The Rectangular Loop Antenna as a Dipole*

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Summary—An integral equation for the current in a rectangular loop of wire is derived for a loop that is driven by two generators located at the centers of one pair of opposite sides. The EMF's are equal in magnitude and in phase in the sense that they maintain currents in the generators that are in the same direction relative to the coordinate system and, therefore, in opposite directions from the point of view of circulation around the loop. An approximate solution is obtained for the distribution of current around the loop and for the driving-point impedance. It is shown that the solution for the rectangle of wire reduces to that of the symmetrically driven folded dipole when one dimension is made electrically small and to a section of transmission line driven simultaneously at both ends when the other dimension is made small. The loop that is electrically small in both directions is also examined.

Introduction

The circuit properties of the rectangular loop antenna have been studied in the past primarily in two special cases: the electrically small loop shown in Fig. 1(a) and the folded dipole shown in Fig. 1(b). An analysis[1] of the former usually depends upon the assumption that the current is essentially uniform in amplitude and phase in a circulatory sense around the loop when this is driven by a generator located at the center of one side. The currents at corresponding points in opposite sides are then equal in magnitude and instantaneously opposite in direction with respect to the space coordinates, so that, by analogy with the balanced open-wire line, they may be called currents in a *transmission-line mode*. Possible currents in opposite pairs of sides that are equal and instantaneously codirectional at corresponding points, currents that belong to what may be called a transverse *dipole mode*, are ignored or neglected in such an analysis. The conventional folded dipole shown in Fig. 1(b) is a rectangular loop that is electrically small in one dimension but not in the other; it is driven by a generator at the center of one of the longer sides. The folded dipole has been analyzed by the method of symmetrical components[2,3] which, in effect, divides the current into two independent parts and permits their separate determination as currents in the antisymmetrical or transmission-line mode, and currents in the symmetrical or dipole mode. The former are excited by equal and opposite generators, the latter by

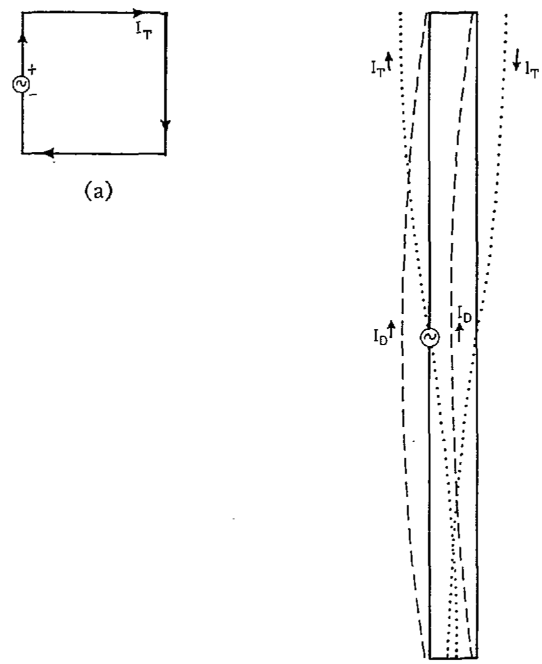

Fig. 1—Extreme types of loop antennas. (a) Conventional loop. Perimeter is small compared to wavelength; currents are equal in amplitude and opposite in direction on opposite sides. (b) Folded dipole. Perimeter is near a wavelength; dipole-mode currents I_D are equal and codirectional. Transmission-line currents I_T are equal and opposite.

equal and codirectional generators at the centers of both of the longer sides. By superposition the generator EMF's add on one side, subtract and cancel on the other.

When a rectangular loop of arbitrary dimensions is driven at the center of one side by a voltage V^e, currents in *both* the transmission-line mode and the dipole mode are excited. The former are maintained by the voltages $\frac{1}{2}V^e$ and $-\frac{1}{2}V^e$, respectively, at the centers of the two longer sides, the latter by two equal voltages $\frac{1}{2}V^e$ at the centers of these sides. As an essential step in the complete analysis of the general rectangular loop as a transmitting and receiving antenna, and in order to determine the circuit properties of the rectangular loop as a dipole antenna in its own right (Fig. 2) it is the purpose of the present study to investigate the currents in, and impedance of a rectangle of dimensions $2c$ and $2d$ constructed of a single turn of wire of radius a and driven at the centers of the sides of length $2d$ by generators with equal and codirectional EMF's. The currents maintained by these generators are in the vertical dipole mode and no others are generated. The loop to be analyzed is shown in Fig. 3. Note that the method used

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[1] See, *e.g.*, R. W. P. King, "Electromagnetic Engineering," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 421–431; 1945.

[2] R. W. P. King, H. R. Minno, and A. H. Wing, Jr., "Transmission Lines, Antennas, and Wave Guides," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 224–226; 1945.

[3] R. W. P. King, "Theory of Linear Antennas," Harvard University Press, Cambridge, Mass., pp. 334–361; 1956.

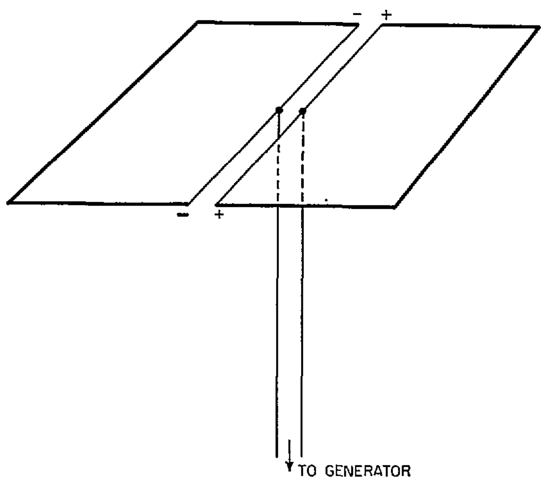

Fig. 2—Horizontal loop as a dipole antenna maintaining a horizontally polarized electric field.

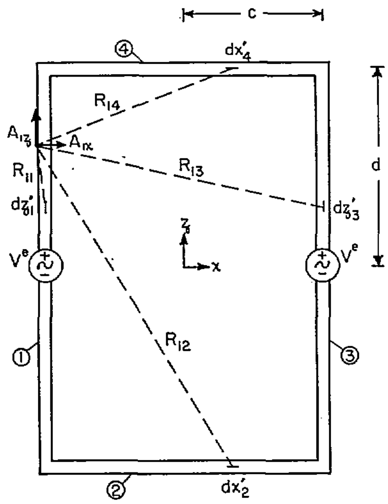

Fig. 3—Rectangular loop excited in the vertical dipole mode.

in the previous analysis[2,3] of the symmetrical or dipole mode specifically in the folded dipole cannot be applied in general, since it assumes that the pair of sides of length $2c$ is so short that it contributes negligibly to the problem. In the proposed analysis the sides of length $2c$ are not so restricted. They may be electrically long or short, and longer or shorter than the other pair of sides that contain the generators.

