Transmission-Line Properties of a Strip on a Dielectric Sheet on a Plane

HAROLD A. WHEELER, FELLOW, IEEE

***Abstract*—The subject is a strip line formed of a strip and a parallel ground plane separated by a dielectric sheet (commonly termed "microstrip"). Building on the author's earlier papers [1], [2], all the significant properties are formulated in explicit form for practical applications. This may mean synthesis and/or analysis. Each formula is a close approximation for all shape ratios, obtained by a gradual transition between theoretical forms for the extremes of narrow and wide strips. The effect of thickness is formulated to a second-order approximation. Then the result is subjected to numerical differentiation for simple evaluation of the magnetic-loss power factor from the skin depth.**

The transition formulas are tested against derived formulas for overlapping narrow and wide ranges of shape. Some of these formulas are restated from the earlier derivations and others are derived herein. The latter include the second-order approximation for a narrow thin strip, and a close approximation for a narrow or wide square cross section in comparison with a circular cross section.

Graphs are given for practical purposes, showing the wave resistance and magnetic loss for a wide range of shape and dielectric. For numerical reading, the formulas are suited for programming on a digital pocket calculator.

I. Introduction

ONE FORM of strip line is naturally suited for the simplest fabrication in a printed circuit. It is the familiar type made of a dielectric sheet with a shield-plane conductor bonded on the bottom side and a pattern of strip lines on the top side.

The purpose of this paper is to present some improved formulas and graphs, including not only the wave resistance but also the losses. The effect of strip thickness is simply formulated to enable the evaluation of magnetic loss.

In the vernacular, this type of line is termed "microstrip," a term which is avoided in this scientific article because it is commonly used without a clear definition and is not self-descriptive. Apparently it was intended to be a short designation for "microwave strip line." The "microwave" description is ambiguous and only partially relevant. Furthermore it does not distinguish from the "sandwich" form of a microwave strip line.

Here also the descriptive term "wave resistance" is used in preference to the nondescriptive term "characteristic impedance."

The subject strip line may be described as half-shielded, by the ground plane on one side, as distinguished from the sandwich type, which is fully shielded, by ground planes on both sides. The half-shielding is adequate for some practical purposes, because the external field is relatively weak and does decay with distance.

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The author is with the Hazeltine Corporation, Greenlawn, NY 11740.

A peculiarity of the half-shielded line is the mixture of two different dielectrics. One is the material of the sheet between the strip and plane. The other is the air above the sheet. The simple rules of conformal mapping are restricted to a uniform dielectric or to some discrete boundaries that are different from the subject configuration. Various other approaches have been directed to this problem.

The first close approximation for this strip line with mixed dielectric was published by the author in 1965 [2]. It is based on some rigorous derivations for a thin strip by conformal mapping. These are supplemented by some logical concepts for interpolation between the extremes of dielectric. The uncertainties of interpolation are small enough to meet design requirements within practical tolerances. The result is a collection of formulas and charts which are complete for the wave resistance of a thin strip.

The loss power factor (PF $= 1/Q$) in a strip line has components of electric loss in the dielectric and magnetic loss in the conductor boundaries. These were not treated in the early paper but have been addressed by some other authors in the meantime.

In the frequency range where a strip line may have a length comparable with the wavelength, the magnetic loss is usually the dominant component. It is largely dependent on the strip thickness, so the formulas for a thin strip do not suffice. This loss PF can be evaluated from knowledge of the inductance of the line, which is independent of the dielectric. This evaluation can be made with the aid of the "incremental-inductance rule," published by the author in 1942 [3]. Other authors have applied this rule to the formulas of the early papers [13], [17] with the first-order thickness effect stated therein.

In the sandwich line, it has been simpler to evaluate its properties, for various reasons. First, the homogeneity of the dielectric avoids the problem of mixed dielectric, which is relevant for wave resistance. Second, the symmetry and two-sided shielding cause much greater decay of a field with distance. The symmetry simplifies the evaluation of the thickness effects, so those have been published, including the magnetic-loss PF [8]. These give an indication of trends in the subject line, but not quantitative values.

As in most of the previous articles, only the lowest mode of wave propagation in the line shall be considered, and, furthermore, only at frequencies so low that there is negligible interaction between the electric and magnetic fields. This is valid if the transverse dimensions are much less than half the wavelength in the dielectric. This mode may be termed the "quasi-TEM" mode, ignoring second-order effects of dispersion and surface-wave phenomena.

After the following list of symbols, the configuration will be defined and the scope of this article will be indicated.

II. Symbols

The units are MKS rationalized (meters, ohms, etc.).

k	=	dielectric constant of the sheet of material separating the strip and the ground plane.
k'	=	$1 + q(k - 1)$ = effective dielectric constant of all space around the strip.
q	=	$(k' - 1)/(k - 1)$ = effective filling fraction of the dielectric material.
R_c	=	$377 = 120\pi$ = wave resistance of a square area of free space or air.
R	=	wave resistance of the transmission line formed by the strip and the ground plane (of perfect conductor) separated by a sheet of dielectric k.
R_1	=	R without dielectric ($k = 1$).
R_δ	=	R_1 subject to skin depth δ in a real conductor.
R/R_1	=	$1/\sqrt{k'} = \lambda_g/\lambda_0$ = speed ratio in mixed dielectric k' relative to free space or air.
w	=	width of the strip conductor.
h	=	height (separation) of the strip from the ground plane.
h	=	thickness of the dielectric sheet.
t	=	thickness of the strip conductor.
w'	=	effective width of a strip with some thickness.
w'	=	width of an equivalent thin strip ($t \to 0$).
Δw	=	$w' - w$ = width adjustment for thickness.
$\Delta w'$	=	width adjustment with mixed dielectric k'.
δ	=	skin depth in the conductor.
p	=	$1/Q$ = magnetic PF of the strip line.
p_k	=	electric PF of the dielectric material k.
p'	=	effective PF of mixed dielectric k'.
P	=	$p \div \delta/h = ph/\delta$ = normalized p.
α	=	rate of attenuation (nepers/meter).
λ_0	=	wavelength in free space or air.
λ_g	=	guide wavelength in mixed dielectric k'.
e	=	2.718 = base of natural logarithms.
$\exp x$	=	e^x = natural exponential function.
$\ln x$	=	$\log_e x$ = natural logarithm.
acosh x	=	anticosh $x = \cosh^{-1} x$.
asinh x	=	antisinh $x = \sinh^{-1} x$.
asin x	=	antisin $x = \sin^{-1} x$.

The following table translates some symbols from the author's earlier papers.

Here	[1] [2]
$w, h, t, \Delta w$	$2a, b, \Delta b, 2\Delta a$
R of 1 strip	R of 2 strips (twice as great)
(A-)(B-)	( )( ) formulas

III. A Strip Line on a Dielectric Sheet on a Plane

Fig. 1(a) shows the cross section of the subject line. It corresponds to the 1965 article [2] except for the translation to "practical" parameters. The latter are the wave resistance

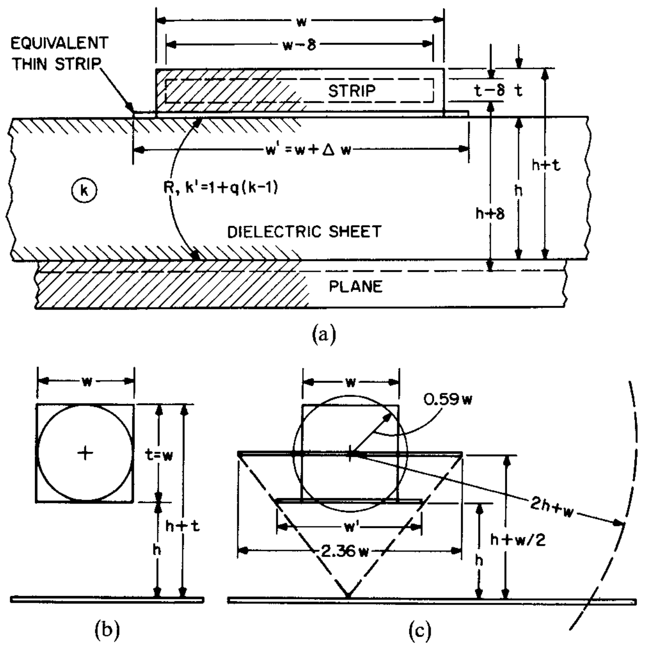

Fig. 1. A strip line parallel to a plane. (a) Rectangular cross section. (b) Cross-section square or inscribed circle. (c) Cross-section small square or equivalent circle.

R of the asymmetric model (single strip and ground plane) and the descriptive dimensions w,h,t. Here the thickness is featured, and the equivalence between a practical strip and a wider theoretical thin strip (a perfect conductor with a thickness approaching zero). This equivalence is described in terms of the width adjustment Δw.

For evaluation of the magnetic-loss PF, the skin effect is indicated in dashed lines. These boundaries are recessed by one half the skin depth ($\delta/2$) so they indicate the actual center of current. The actual boundary is the theoretical current center in a perfect conductor. The change between one and the other is involved in the computation of the magnetic PF. It is assumed that all conductive boundaries are nonmagnetic and have equal conductivity and skin depth.

As an extreme case of strip thickness, a square cross section is introduced, as shown in Fig. 1(b) and (c). These are related to a circular cross section in either of two ways, the inscribed circle (b) or the equivalent circle (c). Each is found to be helpful in some studies, mainly because the circle yields to simple exact formulation for comparison with an approximation for the square. Both will be used for reference.

IV. Scope

The thrust of this article is to enable explicit synthesis of a line to meet some specifications. This is achieved for various sequences. The wave resistance R is related to the dielectric k and the shape. On the other hand, the magnetic PF can be decreased by increasing the size, while the shape has a lesser effect. The PF is usually a tolerance rather than a requisite. The wave-speed ratio is taken not to be specified, but rather evaluated after synthesis of a design of a cross section.

