenient to use the expression for polarization in material coordinates,

$$\Pi(C, E) = -\rho_0 \frac{\partial \Phi}{\partial E}(C, E) = JF^{-1}p \quad (62)$$

$$= JF^{-1}(\epsilon - \epsilon_0)e = (\epsilon - \epsilon_0)JC^{-1}E, \quad (63)$$

where we have used $E = F^T e$. For isotropic materials, we can reformulate the energy function $\Phi(C, E)$ in terms of six invariants, i.e.

$$\Phi(C, E) = U(I_1, I_2, I_3, I_4, I_5, I_6), \quad (64)$$

where the invariants I_k , $k = 1, \dots, 6$ are given by

$$I_1 = \text{tr } C, \quad I_2 = \text{tr } C^*, \quad I_3 = \det C, \quad (65)$$

$$I_4 = \text{tr } (CE \otimes E), \quad I_5 = \text{tr } (C^2 E \otimes E), \quad I_6 = \text{tr } (E \otimes E). \quad (66)$$

Therefore,

$$\Pi = -\rho_0 \frac{\partial \Phi}{\partial E} = -\rho_0 \sum_{i=1}^6 U_i \frac{\partial I_i}{\partial E}, \quad (67)$$

where $U_i \equiv \partial U / \partial I_i$ are the partial derivatives of the energy with respect to the invariants. Using equation (67) and computing the appropriate E -derivatives of the I_k gives the constitutive equation for polarization

$$\Pi = -2\rho_0(U_4 C + U_5 C^2 + U_6 I)E. \quad (68)$$

Equating (68) to the result calculated from the linear dielectric relation (63) along with use of the Cayley–Hamilton theorem in the form

$$C^{-1} = \frac{1}{I_3} (C^2 - I_1 C + I_2 I)$$

shows that we must have

$$-2\rho_0((U_4 + I_1 U_5)C^2 + (U_6 - U_5 I_2)C + U_5 I_3 I)E = (\epsilon - \epsilon_0)JE. \quad (69)$$

Matching coefficients of I , C , C^2 for incompressible materials, for which $J = I_3 = 1$, gives

$$U_4 = -I_1 U_5, \quad U_5 = -\frac{1}{2} \frac{\epsilon - \epsilon_0}{\rho_0}, \quad U_6 = U_5 I_2. \quad (70)$$

Additional conditions are obtained from the required commutativity $U_{ij} = U_{ji}$, where $U_{ij} \equiv \partial^2 U / \partial I_i \partial I_j$. Using equation (70) gives

$$U_{26} = U_{62} = U_5, \quad U_{14} = U_{41} = -U_5, \quad (71)$$

so that $U_5 = -(\epsilon - \epsilon_0)/(2\rho_0) \equiv c$, and assigning linear dependence on I_1, I_2 gives $U_1 \equiv c_1, U_2 \equiv c_2$, where c, c_1, c_2 are constants. We thus obtain the simplest strain energy function consistent with linear dielectric behavior as

$$U = cI_5 - I_1I_4c + I_2I_6c + c_1I_1 + c_2I_2 + K, \quad (72)$$

where K is an arbitrary constant. Setting $U = 0$ at $C = I, E = 0$ fixes the constant K , giving

$$U = c(I_5 - I_1I_4 + I_2I_6) + c_1(I_1 - 3) + c_2(I_2 - 3). \quad (73)$$

This is similar to the Mooney–Rivlin material model for incompressible materials,

$$U(I_1, I_2) = c_1(I_1 - 3) + c_2(I_2 - 3),$$

with additional terms owing to the electromagnetic field interaction. It is common to express the parameters c_1, c_2 in terms of the shear modulus as (Liu, 2002) $c_1 = G\delta/2, c_2 = G(1 - \delta)/2$, where G is the shear modulus and $\delta \in (0, 1)$ is a dimensionless material parameter, typically with values around 0.9 (Liu, 2002).

The Piola–Kirchhoff stress is given by directly implementing equation (19), which, with equation (73), gives

$$\begin{aligned} P = & -pF^* + 2\rho_0F [(U_1 + U_2I_1)I - U_2C \\ & + U_4E \otimes E + U_5(E \otimes CE + CE \otimes E)], \end{aligned} \quad (74)$$

where we have added the constraint pressure p to account for incompressibility of the material. After using $U_1 = c_1, U_2 = c_2$, and equations (70), for $U_i, i = 4, 5, 6$, we get

$$P = P_m + P_e,$$

where P_m is the contribution to the Piola stress unrelated to electromagnetic interaction and P_e is due to the electric field, given by

$$P_m \equiv -pF^* + 2\rho_0F [(c_1 + c_2I_1)I - c_2C], \quad (75)$$

$$P_e \equiv 2\rho_0cF [(I_1I_6 - I_4)I - I_6C - I_1E \otimes E + CE \otimes E + E \otimes CE]. \quad (76)$$

4.2 Comment on the Maxwell Stress Tensor

We previously noted that in many publications on dielectrics in electric fields, the additional stress due to the electric field is formed like the Maxwell stress, replacing the permittivity of free space ϵ_0 with the material permittivity ϵ . In our formulation, the additional stress P_e due to the electric field does not appear to be related to the Maxwell stress T_M in such a

relationship. Following that approach, we would have obtained for the field related stresses

$$P_e = \epsilon \left(e \otimes e - \frac{1}{2} |e|^2 I \right) F^* = \epsilon F \left(C^{-1} E \otimes C^{-1} E - \frac{1}{2} \sqrt{E \cdot C^{-1} E} C^{-1} \right), \quad (77)$$

which bears little resemblance to equation (76). The difference is that we allow the traction response to depend on the electric field, i.e. $t(F, e; n)$, where previous works use the traction as $t(F; n)$ and associate all electric field effects in the balance of linear momentum equation to the Maxwell stress tensor T_M . Deciding whether or not to include the electric field terms at the constitutive level or to regard them as only arising from the Lorentz force is a decision to be made depending on the nature of the system. The hope is that the viewpoint presented in this work and by Kovetz (2000) will be adopted by the community, as we feel that this approach to describing electromechanical coupling is better supported by thermodynamics principles and balance equations.

5 Example Problem

In this section an example problem is posed, and solution given, using the membrane model equations and constitutive formulation derived in the previous sections. We consider an axisymmetric membrane across which a pressure differential is applied, in addition to an electric field produced by fixing charged electrodes to the lateral surfaces of the membrane. We present the solution to this problem by using a semi-inverse approach; we assume a functional form for the allowable deformation and seek equilibrium solutions within this restricted set of deformations. The principal unknowns are the position of the membrane in the physical configuration, denoted y , and the electric field in the membrane, denoted e .

5.1 Problem Geometry

We consider the geometry depicted in Figure 8. The membrane of thickness h is fixed along the outer radius at $R = R_0$, and idealized, freely deformable electrodes are painted on the top and bottom surfaces, with equal and opposite charge densities $\pm \Sigma$.

Position vectors in the reference configuration will be assigned in a cylindrical polar coordinate system as $x = R E_R(\Phi) + \zeta k$, where R, Φ, ζ are the traditional cylindrical polar coordinates. Position coordinates in the physical configuration will likewise be assigned in a cylindrical polar coordinate system as u, ϕ, z , with material point position given in convected coordinate

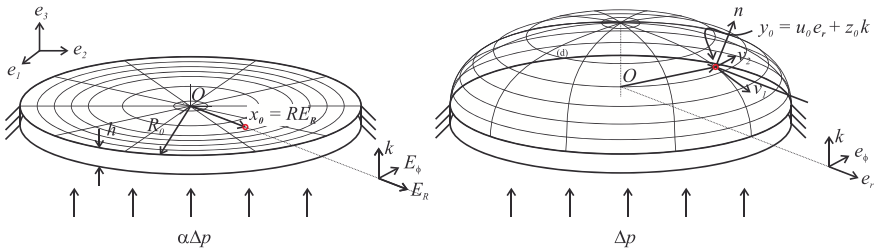


Figure 8. Axisymmetric geometry for the electroelastic membrane deformation problem.

convention by

$$y(R, \Phi, \zeta) = u(R, \Phi, \zeta)e_r(\phi(R, \Phi, \zeta)) + z(R, \Phi, \zeta)k.$$

5.2 Kinematics

Due to the geometric axisymmetry in the present problem (Figure 8) and the nature of the lateral boundary loading and electrode configuration, we consider solutions of the deformed membrane that remain axisymmetric. Therefore, the general functional form for the position vector in the physical configuration is reduced to

$$y(R, \zeta) = u(R, \zeta)e_r(\phi) + z(R, \zeta)k, \tag{78}$$

where e_r is the radial basis vector for cylindrical polar coordinates and $\phi = \Phi$. This can be expanded about $\zeta = 0$ to give

$$y(R, \zeta) \approx \left(u_0(R) + \zeta \frac{\partial u}{\partial \zeta} \right) e_r(\phi) + \left(z_0(R) + \zeta \frac{\partial z}{\partial \zeta} \right) k = y_0 + \zeta y'_0, \tag{79}$$

where

$$y'_0 = \frac{\partial u}{\partial \zeta} e_r + \frac{\partial z}{\partial \zeta} k.$$

For completeness, the deformation gradient F can be constructed by explicitly differentiating y , giving

$$dy \approx \left(\frac{\partial u_0}{\partial R} dR + [y'_0]_r d\zeta \right) e_r(\phi) + u_0 e_\phi d\Phi + \left(\frac{\partial z_0}{\partial R} dR + [y'_0]_k \zeta \right) k, \tag{80}$$

where $[y'_0]_r \equiv y'_0 \cdot e_r$ and $[y'_0]_k \equiv y'_0 \cdot k$.

Using the kinematic relation $dy = Fdx$ along with the differentials in the material coordinates

$$dR = dx \cdot E_R, \quad d\zeta = dx \cdot k, \quad d\Phi = \frac{1}{R} dx \cdot E_\Phi \quad (81)$$

in equation (80) gives the midsurface deformation gradient F_0 as

$$\begin{aligned} F_0 &= \left(\frac{\partial u_0}{\partial R} e_r + \frac{\partial z_0}{\partial R} k \right) \otimes E_R + \frac{u_0}{R} e_\phi \otimes E_\Phi + ([y'_0]_r e_r + [y'_0]_k k) \otimes k \\ &= \nabla y_0 + y'_0 \otimes k, \end{aligned} \quad (82)$$

where

$$\nabla y_0 \equiv \left(\frac{\partial u_0}{\partial R} e_r + \frac{\partial z_0}{\partial R} k \right) \otimes E_R + \frac{u_0}{R} e_\phi \otimes E_\Phi \quad (83)$$

is the two-dimensional gradient operator on the midsurface plane of the membrane.

It will be convenient to introduce a different representation for ∇y_0 by defining orthonormal vectors $v_{1,2}$ and stretch ratios $\lambda_{1,2}$ in terms of u_0, z_0 as

$$v_1 \equiv \frac{\frac{\partial u_0}{\partial R} e_r + \frac{\partial z_0}{\partial R} k}{\left| \frac{\partial u_0}{\partial R} e_r + \frac{\partial z_0}{\partial R} k \right|}, \quad v_2 \equiv e_\phi, \quad \lambda_1 \equiv \sqrt{\left(\frac{\partial u_0}{\partial R} \right)^2 + \left(\frac{\partial z_0}{\partial R} \right)^2}, \quad \lambda_2 \equiv \frac{u_0}{R}, \quad (84)$$

and to assign $u_1 \equiv E_R, u_2 \equiv E_\Phi$ as the basis vectors in the cylindrical polar system. Then we can rewrite equation (82) as

$$F_0 = \lambda_1 v_1 \otimes u_1 + \lambda_2 v_2 \otimes u_2 + y'_0 \otimes k. \quad (85)$$

This will be a useful reformulation when it comes to solving the actual system of equations (21).

Note that $F_0 u_1 = \lambda_1 v_1$ and $F_0 u_2 = \lambda_2 v_2$, so that v_1, v_2 are in the tangent plane to the midsurface of the membrane. The introduction of a final coordinate to describe the tangent plane of the midsurface of the membrane will be useful later (shown in Figure 9).

Define θ such that $v_1 = e_r \cos \theta - k \sin \theta$, so that from equation (84) we obtain

$$\cos \theta = \frac{1}{\lambda_1} \frac{\partial u_0}{\partial R}, \quad \sin \theta = -\frac{1}{\lambda_1} \frac{\partial z_0}{\partial R}. \quad (86)$$

In addition we now have the relation

$$\frac{\partial \lambda_2}{\partial R} = \frac{1}{R} \frac{\partial u_0}{\partial R} - \frac{u_0}{R^2} = \frac{1}{R} (\lambda_1 \cos \theta - \lambda_2).$$

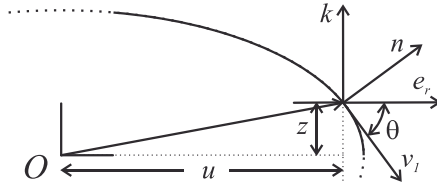


Figure 9. Coordinate system used to describe the kinematics of the membrane. The tangent plane is spanned by v_1, v_2 . The new coordinate is the angle θ that the tangent plane makes with e_r , the radial unit vector in the cylindrical polar system.

The boundary conditions for position are that the membrane is fixed at the outer radius, i.e.

$$y_0|_{R=R_0} = R_0 e_r,$$

and that the membrane is symmetric about the center at $R = 0$, so that

$$\left(\frac{\partial z_0}{\partial R} \right)_{R=0} = 0.$$

In terms of the coordinate θ , the symmetry condition can be expressed as $\theta|_{R=0} = 0$. The electrostatic boundary condition on the electric displacement field occurs at the outer radial edge of the membrane, where we have $1D_0 \cdot \nu_0 = 0$. By symmetry, we have that the electric field is acting along the k direction at $R = 0$, i.e. $e(0) \cdot e_r = 0$.

The unknowns are the four position coordinates u_0, z_0, y'_0 , the two components of the electric field e , and p , the constraint pressure.

5.3 Non-dimensionalization

We non-dimensionalize the governing equations (58), (59) using as length scale the fixed outer radius R_0 of the membrane and using the shear modulus G as a mechanical scaling, so that $\tilde{\nabla} \equiv R_0 \nabla$ is the non-dimensional divergence operator, $\tilde{P} \equiv \hat{P}/G$ is the non-dimensional Piola stress tensor and $\tilde{R} = R_0/R$ is the non-dimensional radial position. Introducing $\tilde{\nabla}, \tilde{P}$ to (58), and using (53) gives

$$\tilde{\nabla} \cdot (\tilde{P}_0 1) = -\Pi \alpha n, \quad \tilde{P}_0 k = 0, \tag{87}$$

where

$$\Pi \equiv \left(\frac{R_0}{h} \right) \left(\frac{\delta p}{G} \right)$$

is the dimensionless lateral boundary loading parameter.

Resolving the non-dimensional versions of equations (58), (59) onto a different basis will lead to some simplifications. We use the orthonormal basis v_1, v_2, v_3 introduced in Section 5.2, where $v_1 = e_r \cos \theta - k \sin \theta$, $v_2 = e_\phi$ are in the midsurface tangent plane in the physical configuration (see Figure 9), and we define v_3 to be $v_3 \equiv v_1 \times v_2 = n$. Projecting onto this basis, equations (58)₁, (59)₁ become

$$\tilde{\nabla} \cdot \tilde{P}_0 \mathbf{1} \cdot v_1 = \frac{\partial \tilde{P}_1}{\partial \tilde{R}} + \frac{\tilde{P}_1 - \tilde{P}_2}{\tilde{R}} \cos \theta = 0, \quad (88)$$

$$\tilde{\nabla} \cdot \tilde{P}_0 \mathbf{1} \cdot v_3 = - \left(\tilde{P}_1 \frac{\partial \theta}{\partial \tilde{R}} + \frac{\tilde{P}_2}{\tilde{R}} \sin \theta \right) = -\Pi \alpha, \quad (89)$$

$$\tilde{\nabla} \cdot \mathbf{1} D_0 = \frac{\partial D_r}{\partial \tilde{R}} + \frac{D_r}{\tilde{R}} = 0, \quad (90)$$

where

$$\tilde{P}_1 \equiv \tilde{P} \cdot v_1 \otimes u_1, \quad \tilde{P}_2 \equiv \tilde{P} \cdot v_2 \otimes u_2, \quad D_r \equiv D \cdot u_1.$$

5.4 Solution and Numerical Formulation

The algebraic conditions

$$\tilde{P}_0 k = 0, \quad D_0 k = -\Sigma, \quad \det F = 1 \quad (91)$$

were solved symbolically for $p, y'_0 \cdot v_1, y'_0 \cdot v_3$ and $E \cdot k$ using *Mathematica*, giving

$$y'_0 \cdot v_1 = \frac{\Sigma[e]_1}{2c_1 + 2c_2 \lambda_2^2}, \quad y'_0 \cdot v_3 = \frac{1}{\lambda_1 \lambda_2}, \quad (92)$$

$$E_k = \frac{\Sigma}{(2\rho_0 c - \epsilon_0) \lambda_1^2 \lambda_2^2} + \frac{\Sigma[e]_1^2}{2c_1 + 2c_2 \lambda_2^2}, \quad (93)$$

where $[e]_1 \equiv e \cdot v_1$ is the electric field in the current configuration projected onto the membrane tangent plane and $E_k \equiv F^T e \cdot k$ is the through thickness component of the electric field in material coordinates. The symbolic expression for p is too lengthy to print here.

The differential equations were also simplified, giving the three ODEs

$$\frac{\partial \lambda_1}{\partial \tilde{R}} = f_1(c_1, c_2, c, \lambda_1, \lambda_2, p, y'_0, e), \quad (94)$$

$$\frac{\partial \theta}{\partial \tilde{R}} = f_2(c_1, c_2, c, \lambda_1, \lambda_2, p, y'_0, e), \quad (95)$$

$$\frac{\partial [e]_1}{\partial \tilde{R}} = f_3(c_1, c_2, c, \lambda_1, \lambda_2, p, y'_0, e), \quad (96)$$

in which the right hand sides are too unwieldy to be recorded here.

The boundary conditions used for the dimensionless problem are

$$\theta(0) = 0, \quad \tilde{u}_0(1) = 1, \quad [e]_1(0) = 0, \quad (97)$$

with $\lambda_1(0) = \lambda_2(0)$ to be determined.

In our study we used as representative material constants $G = 10^6$ MPa, $\delta = 0.9$, $\epsilon = 10 \epsilon_0$ and $\rho_0 = 1000 \text{ kgm}^{-3}$.

5.5 Results

The shooting method is used for the solution of the coupled ODEs (94)–(96). The initial conditions for the stretch ratios λ_1, λ_2 at the center of the membrane $\tilde{R} = 0$ are found by using Newton–Raphson iterations and forward Euler integration from $\tilde{R} = 0$ to $\tilde{R} = 1$ until the boundary condition $\tilde{u}_0(1) = 1$ at the fixed outer radius is met.

Equilibrium solutions for the deformed midsurface, \tilde{u}_0, \tilde{z}_0 , are plotted in Figures 10 and 11. In Figure 10 the load parameter Π is increased from 1000 to 4000 in linearly spaced increments at a constant electric field of 10^5 Vm^{-1} . In Figure 11 the equilibrium midsurface is plotted for values of electric field E_k from 10^5 to 10^7 Vm^{-1} in linear increments and at a fixed pressure of $\Pi = 3.3 \times 10^3$. Note that in Figure 11 a volume is displaced by the membrane as the electric field is ramped up or down. It has been proposed (Mockensturm and Goulbourne, 2004) that the cycling of the imposed electric field be used as means of actuating a dielectric pump.

The results from these two tests are compiled into the surface plot in Figure 12, which shows the interdependence between the loading and the electric field intensity along with the volume encompassed by the membrane.

In all tests the electric field along the membrane $e \cdot v_1$ was found to be zero, so that the final algebraic relations for the director and electric field become

$$y'_0 \cdot v_1 = 0, \quad y'_0 \cdot v_3 = \frac{1}{\lambda_1 \lambda_2}, \quad E_k = -\frac{\Sigma}{\epsilon \lambda_1^2 \lambda_2^2}.$$

These results give that the electric field is acting normal to the membrane, and the director y'_0 is normal to the midsurface, as expected.

6 Conclusion

In this work we have attempted to give an accessible treatment of continuum electrostatics for material geometries which are amenable to reduced dimension analysis. This geometry is typically employed for problems of practical importance, and the membrane equations have immediate applications. We

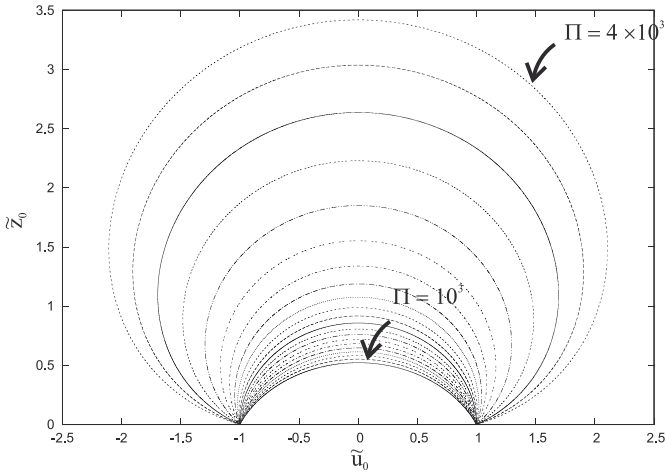


Figure 10. Equilibrium configurations for the membrane midsurface with $E_k = 10^5 \text{ Vm}^{-1}$ and increasing loading parameter Π from 1000 to 4000 in linear increments.

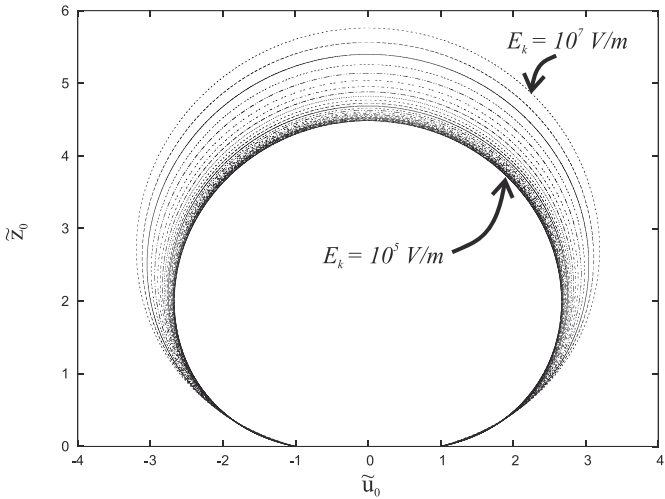


Figure 11. Equilibrium configurations for the membrane midsurface with $\Pi = 3300$ and increasing electric field intensity from $E_k = 10^5$ to 10^7 Vm^{-1} in linearly spaced increments.

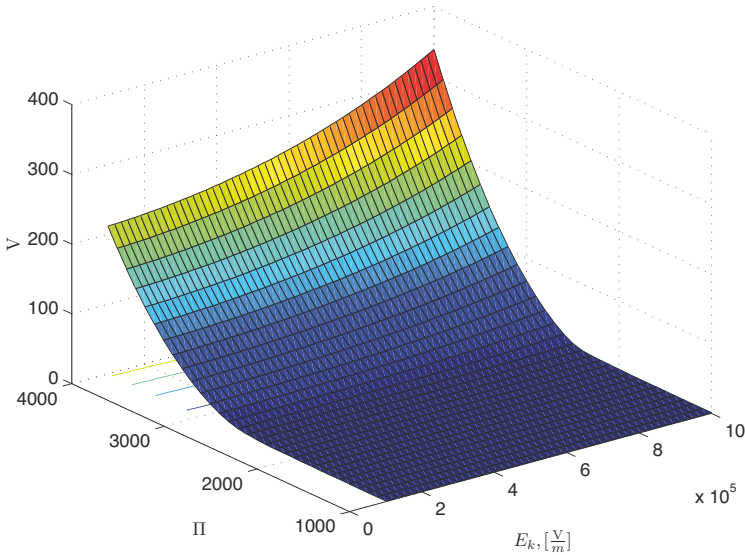


Figure 12. Equilibrium volumes of the inflated membrane, dependence upon both loading factor Π and electric field E_k is shown (plot produced using *MATLAB*).

highlighted the discussion on the distinction between the Maxwell stress tensor and electric field related terms in the constitutive response of the material, in efforts to unify several conventions present in the field. We also gave a constitutive formulation of the elastic energy for a Mooney–Rivlin material model, extended to incorporate electromagnetic behavior. The proposed model for the stored energy function was derived by matching linear dielectric polarization behavior with appropriate derivatives of the stored energy. The membrane model and constitutive formulation were implemented for an axisymmetric geometry and found to exhibit expected behavior.

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Computational Nonlinear Electro-Elasticity – Getting Started –

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Abstract. The intention of this contribution is to provide the basic ingredients needed for the formulation and computation of nonlinear problems in electro-elasticity. Thus, firstly the underlying variational setting of nonlinear electro-elasticity is outlined. Then, secondly the appropriate discretization in terms of the finite element method combined with the boundary element method together with the corresponding solution method are discussed in much detail. Finally the solution of some nonlinear boundary value problems demonstrates the applicability of the derived methods and highlights the characteristic features of coupled problems in nonlinear electro-elasticity.

1 Introduction

In the last few years there has been an ever growing interest for smart materials that exhibit geometrically nonlinear deformations and that change their mechanical behavior in response to the application of electric (or magnetic) fields, see, e.g., the overview in Bar-Cohen (2002) and modelling approaches in the works by Brigdanov and Dorfmann (2003), Dorfmann and Brigdanov (2004) and Dorfmann and Ogden (2003, 2004a,b, 2005a,b). Electronic electro-active polymers (EEAP) are considered to belong to this type of materials¹. For decades, EEAP have been known to have the capability

¹EEAP belong to one of the two subclasses of so-called *electro-active polymers* (EAP) with the other subclass being called ionic electro-active polymers (IEAP). While EEAP are driven by electrostatic forces, IEAP are driven by the diffusion of ions inside the material. The major advantage of IEAP is the requirement for low drive voltages. However, they have slow response and there is the need to maintain their wetness. Besides, it is difficult to sustain current induced displacements, see the overview in Bar-Cohen (2002).

of changing shape and size in response to electric stimulation. However, applications of EEAP caught attention only recently with the discovery of new materials capable of inducing very large deformation. As an alternative to materials that are commonly used for actuators in adaptive structures like piezoelectric ceramics, piezoelectric composites, shape memory metals and alloys, magneto- and electro-rheological fluids, the emerging EEAP offer the possibility to develop lightweight, inexpensive, resilient, damage tolerant, noiseless and agile robotic systems. Although EEAP require high voltages, the advantages such as rapid response, the ability to induce relatively large actuation forces, the ability to operate in room conditions for a long period of time and, most importantly, the ability to hold the induced displacement under activation of a voltage make them ideally suited candidates for various types of actuators².

Potential applications of EEAP include artificial muscles, robotic systems such as mobile mini- and micro-robots, micro-pumps, micro-valves, micro-air vehicles, disk drives, prosthetic devices and flat panel loudspeakers, see the non-exhaustive list of examples contained in the works by Heydt et al. (1998), Eckerle et al. (2001), Kim et al. (2001), Kofod (2001), Kofod et al. (2001), Pelrine et al. (2002), Sommer-Larsen et al. (2002), Wingert et al. (2002), Carpi et al. (2003), Goulbourne et al. (2003), Pei et al. (2003), Lacour et al. (2004), Trujillo et al. (2004), Kofod and Sommer-Larsen (2005), Zhang et al. (2005), Loverich et al. (2006), Wingert et al. (2006), Zhang et al. (2006).

When hysteresis can be neglected, the behavior of EEAP can be modelled by the continuum (field) theory of *nonlinear electro-elasticity*. In general, the behavior of electric (and magnetic) fields, as well as their interactions with matter are governed by Maxwell's equations, see, e.g., the monographs by Maugin (1988), Eringen and Maugin (1990), Griffith (1998), Jackson (1999) and Kovetz (2000). Confining to the quasi-static case of nonlinear electro-elasticity, the electric field acting on matter is governed by Gauss' law for electricity, a relationship describing the link between the electric

² Thereby, the most widespread idea to develop polymer based actuators is to exploit the deformation of EEAP films under electrostatic activation. Actuators of this type may be formed by placing an EEAP thin film between two compliant electrodes. Under activation by a sufficiently high electric voltage applied to the electrodes, the elastic film compresses in thickness and expands in area due to the electrostatic forces between the electrodes. Using the same principle, rolled actuators are made by rolling a biaxially pre-strained double-side coated polymer film around a compressed metal coil spring. As soon as the polymer film is activated, the spring elongates longitudinally due to the deformation of the polymer film. When the film is deactivated the spring contracts back to the normal state.

field (weighted by the vacuum permittivity) together with the polarization or collectively the dielectric displacement and the free charge density. Because of the electric polarization, the electric field exerts a (ponderomotive) body force on matter that can be considered as a function of the electric field and the polarization. With this ponderomotive body force included, the balance equation of linear momentum is the same as that of ordinary nonlinear elasticity except the fact that the ordinary Cauchy stress tensor is not anymore symmetric. Besides the non-symmetric property of the ordinary Cauchy stress, it is noted that difficulties also appear in dealing with the jump conditions for the ordinary Cauchy stress at the boundary of the considered body or across a surface of discontinuity within the body. This is due to the fact that on the one hand the jump of the ordinary Cauchy stress across a surface must balance both mechanical tractions and electrically induced Maxwell tractions. On the other hand, any traction measured by mechanical means is related to the contribution of both mechanical and electrical effects, since no available experiment can separate the effects of the ordinary Cauchy and Maxwell stresses unambiguously, see also the discussion by McMeeking and Landis (2005). This leads to the definition of the so-called total stress tensor, which is the combination of the ordinary Cauchy stress and the Maxwell stress. By assuming the existence of a total energy density that depends on the deformation gradient and on the electric field (see, e.g., Brigdanov and Dorfmann, 2003; Dorfmann and Brigdanov, 2004; Dorfmann and Ogden, 2003, 2004a,b, 2005a,b) a variational formulation of the problem can eventually be formulated.

Based on the variational setting this contribution aims in the representation of a discretization method suited for problems in nonlinear electro-elasticity as developed in Vu et al. (2007) and Vu and Steinmann (2010).

2 Variational Setting of Nonlinear Electro-Elasticity

The starting point for the variational setting of geometrically nonlinear, quasi-static electro-elasticity is the definition of an appropriate energy functional $I = I(\varphi, \boldsymbol{\varphi})$ in terms of the unknown *electric potential* $\varphi = \varphi(\mathbf{X})$ and the *deformation map* $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{X})$ that are parameterized in terms of the material coordinates \mathbf{X} ³

$$I = \int_{S_0} E_0(\mathbf{F}, \mathbb{E}) \, dV + \int_{\mathcal{B}_0} U_0(\varphi, \mathbf{F}, \varphi, \mathbb{E}) \, dV + \int_{\partial\mathcal{B}_0} u_0(\varphi, \varphi) \, dA. \quad (1)$$

³Even if there is no matter outside the body and thus there exists no *physical* deformation map $\boldsymbol{\varphi}$ in free space that justifies a material setting, we may imagine a *fictitious* deformation map $\boldsymbol{\varphi}$ that extends $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{X})$ from the body into its exterior.

Here we⁴ consider a material body consisting of matter occupying its material configuration \mathcal{B}_0 surrounded by free space that occupies the material configuration \mathcal{S}_0 , see Figure 1. It goes without saying that due to the deformation of the body the matter and the free space will occupy spatial configurations \mathcal{B}_t and \mathcal{S}_t , however with $\mathcal{B}_0 \cup \mathcal{S}_0 = \mathcal{B}_t \cup \mathcal{S}_t$. The boundary of the body $\partial\mathcal{B}_0$ in the material configuration defines the interface between free space and matter. For the sake of simplicity of exposition we shall ignore any energy contributions from the boundary of the free space $\partial\mathcal{S}_\infty$ at infinity in this contribution.

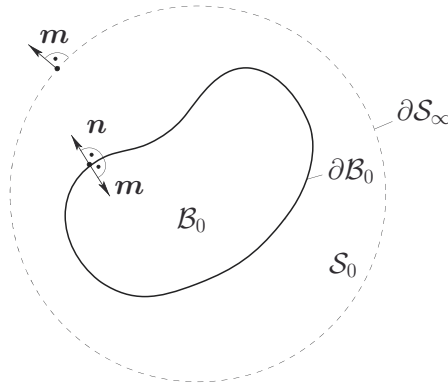


Figure 1. Material body consisting of matter occupying its material configuration \mathcal{B}_0 surrounded by free space that occupies the material configuration \mathcal{S}_0 . The boundary of the body $\partial\mathcal{B}_0$ in the material configuration defines the interface between free space and matter.