The Integral Equations

The integral equations for the currents in the four sides of the loop may be obtained from the boundary conditions obeyed by the tangential component of the electric field at the highly conducting surfaces of the rectangle of wire. For sides 1 and 2 they may be expressed in terms of the scalar and vector potentials ϕ and $\boldsymbol{A}$ as follows:

$$E_{1z} = \frac{-\partial\phi_{1(x)}}{\partial z} - \frac{j\omega}{\beta_0^2}\left(\frac{\partial^2 A_{1z}}{\partial z^2} + \beta_0^2 A_{1z}\right) \doteq 0 \qquad \text{(1a)}$$

$$E_{2x} = \frac{-\partial\phi_{2(z)}}{\partial x} - \frac{j\omega}{\beta_0^2}\left(\frac{\partial^2 A_{2x}}{\partial x^2} + \beta_0^2 A_{2x}\right) \doteq 0 \qquad \text{(1b)}$$

where $\beta_0 = \omega/v_0 = 2\pi/\lambda_0$ and $v_0 = 1/\sqrt{\epsilon_0\mu_0} \doteq 3\times 10^8$ meters/sec. The vector and scalar potentials at arbitrary points *on the surface* of conductors 1 and 2 have the following forms. Note that the symmetry relations,

$$A_3 = A_1, \qquad A_4 = -A_2, \qquad \text{(2)}$$

have been invoked.

$$A_1 = \hat{x}A_{1x} + \hat{z}A_{1z}, \qquad \phi_1 = \phi_{1(x)} + \phi_{1(z)};$$

$$\phi_{1(x)} = \frac{j\omega}{\beta_0^2}\frac{\partial A_{1x}}{\partial x}, \qquad \phi_{1(z)} = \frac{j\omega}{\beta_0^2}\frac{\partial A_{1z}}{\partial z}; \qquad \text{(3a)}$$

$$A_2 = \hat{x}A_{2x} + \hat{z}A_{2z}, \qquad \phi_2 = \phi_{2(x)} + \phi_{2(z)};$$

$$\phi_{2(x)} = \frac{j\omega}{\beta_0^2}\frac{\partial A_{2x}}{\partial x}, \qquad \phi_{2(z)} = \frac{j\omega}{\beta_0^2}\frac{\partial A_{2z}}{\partial z}; \qquad \text{(3b)}$$

where

$$A_{1z} = \frac{1}{4\pi\nu_0}\int_{-d}^{d} I_{1z}(z')\mathcal{K}_{1A}(z, z')dz',$$

$$\phi_{1(x)} = \frac{1}{4\pi\epsilon_0}\int_{-c}^{c} q_2(x')\mathcal{K}_{1B}(z, x')dx' \qquad \text{(4a)}$$

$$A_{2x} = \frac{1}{4\pi\nu_0}\int_{-c}^{c} I_{2x}(x)\mathcal{K}_{2A}(x, x')dx',$$

$$\phi_{2(z)} = \frac{1}{4\pi\epsilon_0}\int_{-d}^{d} q_1(z')\mathcal{K}_{2B}(x, z')dz'. \qquad \text{(4b)}$$

The notation $\nu_0 = 1/\mu_0$ is used, where $\mu_0 = 4\pi\times 10^{-7}$ farad/m. The kernels are defined as follows:

$$\mathcal{K}_{1A}(z, z') = \mathcal{K}_{11}(z, z') + \mathcal{K}_{13}(z, z'),$$

$$\mathcal{K}_{1B}(z, x') = \mathcal{K}_{12}(z, x') - \mathcal{K}_{14}(z, x'), \qquad \text{(5a)}$$

$$\mathcal{K}_{2A}(x, x') = \mathcal{K}_{22}(x, x') - \mathcal{K}_{24}(x, x'),$$

$$\mathcal{K}_{2B}(x, z') = \mathcal{K}_{21}(x, z') + \mathcal{K}_{23}(x, z'), \qquad \text{(5b)}$$

where

$$\mathcal{K}_{ij}(u, v') = \frac{e^{-j\beta_0 R_{ij}}}{R_{ij}}\cdot \qquad \text{(5c)}$$

The several distances are defined as follows:

$$R_{11} = \sqrt{(z'-z)^2 + a^2} \qquad R_{13} = \sqrt{(z'-z)^2 + 4d^2}, \qquad \text{(6a)}$$

$$R_{12} = \sqrt{(d+z)^2 + (c+x')^2}, \quad R_{14} = \sqrt{(d-z)^2 + (c+x')^2}, \qquad \text{(6b)}$$

$$R_{22} = \sqrt{(x'-x)^2 + a^2}, \qquad R_{24} = \sqrt{(x'-x)^2 + 4c^2}, \qquad \text{(6c)}$$

$$R_{21} = \sqrt{(c+x)^2 + (d+z')^2}, \quad R_{23} = \sqrt{(c-x)^2 + (d+z')^2}. \qquad \text{(6d)}$$

The solutions of (1a) and (1b) for A_{1z} and A_{2x}, respectively, may be expressed as sums of trigonometric functions and a particular integral. They may then be combined with (4a) and (4b) to obtain the following integral equations in which $\zeta_0 = \sqrt{\mu_0\epsilon_0}$:

$$4\pi\nu_0 A_{1z}(z) = \int_{-d}^{d} I_{1z}(z')\mathcal{K}_{1A}(z, z')dz'$$
$$= \frac{-j4\pi}{\zeta_0}[C_{1z}\cos\beta_0 z + C_{2z}\sin\beta_0 z - \theta_{1A}(z)], \quad (7a)$$
$$4\pi\nu_0 A_{2x}(x) = \int_{-c}^{c} I_{2x}(x')\mathcal{K}_{2A}(x, x')dx'$$
$$= \frac{-j4\pi}{\zeta_0}[C_{1x}\cos\beta_0 x + C_{2x}\sin\beta_0 x - \theta_{2A}(x)]. \quad (7b)$$

The particular integrals in (7a) and (7b) are

$$\theta_{1A}(z) = \beta_0\int_0^z \phi_{1(z)}(w)\cos\beta_0(z-w)dw$$
$$- \phi_{1(z)}(0)\sin\beta_0 z, \quad (8a)$$
$$\theta_{2A}(x) = \beta_0\int_0^x \phi_{2(z)}(w)\cos\beta_0(x-w)dw$$
$$- \phi_{2(z)}(0)\sin\beta_0 x. \quad (8b)$$

As a consequence of geometrical and electrical symmetry and the assumed driving conditions, the following relations obtain:

$$A_{1z}(-z) = A_{1z}(z),\ A_{1x}(-z) = -A_{1x}(z) \quad (9a)$$
$$\phi_1(-z) = -\phi_1(z),\ \phi_{1(z)}(0) = 0,$$
$$\lim_{z\to 0}[\phi_1(z) - \phi_1(-z)] = 2\phi_{1(z)}(0) = V^e \quad (9b)$$
$$I_1(-z) = I_1(z),\ q_1(-z) = -q_1(z) \quad (9c)$$
$$A_{2x}(-x) = -A_{2x}(x),\ A_{2z}(-x) = A_{2z}(x),\ A_{2x}(0) = 0 \quad (9d)$$
$$\phi_2(-x) = \phi_2(x) \quad (9e)$$
$$I_{2x}(-x) = -I_{2x}(x),\ q_2(-x) = q_2(x). \quad (9f)$$