Some graphs are introduced here, for reference in various sections. They present the relations needed for the purposes

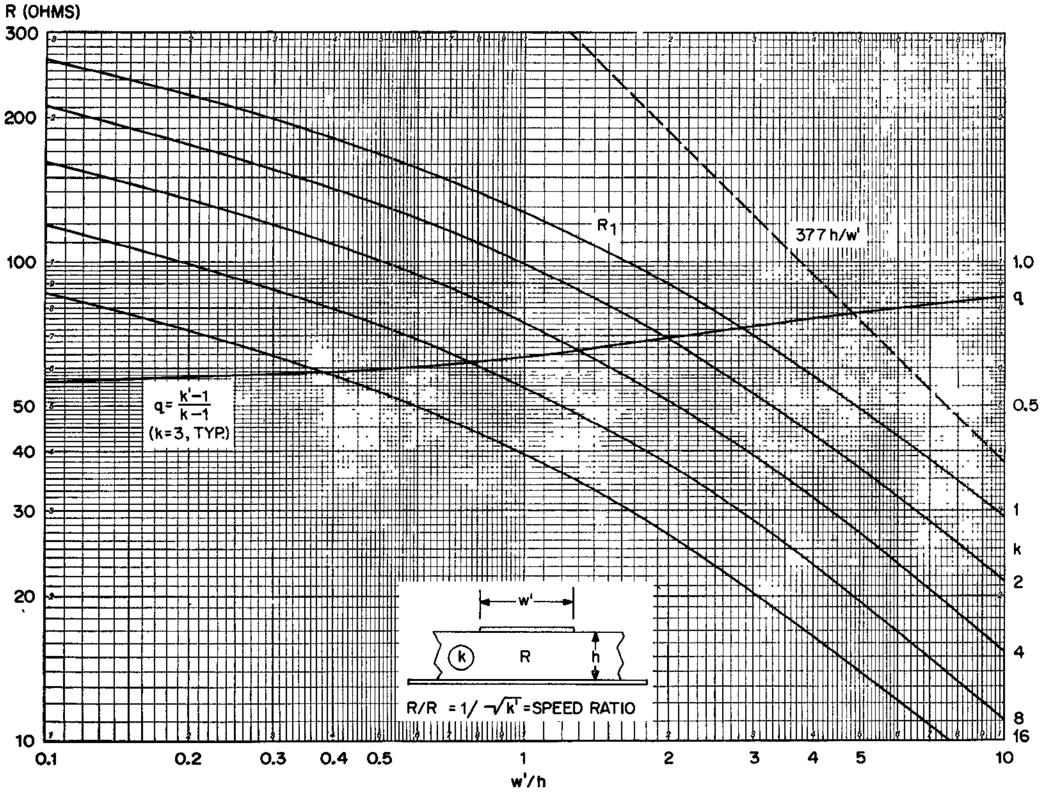

Fig. 2. The wave resistance of a thin strip on a dielectric sheet on a plane.

of practical design, and can be read close enough for ordinary purposes. The formulas to be given are intended as an alternative to the graphs, and also to give further insight into the relations. The formulas are designed for programming in a small digital calculator such as the HP-25 or HP-65.

Fig. 2 is a graph of the wave resistance of a thin strip, the same as previously published [2]. It is made with closest approximation by complete computation for overlapping ranges of narrow and wide strips. An alternative to this graph is the simple empirical formulas to be given for the entire range of width. The wave-speed ratio (relative to air or free space) for any width ratio is equal to the ratio of wave resistance with and without dielectric (R/R_1).

The effective filling fraction q of the dielectric is also graphed on Fig. 2 for a mean value of the dielectric constant ($k = 3$). It enables an alternative computation of the effective dielectric constant k' and the resulting speed ratio $(1/\sqrt{k'})$.

Fig. 3 is a graph of the thickness effect on the wave resistance without dielectric. The relative effect is less with dielectric, so the indicated effect is an upper bound. It is a small effect with respect to wave resistance but has a greater effect on the magnetic PF. This is generally similar to the first-order effect of thickness as previously stated [2] but is refined and extended to include the second-order effect in some degree.

Fig. 4 is a graph of the normalized magnetic PF ($P = p \div \delta/h$) as evaluated from the thickness effect. The magnetic PF is independent of the dielectric and its normalized value is independent of the size. The thickness parameter t/h is chosen as being a property of the laminate, specifically the thickness ratio of the conductive strip and the dielectric sheet.

New formulas are presented here in the main text without derivation. Most of them are empirical formulas providing a gradual transition between narrow and wide extremes. These are tested against the derived close approximations for overlapping narrow and wide ranges, which are reviewed in Appendix VI. Some derivations, not previously available, are given in Appendixes IV and V. Special emphasis is placed on some formulas which are "reversible" in the sense that a formula can be expressed explicitly in a simple form for either analysis or synthesis.

V. A Thin Strip Without Dielectric

The 1964 paper [1] gave the derivation for a wide thin strip without dielectric, and, incidentally, also gave formulas for a narrow thin strip. These together covered any width. Explicit formulas were given for both purposes, analysis and synthesis.

Recent studies yielded the discovery that the "narrow" formula could be put into a form which would also be asymptotic to the "wide" formula. This is accomplished while retaining its principal features for "narrow" approximation. Furthermore, this has been so arranged that the formula is "reversible." By this is meant that an explicit formula for either analysis or synthesis can be converted to an explicit formula for the other. This conversion is permitted no complication beyond the solution of a quadratic equation. The resulting formulas are empirical in the sense that they must be tested against derived formulas in the "wide" range and in the overlap of "wide" and "narrow." For

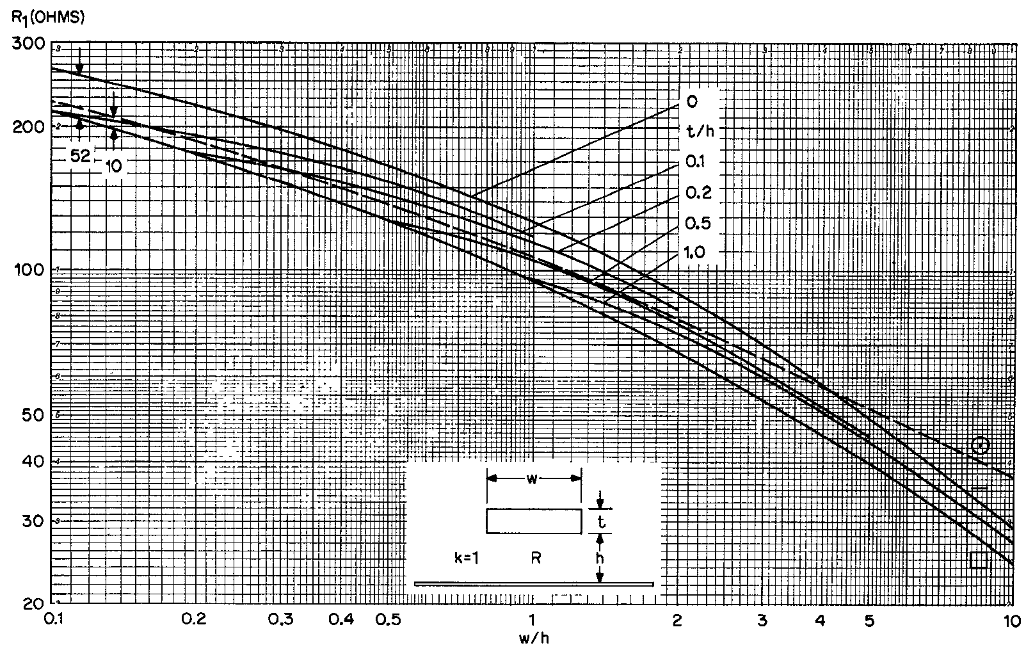

Fig. 3. The wave resistance of a strip without dielectric, showing the effect of thickness.

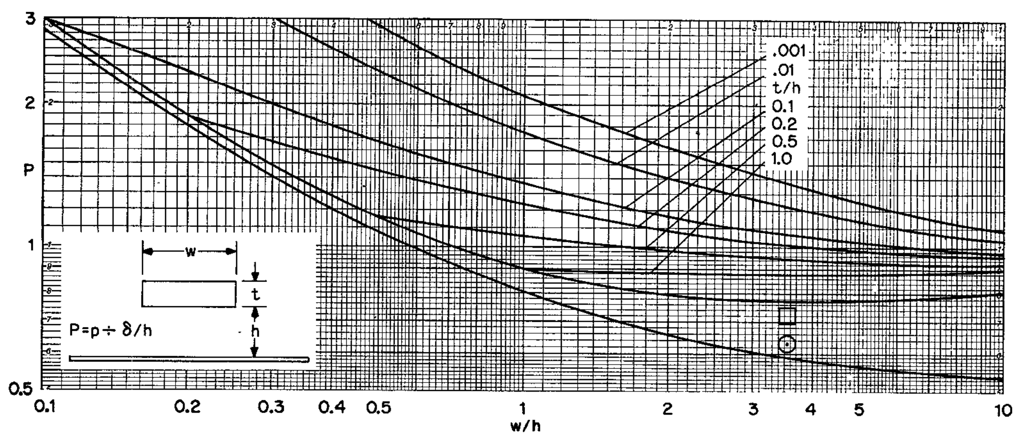

Fig. 4. The magnetic power factor of a strip, showing the effect of thickness.

the thin strip without dielectric, such derived formulas are the subject of the 1964 paper. The resulting formulas herein are based on "narrow" derivations which are relatively simple although the derivation of the second term has not been published before. It is based on a two-wire second-order approximation for a narrow thin strip. The overlap of the "narrow" and "wide" derivations is indicated, by stating the small error of either at the transition.

The new formulas are to be generalized for dielectric, but they are first given here in simplest form to show some features. Reversible formulas are here shown first for synthesis and then for analysis:

$$w'/h = 8\frac{\sqrt{(\exp R_1/30 - 1) + \pi^2/4}}{(\exp R_1/30 - 1)} \tag{1}$$

$$R_1 = 30 \ln \{1 + \tfrac{1}{2}(8h/w')[(8h/w') + \sqrt{(8h/w')^2 + \pi^2}]\} \tag{2}$$

where the error is $<0.01R_1$.

The following are the asymptotic forms for the narrow and wide extremes: narrow:

$$w'/h = 8(\exp - R_1/60)[1 + 1.73(\exp - R_1/60)^2]$$

$$R_1 = 60 \ln (8h/w' + 1.73(w'/8h)) \tag{3}$$

wide:

$$w'/h = 120\pi/R_1;\ R_1 = 120\pi(h/w'). \tag{4}$$

In the "narrow" formula, the second-order term has the proper form but its coefficient is compromised (1.73 instead of 2) to accomplish asymptotic "wide" behavior.

The asymptotic behavior at both extremes could be accomplished by any of several variants, yielding somewhat different behavior in the transition region. The form chosen was found to give close enough approximation with the minimum number of terms.

The form of these transitional approximate formulas shows some points of similarity to the exact formulas for a round wire near a plane, which are to be given here.

VI. Square or Circular Cross Section

As an extreme departure from the thin strip, a square or circular cross section is considered, still without dielectric. Fig. 1(b) shows a square or an inscribed circle as the cross section, with the description in the same terms as Fig. 1(a) ($t/w = 1$). It is noted that the distance from the plane is described by the separation height h, not by the distance to center (which is $h + t/2$). Hence it is compatible with separation by a dielectric sheet.

For the narrow case, simple formulations for a square wire and the equivalent round wire are known. See Appendix III. Fig. 1(c) shows this relation and the radius $(2h + w)$ of the outer circle equivalent to the plane.

For the wide case, the exact formula is known for the round wire but not for the square one. Therefore a close approximation for the square wire has been derived and is presented in Appendix IV.

For the round wire, the exact formula for any width ratio is known in simple reversible form. By modifying this form, a reversible empirical formula has been derived for the square shape. These formulas are presented here. (R_1 without dielectric is here simplified to R, because here there is no need for this distinction.)

