Polarization in Antennas and Radar





POLARIZATION IN ANTENNAS AND RADAR

HAROLD MOTT

Department of Electrical Engineering University of Alabama

A Wiley-Interscience Publication JOHN WILEY & SONS New York Chichester Brisbane Toronto Singapore

Copyright © 1986 by John Wiley & Sons, Inc.

All rights reserved. Published simultaneously in Canada.

Reproduction or translation of any part of this work beyond that permitted by Section 107 or 108 of the 1976 United States Copyright Act without the permission of the copyright owner is unlawful. Requests for permission or further information should be addressed to the Permissions Department, John Wiley & Sons, Inc.

Library of Congress Cataloging in Publication Data:

Mott, Harold.

Polarization in antennas and radar.

"A Wiley-Interscience publication." Bibliography: p. Includes Index. 1. Radar. 2. Antennas (Electronics) 3. Electromagnetic waves--Polarization. I. Title. 621.38'028'3 TK6580.M68 1986

ISBN 0-471-01167-3

86-1349

Printed in the United States of America

To my mother, my wife Betty, and my son John

PREFACE

It is my belief that the concepts of wave polarization, even though they are not difficult to understand, are not known as widely as they should be. Introductory electromagnetics texts normally restrict discussions of wave propagation to linearly polarized waves, and the more advanced texts that are widely used in this country devote only little attention to elliptical waves. Antenna textbooks, with that of Kraus a notable exception, either ignore the questions of polarization and polarization match or treat them too lightly. Principles of Optics by Born and Wolf has a good discussion of wave polarization, and Clarke and Grainger (Polarized Light and Optical Measurement) treat polarized light extensively. Beckman (The Depolarization of Electromagnetic Waves and The Scattering of Electromagnetic Waves from Rough Surfaces, the latter book written with A. Spizzichino) discusses polarization changes caused by scattering. None of these books is readily usable by many who must deal with polarization problems, particularly those associated with radar and antennas, and they must gather their information from several sources, with incomplete coverage and inconsistent notation.

Polarization effects can provide significant identification information about radar targets, and the use of an optimum polarization can increase target cross sections, decrease rain and ground clutter, and ameliorate the effects of jamming on radar and communication systems. Adaptive arrays can be used for polarization adaptivity as well as for beam forming and null steering, and an example of such use is given in this book. Other examples can be cited for microwave and millimeter-wave radar and communications, and for optics. There is a clear need for a wider dissemination of the methods of treating polarization problems, and I hope to answer that need with this book.

The text is intended for use at the graduate or advanced undergraduate level for engineering and physics students and as a reference for radar and

PREFACE

communications engineers. It is expected to be useful also for those working with coherent light sources. It has been used for a one-semester course for graduate students and seniors, with good results. The necessary background is a good understanding of calculus and vector analysis, some knowledge of matrices, and a knowledge of electromagnetics equivalent to that acquired by completing the study of a good undergraduate text. The chapter on antennas will provide a satisfactory base for those without such a background. The chapter necessarily omits a discussion of many types of antennas and of many of the methods for determining radiation patterns and impedances. It does, however, include definitions of the more important antenna parameters (from the 1983 IEEE Standard) and developments of the equations for using them in a communications or radar system. In fact, Chapters 1 through 4 can be used for a one-semester course in antennas for students interested in the use of antennas rather than in their design, if it is supplemented by material on arrays. Chapter 3 discusses polarization matching for antennas in more detail than any of the standard antennas texts, and it analyzes transmission between antennas that are not pointing at each other and are not polarization matched. Euler angle transformations between coordinate systems are given for use in the analysis. Such a complete discussion is not commonly included in antennas texts, many of which are oriented toward design rather than use of antennas. The analysis will also be useful for optical and infrared systems. Chapter 4 describes polarization properties of several antennas and appropriate test antennas for use with them, to determine the degree to which they meet design criteria. Chapter 5 describes methods for generating waves with any desired polarization and analyzes a system that is polarization adaptable.