The continuity of both current and scalar potential at the corners demands that

$$I_{1z}(-d) = -I_{2x}(-c), \quad \phi_1(-d) = \phi_2(-c). \quad (10)$$

It follows directly from (7b) with (9d) that

$$C_{1x} = 0. \quad (11)$$

Expressions for the scalar potentials corresponding to (7a) and (7b) for the vector potentials but specialized to satisfy (9a)–(9f) and (11) are readily obtained. Thus,

$$\phi_1(z) = -C_{1z}\sin\beta_0 z + C_{2z}\cos\beta_0 z + \theta_{1V}(z), \quad z \geqq 0, \quad (12)$$
$$\phi_1(z) = -C_{1z}\sin\beta_0 z - C_{2z}\cos\beta_0 z + \theta_{1V}(z), \quad z \leqq 0, \quad (13)$$

where

$$\theta_{1V}(z) = \beta_0\int_0^z \phi_{1(z)}(w)\sin\beta_0(z-w)dw. \quad (14)$$

Similarly, and for positive and negative values of x,

$$\phi_2(x) = C_{2x}\cos\beta_0 x - \frac{1}{\beta_0}\frac{\partial\theta_{2A}(x)}{\partial x} + \phi_{2(z)}(x)$$
$$= C_{2x}\cos\beta_0 x + \theta_{2V}(x), \quad (15)$$

where

$$\theta_{2V}(x) = \beta_0\int_0^x \phi_{2(z)}(w)\sin\beta_0(x-w)dw$$
$$+ \phi_{2(z)}(0)\cos\beta_0 x. \quad (16)$$

The driving voltage as defined in (9b) now may be introduced. With (12) and (13) it follows that

$$C_{2z} = \tfrac{1}{2}V^e. \quad (17)$$

The simultaneous integral equations for the currents I_{1z} and I_{2x} are:

$$\int_{-d}^{d} I_{1z}(z')\mathcal{K}_{1a}(z, z')dz'$$
$$= \frac{-j4\pi}{\zeta_0}\left[C_{1z}\cos\beta_0 z + \frac{1}{2}V^e\sin\beta_0|z| - \theta_{1A}(z)\right] \quad (18a)$$
$$\int_{-c}^{c} I_{2x}(x)\mathcal{K}_{2A}(x, x')dx' = \frac{-j4\pi}{\zeta_0}[C_{2x}\sin\beta_0 x - \theta_{2A}(x)] \quad (18b)$$

where C_{1z} and C_{2x} may be evaluated by requiring continuity of current and scalar potential at the corners, as specified in (10). Note that with the equations of continuity,

$$\frac{\partial I_{1z}}{\partial z} + j\omega q_1 = 0, \qquad \frac{\partial I_{2x}}{\partial x} + j\omega q_2 = 0, \quad (19)$$

$\theta_{1A}(z)$ is a function of $I_{2A}(x)$, and $\theta_{2A}(x)$ is a function of I_{1z}.

Approximate Solutions of the Integral Equations

The simultaneous integral equations (18a) and (18b) may be solved approximately for the currents $I_1(z) = I_3(z)$ and $I_2(x) = -I_4(x)$ by a method of iteration resembling that used in the analysis of coupled parallel antennas.[4] The present problem is complicated by the presence of two equations rather than one. In devising a suitable iteration procedure two points are noteworthy. First, the mutual terms interrelating the two equations are limited to the particular integrals which take account of capacitive coupling between adjacent, mutually perpendicular sides. Since this effect is significant primarily near the corners, it may be assumed that it is not of primary significance in determining the distributions of current and may be included in the first correction term. The second point to be noted is that the oscillation of charges and currents in the rectangle when excited in the dipole mode as shown in Fig. 3 must correspond roughly to that in two parallel center-driven antennas each of length $2(c+d)$. This follows from the fact that the currents are continuous at the corners and vanish at the centers $x=0$, $z=\pm d$, of the two sides without generators. This suggests that (18a) and (18b) may be expressed as follows:

[4] *Ibid.*, p. 264 ff.

$$\int_{-(d+c)}^{d+c} I_{1z}(z')\mathcal{K}_{11}(z, z')dz' + \int_{-d}^{d} I_{1z}(z')\mathcal{K}_{13}(z, z')dz'$$
$$- \left(\int_{-(d+c)}^{-d} \int_{d}^{d+c}\right) I_{1z}(z')\mathcal{K}_{24}(z, z)dz'$$
$$= \frac{-j4\pi}{\zeta_0}\left[C_{1z}\cos\beta_0 z + \frac{1}{2}V^e \sin\beta_0 |z| - \theta_{1A}(z)\right] + \mathcal{G}_1(z) \quad \text{(20a)}$$

$$\int_{-(d+c)}^{d+c} I_{2x}(x)\mathcal{K}_{22}(x, x')dx' - \int_{-c}^{c} I_{2x}(x')\mathcal{K}_{24}(x, x')dx'$$
$$- \left(\int_{-(d+c)}^{-c} + \int_{c}^{d+c}\right) I_{2x}(x')\mathcal{K}_{13}(x, x')dx'$$
$$= \frac{-j4\pi}{\zeta_0}\left[C_{2x}\sin\beta_0 x - \theta_{2A}(x)\right] + \mathcal{G}_2(x) \quad \text{(20b)}$$

where

$$\mathcal{G}_1(z) = \left(\int_{-(d+c)}^{-d} + \int_{d}^{d+c}\right) I_{1z}(z')\mathcal{K}_{2A}(z, z')dz' \quad \text{(21a)}$$

$$\mathcal{G}_2(z) = \left(\int_{-(d+c)}^{-c} + \int_{c}^{d+c}\right) I_{2x}(x')\mathcal{K}_{1A}(x, x')dx'. \quad \text{(21b)}$$

Note that $\mathcal{K}_{2A}(z, z')$, $\mathcal{K}_{1A}(x, x')$, R_{24} in (20a), and R_{13} in (20b) are obtained from (5) and (6) with x substituted for z and vice versa.

In the integral on the left in (20a), $I_{1z}(z')$ is the actual current in side 1 of the rectangle in the range $-d \leqq z' \leqq d$. In the ranges $-(c+d) \leqq z \leqq -d$ and $d \leqq z \leqq (c+d)$, the current is the fictitious extension in the z direction of currents actually existing in the top and bottom of the loop as $I_{2x}(x')$ and $I_{4x}(x')$. Similarly, in the integral on the left in (20b), $I_{2x}(x')$ actually exists only in the range $-c \leqq x \leqq c$. Outside this range the currents are the fictitious extension in the x direction of actual currents in the vertical sides in the z direction. This is shown schematically in Fig. 4 for sides 1 and 2. The addition of the integrals $\mathcal{G}_1(z)$ and $\mathcal{G}_2(x)$, respectively, to both sides of (21a) and (21b) modifies the left sides (which are proportional to the tangential components of the vector potential) in a manner to improve the constancy of the ratio of vector potential to current especially near the corners where large deviations occur. Note that whereas the change in the direction of the current at a corner, for example at $z=-d$, $x=-c$, can involve no great modification in its amplitude or distribution as compared with the current at the corresponding point $z=-d$ in two parallel antennas of length $2(c+d)$ when driven by equal generators in phase, this is not true of the component of the vector potential tangent to the conductor. The currents in the conductor on the two sides of the right-angle bend do not contribute to the same component of the vector potential as they do when there is no bend and the conductor is straight.