For round wire without dielectric the exact formulas are as follows:

$$\begin{aligned} w/h &= \frac{2}{\cosh R/60 - 1} = \frac{1}{(\sinh R/120)^2} \\ &= \left(\frac{2}{\exp R/120 - \exp - R/120}\right)^2 \\ &= \frac{4}{\exp R/60 + \exp - R/60 - 2} \\ &= \frac{4 \exp - R/60}{(1 - \exp - R/60)^2} = \frac{4 \exp R/60}{(\exp R/60 - 1)^2} \end{aligned} \tag{5}$$

$$\begin{aligned} R &= 60 \operatorname{acosh}(2h/w + 1) = 120 \operatorname{asinh} \sqrt{h/w} \\ &= 60 \ln \left[(2h/w + 1) + \sqrt{(2h/w + 1)^2 - 1}\right] \\ &= 120 \ln \left(\sqrt{h/w} + \sqrt{h/w + 1}\right). \end{aligned} \tag{6}$$

For square wire without dielectric the approximate formulas are as follows:

$$\begin{aligned} w/h &= \frac{1/0.59}{\exp R/60 - 0.2} \frac{2 + \exp - R/60}{1 - \exp - R/60} \\ &= \frac{1}{0.118} \frac{2 + \exp - R/60}{5 \exp R/60 + \exp - R/60 - 6} \end{aligned} \tag{7}$$

$$\begin{aligned} R = 60 \ln \Bigg[&\left(\frac{h}{0.59w} + 1.1\right) - 0.5 \\ &+ \sqrt{\left(\frac{h}{0.59w} + 1.1\right)^2 - 1.05} \Bigg]. \end{aligned} \tag{8}$$

The relative error is < 0.025 or $< (0.005R + 0.5\,\Omega)$. If $R \to 0$, $w/h \to 381/R$ (near $377/R$). Each of these formulas is asymptotic in the first- and second-order terms for "narrow" and the first-order term for "wide."

These formulas are intended mainly for the magnetic-loss PF, for which there is no effect of dielectric, and only the "analysis" form (R of w/h) is used. The synthesis form (w/h of R) is shown mainly for academic interest, since it formed the basis for the empirical formulas for the square wire over the entire range of width ratio.

In Fig. 1(c), in addition to the equivalent circular and square cross sections, there are shown some equivalent thin strips. A round wire far from the plane has an equivalent concentric thin strip whose width is double the wire diameter ($2.36w$). If not so far from the plane, there is a thin strip of lesser width (w') which is equivalent by the following two tests:

a) height above the plane equal to that of the lower side of the square;
b) equal wave resistance.

The indicated geometric proportionality of the two strip widths is of interest in kind but not in degree, because their difference becomes substantial for a square so wide that the simple rules of equivalence are failing.

The lesser thin strip, compared with the square, determines the width adjustment here associated with the thickness of the square.

VII. A Thin Strip with Dielectric

The 1965 paper [2] gave the derivation for a thin strip with dielectric. Two sets of formulas covered wide and narrow strips, with close agreement in the transition region. The reversible formulas given above are here adapted to dielectric. Asymptotic behavior is achieved for the following conditions:

a) narrow strip, low-k and high-k extremes, with a logical interpolation therebetween;
b) wide strip, all k.

The resulting empirical formulas are found to track the derived formulas over the entire range of width and dielectric:

$$w'/h = 8 \frac{\sqrt{\left[\exp\left(\frac{R}{42.4}\sqrt{k+1}\right) - 1\right]\frac{7 + 4/k}{11} + \frac{1 + 1/k}{0.81}}}{\left[\exp\left(\frac{R}{42.4}\sqrt{k+1}\right) - 1\right]} \tag{9}$$

$$\begin{aligned} R = \frac{42.4}{\sqrt{k+1}} \ln \Bigg\{ 1 + &\left(\frac{4h}{w'}\right)\Bigg[\left(\frac{14 + 8/k}{11}\right)\left(\frac{4h}{w'}\right) \\ &+ \sqrt{\left(\frac{14 + 8/k}{11}\right)^2\left(\frac{4h}{w'}\right)^2 + \frac{1 + 1/k}{2}\pi^2}\Bigg]\Bigg\}. \end{aligned} \tag{10}$$

The error is $< 0.02R$ (or $< 0.01R$ over most of range).

The analytic form gives R (for k) and R_1 (for $k = 1$), from which the speed ratio is

$$R/R_1 = 1/\sqrt{k'} < 1. \tag{11}$$

Therefore no other formula is needed for the speed ratio. The simpler formula (2) for R_1 may be used, but that is no advantage if the more general formula is recorded in a program for numerical computation.

While the effective filling fraction q [2] of the dielectric is not required in the procedures given here for design computations, it is a matter of some interest. Particularly, it is a factor in the electric-loss PF to be formulated. Schneider [14] has given an ingenious simple empirical formula, based on [2], which is close enough for practical purposes:

$$q = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + 10h/w}}\right). \tag{12}$$

Compared with the derived formulas for narrow and wide ranges (for a mean value, $k = 3$) the departure is <0.02. It is a simple transition between the bounds $(\frac{1}{2},1)$. It lacks the shape that is peculiar to either extreme, which is contained in the derived formulas.

VIII. Strip Thickness and the Loss Power Factor

The earlier papers did not make any attempt to evaluate conductor loss, because it is not determined in the limit of a thin strip. However, there was given a width adjustment for the edge effect of a small thickness. From this adjustment, some other authors have formulated the losses to be expected, and their reduction by thickness [13], [17].

This subject has been reviewed. The width adjustment has been verified for small thicknesses of a narrow strip, and has been formulated more closely for a wide strip. A single formula is given here for the entire range of width. Also it is adapted to moderately large thicknesses (up to a square cross section for a narrow strip).

The loss PF (PF $= 1/Q$) of the magnetic field (bounded by the conductors) is evaluated by the rule proposed by the author in 1942 [3]. This "incremental-inductance rule" is based on differentiation of the inductance relative to the skin depth δ in the conductor boundaries, as indicated in Fig. 1. In a transmission line with perfect boundaries and no dielectric material, the inductance is proportional to the wave resistance. Only the relative change is significant, so the rule is here applied to the wave resistance R_1. This avoids the nuisance of magnetic units and surface resistivity.

A great simplification is now available by numerical differentiation. This was not available in the slide-rule computations of earlier days so analytical differentiation was necessary, however cumbersome. It was used by the other authors. It is no longer needed. What is needed is an analytic formula giving the wave resistance in terms of all dimensions but without dielectric.

The edge effect related to the strip thickness is here described in terms of the extra width Δw of a thin strip having equal wave resistance R_1 without dielectric. This is indicated in Fig. 1.

The first-order effect of a small thickness is given in the 1965 paper, for the extreme cases of narrow and wide strips. Three advances are here presented:

a) a refinement for the wide strip (Appendixes I and II),
b) a unified formula for the entire range of width,
c) a second approximation for greater thickness, within some restrictions.

The resulting formula is expressed in terms of the actual width w or the equivalent-thin-strip width w'. As mentioned above, these relations are based on free space, without dielectric:

$$\frac{\Delta w}{t} = \frac{1}{\pi} \ln \frac{4e}{\sqrt{\left(\frac{t}{h}\right)^2 + \left(\frac{1/\pi}{w/t + 1.10}\right)^2}} \tag{13}$$

or

$$\frac{1}{\pi} \ln \frac{4e}{\sqrt{\left(\frac{t}{h}\right)^2 + \left(\frac{1/\pi}{w'/t - 0.26}\right)^2}}. \tag{14}$$

This adjustment enables a width conversion either way between equivalent strips with or without thickness.

The development of this formula for the wide and intermediate regions has been enabled by complete computation of a few examples (Appendix II). These were accomplished by the technique of conformal mapping. Specifically, a few shapes (w,h,t) of rather small thickness were evaluated by numerical integration of the space gradient. This process is laborious and required some ingenuity near some bounds of integration.

Three examples so evaluated were sufficient to indicate two features implicit in this formula.

a) For a wide strip, the previous formula (1965) is refined in respect to its second-order effect. The ratio previously included as $2h/t$ is here changed to $4h/t$. The former ratio was based on unlimited width, and the change is an adaption to the limited width.
b) The "narrow" and "wide" formulas appear to be upper bounds, as would be expected. Furthermore, the quadratic sum of the two inverse ratios fits the sample points.

The adaptation of this formula for a greater thickness has been enabled by derivations for a square cross section. The extra numbers ($+1.10$ or -0.26) are chosen to match the square condition ($t = w$) for a narrow strip. The formula is a close approximation for moderate thicknesses ($t < h$) of a wide strip. (Another formula has been derived for a wide strip of square cross section, Appendix IV.)

For loss computation, the actual width and thickness (w,t) are converted to the width of an equivalent thin strip ($w' = w + \Delta w$). Then the thin-strip formula (R_1 of w') can be used for differentiation with respect to the actual dimensions (w,h,t).

As indicated in Fig. 1, each dimension is incremented by $\pm\delta$ and the same formula is used again to obtain R_δ. Then the (small) loss PF is computed by the incremental-inductance rule:

$$p = \frac{R_\delta - R_1}{R_\delta} = 1 - R_1/R_\delta = \ln R_\delta/R_1 \ll 1. \tag{15}$$

A normalized form for loss PF is proposed, which gives the effect of shape, independent of the size, frequency, and conductor material. It is normalized to the height h:

$$P = p \div (\delta/h) = p(h/\delta) \qquad p = P(\delta/h). \tag{16}$$

The reference (δ/h) is the nominal PF of a very wide strip.

In computing the normalized PF P, the value of the skin depth is immaterial if it is sufficiently small to approach the limiting behavior of the skin effect (which is usually of interest). Also it must not approach the sensitivity of the computer. In a computer giving ten decimal places, a fair compromise is $\delta/h = 0.0001$. Then the skin effect is well represented if all dimension ratios exceed 0.001.

For evaluation of a resonator made of a strip line, the loss PF (or dissipation factor or $1/Q$) is usually the most significant factor. The wave R is incidentally relevant in the circuit application of the resonator. The loss PF of the magnetic field is evaluated by the simplest formulas (R_1 and Δw without dielectric). For any shape, the value of P enables a computation of the size of the cross section to realize a value of p:

$$h = P\delta/p = P\delta Q. \tag{17}$$

The graphs in Fig. 4 show the loss PF in terms of P for a wide range of shapes. The common reference is the height h and the thickness ratio t/h because they may be fixed by a dielectric sheet and a conductive sheet bonded thereto.

For small thicknesses, the loss PF exceeds the reference value, as would be expected. Also the amount of excess is greater for lesser thickness, as a result of the current concentration at the edges. For example, reducing the thickness from square to $t/h = 0.02$ may double the PF (in the moderately narrow range).

An unexpected result is the loss PF being less than the reference value for a wide strip of substantial thickness. This happens because part of the magnetic energy is beyond the region bounded by the height. This part has boundaries further apart, and hence a lesser value of loss PF.