The free space electric energy density E_0 per referential unit volume or rather, in order to explain the sign, the free space *electric Lagrange density* is defined as

$$E_0 = -\frac{1}{2} \mathbb{E} \cdot \mathbb{D}^\epsilon. \quad (2)$$

Here \mathbb{E} denotes the referential *electric field* in terms of the material gradient of the electric potential φ , i.e. $\mathbb{E} = -\nabla_X \varphi$, and \mathbb{D}^ϵ is the referential free space *dielectric displacement*, i.e. $\mathbb{D}^\epsilon = \epsilon_0 J \mathbf{B} \cdot \mathbb{E}$, with $\epsilon_0 = 8.854 \cdot 10^{-12} \text{ C}^2/[\text{Nm}^2]$ the vacuum permittivity, the *jacobian* $J = \det \mathbf{F}$ and the *inverse Cauchy-Green strain* $\mathbf{B} = [\mathbf{F}^t \cdot \mathbf{F}]^{-1}$. Furthermore \mathbf{F} denotes the *deformation gradient*, i.e. the material gradient of the nonlinear deformation

⁴That is the reader and the author.

map $\mathbf{F} = \nabla_X \boldsymbol{\varphi}$. The referential quantities follow by appropriate pull-back operations (i.e. either by chain rule or the Piola transformation) from their spatial counterparts $\boldsymbol{\mathfrak{e}} = -\nabla_x \varphi$ and $\mathrm{d}\ell^\epsilon = \epsilon_0 \boldsymbol{\mathfrak{e}}$. Thus the free space electric Lagrange density E_t per spatial unit volume may alternatively be expressed as a quadratic form in $\boldsymbol{\mathfrak{e}}$

$$E_t = -\frac{1}{2} \boldsymbol{\mathfrak{e}} \cdot \mathrm{d}\ell^\epsilon = -\frac{1}{2} \epsilon_0 \boldsymbol{\mathfrak{e}} \cdot \boldsymbol{\mathfrak{e}}. \quad (3)$$

The energy density U_0 related to the matter, per referential unit volume, consists of internal W_0 and external V_0 contributions

$$U_0(\boldsymbol{\varphi}, \mathbf{F}, \varphi, \mathbb{E}) = W_0(\mathbf{F}, \mathbb{E}) + V_0(\boldsymbol{\varphi}, \varphi). \quad (4)$$

The internal contribution W_0 to the energy density in turn consists of the electric energy density E_0 and an energy density ψ_0 associated with matter that is sometimes denoted as *electric free (Gibbs) enthalpy density*⁵. For isothermal conditions at a reference temperature $\theta = \theta_{\text{ref}}$ the internal contribution W_0 to the energy density, sometimes denoted as the total energy density, reads as follows

$$W_0(\mathbf{F}, \mathbb{E}) = E_0(\mathbf{F}, \mathbb{E}) + \psi_0(\mathbf{F}, \theta, \mathbb{E})|_{\theta=\theta_{\text{ref}}}. \quad (5)$$

Finally, besides an energy density in the bulk, we allow for an energy density u_0 , per referential surface area, at the boundary of the considered body

$$u_0 = u_0(\boldsymbol{\varphi}, \varphi). \quad (6)$$

The external contribution V_0 to the energy density in the bulk and the energy density at the boundary do exclusively depend on the unknown solution fields, i.e. the deformation map $\boldsymbol{\varphi}$ and the electric potential φ . Based on the above energy densities the *total Piola stress* \mathbf{P}^{tot} and the referential *dielectric displacement* \mathbb{D} are defined as

$$\mathbf{P}^{\text{tot}} = \frac{\partial U_0}{\partial \mathbf{F}} \quad \text{and} \quad \mathbb{D} = -\frac{\partial U_0}{\partial \mathbb{E}}. \quad (7)$$

The referential dielectric displacement in matter is expressed as the sum of two terms $\mathbb{D} = \mathbb{P} + \mathbb{D}^\epsilon$, i.e. the referential *polarization* \mathbb{P} and the referential free space dielectric displacement \mathbb{D}^ϵ with

$$\mathbb{P} = -\frac{\partial \psi_0}{\partial \mathbb{E}} \quad \text{and} \quad \mathbb{D}^\epsilon = -\frac{\partial E_0}{\partial \mathbb{E}}. \quad (8)$$

⁵The electric free enthalpy density ψ_0 results from a double Legendre transformation $\psi_0(\mathbf{F}, \theta, \mathbb{E}) = \epsilon_0(\mathbf{F}, \sigma_0, \mathbb{P}) - \theta \sigma_0 - \mathbb{E} \cdot \mathbb{P}$ of the internal energy density $\epsilon_0(\mathbf{F}, \sigma_0, \mathbb{P})$ that exchanges (i) the entropy density σ_0 for the absolute temperature θ and (ii) the referential polarization \mathbb{P} for the referential electric field \mathbb{E} .

Observe that no polarization \mathbb{P} exists in free space, whereas \mathbb{D}^ϵ is defined everywhere. Likewise, the total Piola stress in matter follows as the sum of three terms $\mathbf{P}^{\text{tot}} = [\mathbf{P} + \mathbf{P}^{\text{pol}}] + \mathbf{P}^{\text{max}}$, i.e. the ordinary⁶ Piola stress \mathbf{P} and the *polarization stress* \mathbf{P}^{pol} together with the *Maxwell stress* \mathbf{P}^{max} , i.e.

$$\mathbf{P} + \mathbf{P}^{\text{pol}} = \frac{\partial \psi_0}{\partial \mathbf{F}} \quad \text{and} \quad \mathbf{P}^{\text{max}} = \frac{\partial E_0}{\partial \mathbf{F}}. \quad (9)$$

Observe that the Maxwell stress \mathbf{P}^{max} also exists in free space with the same format whereas the ordinary Piola stress \mathbf{P} and the polarization stress \mathbf{P}^{pol} do vanish outside matter. It is sometimes convenient to consider the re-parametrization ϕ_0 of the electric free enthalpy density ψ_0 in terms of the spatial electric field \mathfrak{e}

$$\phi_0 = \phi_0(\mathbf{F}, \mathfrak{e}) \quad \text{with} \quad \psi_0(\mathbf{F}, \mathbb{E}) = \phi_0(\mathbf{F}, \mathbb{E} \cdot \mathbf{F}^{-1}). \quad (10)$$

Based on ϕ_0 the various quantities introduced so far take the following enlightening representations

$$\mathbf{P} = \frac{\partial \phi_0}{\partial \mathbf{F}}, \quad \mathbf{P}^{\text{pol}} = \frac{\partial \phi_0}{\partial \mathfrak{e}} \cdot \frac{\partial \mathfrak{e}}{\partial \mathbf{F}} = \mathfrak{e} \otimes \mathbb{P}, \quad \mathbb{P} = -\frac{\partial \phi_0}{\partial \mathfrak{e}} \cdot \frac{\partial \mathfrak{e}}{\partial \mathbb{E}} = \mathbb{P} \cdot \text{cof} \mathbf{F}. \quad (11)$$

Here we implicitly defined the spatial polarization $\mathbb{P} = -\partial \phi_t / \partial \mathfrak{e}$ with $\phi_t = J^{-1} \phi_0$ the re-parameterized electric free enthalpy density per unit volume of the spatial configuration of the body. Since the electric energy density E_0 is given without ambiguity, the Maxwell stress \mathbf{P}^{max} takes the following explicit format

$$\mathbf{P}^{\text{max}} = E_0 \mathbf{F}^{-t} + \mathfrak{e} \otimes \mathbb{D}^\epsilon. \quad (12)$$

The corresponding Piola transformation defines the well-known spatial description (Cauchy-type) version of the Maxwell stress

$$\boldsymbol{\sigma}^{\text{max}} = E_t \mathbf{i} + \mathfrak{e} \otimes \mathfrak{d}^\epsilon. \quad (13)$$

Here, \mathbf{i} denotes the second-order spatial unit tensor with coefficients δ_{ij} . The partial derivatives of the energy density U_0 with respect to the unknown solution fields, i.e. the deformation map $\boldsymbol{\varphi}$ and the electric potential φ render finally the following quantities

$$\mathbf{b}_0 = -\frac{\partial U_0}{\partial \boldsymbol{\varphi}}, \quad \mathbf{t}_0 = -\frac{\partial u_0}{\partial \varphi}, \quad \mathfrak{d}_0^f = \frac{\partial U_0}{\partial \varphi}, \quad \widehat{\mathfrak{d}}_0^f = \frac{\partial u_0}{\partial \varphi}. \quad (14)$$

⁶The ordinary Piola stress \mathbf{P} is in equilibrium with the mechanical body forces \mathbf{b}_0 and the ponderomotive (Lorentz) body force $\mathbf{b}_0^{\text{pon}} = \text{Div}(\mathbf{P}^{\text{pol}} + \mathbf{P}^{\text{max}})$ in \mathcal{B}_0 , i.e. $\text{Div} \mathbf{P} = -\mathbf{b}_0 - \mathbf{b}_0^{\text{pon}}$, and mechanical surface tractions \mathbf{t}_0 and ponderomotive surface tractions $\mathbf{t}_0^{\text{pon}} = \llbracket \mathbf{P}^{\text{pol}} + \mathbf{P}^{\text{max}} \rrbracket \cdot \mathbf{N}$ in $\partial \mathcal{B}_0$, i.e. $\mathbf{P} \cdot \mathbf{N} = \mathbf{t}_0 + \mathbf{t}_0^{\text{pon}}$.

Thereby, \mathbf{b}_0 and \mathbf{t}_0 denote the *mechanical body force* per unit volume in \mathcal{B}_0 and the *mechanical surface traction* per unit area in $\partial\mathcal{B}_0$, respectively, whereas ϱ_0^f and $\widehat{\varrho}_0^f$ represent the *density of free charges* per unit volume in \mathcal{B}_0 and the *density of free charges* per unit area in $\partial\mathcal{B}_0$, respectively. Observe that \mathbf{b}_0 and ϱ_0^f do not exist in free space. With these preliminaries, the stationary point of the energy functional $I = I(\varphi, \boldsymbol{\varphi})$ with respect to variations $\delta\varphi$ in the deformation map renders

$$\int_{\mathcal{S}_0 \cup \mathcal{B}_0} \nabla_X \delta\varphi : \mathbf{P}^{\text{tot}} \, dV - \int_{\mathcal{B}_0} \delta\varphi \cdot \mathbf{b}_0 \, dV - \int_{\partial\mathcal{B}_0} \delta\varphi \cdot \mathbf{t}_0 \, dA = 0. \quad (15)$$

Here we set $\mathbf{P}^{\text{tot}} = \mathbf{P}^{\text{max}}$ in \mathcal{S}_0 to abbreviate writing. Assuming sufficient smoothness and taking into account the arbitrariness of $\delta\varphi$ renders as Euler equation the mechanical equilibrium conditions⁷

$$\text{Div} \mathbf{P}^{\text{tot}} \doteq -\mathbf{b}_0 \quad \text{in } \mathcal{B}_0 \quad \text{and} \quad \text{Div} \mathbf{P}^{\text{max}} \doteq \mathbf{0} \quad \text{in } \mathcal{S}_0. \quad (16)$$

Likewise the corresponding Neumann boundary condition follows as

$$[[\mathbf{P}^{\text{tot}}]] \cdot \mathbf{N} \doteq -\mathbf{t}_0 \quad \text{on } \partial\mathcal{B}_0. \quad (17)$$

Here \mathbf{N} is the referential outward pointing normal to the boundary of the body, i.e. pointing from matter to free space, and the jump of a quantity (\bullet) at $\partial\mathcal{B}_0$ is defined as $[[(\bullet)]] = (\bullet)_{\text{free space}} - (\bullet)_{\text{matter}}$. Likewise the stationary point of the energy functional $I(\varphi, \boldsymbol{\varphi})$ with respect to variations $\delta\varphi$ in the electric potential renders

$$\int_{\mathcal{S}_0 \cup \mathcal{B}_0} \nabla_X \delta\varphi \cdot \mathbb{D} \, dV + \int_{\mathcal{B}_0} \delta\varphi \varrho_0^f \, dV + \int_{\partial\mathcal{B}_0} \delta\varphi \widehat{\varrho}_0^f \, dA = 0. \quad (18)$$

Here we set $\mathbb{D} = \mathbb{D}^\epsilon$ in \mathcal{S}_0 to abbreviate writing. Assuming sufficient smoothness and taking into account the arbitrariness of $\delta\varphi$ renders as Euler equation the electric charge conservation conditions or rather the Gauss law of electricity

$$\text{Div} \mathbb{D} \doteq \varrho_0^f \quad \text{in } \mathcal{B}_0 \quad \text{and} \quad \text{Div} \mathbb{D}^\epsilon \doteq 0 \quad \text{in } \mathcal{S}_0. \quad (19)$$

Moreover the corresponding Neumann boundary condition follows as

$$[[\mathbb{D}]] \cdot \mathbf{N} \doteq \widehat{\varrho}_0^f. \quad (20)$$

⁷Note that the condition $\text{Div} \mathbf{P}^{\text{max}} \doteq \mathbf{0}$ in \mathcal{S}_0 is consistent with the relation $\text{Div} \mathbb{D}^\epsilon = 0$ and $\nabla_X \mathbb{E} = [\nabla_X \mathbb{E}]^t$ or rather $\text{Curl} \mathbb{E} = \mathbf{0}$ in \mathcal{S}_0 .

3 Reformulation of Euler Equations

3.1 Weak Form for the Material Body

For the material body only, the stationary conditions for the energy functional $I(\varphi, \boldsymbol{\varphi})$ with respect to variations $\delta\varphi$, i.e. the mechanical Euler equations may be recast into weak form by testing with $\delta\varphi$

$$R^u = \int_{\partial\mathcal{B}_0} \delta\varphi \cdot [\mathbf{t}_0 + \mathbf{t}_0^{\max}] dA + \int_{\mathcal{B}_0} \delta\varphi \cdot \mathbf{b}_0 dV - \int_{\mathcal{B}_0} \nabla_X \delta\varphi : \mathbf{P}^{\text{tot}} dV. \quad (21)$$

Here $\mathbf{t}_0^{\max} = \mathbf{P}^{\max} \cdot \mathbf{N}$ is the *Maxwell traction* exerted by the electric field in free space onto the material body.

Note that the integrals in the above residual statement extend over the material configuration, thus all densities refer to unit volume in \mathcal{B}_0 or unit area in $\partial\mathcal{B}_0$, respectively. By (i) transforming the integration domain according to, e.g., $dv = J dV$ and (ii) by exchanging the material gradient ∇_X of $\delta\varphi$ by its spatial gradient with $\nabla_X \delta\varphi = \nabla_x \delta\varphi \cdot \mathbf{F}$ we may rewrite the above residual statement as

$$R^u = \int_{\partial\mathcal{B}_t} \delta\varphi \cdot [\mathbf{t}_t + \mathbf{t}_t^{\max}] da + \int_{\mathcal{B}_t} \delta\varphi \cdot \mathbf{b}_t dv - \int_{\mathcal{B}_t} \nabla_x \delta\varphi : \boldsymbol{\sigma}^{\text{tot}} dv. \quad (22)$$

Here we set $\mathbf{t}_0 dA = \mathbf{t}_t da$ and $\mathbf{b}_0 dV = \mathbf{b}_t dv$ and introduced the *total Cauchy stress* $\boldsymbol{\sigma}^{\text{tot}}$ with $\boldsymbol{\sigma}^{\text{tot}} dv = \mathbf{P}^{\text{tot}} \cdot \mathbf{F}^t dV$.

The Maxwell traction $\mathbf{t}_t^{\max} = \boldsymbol{\sigma}^{\max} \cdot \mathbf{n}$ follows from the Cauchy-type Maxwell stress $\boldsymbol{\sigma}^{\max} = -\epsilon_0 \mathbb{e} \cdot \mathbb{e} / 2 \mathbf{i} + \epsilon_0 \mathbb{e} \otimes \mathbb{e}$ as defined in free space.

The spatial electric field may be split into normal and tangential contributions $\mathbb{e} = [\mathbb{e} \cdot \mathbf{n}] \mathbf{n} + [\mathbb{e} \cdot \mathbf{t}] \mathbf{t}$, whereby \mathbf{t} denotes the tangent unit vector to the surface $\partial\mathcal{B}_t$. Recall that \mathbb{e} is continuous across $\partial\mathcal{B}_t$ in tangent direction, i.e. $[[\mathbb{e}]] \cdot \mathbf{t} = 0$. Next we express \mathbb{e} in terms of the electric potential $\varphi = -\nabla_x \varphi = -\nabla_n \varphi \mathbf{n} - \nabla_t \varphi \mathbf{t}$ with obvious meaning for the operators ∇_n and ∇_t . Incorporating the boundary condition $q_t = -\mathbb{d}^\epsilon \cdot \mathbf{n} = \epsilon_0 \nabla_n \varphi$, with q_t an independent flux variable (see below), we obtain $\nabla_n \varphi = q_t / \epsilon_0$. Thus we may eventually express the Maxwell traction exerted by the electric field on the boundary of the body by

$$\mathbf{t}_t^{\max} = \frac{q_t^2}{2\epsilon_0} \mathbf{n} - \frac{\epsilon_0}{2} [\nabla_t \varphi]^2 \mathbf{n} + q_t \nabla_t \varphi \mathbf{t}. \quad (23)$$

Along the same lines, for the material body only, the stationary conditions for the energy functional $I(\varphi, \boldsymbol{\varphi})$ with respect to variations $\delta\varphi$, i.e. the electrical Euler equations may be recast into weak form by testing with $\delta\varphi$

$$R^\varphi = - \int_{\partial\mathcal{B}_0} \delta\varphi [\hat{\varrho}_0^f + q_0] dA - \int_{\mathcal{B}_0} \delta\varphi \varrho_0^f dV - \int_{\mathcal{B}_0} \nabla_X \delta\varphi \cdot \mathbb{D} dV. \quad (24)$$

Recall that $q_0 = -\mathbb{D}^\epsilon \cdot \mathbf{N}$ denotes the independent flux variable, now expressed per referential unit area. Again this statement may be re-expressed by (i) transforming the integration domain and (ii) by exchanging the material gradient ∇_X of $\delta\varphi$ by its spatial gradient with $\nabla_x \delta\varphi = \nabla_x \tilde{\varphi} \cdot \mathbf{F}$

$$R^\varphi = - \int_{\partial \mathcal{B}_t} \delta\varphi [\tilde{\varrho}_t^f + q_t] da - \int_{\mathcal{B}_t} \delta\varphi \varrho_t^f dv - \int_{\mathcal{B}_t} \nabla_x \delta\varphi \cdot d dv. \quad (25)$$

Here we used the obvious relations $\tilde{\varrho}_0^f dA = \tilde{\varrho}_t^f da$, $\varrho_0^f dV = \varrho_t^f dv$ and $d dv = \mathbb{D} \cdot \mathbf{F}^t dV$. Note that $q_t = -d^\epsilon \cdot \mathbf{n}$ is an independent flux variable that we have to solve for and that denotes the contribution from the free space.

3.2 Linearization of the Weak Form and Tangent Moduli

Suppose next that we want to satisfy the residual statements $R^u = 0$ and $R^\varphi = 0$. To this end we may conceive a Newton-like procedure with $R^u + \Delta R^u = 0$ and $R^\varphi + \Delta R^\varphi = 0$ whereby $\Delta R^u = -L^{uu} - L^{u\varphi}$ and $\Delta R^\varphi = -L^{\varphi u} - L^{\varphi\varphi}$ define the linearization of the residuals with respect to the unknown solution fields φ, φ . Thus, based on the above residual statements we obtain the following representations for the linearization of the *internal* contributions to the weak form

$$\begin{aligned} L_{\text{int}}^{uu} &= \int_{\mathcal{B}_0} \nabla_X \delta\varphi : \mathbf{E} : \nabla_X \Delta\varphi dV, \\ L_{\text{int}}^{u\varphi} &= \int_{\mathcal{B}_0} \nabla_X \delta\varphi : \mathbf{P}^T \cdot \nabla_X \Delta\varphi dV, \\ L_{\text{int}}^{\varphi u} &= \int_{\mathcal{B}_0} \nabla_X \delta\varphi \cdot \mathbf{P} : \nabla_X \Delta\varphi dV, \\ L_{\text{int}}^{\varphi\varphi} &= - \int_{\mathcal{B}_0} \nabla_X \delta\varphi \cdot \mathbf{D} \cdot \nabla_X \Delta\varphi dV. \end{aligned} \quad (26)$$

Thereby the fourth-order referential *elasticity tensor* \mathbf{E} follows as the derivative of the total Piola stress \mathbf{P}^{tot} with respect to the deformation gradient \mathbf{F} or, taking into account the constitutive law for \mathbf{P}^{tot} , as part of the *hessian* of the total energy density W_0

$$\mathbf{E} = \frac{\partial \mathbf{P}^{\text{tot}}}{\partial \mathbf{F}} = \frac{\partial^2 W_0}{\partial \mathbf{F} \otimes \partial \mathbf{F}}. \quad (27)$$

Likewise the second-order referential *dielectricity tensor* \mathbf{D} follows as the derivative of the referential dielectric displacement \mathbb{D} with respect to the referential electric field \mathbb{E} or, taking into account the constitutive law for \mathbb{D} , as part of the hessian of the total energy density W_0

$$\mathbf{D} = \frac{\partial \mathbb{D}}{\partial \mathbb{E}} = - \frac{\partial^2 W_0}{\partial \mathbb{E} \otimes \partial \mathbb{E}}. \quad (28)$$

Finally the third-order referential *piezoelectricity tensor* \mathbf{P} follows either as the derivative of the referential dielectric displacement \mathbb{D} with respect to the deformation gradient \mathbf{F} or alternatively as the derivative of the total Piola stress \mathbf{P}^{tot} with respect to the referential electric field \mathbb{E}

$$\mathbf{P} = \frac{\partial \mathbb{D}}{\partial \mathbf{F}} = -\frac{\partial^2 W_0}{\partial \mathbb{E} \otimes \partial \mathbf{F}}, \quad \mathbf{P}^T = -\frac{\partial \mathbf{P}^{\text{tot}}}{\partial \mathbb{E}} = -\frac{\partial^2 W_0}{\partial \mathbf{F} \otimes \partial \mathbb{E}}. \quad (29)$$

Here the the superscript T refers to a transposition of a third-order tensor $[\mathbf{P}]_{IjK} = P_{IjK}$ that exchanges the first and the two last indices into $[\mathbf{P}^T]_{jKI} = P_{IjK}$. All integrals in the linearizations $L_{\text{int}}^{uu}, L_{\text{int}}^{u\varphi}, L_{\text{int}}^{\varphi u}, L_{\text{int}}^{\varphi\varphi}$ extend over the material configuration and involve exclusively the material gradient operator ∇_X . However, by (i) transforming the integration domain according to $dv = J dV$ and (ii) by exchanging the material gradient operator ∇_X with the spatial gradient operator ∇_x we may rewrite the linearization of the internal contributions to the residual statements as follows

$$\begin{aligned} L_{\text{int}}^{uu} &= \int_{\mathcal{B}_t} \nabla_x \delta\varphi : \mathbf{e} : \nabla_x \Delta\varphi \, dv, \\ L_{\text{int}}^{u\varphi} &= \int_{\mathcal{B}_t} \nabla_x \delta\varphi : \mathbf{p}^T \cdot \nabla_x \Delta\varphi \, dv, \\ L_{\text{int}}^{\varphi u} &= \int_{\mathcal{B}_t} \nabla_x \delta\varphi \cdot \mathbf{p} : \nabla_x \Delta\varphi \, dv, \\ L_{\text{int}}^{\varphi\varphi} &= - \int_{\mathcal{B}_t} \nabla_x \delta\varphi \cdot \mathbf{d} \cdot \nabla_x \Delta\varphi \, dv. \end{aligned} \quad (30)$$

In this representation the fourth-order spatial elasticity tensor \mathbf{e} follows from a push-forward of the referential elasticity tensor \mathbf{E} with the help of the deformation gradient \mathbf{F} as

$$J\mathbf{e} = [\mathbf{i} \overline{\otimes} \mathbf{F}] : \mathbf{E} : [\mathbf{i} \overline{\otimes} \mathbf{F}^t], \quad J\mathbf{e}_{ijkl} = F_{jI} E_{iJkL} F_{lL}. \quad (31)$$

Recall the \mathbf{i} denotes the second-order spatial unit tensor with coefficients δ_{ij} ; the special dyadic product $\overline{\otimes}$ orders indices so that, e.g., $[\mathbf{i} \overline{\otimes} \mathbf{F}]_{ijkl} = \delta_{ik} F_{jL}$. Moreover, in our index notation lower case and upper case indices refer to the spatial and material base vectors, respectively. Furthermore the second-order spatial dielectricity tensor \mathbf{d} follows from a push-forward of the referential dielectricity tensor \mathbf{D}

$$J\mathbf{d} = \mathbf{F} \cdot \mathbf{D} \cdot \mathbf{F}^t, \quad J\mathbf{d}_{ij} = F_{iI} D_{IJ} F_{jJ}. \quad (32)$$

Finally the third-order spatial piezoelectricity tensor \mathbf{p} follows from a push-forward of the referential piezoelectricity tensor \mathbf{P}

$$J\mathbf{p} = \mathbf{F} \cdot \mathbf{P} : [\mathbf{i} \overline{\otimes} \mathbf{F}^t], \quad J\mathbf{p}_{ijk} = F_{iI} P_{IjK} F_{kK}. \quad (33)$$

Material and geometric parts of tangent moduli. It is sometimes convenient to introduce the symmetric *total Piola-Kirchhoff stress* \mathbf{S}^{tot} from

$$\mathbf{P}^{\text{tot}} = \mathbf{F} \cdot \mathbf{S}^{\text{tot}}. \quad (34)$$

Thus the linearization of the total Piola stress may be decomposed into

$$\Delta \mathbf{P}^{\text{tot}} = \mathbf{F} \cdot \Delta \mathbf{S}^{\text{tot}} + \Delta \mathbf{F} \cdot \mathbf{S}^{\text{tot}}. \quad (35)$$

Here the linearization $\Delta \mathbf{S}^{\text{tot}}$ of the total Piola-Kirchhoff stress may be expressed in terms of the fourth-order referential elasticity tensor \mathbf{C} and the third-order referential piezoelectricity tensor $\mathbf{\Pi}$

$$\Delta \mathbf{S}^{\text{tot}} = \mathbf{C} : \frac{1}{2} \Delta \mathbf{C} - \mathbf{\Pi}^T : \Delta \mathbb{E}. \quad (36)$$

It is easy to show from the chain rule that \mathbf{S}^{tot} follows from the derivative of the total energy density W_0 with respect to the *Cauchy-Green strain* $\mathbf{C} = \mathbf{F}^t \cdot \mathbf{F}$ as

$$\mathbf{S}^{\text{tot}} = 2 \frac{\partial W_0}{\partial \mathbf{C}}. \quad (37)$$

Thus the corresponding fourth-order referential elasticity tensor \mathbf{C} and the third-order referential piezoelectricity tensor $\mathbf{\Pi}$ follow in turn as part of the hessian of the total energy density W_0 as

$$\mathbf{C} = 2 \frac{\partial \mathbf{S}^{\text{tot}}}{\partial \mathbf{C}} = 4 \frac{\partial^2 W_0}{\partial \mathbf{C} \otimes \partial \mathbf{C}} \quad \mathbf{\Pi} = 2 \frac{\partial \mathbb{D}}{\partial \mathbf{C}} = -2 \frac{\partial^2 W_0}{\partial \mathbb{E} \otimes \partial \mathbf{C}}. \quad (38)$$

With these preliminaries at hand we then identify the typical decomposition of the referential elasticity tensor \mathbf{E} into *material* and *geometric* parts \mathbf{E}^{mat} and \mathbf{E}^{geo}

$$\Delta \mathbf{P}^{\text{tot}} = [\mathbf{E}^{\text{mat}} + \mathbf{E}^{\text{geo}}] : \Delta \mathbf{F} - \mathbf{P}^T \cdot \Delta \mathbb{E}. \quad (39)$$

Thereby the material part \mathbf{E}^{mat} of the referential elasticity tensor is given in terms of \mathbf{C}

$$\mathbf{E}^{\text{mat}} = [\mathbf{F} \overline{\otimes} \mathbf{I}] : \mathbf{C} : [\mathbf{F}^t \overline{\otimes} \mathbf{I}], \quad \mathbf{E}_{iJkL}^{\text{mat}} = F_{iI} \mathbf{C}_{IJKL} F_{kK}, \quad (40)$$

whereas its geometric part \mathbf{E}^{geo} has the flavor of a pre-stress and is represented in terms of \mathbf{S}^{tot}

$$\mathbf{E}^{\text{geo}} = \mathbf{i} \overline{\otimes} \mathbf{S}^{\text{tot}}, \quad \mathbf{E}_{iJkL}^{\text{geo}} = \delta_{ik} S_{JL}^{\text{tot}}. \quad (41)$$

To complete the picture the third-order piezoelectricity tensors \mathbf{P} and $\mathbf{\Pi}$ are finally related as

$$\mathbf{P} = \mathbf{\Pi} : [\mathbf{F}^t \overline{\otimes} \mathbf{I}], \quad P_{IjK} = \Pi_{IJK} F_{jJ}. \quad (42)$$

In the above \mathbf{I} denotes the second-order referential unit tensor with coefficients δ_{IJ} . Quite in analogy to the previous discussion the decomposition of the elasticity tensor into material and geometric parts may be recast in a spatial description

$$\mathbf{e} = \mathbf{e}^{\text{mat}} + \mathbf{e}^{\text{geo}}. \quad (43)$$

Here the material part \mathbf{e}^{mat} of the spatial elasticity tensor \mathbf{e} follows from a push-forward of its referential counterpart \mathbf{E}^{mat} with the help of the deformation gradient \mathbf{F} as

$$J\mathbf{e}^{\text{mat}} = [\mathbf{F} \overline{\otimes} \mathbf{F}] : \mathbf{C} : [\mathbf{F}^t \overline{\otimes} \mathbf{F}^t], \quad J\mathbf{e}_{ijkl}^{\text{mat}} = F_{iI} F_{jJ} C_{IJKL} F_{kK} F_{lL}. \quad (44)$$

Moreover, the geometric part \mathbf{e}^{geo} of the spatial elasticity tensor \mathbf{e} is expressed in terms of the total Cauchy stress $\boldsymbol{\sigma}^{\text{tot}}$ as

$$\mathbf{e}^{\text{geo}} = \mathbf{i} \overline{\otimes} \boldsymbol{\sigma}^{\text{tot}}, \quad \mathbf{e}_{ijkl}^{\text{geo}} = \delta_{ik} \sigma_{jl}^{\text{tot}}. \quad (45)$$

Our discussion of tangent moduli is eventually completed by stating the push-forward of the third-order piezoelectricity tensor $\mathbf{\Pi}$

$$J\mathbf{p} = \mathbf{F} \cdot \mathbf{\Pi} : [\mathbf{F}^t \overline{\otimes} \mathbf{F}^t], \quad J\mathbf{p}_{ijk} = F_{iI} \Pi_{IJK} F_{jJ} F_{kK}. \quad (46)$$

3.3 Boundary Integral Equation for the Free Space

In the free space solution domain \mathcal{S}_t the only unknown field variable is the electric potential $\varphi = \varphi(\mathbf{x})$ that is governed by the following Laplace equation

$$\Delta_x \varphi = 0 \quad \text{in } \mathcal{S}_t, \quad (47)$$

with Δ_x the spatial Laplace operator, and with Neumann boundary data

$$\nabla_m \varphi = -\frac{q_t}{\epsilon_0} \quad \text{on } \partial \mathcal{S}_t. \quad (48)$$

Here \mathbf{m} denotes the spatial outward pointing normal to the boundary $\partial \mathcal{S}_t$ of the free space, $\nabla_m \varphi = \nabla_x \varphi \cdot \mathbf{m}$ and the (unknown) flux q_t is related by the appropriate jump condition $\llbracket \mathbf{d} \rrbracket \cdot \mathbf{n} = -\widehat{\varrho}_t^{\text{f}}$ to the corresponding flux from the matter and the density of free charges at the surface by $-q_t = \mathbf{d} \cdot \mathbf{n} + \widehat{\varrho}_t^{\text{f}} = \mathbf{d}^\epsilon \cdot \mathbf{n}$. Since we identified the interface between free space and matter by $\partial \mathcal{B}_t = \partial \mathcal{S}_t \setminus \partial \mathcal{S}_\infty$ the normal to the free space boundary is

related to the normal to the boundary of the material body by $\mathbf{m} = -\mathbf{n}$ on $\partial\mathcal{B}_t \cap \partial\mathcal{S}_t$.

The starting point for the derivation of the boundary integral equation is the weighted format of the governing Laplace equation

$$\int_{\mathcal{S}_t} G_\xi(\mathbf{x}) \Delta_x \varphi(\mathbf{x}) \, dv = 0. \tag{49}$$

For reasons that become obvious later we selected for the weight (or test) function the *fundamental solution* G_ξ to the Laplace equation. Thereby the fundamental solution $G_\xi = G_\xi(\mathbf{x})$ corresponding to the Laplace equation follows from the related auxiliary problem

$$\Delta_x G_\xi = -\delta_\xi \quad \text{in } \mathcal{S}_t. \tag{50}$$

Here $\delta_\xi = \delta(\mathbf{x} - \boldsymbol{\xi})$ denotes the Dirac distribution at the *source point* $\boldsymbol{\xi}$. Next, by applying partial integration twice (or rather Green’s second identity) the weighted statement in the above equation is re-expressed as

$$\begin{aligned} \int_{\mathcal{S}_t} G_\xi(\mathbf{x}) \Delta_x \varphi(\mathbf{x}) \, dv &= \int_{\mathcal{S}_t} \varphi(\mathbf{x}) \Delta_x G_\xi(\mathbf{x}) \, dv \\ &+ \int_{\partial\mathcal{S}_t} [G_\xi(\mathbf{x}) \nabla_m \varphi(\mathbf{x}) - \varphi(\mathbf{x}) \nabla_m G_\xi(\mathbf{x})] \, da. \end{aligned} \tag{51}$$

By exploiting the properties of (i) the fundamental solution, $\Delta_x G_\xi = -\delta_\xi$, where $G_\xi = G_\xi(\mathbf{x})$ and (ii) the Dirac distribution $\delta_\xi = \delta(\mathbf{x} - \boldsymbol{\xi})$ the volume integral on the right hand side is re-written as

$$\int_{\mathcal{S}_t} \varphi(\mathbf{x}) \Delta_x G_\xi(\mathbf{x}) \, dv = - \int_{\mathcal{S}_t} \varphi(\mathbf{x}) \delta(\mathbf{x} - \boldsymbol{\xi}) \, dv = -\varphi(\boldsymbol{\xi}). \tag{52}$$

As a consequence we obtain a representation for the solution $\varphi(\boldsymbol{\xi})$ at the source point $\boldsymbol{\xi}$ located within the (open) solution domain $\mathcal{S}_t \setminus \partial\mathcal{S}_t$ expressed in terms of integrals extending over the total boundary $\partial\mathcal{S}_t = \partial\mathcal{B}_t \cup \partial\mathcal{S}_\infty$ of the free space

$$\varphi(\boldsymbol{\xi}) = \int_{\partial\mathcal{S}_t} [G_\xi(\mathbf{x}) \nabla_m \varphi(\mathbf{x}) - \varphi(\mathbf{x}) \nabla_m G_\xi(\mathbf{x})] \, da. \tag{53}$$

The interest lies, however, in a boundary integral expression for the solution $\varphi(\boldsymbol{\xi})$ at the source point $\boldsymbol{\xi} \in \partial\mathcal{S}_t$ located on the boundary $\partial\mathcal{S}_t$ to the solution domain \mathcal{S}_t . This is achieved by (i) positioning the source point $\boldsymbol{\xi}$ at the boundary, (ii) adding a spherical (circular) surrounding \mathcal{S}_ϵ with radius

ϵ centered at ξ so that $\mathcal{S}_t^* = \mathcal{S}_t \cup \mathcal{S}_\epsilon$ with $\partial\mathcal{S}_t^* = [\partial\mathcal{S}_t \cup \partial\mathcal{S}_\epsilon] \setminus [\partial\mathcal{S}_t \cap \partial\mathcal{S}_\epsilon]$, see Figure 2 and (iii) taking the limit for $\epsilon \rightarrow 0$

$$\varphi(\xi) = \lim_{\epsilon \rightarrow 0} \int_{\partial\mathcal{S}_t^*} [G_\xi(\mathbf{x})\nabla_m\varphi(\mathbf{x}) - \varphi(\mathbf{x})\nabla_m G_\xi(\mathbf{x})] da. \tag{54}$$

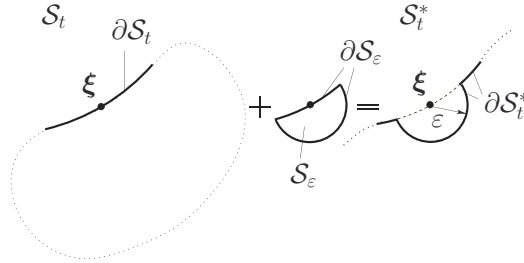


Figure 2. Addition of a spherical (circular) surrounding \mathcal{S}_ϵ with radius ϵ centered at ξ so that $\mathcal{S}_t^* = \mathcal{S}_t \cup \mathcal{S}_\epsilon$ with $\partial\mathcal{S}_t^* = [\partial\mathcal{S}_t \cup \partial\mathcal{S}_\epsilon] \setminus [\partial\mathcal{S}_t \cap \partial\mathcal{S}_\epsilon]$. We consider the limit for $\epsilon \rightarrow 0$

Then by decomposing $\partial\mathcal{S}_t^*$ into its individual contributions and by carefully considering the different types of singularities⁸ stemming from the (known) fundamental solution G_ξ we obtain eventually

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\partial\mathcal{S}_t^*} G_\xi(\mathbf{x})\nabla_m\varphi(\mathbf{x}) da &\rightarrow \int_{\partial\mathcal{S}_t} G_\xi(\mathbf{x})\nabla_m\varphi(\mathbf{x}) da, & (55) \\ \lim_{\epsilon \rightarrow 0} \int_{\partial\mathcal{S}_t^* \setminus [\partial\mathcal{S}_t \cap \partial\mathcal{S}_\epsilon]} \varphi(\mathbf{x})\nabla_m G_\xi(\mathbf{x}) da &\rightarrow \int_{\partial\mathcal{S}_t} \varphi(\mathbf{x})\nabla_m G_\xi(\mathbf{x}) da, \\ \lim_{\epsilon \rightarrow 0} \int_{\partial\mathcal{S}_\epsilon \setminus [\partial\mathcal{S}_t \cap \partial\mathcal{S}_\epsilon]} \varphi(\mathbf{x})\nabla_m G_\xi(\mathbf{x}) da &\rightarrow \varphi(\xi) \lim_{\epsilon \rightarrow 0} \int_{\partial\mathcal{S}_\epsilon \setminus [\partial\mathcal{S}_t \cap \partial\mathcal{S}_\epsilon]} \nabla_m G_\xi(\mathbf{x}) da. \end{aligned}$$

⁸To analyze further the individual terms in the above integral it is helpful to recall the fundamental solution G_ξ for the Laplace equation in two and three dimensions together with its first two derivatives $\partial_r G_\xi$ and $\partial_{rr}^2 G_\xi$ and the classification of singularities of integrands:

	G_ξ	$\partial_r G_\xi$	$\partial_{rr}^2 G_\xi$	da	Weak	Strong	Hyper
2d	$\frac{1}{2\pi} \ln r^{-1}$	$-\frac{1}{2\pi} r^{-1}$	$\frac{1}{2\pi} r^{-2}$	$O(r)$	$O(\ln r^{-1})$	$O(r^{-1})$	$O(r^{-2})$
3d	$\frac{1}{4\pi} r^{-1}$	$-\frac{1}{4\pi} r^{-2}$	$\frac{1}{2\pi} r^{-3}$	$O(r^2)$	$O(r^{-1})$	$O(r^{-2})$	$O(r^{-3})$

Here $r = |\mathbf{r}|$ with $\mathbf{r} = \mathbf{x} - \xi$ denotes the radial distance of the field point \mathbf{x} from the source point ξ .