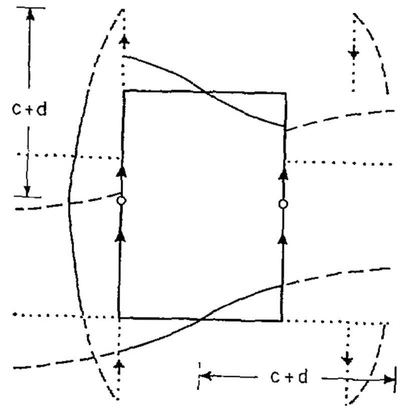

Fig. 4—Actual and fictitious conductors and currents used in establishing the integral equation in a form suitable for iteration.

The solutions of (21) by iteration may be carried out much as in the case of two parallel antennas.[4] For this purpose let

$$g_1(z, z') \doteq \frac{I_{1z}(z')}{I_{1z}(z)}; \qquad g_2(x, x') \doteq \frac{I_{2x}(x')}{I_{2x}(x)} \quad \text{(22)}$$

be approximate relative distributions of current. Also let the following functions be defined:

$$\Psi_1(z) = \Psi_s + \gamma_1(z) = \int_{-(d+c)}^{d+c} g_1(z, z')\mathcal{K}_{11}(z, z')dz'$$
$$+ \int_{-d}^{d} g_1(z, z')\mathcal{K}_{13}(z, z')dz'$$
$$- \left(\int_{-(d+c)}^{-d} + \int_{d}^{d+c}\right) g_1(z, z')\mathcal{K}_{24}(z, z')dz' \quad \text{(23a)}$$

$$\Psi_2(x) = \Psi_s + \gamma_2(x) = \int_{-(d+c)}^{d+c} g_2(x, x')\mathcal{K}_{22}(x, x')dx'$$
$$- \int_{-c}^{c} g_2(x, x')\mathcal{K}_{24}(x, x')dx'$$
$$+ \left(\int_{-(d+c)}^{c} + \int_{c}^{d+c}\right) g_2(x, x')\mathcal{K}_{13}(x, x')dx' \quad \text{(23b)}$$

where Ψ_s is an appropriately defined magnitude and $\gamma_1(z)$ and $\gamma_2(x)$ are the necessary and presumably small functions required to make (23) exact. If $g_1(z, z')$ and $g_2(x, x')$ can be so chosen that they are good approximations of the actual current distributions, the following integrals are small:

$$\mathfrak{D}_1(z) = \int_{-(d+c)}^{d+c} [I_{1z}(z') - I_{1z}(z)g_1(z, z')]\mathcal{K}_{11}(z, z')dz'$$
$$- \int_{-d}^{d} [I_{1z}(z') - I_{1z}(z)g_1(z, z')]\mathcal{K}_{13}(z, z')dz'$$
$$- \left(\int_{-(d+c)}^{d} + \int_{d}^{d+\lambda}\right) [I_{1z}(z')$$
$$- I_{1z}(z)g_1(z, z')]\mathcal{K}_{24}(z, z')dz' \quad \text{(24a)}$$

$$\mathfrak{D}_2(x) = \int_{(d+c)}^{d+c} [I_{2x}(x') - I_{2x}(x)g_2(x, x')]\mathcal{K}_{22}(x, x')dx'$$
$$- \int_{-c}^{c} [I_{2x}(x') - I_{2x}(x)g_2(x, x')]\mathcal{K}_{24}(x, x')dx'$$
$$- \left(\int_{-(d+c)}^{-c} + \int_{c}^{d+c}\right)[I_{2x}(x')$$
$$- I_{2x}(x)g_2(x, x')]\mathcal{K}_{13}(x, x')dx'. \quad (24b)$$

With (23) and (24), the integral equations of (20) may be rearranged as follows:

$$I_{1z} = \frac{-j4\pi}{\zeta_0\Psi_s}\left[C_{1z}\cos\beta_0 z + \frac{1}{2}V^e\sin\beta_0|z|\right] + \frac{L_1(z)}{\Psi_s} \quad (25a)$$

$$I_{2x}(x) = \frac{-j4\pi}{\zeta_0\Psi_s}C_{2x}\sin\beta_0 x + \frac{L_{2(x)}}{\Psi_s} \quad (25b)$$

where

$$L_1(z) = \frac{j4\pi}{\zeta_0}\theta_{1A}(z) + P_1(z), \quad (26a)$$

$$L_2(x) = \frac{j4\pi}{\zeta_0}\theta_{2A}(x) + P_2(x), \quad (26b)$$

and

$$P_1(z) = \mathcal{J}_1(z) - \mathfrak{D}_1(z) - I_{1z}(z)\gamma_1(z)$$
$$= I_{1z}(z)\Psi_s - \int_{-d}^{d} I_{1z}(z')\mathcal{K}_{1A}(z, z')dz'. \quad (26c)$$

The functions $\theta_{1A}(z)$ and $\theta_{2A}(x)$ are defined in (8). In order to assure *exact* continuity of current at the corners in the form $I_{1z}(-d) = -I_{2x}(-c)$ even in approximate expressions, it is advantageous to subtract the quantity $0 = I_{1z}(-d) + I_{2x}(-c)$ from (25a). The result is

$$I_{1z}(z) = \frac{-j4\pi}{\zeta_0\Psi_s}\left[C_{1z}(\cos\beta_0 z - \cos\beta_0 d) + \frac{1}{2}V^e(\sin\beta_0|z| - \sin\beta_0 d) + C_{2x}\sin\beta_0 c\right]$$
$$+ \frac{1}{\Psi_s}[L_1(z) - L_1(-d) - L_2(-c)]. \quad (27)$$

Note that (25) or (27) and (25b) are still integral equations for the currents since these occur on the right in $L_1(z)$ and $L_2(x)$ under the signs of integration. However, the expressions are now so arranged that if a proper choice is made of the distribution functions $g_1(z, z')$ and $g_2(x, x')$ and of the parameter Ψ_s, the sum of the terms on the right in which the currents occur is small compared with the zeroth-order terms. Accordingly, (27) and (25b) are in forms appropriate for iteration. Suggested zeroth-order currents and charges are:

$$[I_{1z}(z)]_0 = \frac{-j4\pi}{\zeta_0\Psi_s}\left[C_{1z}(\cos\beta_0 z - \cos\beta_0 d) + \frac{1}{2}V^e(\sin\beta_0|z| - \sin\beta_0 d) + C_{2x}\sin\beta_0 c\right], \quad (28a)$$

$$[q^1(z)]_0 = \frac{4\pi\epsilon_0}{\Psi_s}\left[-C_{1z}\sin\beta_0 z + \frac{1}{2}V^e\cos\beta_0 z\right], \quad 0 \leqq z \leqq d, \quad (28b)$$

$$[q_1(z)]_0 = \frac{-4\pi\epsilon_0}{\Psi_s}\left[C_{1z}\sin\beta_0 z + \frac{1}{2}V^e\cos\beta_0 z\right], \quad -d \leqq z \leqq 0, \quad (28c)$$

$$[I_{2x}(x)]_0 = \frac{-j4\pi}{\zeta_0\Psi_s}C_{2x}\sin\beta_0 x, \quad (29a)$$

$$[q_2(x)]_0 = \frac{4\pi\epsilon_0}{\Psi_s}C_{2x}\cos\beta_0 x. \quad (29b)$$

The substitution of these zeroth-order currents and charges in the several parts of $L_1(z)$ and $L_2(x)$ as defined in (26), and the use of (23) leads to the following first-order integrals:

$$F_{11}(z) = \Psi_s(\cos\beta_0 z - \cos\beta_0 d) - [C_a(d, z) + C_{2c}(d, z)] + [E_a(d, z) + E_{2c}(d, z)]\cos\beta_0 d \quad (30a)$$

$$G_{11}(z) = \Psi_s(\sin\beta_0|z| - \sin\beta_0 d) - [S_a(d, z) + S_{2c}(d, z)] + [E_a(d, z) + E_{2c}(d, z)]\sin\beta_0 d \quad (30b)$$

$$H_{11}(z) = [\Psi_s - E_a(d, z) - E_{2c}(d, z)]\sin\beta_0 c \quad (30d)$$

$$G_{21}(x) = \Psi_s\sin\beta_0 x - \mathcal{S}_a(c, x) + \mathcal{S}_{2d}(c, x) \quad (30c)$$

where the following functions are involved:

$$C_i(h, z) = \int_0^h \cos\beta_0 z'\left[\frac{e^{-j\beta_0 R_{1i}}}{R_{1i}} + \frac{e^{-j\beta_0 R_{2i}}}{R_{2i}}\right]dz', \quad (31a)$$

$$S_i(h, z) = \int_0^h \sin\beta_0 z'\left[\frac{e^{-j\beta_0 R_{1i}}}{R_{1i}} + \frac{e^{-j\beta_0 R_{2i}}}{R_{2i}}\right]dz', \quad (31b)$$

$$E_i(h, z) = \int_0^h \left[\frac{e^{-j\beta_0 R_{1i}}}{R_{1i}} + \frac{e^{-j\beta_0 R_{2i}}}{R_{2i}}\right]dz', \quad (31c)$$

$$\mathcal{S}_i(h, z) = \int_0^h \sin\beta_0 z'\left[\frac{e^{-j\beta_0 R_{1i}}}{R_{1i}} - \frac{e^{-j\beta_0 R_{2i}}}{R_{2i}}\right]dz'. \quad (31d)$$

In these expressions

$$R_{1i} = \sqrt{(z' - z)^2 + i^2}; \qquad R_{2i} = \sqrt{(z' + z)^2 + i^2}. \quad (32)$$

The integral functions in (31) may be expressed in terms of the tabulated generalized sine and cosine integral functions.[5,6] Other integrals involved in $L_1(z)$ and

[5] *Ibid.*, pp. 97, 274.
[6] "Tables of Generalized Sine- and Cosine-Integral Functions," Harvard University Press, Cambridge, Mass.; 1948.

$L_2(x)$ are

$$f_{11}(z) = \int_0^z \rho_{11}(w) \cos \beta_0(z - w)\beta_0 dw, \tag{33a}$$

$$f_{21}(x) = \int_0^x \rho_{21}(w) \cos \beta_0(x - w)\beta_0 dw - \rho_{21}(0) \sin \beta_0 x, \tag{33b}$$

$$p_{21}(x) = \int_0^x \sigma_{21}(w) \cos \beta_0(x - w)\beta_0 dw - \sigma_{21}(0) \sin \beta_0 x. \tag{33c}$$

The quantities ρ_{11}, ρ_{21}, and σ_{21} introduced in (33) are defined as follows. Note that $\mathcal{K}_{1B}(z, x')$ and $\mathcal{K}_{2B}(x, z')$ are defined in (5).

$$\rho_{11}(z) = \int_{-c}^{c} \cos \beta_0 x' \mathcal{K}_{1B}(z, x') \tag{34a}$$

$$\rho_{21}(x) = \int_0^d \cos \beta_0 z'[\mathcal{K}_{2B}(x, z') - \mathcal{K}_{2B}(x, -z')]dz' \tag{34b}$$

$$\sigma_{21}(x) = \int_{-d}^{d} \sin \beta_0 z' \mathcal{K}_{2B}(x, z')dz'. \tag{34c}$$

These integrals can also be expressed in terms of the tabulated generalized sine and cosine integrals. In terms of the integrals (31a)–(31c) and (33) the first-order expressions for $L_1(z)$ and $L_2(x)$ are

$$[L_1(z)]_1 = \frac{-j4\pi}{\zeta_0\Psi_s}\left\{C_{1z}F_{11}(z) + \frac{1}{2}V^eG_{11}(z) + C_{2x}[H_{11}(z) - f_{11}(z)]\right\} \tag{35a}$$

$$[L_2(x)]_1 = \frac{-j4\pi}{\zeta_0\Psi_s}\left\{C_{2x}G_{21}(x) + C_{1z}p_{21}(x) - \frac{1}{2}V^ef_{21}(x)\right\}. \tag{35b}$$

If these values are substituted in (27) and (25b), the following first-order solutions for the currents are obtained:

$$[I_{1z}(z)]_1 = \frac{-j4\pi}{\zeta_0\Psi_s}\left[C_{1z}\left\{F_{0z} + \frac{1}{\Psi_s}[F_{11z} - p_{21}(-c)]\right\} + \frac{1}{2}V^e\left\{G_{0z} + \frac{1}{\Psi_s}[G_{11z} + f_{21}(-c)]\right\} + C_{2x}\left\{\sin \beta_0 c + \frac{1}{\Psi_s}\cdot[H_{11z} - f_{11z} - G_{21}(-c)]\right\}\right] \tag{36a}$$

$$[I_{2x}(x)]_1 = \frac{-j4\pi}{\zeta_0\Psi_s}\left[C_{2x}\left\{\sin \beta_0 x + \frac{G_{21}(x)}{\Psi_s}\right\} + C_{1z}\frac{p_{21}(x)}{\Psi_s} - \frac{1}{2}V^e\frac{f_{21}(x)}{\Psi_s}\right] \tag{36b}$$

where

$$F_{0z} = \cos \beta_0 z - \cos \beta_0 d,$$
$$G_{0z} = \sin \beta_0 |z| - \sin \beta_0 d, \tag{36c}$$
$$F_{11z} = F_{11}(z) - F_{11}(-d),$$
$$G_{11z} = G_{11}(z) - G_{11}(-d), \tag{36d}$$
$$H_{11z} = H_{11}(z) - H_{11}(-d),$$
$$f_{11z} = f_{11}(z) - f_{11}(-d). \tag{36e}$$

In order to evaluate the constants C_{1z} and C_{2x} in terms of V^e the conditions (10) requiring continuity of scalar potential and current at the corners must be introduced. The first-order expressions for the scalar potential are obtained from (13) and (15) by substituting zeroth-order values of the charges in (14) and (16). The following integrals are involved:

$$h_{11}(z) = \beta_0 \int_0^z \rho_{11}(w) \sin \beta_0(z - w)dw, \tag{37a}$$

$$h_{21}(x) = \beta_0 \int_0^x \rho_{21}(w) \sin \beta_0(x - w)dw + \rho_{21}(0) \cos \beta_0 x, \tag{37b}$$

$$k_{21}(x) = \beta_0 \int_0^x \sigma_{21}(w) \sin \beta_0(x - w)dw + \sigma_{21}(0) \cos \beta_0 x. \tag{37c}$$