In Fig. 4, the two lowest curves give the loss PF for square and circular cross sections of the same width. It is less for the latter, the lower bound for the wide extreme being one half the reference value ($P \rightarrow \frac{1}{2}$). In the narrow region, it is less because of the following.

a) The two shapes are known to have equal skin resistance [9].
b) The circular shape has greater reactance. The proportionate wave resistance is greater by 60 ln 1.18 = 10 Ω; this is denoted, "the rule of 10 Ω."

If the thickness is comparable with the height, the relevant restriction may be the overall height $(h + t)$, perhaps for reasons of clearance space. Also the width may be restricted. Then the thickness ratio has an optimum value. This is found by minimizing a related normalized PF defined as follows:

$$P_{ht} = P\frac{h+t}{h} = P(1 + t/h) = (p/\delta)(h + t) \tag{18}$$

$$\text{Square:} \quad w/h \text{ near } 0.55, \quad \min P_{ht} = 1.65 \tag{19}$$

$$\text{Circular:} \quad w/h \text{ near } 0.50, \quad \min P_{ht} = 1.56. \tag{20}$$

Within specified bounds of the overall height and width (not less than the overall height), the optimum rectangular cross section is one which is bounded by these two dimensions and has a certain thickness $(t/h < 1)$. The extreme optimum is a peculiar rounded shape bounded by these dimensions.

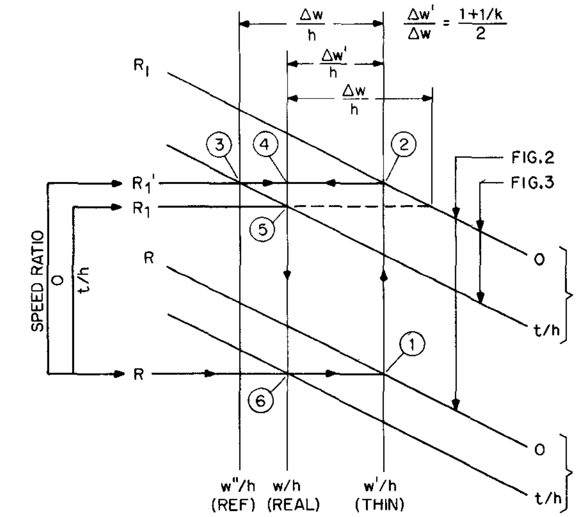

Fig. 5. Relations which determine the speed ratio.

IX. Strip Thickness with Dielectric

The effect of strip thickness is formulated above, but without the effect of dielectric. A width adjustment for thickness may be made in the synthesis for a specified wave resistance with dielectric.

The width of an equivalent thin strip is defined as one which is wider by the amount which yields an equal value of wave resistance. This involves both inductance and capacitance. The width adjustment for the former is independent of dielectric. That for the latter is less for a greater dielectric constant, because the thickness of the edge is somewhat spaced from the dielectric.

To approximate this effect, the width adjustment is divided in two equal parts, and one part is decreased by the factor $1/k$. The modified value becomes:

$$\Delta w' = \frac{1 + 1/k}{2}\Delta w, \qquad w = w' - \Delta w'. \tag{21}$$

The entire width adjustment Δw is effective for wave resistance without dielectric $(k = 1)$ or for inductance alone. For capacitance alone, the entire width adjustment would be decreased by the factor $1/k$.

Fig. 5 shows the behavior of the width adjustment without or with dielectric. Especially it shows its graphic determination from Figs. 2 and 3. The full value of Δw is effective without dielectric, decreasing the wave resistance R_1 equally by decreasing inductance and by increasing capacitance. Its amount is represented by the horizontal separation of the upper pair of curves, both shown in Fig. 3. The effect of dielectric with a thin strip is represented by the separation of the upper curves (R_1, R) in the upper and lower pairs, both shown in Fig. 2. The reduced amount of width adjustment with dielectric is constructed and projected downward to give the lesser horizontal separation of the lower pair of curves.

The speed ratio for a thin strip is R/R_1 from Fig. 2. The interpolation for mixed dielectric, taking account of thickness, gives a greater speed ratio from this construction. The latter locates a point on the lower curve of the lower pair, not shown elsewhere.

As will be seen in a procedure and example to be given, the indicated numerical sequence yields all the quantities from the graphs in Figs. 2 and 3. Or they may be computed in this sequence by these formulas:

Sequence	Formulas
1	(9)
2	(2)
3	(14)
4	(21)
5	(13) (2)
6	(—)

In this sequence, 2–3 is the width adjustment downward from the upperbound for a thin strip. A parallel dashed line shows also the width adjustment upward from a strip with thickness. The former will be used in a synthesis procedure, the latter in analysis. The amount of the adjustment is designated alike in both $(\Delta w/h)$ although it may differ slightly (too little for any practical significance).

X. Attenuation

The rate of attenuation with distance in a transmission line is simply expressed in terms of the average PF (magnetic p and electric p') and the wavelength λ_g in the line:

$$\alpha = \frac{p+p'}{2}\frac{2\pi}{\lambda_g} = \frac{p+p'}{2}\sqrt{k'}\frac{2\pi}{\lambda_0} = \frac{p+p'}{2}\frac{R_1}{R}\frac{2\pi}{\lambda_0}\ (\text{Np/m}). \tag{22}$$

In words, this is the average PF (nepers) per radian length. The magnetic PF is evaluated by the skin effect, as described above.

The electric PF p' of the effective dielectric in the line (k') can be expressed in terms of the various parameters involved:

speed ratio:

$$\lambda_g/\lambda_0 = 1/\sqrt{k'} = R/R_1 \tag{23}$$

filling fraction:

$$q = \frac{k'-1}{k-1} = \frac{1}{k-1}[(R_1/R)^2 - 1] \tag{24}$$

electric PF:

$$p' = \frac{p_k}{1 + \dfrac{1/q - 1}{k}} \tag{25}$$

bounds:

$$p_k > p' > \left\{ \begin{matrix} qp_k \\ \dfrac{p_k}{1+1/k} \end{matrix} \right\} > \tfrac{1}{2}p_k. \tag{26}$$

The electric PF p' is seen to be within $(\frac{3}{4} \pm \frac{1}{4})$ of the dielectric-material PF p_k, and usually it is nearer the upper bound. Therefore the electric PF is only slightly less than that of the material, so the complete formulation is not critical and may be unnecessary. If desired, it can be computed (as above) from R/R_1 and p_k.

Either attenuation or PF may be deduced from the other. However, it is preferable to evaluate the magnetic PF directly, because it is independent of the dielectric and the speed ratio. In a wide range of situations, it represents nearly all of the loss PF.

XI. Procedures for Computation

The formulas are intended for useful applications, which may be theoretical and/or practical. As brought out in the earlier papers, "synthesis" and "analysis" are the alternative objectives, the former for practical design and the latter for evaluation of a configuration (the classical textbook approach). Both are needed here for a practical design to meet some specifications and tests. Therefore a few procedures and examples will be outlined to show the use of these formulas in arriving at a practical design.

The first few procedures start with the synthesis of a line to meet a specification of wave resistance. The subsequent evaluation of speed ratio and skin effect are inherently analysis, but the procedures build on the synthesis.

First Procedure: On a specified printed-circuit board, find the width for a 50-Ω line:

a) specify properties of a dielectric sheet with metal faces: $k = 2.5$, $h = 1$ mm, $t = 0.1$ mm;
b) specify wave resistance: $R = 50\ \Omega$;
c) width of thin strip by (9) or Fig. 2: $w'/h = 2.85$;
d) width adjustment (without dielectric) by (14) or Fig. 3: $\Delta w/h = 0.15$;
e) effect of dielectric by (21): $\Delta w'/h = 0.10$;
f) width by (21): $w/h = w'/h - \Delta w'/h = 2.75$; $w = 2.75$ mm.

Second Procedure: For the same line, evaluate the speed ratio, referring to Fig. 5:

g) no. 1 in sequence, c) above: $w'/h = 2.85$;
h) no. 2, find R_1' of thin strip by (2) or Fig. 2 or 3: $R_1' = 71.5$;
i) no. 3, d) above: $\Delta w/h = 0.15$, $w''/h = w'/h - \Delta w/h = 2.70$;
j) no. 4, e) above: $\Delta w'/h = 0.10$, $w/h = w'/h - \Delta w'/h = 2.75$;
k) no. 5, by Fig. 3: $R_1 = 71$, or can be computed by the "fourth procedure";
l) speed ratio $= R/R_1 = 50/71 = 0.70$.

Third Procedure: For the same line, evaluate the magnetic PF and the attenuation from this cause, referring to Fig. 4, Appendixes VII and VIII:

m) find the normalized PF by Fig. 4: $P = 1.10$; or it may be computed by (16) using a nominal small δ and numerical differentiation;
n) specify the frequency (or wavelength λ_0): $f = 1$ GHz, $\lambda_0 = 0.3$ m;

o) specify the conductivity (or material) of the metal boundaries: copper;
p) evaluate the skin depth by (62) or [7]: $\delta = 2.1\ \mu$m;
q) compute the PF by (16): $p = 0.0023 = 2.3$ mil, $Q = 440$;
r) compute the attenuation rate from PF, speed ratio, etc., by (22): $\alpha = 0.034$ Np/m or 0.30 dB/m.

If used for a long line, the attenuation rate may be significant. If used for a resonator, the PF and speed ratio are relevant.

In the third procedure, if one is concerned with only one example (size, shape, materials, frequency) the actual skin depth may be used directly, then the procedure assumes this order: (n,o,p) (m,q,r).

A lesser PF may be required, or it may be desired to explore the compromise between the loss PF and the height and/or thickness. The first-order relation gives the PF inversely proportional to size (h,t,w). A closer evaluation may require complete computation of various examples, then interpolation.

The following example and procedure are modified to develop from analysis only. In particular, the width adjustment corresponds to the dashed line in Fig. 5.

Another Example: Design a resonator to be made of a square wire bonded to a printed-circuit board. Similarly lettered items refer to the foregoing procedures:

a) $k = 2.5$, $h = 1$ mm, $t = w$;
m) from Fig. 4: near-minimum $P = 0.8$ for $w/h = 2$, $w = t = 2$ mm;
p) $\delta = 2.1\ \mu$m;
q) $p = 0.0017 = 1.7$ mil, $Q = 590$.

The speed ratio can be evaluated by the following procedure. It is found to be $50/67 = 0.75$.

Fourth Procedure: For any configuration, find the speed ratio. For a thin strip, see (11) and Fig. 2 for the simple rule. The following gives the effect of thickness:

a) specify configuration: $k = 2.5$, $w = 2.75$ mm, $h = 1$ mm, $t = 0.1$ mm, $w/h = 2.75$, $t/h = 0.1$;
b) width adjustment (without dielectric) by (14) or Fig. 3: $\Delta w/h = 0.15$, $w'/h = 2.90$;
c) wave resistance (without dielectric) by (2) or Fig. 2 or 3: $R_1 = 71$;
d) effect of dielectric by (21): $\Delta w'/h = 0.7$, $\Delta w/h = 0.10$, $w'/h = 2.85$;
e) wave resistance (with dielectric) by (10) or Fig. 2: $R = 50$;
f) speed ratio: $R/R_1 = 0.70$.