Chapter 6 is a discussion of polarization changes by reflection and transmission. It introduces the scattering matrix and includes scattering matrices for some common reflecting objects. Also presented is the depolarization by reflections from an arbitrarily oriented plane. Chapter 7 develops the theory of partially polarized waves, which is useful in radio astronomy and has applications to jamming in radar. Finally, in Chapter 8, standard techniques (and one nonstandard) for measuring wave polarization are presented.

In the text the polarization ratio P (or its modified form jP) and the circular polarization ratio q are the primary descriptors of a wave or an antenna. The Poincaré sphere is an elegant device for presenting polarization information, and it is useful in developing certain theorems, for example, the theorem that three polarization match factors (amplitudes) between an unknown antenna and three antennas of known polarization suffice to determine the polarization of the unknown antenna. Many of the text developments, therefore, utilize the Poincaré sphere, and since P is a projection, onto a plane, of the polarization point on the sphere, any text equation using P can be easily converted to coordinates on the Poincaré sphere. Description of wave polarization by axial ratio, tilt angle, and

PREFACE

rotation sense of the polarization ellipse is useful for visualization but awkward mathematically. This description is therefore not extensively used in the book.

Some of the material presented here is original, but much of it comes from developments or presentations by others, most notably on the subject of antennas by Kraus, Collin and Zucker, and Balanis, and on the subject of polarization by Rumsey, Sinclair, Deschamps, Born and Wolf, and Beckmann. I wish to acknowledge a great debt to them.

HAROLD MOTT

University, Alabama May 1986

CONTENTS

1. AN INTRODUCTION TO ANTENNAS

1

- 1.1 Introduction, 1
- 1.2 The Vector Potentials, 2
- 1.3 Integral Solutions for the Vector Potentials, 6
- 1.4 Approximations to the Potentials, 8
- 1.5 Far-Zone Fields, 9
- 1.6 Use of the Potential Integrals for Physical Structures, 12
- 1.7 Radiation Pattern, 13
- 1.8 Gain and Directivity, 19
- 1.9 The Dipole Antenna: Fields, 23
- 1.10 Reciprocity Theorem, 27
- 1.11 An Equivalence Theorem, 28
- 1.12 The Dipole Antenna: Input Impedance, 30
- 1.13 Waveguide Opening into Infinite Ground Plane, 37
- 1.14 The Receiving Antenna, 39
- 1.15 Transmission between Antennas, 48
- 1.16 The Radar Equation, 49 References, 51 Problems, 52

2. REPRESENTATION OF WAVE POLARIZATION

- 2.1 Introduction, 54
- 2.2 The General Harmonic Wave, 54
- 2.3 Polarization Ellipse for Plane Waves, 57
- 2.4 Linear and Circular Polarizations, 63
- 2.5 Power Density, 63
- 2.6 Rotation Rate of the Field Vector, 64

- 2.7 Area Sweep Rate, 65
- 2.8 Rotation of & with Distance, 66
- 2.9 The Polarization Ratios, 68
- 2.10 Circular Wave Components, 69
- 2.11 Relationship between P and q, and the Modified Polarization Ratio, 71
- 2.12 Ellipse Characteristics in Terms of q, 72
- 2.13 Ellipse Characteristics in Terms of p and P, 74
- 2.14 Polarization Characteristics for Ranges of p and q, 75
- 2.15 The Transformations p(q) and q(p), 76
- 2.16 The Transformation for u < 0, 80
- 2.17 Polarization Chart as the p Plane, 83
- 2.18 Coincident Points on the q and w Planes, 84
- 2.19 Contours of Constant Axial Ratio and Tilt Angle, 84
- 2.20 Contours of Constant |p|, 85
- 2.21 Contours of Constant ϕ , 88
- 2.22 Stokes Parameters, 92
- 2.23 The Poincaré Sphere, 93
- 2.24 Special Points on the Poincaré Sphere, 94
- 2.25 Other Relationships between the Variables, 97
- 2.26 Mapping the Poincaré Sphere onto a Plane, 98
- 2.27 Mapping onto the q and w Planes, 102 References, 108 Problems, 108