Assuming sufficient smoothness we expanded $\varphi(\mathbf{x})$ in the last expression into a Taylor series $\varphi(\mathbf{x}) = \varphi(\boldsymbol{\xi}) + \nabla_{\boldsymbol{\xi}}\varphi(\boldsymbol{\xi}) \cdot \mathbf{r} + \dots$ (with $\mathbf{r} = \mathbf{x} - \boldsymbol{\xi}$) that we truncated after the first term. As a consequence the *boundary integral equation* for a source point $\boldsymbol{\xi} \in \partial\mathcal{S}_t$ located on the boundary $\partial\mathcal{S}_t$ to the solution domain \mathcal{S}_t reads

$$c(\boldsymbol{\xi})\varphi(\boldsymbol{\xi}) = \int_{\partial\mathcal{S}_t} [G_{\boldsymbol{\xi}}(\mathbf{x})\nabla_m\varphi(\mathbf{x}) - \varphi(\mathbf{x})\nabla_m G_{\boldsymbol{\xi}}(\mathbf{x})] da. \quad (56)$$

Here the term $c(\boldsymbol{\xi})$ depends on the geometry of the boundary at $\boldsymbol{\xi}$ and is defined as

$$c(\boldsymbol{\xi}) = 1 + \lim_{\epsilon \rightarrow 0} \int_{\partial\mathcal{S}_\epsilon \setminus [\partial\mathcal{S}_t \cap \partial\mathcal{S}_\epsilon]} \nabla_m G_{\boldsymbol{\xi}}(\mathbf{x}) da. \quad (57)$$

The need to compute $c(\boldsymbol{\xi})$ is cumbersome in our case since the boundary $\partial\mathcal{S}_t$ is constantly changing with the deformation of the material body. Moreover, the second integral on the right hand side of the boundary integral equation contains a strongly singular integrand that necessitates special attention for its evaluation. However, by considering an auxiliary problem with $\varphi(\mathbf{x}) = \varphi(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \partial\mathcal{S}_t$ we obtain the relation

$$c(\boldsymbol{\xi})\varphi(\boldsymbol{\xi}) = - \int_{\partial\mathcal{S}_t} \varphi(\boldsymbol{\xi})\nabla_m G_{\boldsymbol{\xi}}(\mathbf{x}) da. \quad (58)$$

Inserting this auxiliary result in the original boundary integral equation renders the so-called regularized version of the boundary integral equation that does not necessitate any longer to compute a factor that depends on the geometry. Moreover the singularity in the integrand of the integral on the right hand side of the *regularized boundary integral equation* is reduced by one order

$$\int_{\partial\mathcal{S}_t} G_{\boldsymbol{\xi}}(\mathbf{x})\nabla_m\varphi(\mathbf{x}) da = \int_{\partial\mathcal{S}_t} [\varphi(\mathbf{x}) - \varphi(\boldsymbol{\xi})]\nabla_m G_{\boldsymbol{\xi}}(\mathbf{x}) da. \quad (59)$$

The boundary to the free space is decomposed into the interface with the material body and the boundary at infinity $\partial\mathcal{S}_t = \partial\mathcal{B}_t \cup \partial\mathcal{S}_\infty$. The solution at infinity is assumed constant along $\partial\mathcal{S}_\infty$ and is denoted by φ_∞ , likewise its normal gradient at infinity is assumed to vanish

$$\lim_{\mathbf{x} \rightarrow \partial\mathcal{S}_\infty} \varphi(\mathbf{x}) = \varphi_\infty \quad \text{and} \quad \lim_{\mathbf{x} \rightarrow \partial\mathcal{S}_\infty} \nabla_m\varphi(\mathbf{x}) = 0. \quad (60)$$

Moreover it can be shown that the normal gradient of the fundamental solution at infinity when integrated over $\partial\mathcal{S}_\infty$ (a circle with $da = R d\phi$ and solid angle $\phi = 2\pi$ in two dimensions and a sphere with $da = R^2 d\phi$ and

solid angle $\phi = 4\pi$ in three dimensions, in either case with radius $R \rightarrow \infty$) renders

$$\int_{\partial\mathcal{S}_\infty} \nabla_m G_\xi(\mathbf{x}) \, da = -1. \quad (61)$$

As a consequence the boundary integral equation takes finally a format suited for our application, whereby we recall that the source point $\boldsymbol{\xi} \in \partial\mathcal{B}_t$ is located at the interface between free space and the material body

$$\int_{\partial\mathcal{B}_t} G_\xi(\mathbf{x}) \nabla_m \varphi(\mathbf{x}) \, da = \varphi(\boldsymbol{\xi}) - \varphi_\infty + \int_{\partial\mathcal{B}_t} [\varphi(\mathbf{x}) - \varphi(\boldsymbol{\xi})] \nabla_m G_\xi(\mathbf{x}) \, da. \quad (62)$$

Thus in summary, based on the fundamental solution, we may eventually obtain the boundary integral equation corresponding to the Laplace equation as

$$\varphi(\boldsymbol{\xi}) - \varphi_\infty + \int_{\partial\mathcal{B}_t} [\varphi(\mathbf{x}) - \varphi(\boldsymbol{\xi})] \nabla_m G_\xi(\mathbf{x}) \, da + \int_{\partial\mathcal{B}_t} G_\xi(\mathbf{x}) \frac{q_t(\mathbf{x})}{\epsilon_0} \, da = 0. \quad (63)$$

If the integrated free charge within the material body is equal to zero (as for a dielectricum) we have in addition the *free charge equation*

$$\int_{\mathcal{B}_t} \varrho_t^f \, dv + \int_{\partial\mathcal{B}_t} \widehat{\varrho}_t^f \, da = \int_{\partial\mathcal{B}_t} [\mathfrak{d} \cdot \mathbf{n} + \widehat{\varrho}_t^f] \, da = - \int_{\partial\mathcal{B}_t} q_t \, da = 0. \quad (64)$$

Note that in this formulation the set of unknowns consists of the electric potential $\varphi = \varphi(\mathbf{x})$ and the flux $q_t = q_t(\mathbf{x})$ at $\partial\mathcal{B}_t$ together with the electric potential φ_∞ at infinity.

4 Isotropic Hyperelastic Constitutive Modeling

In the preceding sections the total energy density W_0 was introduced as being parameterized in terms of the deformations gradient \mathbf{F} and the referential electric field \mathbb{E} , i.e. $W_0 = W_0(\mathbf{F}, \mathbb{E})$. The requirement of objectivity or likewise the requirement of observer frame indifference reduces the dependence on \mathbf{F} to a dependence on \mathbf{C} , i.e. $W_0 = W_0(\mathbf{C}, \mathbb{E})$. A further reduction may be obtained in the case of isotropic material behavior. For a material to behave isotropically the total energy density W_0 is required to be an isotropic (scalar valued) function of its tensor and vector valued arguments \mathbf{F} and \mathbb{E} . Due to the representation theorem relevant for this case it eventually turns out that W_0 may only depend on six invariants

$$W_0(\mathbf{F}, \mathbb{E}) \stackrel{obj}{=} W_0(\mathbf{C}, \mathbb{E}) \stackrel{iso}{=} W_0(I_1, I_2, I_3, I_4, I_5, I_6). \quad (65)$$

These six invariants resemble the case of a transversely isotropic material (with the anisotropy axis exchanged by the electric field)

$$I_1 = \mathbf{C} : \mathbf{I}, \quad I_2 = \text{cof} \mathbf{C} : \mathbf{I}, \quad I_3 = \det \mathbf{C}, \quad (66)$$

$$I_4 = [\mathbb{E} \otimes \mathbb{E}] : \mathbf{I}, \quad I_5 = [\mathbb{E} \otimes \mathbb{E}] : \mathbf{C}, \quad I_6 = [\mathbb{E} \otimes \mathbb{E}] : \mathbf{C}^2. \quad (67)$$

The total energy density W_0 includes the electric energy density E_0 . It is thus interesting to note that, based on the Cayley-Hamilton theorem $I_3 \mathbf{B} = \mathbf{C}^2 - I_1 \mathbf{C} + I_2 \mathbf{I}$ with \mathbf{I} the second-order referential unit tensor, e.g. the electric energy E_t per unit volume in \mathcal{B}_t may be expressed in terms of the above invariants

$$E_t = -\frac{\epsilon_0}{2} \mathbb{E} \cdot \mathbb{E} = \frac{\epsilon_0}{2} [I_1 I_5 - I_2 I_4 - I_6] / I_3. \quad (68)$$

The total Piola-Kirchhoff stress \mathbf{S}^{tot} and the referential dielectric displacement \mathbb{D} follow then from the chain rule

$$\mathbf{S}^{\text{tot}} = 2 \sum_{\kappa=1}^6 \frac{\partial W_0}{\partial I_\kappa} \frac{\partial I_\kappa}{\partial \mathbf{C}} \quad \text{and} \quad \mathbb{D} = - \sum_{\kappa=1}^6 \frac{\partial W_0}{\partial I_\kappa} \frac{\partial I_\kappa}{\partial \mathbb{E}}. \quad (69)$$

Clearly, the first derivatives of the invariants I_κ with respect to \mathbf{C} and \mathbb{E} can be computed once and for all without specifying the exact format of the total energy density W_0 . Thus we may collect the following intermediate results

$$\begin{aligned} \frac{\partial I_1}{\partial \mathbf{C}} &= \mathbf{I}, & \frac{\partial I_2}{\partial \mathbf{C}} &= I_2 \mathbf{B} + I_3 \mathbf{B}^2, & \frac{\partial I_3}{\partial \mathbf{C}} &= \text{cof} \mathbf{C}, \\ \frac{\partial I_4}{\partial \mathbf{C}} &= \mathbf{0}, & \frac{\partial I_5}{\partial \mathbf{C}} &= \mathbb{E} \otimes \mathbb{E}, & \frac{\partial I_6}{\partial \mathbf{C}} &= 2[\mathbf{C} \cdot \mathbb{E} \otimes \mathbb{E}]^{\text{sym}}, \\ \frac{\partial I_1}{\partial \mathbb{E}} &= \mathbf{0}, & \frac{\partial I_2}{\partial \mathbb{E}} &= \mathbf{0}, & \frac{\partial I_3}{\partial \mathbb{E}} &= \mathbf{0}, \\ \frac{\partial I_4}{\partial \mathbb{E}} &= 2 \mathbb{E}, & \frac{\partial I_5}{\partial \mathbb{E}} &= 2 \mathbf{C} \cdot \mathbb{E}, & \frac{\partial I_6}{\partial \mathbb{E}} &= 2 \mathbf{C}^2 \cdot \mathbb{E}. \end{aligned}$$

Next, the corresponding fourth-order referential elasticity tensor \mathbf{C} takes the representation

$$\mathbf{C} = 4 \sum_{\kappa=1}^6 \sum_{\lambda=1}^6 \frac{\partial^2 W_0}{\partial I_\kappa \partial I_\lambda} \frac{\partial I_\kappa}{\partial \mathbf{C}} \otimes \frac{\partial I_\lambda}{\partial \mathbf{C}} + 4 \sum_{\kappa=1}^6 \frac{\partial W_0}{\partial I_\kappa} \frac{\partial^2 I_\kappa}{\partial \mathbf{C} \otimes \partial \mathbf{C}}. \quad (70)$$

In analogy the corresponding second-order referential dielectricity tensor \mathbf{D} follows as

$$\mathbf{D} = - \sum_{\kappa=1}^6 \sum_{\lambda=1}^6 \frac{\partial^2 W_0}{\partial I_\kappa \partial I_\lambda} \frac{\partial I_\kappa}{\partial \mathbb{E}} \otimes \frac{\partial I_\lambda}{\partial \mathbb{E}} - \sum_{\kappa=1}^6 \frac{\partial W_0}{\partial I_\kappa} \frac{\partial^2 I_\kappa}{\partial \mathbb{E} \otimes \partial \mathbb{E}}. \quad (71)$$

Finally, the corresponding third-order referential piezoelectricity tensor $\mathbf{\Pi}$ is expressed as

$$\mathbf{\Pi} = -2 \sum_{\kappa=1}^6 \sum_{\lambda=1}^6 \frac{\partial^2 W_0}{\partial I_\kappa \partial I_\lambda} \frac{\partial I_\kappa}{\partial \mathbb{E}} \otimes \frac{\partial I_\lambda}{\partial \mathbf{C}} - 2 \sum_{\kappa=1}^6 \frac{\partial W_0}{\partial I_\kappa} \frac{\partial^2 I_\kappa}{\partial \mathbb{E} \otimes \partial \mathbf{C}}. \quad (72)$$

As before, also the second derivatives of the invariants I_κ with respect to \mathbf{C} and \mathbb{E} can be computed once and for all without specifying the exact format of the total energy density W_0 :

$$\begin{aligned} \frac{\partial^2 I_1}{\partial \mathbf{C} \otimes \partial \mathbf{C}} &= \mathbf{0}, & \frac{\partial^2 I_2}{\partial \mathbf{C} \otimes \partial \mathbf{C}} &= \mathbf{A}_2, & \frac{\partial^2 I_3}{\partial \mathbf{C} \otimes \partial \mathbf{C}} &= I_3[\mathbf{B} \otimes \mathbf{B} - \mathbf{I}_B^{\text{sym}}], \\ \frac{\partial^2 I_4}{\partial \mathbf{C} \otimes \partial \mathbf{C}} &= \mathbf{0}, & \frac{\partial^2 I_5}{\partial \mathbf{C} \otimes \partial \mathbf{C}} &= \mathbf{0}, & \frac{\partial^2 I_6}{\partial \mathbf{C} \otimes \partial \mathbf{C}} &= 2[\mathbf{I}^{\text{sym}} \cdot \mathbb{E} \otimes \mathbb{E}]^{\text{sym}}, \\ \frac{\partial^2 I_1}{\partial \mathbb{E} \otimes \partial \mathbb{E}} &= \mathbf{0}, & \frac{\partial^2 I_2}{\partial \mathbb{E} \otimes \partial \mathbb{E}} &= \mathbf{0}, & \frac{\partial^2 I_3}{\partial \mathbb{E} \otimes \partial \mathbb{E}} &= \mathbf{0}, \\ \frac{\partial^2 I_4}{\partial \mathbb{E} \otimes \partial \mathbb{E}} &= 2\mathbf{I}, & \frac{\partial^2 I_5}{\partial \mathbb{E} \otimes \partial \mathbb{E}} &= 2\mathbf{C}, & \frac{\partial^2 I_6}{\partial \mathbb{E} \otimes \partial \mathbb{E}} &= 2\mathbf{C}^2. \end{aligned}$$

Here $\mathbf{I}_B^{\text{sym}}$ denotes a referential symmetric fourth-order tensor with coefficients $2[\mathbf{I}_B^{\text{sym}}]_{IJKL} = B_{IK}B_{JL} + B_{IL}B_{JK}$ which we may also write in symbolic notation as $2\mathbf{I}_B^{\text{sym}} = \mathbf{B} \overline{\otimes} \mathbf{B} + \mathbf{B} \underline{\otimes} \mathbf{B}$.

To abbreviate the hessian of the invariant I_2 with respect to the Cauchy-Green strain \mathbf{C} we introduced the referential fourth-order tensor \mathbf{A}_2 ⁹. Moreover the mixed second derivatives of the invariants I_κ with respect to \mathbf{C} and \mathbb{E} take the format

$$\begin{aligned} \frac{\partial^2 I_1}{\partial \mathbf{C} \otimes \partial \mathbb{E}} &= \mathbf{0}, & \frac{\partial^2 I_2}{\partial \mathbf{C} \otimes \partial \mathbb{E}} &= \mathbf{0}, & \frac{\partial^2 I_3}{\partial \mathbf{C} \otimes \partial \mathbb{E}} &= \mathbf{0}, \\ \frac{\partial^2 I_4}{\partial \mathbf{C} \otimes \partial \mathbb{E}} &= \mathbf{0}, & \frac{\partial^2 I_5}{\partial \mathbf{C} \otimes \partial \mathbb{E}} &= \mathbb{E} \otimes \mathbf{I} + \mathbf{I} \overline{\otimes} \mathbb{E}, & \frac{\partial^2 I_6}{\partial \mathbf{C} \otimes \partial \mathbb{E}} &= \mathbf{A}_6^T, \\ \frac{\partial^2 I_1}{\partial \mathbb{E} \otimes \partial \mathbf{C}} &= \mathbf{0}, & \frac{\partial^2 I_2}{\partial \mathbb{E} \otimes \partial \mathbf{C}} &= \mathbf{0}, & \frac{\partial^2 I_3}{\partial \mathbb{E} \otimes \partial \mathbf{C}} &= \mathbf{0}, \\ \frac{\partial^2 I_4}{\partial \mathbb{E} \otimes \partial \mathbf{C}} &= \mathbf{0}, & \frac{\partial^2 I_5}{\partial \mathbb{E} \otimes \partial \mathbf{C}} &= 2\mathbb{E} \cdot \mathbf{I}^{\text{sym}}, & \frac{\partial^2 I_6}{\partial \mathbb{E} \otimes \partial \mathbf{C}} &= \mathbf{A}_6. \end{aligned}$$

Here \mathbf{I}^{sym} denotes the referential symmetric fourth-order unit tensor with coefficients $2[\mathbf{I}^{\text{sym}}]_{IJKL} = \delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}$ which we may also write in

⁹ $\mathbf{A}_2 = I_2[\mathbf{B} \otimes \mathbf{B} - \mathbf{I}_B^{\text{sym}}] + 2[\mathbf{B}^2 \otimes \text{cof} \mathbf{C}]^{\text{sym}} - I_3[\mathbf{B}^2 \overline{\otimes} \mathbf{B} + \mathbf{B}^2 \underline{\otimes} \mathbf{B} + \mathbf{B} \overline{\otimes} \mathbf{B}^2 + \mathbf{B} \underline{\otimes} \mathbf{B}^2]$

symbolic notation as $2\mathbf{I}^{\text{sym}} = \mathbf{I}\overline{\otimes}\mathbf{I} + \mathbf{I}\otimes\mathbf{I}$. Moreover we abbreviated the mixed second derivatives of the invariant I_6 by the referential third-order tensor $\mathbf{A}_6 = 2\mathbb{E} \cdot \mathbf{C} \cdot \mathbf{I}^{\text{sym}} + 2[\mathbf{C} \otimes \mathbb{E}] : \mathbf{I}^{\text{sym}}$.

Once these material independent derivatives are implemented correctly, all it needs to consider a new material model is the total energy density W_0 along with the collection of its first and second derivatives with respect to the six invariants:

$$\frac{\partial W_0}{\partial I_\kappa} \quad \text{and} \quad \frac{\partial^2 W_0}{\partial I_\kappa \partial I_\lambda}, \quad \text{with} \quad \kappa, \lambda = 1, \dots, 6. \quad (73)$$

5 Coupled FEM–BEM Discretization Method

5.1 Subdivision of Solution Domain into Patches

In our coupled FEM-BEM (hybrid) discretization method the solution domain $\tilde{\mathcal{B}}_0 = \mathcal{B}_0 \cup \partial\mathcal{B}_0$ (consisting of the material body and the interface to the free space) is approximated by a (polygonal) covering in terms of a subdivision into n_{el} domain and m_{el} external surface patches such that

$$\mathcal{B}_0 \approx \mathcal{B}_0^h = \bigcup_{e=1}^{n_{el}} \mathcal{B}_0^e \quad \text{and} \quad \partial\mathcal{B}_0 \approx \partial\mathcal{B}_0^h = \mathcal{A}_0^h = \bigcup_{e=1}^{m_{el}} \mathcal{A}_0^e. \quad (74)$$

Thereby, n_{el} domain patches (*finite elements*) cover the material body with configuration \mathcal{B}_0 , whereas m_{el} external surface patches (*boundary elements*) cover the interface to the free space with configuration $\partial\mathcal{B}_0$. After the subdivision of the (approximated) open solution domain \mathcal{B}_0^h into n_{el} domain patches we obtain also the set of n_{ed} internal surface patches¹⁰ in the discretization mesh

$$\mathcal{E}_0^h = \bigcup_{k=1}^{n_{ed}} \mathcal{E}_0^k. \quad (75)$$

For a representation of the discretized solution domain refer to Figure 3.

Mechanical problem. Based on the above terminology and definitions the weak form of the mechanical problem, here exemplified in terms of spatial description quantities, is obtained by weighting and integration of the appropriate strong equations on each:

¹⁰The expression *-eder* in polyeder (such as tetraeder, hexaeder and so on) derives from the greek $\acute{\epsilon}\delta\rho\alpha$ (surface).

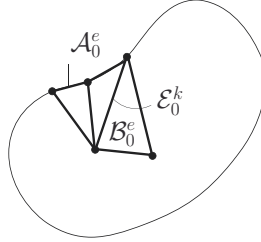


Figure 3. Covering of the discretized (closed) solution domain $\bar{\mathcal{B}}_0^h$ by domain patches \mathcal{B}_0^e and external surface patches \mathcal{A}_0^e . Internal surface patches \mathcal{E}_0^k are arranged between every two domain patches.

i) domain patch

$$\forall e : \int_{\mathcal{B}_t^e} \delta\varphi \cdot [\operatorname{div}\boldsymbol{\sigma}^{\text{tot}} + \mathbf{b}_t] \, dv = 0 \quad \forall \delta\varphi. \quad (76)$$

On each domain patch the mechanical principle of virtual work reads

$$\int_{\mathcal{B}_t^e} \boldsymbol{\epsilon}^\delta : \boldsymbol{\sigma}^{\text{tot}} \, dv = \int_{\partial\mathcal{B}_t^e} \delta\varphi \cdot \boldsymbol{\sigma}^{\text{tot}} \cdot \mathbf{n} \, da + \int_{\mathcal{B}_t^e} \delta\varphi \cdot \mathbf{b}_t \, dv \quad \forall \delta\varphi. \quad (77)$$

Here we abbreviated $\nabla_x^{\text{sym}}\delta\varphi$ as the virtual ‘rate of deformation tensor’ $\boldsymbol{\epsilon}^\delta = \nabla_x^{\text{sym}}\delta\varphi$.

Upon summation of all domain patch contributions we obtain, due to the additivity of integration, for the internal virtual work

$$\sum_{e=1}^{n_{el}} \int_{\mathcal{B}_t^e} \boldsymbol{\epsilon}^\delta : \boldsymbol{\sigma}^{\text{tot}} \, dv = \int_{\mathcal{B}_t^h} \boldsymbol{\epsilon}^\delta : \boldsymbol{\sigma}^{\text{tot}} \, dv. \quad (78)$$

Summation of the contributions of the domain patch boundaries $\partial\mathcal{B}_t^e$ to the external virtual work by counting each internal surface patch twice, with either positive or negative sign for the normal \mathbf{n} , renders

$$\sum_{e=1}^{n_{el}} \int_{\partial\mathcal{B}_t^e} \delta\varphi \cdot \boldsymbol{\sigma}^{\text{tot}} \cdot \mathbf{n} \, da = - \int_{\mathcal{E}_t^h} \delta\varphi \cdot \llbracket \boldsymbol{\sigma}^{\text{tot}} \rrbracket \cdot \mathbf{n} \, da + \int_{\mathcal{A}_t^h} \delta\varphi \cdot \boldsymbol{\sigma}^{\text{tot}} \cdot \mathbf{n} \, da. \quad (79)$$

ii) internal surface patch

$$\forall k : \int_{\mathcal{E}_t^k} \delta\varphi \cdot \llbracket \boldsymbol{\sigma}^{\text{tot}} \rrbracket \cdot \mathbf{n} \, da = 0 \quad \forall \delta\varphi. \quad (80)$$

Upon summation of all internal surface patch contributions we obtain

$$\sum_{k=1}^{n_{ed}} \int_{\mathcal{E}_t^k} \delta\varphi \cdot \llbracket \boldsymbol{\sigma}^{\text{tot}} \rrbracket \cdot \mathbf{n} \, da = \int_{\mathcal{E}_t^h} \delta\varphi \cdot \llbracket \boldsymbol{\sigma}^{\text{tot}} \rrbracket \cdot \mathbf{n} \, da = 0. \quad (81)$$

iii) external surface patch

$$\forall e : \int_{\mathcal{A}_t^e} \delta\varphi \cdot \llbracket \llbracket \boldsymbol{\sigma}^{\text{tot}} \rrbracket \cdot \mathbf{n} + \mathbf{t}_t \rrbracket \, da = 0 \quad \forall \delta\varphi. \quad (82)$$

Upon summation of all external surface patch contributions we obtain

$$\sum_{k=1}^{m_{el}} \int_{\mathcal{A}_t^e} \delta\varphi \cdot \llbracket \llbracket \boldsymbol{\sigma}^{\text{tot}} \rrbracket \cdot \mathbf{n} + \mathbf{t}_t \rrbracket \, da = \int_{\mathcal{A}_t^h} \delta\varphi \cdot \llbracket \llbracket \boldsymbol{\sigma}^{\text{tot}} \rrbracket \cdot \mathbf{n} + \mathbf{t}_t \rrbracket \, da = 0. \quad (83)$$

Due to the previous results the expression in **i)** for the external virtual work is reduced further to

$$\int_{\mathcal{A}_t^h} \delta\varphi \cdot \boldsymbol{\sigma}^{\text{tot}} \cdot \mathbf{n} \, da = \int_{\mathcal{A}_t^h} \delta\varphi \cdot [\mathbf{t}_t + \mathbf{t}_t^{\text{max}}] \, da. \quad (84)$$

As a conclusion, the weak form of the mechanical problem may finally be expressed by considering integrals over \mathcal{B}_t^h and \mathcal{A}_t^h

$$\int_{\mathcal{B}_t^h} \boldsymbol{\epsilon}^\delta : \boldsymbol{\sigma}^{\text{tot}} \, dv = \int_{\mathcal{A}_t^h} \delta\varphi \cdot [\mathbf{t}_t + \mathbf{t}_t^{\text{max}}] \, da + \int_{\mathcal{B}_t^e} \delta\varphi \cdot \mathbf{b}_t \, dv \quad \forall \delta\varphi. \quad (85)$$

Electrical problem (in matter). The weak form of the electrical problem, again exemplified in terms of spatial description quantities, is obtained by weighting and integration of the appropriate strong equations on each:

i) domain patch

$$\forall e : \int_{\mathcal{B}_t^e} \delta\varphi [\text{div} \, \mathbf{d} - \varrho_t^f] \, dv = 0 \quad \forall \delta\varphi. \quad (86)$$

On each domain patch the electrical principle of virtual work reads

$$\int_{\mathcal{B}_t^e} \mathbf{e}^\delta \cdot \mathbf{d} \, dv = - \int_{\partial\mathcal{B}_t^e} \delta\varphi \, \mathbf{d} \cdot \mathbf{n} \, da + \int_{\mathcal{B}_t^e} \delta\varphi \, \varrho_t^f \, dv \quad \forall \delta\varphi. \quad (87)$$

Here we abbreviated the push-forward of the variation of the referential electric field $\delta\mathbb{E}$ as $\mathbf{e}^\delta = -\nabla_x \delta\varphi$. Upon summation of all domain patch

contributions we obtain for the internal virtual work

$$\sum_{e=1}^{n_{el}} \int_{\mathcal{B}_t^e} \mathbb{E}^\delta \cdot \mathbb{d} \, dv = \int_{\mathcal{B}_t^h} \mathbb{E}^\delta \cdot \mathbb{d} \, dv. \quad (88)$$

Summation of the contributions of the domain patch boundaries $\partial\mathcal{B}_t^e$ to the external virtual work renders

$$-\sum_{e=1}^{n_{el}} \int_{\partial\mathcal{B}_t^e} \delta\varphi \, \mathbb{d} \cdot \mathbf{n} \, da = \int_{\mathcal{E}_t^h} \delta\varphi [[\mathbb{d}]] \cdot \mathbf{n} \, da - \int_{\mathcal{A}_t^h} \delta\varphi \, \mathbb{d} \cdot \mathbf{n} \, da. \quad (89)$$

ii) internal surface patch

$$\forall k : \int_{\mathcal{E}_t^k} \delta\varphi [[\mathbb{d}]] \cdot \mathbf{n} \, da = 0 \quad \forall \delta\varphi. \quad (90)$$

Upon summation of all internal surface patch contributions we obtain

$$\sum_{k=1}^{n_{ed}} \int_{\mathcal{E}_t^k} \delta\varphi [[\mathbb{d}]] \cdot \mathbf{n} \, da = \int_{\mathcal{E}_t^h} \delta\varphi [[\mathbb{d}]] \cdot \mathbf{n} \, da = 0. \quad (91)$$

iii) external surface patch

$$\forall e : \int_{\mathcal{A}_t^e} \delta\varphi [[\mathbb{d}]] \cdot \mathbf{n} - \widehat{\varrho}_t^f] \, da = 0 \quad \forall \delta\varphi. \quad (92)$$

Upon summation of all external surface patch contributions we obtain

$$\sum_{e=1}^{m_{el}} \int_{\mathcal{A}_t^e} \delta\varphi [[\mathbb{d}]] \cdot \mathbf{n} - \widehat{\varrho}_t^f] \, da = \int_{\mathcal{A}_t^h} \delta\varphi [[\mathbb{d}]] \cdot \mathbf{n} - \widehat{\varrho}_t^f] \, da = 0. \quad (93)$$

Due to the previous results the expression in **i)** for the external virtual work is reduced further to

$$-\int_{\mathcal{A}_t^h} \delta\varphi \, \mathbb{d} \cdot \mathbf{n} \, da = \int_{\mathcal{A}_t^h} \delta\varphi [\widehat{\varrho}_t^f + q_t] \, da. \quad (94)$$

As a conclusion, the weak form of the electrical problem (in matter) may finally be expressed by considering integrals over \mathcal{B}_t^h and \mathcal{A}_t^h :

$$\int_{\mathcal{B}_t^h} \mathbb{E}^\delta \cdot \mathbb{d} \, dv = \int_{\mathcal{A}_t^h} \delta\varphi [\widehat{\varrho}_t^f + q_t] \, da + \int_{\mathcal{B}_t^h} \delta\varphi \varrho_t^f \, dv \quad \forall \delta\varphi. \quad (95)$$

Electrical problem (in free space). The subdivision of the (approximated) solution domain \mathcal{A}_t^h into m_{el} external surface patches \mathcal{A}_t^e results simply in a summation over patch-wise integrals. Thus we obtain for the boundary integrals corresponding to the Laplace equation

$$\begin{aligned} \int_{\mathcal{A}_t^h} \varphi(\mathbf{x}) \nabla_m G_\xi(\mathbf{x}) \, da &= \sum_{e=1}^{m_{el}} \int_{\mathcal{A}_t^e} \varphi(\mathbf{x}) \nabla_m G_\xi(\mathbf{x}) \, da, \\ \int_{\mathcal{A}_t^h} \nabla_m G_\xi(\mathbf{x}) \, da &= \sum_{e=1}^{m_{el}} \int_{\mathcal{A}_t^e} \nabla_m G_\xi(\mathbf{x}) \, da, \\ \int_{\mathcal{A}_t^h} G_\xi(\mathbf{x}) \frac{q_t(\mathbf{x})}{\epsilon_0} \, da &= \sum_{e=1}^{m_{el}} \int_{\mathcal{A}_t^e} G_\xi(\mathbf{x}) \frac{q_t(\mathbf{x})}{\epsilon_0} \, da. \end{aligned} \quad (96)$$

Likewise we obtain for the boundary integrals corresponding to the free charge equation

$$\int_{\mathcal{A}_t^h} q_t \, da = \sum_{e=1}^{m_{el}} \int_{\mathcal{A}_t^e} q_t \, da. \quad (97)$$

Residual error measure for the weak form. Applying a discretization method is equivalent to committing errors. Thus it is important to measure the error in one way or the other. Based on the above discussion on the appropriate strong equations posed within each domain patch (finite element) and on its boundary, a patch-wise residual for the mechanical weak form may be expressed by

$$[\eta_e^{\text{mec}}]^2 = h_B^2 \int_{\mathcal{B}_t^e} |\text{div} \boldsymbol{\sigma}^{\text{tot}} + \mathbf{b}_t|^2 \, dv + h_\mathcal{E} \int_{\partial \mathcal{B}_t^e} |[\boldsymbol{\sigma}^{\text{tot}}] \cdot \mathbf{n} + \mathbf{t}_t|^2 \, da, \quad (98)$$

with $\mathbf{t}_t = \mathbf{0}$ if $\partial \mathcal{B}_t^e \in \mathcal{E}_t^h$ and $[\boldsymbol{\sigma}^{\text{tot}}] = \boldsymbol{\sigma}^{\text{max}} - \boldsymbol{\sigma}^{\text{tot}}$ if $\partial \mathcal{B}_t^e \in \mathcal{A}_t^h$. Here h_B and $h_\mathcal{E}$ are appropriate scaling factors of dimension length that measure, e.g., the radius of the largest ball inscribable into the element and its edge length, respectively.