The speed ratio is slightly greater than that for a thin strip of the same width.

If resonance (small PF or high Q) is the principal objective (rather than wave resistance) a different procedure may be indicated. The following outline gives some relevant considerations.

a) Choose between a specified printed-circuit material (h,t) and the alternative of an attached thick strip (which may have a square or circular cross section). The latter offers a lesser PF.
b) If a thick strip is to be afforded, specify the bounds of the space (overall height and width, $h + t$ and w).
c) Specify whether the conductor (strip or whatever) is to be supported in contact with a dielectric sheet. If so, specify the height of the latter (h).
d) Subject to these restrictions, choose a cross section giving near-minimum P_{ht} (18).
e) If using a strip of small thickness t, a lesser PF is obtainable by greater width w and greater height h.
f) If using a square cross section in contact with a dielectric sheet ($t/h = w/h$), the least PF is obtainable by a moderately wide shape (say w/h near 3).
g) If using a round wire in contact, a lesser PF is obtainable by greater width (diameter), but little reduction is obtainable beyond a moderate width (say w/h near 3).
h) If using a square or round wire with no need for contact, the least PF is obtainable by a width near one third the overall height.
i) If a rectangular space is specified, with the width not less than the overall height, the least PF obtainable with a rectangular cross section requires some thickness less than one third the overall height.

There is usually not available an explicit formula for the synthesis to realize a specified value of the loss PF. The graphs in Fig. 4 can be applied to this problem. Knowing the skin depth δ and specifying the material (h,t), a value of the loss PF p requires the P computed from (16). In Fig. 4, this value of P determines the shape ratio w/h and hence the width w. If this P is lower than a practical curve, the size (h,t) may be increased to permit a greater value of P.

XII. Conclusion

The transmission-line properties of a strip parallel to a plane, with or without an intervening dielectric sheet, are evaluated in simple formulas, each one adapted for all shape ratios. The formulas relating the width/height ratio with wave resistance are stated explicitly for both analysis and synthesis, with or without dielectric. The wave-speed ratio and the magnetic-loss PF are stated from the viewpoint of analysis, which is usually what is needed.

The advance over previous publications appears mainly in two areas:

a) a relation is expressed explicitly by a single simple formula for the entire range of the shape ratio;
b) the width adjustment for thickness is formulated and used for evaluation of the magnetic loss.

Each formula is an empirical relation obtained by designing a gradual transition between known simple formulas for both extremes of narrow and wide shapes.

All formulas are designed for ease of programming on a pocket calculator such as the HP-25 or HP-65. Particularly, the digital calculator enables the numerical differentiation (for loss evaluation) which is here used to realize a great simplification. While beyond the scope of this article, the writer would welcome inquiries relating to programs for the HP-25, some of which may be available on request.

The subject line, formed by a strip parallel to a plane, has presented problems of evaluation which are much more difficult than those of the strip between two planes (sandwich line). That configuration is symmetrical and the dielectric is homogeneous, so even the thickness effects have yielded to straightforward formulation [8]. The asymmetrical strip line is here formulated in a manner that is competitive, although necessarily involving mixed dielectric.

While there is always room for further progress, the graphs and formulas presented here are complete in that they offer the option of graphical or numerical reading for the all numerical values that may be needed for design purposes. Preliminary estimating is usually aided most by the graphs.

XIII. Acknowledgment

This study has been stimulated by the attempts of other authors, building on the writer's early papers. The stimulation has come partly from a perception of some deficiencies in progress, but more from an appreciation of the constructive efforts and interesting results of a few of the intervening workers. The final impetus was provided by the advent of the HP-25, which offered the computational power best suited for the "close support" essential to such a development.

Appendix I
Behavior of the Width Adjustment for Thickness

Formulas (13) and (14) for the width adjustment are based on some asymptotic relations and a transition therebetween. Asymptotic formulas for narrow and wide extremes were given in the early papers [1], [2]. Here a revision of the wide formula and an integrated formula with a simple form of transition were presented. This appendix is a graphical description of the behavior of this adjustment, for the purpose of visualizing the transition and some associated relations.

Fig. 6(a) shows a graph for a constant ratio of thickness/height (t/h). This may be the practical situation when designing for a printed circuit to be made by etching a conductive sheet bonded to a dielectric sheet. The width adjustment ratio $\pi\Delta w/t$ is plotted on the width ratio w/h. The scales are, respectively, linear and logarithmic, to give straight lines for the sloping graphs.

The normalized form for the width adjustment takes out the principal dependence on thickness, so one can see the variations of the coefficient which is dependent on shape. A higher value indicates a greater coefficient (responsive to thinness) but the amount of the adjustment is still nearly proportional to thickness.

There are two upper bounds (UB's) for this coefficient, based on the narrow and wide asymptotic behavior. The level upper line is based on the wide extreme, the edge-field pattern being influenced mainly by the proximity of the shield plane. The sloping lower line is based on the narrow

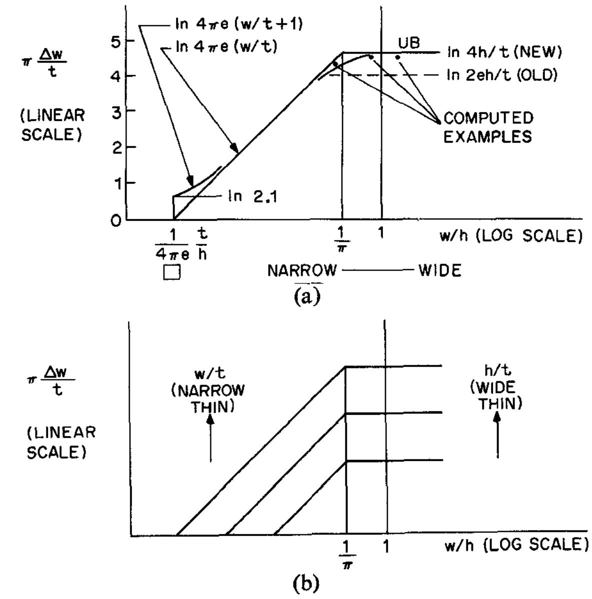

Fig. 6. The behavior of the width adjustment for thickness. (a) Transition between narrow and wide. (b) Family of transitions.

extreme, the edge-field pattern being influenced mainly by the proximity of the two edges.

A smooth transition at the knee is provided by a quadrature combination in formula (13). This is validated by some computations to be described in Appendix II, indicated as three points on the curve. This validation requires a change in the wide formula, from the "old" in [2] to the "new" in (13). The computed points indicated that the level UB should be raised by a factor of two under the logarithm, as seen. (This factor is not exactly determined, but two appears to be the nearest and simplest number that might be indicated, and it may have an exact basis.) This is regarded as a refinement of the previous rule, whose derivation ignored the second-order interaction between the edges far apart. It is noted that the transition occurs in the vicinity of a width ratio somewhat less than unity ($w/h = 1/\pi$).

The asymptotic relations are based on the limiting condition of a thin strip. Formula (13) includes an adaptation ($w/t + 1.1$) which extends the close approximation to the square condition. This introduces another curved transition at the foot of the graph, raising the curve from the "square" point ($t/w = 1$). While beyond the present scope, it is noted that the curve has a minimum near the foot and approaches a higher level (π) in the narrow extreme ($w/t \ll 1$).

Fig. 6(b) is a diagram showing a family of such graphs. For greater thickness, the knee is closer to the foot of the graph, so the two curved transitions would merge as the thickness approaches the square shape. Then their separate descriptions become indefinite, so the validity of formula (13) is further tested on square shapes, as evaluated in Appendixes III and IV.

Appendix II
Small-Thickness Examples by Conformal Mapping

Formula (13) gives the width adjustment for thickness. It is an empirical transition between the narrow and wide

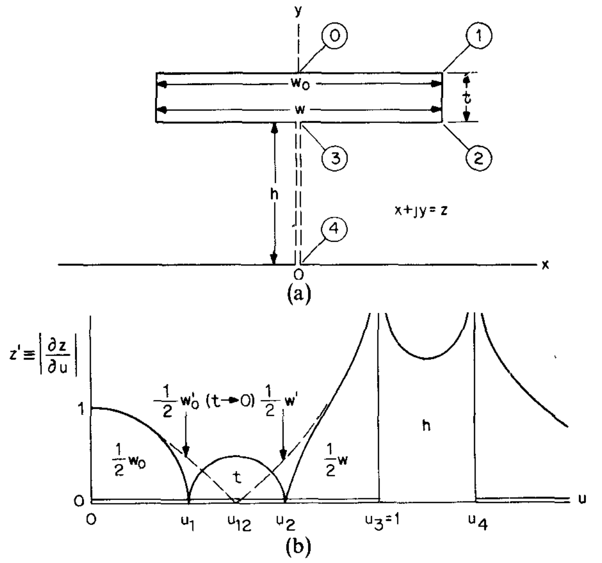

Fig. 7. Conformal mapping of the cross section of the strip line. (a) Contour in space. (b) Space gradient. (Each area equals dimension in space.)

extremes of the asymptotic behavior in the limit of a thin strip. The first-order relations for these extremes have been known [2] but not the behavior in the transition regions. Also there is found a second-order effect requiring a revision for the wide extreme.

The validation of this formula, especially in the transition region, is provided by some complete computations of a few examples by a procedure based on conformal mapping. The contour of the cross section is mapped on a straight line. The space gradient on this line is integrated to evaluate the dimensional ratios on the contour. Rather than implicit elliptic integrals, numerical integration is used. Even that is confronted by difficulties of integration where there is an infinite value and/or slope at either bound ($\infty, \sqrt{\infty}$, or $\sqrt{0}$).

Fig. 7 shows the essentials of the conformal mapping of the cross section of the strip line. The actual contour, Fig. 7(a), is described on the space plane ($x + jy = z$) in terms of the shape dimensions w,h,t whose ratios determine the properties. This contour is mapped on a straight line, Fig. 7(b). On the scale of this line u is graphed the space gradient (or inverse field gradient) on the contour. The area under the space gradient in each interval is equal to the dimension on the contour.

The space gradient is formulated by inspection, as follows:

$$Z' = |\partial z/\partial u| = \left| \frac{[1-(u/u_1)^2][1-(u/u_2)^2]}{[1-(u/u_3)^2][1-(u/u_4)^2]} \right|^{1/2}. \quad (27)$$

Only the area ratios are significant, so the scale is arbitrarily chosen for simplicity.

The analytic integration would involve elliptic integrals. There is a constraint that precludes an explicit solution. The upper and lower faces of the strip must have equal width ($w_0 = w$), to be realized by proportioning one of the critical values on the straight line.