3. POLARIZATION MATCHING OF ANTENNAS

- 3.1 Introduction, 110
- 3.2 Effective Length of an Antenna, 110
- 3.3 Received Voltage, 111
- 3.4 Maximum Received Power, 114
- 3.5 Polarization Match Factor, 117
- 3.6 Polarization Match Factor: Special Cases, 120
- 3.7 Match Factor in Other Forms, 125
- 3.8 Contours of Constant Match Factor, 127
- 3.9 The Poincaré Sphere and Polarization Match Factor, 136
- 3.10 Match Factor Using One Coordinate System, 138
- 3.11 Polarization Match Factor: Misaligned Antennas, 139 References, 146 Problems, 147

4. POLARIZATION CHARACTERISTICS OF SOME ANTENNAS 148

- 4.1 Introduction, 148
- 4.2 Test Antennas for Determining Effect of Polarization, 150

CONTENTS

- 4.3 The Short Dipole, 154
- 4.4 Crossed Dipoles (Turnstile Autennas), 157
- 4.5 Crossed Dipoles with Ground Plane, 159
- 4.6 The Loop Antenna, 161
- 4.7 Loop and Dipole, 162
- 4.8 Waveguide Opening into Infinite Ground Plane, 164
- 4.9 Horns, 167
- 4.10 Paraboloidal Reflector, 170
- 4.11 Narrow-Polarization-Beamwidth Array, 186 References, 189 Problems, 189

5. GENERATION OF GENERAL POLARIZATIONS

- 5.1 Introduction, 191
- 5.2 Simple Waveguide System for Elliptical Polarization, 191
- 5.3 Another Waveguide System, 195
- 5.4 Lossless Power Combiner and Divider System, 196 References, 206

6. POLARIZATION CHANGES BY REFLECTION AND TRANSMISSION

207

191

- 6.1 Linear Polarization, 207
- 6.2 Elliptical Waves, 217
- 6.3 Reflection and Transmission Matrices, 226
- 6.4 Backscattering from a Target: The Scattering Matrix, 228
- 6.5 Scattering Matrix for Circular Wave Components, 235
- 6.6 Relationship of the Scattering Matrix and Polarization Ratio, 237
- 6.7 Scattering Matrices for Some Common Reflectors, 239
- 6.8 Reflections from Arbitrarily Oriented Plane, 244 References, 252

7. PARTIAL POLARIZATION

- 7.1 Introduction, 253
- 7.2 Analytic Signals, 253
- 7.3 Coherency Matrix of a Quasi-Monochromatic Plane Wave, 254

xiii

CONTENTS

- 7.4 Degree of Polarization, 259
- 7.5 Stokes Parameters of Partially Polarized Waves, 262
- 7.6 Polarization Ratio of Partially Polarized Waves, 264
- 7.7 Reception of Partially Polarized Waves, 265 References, 268 Problems, 269

8. POLARIZATION MEASUREMENTS

0.	FULA	IRIZATION MEASUREMENTS	210
	8.1	Introduction, 270	
	8.2	The Linear Component Method, 270	
	8.3	The Circular Component Method, 271	
	8.4	The Polarization Pattern, 272	
	8.5	Power Combiner and Divider System, 274	
	8.6	Polarization Measurement with Unequal Effective	
	07	Lengths, 2/5 Polociation Properties from Amplitude Macoursets 277	
	8.7	Polarization Properties from Amplitude Measurements, 2//	
		References, 280	
		Problems, 280	
Ap	pendix	A. Relation between Effective Length and Gain	282
Ap	pendix	B. Isotropic Antennas and Null-Free Antennas References, 292	284
Ind	lex		293