Likewise a patch-wise residual for the electrical weak form may be measured by

$$[\eta_e^{\text{ele}}]^2 = h_B^2 \int_{\mathcal{B}_t^e} |\text{div} \mathbf{d} - \varrho_t^f|^2 \, dv + h_\mathcal{E} \int_{\partial \mathcal{B}_t^e} |[\mathbf{d}] \cdot \mathbf{n} - \widehat{\varrho}_t^f|^2 \, da, \quad (99)$$

with $\widehat{\varrho}_t^f = 0$ if $\partial \mathcal{B}_t^e \in \mathcal{E}_t^h$ and $[\mathbf{d}] = \mathbf{d}^\epsilon - \mathbf{d}$ if $\partial \mathcal{B}_t^e \in \mathcal{A}_t^h$. The error measures η_e^{mec} and η_e^{ele} may eventually be combined into a single total error measure

η by the help of appropriate weighting factors w^{mec} and w^{ele} , respectively

$$\eta^2 = \sum_{e=1}^{n_{el}} w^{\text{mec}} [\eta_e^{\text{mec}}]^2 + w^{\text{ele}} [\eta_e^{\text{ele}}]^2. \quad (100)$$

5.2 Finite Element Shape Functions

On each finite element $\mathcal{B}_0^e \in \mathcal{B}_0^h$ the referential coordinates \mathbf{X} are approximated in terms of shape functions $N_i^u = N_i^u(\boldsymbol{\eta})$ defined on isoparametric coordinates $\boldsymbol{\eta} \in [-1, +1]^{n_{dm}}$ and n_{en}^u nodal values \mathbf{X}_i

$$\mathbf{X}^h(\boldsymbol{\eta})|_{\mathcal{B}_0^e} = \sum_{i=1}^{n_{en}^u} N_i^u(\boldsymbol{\eta}) \mathbf{X}_i. \quad (101)$$

The elementwise jacobian \mathbf{J}_e of the map $\mathbf{X}^h = \mathbf{X}^h(\boldsymbol{\eta})$ will be needed subsequently in order to calculate gradients with respect to the material coordinates \mathbf{X} based on the chain rule

$$\nabla_{\mathbf{X}}(\bullet) = \nabla_{\boldsymbol{\eta}}(\bullet) \cdot \mathbf{J}_e^{-1} \quad \text{with} \quad \mathbf{J}_e(\boldsymbol{\eta}) = \nabla_{\boldsymbol{\eta}} \mathbf{X}^h(\boldsymbol{\eta}) = \sum_{i=1}^{n_{en}^u} \mathbf{X}_i \otimes \nabla_{\boldsymbol{\eta}} N_i^u. \quad (102)$$

In view of \mathbf{J}_e being a linear map from the tangent to the isoparametric domain, say \square , to the tangent to \mathcal{B}_0^e , the corresponding volume elements transform as $dV = \det \mathbf{J}_e d\square$. It is sometimes convenient to assemble the nodal values \mathbf{X}_i into an element (column) vector of nodal values \mathbf{X}_e so that

$$\mathbf{X}_e = [[\mathbf{X}_1]^t, \dots, [\mathbf{X}_i]^t, \dots, [\mathbf{X}_{n_{en}^u}]^t]^t, \quad \mathbf{X}_i = [X_1, \dots, X_{n_{dm}}]^t. \quad (103)$$

Accordingly, the shape functions may be arranged into a matrix with

$$\mathbf{N}_e^u = [N_1^u, \dots, N_i^u, \dots, N_{n_{en}^u}^u], \quad \mathbf{N}_i^u = [N_i^u]_{n_{dm} \times n_{dm}}. \quad (104)$$

Thus the approximation of the referential coordinates \mathbf{X} takes the alternative, more abbreviated matrix-vector representation

$$\mathbf{X}^h(\boldsymbol{\eta})|_{\mathcal{B}_0^e} = \mathbf{N}_e^u(\boldsymbol{\eta}) \mathbf{X}_e. \quad (105)$$

Next, in the spirit of the isoparametric concept, the trial functions for the deformation map $\boldsymbol{\varphi}$ are approximated by the same shape functions as the referential coordinates \mathbf{X} , whereby the nodal values $\mathbf{d}_i^u = \boldsymbol{\varphi}^h(\mathbf{X}_i) = [X_1 + u_1, \dots, X_{n_{dm}} + u_{n_{dm}}]^t$ now denote the nodal degrees of freedom for the deformation map in terms of the displacement $\mathbf{u} = \boldsymbol{\varphi} - \mathbf{X}$

$$\boldsymbol{\varphi}^h(\boldsymbol{\eta})|_{\mathcal{B}_0^e} = \sum_{i=1}^{n_{en}^u} N_i^u(\boldsymbol{\eta}) \mathbf{d}_i^u = \mathbf{N}_e^u(\boldsymbol{\eta}) \mathbf{d}_e^u. \quad (106)$$

The arrangement of the \mathbf{d}_i^u into an element (column) vector of nodal degrees of freedom for the deformation map $\mathbf{d}_e^u = [[\mathbf{d}_1^u]^t, \dots, [\mathbf{d}_i^u]^t, \dots, [\mathbf{d}_{n_{en}^u}^u]^t]^t$ follows the same pattern as that for the referential coordinates.

In general the number of element nodes n_{en}^φ for the electric potential does not have to agree with the number of element nodes n_{en}^u for the deformation map. Thus the trial functions for the electric potential φ are approximated in terms of nodal values $d_i^\varphi = \varphi^h(\mathbf{X}_i)$ and shape functions $N_i^\varphi = N_i^\varphi(\boldsymbol{\eta})$

$$\varphi^h(\boldsymbol{\eta})|_{\mathcal{B}_e^\varphi} = \sum_{i=1}^{n_{en}^\varphi} N_i^\varphi(\boldsymbol{\eta}) d_i^\varphi = \mathbf{N}_e^\varphi(\boldsymbol{\eta}) \mathbf{d}_e^\varphi. \quad (107)$$

The element (column) vector of nodal degrees of freedom for the electric potential and the corresponding arrangement of the shape functions into a (row) vector follows as

$$\mathbf{d}_e^\varphi = [d_1^\varphi, \dots, d_i^\varphi, \dots, d_{n_{en}^\varphi}^\varphi]^t, \quad \mathbf{N}_e^\varphi = [N_1^\varphi, \dots, N_i^\varphi, \dots, N_{n_{en}^\varphi}^\varphi]. \quad (108)$$

Based on the above approximations for the solution fields, the deformation gradient \mathbf{F} and the referential electric field \mathbb{E} , expressed as gradient operators, follow eventually as

$$\mathbf{F}(\varphi^h)|_{\mathcal{B}_e^\varphi} = \sum_{i=1}^{n_{en}^u} \mathbf{d}_i^u \otimes \nabla_X N_i^u \quad \text{and} \quad \mathbb{E}(\varphi^h)|_{\mathcal{B}_e^\varphi} = - \sum_{i=1}^{n_{en}^\varphi} d_i^\varphi \nabla_X N_i^\varphi. \quad (109)$$

Finally, it goes without saying that the test functions $\delta\varphi$ and $\delta\varphi$ are approximated by the same shape functions as the trial functions in the spirit of the Bubnov-Galerkin method. Thus we obtain for the spatial gradients of the discretized test functions

$$\nabla_x \delta\varphi^h|_{\mathcal{B}_e^\varphi} = \sum_{i=1}^{n_{en}^u} \delta \mathbf{d}_i^u \otimes \nabla_x N_i^u \quad \text{and} \quad \nabla_x \delta\varphi^h|_{\mathcal{B}_e^\varphi} = \sum_{i=1}^{n_{en}^\varphi} \delta d_i^\varphi \nabla_x N_i^\varphi. \quad (110)$$

Next we arrange the six independent entries of the virtual ‘rate of deformation tensor’ $\boldsymbol{\epsilon}^\delta = \nabla_x^{\text{sym}} \delta\varphi$ by the help of the Voigt notation into a vector

$$\boldsymbol{\epsilon}_v^\delta = [\delta u_{1,1}, \delta u_{2,2}, \delta u_{3,3}, \underbrace{[\delta u_{1,2} + \delta u_{2,1}]}_{2\epsilon_{12}^\delta}, \underbrace{[\delta u_{2,3} + \delta u_{3,2}]}_{2\epsilon_{23}^\delta}, \underbrace{[\delta u_{3,1} + \delta u_{1,3}]}_{2\epsilon_{31}^\delta}]^t.$$

Here and in the following, a comma denotes the partial derivative with respect to the spatial coordinates if not stated otherwise. We may then

express the discretized virtual ‘rate of deformation tensor’ in Voigt notation

$$\epsilon_v^\delta(\delta\varphi^h)|_{\mathcal{B}_t^e} = \sum_{i=1}^{n_{en}^u} \mathbf{B}_i^u \delta \mathbf{d}_i^u = \mathbf{B}_e^u \delta \mathbf{d}_e^u. \quad (111)$$

Here we introduced the discrete gradient operator matrix for $n_{dm} = 3$ as

$$\mathbf{B}_e^u = [\mathbf{B}_1^u, \dots, \mathbf{B}_i^u, \dots, \mathbf{B}_{n_{en}^u}^u], \quad \mathbf{B}_i^u = \begin{bmatrix} N_{i,1}^u & 0 & 0 \\ 0 & N_{i,2}^u & 0 \\ 0 & 0 & N_{i,3}^u \\ N_{i,2}^u & N_{i,1}^u & 0 \\ 0 & N_{i,3}^u & N_{i,2}^u \\ N_{i,3}^u & 0 & N_{i,1}^u \end{bmatrix}. \quad (112)$$

The three entries of the push-forward of the variation of the referential electric field $\mathfrak{e}^\delta = -\nabla_x \delta\varphi$ are arranged into vector notation

$$\mathfrak{e}^\delta = -[\delta\varphi_{,1}, \delta\varphi_{,2}, \delta\varphi_{,3}]^t. \quad (113)$$

Thus the discretized version of \mathfrak{e}^δ follows as

$$\mathfrak{e}^\delta(\delta\varphi^h)|_{\mathcal{B}_t^e} = -\sum_{i=1}^{n_{en}^\varphi} \mathbf{B}_i^\varphi \delta d_i^\varphi = -\mathbf{B}_e^\varphi \delta \mathbf{d}_e^\varphi. \quad (114)$$

Again we introduced a discrete gradient operator matrix as

$$\mathbf{B}_e^\varphi = [\mathbf{B}_1^\varphi, \dots, \mathbf{B}_i^\varphi, \dots, \mathbf{B}_{n_{en}^\varphi}^\varphi], \quad \mathbf{B}_i^\varphi = \begin{bmatrix} N_{i,1}^\varphi \\ N_{i,2}^\varphi \\ N_{i,3}^\varphi \end{bmatrix}. \quad (115)$$

5.3 Boundary Element Shape Functions

On each boundary element $\mathcal{A}_t^e \in \mathcal{A}_t^h$ the spatial coordinates \mathbf{x} are approximated in terms of shape functions $M_i^u = M_i^u(\hat{\boldsymbol{\eta}})$ defined on isoparametric coordinates $\hat{\boldsymbol{\eta}} \in [-1, 1]^{n_{dm}-1}$ and m_{en}^u nodal values $\hat{\mathbf{x}}_i$ situated at the boundary $\partial\mathcal{B}_t^h$ of the material body

$$\mathbf{x}^h(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_t^e} = \sum_{i=1}^{m_{en}^u} M_i^u(\hat{\boldsymbol{\eta}}) \hat{\mathbf{x}}_i = \mathbf{M}_e^u(\hat{\boldsymbol{\eta}}) \hat{\mathbf{x}}_e. \quad (116)$$

Here and in the following the matrix-vector representation follows the conventions discussed earlier.

Clearly, in order for the approximation of the geometry by the finite element shape functions $N_i^u = N_i^u(\boldsymbol{\eta})$ discussed previously and the boundary element shape functions $M_i^u = M_i^u(\hat{\boldsymbol{\eta}})$ to be compatible, the boundary element shape functions have to be the trace of the finite element shape functions $M_i^u = N_i^u|_{\mathcal{A}_t^h}$ at \mathcal{A}_t^h . The tangent vectors to the coordinate lines $\hat{\boldsymbol{\eta}}^\alpha$ (with $\alpha = 1, n_{dm} - 1$) or rather the covariant (natural) base vectors to the boundary surface of the material body follow from

$$\mathbf{a}_\alpha(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_t^e} = \sum_{i=1}^{m_{en}^u} \partial_{\hat{\eta}^\alpha} M_i^u(\hat{\boldsymbol{\eta}}) \hat{\mathbf{x}}_i = \mathbf{L}_{e\alpha}^u(\hat{\boldsymbol{\eta}}) \hat{\mathbf{x}}_e. \quad (117)$$

Here $\mathbf{L}_{e\alpha}^u$ is an elementwise matrix of dimensions $n_{dm} \times n_{dm} m_{en}^u$ that essentially incorporates the isoparametric derivatives of the shape functions M_i^u . The surface normal \mathbf{m} (pointing from free space to matter) and the area element da of the boundary to the three dimensional¹¹ material body are computed from

$$\mathbf{m}(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_t^e} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \quad \text{and} \quad da(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_t^e} = |\mathbf{a}_1 \times \mathbf{a}_2| d\hat{\square}. \quad (118)$$

The corresponding area element in the isoparametric domain is denoted by $d\hat{\square} = d\hat{\eta}^1 d\hat{\eta}^2$. In order that \mathbf{m} is the outward pointing normal to the free space the sign of tangent vectors \mathbf{a}_α has to be chosen so that they span a plane with the material body on its ‘right side’. Next, the trial functions for the electric potential φ are approximated in terms of nodal values $\hat{d}_i^\varphi = \varphi^h(\hat{\mathbf{x}}_i)$ and shape functions $M_i^\varphi = M_i^\varphi(\hat{\boldsymbol{\eta}})$

$$\varphi^h(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_t^e} = \sum_{i=1}^{m_{en}^\varphi} M_i^\varphi(\hat{\boldsymbol{\eta}}) \hat{d}_i^\varphi = \mathbf{M}_e^\varphi(\hat{\boldsymbol{\eta}}) \hat{\mathbf{d}}_e^\varphi. \quad (119)$$

The number of boundary element nodes m_{en}^φ for the electric potential does not have to agree with the number of boundary element nodes m_{en}^u for the geometry description. However, in order for the approximation of the electric potential by the finite element shape functions $N_i^\varphi = N_i^\varphi(\boldsymbol{\eta})$ discussed previously and the boundary element shape functions $M_i^\varphi = M_i^\varphi(\hat{\boldsymbol{\eta}})$ to be compatible, the boundary element shape functions have to be the trace of the finite element shape functions $M_i^\varphi = N_i^\varphi|_{\mathcal{A}_t^h}$ at \mathcal{A}_t^h . Finally,

¹¹ In two dimensions $\mathbf{m} = \mathbf{a} \times \mathbf{e}_3 / |\mathbf{a}|$ (with \mathbf{e}_3 perpendicular to the plane) and $da = |\mathbf{a}| d\hat{\square}$ holds with $d\hat{\square} = d\hat{\eta}$ instead, whereby we omitted the index α . Thereby the sign convention is that \mathbf{a} has a direction with the material body on its ‘right side’ so that \mathbf{m} is the outward pointing normal to the free space.

the trial functions for the flux q_t are approximated in terms of nodal values $\widehat{d}_i^q = q^h(\widehat{\mathbf{x}}_i)$ and shape functions $M_i^q = M_i^q(\widehat{\boldsymbol{\eta}})$

$$q_t^h(\widehat{\boldsymbol{\eta}})|_{\mathcal{A}_\varepsilon} = \sum_{i=1}^{m_{en}^q} M_i^q(\widehat{\boldsymbol{\eta}}) \widehat{d}_i^q = \mathbf{M}_e^q(\widehat{\boldsymbol{\eta}}) \widehat{\mathbf{d}}_e^q. \quad (120)$$

Typically the total number of unknowns for the flux variable m_{np}^q has to correspond to the number of collocation points m_{cp} , see the discussion below.

5.4 Discretized Weak Form for the Material Body

Mechanical problem. After inserting the shape functions, the discretized version of the weak form of the mechanical problem that has to hold for all $\delta \mathbf{d}_i^u$ reads for each finite element as

$$\sum_{i=1}^{n_{en}^u} \delta \mathbf{d}_i^u \cdot \left[\int_{\mathcal{B}_\varepsilon^e} [\boldsymbol{\sigma}^{\text{tot}} \cdot \nabla_x N_i^u - \mathbf{b}_t N_i^u] dv - \int_{\partial \mathcal{B}_\varepsilon^e} \boldsymbol{\sigma}^{\text{tot}} \cdot \mathbf{n} N_i^u da \right] = 0. \quad (121)$$

Next, by the help of the Voigt notation, we arrange the total stress tensor into a vector

$$\boldsymbol{\sigma}^{\text{tot}} \rightarrow \boldsymbol{\sigma}_v^{\text{tot}} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^t. \quad (122)$$

Thus the discretized version of the weak form of the mechanical problem that has to hold for all $\delta \mathbf{d}_e^u$ reads in matrix notation

$$[\delta \mathbf{d}_e^u]^t \left[\int_{\mathcal{B}_\varepsilon^e} [[\mathbf{B}_e^u]^t \boldsymbol{\sigma}_v^{\text{tot}} - [\mathbf{N}_e^u]^t \mathbf{b}_t] dv - \int_{\partial \mathcal{B}_\varepsilon^e} [\mathbf{N}_e^u]^t \boldsymbol{\sigma}^{\text{tot}} \cdot \mathbf{n} da \right] = 0. \quad (123)$$

With the definition of the elementwise internal mechanical load vector

$$\mathbf{s}_e^u = \int_{\mathcal{B}_\varepsilon^e} [\mathbf{B}_e^u]^t \boldsymbol{\sigma}_v^{\text{tot}} dv \quad (124)$$

and the definition of the elementwise external mechanical load vector

$$\mathbf{f}_e^u = \int_{\partial \mathcal{B}_\varepsilon^e} [\mathbf{N}_e^u]^t \boldsymbol{\sigma}^{\text{tot}} \cdot \mathbf{n} da + \int_{\mathcal{B}_\varepsilon^e} [\mathbf{N}_e^u]^t \mathbf{b}_t dv \quad (125)$$

and due to the arbitrariness of $\delta \mathbf{d}_e^u$ we may eventually define the elementwise mechanical residual as

$$\mathbf{r}_e^u = \mathbf{f}_e^u - \mathbf{s}_e^u \doteq \mathbf{0}. \quad (126)$$

Next the $n_{en}^u n_{dm}$ local (elementwise) degrees of freedom are assigned to the $n_{np} n_{dm}$ global degrees of freedom (with n_{np} denoting the global number of node points) by the help of Boolean matrices that contain the local versus global numbering of degrees of freedom

$$\mathbf{d}_e^u = \mathbf{a}_e^u \mathbf{d}^u \quad \text{with} \quad \dim \mathbf{a}_e^u = n_{en}^u n_{dm} \times n_{np} n_{dm}. \quad (127)$$

Thus the discretized version of the weak form of the mechanical problem abbreviates as

$$\sum_{e=1}^{n_{el}} [\delta \mathbf{d}_e^u]^t \mathbf{r}_e^u = [\delta \mathbf{d}^u]^t \sum_{e=1}^{n_{el}} [\mathbf{a}_e^u]^t \mathbf{r}_e^u = [\delta \mathbf{d}^u]^t \mathbf{r}^u = 0 \quad \forall \delta \mathbf{d}^u. \quad (128)$$

Here the assembly into the global mechanical residual reads as

$$\mathbf{r}^u = \sum_{e=1}^{n_{el}} [\mathbf{a}_e^u]^t \mathbf{r}_e^u = \mathbf{A} \mathbf{r}_e^u \quad (129)$$

whereby we incorporated the following side condition for the tractions

$$\mathbf{A} \int_{\partial \mathcal{B}_e^e} [\mathbf{N}_e^u]^t \boldsymbol{\sigma}^{\text{tot}} \cdot \mathbf{n} \, da = \mathbf{A} \int_{\mathcal{A}_t^e} [\mathbf{M}_e^u]^t [\mathbf{t}_t + \mathbf{t}_t^{\text{max}}] \, da. \quad (130)$$

Thus the boundary contribution to the global external mechanical load of the material body includes in particular the Maxwell traction

$$\mathbf{A} \int_{\mathcal{A}_t^e} [\mathbf{M}_e^u]^t [\mathbf{t}_t + \mathbf{t}_t^{\text{max}}] \, da. \quad (131)$$

Electrical problem (in matter). Again by the shape functions, the discretized version of the weak form of the electrical problem that has to hold for all δd_i^φ reads for each finite element as

$$\sum_{i=1}^{n_{en}^\varphi} \delta d_i^\varphi \left[\int_{\mathcal{B}_i^e} [-\mathbf{d} \cdot \nabla_x N_i^\varphi - \varrho_i^f N_i^\varphi] \, dv + \int_{\partial \mathcal{B}_i^e} \mathbf{d} \cdot \mathbf{n} N_i^\varphi \, da \right] = 0. \quad (132)$$

Likewise, in matrix notation the discretized version of the weak form of the electrical problem that has to hold for all $\delta \mathbf{d}_e^\varphi$ reads

$$[\delta \mathbf{d}_e^\varphi]^t \left[\int_{\mathcal{B}_i^e} [-[\mathbf{B}_e^\varphi]^t \mathbf{d} - [\mathbf{N}_e^\varphi]^t \varrho_i^f] \, dv + \int_{\partial \mathcal{B}_i^e} [\mathbf{N}_e^\varphi]^t \mathbf{d} \cdot \mathbf{n} \, da \right] = 0. \quad (133)$$

Next we define the elementwise internal electrical load vector

$$\mathbf{s}_e^\varphi = \int_{\mathcal{B}_t^e} [\mathbf{B}_e^\varphi]^t \mathbf{d} \, dv \quad (134)$$

together with the elementwise external electrical load vector

$$\mathbf{f}_e^\varphi = \int_{\partial\mathcal{B}_t^e} [\mathbf{N}_e^\varphi]^t \mathbf{d} \cdot \mathbf{n} \, da - \int_{\mathcal{B}_t^e} [\mathbf{N}_e^\varphi]^t \varrho_t^f \, dv \quad (135)$$

in order to formulate the elementwise electrical residual based on the arbitrariness of $\delta \mathbf{d}_e^\varphi$

$$\mathbf{r}_e^\varphi = \mathbf{f}_e^\varphi - \mathbf{s}_e^\varphi \doteq \mathbf{0}. \quad (136)$$

Again the n_{en}^φ local (elementwise) degrees of freedom are assigned to the n_{np} global degrees of freedom by the help of Boolean matrices

$$\mathbf{d}_e^\varphi = \mathbf{a}_e^\varphi \mathbf{d}^\varphi \quad \text{with} \quad \dim \mathbf{a}_e^\varphi = n_{en}^\varphi \times n_{np}. \quad (137)$$

Consequently the discretized version of the weak form of the electrical problem is recast as

$$\sum_{e=1}^{n_{el}} [\delta \mathbf{d}_e^\varphi]^t \mathbf{r}_e^\varphi = [\delta \mathbf{d}^\varphi]^t \sum_{e=1}^{n_{el}} [\mathbf{a}_e^\varphi]^t \mathbf{r}_e^\varphi = [\delta \mathbf{d}^\varphi]^t \mathbf{r}^\varphi = 0 \quad \forall \delta \mathbf{d}^\varphi. \quad (138)$$

As before the assembly into the global electrical residual follows as

$$\mathbf{r}^\varphi = \sum_{e=1}^{n_{el}} [\mathbf{a}_e^\varphi]^t \mathbf{r}_e^\varphi = \mathbf{A}_{e=1}^{n_{el}} \mathbf{r}_e^\varphi \quad (139)$$

with the following side conditions for the electrical flux

$$\mathbf{A}_{e=1}^{n_{el}} \int_{\partial\mathcal{B}_t^e} [\mathbf{N}_e^\varphi]^t \mathbf{d} \cdot \mathbf{n} \, da = - \mathbf{A}_{e=1}^{m_{el}} \int_{\mathcal{A}_t^e} [\mathbf{M}_e^\varphi]^t [\widehat{\varrho}_t^f + q_t] \, da. \quad (140)$$

Thus the contribution to the global external electrical load of the material body includes in particular the flux q_t

$$- \mathbf{A}_{e=1}^{m_{el}} \int_{\mathcal{A}_t^e} [\mathbf{M}_e^\varphi]^t [\widehat{\varrho}_t^f + q_t] \, da. \quad (141)$$

5.5 Discretized Boundary Integral Equation for the Free Space

Laplace equation. By the help of the shape functions the discretized version of the boundary integral equation reads for each source point $\boldsymbol{\xi}$ as

$$\begin{aligned} c^h(\boldsymbol{\xi})\varphi^h(\boldsymbol{\xi}) - \varphi_\infty + \sum_{e=1}^{m_{el}} \sum_{i=1}^{m_{en}^\varphi} \hat{d}_i^\varphi \int_{\mathcal{A}_i^e} M_i^\varphi \nabla_m G_\xi \, da \\ + \sum_{e=1}^{m_{el}} \sum_{i=1}^{m_{en}^q} \hat{d}_i^q \int_{\mathcal{A}_i^e} M_i^q \frac{G_\xi}{\epsilon_0} \, da = 0. \end{aligned} \quad (142)$$

Here $c^h(\boldsymbol{\xi})$ is the approximation to the geometry factor $c(\boldsymbol{\xi})$ and is computed without involving any shape functions

$$c^h(\boldsymbol{\xi}) = - \int_{\partial\mathcal{S}_\infty \cup \partial\mathcal{B}_t} \nabla_m G_\xi \, da = 1 - \sum_{e=1}^{m_{el}} \int_{\mathcal{A}_i^e} \nabla_m G_\xi \, da. \quad (143)$$

Likewise, in matrix notation the discretized version of the boundary integral equation reads for each source point $\boldsymbol{\xi}$ as

$$\begin{aligned} c^h(\boldsymbol{\xi})\varphi^h(\boldsymbol{\xi}) - \varphi_\infty + \sum_{e=1}^{m_{el}} [\hat{\mathbf{d}}_e^\varphi]^t \int_{\mathcal{A}_i^e} [\mathbf{M}_e^\varphi]^t \nabla_m G_\xi \, da \\ + \sum_{e=1}^{m_{el}} [\hat{\mathbf{d}}_e^q]^t \int_{\mathcal{A}_i^e} [\mathbf{M}_e^q]^t \frac{G_\xi}{\epsilon_0} \, da = 0. \end{aligned} \quad (144)$$

Next we define the elementwise ‘fundamental’ vectors for each source point $\boldsymbol{\xi}$

$$\hat{\mathbf{g}}_{\xi_e}^{\varphi*} = \int_{\mathcal{A}_i^e} [\mathbf{M}_e^\varphi]^t \nabla_m G_\xi \, da \quad \text{and} \quad \hat{\mathbf{g}}_{\xi_e}^q = \int_{\mathcal{A}_i^e} [\mathbf{M}_e^q]^t \frac{G_\xi}{\epsilon_0} \, da \quad (145)$$

in order to abbreviate the discretized version of the boundary integral equation as

$$c^h(\boldsymbol{\xi})\varphi^h(\boldsymbol{\xi}) - \varphi_\infty + \sum_{e=1}^{m_{el}} [\hat{\mathbf{d}}_e^\varphi]^t \hat{\mathbf{g}}_{\xi_e}^{\varphi*} + \sum_{e=1}^{m_{el}} [\hat{\mathbf{d}}_e^q]^t \hat{\mathbf{g}}_{\xi_e}^q = 0. \quad (146)$$

The m_{en}^φ local (elementwise) degrees of freedom for the electric potential and the m_{en}^q local (elementwise) degrees of freedom for the flux, respectively, are assigned to corresponding m_{np}^φ and m_{np}^q global degrees of freedom by the help of Boolean matrices

$$\hat{\mathbf{d}}_e^\varphi = \hat{\mathbf{a}}_e^\varphi \hat{\mathbf{d}}^\varphi \quad \text{and} \quad \hat{\mathbf{d}}_e^q = \hat{\mathbf{a}}_e^q \hat{\mathbf{d}}^q. \quad (147)$$

Consequently, the discretized version of the boundary integral equation is recast for each source point $\boldsymbol{\xi}$ as

$$c^h(\boldsymbol{\xi})\varphi^h(\boldsymbol{\xi}) - \varphi_\infty + [\hat{\mathbf{d}}^\varphi]^t \hat{\mathbf{g}}_\xi^{\varphi*} + [\hat{\mathbf{d}}^q]^t \hat{\mathbf{g}}_\xi^q = 0. \quad (148)$$

Here the assembly into the global ‘fundamental’ vectors for each source point $\boldsymbol{\xi}$ follows as

$$\hat{\mathbf{g}}_\xi^{\varphi*} = \sum_{e=1}^{m_{el}} [\hat{\mathbf{a}}_e^\varphi]^t \hat{\mathbf{g}}_{\xi e}^{\varphi*} = \mathbf{A}_{e=1}^{m_{el}} \hat{\mathbf{g}}_{\xi e}^{\varphi*} \quad \text{and} \quad \hat{\mathbf{g}}_\xi^q = \sum_{e=1}^{m_{el}} [\hat{\mathbf{a}}_e^q]^t \hat{\mathbf{g}}_{\xi e}^q = \mathbf{A}_{e=1}^{m_{el}} \hat{\mathbf{g}}_{\xi e}^q. \quad (149)$$

Next we may further relate the value of $c^h(\boldsymbol{\xi})\varphi^h(\boldsymbol{\xi})$ to the global degrees of freedom $\hat{\mathbf{d}}^\varphi$ by a vector

$$\hat{\mathbf{g}}_\xi^{\varphi**} = c^h(\boldsymbol{\xi})[\hat{\mathbf{a}}_e^\varphi]^t [M_e^\varphi(\boldsymbol{\xi})]^t \quad \text{with} \quad \boldsymbol{\xi} \in \mathcal{A}_t^e \quad (150)$$

that contains essentially the shape functions M_i^φ multiplied with $c^h(\boldsymbol{\xi})$, both evaluated at the source point $\boldsymbol{\xi}$

$$c^h(\boldsymbol{\xi})\varphi^h(\boldsymbol{\xi}) = [\hat{\mathbf{d}}^\varphi]^t \hat{\mathbf{g}}_\xi^{\varphi**}. \quad (151)$$

Moreover we might define a projection \mathbf{p}^φ from the global degrees of freedom \mathbf{d}^φ as used in the finite element setting to the global degrees of freedom $\hat{\mathbf{d}}^\varphi$ as used in the boundary element setting

$$\hat{\mathbf{d}}^\varphi = \mathbf{p}^\varphi \mathbf{d}^\varphi. \quad (152)$$

Thus, the discretized version of the boundary integral equation takes a format that is finally amenable for an assembly with the finite element equations

$$[\mathbf{d}^\varphi]^t \mathbf{g}_\xi^\varphi + [\hat{\mathbf{d}}^q]^t \hat{\mathbf{g}}_\xi^q - \varphi_\infty = 0. \quad (153)$$

Here we abbreviated for the generalized global ‘fundamental’ vector as

$$\mathbf{g}_\xi^\varphi = [\mathbf{p}^\varphi]^t [\hat{\mathbf{g}}_\xi^{\varphi*} + \hat{\mathbf{g}}_\xi^{\varphi**}]. \quad (154)$$

After assembly with the finite element equations the electric potential degrees of freedom at the interface between the material body and the free space $\hat{\mathbf{d}}^\varphi = \mathbf{p}^\varphi \mathbf{d}^\varphi$ are already taken care of. For the determination of the electric potential φ_∞ at infinity we will set up a separate equation for the integrated free charge of the material body. Thus it remains to specify m_{np}^q equations for the determination of the m_{np}^q unknowns for the flux contained in $\hat{\mathbf{d}}^q$. This might be achieved by:

i) evaluating the boundary integral equation at $m_{cp} = m_{np}^q$ collocation points

or

ii) formulating a Galerkin approach, e.g., by using a test function δq^h with the same shape functions as those for q^h .

Here, for simplicity, we shall use the collocation method **i**). Thus the boundary integral equation is required to hold at $m_{cp} = m_{np}^q$ discrete source (collocation) points ξ_c (with $c = 1, m_{cp}$)

$$[\mathbf{d}^\varphi]^t \mathbf{g}_{\xi_c}^\varphi + [\hat{\mathbf{d}}^q]^t \hat{\mathbf{g}}_{\xi_c}^q - \varphi_\infty = 0 \quad \forall \xi_c. \quad (155)$$

By introducing global ‘fundamental’ matrices $\mathbf{g}^\varphi = [\mathbf{g}_{\xi_1}^\varphi, \dots, \mathbf{g}_{\xi_{m_{cp}}}^\varphi]$ and $\hat{\mathbf{g}}^q = [\hat{\mathbf{g}}_{\xi_1}^q, \dots, \hat{\mathbf{g}}_{\xi_{m_{cp}}}^q]$ with columns defined by the global ‘fundamental’ vectors $\mathbf{g}_{\xi_c}^\varphi$ and $\hat{\mathbf{g}}_{\xi_c}^q$ with $c = 1, m_{cp}$ we arrive eventually at the global statement formulated as a residual

$$\mathbf{r}^q = [\mathbf{d}^\varphi]^t \mathbf{g}^\varphi + [\hat{\mathbf{d}}^q]^t \hat{\mathbf{g}}^q - [\mathbf{d}_\infty^\varphi]^t = \mathbf{0}. \quad (156)$$

Here $\mathbf{d}_\infty^\varphi$ denotes a vector with m_{cp} identical entries φ_∞ .