Numerical integration is simple in concept and has been found useful in computing a few examples. Some special

TABLE I
COMPUTED EXAMPLES

No.	1	2	3
R_1	188.5	133.7	88.5
u_4	1.414	1.1	1.01
u_{12}	0.7368	0.7774	0.8412
u_2	0.8867	0.8720	0.8988
u_1	0.5267	0.6520	0.7648
w'/h	0.348	0.879	2.020
w/h	0.237	0 .754	1.894
t/h	0.0755	0.0816	0.0838
$\Delta w/h$	0.111	0.125	0.126
$\Delta w/t$	1.48	1.53	1.51
(14)	1.47	1.55	1.56
dif.	+.01	-.02	-.05

rules have been devised for closer convergence near the singular points which correspond to the angles of the contour. The result is a close approximation in cases where the singular points are not too closely spaced.

The wave resistance is determined by the gaps in the straight line, both sides of center. For the upper half-plane,

$$R = \tfrac{1}{2}R_c \frac{K'(k)}{K(k)} \doteq \frac{\left(\dfrac{1+1.14k^2}{\pi} \ln \dfrac{16}{k^2}\right)^{1-k^2}}{\left(\dfrac{1+1.14(1-k^2)}{\pi} \ln \dfrac{16}{1-k^2}\right)^{k^2}} \quad (28)$$

in which $k = 1/u_4$.

The latter (empirical) formula has a relative error <0.005. It has the correct center value, skew symmetry, and asymptotic behavior at both extremes. A closer simple formula for a wide strip is

$$R = \frac{60\pi^2}{\ln \dfrac{8}{\ln 1/k}} = \frac{60\pi^2}{\ln \dfrac{8}{\ln u_4}}. \quad (29)$$

If $1/k = u_4 < 1.4$, the relative error is $<0.001R$.

Three examples have been computed. They are summarized in Table I, numbered in order of increasing width. The first is a critical shape ($R_1 = 377/2$) while the others are chosen to give a range of widths in the transition region. These three examples have comparable values of the thickness ratio (t/h near 0.08). This ratio is small enough to be representative of small thicknesses ($t/h \ll 1$ and $w/h \ll 1$). Its value is the basis for the graph in Fig. 6(a), and the three points are plotted in relation to the curve of formula (13). The close agreement is regarded as confirmation of that formula (for small thicknesses), especially in the transition region which does not have a clear theoretical basis. This result was the objective of the complete computation of these few examples.

APPENDIX III
EQUIVALENT SQUARE AND CIRCULAR CROSS SECTIONS

The extreme thickness of a narrow strip line is taken to be a square cross section ($t/w = 1$). Therefore the formula for width adjustment contains a constant which assures a close approximation up to this thickness. Its derivation is based on the relations among three equivalent concentric cross sections, the square, the circle, and the thin strip, shown to scale in Fig. 1(c). Their dimensional ratios are such as to give equal values of capacitance and inductance (assuming a small skin depth) and the resulting wave resistance (all in free space).

Starting with the square (of width $= w$), the equivalent circle has a diameter which is greater in the ratio:

$$\sqrt{9/2\pi}\,\frac{\Gamma(5/4)}{\Gamma(7/4)} = 1.1803 = 1/0.8472. \tag{30}$$

The equivalent strip has a width which is double this diameter.

In Fig. 1(c), the large dashed arc is the circular boundary equivalent to the ground plane (radius $= 2h + w$).

Based on these equivalents, a narrow strip of square cross section has the following wave resistance:

$$R = 60 \ln \frac{2h + w}{0.59w} = 60 \ln 1.70(2h/w + 1) = 60 \ln 8h/w' \tag{31}$$

in which

$$w' = \frac{2.36w}{1 + w/2h}.$$

In terms of width and thickness, the corresponding adjustment is

$$\Delta w = w' - w = w\left(\frac{2.36}{1 + t/2h} - 1\right) = w'\left(1 - \frac{1 + t/2h}{2.36}\right). \tag{32}$$

In formula (13) or (14) for Δw, the constant $+1.1$ or -0.26 is inserted to give the correct value for a narrow strip of square cross section.

From another viewpoint, Fig. 1(b) shows cross sections of a square and an inscribed circle (having equal width). In the narrow case, these have wave resistances differing by

$$60 \ln 1.18 = 9.93 \text{ (near } 10\ \Omega). \tag{33}$$

This is denoted, the "10-Ω rule" for these two cross sections. They are known to have equal skin resistances [9] so the loss PF of the circular wire is less in the inverse ratio of its greater reactance and wave resistance.

Appendix IV
Wide Square Cross Section

The wide square cross section is here evaluated in simple terms by invoking a variety of techniques in four regions of each of the two active quadrants. These regions are described in Fig. 8(a). Each is to be evaluated first in terms of normalized capacitance C, which is the simplest concept for the boundaries involved. The dimensions are referred to the height ($h = 1$).

The first region, 1, is taken to be filled with uniform field:

$$C_1 = w/2. \tag{34}$$

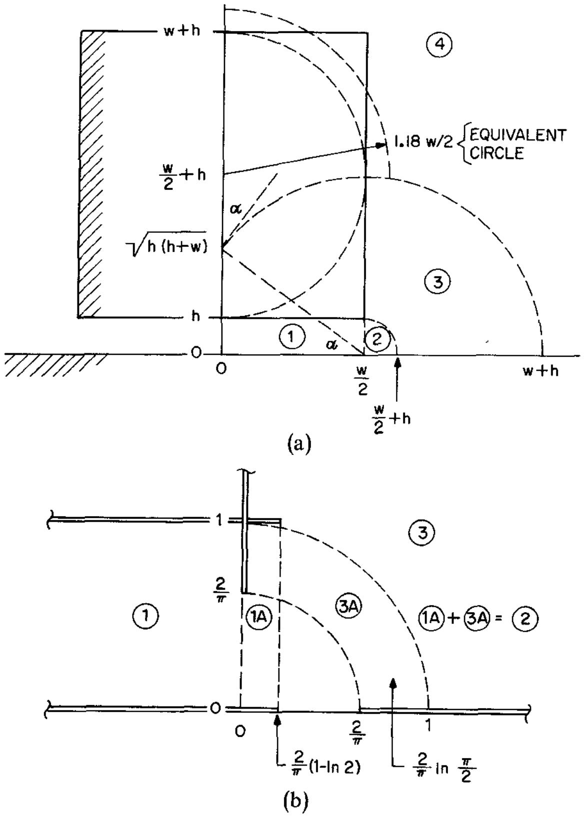

Fig. 8. Derivation for a wide square cross section. (a) The four regions in one quadrant. (b) Analysis of the corner region.

The third region, 3, is taken to be filled with logarithmic field, which is described by radial lines and concentric circles:

$$C_3 = \frac{2}{\pi} \ln (1 + w/2). \tag{35}$$

The second region, 2, is the excess in the transition between 1 and 3, taking into account the nearby distortion in both of those regions. The upper and lower boundaries (corner and straight line) are mapped on parallel straight lines. Then the relative displacement of far points evaluates a "stretch" which represents the excess in the transition. This is divided in two parts for the two directions from the corner. Fig. 8(b) shows this result diagrammatically. Region 2 is represented by 1A and 3A, the respective extensions of the adjacent regions. The validity of this viewpoint resides in the fact that the distortion from the transition decays rapidly in either direction, and also tends to average out. The resulting value of the transition region is

$$C_2 = \frac{2}{\pi}(1 - \ln 2) + \frac{2}{\pi} \ln \frac{\pi}{2} = \frac{2}{\pi}\left(1 - \ln \frac{4}{\pi}\right) = 0.483. \tag{36}$$

The fourth region, 4, is closely related to an inscribed circle, as shown. The region around the inscribed circle would contribute

$$(C_4) = \frac{\alpha}{\pi} \frac{\pi}{\operatorname{acosh}(1 + 2/w)} = \frac{\operatorname{asin} \frac{1}{\sqrt{w+1}}}{\operatorname{asinh} \sqrt{1/w}} < 1. \tag{37}$$

For ease of computation, this ratio can be approximated by

$$(C_4) = \left(\frac{1}{1 + 2/w}\right)^{1/3} < 1. \tag{38}$$

Between the inscribed circle and the equivalent circle (of 1.18 times the radius) the nominal capacitance is that of one quadrant:

$$[C_4] = \frac{\pi/2}{\ln 1.18} = 9.49 = 1/0.105. \tag{39}$$

This is used to increase (C_3) to approximate the capacitance of the square in this quadrant:

$$C_4 = \frac{1}{1/(C_4) - 1/[C_4]} < 1.12. \tag{40}$$

The resulting wave resistance of the two quadrants (restoring w to w/h) is

$$R = \frac{377}{2C_1 + 2C_2 + 2C_3 + 2C_4} = \frac{377}{w/h + 0.966 + \frac{4}{\pi}\ln(1 + w/2h) + \frac{2}{(1 + 2h/w)^{1/3} - 0.1}}. \tag{41}$$

This formula is best for a wide square cross section. The best for narrow is formula (31) for the equivalent round wire shown in Fig. 1(c). Their effective overlap is indicated by their close values for a transition shape ($w/h = 1$):

1) wide (41) above: 94.91;
2) narrow (31): 95.11 (close lower bound);
3) all (13) (2): 95.32.

The intermediate value is believed to be the closest approximation for this case.

Appendix V
Narrow Thin Strip

For a narrow thin strip (without dielectric) there is here derived the second-order approximation stated without proof in the early papers [1], [2]. It is based on a pair of small wires equivalent to the strip. It forms the basis for the simple formulas (1), (2) for any width.

Fig. 9(a) shows a single round wire of unit radius and its known equivalent thin strip whose width is 4 units. It is described on the z plane. It is to be transformed to another plane, $z' = \sqrt{z}$.

This transformation is here performed about one end of the strip cross section, and the result is seen in Fig. 9(b). An equal strip survives but the wire becomes a pair of smaller wires. This pair provides a second-order approximation to the far field of the strip. (This simple equivalence has not been seen by the author in any of the many published exercises in conformal mapping.) It is noted that the smaller wires are not strictly circular in cross section, but that is irrelevant in the use of the concept herein.

Fig. 9(c) shows the thin strip (or equivalent pair of wires) and its image in a ground plane. From this geometry, the mean distance between one pair of current centers and the other pair is increased to

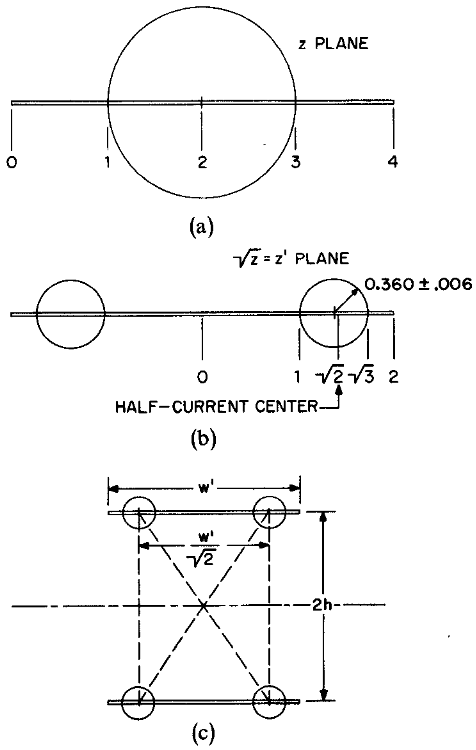

Fig. 9. The pair of small wires equivalent to a thin strip. (a) Thin strip and equivalent round wire. (b) Thin strip and equivalent pair of round wires. (c) Thin strip and its image in a ground plane.

$$2h' = \sqrt{2h\sqrt{(2h)^2 + w^2/2}} = 2h[1 + \tfrac{1}{8}(w/h)^2]^{1/4} = 2h[1 + \tfrac{1}{32}(w/h)^2 - \cdots]. \tag{42}$$

The narrow-strip formula becomes

$$R = 60 \ln 8h'/w = 30 \ln \left[(8h/w)\sqrt{(8h/w)^2 + 8}\right]. \tag{43}$$

This is a reversible formula, giving the following for synthesis:

$$w/h = 8\frac{\sqrt{\sqrt{\exp R/15 + 16} + 4}}{\exp R/30}. \tag{44}$$

For a narrow strip, the 4 term is of the second order ($\exp R/30$) and the 16 is of the fourth order ($\exp R/15$) relative to the first order ($\exp R/60$).