In terms of these integrals the first-order potentials are:

$$[\phi_1(z)]_1 = -C_{1z} \sin \beta_0 z - \frac{1}{2}V^e \cos \beta_0 z + \frac{C_{2x}h_{11}(z)}{\Psi_s}, \quad z \leqq 0, \tag{38a}$$

$$[\phi_2(x)]_1 = C_{2x} \cos \beta_0 x - \frac{1}{\Psi_s}\left[C_{1z}k_{21}(x) + \frac{1}{2}V^eh_{21}(x)\right]. \tag{38b}$$

With (10), (25), (35), and (38) the following pair of equations is obtained for the two constants C_{1z} and C_{2x}:

$$C_{1z}a_{11} + C_{2x}a_{12} = \tfrac{1}{2}V^eb_1 \tag{39a}$$
$$C_{1z}a_{21} + C_{2x}a_{22} = \tfrac{1}{2}V^eb_2 \tag{39b}$$

where, for first-order values,

$$a_{11} = \cos \beta_0 d + \frac{1}{\psi_s}[F_{11}(-d) + p_{21}(-c)], \tag{40a}$$

$$a_{12} = -\beta_0 c + \frac{1}{\Psi_s}[G_{21}(-c) + H_{11}(-d) - f_{11}(-d)], \tag{40b}$$

$$b_1 = -\sin \beta_0 d - \frac{1}{\Psi_s}[G_{11}(-d) - f_{21}(-c)], \tag{40c}$$

$$a_{21} = \sin \beta_0 d + \frac{k_{21}(-c)}{\Psi_s}, \tag{41a}$$

$$a_{22} = -\cos\beta_0 c + \frac{h_{11}(-d)}{\Psi_s}, \tag{41b}$$

$$b_2 = \cos\beta_0 d - \frac{h_{21}(-c)}{\Psi_s}. \tag{41c}$$

It follows directly that

$$C_{1z} = \frac{V^e N_1}{2D}, \qquad C_{2x} = \frac{V^e N_2}{2D}, \tag{42}$$

where

$$\begin{aligned} N_1 &= b_1 a_{22} - b_2 a_{12} \\ &= \sin\beta_0(c+d) + N_{11}/\Psi_s + N_{12}/\Psi_s^2, \end{aligned} \tag{43a}$$

$$N_2 = b_2 a_{11} b_1 a_{21} = 1 + N_{21}/\Psi_s + N_{22}/\Psi_s^2, \tag{43b}$$

$$\begin{aligned} D &= a_{11}a_{22} - a_{12}a_{21} \\ &\quad - [\cos\beta_0(c+d) + D_1/\Psi_s + D_2/\Psi_s^2]. \end{aligned} \tag{43c}$$

The following quantities occur in (43):

$$\begin{aligned} N_{11} &= -h_{11}(-d)\sin\beta_0 d \\ &\quad + [G_{11}(-d) - f_{21}(-c)]\cos\beta_0 c - h_{21}(-c)\sin\beta_0 c \\ &\quad - [G_{21}(-c) + H_{11}(-d) - f_{11}(-d)]\cos\beta_0 d \end{aligned} \tag{44a}$$

$$\begin{aligned} N_{12} &= h_{11}(-d)[-G_{11}(-d) + f_{21}(-c)] \\ &\quad + h_{21}(-c)[G_{21}(-c) + H_{11}(-d) - f_{11}(-d)] \end{aligned} \tag{44b}$$

$$\begin{aligned} N_{21} &= [F_{11}(-d) + p_{21}(-c)]\cos\beta_0 d \\ &\quad + [G_{11}(-d) - f_{21}(-c) + k_{21}(-c)]\sin\beta_0 d \end{aligned} \tag{44c}$$

$$\begin{aligned} N_{22} &= k_{21}(-c)[G_{11}(-d) - f_{21}(-c)] \\ &\quad - h_{21}(-c)[F_{11}(-d) + p_{21}(-c) \end{aligned} \tag{44d}$$

$$\begin{aligned} D_1 &= [F_{11}(-d) + p_{21}(-c)]\cos\beta_0 c - h_{11}(-d)\cos\beta_0 d \\ &\quad + [G_{21}(-c) + H_{11}(-d) - f_{11}(-d)]\sin\beta_0 d \\ &\quad - k_{21}(-c)\sin\beta_0 c \end{aligned} \tag{44e}$$

$$\begin{aligned} D_2 &= -[F_{11}(-d) + p_{21}(-c)]h_{11}(-d) \\ &\quad + [G_{21}(-c) + H_{11}(-d) - f_{11}(-d)]k_{21}(-c). \end{aligned} \tag{44f}$$

If the equations of (43) are substituted for (42) and (36a) and (36b), the following expressions are obtained for the currents if only terms of order $1/\Psi_s$ are retained in both numerator and denominator:

$$[I_{1z}(z)]_1 = \frac{j2\pi V^e}{\zeta_0\Psi_s} \cdot \left[\frac{\sin\beta_0(c+d-|z|) + B_1(z)/\Psi_s}{\cos\beta_0(c+d) + D_1/\Psi_s}\right], \tag{45a}$$

$$[I_{2x}(x)]_1 = \frac{j2\pi V^e}{\zeta_0\Psi_s}\left[\frac{\sin\beta_0 x + M_{21}(x)/\Psi_s}{\cos\beta_0(c+d) + D_1/\Psi_s}\right], \tag{45b}$$

where

$$B_1(z) = M_{11}(z) - M_{11}(-d) - M_{21}(-c) \tag{46a}$$

and

$$\begin{aligned} M_{11}(z) &= N_{11}\cos\beta_0 z - D_1\sin\beta_0|z| + F_{11}(z)\sin\beta_0(c+d) \\ &\quad - G_{11}(z)\cos\beta_0(c+d) + H_{11}(z) - f_{11}(z), \\ M_{21}(x) &= N_{21}\sin\beta_0 x + G_{21}(x) + p_{21}(x)\sin\beta_0(c+d) \\ &\quad + f_{21}(x)\cos\beta_0(c+d). \end{aligned} \tag{46b}$$

Higher-order terms may be obtained by continuing the iteration.

The first-order driving-point impedance is given by

$$[Z_{1\text{in}}]_1 = \frac{-j\zeta_0\Psi_s}{2\pi}\left[\frac{\cos\beta_0(c+d) + D_1/\Psi_s}{\sin\beta_0(c+d) + B_1(0)/\Psi_s}\right]. \tag{47}$$

The corresponding admittance is $[Y_{1\,\text{in}}]_1 = 1/[Z_{1\,\text{in}}]_1$. The coefficient D_1 in (47) is given by (44e). The corresponding value of $B_1(0)$ is obtained from (46a) with $z=0$. It is:

$$\begin{aligned} B_1(0) &= F_{11}(0)\sin\beta_0(c+d) - G_{11}(0)\cos\beta_0(c+d) \\ &\quad + G_{11}(-d)\cos\beta_0 c - G_{21}(-c)\cos\beta_0 d \\ &\quad + H_{11}(0) - H_{11}(-d)\cos\beta_0 d - f_{11}(0) \\ &\quad + f_{11}(-d)\cos\beta_0 d - h_{11}(-d)\sin\beta_0 d \\ &\quad - f_{21}(-c)\cos\beta_0 c - h_{21}(-c)\sin\beta_0 c. \end{aligned} \tag{48a}$$