Free charge equation. In analogy, again by the help of the shape functions, the discretized version of the free charge equation reads

$$\sum_{e=1}^{m_{el}} \sum_{i=1}^{m_{en}^q} \hat{d}_i^q \int_{\mathcal{A}_i^e} M_i^q da = 0. \quad (157)$$

Likewise, in matrix notation the discretized version of the free charge equation is expressed as

$$\sum_{e=1}^{m_{el}} [\hat{\mathbf{d}}_e^q]^t \int_{\mathcal{A}_e^e} [\mathbf{M}_e^q]^t da = 0. \quad (158)$$

Next we define the elementwise vectors

$$\hat{\mathbf{m}}_e^q = \int_{\mathcal{A}_e^e} [\mathbf{M}_e^q]^t da \quad (159)$$

in order to abbreviate the discretized version of the free charge equation

$$\sum_{e=1}^{m_{el}} [\hat{\mathbf{d}}_e^q]^t \hat{\mathbf{m}}_e^q = 0. \quad (160)$$

By the help of the previously introduced elementwise Boolean matrices $\hat{\mathbf{a}}_e^q$ we finally assemble the global vector

$$\hat{\mathbf{m}}^q = \sum_{e=1}^{m_{el}} [\hat{\mathbf{a}}_e^q]^t \hat{\mathbf{m}}_e^q = \mathbf{A} \hat{\mathbf{m}}_e^q \quad (161)$$

and write eventually the global statement for the free charge equation as a residual

$$r^f = [\hat{\mathbf{d}}^q]^t \hat{\mathbf{m}}^q = 0. \quad (162)$$

5.6 Linearization of Discretized Weak Form

Mechanical problem. At the finite element level the linearization of the discretized version of the internal mechanical virtual work

$$\Delta \left[\sum_{i=1}^{n_{en}^u} \int_{\mathcal{B}_i^e} [\delta \mathbf{d}_i^u \otimes \nabla_x N_i^u] : \boldsymbol{\sigma}^{\text{tot}} \, dv \right] = L_{\text{inte}}^{uu} + L_{\text{inte}}^{u\varphi} \quad (163)$$

renders two contributions L_{inte}^{uu} and $L_{\text{inte}}^{u\varphi}$. Based on the tangent moduli introduced earlier these two contributions read

$$\begin{aligned} L_{\text{inte}}^{uu} &= \sum_{i=1}^{n_{en}^u} \sum_{j=1}^{n_{en}^u} \int_{\mathcal{B}_i^e} [\delta \mathbf{d}_i^u \otimes \nabla_x N_i^u] : \mathbf{e} : [\Delta \mathbf{d}_j^u \otimes \nabla_x N_j^u] \, dv, \quad (164) \\ L_{\text{inte}}^{u\varphi} &= \sum_{i=1}^{n_{en}^u} \sum_{j=1}^{n_{en}^\varphi} \int_{\mathcal{B}_i^e} [\delta \mathbf{d}_i^u \otimes \nabla_x N_i^u] : \mathbf{p}^T \cdot [\Delta \mathbf{d}_j^\varphi \quad \nabla_x N_j^\varphi] \, dv. \end{aligned}$$

Alternatively the linearization of the discretized version of the internal mechanical virtual work is recast in matrix representation as

$$\Delta \left[[\delta \mathbf{d}_e^u]^t \int_{\mathcal{B}_e^e} [\mathbf{B}_e^u]^t \boldsymbol{\sigma}_v^{\text{tot}} \, dv \right] = L_{\text{inte}}^{uu} + L_{\text{inte}}^{u\varphi}. \quad (165)$$

Before proceeding we also need to assemble the fourth-order spatial elasticity tensor \mathbf{e} in Voigt notation

$$\mathbf{e} \rightarrow \mathbf{e}_v = \begin{bmatrix} \mathbf{e}_{1111} & \mathbf{e}_{1122} & \mathbf{e}_{1133} & \mathbf{e}_{11(12)} & \mathbf{e}_{11(23)} & \mathbf{e}_{11(31)} \\ \mathbf{e}_{2211} & \mathbf{e}_{2222} & \mathbf{e}_{2233} & \mathbf{e}_{22(12)} & \mathbf{e}_{22(23)} & \mathbf{e}_{22(31)} \\ \mathbf{e}_{3311} & \mathbf{e}_{3322} & \mathbf{e}_{3333} & \mathbf{e}_{33(12)} & \mathbf{e}_{33(23)} & \mathbf{e}_{33(31)} \\ \mathbf{e}_{(12)11} & \mathbf{e}_{(12)22} & \mathbf{e}_{(12)33} & \mathbf{e}_{(12)(12)} & \mathbf{e}_{(12)(23)} & \mathbf{e}_{(12)(31)} \\ \mathbf{e}_{(23)11} & \mathbf{e}_{(23)22} & \mathbf{e}_{(23)33} & \mathbf{e}_{(23)(12)} & \mathbf{e}_{(23)(23)} & \mathbf{e}_{(23)(31)} \\ \mathbf{e}_{(31)11} & \mathbf{e}_{(31)22} & \mathbf{e}_{(31)33} & \mathbf{e}_{(31)(12)} & \mathbf{e}_{(31)(23)} & \mathbf{e}_{(31)(31)} \end{bmatrix}. \quad (166)$$

Here, the notation $\mathbf{e}_{11(12)}$ denotes the symmetric index combination with $2\mathbf{e}_{11(12)} = \mathbf{e}_{1112} + \mathbf{e}_{1121}$.

Likewise we abbreviate the third-order spatial piezoelectricity tensor \mathbf{p} in Voigt notation

$$\mathbf{p} \rightarrow \mathbf{p}_v = \begin{bmatrix} \mathfrak{p}_{111} & \mathfrak{p}_{122} & \mathfrak{p}_{133} & \mathfrak{p}_{1(12)} & \mathfrak{p}_{1(23)} & \mathfrak{p}_{1(31)} \\ \mathfrak{p}_{211} & \mathfrak{p}_{222} & \mathfrak{p}_{233} & \mathfrak{p}_{2(12)} & \mathfrak{p}_{2(23)} & \mathfrak{p}_{2(31)} \\ \mathfrak{p}_{311} & \mathfrak{p}_{322} & \mathfrak{p}_{333} & \mathfrak{p}_{3(12)} & \mathfrak{p}_{3(23)} & \mathfrak{p}_{3(31)} \end{bmatrix}. \quad (167)$$

Thus the two contributions to the linearization of the discretized version of the internal mechanical virtual work are eventually expressed in matrix representation as

$$\begin{aligned} L_{\text{int}e}^{uu} &= [\delta \mathbf{d}_e^u]^t \int_{\mathcal{B}_t^e} [\mathbf{B}_e^u]^t \mathbf{e}_v \mathbf{B}_e^u \, dv \Delta \mathbf{d}_e^u = [\delta \mathbf{d}_e^u]^t \mathbf{k}_e^{uu} \Delta \mathbf{d}_e^u, \\ L_{\text{int}e}^{u\varphi} &= [\delta \mathbf{d}_e^u]^t \int_{\mathcal{B}_t^e} [\mathbf{B}_e^u]^t \mathbf{p}_v^t \mathbf{B}_e^\varphi \, dv \Delta \mathbf{d}_e^\varphi = [\delta \mathbf{d}_e^u]^t \mathbf{k}_e^{u\varphi} \Delta \mathbf{d}_e^\varphi. \end{aligned} \quad (168)$$

Here \mathbf{k}_e^{uu} and $\mathbf{k}_e^{u\varphi}$ denote partitions of the element stiffness matrix. Finally we may sum the element-wise contributions to the linearized internal mechanical virtual work over, e.g., all the n_{el} finite elements covering the material body to obtain

$$\begin{aligned} \sum_{e=1}^{n_{el}} [\delta \mathbf{d}_e^u]^t \mathbf{k}_e^{uu} \Delta \mathbf{d}_e^u &= [\delta \mathbf{d}^u]^t \mathbf{k}^{uu} \Delta \mathbf{d}^u, \\ \sum_{e=1}^{n_{el}} [\delta \mathbf{d}_e^u]^t \mathbf{k}_e^{u\varphi} \Delta \mathbf{d}_e^\varphi &= [\delta \mathbf{d}^u]^t \mathbf{k}^{u\varphi} \Delta \mathbf{d}^\varphi. \end{aligned} \quad (169)$$

Here \mathbf{k}^{uu} and $\mathbf{k}^{u\varphi}$ denote partitions of the global stiffness matrix that follow from an assembly by the help of the previously introduced Boolean matrices

$$\begin{aligned} \mathbf{k}^{uu} &= \sum_{e=1}^{n_{el}} [\mathbf{a}_e^u]^t \mathbf{k}_e^{uu} \mathbf{a}_e^u = \mathbf{A} \mathbf{k}_e^{uu}, \\ \mathbf{k}^{u\varphi} &= \sum_{e=1}^{n_{el}} [\mathbf{a}_e^u]^t \mathbf{k}_e^{u\varphi} \mathbf{a}_e^\varphi = \mathbf{A} \mathbf{k}_e^{u\varphi}. \end{aligned}$$

Contribution from the Boundary Finally the linearization of the (solution dependent) mechanical external virtual work at the boundary renders at the global level

$$\Delta \left[\sum_{e=1}^{m_{el}} [\delta \hat{\mathbf{d}}_e^u]^t \int_{\mathcal{A}_t^e} [\mathbf{M}_e^u]^t [\mathbf{t}_t + \mathbf{t}_t^{\max}] \, da \right] = L_{\text{ext}}^{uu} + L_{\text{ext}}^{u\varphi} + L_{\text{ext}}^{uq}. \quad (170)$$

Based on the discussions and abbreviations in the following section (and skipping the tedious detailed derivations) these terms expand in terms of boundary contributions to the global stiffness matrix

$$\begin{aligned} L_{\text{ext}}^{uu} &= [\delta \mathbf{d}^u]^t \hat{\mathbf{k}}^{uu} \Delta \mathbf{d}^u, \\ L_{\text{ext}}^{u\varphi} &= [\delta \mathbf{d}^u]^t \hat{\mathbf{k}}^{u\varphi} \Delta \mathbf{d}^\varphi, \\ L_{\text{ext}}^{uq} &= [\delta \mathbf{d}^u]^t \hat{\mathbf{k}}^{uq} \Delta \hat{\mathbf{d}}^q. \end{aligned} \quad (171)$$

Electrical problem (in matter). In analogy the linearization of the discretized version of the internal electrical virtual work

$$\Delta \left[\sum_{i=1}^{n_{en}^\varphi} \int_{\mathcal{B}_i^e} [\delta d_i^\varphi \nabla_x N_i^\varphi] \cdot \mathbf{d} \, dv \right] = L_{\text{inte}}^{\varphi u} + L_{\text{inte}}^{\varphi\varphi} \quad (172)$$

renders two contributions $L_{\text{inte}}^{\varphi u}$ and $L_{\text{inte}}^{\varphi\varphi}$ at the finite element level. With the help of the tangent moduli introduced earlier these two contributions read

$$\begin{aligned} L_{\text{inte}}^{\varphi u} &= \sum_{i=1}^{n_{en}^\varphi} \sum_{j=1}^{n_{en}^u} \int_{\mathcal{B}_i^e} [\delta d_i^\varphi \nabla_x N_i^\varphi] \cdot \mathbf{p} : [\Delta \mathbf{d}_j^u \otimes \nabla_x N_j^u] \, dv, \\ L_{\text{inte}}^{\varphi\varphi} &= - \sum_{i=1}^{n_{en}^\varphi} \sum_{j=1}^{n_{en}^\varphi} \int_{\mathcal{B}_i^e} [\delta d_i^\varphi \nabla_x N_i^\varphi] \cdot \mathbf{d} \cdot [\Delta d_j^\varphi \quad \nabla_x N_j^\varphi] \, dv. \end{aligned} \quad (173)$$

Again in analogy we may represent the linearization of the discretized version of the internal electrical virtual work alternatively in matrix representation

$$\Delta \left[[\delta \mathbf{d}_e^\varphi]^t \int_{\mathcal{B}_e^e} [\mathbf{B}_e^\varphi]^t \mathbf{d} \, dv \right] = L_{\text{inte}}^{\varphi u} + L_{\text{inte}}^{\varphi\varphi}. \quad (174)$$

Recall the arrangement of coefficients for the second-order dielectricity tensor \mathbf{d}

$$\mathbf{d} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}. \quad (175)$$

Based on the appropriate Voigt notation the two contributions to the linearization of the discretized version of the internal electrical virtual work thus follow in matrix representation as

$$\begin{aligned} L_{\text{inte}}^{\varphi u} &= [\delta \mathbf{d}_e^\varphi]^t \int_{\mathcal{B}_e^e} [\mathbf{B}_e^\varphi]^t \mathbf{p}_v \mathbf{B}_e^u \, dv \Delta \mathbf{d}_e^u = [\delta \mathbf{d}_e^\varphi]^t \mathbf{k}_e^{\varphi u} \Delta \mathbf{d}_e^u, \\ L_{\text{inte}}^{\varphi\varphi} &= - [\delta \mathbf{d}_e^\varphi]^t \int_{\mathcal{B}_e^e} [\mathbf{B}_e^\varphi]^t \mathbf{d} \mathbf{B}_e^\varphi \, dv \Delta \mathbf{d}_e^\varphi = [\delta \mathbf{d}_e^\varphi]^t \mathbf{k}_e^{\varphi\varphi} \Delta \mathbf{d}_e^\varphi. \end{aligned} \quad (176)$$

Here $\mathbf{k}_e^{\varphi u}$ and $\mathbf{k}_e^{\varphi\varphi}$ denote the remaining partitions of the element stiffness matrix. Summing up the elementwise contributions to the linearized internal electrical virtual work over, e.g., all the n_{el} finite elements covering the material body renders

$$\begin{aligned} \sum_{e=1}^{n_{el}} [\delta \mathbf{d}_e^\varphi]^t \mathbf{k}_e^{\varphi u} \Delta \mathbf{d}_e^u &= [\delta \mathbf{d}^\varphi]^t \mathbf{k}^{\varphi u} \Delta \mathbf{d}^u, \\ \sum_{e=1}^{n_{el}} [\delta \mathbf{d}_e^\varphi]^t \mathbf{k}_e^{\varphi\varphi} \Delta \mathbf{d}_e^\varphi &= [\delta \mathbf{d}^\varphi]^t \mathbf{k}^{\varphi\varphi} \Delta \mathbf{d}^\varphi. \end{aligned} \quad (177)$$

Here $\mathbf{k}^{\varphi u}$ and $\mathbf{k}^{\varphi\varphi}$ denote partitions of the global stiffness matrix that follow again from an assembly by the help of the previously introduced Boolean matrices

$$\begin{aligned} \mathbf{k}^{\varphi u} &= \sum_{e=1}^{n_{el}} [\mathbf{a}_e^\varphi]^t \mathbf{k}_e^{\varphi u} \mathbf{a}_e^u = \mathbf{A} \mathbf{k}_e^{\varphi u}, \\ \mathbf{k}^{\varphi\varphi} &= \sum_{e=1}^{n_{el}} [\mathbf{a}_e^\varphi]^t \mathbf{k}_e^{\varphi\varphi} \mathbf{a}_e^\varphi = \mathbf{A} \mathbf{k}_e^{\varphi\varphi}. \end{aligned} \quad (178)$$

Contribution from the Boundary Finally the linearization of the (solution dependent) electrical external virtual work at the boundary renders at the global level

$$-\Delta \left[\sum_{e=1}^{m_{el}} [\delta \hat{\mathbf{d}}_e^\varphi]^t \int_{\mathcal{A}_i^e} [\mathbf{M}_e^\varphi]^t [\hat{\varrho}_t^f + q_t] da \right] = L_{\text{ext}}^{\varphi u} + L_{\text{ext}}^{\varphi\varphi} + L_{\text{ext}}^{\varphi q}. \quad (179)$$

Based on the discussions and abbreviations in the following section (and skipping the tedious detailed derivations) these terms expand in terms of boundary contributions to the global stiffness matrix

$$L_{\text{ext}}^{\varphi u} = [\delta \mathbf{d}^\varphi]^t \hat{\mathbf{k}}^{\varphi u} \Delta \mathbf{d}^u, \quad L_{\text{ext}}^{\varphi\varphi} = 0, \quad L_{\text{ext}}^{\varphi q} = [\delta \mathbf{d}^\varphi]^t \hat{\mathbf{k}}^{\varphi q} \Delta \hat{\mathbf{d}}^q. \quad (180)$$

5.7 Linearization of Discretized Boundary Integral Equation

Usually the boundary element method is applied to linear problems whereby the linearization of the discretized equations is not an issue. Here, however, even if the equations that we solve are linear, the solution domain, i.e. the interface $\partial\mathcal{B}_t$ between the material body and the free space, is varying with the deformation of the material body. Thus we have to linearize the discretized boundary integral equation with respect to its solution domain, a quite cumbersome endeavor as it will turn out in the sequel.

Laplace equation. At the global level, i.e. after assembly of all element-wise contributions the linearization of the discretized version of the boundary integral equation reads

$$\Delta \mathbf{r}^q = [\Delta \mathbf{d}^{\varphi}]^t \mathbf{g}^{\varphi} + [\Delta \hat{\mathbf{d}}^q]^t \hat{\mathbf{g}}^q + [\mathbf{d}^{\varphi}]^t \Delta \mathbf{g}^{\varphi} + [\hat{\mathbf{d}}^q]^t \Delta \hat{\mathbf{g}}^q - [\Delta \mathbf{d}_{\infty}^{\varphi}]^t. \quad (181)$$

Anticipating the appropriate linearizations and ordering terms we might eventually write

$$\Delta \mathbf{r}^q = \mathbf{k}^{q\varphi} \Delta \mathbf{d}^{\varphi} + \mathbf{k}^{qq} \Delta \hat{\mathbf{d}}^q + \mathbf{k}^{qu} \Delta \mathbf{d}^u + \mathbf{k}^{qf} \Delta \mathbf{d}^f. \quad (182)$$

Here we set $\varphi_{\infty} = \mathbf{d}^f$ for notational convenience, moreover we abbreviated for the matrices $\mathbf{k}^{q\varphi}$, \mathbf{k}^{qq} and \mathbf{k}^{qu} of dimensions $m_{cp} \times m_{np}^{\varphi}$, $m_{cp} \times m_{cp}$ and $m_{cp} \times n_{np} n_{dm}$, respectively,

$$\mathbf{k}^{q\varphi} = [\mathbf{g}^{\varphi}]^t, \quad \mathbf{k}^{qq} = [\hat{\mathbf{g}}^q]^t, \quad \mathbf{k}^{qu} = [\mathbf{d}^{\varphi}]^t \mathbf{h}^{\varphi u} + [\hat{\mathbf{d}}^q]^t \mathbf{h}^{qu}. \quad (183)$$

The urgent reader might want to jump over the admittedly tedious details of the linearizations in the following parts **a)** to **c)**. Otherwise, in order to derive the $\mathbf{h}^{\varphi u}$ and \mathbf{h}^{qu} terms we have to start from the boundary element level:

a) We have to provide first the linearization of the spatial covariant base vectors

$$\Delta \mathbf{a}_{\alpha}(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_e^{\hat{\boldsymbol{\eta}}}} = \mathbf{L}_{e\alpha}^u(\hat{\boldsymbol{\eta}}) \Delta \hat{\mathbf{d}}_e^u. \quad (184)$$

Here we took into account that the element-wise nodal coordinates $\hat{\mathbf{x}}_e$ change like the corresponding nodal degrees for the deformation map $\hat{\mathbf{d}}_e^u = \mathbf{p}_e^u \mathbf{d}_e^u$, whereby \mathbf{p}_e^u denotes an element-wise projection from finite element to boundary element degrees of freedom. Thus the linearized area element Δda at the boundary of the three dimensional ¹² material body is computed by

$$\frac{\Delta da(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_e^{\hat{\boldsymbol{\eta}}}}}{da} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|^2} \cdot [\Delta \mathbf{a}_1 \times \mathbf{a}_2 + \mathbf{a}_1 \times \Delta \mathbf{a}_2] = \mathbf{A}_e^u(\hat{\boldsymbol{\eta}}) \Delta \hat{\mathbf{d}}_e^u. \quad (185)$$

Here \mathbf{A}_e^u is an element-wise vector of dimension $n_{dm} m_{en}^u$ that incorporates the isoparametric derivatives of the shape functions M_i^u . Likewise the linearization of the surface normal \mathbf{m} is expressed in terms of the linearization of the spatial covariant base vectors

$$\Delta \mathbf{m}(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_e^{\hat{\boldsymbol{\eta}}}} = \frac{\mathbf{i} - \mathbf{m} \otimes \mathbf{m}}{|\mathbf{a}_1 \times \mathbf{a}_2|} \cdot [\Delta \mathbf{a}_1 \times \mathbf{a}_2 + \mathbf{a}_1 \times \Delta \mathbf{a}_2] = \mathbf{S}_e^u(\hat{\boldsymbol{\eta}}) \Delta \hat{\mathbf{d}}_e^u. \quad (186)$$

¹²In two dimensions $da = \mathbf{a} \cdot \Delta \mathbf{a} / |\mathbf{a}|^2 da$ holds instead; the index α is omitted.

Here \mathbf{S}_e^u is an element-wise matrix of dimensions $n_{dm} \times n_{dm} m_{en}^u$ that incorporates the isoparametric derivatives of the shape functions M_i^u .

Recall next $\mathbf{r} = \mathbf{x} - \boldsymbol{\xi}$ to obtain

$$\Delta \mathbf{r}^h(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_t^e} = \mathbf{M}_e^u(\hat{\boldsymbol{\eta}}) \Delta \hat{\mathbf{d}}_e^u - \Delta \boldsymbol{\xi}^h. \quad (187)$$

Thus the linearization of $r = |\mathbf{r}|$, i.e. the linearization of the radial distance of the field point \mathbf{x} from the source point $\boldsymbol{\xi}$ is expressed as

$$\Delta r^h(\hat{\boldsymbol{\eta}})|_{\mathcal{A}_t^e} = \frac{\mathbf{r}^h}{r^h} \cdot \Delta \mathbf{r}^h = \mathbf{R}_e^u(\hat{\boldsymbol{\eta}}) \Delta \hat{\mathbf{d}}_e^u - \Delta \boldsymbol{\xi}_r^h. \quad (188)$$

Here \mathbf{R}_e^u is an element-wise vector of dimension $n_{dm} m_{en}^u$ that incorporates the shape functions M_i^u and the radial distance vector \mathbf{r}^h . Note that

$$\Delta \boldsymbol{\xi}_r^h = \frac{\mathbf{r}^h}{r^h} \cdot \Delta \boldsymbol{\xi}^h \quad (189)$$

has a different expansion depending on whether the source point $\boldsymbol{\xi}^h$ is located within or without the boundary element \mathcal{A}_t^e . Finally we may relate $\Delta \boldsymbol{\xi}^h$ to the global degrees of freedom $\hat{\mathbf{d}}^u$ by

$$\Delta \boldsymbol{\xi}^h = \mathbf{M}_e^u(\boldsymbol{\xi}^h) \hat{\mathbf{a}}_e^u \Delta \hat{\mathbf{d}}^u \quad \text{with} \quad \boldsymbol{\xi}^h \in \mathcal{A}_t^e. \quad (190)$$

b.1) With the above preliminaries at hand we may compute the linearization of $\hat{\mathbf{g}}_{\xi_e}^{\varphi*}$ as

$$\Delta \hat{\mathbf{g}}_{\xi_e}^{\varphi*} = \hat{\mathbf{h}}_{\xi_e}^{\varphi u*} \Delta \hat{\mathbf{d}}_e^u - \hat{\boldsymbol{\xi}}_{\xi_e}^{\varphi}. \quad (191)$$

Here we introduced appropriate abbreviations

$$\hat{\mathbf{h}}_{\xi_e}^{\varphi u*} = \int_{\mathcal{A}_t^e} [\mathbf{M}_e^{\varphi}]^t \mathbf{G}_{\xi_e}^u \, da, \quad \hat{\boldsymbol{\xi}}_{\xi_e}^{\varphi} = \int_{\mathcal{A}_t^e} [\mathbf{M}_e^{\varphi}]^t \Delta \boldsymbol{\xi}_m^h \, da \quad (192)$$

with $\mathbf{G}_{\xi_e}^u = \mathbf{G}_{\xi_e}^{u*} + \mathbf{G}_{\xi_e}^{u**} + \mathbf{G}_{\xi_e}^{u***}$ ¹³.

¹³Since $\nabla_m G_{\xi} = \nabla_r G_{\xi} \cdot \mathbf{m}$ we have that $\Delta \nabla_m G_{\xi} = \mathbf{m} \cdot \Delta \nabla_r G_{\xi} + \nabla_r G_{\xi} \cdot \Delta \mathbf{m}$. The first term expands into

$$\mathbf{m} \cdot \Delta \nabla_r G_{\xi} = \mathbf{m} \cdot \left[\left[\partial_{rr}^2 G_{\xi} - \frac{\partial_r G_{\xi}}{r} \right] \frac{\mathbf{r}}{r} \otimes \frac{\mathbf{r}}{r} + \frac{\partial_r G_{\xi}}{r} \mathbf{i} \right] \cdot \Delta \mathbf{r} = \boldsymbol{\gamma} \cdot \Delta \mathbf{r}.$$

With the expansion for $\Delta \mathbf{r}$ and $\Delta \boldsymbol{\xi}_m^h = \boldsymbol{\gamma} \cdot \Delta \boldsymbol{\xi}$ we may abbreviate the first term $\mathbf{m} \cdot \Delta \nabla_r G_{\xi} = \mathbf{G}_{\xi_e}^{u*} \Delta \hat{\mathbf{d}}_e^u - \Delta \boldsymbol{\xi}_m^h$. With the expansion for $\Delta \mathbf{m}$ the second term abbreviates as $\nabla_r G_{\xi} \cdot \Delta \mathbf{m} = \mathbf{G}_{\xi_e}^{u**} \Delta \hat{\mathbf{d}}_e^u$. Finally the changing area element da is taken care of by $\mathbf{G}_{\xi_e}^{u***} = \nabla_m G_{\xi} A_e^u$.

By the assembly of all boundary elements and appropriate assignments following the discussions in the above¹⁴ we write further for the linearization of the global ‘fundamental’ vector $\hat{\mathbf{g}}_\xi^{\varphi*}$ at each source point ξ

$$\Delta \hat{\mathbf{g}}_\xi^{\varphi*} = \sum_{e=1}^{m_{el}} \mathbf{A}_e \Delta \hat{\mathbf{g}}_{\xi_e}^\varphi = \hat{\mathbf{h}}_\xi^{\varphi u*} \Delta \hat{\mathbf{d}}^u. \quad (193)$$

b.2) Next, we linearize $\Delta \hat{\mathbf{g}}_\xi^{\varphi**}$ by first considering

$$\Delta c^h(\xi) = - \sum_{e=1}^{m_{el}} \int_{\mathcal{A}_e^t} \mathbf{G}_{\xi_e}^u da \Delta \hat{\mathbf{d}}_e^u + \sum_{e=1}^{m_{el}} \int_{\mathcal{A}_e^t} \Delta \xi_m^h da = c_\xi^t \Delta \hat{\mathbf{d}}^u. \quad (194)$$

Thus we may express the linearization of $\hat{\mathbf{g}}_\xi^{\varphi**}$ (with $\xi \in \mathcal{A}_e^t$) as

$$\Delta \hat{\mathbf{g}}_\xi^{\varphi**} = [\hat{\mathbf{a}}_e^\varphi]^t [\mathbf{M}_e^\varphi(\xi)]^t \Delta c^h(\xi) = \hat{\mathbf{h}}_\xi^{\varphi u**} \Delta \hat{\mathbf{d}}^u. \quad (195)$$

b.3) Finally we may write for the linearization of the generalized global ‘fundamental’ vector

$$\Delta \mathbf{g}_\xi^\varphi = [\mathbf{p}^\varphi]^t [\Delta \hat{\mathbf{g}}_\xi^{\varphi*} + \Delta \hat{\mathbf{g}}_\xi^{\varphi**}] = \mathbf{h}_\xi^{\varphi u} \Delta \mathbf{d}^u. \quad (196)$$

By evaluating the linearization of the global ‘fundamental’ vector $\Delta \mathbf{g}_{\xi_c}^\varphi$ at each collocation point ξ_c and arranging everything nicely in a ‘hexaedrix’ (the algebraic equivalent to a third order tensor) of dimension $m_{np} \times m_{cp} \times n_{dm} m_{np}$ we finally obtain

$$\Delta \mathbf{g}^\varphi = \mathbf{h}^{\varphi u} \Delta \mathbf{d}^u. \quad (197)$$

c) With the above preliminaries at hand we may further compute the linearization of $\hat{\mathbf{g}}_{\xi_e}^q$ as

$$\Delta \hat{\mathbf{g}}_{\xi_e}^q = \hat{\mathbf{h}}_{\xi_e}^{qu} \Delta \hat{\mathbf{d}}_e^u - \hat{\xi}_{\xi_e}^q. \quad (198)$$

Here we introduced appropriate abbreviations

$$\hat{\mathbf{h}}_{\xi_e}^{qu} = \frac{1}{\epsilon_0} \int_{\mathcal{A}_e^t} [\mathbf{M}_e^q]^t [\partial_r G_\xi \mathbf{R}_e^u + G_\xi \mathbf{A}_e^u] da, \quad \hat{\xi}_{\xi_e}^q = \frac{1}{\epsilon_0} \int_{\mathcal{A}_e^t} [\mathbf{M}_e^q]^t \partial_r G_\xi \Delta \xi_r^h da.$$

¹⁴After assembly we have that

$$\hat{\mathbf{h}}_\xi^{\varphi u*} = \sum_{e=1}^{m_{el}} \mathbf{A}_e \hat{\mathbf{h}}_{\xi_e}^{\varphi u*} - \hat{\kappa}_\xi^{\varphi u} \quad \text{with} \quad \hat{\xi}_\xi^\varphi = \sum_{e=1}^{m_{el}} \mathbf{A}_e \hat{\xi}_{\xi_e}^\varphi = \hat{\kappa}_\xi^{\varphi u} \Delta \hat{\mathbf{d}}^u.$$

By the assembly of all boundary elements and appropriate assignments following the discussions in the above¹⁵ we write further for the linearization of the global ‘fundamental’ vector at each source point $\boldsymbol{\xi}$

$$\Delta \hat{\boldsymbol{g}}_{\boldsymbol{\xi}}^q = \mathbf{A}_{e=1}^{mel} \Delta \hat{\boldsymbol{g}}_{\boldsymbol{\xi}_e}^q = \hat{\boldsymbol{h}}_{\boldsymbol{\xi}}^{qu} \Delta \hat{\boldsymbol{d}}^u. \quad (199)$$

By evaluating the linearization of the global ‘fundamental’ vector $\Delta \hat{\boldsymbol{g}}_{\boldsymbol{\xi}_c}^q$ at each collocation point $\boldsymbol{\xi}_c$ and arranging everything nicely in a ‘hexaedrix’ of dimension $m_{cp} \times m_{cp} \times n_{dm} m_{np}$ we finally obtain

$$\Delta \hat{\boldsymbol{g}}^q = \hat{\boldsymbol{h}}^{qu} \Delta \hat{\boldsymbol{d}}^u = \boldsymbol{h}^{qu} \Delta \hat{\boldsymbol{d}}^u. \quad (200)$$

Free charge equation. At the global level, i.e. after assembly of all elementwise contributions the linearization of the discretized version of the free charge equation reads

$$\Delta r^f = [\Delta \hat{\boldsymbol{d}}^q]^t \hat{\boldsymbol{m}}^q + [\hat{\boldsymbol{d}}^q]^t \Delta \hat{\boldsymbol{m}}^q. \quad (201)$$

Anticipating the appropriate linearizations and ordering terms we might eventually write

$$\Delta r^f = \boldsymbol{k}^{fq} \Delta \hat{\boldsymbol{d}}^q + \boldsymbol{k}^{fu} \Delta \hat{\boldsymbol{d}}^u. \quad (202)$$

Here we abbreviated for the matrices \boldsymbol{k}^{fq} and \boldsymbol{k}^{fu} of dimensions $1 \times m_{cp}$ and $1 \times n_{np} n_{dm}$, respectively

$$\boldsymbol{k}^{fq} = [\hat{\boldsymbol{m}}^q]^t, \quad \boldsymbol{k}^{fu} = [\hat{\boldsymbol{d}}^q]^t \boldsymbol{n}^{qu}. \quad (203)$$

Again, the urgent reader might want to jump over the following details of the linearization. Otherwise, in order to derive the \boldsymbol{n}^{qu} term we have to start from the boundary element level: with the above preliminaries at hand we may compute the linearization of $\hat{\boldsymbol{m}}_e^q$ as

$$\Delta \hat{\boldsymbol{m}}_e^q = \hat{\boldsymbol{n}}_e^{qu} \Delta \hat{\boldsymbol{d}}_e^u. \quad (204)$$

Here we introduced the abbreviation

$$\hat{\boldsymbol{n}}_e^{qu} = \int_{\mathcal{A}_i^e} [\boldsymbol{M}_e^q]^t \boldsymbol{A}_e^u da. \quad (205)$$

¹⁵After assembly we have that

$$\hat{\boldsymbol{h}}_{\boldsymbol{\xi}}^{qu} = \mathbf{A}_{e=1}^{mel} \hat{\boldsymbol{h}}_{\boldsymbol{\xi}_e}^{qu} - \hat{\boldsymbol{\kappa}}_{\boldsymbol{\xi}}^{qu} \quad \text{with} \quad \hat{\boldsymbol{\kappa}}_{\boldsymbol{\xi}}^q = \mathbf{A}_{e=1}^{mel} \hat{\boldsymbol{\kappa}}_{\boldsymbol{\xi}_e}^q = \hat{\boldsymbol{\kappa}}_{\boldsymbol{\xi}}^{qu} \Delta \hat{\boldsymbol{d}}^u.$$

By the assembly of all boundary elements we finally write for the linearization of the global vector $\hat{\mathbf{m}}^q$

$$\Delta \hat{\mathbf{m}}^q = \mathbf{A} \Delta \hat{\mathbf{m}}_e^q = \hat{\mathbf{n}}^{qu} \Delta \hat{\mathbf{d}}^u. \quad (206)$$

Global problem. After assembly and after considering the appropriate number of Dirichlet boundary conditions the previously derived partitions of the global residual¹⁶ $\mathbf{r}^u, \mathbf{r}^\varphi, \mathbf{r}^q, \mathbf{r}^f$ and the corresponding partitions of the global stiffness matrix are employed in a typical Newton-Raphson solution scheme

$$\mathbf{k} \Delta \mathbf{d} = \mathbf{r}, \quad (207)$$

whereby each iteration step proceeds along the following pattern

$$\begin{aligned} \mathbf{d} &\leftarrow \mathbf{d} + \mathbf{k}^{-1} \mathbf{r}, \\ \mathbf{r} &= \mathbf{r}(\mathbf{d}), \\ |\mathbf{r}| &\leq \text{tol?} \end{aligned} \quad (208)$$

If everything is fine, i.e. the residual is correctly implemented and has been properly linearized, this scheme converges quadratically¹⁷ close to the solution. Thereby the tolerance tol depends of course on the precision for the computer representation of floating point numbers and the condition number of the iteration matrix, but the experience from countless computational experiments indicates that $\log(\text{tol}) = -8$ is a quite decent value. In the above the global mechanical and electrical degrees of freedom and the related partitions of the global residuals are arranged as follows

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}^u \\ \mathbf{d}^\varphi \\ \hat{\mathbf{d}}^q \\ \mathbf{d}^f \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} \mathbf{r}^u \\ \mathbf{r}^\varphi \\ \mathbf{r}^q \\ \mathbf{r}^f \end{bmatrix}. \quad (209)$$

Correspondingly, the partitions of the global stiffness matrix follow as

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}^{uu} + \hat{\mathbf{k}}^{uu} & \mathbf{k}^{u\varphi} + \hat{\mathbf{k}}^{u\varphi} & \mathbf{k}^{uq} + \hat{\mathbf{k}}^{uq} & \mathbf{0} \\ \mathbf{k}^{\varphi u} + \hat{\mathbf{k}}^{\varphi u} & \mathbf{k}^{\varphi\varphi} & \mathbf{k}^{\varphi q} + \hat{\mathbf{k}}^{\varphi q} & \mathbf{0} \\ \mathbf{k}^{qu} & \mathbf{k}^{q\varphi} & \mathbf{k}^{qq} & \mathbf{k}^{qf} \\ \mathbf{k}^{fu} & \mathbf{0} & \mathbf{k}^{fq} & \mathbf{0} \end{bmatrix}. \quad (210)$$

¹⁶Note that we from now on \mathbf{r} denotes the global residual vector!