A modification of the above formula gives a simpler form which has a linear slope for $R \to 0$, while retaining the second-order approximation:

$$w/h = 8\frac{\sqrt{\exp R/30 + 2}}{\exp R/30 - 1}. \tag{45}$$

In the limit, $w/h \to (\sqrt{3})240/R = 416/R$.

This is not far from the desired $377/R$. The latter result can be obtained by substituting a slightly lesser value ($\pi^2/4 - 1 = 1.467$) for the constant 2. Asymptotic behavior for a wide strip is then realized at the cost of a slight deficiency in the second-order term. The result is the simple formula (1) giving a close approximation for any width.

The two preceding formulas are extremely close (the relative difference is <0.0002) in the narrow range ($w/h < 2$). The first term of error is proportional to $(w/h)^3$. Their average has a relative error <0.003 if $w/h < 4$. The second formula is closer for greater widths, and therefore probably for all widths. The corresponding formula for analysis is (2) except change π^2 to 12.

This exercise is a striking example of the technique of higher order approximation and its application to obtain a simple and versatile empirical formula with support from various theoretical relations.

Appendix VI
Previously Derived Formulas

Any empirical formula must be validated by comparison with derived formulas. These are typically more complicated and/or restricted as to the range of the width ratio.

Here are some such formulas for a thin strip, selected from the earlier papers, (A-#) referring to the first [1] and (B-#) referring to the second [2]. They are stated in a form that is convenient for computation, to provide a comparison test for the more recent empirical formulas covering the entire range of the width ratio. They are converted to the dimensions used herein (w, h, etc.).

Every one of the derived formulas is essentially the first few terms of a series converging in the extreme of a narrow or wide shape (and small thickness). Therefore a "narrow" or "wide" identity is necessary. The transition between the two occurs for a shape which may be "borderline" for close approximation by either. Hopefully the two kinds will overlap to give a coverage for all shapes. As stated in the early papers and as supported by more recent studies, the transition occurs near $w/h = 1$. The more sophisticated formulas give substantial overlap.

For a narrow thin strip without dielectric, the second-order approximation was not supported by a derivation. One is given here in Appendix VI. Formula (45) and the corresponding modification of (2) are presented as the closest approximation known to date. It provides overlap of the wide range. The relative error is $<0.003R$ if $w/h < 2$.

For a wide thin strip without dielectric, the first paper yields a remarkably close approximation with overlap of the narrow range. The synthesis form is an explicit formulation. Specify

$$R_1 < 60\pi = 188$$

(A-1), (A-45)
$$d' = \frac{\pi}{2} R_c/R_1 = 592/R_1 > \pi \qquad (46)$$

(A-67)
$$d = d' + (2d')^2 \exp - (2d') > \pi \qquad (47)$$

(A-10)
$$c = \sqrt{(d-1)^2 - 1} = \sqrt{d(2d-1)} \qquad (48)$$

(A-68)
$$w/h = \frac{2}{\pi}[c - \operatorname{acosh}(d-1)] = \frac{2}{\pi}[c - \ln(c+d-1)] > 0.3. \qquad (49)$$

The relative error is $<0.001R_1$.

The two preceding approximations give a large overlap. They are closest near $R_1 = 126$ or $w/h = 1$, where the relative difference is $0.0005R_1$. For the graphs in Fig. 2, the computation of any one point is made with the formula judged to be the closer of the two; if so, its relative error is less than this amount.

A comparison of these two formulas can be made in explicit form by the following sequence:

1) wide (49): w/h from R_{1w};
2) narrow (2) (modified): R_{1n} from w/h;
3) ratio: $R_{1n}/R_{1w} = 1 +$ relative difference. (50)

The relative difference of R_1 is the significant comparison.

A thin strip with dielectric likewise has different formulations for narrow and wide. The effective dielectric constant k' depends on the shape w/h and on the dielectric k. One sequence can be used for explicit formulations in any case:

1) specify: R_1;
2) compute (45): w/h;
3) specify: k;
4) compute (53), (57), (52): q, k', $R = R_1/\sqrt{k'}$;
5) graph: R for k, w/h. (51)

The "effective filling fraction" [2], defined as follows, depends mainly on the shape and less on the dielectric:

$$q = \frac{k'-1}{k-1}, \qquad k' = 1 + q(k-1). \qquad (52)$$

Because it has only second-order dependence on the dielectric, a simple formula for a mean value of k is sufficient for practical purposes. A mean value ($k = 3$) is chosen because it places the effective dielectric k' midway between the extremes (for $1 < k < \infty$) and within the midrange of practical values. Some formulas will be stated for this mean value, with a supplemental term which may be ignored, having a factor $(1/k - \frac{1}{3})$. It is graphed in Fig. 2 in terms of w/h directly and R_1 indirectly.

The shape dependence of the filling fraction was derived in terms of the wave resistance without dielectric (R_1) and is most simply expressed in those terms. This R_1 and the actual shape w/h are related by various formulas. The filling fraction is here expressed in very simple form from the previous derivations for narrow and wide.

For a narrow thin strip with dielectric, the effective dielectric constant is formulated as follows. The shape is introduced in terms of the wave resistance without dielectric (R_1):

(B-32), (B-44)
$$q = \frac{1}{2} + \frac{30}{R_1}\left(\ln\frac{\pi}{2} + \frac{1}{k}\ln\frac{4}{\pi}\right) = \frac{1}{2} + \frac{16}{R_1}\left[1 + 0.453\left(\frac{1}{k} - \frac{1}{3}\right)\right] \qquad (53)$$

(B-32), (B-45)
$$k' = \frac{k+1}{2} + \frac{60}{R_1}\frac{k-1}{2}\left(\ln\frac{\pi}{2} + \frac{1}{k}\ln\frac{4}{\pi}\right). \tag{54}$$

The relative error is <0.01 of k' if R' is >70; w/h is <3; q is <0.72.

For a wide thin strip with dielectric, the effective dielectric constant is formulated in terms of parameters defined above and here:

(B-8) $d = \frac{\pi}{2} R_c/R_1 = 592/R_1 > \pi, \qquad R_1 < 188$

(A-14), (A-16), (B-16)
$$s' = 0.732[\operatorname{acosh}(d-1) - \operatorname{acosh}(0.358d + 0.598)] \tag{55}$$

(B-25)
$$s'' = \ln 4 - 1 - 1/(2d-1) = 0.386 - 1/(2d-1) \tag{56}$$

(B-4)
$$q = 1 - \frac{1}{d}\left[\operatorname{acosh}(d-1) - s'' + \frac{s''-s'}{k}\right]. \tag{57}$$

The relative error is <0.01 (estimated).

For the mean case ($k = 3$), this result is approximated very closely by the simple formula
$$q = 1 - \frac{R_1}{592}\ln\frac{710}{R_1}. \tag{58}$$

The overlap between the two simple formulas (53) and (58) occurs near $R_1 = 100$ or $w/h = 1.5$.

The narrow and wide simple formulas for the mean case can be integrated and supplemented by an adjustment for any k, as follows:
$$q = \frac{1}{7}\left[1 + \frac{6}{1 + \frac{R_1}{507}\ln\left(\frac{710}{R_1} + 1\right)}\right] - \frac{0.2}{\frac{R_1}{220} + \frac{220}{R_1}} + \left(\frac{1}{k} - \frac{1}{3}\right)\frac{0.05}{\frac{R_1}{90} + \frac{90}{R_1}}. \tag{59}$$

The relative error is <0.01 of k'. The first term is a transition between the narrow and wide extremes. The second term is a very close adjustment for the intermediate range. The last term is negligible in the practical effect on k', so it serves mainly to indicate the weakness of the dependence on k.

One simple example is here reviewed in Table II as a test of various derivations for a thin strip without dielectric. It is a shape ($w/h = 1$) which is in the region of transition between narrow and wide approximations. The wave resistance R_1 is based on free space (120π). The items are listed in order of increasing error from the first. The derivation is described with respect to its development from the extreme of narrow and/or wide strip. The following notes give further comments.

TABLE II
COMPARISON OF FORMULAS FOR ONE EXAMPLE

Identification	Derivation	R_1 (ohms)	Error
(S) [14]	unrestricted	126.553	0
(W.1) (43)	narrow	126.533	-.020
(W.2) (2) (modified)	narrow	126.528	-.025
(W.3) [1] (A-68)	wide	126.473	-.042
(W.4) [1] (A-66)	narrow	126.641	+.088
(W.5) (2)	narrow-wide	126.310	-.243
(K) [16]	wide	127.857	+1.304
(W.6) [1] (A-71)	wide	124.424	-2.129

(S) Schneider's example is derived rigorously from elliptic integrals and is taken to be "exact" for purposes of comparison. Its relation to the other items tends to confirm its validity.

(W.1) This is the derivation based on the pair of wires equivalent to a narrow thin strip. It is the closest approximation (the relative error is <0.0002).

(W.2) This is similar to (W.1) but modified to a form suitable for matching the wide extreme. It is the reverse of formula (45).

(W.3) (W.4) These are the closest approximations given in the 1964 paper. They bracket the correct value within a relative difference of ± 0.0007. Their computation is much easier than (S).

(W.5) This is the only item providing a rather close approximation over the entire range of shape.

(K) Kaden's "wide" formula is an approximation to his derivation from elliptic integrals. The error (about 1 percent) indicates that this shape is "borderline" for his approximation. It is comparable with (W.6) in its explicit form and in simplicity, and gives a closer approximation.

This concludes a summary of the earlier formulas, and some more recent, as required for the above procedure. They are adequate for a set of close computations for a thin strip with any dielectric. These may be used for checking any empirical formula such as those proposed herein. They are used for the graphs in Fig. 2.

APPENDIX VII
COMPUTATION OF LOSS BY NUMERICAL DIFFERENTIATION

In practical applications of a strip line, the PF of conductor loss is usually determined by the skin effect. Some simple rules are applicable if the skin depth δ is much less than the least transverse dimension. One is the "incremental-inductance rule" stated by the author [3]. It relates the skin loss with the inductance, by a formula based on differentiation.