If use is made of (30) in (48a) and (44e) the following formulas are obtained:

$$\begin{aligned} B_1(0) &= [\Psi_s - C_a(d,0) - C_{2c}(d,0) + E_a(d,-d) \\ &\quad + E_{2c}(d,-d)]\sin\beta_0(c+d) \\ &\quad + [S_a(d,0) + S_{2c}(d,0)]\cos\beta_0(c+d) \\ &\quad - [S_a(d,-d) + S_{2c}(d,-d)]\cos\beta_0 c \\ &\quad + [\mathcal{S}_a(c,-c) - \mathcal{S}_{2d}(c,-c)]\cos\beta_0 d \\ &\quad - f_{11}(0) + f_{11}(-d)\cos\beta_0 d - h_{11}(-d)\sin\beta_0 d \\ &\quad - f_{21}(-c)\cos\beta_0 c - h_{21}(-c)\sin\beta_0 c, \end{aligned} \tag{48b}$$

$$\begin{aligned} D_1 &= [E_a(d,-d) + E_{2c}(d,-d)]\cos\beta_0(c+d) \\ &\quad - [C_a(d,-d) + C_{2a}(d,-d)]\cos\beta_0 c \\ &\quad - [\mathcal{S}_a(c,-c) - \mathcal{S}_{2d}(c,-c)]\sin\beta_0 d \\ &\quad - f_{11}(-d)\sin\beta_0 d - h_{11}(-d)\cos\beta_0 d \\ &\quad + p_{21}(-c)\cos\beta_0 c - k_{21}(-c)\sin\beta_0 c. \end{aligned} \tag{48c}$$

If the real and imaginary parts of D_1 and $B_1(0)$ are separated and the notation $D_1 = D_1^{\mathrm{I}} + jD_1^{\mathrm{II}}$, $B_1(0) = B_1^{\mathrm{I}} + jB_1^{\mathrm{II}}$, is introduced, the impedance $[Z_{1\,\text{in}}]_1 = [R_{1\,\text{in}}]_1 + j[X_{1\,\text{in}}]_1$ may be separated into its resistive and reactive parts as follows:

$$[R_{1\,\text{in}}]_1 = \frac{\zeta_0}{2\pi}\left\{\frac{D_1^{\mathrm{II}}\sin\beta_0(c+d) - B_1^{\mathrm{II}}\cos\beta_0(c+d) + [D_1^{\mathrm{II}}B_1^{\mathrm{I}} - D_1^{\mathrm{I}}B_1^{\mathrm{II}}]/\Psi_s}{[\sin\beta_0(c+d) + B_1^{\mathrm{I}}/\Psi_s]^2 + [B_1^{\mathrm{II}}/\Psi_s]^2}\right. \tag{49a}$$

$$[X_{1\,\text{in}}]_1 = \frac{-\zeta_0}{2\pi}\left\{\frac{[\sin\beta_0(c+d) + B_1^{\mathrm{I}}/\Psi][\cos\beta_0(c+d) + D_1^{\mathrm{I}}/\Psi] + B_1^{\mathrm{II}}D_1^{\mathrm{II}}/\Psi^2}{[\sin\beta_0(c+d) + B_1^{\mathrm{I}}/\Psi_s]^2[B_1^{\mathrm{II}}/\Psi_s]^2}\right.. \tag{49b}$$

These are the final first-order formulas.

The Expansion Parameter

The selection of an appropriate expansion parameter Ψ_s depends upon the evaluation of $\Psi_1(z)$ and $\Psi_2(x)$ as defined in (23). As shown for the comparable problem in the analysis of the linear antenna[3] very satisfactory results are obtained with zeroth-order distribution functions. In the case at hand this means that the distribution functions,

$$g_1(z, z') = \frac{\sin \beta_0(c + d - |z|)}{\sin \beta_0(c + d - |z|)},$$
$$g_2(x, x') = \frac{\sin \beta_0 x'}{\sin \beta_0 x}, \tag{50}$$

are to be substituted in (23a) and (23b). The result for $\Psi_1(z)$ with $z \geqq 0$ and the definitions (31) is:

$$\begin{aligned}\Psi_1(z) = \csc \beta_0(c + d - z)\{&\sin \beta_0(c + d)[C_a(c + d, z) \\ &+ C_{2c}(d, z) - C_{2d}(c + d, z) + C_{2d}(d, z)] \\ &- \cos \beta_0(c + d)[S_a(c + d, z) + S_{2c}(d, z) \\ &- S_{2d}(c + d, z) + C_{2d}(d, z)]\}.\end{aligned} \tag{51}$$

Correspondingly with (31d) the result for $\Psi_2(x)$ is:

$$\begin{aligned}\Psi_2(x) = \csc \beta_0 x[&\mathrm{S}_a(c + d, x) - \mathrm{S}_{2d}(c, x) \\ &+ \mathrm{S}_{2c}(c + d, x) - \mathrm{S}_{2c}(c, x)].\end{aligned} \tag{52}$$

Since both $\Psi_1(z)$ and $\Psi_2(x)$ are proportional to the ratio of zeroth-order vector potential to zeroth-order current at points along conductors that have the same radii and similar distributions of current, the magnitude of $\Psi_1(z)$ and $\Psi_2(x)$ should be essentially constant and equal except as modified by asymmetries. In general, a good choice of Ψ is the magnitude of $\Psi(z)$ at a point z where the vector potential and the current both have maxima. In the presently considered case of the loop excited in the dipole mode, an even better choice owing to more complete symmetry is at the centers of the sides without generators (with fictitious extensions) where both the vector potential and the current vanish, but where their ratio has a definite and constant value. Thus, let $\Psi_s = |\lim_{x\to 0} \Psi_2(0)|$. This function is readily evaluated directly from the integrals by differentiating the indeterminate form when expressed as 0/0. An integration by parts in the numerator leads to the following formula for the expansion parameter Ψ_s of the symmetrical or dipole mode in the rectangle:

$$\begin{aligned}\Psi_s = \Psi_2 = \Bigg| &C_a(c + d, 0) + C_b(c + d, 0) - C_{2d}(c, 0) \\ &- C_b(c, 0) - \frac{2}{\beta_0}\left\{\sin \beta_0(c + d)\left[\frac{e^{-j\beta_0 R_1}}{R_1} + \frac{e^{-j\beta_0 R_3}}{R_3}\right]\right. \\ &\left. - \sin \beta_0 c\left[\frac{e^{-j\beta_0 R_2}}{R_2} + \frac{e^{-j\beta_0 R_4}}{R_4}\right]\right\}\Bigg|\end{aligned} \tag{53}$$

where

$$b = \sqrt{4c^2 + a^2} \tag{54a}$$
$$R_1 = \sqrt{(c + d)^2 + a^2}, \qquad R_2 = \sqrt{c^2 + 4d^2 + a^2}$$
$$R_3 = \sqrt{(c + d)^2 + 4c^2 + a^2}, \quad R_4 = \sqrt{5c^2 + a^2}. \tag{54b}$$

Note that in (54) a^2 usually is negligible except when c or d becomes very small. The difference functions $\gamma_1(z)$ and $\gamma_2(x)$ are given by

$$\gamma_1(z) = \Psi_1(z) - \Psi_s, \quad \gamma_2(x) = \Psi_2(x) - \Psi_s. \tag{55}$$

With the expansion parameter Ψ_s as defined in (53), substituted in (45) for the distributions of current, and in (47) or (49a) and (49b) for the impedance, the first-order circuit properties of the rectangular loop of arbitrary size are determined when it is driven so that only the vertical dipole mode is excited. Important special cases must still be considered.