¹⁷In this context quadratic convergence denotes a situation in which the euclidean norm of the residual $r = |\mathbf{r}|$ reduces from iteration i to iteration $i+1$ so that $\log r_{i+1} \approx 2 \log r_i$ once $\log r_i < 0$.

It goes without saying that various options beyond a monolithic solution are available for this coupled set of algebraic equations.

6 Computational Examples

The following simulations highlight some prominent features of the coupled electro-mechanical response of electro-elastic bodies and demonstrate the applicability and the accuracy of the numerical methods developed in the above. To this end we consider a square specimen of dimension $60\mu m \times 60\mu m \times 10\mu m$ with a circular hole in its center of radius $40\mu m$. The specimen is subjected to an externally applied electric potential in both its upper and lower sides. The simulations are performed for two cases:

Case 1: Firstly, we will study an example in which the effect of the surrounding free space can be neglected and the results obtained with both the FEM (in which only the material body is taken into account) and the coupled BEM–FEM (in which free space is taken into account additionally) do not exhibit significant differences.

Case 2: Secondly, we will study an example in which the effect of the surrounding free space has a significant influence on the electro-mechanical response of the material and can therefore not be neglected. Thereby, the effect of the surrounding free space can be taken into account by invoking the coupled BEM–FEM as developed in the above.

6.1 Case 1: Free Space may be Neglected

In the first case, the electric potential is prescribed at the lower and upper side of the specimen as: $\varphi_{lower} = -0.03$ V and $\varphi_{upper} = +0.03$ V, respectively. The plate is assumed to be made of a compressible Neo-Hooke-type material with total energy density

$$W_0 = 0.5\mu[I_1 - \ln I_3 - n_{dm}] + 2\lambda[\ln I_3]^2 + \alpha I_4 + \beta I_5 - 0.5J\epsilon \cdot \mathbf{e}$$

and with material parameters $\mu = 10$ MPa, $\lambda = 5$ MPa, $\alpha = 10$ Pa m^2/V^2 , $\beta = 6$ Pa m^2/V^2 and $\epsilon = \epsilon_0$ in free space while $\epsilon = 5\epsilon_0$ in matter. Recall the electrical permittivity of vacuum $\epsilon_0 = 8.85 \cdot 10^{-12}$ Pa m^2/V^2 . The purely elastic response of the material under consideration depends on the parameters μ and λ , which are the ordinary two Lamé coefficients. When the material is subjected to an electric field, the total energy density infers in particular that the material will exhibit a nonlinear coupling behavior through the invariant I_5 . It is important to mention that the material

constants α and β would have to be determined experimentally. However, there is still a significant lack of experimental results, and therefore real material properties are not available at this point. For the purpose of testing the robustness of our numerical implementation, in this example we will assume that the material of which the plate is made has the previously mentioned properties.

The discretization generated for the 2d simulations has 2176 nodes and 2048 quadrilateral finite elements. In the case in which the free space is additionally considered, the external and internal boundary of the specimen is discretized into 256 boundary elements with linear electric potential and constant flux. In the 3d simulation, 2880 nodes corresponding to 2048 hexahedral finite elements were used. In all cases, the mid-line (or mid-plane in 3d) was constrained in the vertical direction for reasons of symmetry and the mid-left point had a double constraint to avoid the displacement of the whole plate in the horizontal direction. In the 3d example an extra constraint in the out-of-plane direction was applied to the mid-right node to avoid rotation.

Comparing the results of the simulations in Figures 4 and 5, we note that they are practically identical.

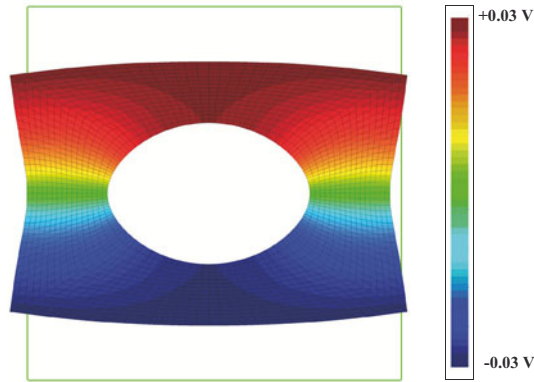


Figure 4. Deformation and distribution of electric potential of a square specimen with a circular hole loaded by a electric potential of $\varphi_{lower} = -0.03$ V and $\varphi_{upper} = +0.03$ V applied at its lower and its upper sides: 2d result obtained by FEM (in which only the material body is taken into account).

This, of course, demonstrates that the effect of the surrounding free space can indeed be neglected in this case. Finally the simulation in Figure 6 is

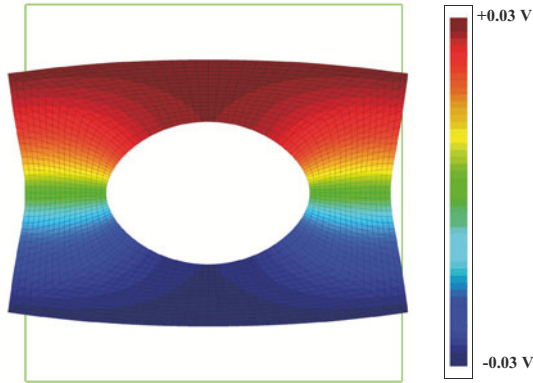


Figure 5. Deformation and distribution of electric potential of a square specimen with a circular hole loaded by a electric potential of $\varphi_{lower} = -0.03$ V and $\varphi_{upper} = +0.03$ V applied at its lower and its upper sides: 2d result obtained by coupled FEM-BEM (in which free space is additionally taken into account).

the 3d version of the previous examples and supports the results obtained in 2d.

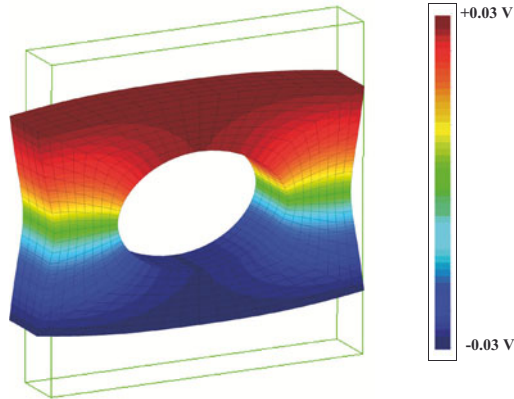


Figure 6. Deformation and distribution of electric potential of a square specimen with a circular hole loaded by a electric potential of $\varphi_{lower} = -0.03$ V and $\varphi_{upper} = +0.03$ V applied at its lower and its upper sides: 3d result obtained by FEM.

6.2 Case 2: Free Space can not be Neglected

In the second case, a different set material parameters was chosen with $\mu = 0.1$ MPa, $\lambda = 0.05$ MPa, $\alpha = 0.2\epsilon_0 = 1.77 \cdot 10^{-12}$ Pa m²/V², $\beta = 2\epsilon_0 = 1.77 \cdot 10^{-11}$ Pa m²/V², the values for ϵ were set as in the above. The huge difference in the values for α and β , when compared to the values used for the first example, is the reason why we can not neglect the effect of the surrounding free space in the present example. The potential applied on the lower and upper sides of the specimen is $\varphi_{lower} = -500$ V and $\varphi_{upper} = +500$ V, respectively. The mesh and the boundary conditions are identical to the one used in the previous examples.

This problem was solved twice: i) FEM simulation ignoring the surrounding free space and ii) coupled FEM–BEM simulation including free space. As a consequence of the chosen material parameter, the results displayed in Figures 7 and 8 exhibit a significant difference. The specimen suffers a markedly different deformation in both cases, demonstrating that in the present case we can not neglect the influence of the surrounding free space. Obviously a coupled BEM–FEM analysis is mandatory in this case in order to obtain correct results.

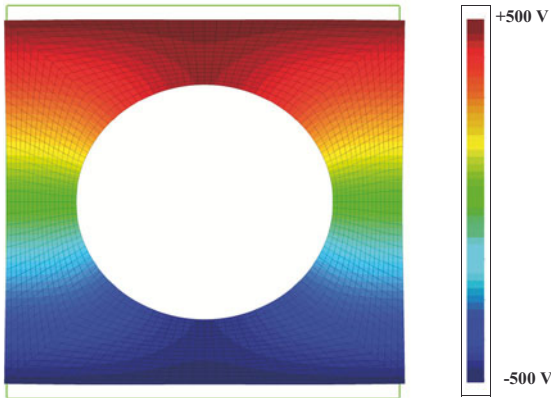


Figure 7. Deformation and distribution of electric potential of a square specimen with a circular hole loaded by a electric potential of $\varphi_{lower} = -500$ V and $\varphi_{upper} = +500$ V applied at its lower and its upper sides: 2d result obtained by FEM (in which only the material body is taken into account).

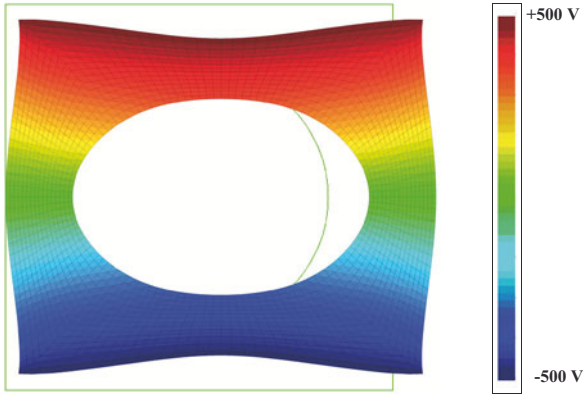


Figure 8. Deformation and distribution of electric potential of a square specimen with a circular hole loaded by a electric potential of $\varphi_{lower} = -500$ V and $\varphi_{upper} = +500$ V applied at its lower and its upper sides: 2d result obtained by coupled FEM-BEM (in which free space is additionally taken into account).

7 Conclusion

This contribution discussed the essential topics needed for the formulation and computation of nonlinear problems in electro-elasticity. Thereby the underlying variational setting of nonlinear electro-elasticity is exploited for an the appropriate discretization in terms of the finite element method combined with the boundary element method. The corresponding solution methods have been discussed in much detail. In particular the computational analysis of nonlinear boundary value problems highlighted some key features of coupled problems in nonlinear electro-elasticity.

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Electro-Mechanical Response of Nematic Elastomers: an Introduction

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Abstract. We review in these lecture notes some of our recent work on modeling the response of nematic elastomers to applied mechanical loads and/or to electric fields, both in the static and in the dynamic regime. Our aim is to compare theoretical results based on mathematical analysis and on numerical simulations with the available experimental evidence, in order to examine critically the various recent accomplishments, and some challenging problems that remain open. Nematic elastomers combine the electro-optical properties and rotational degrees of freedom of nematic liquid crystals with the mechanical properties and translational degrees of freedom of entropic rubbery solids. The rich behavior they exhibit, the interesting applications they seem to make possible, the breadth and depth of recent breakthroughs at the experimental, theoretical, and computational level make nematic elastomers an exciting model system for advanced research in mechanics.

1 Introduction

In these lecture notes we focus on the electro-mechanical behavior of *one* specific material: nematic elastomers. It is a new material, so our understanding of it is still incomplete. Among its distinguishing features are large spontaneous deformations, actuation by many different means including electric fields, and mechanical compliance. This makes it suitable for fast soft actuators and, in particular, for new applications such as artificial muscles, which are currently of great technological interest. The reader is referred to the monograph by Warner and Terentjev (2003) for a detailed account of the chemistry and physics of nematic elastomers, and for an extensive list of references.

The mechanism for electro-mechanical coupling is the anisotropy of dielectric constants, as it is typical for liquid crystals. Nematic Liquid Crystal

Displays (LCDs), which represent one of the biggest market arenas for technological devices based on electro-mechanical coupling, exploit precisely this mechanism. Indeed, a localized applied voltage is able to change the local orientation of nematic molecules, which in turn results in a change of optical properties: the material can change from being transparent to opaque when sandwiched between crossed polarizers, giving rise to a very reliable optical micro-shutter. Individual pixels of LCDs are realized in this fashion. We notice that the mechanism for electro-mechanical coupling based on dielectric anisotropy is different from those based on either permanent or induced polarization, which occur in ferroelectric and piezoelectric materials, respectively. Indeed, nematic elastomers are neither ferroelectric nor piezoelectric.

Nematic elastomers provide a counterpart in the world of rubbery solids to nematic liquid crystals. Thanks to the coupling with nematic degrees of freedom, their entropic elasticity can be activated by temperature changes (similarly to what happens in shape-memory alloys, SMAs), electric fields (like in electro-active polymers, EAPs), or by irradiation with UV light. The lessons one can learn by studying this fascinating model material may provide very useful insight on the behavior of many other interesting systems.

2 Molecular Structure and Macroscopic Response

Nematic elastomers consist of cross-linked networks of polymeric chains containing nematic mesogens. The three main chemical constituents of this assembly are a polymer backbone, nematic mesogens, and cross-linkers.

The *polymer backbone* results from the repeat of monomers containing tetra-valent atoms, such as Carbon (C) or Silicon (Si), that are able to form long and flexible chains. In these geometries, two bonds are used to construct a connected chain, while two more bonds are free and available for attachment of side units (see Figure 1).

Nematic mesogens are rigid rod-like molecules containing benzenic rings. They are responsible for the establishment of nematic order at sufficiently low temperatures. The isotropic-to-nematic transition is a phase transformation determined by the alignment of the nematic mesogens, and accompanied by a change of the optical properties of the system (which becomes anisotropic). At the same time, the material tends to become transparent. Nematic mesogens can either be part of the backbone (main-chain nematic elastomer) or be attached sideways (side-chain nematic elastomers). The possibility of attachment typically comes from the presence of a double carbon bond $C=C$ which can open up into $-C-C-$ leaving the unsaturated ends

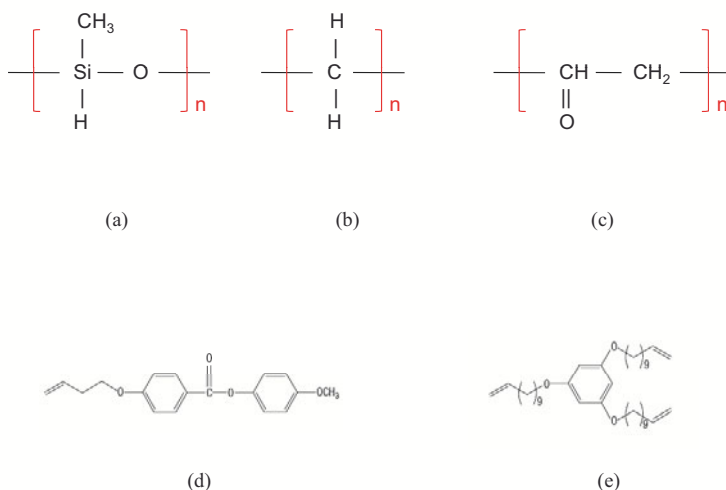


Figure 1. Basic chemistry of nematic elastomers. On the top row, some typical polymer backbones: methyl-siloxane, an example of polysiloxane (a), a $(\text{CH}_2)_n$ chain (b), and polyacrilate (c). On the bottom row, a bi-phenil side-chain nematic mesogen (d) and a tri-functional cross-linker (e).

free for bonding.

Depending on whether the $\text{C}=\text{C}$ unit is at one end or in the central part of the nematic mesogen, this will orient parallel or perpendicular to the backbone giving rise to prolate or oblate structures. When the isotropic-to-nematic phase transition takes place, the alignment of nematic mesogens causes a distortion of the polymer backbone to which they are attached. We will be mostly concerned with the prolate case, in which the polymer chains tend to elongate along the direction of alignment of the nematic mesogens.

Cross-linkers are flexible chains containing double $\text{C}=\text{C}$ bonds at both ends. Hence they are able to attach to two distinct polymer chains, connecting them. This is what turns an ensemble of disjoint polymer chains (a polymeric liquid) into a percolating network able to transmit static shear stresses (an elastomer, or rubbery solid). The combination of polymer backbone, nematic mesogens, and cross-linkers leads to a system in which the orientational degrees of freedom and the associated optical-elastic properties typical of nematic liquid crystals (dielectric anisotropy, Frank curva-

ture elasticity associated with spatial variations of nematic order) appear in combination with the mechanical properties and the translational degrees of freedom exhibited by an elastic solid (deformation gradients, rubber elasticity, shear moduli).

The coupling between nematic orientational order and rubber entropic elasticity has profound consequences. The alignment of nematic mesogens in a neighborhood of a point x along an average direction $\pm \mathbf{n}(x)$, where \mathbf{n} is a unit vector field called *nematic director*, induces a spontaneous distortion of the polymer chains described by

$$\mathbf{V}_{\mathbf{n}} = a^{1/3} \mathbf{N} + a^{-1/6} (\mathbf{I} - \mathbf{N}), \quad (1)$$

where $a > 1$ (prolate case), \mathbf{I} is the identity, and $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$. Here $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of the vectors \mathbf{a} and \mathbf{b} with components $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. Tensor \mathbf{N} is closely related to de Gennes' order tensor \mathbf{Q} . Here we are using the framework of Frank-type theories, in which order is constrained to be uniaxial and the degree of order is fixed. Then, one has $\mathbf{Q} = s(\mathbf{N} - \frac{1}{3}\mathbf{I})$, with $s > 0$ constant, and the descriptions of nematic order in terms of either \mathbf{Q} or \mathbf{N} are equivalent. The material parameter a , which in the oblate case is smaller than one, gives the amount of spontaneous elongation along \mathbf{n} accompanying the isotropic-to-nematic phase transformation. It is a combined measure of the degree of order and of the strength of the nematic-elastic coupling, and it is in principle a function of temperature. We will ignore this, as we will be working at a fixed, constant temperature, well below the isotropic-to-nematic transition temperature T_{IN} . Tensor $\mathbf{V}_{\mathbf{n}}$ represents a volume-preserving uniaxial stretch along the *current* direction of the director \mathbf{n} .

The spontaneous distortion (1) can be very large (up to 300% in some main-chain elastomers) and it is easily observable when the temperature of the elastomer is lowered below T_{IN} starting from a temperature above T_{IN} (at which the material behaves like a standard rubber). Working at fixed $T < T_{IN}$, one way of observing (1) is to apply an electric field to a mechanically unconstrained sample (e.g., a nematic gel surrounded by silicon oil, inside a capacitor with transparent electrodes). As is well known from ordinary nematic liquids, due to the anisotropy of the dielectric tensor (we assume here that the material has positive dielectric anisotropy ε_a), a sufficiently strong applied voltage tends to align the director with the electric field \mathbf{E} , i.e. $\mathbf{n} = \pm \mathbf{E}/|\mathbf{E}|$. The quantitative details of this coupling will be described in Section 7. Suffice it to say here that by rotating the applied field one may induce rotations of \mathbf{n} and observe the macroscopic shape changes of the sample accompanying this process. Also, simultaneous birefringence measurements can be used to determine directly the dependence of \mathbf{n} on

the applied electric field. It turns out that the correlation between observed deformations and measured \mathbf{n} follows equation (1) to a remarkable level of accuracy; see Fukunaga et al. (2008).

A more subtle consequence of (1) emerges in stretching experiments in the absence of applied electric fields. Again, the temperature is fixed at a constant value below T_{IN} . The sample is prepared so that the director is spatially uniform, say \mathbf{n} aligned with \mathbf{e}_3 , the third unit vector of the canonical basis, and its initial state is the natural one corresponding to $\mathbf{n} = \mathbf{e}_3$. This means that polymer chains are elongated along the direction of \mathbf{e}_3 , with stretch $a^{1/3} > 1$ along \mathbf{e}_3 with respect to the reference configuration. Imagine now that the sample, a thin film with thickness direction parallel to \mathbf{e}_1 , is stretched along \mathbf{e}_2 , with rigid clamps applied on the two edges perpendicular to \mathbf{e}_2 . Experiments show that the force–stretch diagram is unusually soft, with an extended flat plateau following a small region of initially hard response.

We will refer in what follows to the idealized case in which this initially hard regime is not present as the *ideally soft* case. The interpretation of this unusual softness is that the sample accommodates the externally imposed deformations by reorienting the director along the direction of maximal stretch, hence storing less elastic energy. This is confirmed by optical microscopy under crossed polarizers, which reveals a texture of opaque and transparent bands parallel to \mathbf{e}_2 . The existence of this optical contrast shows that the director reorientation process occurs in a spatially nonuniform manner (*stripe-domain patterns*); in view of the coupling implied by (1), oscillating shears are triggered by the oscillations of the nematic director. This means that nematic elastomers exhibit material instabilities (co-operative elastic shear banding, which is fully reversible; see Sections 5 and 6) as a consequence of the spontaneous distortion (1) accompanying the symmetry breaking transformation from the high temperature isotropic phase to the low temperature nematic phase.

3 Warm-up in Finite Dimensions

Consider the following model mechanical system, lying in the plane $\{\mathbf{e}_1, \mathbf{e}_2\}$, with origin \mathbf{O} . It is made of two rigid links \mathbf{OQ}' , and $\mathbf{Q}'\mathbf{Q}$, each of length $\frac{1}{2}r > 0$, and of an extensible spring \mathbf{QP} with stiffness $k > 0$. There are frictionless joints in \mathbf{O} , \mathbf{Q}' , and \mathbf{Q} , so that \mathbf{O} is fixed and only relative rotations are allowed in \mathbf{Q}' and \mathbf{Q} . A force $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2$ acts on the free end \mathbf{P} , and all points are constrained to lie in the half-plane $x_2 \geq 0$.

We are interested in the following problem. Given an arbitrary force \mathbf{F}

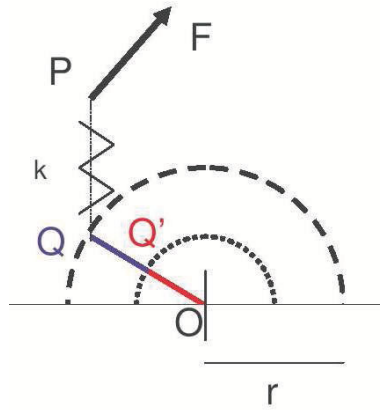


Figure 2. A finite-dimensional model system.

with $\mathbf{F} \cdot \mathbf{e}_2 \geq 0$, find the configurations of the system minimizing its energy

$$\mathcal{E}(\mathbf{P}, \mathbf{Q}) = \frac{k}{2} |\mathbf{P} - \mathbf{Q}|^2 - \mathbf{F} \cdot \mathbf{P}. \tag{2}$$

Once this problem is solved for every \mathbf{F} , we can imagine to fix the direction of \mathbf{F} , say $\mathbf{F} = F\mathbf{e}$, and to vary its intensity F . By plotting the component along \mathbf{e} of the solution $\mathbf{P} - \mathbf{O}$ of the minimization problem against the value F of the corresponding force we may obtain a force–stretch diagram summarizing the essentials of the mechanical response of the system to the prescribed applied loads.

It is interesting to notice that, since \mathbf{OQ}' and $\mathbf{Q}'\mathbf{Q}$ are inextensible, the configuration of the whole system is uniquely identified by the position of points \mathbf{P} and \mathbf{Q} . Point \mathbf{Q} is, however, an internal variable in the sense that no external force is directly applied to it. Moreover, in view of the constraints present on the system, the set of admissible positions for point \mathbf{Q} is

$$\mathcal{A} := \{\mathbf{Q} \in \mathbb{R}^2 : |\mathbf{Q} - \mathbf{O}| \leq r, \mathbf{Q} \cdot \mathbf{e}_2 \geq 0\}. \tag{3}$$

We may obtain the solution to the problem above in two steps. First we minimize out the internal variable \mathbf{Q} . Indeed

$$\min_{\mathbf{P}, \mathbf{Q}} \mathcal{E}(\mathbf{P}, \mathbf{Q}) = \min_{\mathbf{P}} \left(\min_{\mathbf{Q}} \frac{k}{2} |\mathbf{P} - \mathbf{Q}|^2 - \mathbf{F} \cdot \mathbf{P} \right). \tag{4}$$

We set

$$\mathcal{E}_{\text{eff}}(\mathbf{P}) = \min_{\mathbf{Q}} \frac{k}{2} |\mathbf{P} - \mathbf{Q}|^2 = \frac{k}{2} |\mathbf{P} - \mathbf{Q}_{\mathbf{P}}|^2, \tag{5}$$

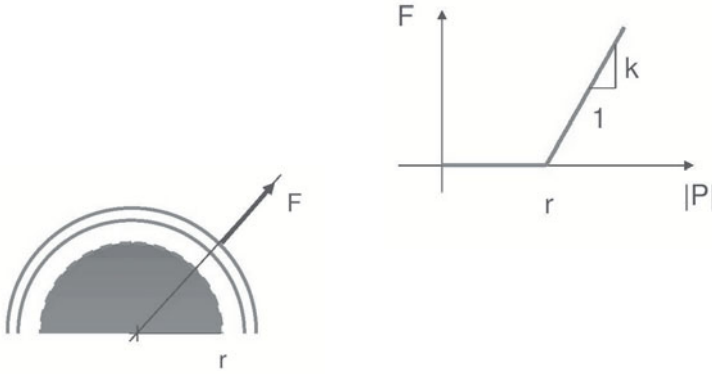


Figure 3. Level curves of the energy (left) and the force response (right) of the finite-dimensional model system.

where \mathbf{Q}_P is the orthogonal projection of \mathbf{P} onto the closed convex set \mathcal{A} . Notice that \mathbf{Q}_P coincides with \mathbf{P} if $\mathbf{P} \in \mathcal{A}$.

Granted (4) and (5), we can perform the second step in our minimization problem

$$\min_{\mathbf{P}, \mathbf{Q}} E(\mathbf{P}, \mathbf{Q}) = \min_{\mathbf{P}} (\mathcal{E}_{\text{eff}}(\mathbf{P}) - \mathbf{F} \cdot \mathbf{P}). \tag{6}$$

If we consider a stretching experiment starting from $\mathbf{Q} = \mathbf{O}$, an equilibrium configuration under zero force, we obtain a zero force response with \mathbf{Q} moving along a segment parallel to \mathbf{e} until $\mathbf{Q} - \mathbf{O} = r\mathbf{e}$. The force response to further extension is linear, given by $k(|\mathbf{P} - \mathbf{O}| - r)$. In other words, we can obtain the force response by differentiating \mathcal{E}_{eff} .

In spite of its simplicity, the model finite-dimensional system described in this section provides some interesting guidance for our future developments. For example, it shows that in spite of non-uniqueness of the minimal energy configuration of the two rigid links in the determination of (5) (notice that \mathbf{Q}' is not uniquely defined by \mathbf{Q}_P in (6) if $|\mathbf{P} - \mathbf{O}| < r$), the effective energy itself, \mathcal{E}_{eff} , and (hence) the force-stretch diagram are unique. Moreover, the example raises the question of dynamic accessibility of the energy-minimizing states. Indeed, if after having reached the linear regime in the extension experiment we reversed the sign of the force, a buckling instability would occur at $|\mathbf{Q} - \mathbf{O}| = r$. Following one of the buckling branches one can return to the initial configuration $\mathbf{Q} = \mathbf{O}$. Following the symmetric path we would hit the constraint $x_2 \geq 0$ preventing us from reaching $\mathbf{Q} = \mathbf{O}$.

The notion of effective energy will appear in what follows in two different circumstances, in particular in Section 6. One is the energy density $W_{\text{eff}}(\mathbf{F})$ arising from optimizing over the nematic degrees of freedom the energy density $W(\mathbf{F}, \mathbf{n})$ at fixed deformation gradient \mathbf{F} . Another one is the coarse-graining of the energy W_{eff} over elastic degrees of freedom oscillating at fine length-scales (microstructures), in order to compute its quasi-convex envelope $W_{\text{eff}}^{\text{qc}}$.

4 Elastic Energy Densities for Nematic Elastomers

This section is mostly based on DeSimone and Teresi (2009), to which the reader is referred for further details. We will denote by $\mathbf{F} = \nabla \mathbf{y}$ the gradient of the deformation with respect to the reference configuration, chosen as the one the sample would exhibit if stress-free in the high-temperature isotropic state. Moreover, we denote by $J = \det \mathbf{F}$ the determinant of the deformation gradient \mathbf{F} . In our discussion we focus on the most basic (and fundamental) expression for the elastic energy density stored by a nematic elastomer. This is based on the trace formula of Bladon et al. (1993) which, after a change of variables first proposed by DeSimone (1999), becomes

$$\begin{aligned} W(\mathbf{F}, \mathbf{N}) &= \frac{1}{2} \mu \mathbf{B}^e \cdot \mathbf{I}, \quad \det \mathbf{B}^e = J^2 = 1, \\ \mathbf{B}^e(\mathbf{F}, \mathbf{N}) &= \mathbf{B} \mathbf{L}^{-1} = \mathbf{F} \mathbf{F}^T \mathbf{L}^{-1}(\mathbf{N}), \end{aligned} \quad (7)$$

where

$$\mathbf{L}(\mathbf{N}) := a^{2/3} \mathbf{N} + a^{-1/3} (\mathbf{I} - \mathbf{N}) = \mathbf{V}_{\mathbf{n}}^2 \quad (8)$$

and $\mathbf{V}_{\mathbf{n}}$ is the spontaneous stretch defined in (1). The second line in (7) emphasizes that, according to the trace formula, the part of the deformation responsible for storage of elastic energy (the elastic part in a multiplicative decomposition, in the same spirit of the Kröner-Lee multiplicative decomposition in finite plasticity) is $\mathbf{B}^e = \mathbf{B} \mathbf{L}^{-1}$. To the best of our knowledge, this seemingly obvious observation has not been made before DeSimone and Teresi (2009) in spite of the fact that it has profound implications.

Proposition 1 in the Appendix shows that, given an arbitrary current orientation of the nematic director \mathbf{N} , (7) is minimized at the energy level $\frac{3}{2} \mu$, which is independent of \mathbf{N} , by any deformation p_0 with $\nabla p_0 \nabla p_0^T = \mathbf{L}(\mathbf{N})$. By the polar decomposition theorem, ∇p_0 is then of the form

$$\nabla p_0 = \mathbf{L}^{1/2}(\mathbf{N}) \mathbf{Q}, \quad (9)$$

where \mathbf{Q} is an arbitrary rotation. Every pair $(\nabla p_0, \mathbf{N})$ is a natural, stress-free state for a material described by the energy density W above.

Formula (7) lends itself to easy and useful generalizations. Expressions for the energy density, which are more suitable to study the regime of high

stresses, can be obtained by replacing (7) with

$$W(\mathbf{F}, \mathbf{N}) = W_{iso}(\mathbf{B}^e(\mathbf{F}, \mathbf{N})), \quad J = 1, \tag{10}$$

where one may choose for $W_{iso}(\mathbf{B}^e)$ any of the available functional forms used to model isotropic incompressible elastic materials, which have a strict global minimum at $\mathbf{B}^e = \mathbf{I}$. Formula (7) corresponds to the neo-Hookean expression; a few other alternative examples are listed in DeSimone and Teresi (2009). We quote here, in particular, the Ogden form

$$\begin{aligned} & \sum_{i=1}^N a_i \text{tr}(\mathbf{B}^e)^{\gamma_i/2} + \sum_{j=1}^M b_j \text{tr}(\text{cof } \mathbf{B}^e)^{\delta_j/2} \\ &= \sum_{i=1}^N a_i (v_1^{\gamma_i} + v_2^{\gamma_i} + v_3^{\gamma_i}) + \sum_{j=1}^M b_j ((v_2 v_3)^{\delta_j} + (v_3 v_1)^{\delta_j} + (v_1 v_2)^{\delta_j}), \end{aligned}$$

where v_k denotes the k -th principal stretch, i.e. the square root of the k -th eigenvalue of \mathbf{B}^e .

Extensions of Formula (7) to the compressible case are also straightforward, by setting

$$\tilde{W}(\mathbf{F}, \mathbf{N}) = W_{iso}(\mathbf{B}_s^e(\mathbf{F}, \mathbf{N})) + W_{vol}(J), \quad \mathbf{B}_s^e = J^{-2/3} \mathbf{B}^e. \tag{11}$$

Here $W_{vol}(s)$ is a non-negative, strictly convex function which is finite only for $s > 0$, vanishes only at $s = 1$, and diverges to $+\infty$ as s tends to either 0 or $+\infty$. This modification leaves the energy-well structure unchanged, because the minimizers of (11) are clearly the same as for (7). Since in what follows we will be only interested in the behavior of the energy in a neighborhood of a natural state, a quadratic expansion of W_{vol} may suffice, leading to the following model expression for the compressible isotropic case

$$\tilde{W}(\mathbf{F}, \mathbf{N}) = \frac{1}{2} \mu \mathbf{B}_s \cdot \mathbf{L}^{-1}(\mathbf{N}) + \frac{1}{2} \kappa (\sqrt{\det \mathbf{B}} - 1)^2. \tag{12}$$

Another important generalization is discussed in detail in DeSimone and Teresi (2009), and it consists of adding some anisotropic corrections to the isotropic energies described above. The two most basic ones are given below. The first one is

$$\tilde{W}_\beta(\mathbf{F}, \mathbf{N}) = \frac{1}{2} \mu_\beta \mathbf{C}_s \cdot \mathbf{L}_a^{-1} + \tilde{W}(\mathbf{F}, \mathbf{N}), \tag{13}$$

where $\mathbf{C}_s := (\det \mathbf{C})^{-2/3} \mathbf{C}$ and

$$\mathbf{L}_a := \mathbf{L}(\mathbf{N}_a) = a^{2/3} \mathbf{N}_a + a^{-1/3} (\mathbf{I} - \mathbf{N}_a)$$

with $\mathbf{N}_a := \mathbf{n}_a \otimes \mathbf{n}_a$ and \mathbf{n}_a a unit vector along the axis of anisotropy in the reference configuration.

The second model anisotropic expression is

$$\tilde{W}_\alpha(\mathbf{F}, \mathbf{N}) = \frac{1}{2}\mu_\alpha(1 - \mathbf{N} \cdot \mathbf{N}^*(\mathbf{F})) + \tilde{W}(\mathbf{F}, \mathbf{N}), \quad (14)$$

where

$$\mathbf{N}^* := \mathbf{n}^* \otimes \mathbf{n}^*, \quad \mathbf{n}^* = \mathbf{n}^*(\mathbf{F}) := \mathbf{F}\mathbf{n}_a/|\mathbf{F}\mathbf{n}_a|, \quad (15)$$

and \mathbf{n}^* gives the current orientation of the axis of anisotropy \mathbf{n}_a . A somewhat related model, based on the notion of nonlinear relative rotations, has been proposed in Menzel et al. (2009).