In a transmission line made of perfect conductors, the wave resistance without dielectric (R_1) is uniquely related to the inductance, so that formula may be used instead. Then the loss PF of the skin effect may be expressed as follows:

$$p = \Delta L/L = \frac{R_\delta - R_1}{R_\delta} = 1 - R_1/R_\delta \ll 1. \tag{60}$$

The incremental-inductance rule is here represented by the relative increment of inductance ($\Delta L/L$) that would be caused by removing a thickness ($\delta/2$) from the face of every conductor bounding the field. The wave resistance (R_1) for a perfect conductor would be increased in the same ratio if the boundary were modified in the same manner. Then this change (from R_1 to R_δ) is used to compute the PF.

The elegant application of the incremental-inductance rule was stated in terms of the analytic differentiation of an inductance formula in terms of a simple continuous function. In its more general application, such a formula may not be available. The rule is still useful if the inductance variation is formulated continuously over a range of dimensions. Such a formula can be subjected to analytic differentiation, but the resulting expression may be much more complicated than the inductance formula from which it is derived. This has been the experience of some workers who have taken this approach in evaluating the loss PF in a strip line [13], [17].

In the meantime, the advent of the digital calculator has opened up a new opportunity for the differentiation of a complicated formula. It enables a close approximation of the derivative by computing a small finite difference. The basic formula is used twice, which requires little more effort in a programmable calculator. The versatility of this procedure reaches a peak in the Hewlett-Packard pocket models, HP-25 and HP-65. The convenience and availability of the HP-25 provided the author with the tools and incentive to prepare this paper.

Having stated the objective of numerical differentiation by finite differences, the programming is routine. Fig. 10 shows the flow chart of one such program. It serves to bring out the application of some features available in the HP-25. It includes as the principal subroutine, some formula for R_1 in terms of the transverse dimensions. After incrementing the dimensions in accord with the skin effect, this subroutine is traversed a second time for R_δ. The relative increase is interpreted as the PF in direct or normalized form. The following features of the program are notable.

The skin depth δ may be evaluated and then utilized to give the PF p for any example. The more versatile normalized PF P may be obtained by arbitrarily choosing a small difference (say $\delta = 0.0001$). Then the result approaches the analytic derivative. The value of the difference cancels out in the normalized form. The number of decimal places in the small difference may be somewhat less than one half the number available in the computation. The skin depth or small difference is entered once in one register (R4) where it need not be renewed unless a change is desired.

The dimensions w,h,t are entered in assigned registers R0,R1,R2. For the second computation, each dimension is incremented by $\pm\delta$. Each dimension is between two conductor faces so the removal of $\delta/2$ on each face requires that

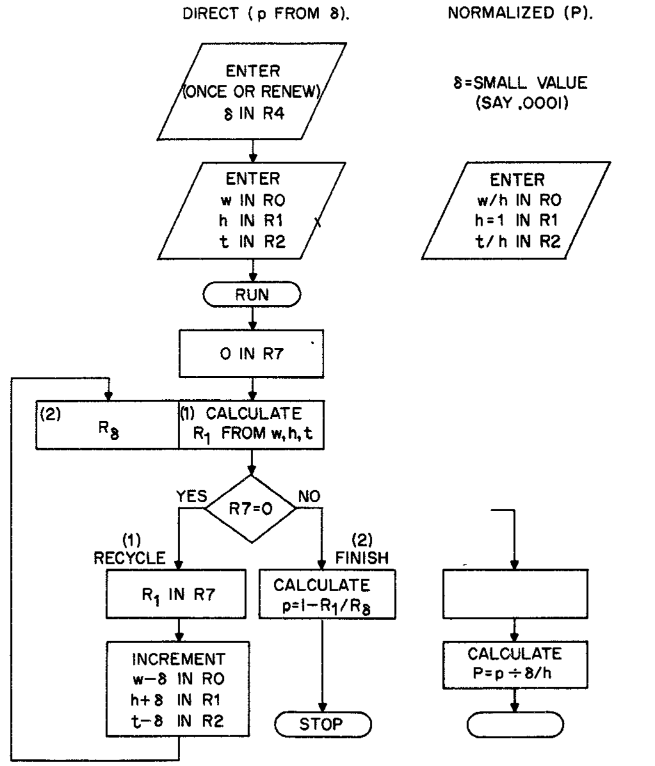

Fig. 10. Flow chart for computing the loss power factor by numerical differentiation.

the dimension be changed by δ, either increased or decreased. The increment is entered by arithmetic in the register.

Conditional branching is required at the end of each execution of the R_1 subroutine. One register (R7) is vacated until the end of the first execution, then occupied by R_1, which signals the end of the second. This serves also to retain R_1 for comparison with R_δ.

If the program storage is inadequate for the principal subroutine and also the transitional subroutines, one or more of the latter is easily performed manually.

In Fig. 10, the right-hand column of notes describe the program changes for the normalized form.

Appendix VIII
Formulas for the Skin Depth

Here are some formulas for the skin depth in nonmagnetic conductors [3], [7]:

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} = \frac{1}{\sqrt{\pi f \mu_0 \sigma}} = \sqrt{\frac{\lambda G_0}{\pi\sigma}} = \sqrt{\frac{\lambda\rho}{\pi R_0}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\lambda}{30\sigma}} = \frac{1}{2\pi}\sqrt{\frac{\lambda\rho}{30}}$$

$$= 0.0291\sqrt{\lambda/\sigma} = \frac{1}{34.4}\sqrt{\lambda/\sigma}. \tag{61}$$

In copper ($\sigma = 58$ Mmho/m):

$$\delta_c = 3.81\sqrt{\lambda}\ \mu\text{m} = \sqrt{\frac{4.36\ \text{kHz}}{f}}\ \text{mm} = \sqrt{\frac{4.36\ \text{GHz}}{f}}\ \mu\text{m} = \frac{66}{\sqrt{f}}\ \text{mm} = \frac{0.066}{\sqrt{f\,(\text{MHz})}}\ \text{mm}. \tag{62}$$

Symbols used in (61) and (62) are defined below.

ω = $2\pi f$ = radian frequency (radians/second).
λ = wavelength in free space (meters).
R_0 = $1/G_0$ = wave resistance of a plane wave in a square area in free space (ohms).
μ_0 = magnetivity (magnetic permeability) in free space (henries/meter).
σ = $1/p$ = conductivity in copper (mhos/meter).
R_0 = $377 = 120\pi\ \Omega$.
μ_0 = $0.4\pi = 1.257\ \mu$H/m.

Appendix IX
Recent Article on Wide Strip with Thickness

Subsequent to the preparation of this paper, the author has seen a recent article related to the subject [19]. W. H. Chang has described an ingenuous and powerful approximation based on conformal mapping. To that extent, it has something in common with the author's 1964 paper [1]. Some thickness is accommodated at the expense of some refinements in other respects. The result is a very useful approximation for wide strips with thickness. To yield this in analytic form is a major achievement. Moreover, it appears that his result may be closely bracketed by further appreciation of his approximation.

Relevant to the present paper, Chang gives a table of examples computed from his formula and also by a numerical approximation from W. J. Weeks [18]. The agreement is very close. Most of those examples fall within the range of validity of the present paper, formulas (1), (2), (13), (14) without dielectric. The agreement is well within $0.01R$. The formulas herein offer a close approximation for any width. They are based on a thin strip with width adjustment for thickness. Chang's formulas for a wide strip are remarkable for including the width and the thickness in one formula.

The small-thickness examples reported in Appendix II are in the range of marginal approximation by Chang, so they have not been compared.

References

[1] H. A. Wheeler, "Transmission-line properties of parallel wide strips by a conformal-mapping approximation," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-12, pp. 280–289, May 1964.

[2] ——, "Transmission-line properties of parallel strips separated by a dielectric sheet," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 172–185, Mar. 1965.

[3] ——, "Formulas for the skin effect," *Proc. IRE*, vol. 30, pp. 412–424, Sept. 1942. (Skin loss by the "incremental-inductance rule.")

[4] F. Oberhettinger and W. Magnus, *Applications of Elliptic Functions in Physics and Technology.* New York: Springer, 1949. (Wave resistance of coplanar strip, p. 63. Tables of K'/K, p. 114.)

[5] E. Weber, *Electromagnetic Fields—Theory and Applications—Mapping of Fields.* New York: Wiley, 1950. (Thickness, p. 347.)

[6] H. A. Wheeler, "Transmission-line impedance curves," *Proc. IRE*, vol. 38, pp. 1400–1403, Dec. 1950.

[7] ——, "Universal skin-effect chart for conducting materials," *Electronics*, vol. 25, no. 11, pp. 152–154, Nov. 1952.

[8] S. Cohn, "Problems in strip transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-3, pp. 119–126, Mar. 1955. (Sandwich line, thickness and loss.)

[9] H. A. Wheeler, "Skin resistance of a transmission-line conductor of polygon cross section," *Proc. IRE*, vol. 43, pp. 805–808, July 1955.

[10] D. S. Lerner (Wheeler Labs., Inc.), unpublished notes, Nov. 1963. (Very wide strip, width adjustment for thickness. Half-shielded type compared with sandwich type.)

[11] M. Caulton, J. J. Hughes, and H. Sobol, "Measurements on the properties of microstrip transmission lines for microwave integrated circuits," *RCA Rev.*, vol. 27, pp. 377–391, Sept. 1966.

[12] H. Sobol, "Extending IC technology to microwave equipment," *Electronics*, vol. 40, no. 6, pp. 112–124, Mar. 1967. (A simple empirical formula, remarkably close for a practical range.)

[13] R. A. Pucel, D. J. Masse, and C. P. Hartwig, "Losses in microstrip," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 342–350, June 1968. (By analytic differentiation.)

[14] M. V. Schneider, "Microstrip lines for microwave integrated circuits," *Bell Syst. Tech. J.*, vol. 48, pp. 1421–1444, May 1969.

[15] ——, "Dielectric loss in integrated microwave circuits," *Bell Syst. Tech. J.*, vol. 48, pp. 2325–2332, Sept. 1969.

[16] H. Kaden, "Advances in microstrip theory," *Siemens Forsch. u. Entwickl. Ber.*, vol. 3, pp. 115–124, 1974. (Computation of wave resistance and loss for a wide strip of small thickness.)

[17] R. Mittra and T. Itoh, "Analysis of microstrip transmission lines," in *Advances in Microwaves*, vol. 8. New York: Academic Press, 1974, pp. 67–141. (Latest review, many references.)

[18] W. J. Weeks, "Calculation of coefficients of capacitance for multiconductor transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 35–43, Jan. 1970. (Numerical method for a rectangular cross section with a parallel plane.)

[19] W. H. Chang, "Analytical IC metal-line capacitance formulas," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 608–611, Sept. 1976. (Conformal mapping approximation for a wide strip with thickness. Examples compared with numerical method of Weeks [18] using his program.)