Folded Dipole and Transmission Line

When the dimension $2d$ or $2c$ of the rectangle of wire is kept electrically small ($\beta_0 d \ll 1$ or $\beta_0 c \ll 1$) while the other dimension is unrestricted, the loop becomes on the one hand a symmetrically driven folded dipole, on the other hand a section of transmission line driven simultaneously at both ends by codirectional generators. Both of these special cases have been analyzed: the former (see King[5,7]) by neglecting corner effects and treating the two sides of the long and narrow rectangle as two closely spaced symmetrically driven dipoles; the latter in terms of transmission-line theory for the reactance and the Poynting vector theorem for the radiation resistance.[8,9]

It is readily verified that when c is small (53) becomes

$$\Psi_s = 2\left| C_a(d, 0) - \ln \frac{2c}{a} - \frac{\sin \beta_0 d}{\beta_0 d} e^{-j\beta_0 d} \right|. \tag{56}$$

This is essentially equivalent to the expansion parameter Ψ_s for two closely spaced symmetrically driven antennas or for the folded dipole.[5,7] Moreover, if the capacitive coupling at the corners is neglected when c is sufficiently small by setting the functions $f_{11}(z)$, $f_{21}(x)$, and $p_{21}(x)$ as defined in (33) equal to zero, (46b) and (48c) for $M_{11}(z)$ and D_1 reduce essentially to the corresponding functions characteristic of the symmetrically driven pair of parallel antennas. Small differences are a consequence of the definition of Ψ_s in (53) in terms of $\Psi_2(x)$ instead of $\Psi_1(z)$. It follows that (47) reduces to the formula for the folded dipole.

[7] King, "Theory of Linear Antennas," *op. cit.*, pp. 267–270, 335–337.

[8] J. E. Storer and R. W. P. King, "Radiation resistance of two-wire line," Proc. IRE, vol. 39, pp. 1408–1412; November, 1951.

[9] R. W. P. King, "Transmission-Line Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1955.

When the dimension $2d$ is small compared with the wavelength and c so that $\beta_0 d \ll 1$, $c^2 \gg d^2$, the expansion parameter ψ_s reduces to

$$\psi_s = 2 \ln \frac{2d}{a} \cdot \tag{57}$$

The leading term in the reactance is

$$X_{\text{in}} = -R_c \cot \beta_0 c \tag{58a}$$

where

$$R_c = \frac{\zeta_0 \Psi_s}{2\pi} = \frac{\zeta_0}{\pi} \ln \frac{2d}{a}, \tag{58b}$$

is the familiar expression for the characteristic impedance of a lossless two-wire line with wire spacing $2d$.

The leading term in the resistance has not been evaluated in general when $\beta_0 d \ll 1$ and $\beta_0 c$ is unrestricted. However, the special case when both $\beta_0 d$ and $\beta_0 c$ are small is considered below.

The Electrically Small Loop as a Dipole

An important special case is the electrically small rectangular loop defined by the inequality

$$\beta_0^2 (c + d)^2 \ll 1. \tag{59}$$

The general formula for the expansion parameter reduces to the following approximate form:

$$\begin{aligned} \Psi_s \doteq 2 \bigg[& \sinh^{-1} \frac{c+d}{a} + \sinh^{-1} \frac{c+d}{\sqrt{4c^2 + a^2}} \\ & - \sinh^{-1} \frac{c}{2d} - \sinh^{-1} \frac{c}{\sqrt{4c^2 + a^2}} \bigg] \\ & - (c+d) \left(\frac{1}{R_1} + \frac{1}{R_3} \right) + c \left(\frac{1}{R_2} + \frac{1}{R_4} \right) \end{aligned} \tag{60}$$

where the R's are defined in (54). It is readily verified that when d is small compared with c, (60) reduces to (57). Alternatively, when d is large compared with c but small enough to satisfy the inequality $\beta_0^2 d^2 \ll 1$, (60) and (56) both give

$$\Psi_s = 2 \left[2 \ln \frac{2d}{a} - \ln \frac{2c}{a} - 2 \right] \tag{61}$$

in agreement with the value found in the literature[10] for an electrically short two-element cage antenna

Approximate expressions for the resistance and reactance may be obtained by simplifying (49a) and (49b). With (59) it is clear that the small integrals (33) are predominately real, so that the leading terms in B_1^{II} and D_1^{II} are:

$$B_1^{\text{II}} \doteq 2\beta_0^4 d^3 \left(\frac{d}{3} + c \right) + \frac{4}{3} \beta_0^4 d^2 c^2;$$

$$D_1^{\text{II}} \doteq 4\beta_0^3 d^3 \left(\frac{d}{3} + c \right). \tag{62}$$

With these values the leading terms in the resistance and reactance are:

$$R_{1\text{in}} \doteq \frac{\zeta_0}{3\pi} \beta_0^2 d^2 \left[\frac{(d + 3c)(d + 2c) - 4c^2}{(d + c)^2} \right] \tag{63}$$

$$X_{\text{in}} \doteq \frac{\zeta_0 \Psi_s}{2\pi \beta_0 (c + d)} \tag{64}$$

where Ψ_s is given by (60) in general, and by (57) or (61) in special cases.

Note that with $c \ll d$,

$$R_{1\text{in}} \doteq \frac{\zeta_0}{3\pi} \beta_0^2 d^2 = 40 \beta_0^2 d^2 \text{ ohms} \tag{65}$$

in agreement with the approximate formula

$$R_{1\text{in}} \doteq \frac{\zeta_0}{6\pi} \beta_0^2 d^2 = 20 \beta_0^2 d^2 \text{ ohms}$$

for the isolated dipole of half-length d. The factor 2 is explained by the fact that the symmetrical impedances of two parallel dipoles driven in phase by two generators are in zeroth order, double the value for a single isolated antenna. Alternatively, when $d \ll c$ the rectangle becomes a transmission line of length $2c$ and with spacing $b = 2d$. The line is driven at each end so that the currents vanish at the centers of the long sides. In this case

$$R_{1\text{in}} = \frac{2\zeta_0}{3\pi} \beta_0^2 d^2 = 80 \beta_0^2 d^2 \text{ ohms} = 20 \beta_0^2 b^2 \text{ ohms}. \tag{66}$$

This is the resistance seen by each generator. It is equal to the resistance of a short end-loaded dipole of half-length d with an essentially uniform current. Contributions to the radiation from the equal and opposite currents in the electrically short sections of line is of higher order than contributions from the short ends.

Conclusion

The circuit properties of the rectangular loop antenna have been determined when the loop is driven in a transverse mode by equal and codirectional generators at the centers of one pair of parallel sides. First-order expressions for the currents and the identical input impedances at the two driving points are given in a form that involves only tabulated functions. It is shown that the new formula for the impedance is consistent with previously available formulas for the symmetrically driven folded dipole and for the transmission line.

[10] King, "Theory of Linear Antennas," *op. cit.*, p. 274.