Finally, we consider the analogues of the energy densities described above in the framework of a geometrically linear theory. These are derived in DeSimone and Teresi (2009) by Taylor expansion. Assume that $a^{1/3} = 1 + \gamma$, with $0 < \gamma \ll 1$. We then have

$$\mathbf{L}^{-1}(\mathbf{N}) = \mathbf{I} - \gamma(3\mathbf{N} - \mathbf{I}) + 3\gamma^2\mathbf{N}. \quad (16)$$

Assume moreover that $\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$, where $\mathbf{u}(x) = \mathbf{y}(x) - x$ is the displacement, and $|\nabla\mathbf{u}| = \varepsilon \ll 1$. We then have $\mathbf{B} = \mathbf{I} + 2\mathbf{E} + o(\varepsilon^2)$, where \mathbf{E} is the symmetric part of the displacement gradient (linear strain), and

$$\begin{aligned} \mathbf{B}_s &= (\det(\mathbf{I} + 2\mathbf{E}))^{-1/3}(\mathbf{I} + 2\mathbf{E}) \\ &= \mathbf{I} + 2\mathbf{E}_d + \frac{2}{3}((\mathbf{E} \cdot \mathbf{E} + \frac{1}{3}(\text{tr}(\mathbf{E}))^2)\mathbf{I} - 2\text{tr}(\mathbf{E})\mathbf{E}) + o(\varepsilon^2), \end{aligned} \quad (17)$$

where \mathbf{E}_d is the deviatoric part of \mathbf{E} ; see DeSimone and Teresi (2009). It follows from (16) and (17) that

$$\mathbf{B}_s \cdot \mathbf{L}^{-1} = 3 + 2(\mathbf{E}_d - \mathbf{E}_0(\mathbf{N})) \cdot (\mathbf{E}_d - \mathbf{E}_0(\mathbf{N})) + o(\varepsilon^2, \gamma^2, \varepsilon\gamma), \quad (18)$$

where

$$\mathbf{E}_0(\mathbf{n}) = \frac{3}{2}\gamma(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}) \quad (19)$$

represents the small strain counterpart of the spontaneous strain $\mathbf{V}_\mathbf{n}$ given in (1). Finally, we have that

$$(\sqrt{\det \mathbf{B}} - 1)^2 = (\text{tr} \mathbf{E})^2 + o(\varepsilon^2). \quad (20)$$

The calculations above show that, modulo additive constants, the small strain counterpart of \tilde{W} is given by the following expression

$$\tilde{\Phi}(\mathbf{E}, \mathbf{N}) = \mu|\mathbf{E}_d - \mathbf{E}_0(\mathbf{N})|^2 + \frac{1}{2}\kappa(\text{tr} \mathbf{E})^2. \quad (21)$$

The incompressible version is obtained by formally setting $\kappa = +\infty$, so that

$$\Phi(\mathbf{E}, \mathbf{N}) = \mu|\mathbf{E}_d - \mathbf{E}_0(\mathbf{N})|^2, \quad \text{tr} \mathbf{E} = \text{div} \mathbf{u} = 0. \quad (22)$$

It is worth comparing the expressions $\mathbf{B}^e = \mathbf{B}\mathbf{L}^{-1}(\mathbf{N})$ and $\mathbf{E}^e = \mathbf{E} - \mathbf{E}_0(\mathbf{N})$, which describe the relative deformation between the current one and the preferred one associated with \mathbf{N} . The first expression does this through the composition with an inverse, as should be expected in nonlinear kinematics; the second one through a difference, as is appropriate in linear kinematics. In both cases, it is only this relative deformation (the elastic part of the appropriate strain measure) that contributes to storage of elastic energy. A rigorous proof that (the quasiconvexification of) (22) gives the correct small-strain limit of (12) (in the sense of Gamma-convergence) is provided in Agostiniani and DeSimone (2010).

The expansion of W_β works similarly, and one obtains

$$\tilde{\Phi}_\beta(\mathbf{E}, \mathbf{N}) = \tilde{\Phi}(\mathbf{E}, \mathbf{N}) + \mu_\beta |\mathbf{E}_d - \mathbf{E}_0(\mathbf{N}_a)|^2 \quad (23)$$

as the small strain counterpart of \tilde{W}_β . The small-strain approximation of \tilde{W}_α is instead more complicated, and we only report here a simplified expression valid in the regime where director rotations are large, while strains are small

$$\tilde{\Phi}_\alpha(\mathbf{E}, \mathbf{N}) = \tilde{\Phi}(\mathbf{E}, \mathbf{N}) + \frac{1}{2}\mu_\alpha(1 - \mathbf{N} \cdot \mathbf{N}_a), \quad (24)$$

where $\tilde{\Phi}$ is given in (21). This energy has been used in Fukunaga et al. (2008) to analyze the response of a free-standing film of a swollen nematic elastomer, to which an electric field is applied in order to drive the director away from its initial direction $\mathbf{n} = \mathbf{n}_a$. In the experiments, a finite critical field needs to be overcome in order to trigger director rotation. Measuring the equilibrium angle between \mathbf{n} and \mathbf{n}_a as a function of the applied electric field provides an experimental validation of (24) and a way of determining the value of the material parameter μ_α . It turns out that, when the field is removed, the director relaxes back to its preferred orientation \mathbf{n}_a . When $\mu_\alpha = 0$, the spring-back mechanism is suppressed and the critical field needed to start director reorientation is zero (Fukunaga et al., 2008, eq. (24)). Interestingly, if one describes anisotropy using (23) instead of (24) then the spring-back mechanism is suppressed.

5 Material Instabilities

We choose a reference frame so that \mathbf{n}_a is along the third coordinate axis and set

$$\mathbf{n}(\theta) = \begin{bmatrix} 0 \\ \sin \theta \\ \cos \theta \end{bmatrix}, \quad \mathbf{n}_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (25)$$

where θ is the angle between \mathbf{n} and \mathbf{n}_a . The state with $\theta = 0$ and $\mathbf{F} = \mathbf{I}_a^{1/2}$ is a global minimizer for all the energies introduced above. We are interested in the stability with respect to superposed shears of equilibrium states with $\theta = 0$, both in the initial configuration and in those obtained by (moderately) stretching the material in a direction perpendicular to \mathbf{n}_a . For this purpose, we consider the deformations

$$\mathbf{F}(\delta; \lambda) = \begin{bmatrix} a^{-1/6} & 0 & 0 \\ 0 & \lambda & \delta \\ 0 & 0 & a^{1/6}/\lambda \end{bmatrix} \quad (26)$$

with λ a fixed stretching parameter varying in a right neighborhood of $a^{-1/6}$. More precisely, we will take $\lambda \in [a^{-1/6}, a^{1/12})$.

By substituting $\mathbf{F}(\delta; \lambda)$ and $\mathbf{n}(\theta)$ in the various expressions of the energy, equations (12)–(14), we obtain three energies of the form $f(\delta, \theta; \lambda)$. In all cases $\partial f/\partial\delta$ and $\partial f/\partial\theta$ vanish at $\delta = \theta = 0$. Thus $\delta = \theta = 0$ is always an equilibrium configuration (this is easily seen by symmetry under $\pm\delta$ and $\pm\theta$) and we obtain expansions to second order of the following form

$$f(\delta, \theta; \lambda) = f(0, 0; \lambda) + \frac{1}{2} (G_{\delta\delta}\delta^2 + 2G_{\delta\theta}\delta\theta + G_{\theta\theta}\theta^2), \quad (27)$$

where

$$G_{\delta\delta}(\lambda) = \frac{\partial^2 f}{\partial\delta^2}(0, 0; \lambda), \quad G_{\delta\theta}(\lambda) = \frac{\partial^2 f}{\partial\delta\partial\theta}(0, 0; \lambda), \quad G_{\theta\theta}(\lambda) = \frac{\partial^2 f}{\partial\theta^2}(0, 0; \lambda). \quad (28)$$

The equilibrium value θ_0 of θ as a function of δ is obtained from

$$G_{\delta\theta}\delta + G_{\theta\theta}\theta = 0 \Rightarrow \theta_0(\delta) = -\frac{G_{\delta\theta}}{G_{\theta\theta}}\delta, \quad (29)$$

and substituting this into (27) we get

$$f(\delta, \theta_0(\delta); \lambda) - f(0, 0; \lambda) = \frac{1}{2}G(\lambda)\delta^2, \quad (30)$$

where we have set

$$G(\lambda) = \left(G_{\delta\delta} - \frac{G_{\delta\theta}^2}{G_{\theta\theta}} \right). \quad (31)$$

Depending on whether $G(\lambda) > 0$, $G(\lambda) = 0$, or $G(\lambda) < 0$, we have that the equilibrium state ($\delta = 0, \theta = 0$) is stable, neutrally stable, or unstable with respect to superposed shears. The special case $\lambda = a^{-1/6}$ reproduces the analysis in de Gennes (1980): simple shear from the natural state corresponding to $\mathbf{N} = \mathbf{N}_r$. Small shears superposed to large stretches have been

considered also in Ye et al. (2007), and the case of small shears superposed to large deformations arising in uniaxial extension experiments has been considered in Biggins et al. (2008).

We now compute $G(\lambda)$ for the three model energies \tilde{W} , \tilde{W}_β , \tilde{W}_α , given by (12), (13), (14), respectively. In the isotropic case, inserting $\mathbf{n}(\theta)$ and $\mathbf{F}(\delta; \lambda)$ into (27) (where we replace f by \tilde{W} or by W : since $\det \mathbf{F}(\delta; \lambda) \equiv 1$ this makes no difference), we obtain

$$\begin{aligned} G_{\delta\delta}(\lambda) &= \mu a^{1/3}, & G_{\delta\theta}(\lambda) &= -\mu a^{1/3} \left(\frac{a-1}{a}\right) \frac{a^{1/6}}{\lambda}, \\ G_{\theta\theta}(\lambda) &= \mu a^{1/3} \left(\frac{a-1}{a}\right) \left(\frac{a^{1/3}}{\lambda^2} - \lambda^2\right). \end{aligned} \quad (32)$$

Thus, by (31), we have

$$G(\lambda) = \mu a^{1/3} (1 - g(\lambda)), \quad (33)$$

where

$$g(\lambda) := \frac{a-1}{a} \frac{a^{1/3}}{a^{1/3} - \lambda^4}. \quad (34)$$

Since $g(\lambda) = 1$ for $\lambda = a^{-1/6}$, and $g(\lambda)$ is strictly increasing in the interval $[a^{-1/6}, a^{1/12})$, we conclude that

$$G(a^{-1/6}) = 0, \quad \text{and} \quad G(\lambda) < 0, \quad \text{for every } \lambda \in (a^{-1/6}, a^{1/12}). \quad (35)$$

Considering energy \tilde{W}_β we obtain

$$\begin{aligned} G_{\delta\delta}^\beta(\lambda) &= \mu a^{1/3} + \frac{\beta\mu}{a^{2/3}}, & G_{\delta\theta}^\beta(\lambda) &= -\mu a^{1/3} \left(\frac{a-1}{a}\right) \frac{a^{1/6}}{\lambda}, \\ G_{\theta\theta}^\beta(\lambda) &= \mu a^{1/3} \left(\frac{a-1}{a}\right) \left(\frac{a^{1/3}}{\lambda^2} - \lambda^2\right), \end{aligned} \quad (36)$$

so that, by (31), we have

$$G^\beta(\lambda) = \mu a^{1/3} \left(1 - g(\lambda) + \frac{\beta}{a}\right). \quad (37)$$

Since $g(\lambda)$ is strictly increasing in the interval $[a^{-1/6}, a^{1/12})$ starting from the value $g(a^{-1/6}) = 1$, and it diverges as $\lambda \rightarrow a^{1/12}$, we conclude that there exists $\lambda_c^\beta \in (a^{-1/6}, a^{1/12})$ such that

$$G^\beta(\lambda) > 0, \quad \text{for } \lambda \in [a^{-1/6}, \lambda_c^\beta), \quad \text{and} \quad G^\beta(\lambda) < 0, \quad \text{for } \lambda \in (\lambda_c^\beta, a^{1/12}). \quad (38)$$

The critical stretch λ_c^β is obtained by solving $g(\lambda_c^\beta) = 1 + \beta/a$, yielding

$$\lambda_c^\beta = a^{1/12} \left(\frac{\beta + 1}{\beta + a} \right)^{1/4}. \quad (39)$$

As β increases from 0 to ∞ , λ_c^β increases from $a^{-1/6}$ to $a^{1/12}$. Repeating the same procedure for energy \tilde{W}_α given by (14) we obtain

$$G^\alpha(\lambda) = \mu a^{1/3} \left[1 - \frac{1}{a} \frac{a^{1/3}(a-1) + \alpha a^{2/3} \lambda^2 + \alpha a^{1/3} \lambda^6}{a^{1/3} + \alpha a^{2/3} \lambda^2 / (a-1) - \lambda^4} \right]. \quad (40)$$

Again, it turns out that there exists $\lambda_c^\alpha \geq a^{-1/6}$ such that

$$G^\alpha(\lambda) > 0, \quad \text{for } \lambda < \lambda_c^\alpha, \quad \text{and} \quad G^\alpha(\lambda) < 0, \quad \text{for } \lambda > \lambda_c^\alpha. \quad (41)$$

The critical stretch λ_c^α is an increasing function of α and, as α increases from 0 to ∞ , λ_c^α increases from $a^{-1/6}$ to the value

$$\lambda_c^\alpha = \frac{1}{(a-1)^{1/4}} a^{1/12}, \quad \alpha = \infty. \quad (42)$$

The corresponding values of $G^\alpha(\lambda)$ are

$$G^\alpha(\lambda) = \mu a^{1/3} \left[1 - \frac{a-1}{a} \frac{a^{1/3}}{a^{1/3} - \lambda^4} \right], \quad \alpha = 0, \quad (43)$$

$$G^\alpha(\lambda) = \mu a^{1/3} \left[1 - \frac{a-1}{a} \frac{a^{1/3} + \lambda^4}{a^{1/3}} \right], \quad \alpha = +\infty. \quad (44)$$

If the anisotropy parameter a is sufficiently large, say, $a \geq 2$, then the value of λ_c^α for $\alpha = +\infty$ is not larger than $a^{1/12}$ and we have that $\lambda_c^\alpha \leq a^{1/12}$ for all $\alpha \geq 0$. Using the values $\alpha = 1$ and $a = 2$ we obtain

$$0.89 = a^{-1/6} < \lambda_c^\alpha = 0.9637 < a^{1/12} = 1.06, \quad \alpha = 1, \quad a = 2. \quad (45)$$

The shear moduli calculated above, which become negative for certain values of the stretching parameter λ , show that the isotropic energy W leads to material instabilities: uniformly stretched states become unstable to superposed shears. In other words, the stripe-domain instabilities discussed in Section 2, and analyzed in detail in the literature on nematic elastomers (see Verwey et al., 1996, the discussion in Warner and Terentjev, 2003, Chapter 7, and the analysis in DeSimone and Dolzmann, 2002, Conti et al., 2002a, and Conti et al., 2002c) represent a form of elastic, reversible, shear band instability.

Indeed, consider the case of a sample which is uniformly stretched, starting from the natural state corresponding to $\mathbf{N} = \mathbf{N}_a = \mathbf{e}_3 \otimes \mathbf{e}_3$, according to the deformation gradient

$$\mathbf{F}(0; \lambda) = \begin{bmatrix} a^{-1/6} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & a^{1/6}/\lambda \end{bmatrix}, \quad (46)$$

with $\lambda \geq a^{-1/6}$. The occurrence of shear-like instabilities can be detected from the stability condition (35), which shows that the state $(\mathbf{F}(0; \lambda), \mathbf{N}_a)$ is unstable for every $\lambda > a^{-1/6}$.

The anisotropic corrections impart to the material a positive shear modulus up to a critical stretch λ_c . At this critical stretch, the modulus for shearing in planes containing \mathbf{n}_r vanishes, and a stripe domain instability with alternating shears becomes the mode of response of lowest energy to further stretching. This scenario is consistent both with the theoretical analyses in Golubović and Lubensky (1989) and Warner and Terentjev (2003), and with the experimental results in Rogez et al. (2006): with the anisotropic corrections, the soft mode of response of the ideally soft limit is latent in the initial configuration, and it is activated at a sufficiently large imposed stretch.

It is interesting to observe that this very transparent picture emerges naturally from a simple analysis of two fully nonlinear anisotropic energies, and from the geometric structure of the associated energy landscape. Figures 4 and 5 provide a concrete representation of such energy landscapes through the level curves of the functions

$$f(\delta, \lambda) := \min_{\theta} \tilde{W}(\mathbf{F}(\delta; \lambda), \mathbf{N}(\theta)) - \frac{3}{2}\mu, \quad (47)$$

$$f_{\beta}(\delta, \lambda) := \min_{\theta} \tilde{W}_{\beta}(\mathbf{F}(\delta; \lambda), \mathbf{N}(\theta)) - \frac{3}{2}\mu(1 + \beta) \quad (48)$$

obtained by evaluating energies (12) and (13) on states described by (25) and (26), and optimizing with respect to θ . The functions $G(\lambda)$ and $G_{\beta}(\lambda)$ used in this Section (and also in Section 8 for the interpretation of the key experimental evidence available on nematic elastomers) give the curvature of the graphs of (47) and (48) along the line $\delta = 0$, and they enable us to identify the material instabilities associated with the non-convexity of energies (12) and (13).

The results discussed above are fully consistent with Ye et al. (2007) and Biggins et al. (2008), where the effects of compositional fluctuations or of the aligning fields arising with the cross-linking process are discussed. We notice in addition that the analysis of the stability of equilibria with $\theta = \pi/2$

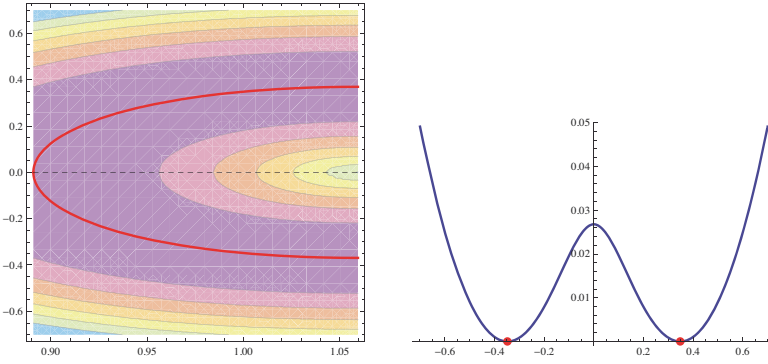


Figure 4. Energy landscape for the (ideally soft) isotropic energy (47) with $a = 2$ and $\mu = 1$. Equally spaced level curves in a plane (λ, δ) (left); graph of the section at $\lambda = 1$ (right). Energy minimizing states are shown by the thick red curve (left) and the red dots (right).

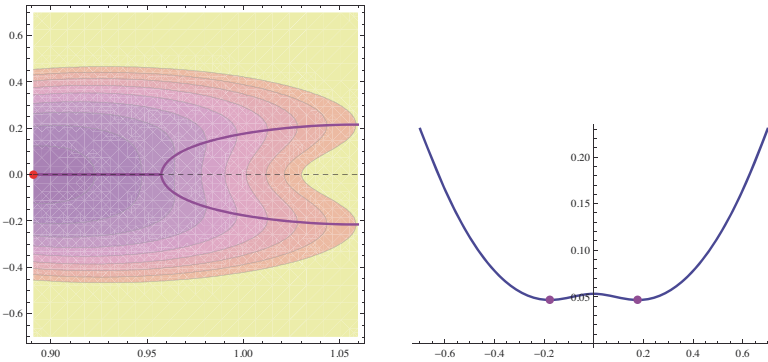


Figure 5. Energy landscape for the anisotropic energy (48) with $a = 2$, $\mu = 1$, and $\beta = 1$. Equally spaced level curves in a plane (λ, δ) (left); graph of the section at $\lambda = 1$ (right). The unique energy minimizing state is shown by the red dot; local minimizers at constant λ are shown by the thick purple curve (left) and the purple dots (right).

in a neighborhood of $\lambda = a^{1/3}$ (the stretch defining the upper limit of the plateau in the ideally soft case – see next section) is completely analogous to the one we have explicitly performed here, leading to similar instabilities and to another critical stretch defined by a vanishing shear modulus. Moreover, while our quantitative analysis is based on some simple concrete energy

expressions, the qualitative picture that emerges is much more general, and it will be shared by a much larger class of energies.

The energy landscapes in Figures 4 and 5 enable us also to unfold the bifurcation occurring at fixed imposed stretch λ , and to anticipate the ensuing post-critical behavior. Indeed, the intersection of a vertical line through $(\lambda, 0)$ with the pitchforks in the graphs identifies two co-operative shears $\pm\delta(\lambda)$, which are kinematically compatible and average to zero if occurring in bands of equal width. With these two opposite shears, we can uniquely associate two symmetric orientations $\pm\theta(\lambda)$ of the nematic director, where $\theta(\lambda)$ is the minimizer in (47) or (47) corresponding to $\mathbf{F}(\delta(\lambda); \lambda)$. These two orientations of the nematic director give rise to the optical contrast observed in the stripe-domain instability. A more complete analysis of this post-bifurcation mode of response, based on co-operative elastic shear banding, will be the object of the next section.

6 Effective Energy: Coarse-graining and Quasi-convexification

We return now to the basic expression (7) for the elastic energy density in the incompressible case. For fixed \mathbf{F} , we minimize with respect to \mathbf{n} to obtain the effective energy

$$W_{\text{eff}}(\mathbf{F}) = \min_{|\mathbf{n}|=1} (W(\mathbf{F}, \mathbf{N}) - \frac{3}{2}\mu). \tag{49}$$

More explicitly,

$$W_{\text{eff}}(\mathbf{F}) = \begin{cases} \frac{1}{2}\mu a^{1/3} (\lambda_1^2(\mathbf{F}) + \lambda_2^2(\mathbf{F}) + a^{-1}\lambda_3^2(\mathbf{F}) - 3a^{-1/3}) & \text{if } \det \mathbf{F} = 1 \\ +\infty & \text{else,} \end{cases} \tag{50}$$

where the $\lambda_i(\mathbf{F})$ are the ordered principal stretches (in particular, $\lambda_3 = \lambda_{max}$). We remark that, if one evaluates (50) on deformation gradients $\mathbf{F}(\delta; \lambda)$ of the form (26), one obtains precisely the graph of Figure 4. In other words, $W_{\text{eff}}(\mathbf{F}(\delta; \lambda)) = f(\delta, \lambda)$, where f is given by (47). Moreover, the \mathbf{n} that achieves the minimum in (49) is the eigenvector \mathbf{n}_{opt} associated with the largest eigenvalue of $\mathbf{F}\mathbf{F}^T$:

$$\mathbf{F}\mathbf{F}^T \mathbf{n}_{opt} = \lambda_{max}^2(\mathbf{F})\mathbf{n}_{opt}. \tag{51}$$

The shear banding instabilities described in the previous section are related to the non-convexity of the energy landscape, as Figure 4 illustrates rather clearly. A useful notion of material stability is the quasiconvexity of

the governing energy density. This is an infinite-dimensional analogue of the patch-test for finite elements. It means that an affine state of deformation \mathbf{F} gives the minimal energy state in a sample if one prescribes at its boundary affine displacement boundary conditions compatible with \mathbf{F} . As discussed in the previous section, (50) cannot be quasiconvex because it can be lowered by development of shear bands.

The quasiconvex envelope of W_{eff}

$$W_{\text{eff}}^{\text{qc}}(\mathbf{F}) = \inf_{\mathbf{y}} \left\{ \frac{1}{|\Omega|} \int_{\Omega} W_{\text{eff}}(\nabla \mathbf{y}(x)) dx : \mathbf{y}(x) = \mathbf{F}x \text{ on } \partial\Omega, \det \nabla \mathbf{y}(x) = 1 \right\} \quad (52)$$

coarse-grains the energetics of the system: it gives the minimum energy needed to produce the macroscopic deformation \mathbf{F} , optimized over all possible admissible microstructures $\mathbf{y}(x)$. The infimum in (52) is taken over all functions \mathbf{y} that are Lipschitz-continuous. Note also that the domain Ω , whose volume we denote by $|\Omega|$, plays here the role of a representative volume element: it can be verified that $W_{\text{eff}}^{\text{qc}}$ does not depend on Ω . The use of $W_{\text{eff}}^{\text{qc}}$ in numerical computations allows one to resolve only the macroscopic length scale, with the (possibly infinitesimal) microscopic scale already accounted for in $W_{\text{eff}}^{\text{qc}}$. Clearly, this approach gives only average information on the fine phase mixtures and focuses on the macroscopic response of the system.

An explicit formula for the quasi-convex envelope of (50) has been derived in DeSimone and Dolzmann (2002). For volume-preserving deformation gradients it reads

$$W_{\text{eff}}^{\text{qc}}(\mathbf{F}) = \begin{cases} 0 & \text{(phase L) if } \lambda_1 \geq a^{-1/6} \\ W_{\text{eff}}(\mathbf{F}) & \text{(phase S) if } a^{-1/2} \lambda_3^2 \lambda_1 > 1 \\ \frac{1}{2} \mu a^{1/3} (\lambda_1^2 + 2a^{-1/2} \lambda_1^{-1} - 3a^{-1/3}) & \text{(phase I) else,} \end{cases} \quad (53)$$

while $W_{\text{eff}}^{\text{qc}}(\mathbf{F}) = +\infty$ if $\det \mathbf{F} \neq 1$. Here the labels L, S, and I refer to the fact that the resulting material response is liquid-like, solid-like, or of an intermediate type; see Figure 6 and the discussion below.

The formula above gives a very precise picture of the macroscopic mechanical response resulting from our model, and of its microscopic origin. There are three regimes in (53), arising from the collective behavior of energetically optimal fine phase mixtures. They represent three different modes of macroscopic mechanical response, corresponding to three different patterns of microscopic decomposition of the macroscopic deformation gradient \mathbf{F} . Phase L describes a liquid-like response (at least within the ideally soft approximation underlying expression (50) for the microscopic energy density; a more realistic semi-soft case is discussed in Conti et al., 2002b). All

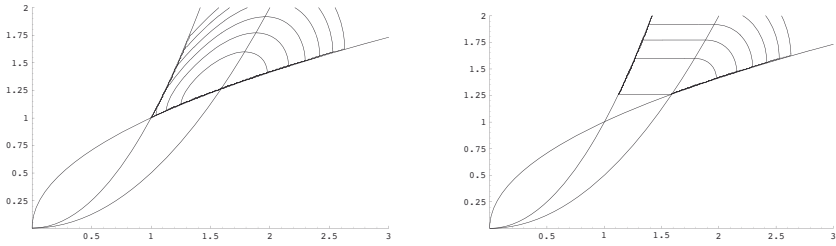


Figure 6. Level curves of the energy W_{eff} given by (50) (left) and of its quasiconvex envelope $W_{\text{eff}}^{\text{qc}}$ given by (53) (right).

gradients falling in this region of the phase diagram, which is the zero level set of W_{qc} , can be sustained at zero internal stress.

To resolve microscopically the whole of phase L (in particular, to resolve the deformation gradient $\mathbf{F} = \text{Id}$) it is necessary to allow for relatively complex microstructures (layers-within-layers). Phase S describes a solid-like response in which fine phase mixtures are ruled out. As a consequence, in this regime the coarse-grained macroscopic energy W_{qc} reproduces the microscopic energy W_{eff} with no changes. Finally, gradients in the intermediate phase I can transmit stresses (unlike phase L) through microstructure formation (unlike phase S). The microscopic patterns required to resolve phase I have a relatively simple geometry (laminates, or simple-layers) Patterns of this kind have been frequently observed experimentally after being first reported in Kundler and Finkelmann (1995). The first attempt to explain them through elastic energy minimization is in Verwey et al. (1996).

The expression (53) for the energy density has been used in Conti et al. (2002a) for the numerical simulation of stretching experiments of sheets of nematic elastomer held between two rigid clamps. The simulations are designed to reproduce the classical experimental setting of Kundler and Finkelmann (1995), where stripe-domain patterns were first observed.

The specimen is a thin sheet of nematic elastomer. We choose a reference frame with axis x_1 parallel to the thickness direction. Moreover, we assume that the specimen is prepared with the director uniformly aligned along x_3 , and is then stretched along x_2 . By reorienting the director from the x_3 to the x_2 direction, the material can accommodate the imposed stretches without storing elastic energy. As it is well known (see, e.g., Warner and Terentjev, 2003), a uniform rotation of the director would induce large shears, which are incompatible with the presence of the clamps. Director reorientation

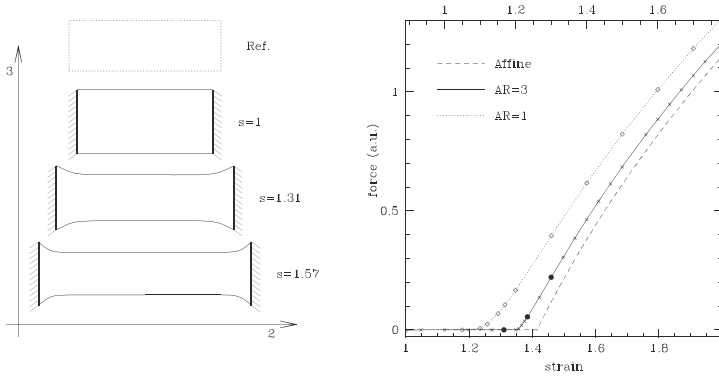


Figure 7. Numerical simulation of stretching experiments on thin sheets of nematic elastomers: geometry (left) and force–stretch diagrams for several aspect ratios AR (right). The panel on the left shows four configurations, namely, reference, initial, and the two at stretches $s=1.31$ and $s=1.57$ for the geometry with $AR=3$. On the corresponding force–stretch curve on the right panel, full dots mark the representative points of configurations shown in Figure 8 (adapted from Conti et al., 2002a).

occurs instead with the development of spatial modulations shaped as bands parallel to the x_2 axis. This is the origin of the striped texture observed in the experiments.

The numerical simulations allow us to analyze the stretching experiments in more detail. If the clamps do not allow lateral contraction, the reorientation of the director towards the direction of the imposed stretch is severely hindered. This constraint is stronger near the clamps, and it decays away from them producing two interesting effects. On the one hand, the induced microstructures are spatially inhomogeneous, with director reorientation occurring more rapidly in the regions far away from the clamps. On the other hand, the stress–strain response shows a marked dependence on the geometry of the sample, with the influence of the clamps becoming less pronounced as the aspect ratio length/width increases. These effects are documented in Figure 7 and Figure 8, which show good qualitative agreement with both the experimental results from the Cavendish Laboratories (Warner and Terentjev, 2003), and with the X-ray scattering measurements (Zubarev et al., 1999).

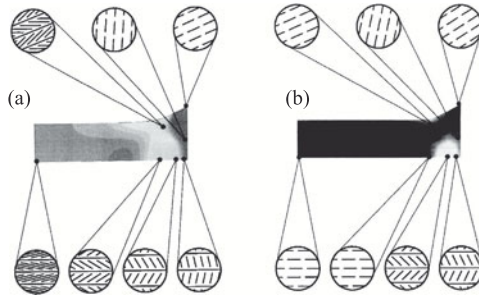


Figure 8. Numerical simulation of stretching experiments on thin sheets of nematic elastomers, based on the coarse-grained energy $W_{\text{eff}}^{\text{qc}}$, at stretches $s=1.31$ (a), and $s=1.38$ (b). Only one-quarter of the sample is shown since the rest of the solution can be obtained by symmetry. The circular insets display energetically optimal microstructures at some selected locations within the sample. The sticks give the local orientation of the principal direction of maximal stretch, i.e., the orientation of the nematic director (adapted from Conti et al., 2002a).

The stripe domain patterns appearing in Figure 8 are all simple laminates, either in phase L or in phase I. Focussing on the point at the center of the sample (the bottom left corner in the plots of the deformed shape), the material is in phase L as long as no force is transmitted at the clamps. The computed deformation gradient is

$$\mathbf{F}_\lambda = \begin{pmatrix} a^{-1/6} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & a^{1/6}/\lambda \end{pmatrix}, \tag{54}$$

with λ varying from $a^{-1/6}$ to $a^{1/3}$. This is resolved by a simple laminate in which the deformation gradient oscillates between the values

$$\mathbf{F}_\lambda^\pm = \begin{pmatrix} a^{-1/6} & 0 & 0 \\ 0 & \lambda & \pm\delta \\ 0 & 0 & a^{1/6}/\lambda \end{pmatrix} \tag{55}$$

in stripes perpendicular to x_3 . The value of $\delta = \delta(\lambda)$ is obtained from $\delta^2 = (a^{2/3} - \lambda^2)(1 - a^{-1/3}\lambda^{-2})$, which ensures that \mathbf{F}_λ^\pm has the characteristic principal stretches giving $W_{\text{eff}}(\mathbf{F}_\lambda^\pm) = 0$. Notice that the kinematic

compatibility condition $\mathbf{F}_\lambda^+ - \mathbf{F}_\lambda^- = \mathbf{a} \otimes \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the reference normal to the stripes and \mathbf{a} is a shear vector, is satisfied with $\mathbf{a} = 2\delta(\lambda)\mathbf{e}_2$ and $\hat{\mathbf{n}} = \mathbf{e}_3$. This guarantees the existence of a continuous map \mathbf{y} such that either $\nabla\mathbf{y}(\mathbf{x}) = \mathbf{F}_\lambda^+$ or $\nabla\mathbf{y}(\mathbf{x}) = \mathbf{F}_\lambda^-$, with $\nabla\mathbf{y}$ constant in layers with normal \mathbf{e}_3 . The deformation patterns given by (55) characterize the systems of shear bands resolving the post-critical behavior of the material following the shear band instability described in the previous sections. Associated with that one finds a modulated pattern $\mathbf{n}_{opt}(\mathbf{F}_\lambda^\pm)$ for the nematic director, where \mathbf{n}_{opt} is given by (51).

Force starts being transmitted through the sample when the deformation gradient in the central point moves to the region I of the phase diagram. The computed deformation gradient is now of the form

$$\mathbf{F}_1(\lambda_1) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1/\lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad (56)$$

where $\lambda_3 > a^{1/3}$ forces $\lambda_1 < a^{-1/6}$. This is resolved by simple laminates similar to the ones above. The deformation gradient oscillates between the values

$$\mathbf{F}_1^\pm(\lambda_1) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1/\lambda_1\lambda_3 \pm \delta \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad (57)$$

in stripes perpendicular to x_3 , and $\delta = \delta(\lambda_1)$ is computed by requiring that the principal stretches be those giving the minimal energy at given λ_1 , namely $(\lambda_1, a^{-1/4}\lambda_1^{-1/2}, a^{1/4}\lambda_1^{-1/2})$; see Conti et al. (2002a). The associated nematic texture is again obtained from $\mathbf{n}_{opt}(\mathbf{F}_1^\pm(\lambda_1))$, with \mathbf{n}_{opt} given by (51).

A relaxation result providing the small strain analog of (53) has been obtained in Cesana (2010). Anisotropic corrections leading to more realistic force–stretch curves in which the soft plateau occurs at small but finite levels of force are discussed in Conti et al. (2002b).

7 Dynamics under an Applied Electric Field

In order to move the first steps towards modeling the dynamic response of nematic elastomers to applied electric fields, we follow DeSimone et al. (2007) and use a simpler, geometrically linear theory. This small-strain approximation has been used to study the equilibrium response to applied electric fields in Cesana and DeSimone (2009). The same approach has been used quite successfully in Fukunaga et al. (2008) to reproduce the exper-

imentally measured dynamic response of nematic gels to applied electric fields.

We consider a sample of a nematic gel occupying a region \mathcal{B} inside a cell Ω . The part $\Omega \setminus \mathcal{B}$ of the cell is occupied by an isotropic dielectric (typically, silicon oil). We denote by \mathbf{u} and \mathbf{n} the displacement and the nematic director in \mathcal{B} , and by φ the electric potential in Ω . As usual, \mathbf{n} is parametrized through a rotation field \mathbf{R} such that $\mathbf{n} = \mathbf{R}\mathbf{n}_r$, where \mathbf{n}_r is a (fixed) reference orientation.

The governing equations of our model are Gauss' law for an anisotropic dielectric, the standard balance of linear momentum for a viscoelastic solid, and an evolution equation modeling a viscous-like dynamics for the director rotation. They read as

$$\operatorname{div}(\mathbf{d}) = 0 \tag{58}$$

in Ω , and

$$\operatorname{div}(\mathbf{S}) = 0, \tag{59}$$

$$\begin{aligned} \eta_n(\dot{\mathbf{R}}\mathbf{R}^\top - \mathbf{W}_{\dot{\mathbf{u}}}) &= [\mathbf{S}, \mathbf{E}_0] + \frac{1}{2} \varepsilon_o \varepsilon_a [\nabla\varphi \otimes \nabla\varphi, \mathbf{n} \otimes \mathbf{n}] \\ &+ \operatorname{skw}(\operatorname{div}(k_F \nabla\mathbf{n}) \otimes \mathbf{n}) \end{aligned} \tag{60}$$

in \mathcal{B} . They are supplemented by suitable initial and boundary conditions, adapted to the specific experimental set-up one is trying to model. Here, in (58), the electric displacement \mathbf{d} is given by

$$\mathbf{d} = -\varepsilon_o \mathbb{D}\nabla\varphi, \tag{61}$$

with

$$\mathbb{D}\nabla\varphi = \begin{cases} \varepsilon_\perp \nabla\varphi + \varepsilon_a (\nabla\varphi \cdot \mathbf{n}) \mathbf{n} & \text{in } \mathcal{B} \\ \varepsilon_c \nabla\varphi & \text{in } \Omega \setminus \mathcal{B}, \end{cases} \tag{62}$$

where $\varepsilon_o > 0$ is the free space permittivity, ε_\parallel and ε_\perp are the relative permittivities of the gel in the directions parallel and perpendicular to \mathbf{n} , $\varepsilon_a = \varepsilon_\parallel - \varepsilon_\perp$ is the dielectric anisotropy, and ε_c is the relative permittivity of the isotropic dielectric occupying the region $\Omega \setminus \mathcal{B}$.

Moreover, in (59) and (60), $\mathbf{E}_0 = \mathbf{E}_0(\mathbf{n})$ is the spontaneous strain associated with the isotropic-to-nematic transformation

$$\mathbf{E}_0(\mathbf{n}) = \frac{3}{2} \gamma (\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} \mathbf{I}), \tag{63}$$

while the stress \mathbf{S} is given by

$$\mathbf{S} = \mathbb{C}(\mathbf{E}_{\mathbf{u}} - \mathbf{E}_0) + \eta_g \mathbf{E}_{\dot{\mathbf{u}}}, \tag{64}$$

where

$$\mathbf{E}_{\mathbf{u}} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top), \quad \mathbf{E}_{\dot{\mathbf{u}}} = \frac{1}{2} (\nabla \dot{\mathbf{u}} + (\nabla \dot{\mathbf{u}})^\top), \quad (65)$$

\mathbb{C} is the (positive definite) tensor of elastic moduli, and $\eta_g > 0$ is the viscosity of the gel. In principle, one would like to assume for $\mathbb{C} = \mathbb{C}(\mathbf{n})$ the symmetry of a transversely isotropic solid with distinguished axis \mathbf{n} , so that the Cartesian components of \mathbb{C} are all described in terms of five independent scalars. Since a detailed experimental characterization of these parameters is not available, whenever quantitative information on them is needed for our analysis, we make the simplifying assumptions $C_{33} = C_{11}$, $C_{12} = C_{13}$, and $C_{66} = C_{44} = (C_{11} - C_{12})/2$ (see, e.g., Ikeda, 1990, Ch. 3); here we are using Voigt's notation for the components of \mathbb{C} , and assuming that \mathbf{n} is directed along the third coordinate axis. In this case, \mathbb{C} becomes isotropic, denoted by \mathbb{C}_{iso} , and the values of the Young modulus Y and the Poisson ratio ν suffice to fully characterize \mathbb{C}_{iso} .

Finally, in (60), $\eta_n > 0$ denotes a parameter describing the rotational viscosity of the director, $\dot{\mathbf{R}}$ denotes the time rate of \mathbf{R} , $\mathbf{W}\dot{\mathbf{u}}$ is the skew-symmetric part of the velocity gradient $\nabla \dot{\mathbf{u}}$, k_F is the Frank constant (giving the strength of curvature elasticity in the one-constant approximation adopted here), $\text{skw}(\mathbf{A}) = (\mathbf{A} - \mathbf{A}^\top)/2$ denotes the skew-symmetric part of the matrix \mathbf{A} , and $[\mathbf{A}, \mathbf{B}] = \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}$ is the commutator of the matrices \mathbf{A} and \mathbf{B} .

The model above is derived as follows. We introduce the total energy functional

$$\begin{aligned} \mathcal{E} = & \frac{1}{2} \int_{\mathcal{B}} (k_F |\nabla \mathbf{n}|^2 + \mathbb{C} (\mathbf{E}_{\mathbf{u}} - \mathbf{E}_0) \cdot (\mathbf{E}_{\mathbf{u}} - \mathbf{E}_0)) \\ & - \frac{1}{2} \int_{\Omega} (\varepsilon_o (\mathbb{D} \nabla \varphi) \cdot \nabla \varphi) - \int_{\partial_s \mathcal{B}} (\mathbf{s}_{ext} \cdot \mathbf{u}), \end{aligned} \quad (66)$$

where the first integrand contains Frank's curvature energy and the elastic energy, the second one is the total electric energy including the energy needed to maintain the constant voltage difference V across the cell (see de Gennes, 1993, eq. (3.67), and Stewart, 2004, eq. (2.86)), and the third one is the potential energy of the loading device exerting an external force per unit area, denoted by \mathbf{s}_{ext} , on the loaded part $\partial_s \mathcal{B}$ of the boundary of \mathcal{B} . When $\mathbb{C} = \mathbb{C}_{iso}$ the elastic energy term in (66) reduces to Φ given by (21).

Equations (58) and (59) are standard. The first one arises by assuming instantaneous relaxation to equilibrium of the electric potential and a viscoelastic dynamics for the elastic displacement

$$0 = \frac{\delta \mathcal{E}}{\delta \varphi}, \quad \frac{\delta \mathcal{D}}{\delta \dot{\mathbf{u}}} = -\frac{\delta \mathcal{E}}{\delta \mathbf{u}}, \quad (67)$$

where the operator δ is used to denote the variational derivatives of the energy functional \mathcal{E} with respect to φ and \mathbf{u} , and the variational derivative of the viscous dissipation \mathcal{D}

$$\mathcal{D} = \eta_m |\dot{\mathbf{R}}\mathbf{R}^\top - \mathbf{W}_{\dot{\mathbf{u}}}|^2 + \eta_g |\mathbf{E}_{\dot{\mathbf{u}}}|^2 \tag{68}$$

with respect to $\dot{\mathbf{u}}$. Straightforward manipulations show that $(67)_2$ is equivalent to (59) supplemented by the constitutive assumptions (63)–(64). Similarly, (60) follows from

$$\frac{\delta \mathcal{D}}{\delta \dot{\mathbf{n}}} = -\frac{\delta \mathcal{E}}{\delta \mathbf{n}}, \tag{69}$$

(notice that $\dot{\mathbf{R}}\mathbf{R}^\top \mathbf{n} = \dot{\mathbf{n}} = \boldsymbol{\omega} \times \mathbf{n}$, where $\boldsymbol{\omega}$ is the director angular velocity) which states that the dynamics is such that the “viscous” dissipation rate accompanying the director evolution balances exactly the energy release rate driving the process.

The structure of equation (60) reveals in a rather transparent way the conditions such that a spatially uniform director field \mathbf{n} be in equilibrium. In particular, the condition $[\mathbf{S}, \mathbf{E}_0] = 0$ is satisfied if and only if the stress \mathbf{S} and the spontaneous distortion $\mathbf{E}_0(\mathbf{n})$ have the same principal directions (see Gurtin, 1981, p. 12).

The model described above has been used in Fukunaga et al. (2008) to understand experiments performed on a free-standing film in which an applied field perpendicular to the initial orientation of the nematic director is switched on suddenly, maintained until the system reaches equilibrium, and then switched off. The comparison between the predicted relaxation times of \mathbf{n} and \mathbf{u} following switch on and switch off and the experimental measurements is in Figure 9. In order for this agreement to be possible, we need to use for the elastic energy the anisotropic expression $\tilde{\Phi}_\alpha$ (24) instead of either $\tilde{\Phi}(\mathbf{E}, \mathbf{N}) = \mathbb{C}_{iso}(\mathbf{E} - \mathbf{E}_0) \cdot (\mathbf{E} - \mathbf{E}_0)/2$ or $\tilde{\Phi}_\beta(\mathbf{E}, \mathbf{N}) = \tilde{\Phi}(\mathbf{E}, \mathbf{N}) + \mu_\beta |\mathbf{E}_d - \mathbf{E}_0(\mathbf{N}_a)|^2$. In spite of the anisotropic correction, this last expression does not provide a spring-back mechanism pushing the director back to the initial orientation \mathbf{n}_a when the electric field is switched off.

The dynamic model can be used also in the absence of applied electric fields to investigate rate effects in the force-stretch curves, and whether the response curves obtained in Section 6 by global energy minimization are also dynamically accessible in the limit of vanishingly small loading rates (DeSimone and Teresi, 2010). Interestingly, one may study in this way the dynamic pathways originating from an unstable state and leading to a new stable state. A stretching experiment giving a dynamic analogue of the one shown in Figure 7 is presented in Figure 10. A snapshot of dynamic simulations leading to formation of stripe domains is shown in Figure 11.

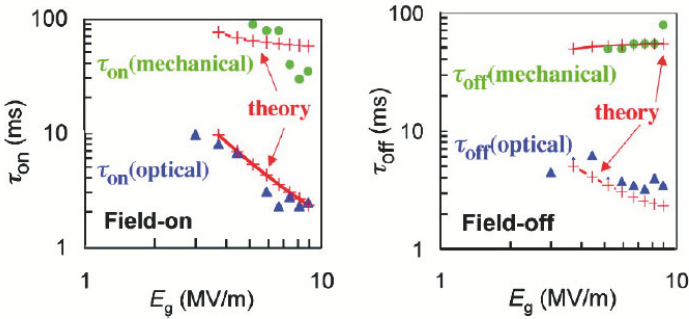


Figure 9. Characteristic relaxation times of \mathbf{n} (optical) and \mathbf{u} (mechanical) following to switch-on and switch-off of an electric field. Adapted from Fukunaga et al. (2008).

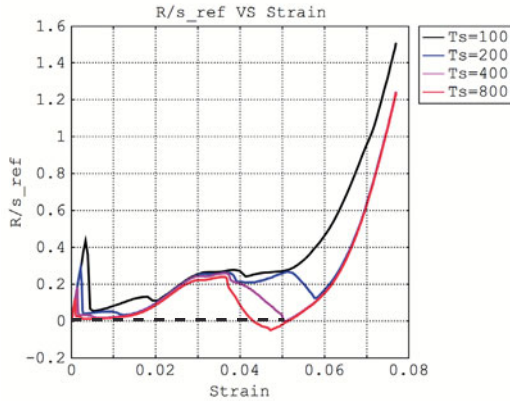


Figure 10. Dynamic force-strain response under purely mechanical stretching. The dashed line gives the response curve corresponding to global energy minimizers. Adapted from DeSimone and Teresi (2010).

8 Comparison with Key Experimental Results

We now compare the predictions of the various models discussed above with the experimental evidence coming from three benchmark experiments: purely mechanical stretching and shearing, and electric-field-induced rotation of the nematic director in a free-standing film.

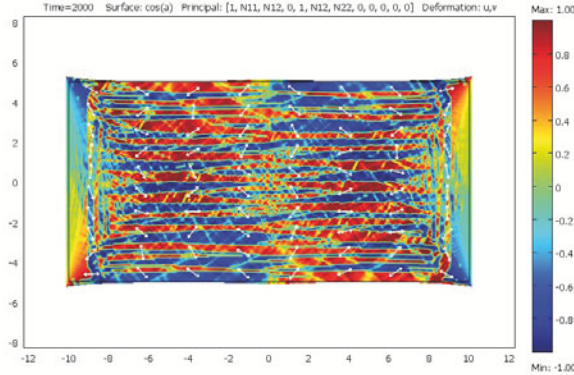


Figure 11. A snapshot from numerical simulations of dynamic stretching experiments at slow stretching rates, leading to formation of stripe domains. Adapted from DeSimone and Teresi (2010).

8.1 Stretch

Consider a stretching experiment starting from the natural state corresponding to $\mathbf{N} = \mathbf{N}_r = \mathbf{e}_3 \otimes \mathbf{e}_3$ and described by the deformation gradient

$$\mathbf{F}(0; \lambda) = \begin{bmatrix} a^{-1/6} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & a^{1/6}/\lambda \end{bmatrix}, \tag{70}$$

with $\lambda \geq a^{-1/6}$. Here \mathbf{n}_r denotes an arbitrary reference orientation when dealing with the isotropic material; it will be chosen as $\mathbf{n}_r = \mathbf{n}_a$ when dealing with one of the anisotropic ones. The deformation described by (70) is a plane-strain extension or, in Treloar’s terminology, a pure shear. As long as the state $(\mathbf{F}(0; \lambda), \mathbf{N} = \mathbf{N}_r)$ is a stable equilibrium, the stress response can be obtained from $W(\mathbf{F}(0; \lambda), \mathbf{N}_r)$ by differentiating with respect to λ . This leads to

$$\sigma(\lambda) = \mu \left(a^{1/3} \lambda - \frac{1}{a^{1/3} \lambda^3} \right), \tag{71}$$

where σ denotes the normal stress difference $S_{22} - S_{33}$ measured in terms of nominal (or first Piola-Kirchhoff) stresses.

As already discussed in the previous sections, the isotropic energy W leads to a stripe-domain instability: already at $\lambda = a^{-1/6}$, the homogeneous state $(\mathbf{F}(0; \lambda), \mathbf{N} = \mathbf{N}_r)$ loses stability in favor of nonhomogeneous

patterns with alternating shears having the same average deformation as (70) but lower energies than the uniformly deformed state (70). These alternating shears play a crucial role in the calculation of the coarse-grained energy (the quasiconvex envelope) performed in Section 6. The analogy between this mode of response and mechanical twinning in materials exhibiting martensitic transformations has been first pointed out in DeSimone (1999). Formula (71) does not apply and, thanks to the development of alternating shear bands of the form (55), the system can accommodate any stretch $\lambda \in [a^{-1/6}, a^{1/3}]$ at zero stress $\sigma(\lambda) \equiv 0$, thus exhibiting an ideally soft response.

Applying a similar argument to energy W_β we obtain instead

$$\sigma^\beta(\lambda) = \mu(1 + \beta) \left(a^{1/3} \lambda - \frac{1}{a^{1/3} \lambda^3} \right), \quad \lambda \in [a^{-1/6}, \lambda_c^\beta]. \quad (72)$$

This implies that the material will show a hard response up to the critical stretch λ_c^β . Then a softer mode of response, accompanied by the emergence of non-homogeneous deformation patterns relying on alternating shears of the form $\mathbf{F}(\pm\delta; \lambda)$ given by (26), becomes energetically advantageous and dynamically accessible. The value λ_c^β is clearly an upper bound for the onset of the instability because, in a real system, imperfections may trigger the instability well before λ_c^β is reached. Applying the same argument to \tilde{W}_α we obtain exactly the same scenario of a hard response only up to a threshold given by

$$\sigma^\alpha(\lambda) = \mu \left(a^{1/3} \lambda - \frac{1}{a^{1/3} \lambda^3} \right), \quad \lambda \in [a^{-1/6}, \lambda_c^\alpha]. \quad (73)$$

Estimates of the critical stretches λ_c^β , λ_c^α for meaningful values of the material parameters are given in (39) and (45). For stretches exceeding the critical value for the stability of a homogeneously stretched state, numerical simulations are needed in order to resolve the complex, non-homogeneous response.

8.2 Shear

We move now to simple shear experiments. Starting from the natural state corresponding to $\mathbf{N} = \mathbf{N}_r = \mathbf{e}_3 \otimes \mathbf{e}_3$, we consider simple shears of magnitude proportional to δ in a plane containing \mathbf{n}_r (see Figure 12)

$$\mathbf{F}(\delta; a^{-1/6}) = \begin{bmatrix} a^{-1/6} & 0 & 0 \\ 0 & a^{-1/6} & \delta \\ 0 & 0 & a^{1/3} \end{bmatrix}, \quad (74)$$

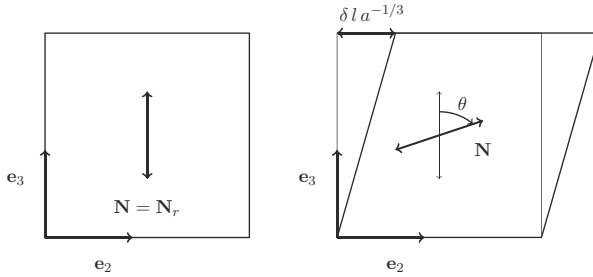


Figure 12. Shear experiment corresponding to (74) on a sample of initial size $h \times l \times l$.

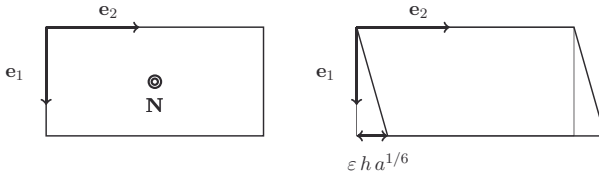


Figure 13. Shear experiment corresponding to (75) on a sample of initial size $h \times l \times l$.

and simple shears of magnitude proportional to ε in a plane perpendicular to \mathbf{n}_r (see Figure 13)

$$\tilde{\mathbf{F}}(\varepsilon; a^{-1/6}) = \begin{bmatrix} a^{-1/6} & 0 & 0 \\ \varepsilon & a^{-1/6} & 0 \\ 0 & 0 & a^{1/3} \end{bmatrix}. \tag{75}$$

In this second case, it turns out that $\mathbf{N} = \mathbf{N}_r$ is always an equilibrium and we obtain energy expressions of the form

$$f(\varepsilon, 0; \lambda) = f(0, 0; \lambda) + \frac{1}{2} \tilde{G} \varepsilon^2, \tag{76}$$

where

$$\tilde{G} = \mu a^{1/3}, \quad \tilde{G}^\beta = \mu(1 + \beta)a^{1/3}, \quad \tilde{G}^\alpha = \mu a^{1/3}. \tag{77}$$

The moduli for shears in a plane containing \mathbf{n}_r (recall that $\mathbf{n}_r = \mathbf{n}_a$ in anisotropic cases) follow from (33), (37), (40), and are given by

$$G = G(a^{-1/6}) = 0, \quad G^\beta = G^\beta(a^{-1/6}) = \frac{\beta \mu}{a^{2/3}}, \tag{78}$$

$$G^\alpha = G^\alpha(a^{-1/6}) = \alpha\mu \frac{1}{a^{2/3}(1 + a^2 - 2a + a\alpha)}. \quad (79)$$

From (77)₂ and (78)₂ it follows that

$$G^\beta = \frac{1}{a} \left(\frac{\beta}{1 + \beta} \right) \tilde{G}^\beta, \quad (80)$$

so that, as β increases from 0 to $+\infty$, G^β increases from 0 to \tilde{G}^β/a . Using a typical value $a = 2$ for a , we obtain that G^β can be as large as half of \tilde{G}^β , provided that β is large enough. From (77)₃ and (79) it follows that

$$G^\alpha = \frac{\alpha}{a + a^3 - 2a^2 + a^2\alpha} \tilde{G}^\alpha, \quad (81)$$

so that, as α increases from 0 to $+\infty$, G^α increases from 0 to \tilde{G}^α/a^2 . For $\alpha = 1$ and $a = 2$ we deduce from (81) that $G^\alpha = \frac{1}{6}\tilde{G}^\alpha$.

These results show that the relatively large moduli reported in Rogez et al. (2006) for shears in planes containing \mathbf{n}_a are not incompatible with the theoretical estimates, provided that the anisotropy parameters α and β have large enough values.

8.3 Electric Field Applied to a Free-standing Film

The experiments reported in Fukunaga et al. (2008) provide another important conceptual benchmark. By applying an electric field to a free-standing film of a swollen nematic elastomer, in such a way that the electric field drives the director away from its initial orientation $\mathbf{n}_r = \mathbf{n}_a$, one obtains a very clean set-up where many features of the mechanics of nematic elastomers can be addressed unambiguously. The experiments confirm the power of formula (1) in reading correctly the coupling between mechanical deformations and nematic order, as shown by the good match between birefringence and strain at steady state as functions of the applied voltage; see Figure 14.

Moreover, the experiments show that a finite critical field needs to be overcome in order to trigger director rotation, and that the director springs back to the initial orientation when the electric field is removed. Both these phenomena are a direct manifestation of anisotropy. Interestingly, of the two proposed anisotropic formulas, only (24) seems capable of capturing spring-back while, with (23), the spring-back mechanism is suppressed.

8.4 Discussion

Our analysis shows that the three experimental findings:

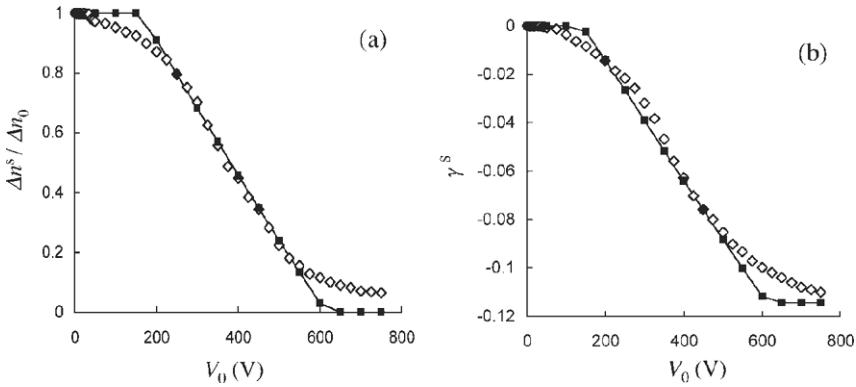


Figure 14. Birefringence (a) and strain (b) at steady state as functions of the applied voltage. Birefringence gives a direct measurement of \mathbf{n} and the correlation between the two curves is precisely the one implied by eq. (23). Adapted from Fukunaga et al. (2008).

- existence of a finite threshold before the emergence of a softer mode of response to stretching,
- absence of a vanishingly small shear modulus in simple shear experiments starting from the natural state corresponding to the director orientation at cross-linking,
- existence of a finite threshold in electric-field induced rotation of the director in free-standing nematic gels,

are all related manifestations of the anisotropy imprinted in the material by memory of the cross-linking state, where $\mathbf{N} = \mathbf{n}_a \otimes \mathbf{n}_a$. Simple anisotropic corrections to the basic trace formula (7), which represents their isotropic, or ideally soft limit, are able to reproduce (at least qualitatively) the available experimental evidence, and hence “explain” it.

We close by emphasizing again that the model anisotropic energies discussed here should not be considered as immediate tools for the faithful reproduction of the experimentally measured response of any specific sample. They are conceptual models. But, as a wise man once said, nothing is more practical than a good theory.

Appendix: Alignment Energies

In this appendix we discuss some examples of alignment energies, namely energies whose minimization enforces alignment with a given vector \mathbf{n}_0 or a

given tensor \mathbf{L} . In parametrizing the set of unit vectors it will be useful to remember that an arbitrary unit vector \mathbf{n} can be represented through the action of a rotation $\mathbf{R} \in \mathbb{R}ot$ acting on a fixed reference unit vector \mathbf{n}_0 (for which one can take, e.g., one of the unit vectors of the canonical basis).

Let \mathbf{n}_0 , with $|\mathbf{n}_0| = 1$, be a given unit vector, let \mathbf{n} be an arbitrary unit vector, and consider the energy density

$$f(\mathbf{n}) = -\frac{1}{2}k\mathbf{n} \cdot \mathbf{n}_0 = -\frac{1}{2}k \cos^2 \theta, \quad k > 0, \tag{82}$$

where θ is the smallest angle between $\pm\mathbf{n}$ and $\pm\mathbf{n}_0$. Since $\cos^2 \theta \leq 1$, we have that $f \geq -\frac{1}{2}k$ with equality achieved only by $\mathbf{n} = \pm\mathbf{n}_0$. An important example is provided by the electrostatic energy density of an anisotropic dielectric in the case of positive dielectric anisotropy $\varepsilon_a > 0$. Indeed, the electrostatic energy density reads

$$f_{ele}(\mathbf{n}) = -\frac{1}{2}\varepsilon_o|\mathbf{E}|^2 (\varepsilon_{\perp} + \varepsilon_a(\mathbf{n} \cdot \mathbf{n}_0)), \quad \mathbf{n}_0 = \mathbf{E}/|\mathbf{E}|, \tag{83}$$

where \mathbf{E} is the electric field and ε_o , ε_{\perp} , and ε_a are dielectric constants. For $\varepsilon_a > 0$, $f_{ele}(\mathbf{n})$ is minimized by $\mathbf{n} = \pm\mathbf{E}/|\mathbf{E}|$. This shows that energy (83) enforces a *quadrupolar* coupling between director \mathbf{n} and electric field \mathbf{E} .

We now move to energies encoding the alignment effect on the state of deformation \mathbf{F} due to a spontaneous or an externally imposed uniaxial tensor field \mathbf{L} . The typical example is

$$\mathbf{L} = \mathbf{R}\mathbf{L}_r\mathbf{R}^T, \quad \mathbf{L}_r = \mathbf{V}_r^2 = a^{2/3}\mathbf{N}_r + a^{-2/6}(\mathbf{I} - \mathbf{N}_r), \tag{84}$$

where $\mathbf{N}_r = \mathbf{n}_r \otimes \mathbf{n}_r$ and \mathbf{n}_r is a fixed reference unit vector. We collect the relevant material in the following proposition.

Proposition 1 *Let \mathbf{B} and \mathbf{L} be in the set of symmetric and positive definite $n \times n$ matrices with determinant equal to d , denoted by \mathbb{P}_d , and consider the scalar function*

$$f(\mathbf{B}, \mathbf{L}) := \mathbf{B} \cdot \mathbf{L}^{-1} = \text{tr}(\mathbf{B}\mathbf{L}^{-1}).$$

Then, for every \mathbf{B} and \mathbf{L} in \mathbb{P}_d ,

$$f(\mathbf{B}, \mathbf{L}) \geq n, \quad \text{with equality only if } \mathbf{B} = \mathbf{L}, \tag{85}$$

so that

$$\min_{\mathbf{B} \in \mathbb{P}_d} f(\mathbf{B}, \mathbf{L}) = f(\mathbf{B}_0(\mathbf{L}), \mathbf{L}) = n, \quad \text{where } \mathbf{B}_0(\mathbf{L}) = \mathbf{L}. \tag{86}$$

Assume further that \mathbf{L} is of the form $\mathbf{L} = \mathbf{L}(\mathbf{R}) = \mathbf{R}\mathbf{L}_r\mathbf{R}^T$, where $\mathbf{R} \in \mathbb{R}ot$ is an arbitrary rotation and \mathbf{L}_r is (a constant matrix) of the form

$$\mathbf{L}_r = a^{2/n}\mathbf{n}_r \otimes \mathbf{n}_r + a^{-2/[(n-1)n]}(\mathbf{I} - \mathbf{n}_r \otimes \mathbf{n}_r) \tag{87}$$

with $a > 1$, \mathbf{n}_r a fixed unit vector, and \mathbf{I} the identity. Then

$$\begin{aligned}
 f_{\text{opt}}(\mathbf{B}) &:= \min_{\mathbf{R} \in \text{Rot}} f(\mathbf{B}, \mathbf{R}\mathbf{L}_r\mathbf{R}^T) = f(\mathbf{B}, \mathbf{R}_0(\mathbf{B})\mathbf{L}_r\mathbf{R}_0^T(\mathbf{B})) = \\
 &= a^{2/[(n-1)n]} \left[\text{tr}(\mathbf{B}) - (1 - a^{-2/(n-1)})\lambda_{\max}^2(\mathbf{B}) \right], \tag{88}
 \end{aligned}$$

where the minimizer $\mathbf{R}_0(\mathbf{B})$ is a rotation that maps \mathbf{n}_r onto the eigenvector of \mathbf{B} corresponding to its largest eigenvalue $\lambda_{\max}^2(\mathbf{B})$. Finally,

$$\min_{\mathbf{B} \in \mathbb{P}_1} f_{\text{opt}}(\mathbf{B}) = n, \tag{89}$$

attained by any matrix $\mathbf{B} \in \mathbb{P}_1$ whose largest eigenvalue is $\lambda_{\max}^2(\mathbf{B}) = a^{2/n}$ and whose other eigenvalues are all equal to $a^{-2/[(n-1)n]}$.

Proof. Writing \mathbf{B} and \mathbf{L}^{-1} in spectral form we have

$$\begin{aligned}
 \mathbf{B} \cdot \mathbf{L}^{-1} &= \sum_{i=1}^n \lambda_i^2(\mathbf{B})\mathbf{b}_i \otimes \mathbf{b}_i \cdot \sum_{j=1}^n \lambda_j^2(\mathbf{L}^{-1})\mathbf{l}_j \otimes \mathbf{l}_j = \\
 &= \sum_{i,j=1}^n \lambda_i^2(\mathbf{B})\lambda_j^2(\mathbf{L}^{-1})(\mathbf{b}_i \cdot \mathbf{l}_j)^2 \geq \sum_{i=1}^n \lambda_i^2(\mathbf{B})\lambda_i^2(\mathbf{L}^{-1})(\mathbf{b}_i \cdot \mathbf{l}_i)^2
 \end{aligned}$$

with equality holding only if $(\mathbf{b}_i \cdot \mathbf{l}_j)^2 = 0$ for $i \neq j$, i.e. only if \mathbf{B} and \mathbf{L}^{-1} share their eigenspaces, in which case they commute. Since we want to minimize $\mathbf{B} \cdot \mathbf{L}^{-1}$, we restrict attention to this case in the rest of the proof.

Let $\mathbf{A} := \mathbf{B}\mathbf{L}^{-1}$. Since $\mathbf{B}, \mathbf{L} \in \mathbb{P}_d$ and $\mathbf{B}\mathbf{L}^{-1} = \mathbf{L}^{-1}\mathbf{B}$, then $\mathbf{A} \in \mathbb{P}_1$. Denoting by $\lambda_i^2(\mathbf{A})$ its eigenvalues, and using the well-known inequality between arithmetic and geometric means, we have

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i^2(\mathbf{A}) \geq n \left(\prod_{i=1}^n \lambda_i^2(\mathbf{A}) \right)^{1/n} = n (\det \mathbf{A})^{1/n} = n \tag{90}$$

where the inequality is always strict unless $\lambda_i^2(\mathbf{A}) = 1$ for all i , or $\mathbf{A} = \mathbf{I}$. This proves (85) and hence (86).

Observe now that

$$\mathbf{L}^{-1}(\mathbf{R}) = \mathbf{R}\mathbf{L}_r^{-1}\mathbf{R}^T, \quad \mathbf{L}_r^{-1} = a^{2/[(n-1)n]} \left[\mathbf{I} - (1 - a^{-2/(n-1)})\mathbf{n}_r \otimes \mathbf{n}_r \right], \tag{91}$$

and therefore

$$\mathbf{B} \cdot \mathbf{L}^{-1}(\mathbf{R}) = a^{2/[(n-1)n]} \left[\text{tr}(\mathbf{B}) - (1 - a^{-2/(n-1)})\mathbf{B}\mathbf{R}\mathbf{n}_r \cdot \mathbf{R}\mathbf{n}_r \right]. \tag{92}$$

Since $1 - a^{-2/(n-1)} > 0$, this is minimized when $\mathbf{B}\mathbf{R}\mathbf{n}_r \cdot \mathbf{R}\mathbf{n}_r$ is maximal, i.e. when \mathbf{R} maps \mathbf{n}_r onto the eigenvector corresponding to the maximal eigenvalue $\lambda_{\max}^2(\mathbf{B})$ of \mathbf{B} . This establishes (88).

Finally, (89) follows by exchanging the order of minimization in \mathbf{B} and \mathbf{L} , in view of (86). We also give a more direct proof, which is instructive. To this end, we order the eigenvalues of \mathbf{B} so that $\lambda_n^2(\mathbf{B}) = \lambda_{\max}^2(\mathbf{B})$. Using again the inequality between arithmetic and geometric means, we have

$$\begin{aligned} \operatorname{tr}(\mathbf{B}) - (1 - a^{-2/(n-1)})\lambda_{\max}^2(\mathbf{B}) &= a^{-2/(n-1)}\lambda_{\max}^2(\mathbf{B}) + \sum_{i=1}^{n-1} \lambda_i^2(\mathbf{B}) \\ &\geq n \left(a^{-2/(n-1)} \prod_{i=1}^n \lambda_i^2(\mathbf{B}) \right)^{1/n} = na^{-2/[(n-1)n]}, \end{aligned}$$

with equality only if $a^{-2/(n-1)}\lambda_{\max}^2(\mathbf{B}) = \lambda_i^2(\mathbf{B})$, $i = 1, \dots, n-1$. Since $1 = \det \mathbf{B} = \lambda_{\max}^2(\mathbf{B})(\lambda_i^2(\mathbf{B}))^{(n-1)}$, this is possible if and only if $\lambda_{\max}^2(\mathbf{B}) = a^{1/n}$ and $\lambda_i^2(\mathbf{B}) = a^{-2/[(n-1)n]}$ for $i = 1, \dots, n-1$. This establishes (89) and completes the proof. \square

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