370

xiv

POLARIZATION IN ANTENNAS AND RADAR

CONTENTS

- 7.4 Degree of Polarization, 259
- 7.5 Stokes Parameters of Partially Polarized Waves, 262
- 7.6 Polarization Ratio of Partially Polarized Waves, 264
- 7.7 Reception of Partially Polarized Waves, 265 References, 268 Problems, 269

8. POLARIZATION MEASUREMENTS

- 8.1 Introduction, 270
- 8.2 The Linear Component Method, 270
- 8.3 The Circular Component Method, 271
- 8.4 The Polarization Pattern, 272
- 8.5 Power Combiner and Divider System, 274
- 8.6 Polarization Measurement with Unequal Effective Lengths, 275
- 8.7 Polarization Properties from Amplitude Measurements, 277 References, 280 Problems, 280

Appendix A. Relation between Effective Length and Gain

Appendix B. Isotropic Antennas and Null-Free Antennas References, 292

Index



AN INTRODUCTION TO ANTENNAS

1.1. INTRODUCTION

This chapter introduces the concepts of antenna pattern, antenna input impedance, gain, effective area of a receiving antenna, losses, the relationship between gain and effective area, the Friis transmission formula, and a simple form of the radar equation. It is intended to impart a sufficient knowledge of these concepts so that a reader without previous study in antennas can readily understand the antenna-related developments in the remainder of the text. It also can serve as a review for those with some knowledge of antenna theory. Those with greater experience might well go to the next chapter.

Antenna concepts cannot be introduced without illustrative examples, so a linear wire antenna, of which the half-wave dipole is the best-known example, will be used to illustrate many of the ideas. Radiation from apertures will also be discussed. Other types of antennas cannot be covered in what must be a relatively short chapter. In addition, it is not the purpose of this chapter to cover all of the techniques for finding impedances, current distributions, and fields of even the antenna types we will discuss. The reader without a thorough background is referred to the excellent texts by Kraus, Elliott, and Collin and Zucker [1-3].

The task of designing an antenna system appears to be formidable, but fortunately it can be broken into simpler tasks that are more easily understood and carried out. The separation is convenient and enlightening. Consider two antennas as shown in Fig. 1.1: a transmitting antenna (1) connected to a generator and a receiving antenna (2) connected to a receiver (which we may treat as a load impedance).

1. To the generator, antenna 1 appears to be a load impedance or



FIGURE 1.1. Transmitting and receiving antennas.

admittance (perhaps transformed by a connecting transmission line or waveguide). We wish to find this impedance for matching purposes and to determine the power accepted by the antenna.

2. A portion of the power accepted by the antenna is radiated and a portion is dissipated as heat. We need to find the total power radiated.

3. The transmitting antenna does not radiate equally in all directions. We must determine the directional characteristics of the antenna and find the power density (Poynting vector magnitude) at the receiving antenna.

4. To the receiver, antenna 2 appears to be a voltage or current source, with the source value partly determined by the incident power density. Antenna 2 also appears to have some internal impedance. We must determine the source value and the receiving antenna internal impedance so that power to the receiver load can be found.

5. The path from antenna 1 to antenna 2 may not be direct but may involve a reflection from ground (multipath) or a target (as in radar). Strictly speaking, these are not antenna problems, but since this book is concerned with radar applications, we will be interested in appropriate descriptions of these phenomena.

6. The power to the receiver in Fig. 1.1 depends on the polarization characteristics of both antennas and any change in polarization during propagation between the antennas. (This is particularly true in radar.) We will defer the development of polarization properties to later chapters, to the greatest extent possible, in order to better construct the foundations of the development.

7. In general, the factors enumerated above depend on frequency, and antennas consequently have a finite bandwidth determined by impedance, radiation pattern, and so on. If all antenna, generator, target, and load parameters are known as functions of frequency, the bandwidth determination is straightforward, and we will devote little attention to it here.

1.2. THE VECTOR POTENTIALS

The Maxwell equations in time-invariant form, generalized to include magnetic sources, are

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} \qquad (a) \qquad \nabla \cdot \mathbf{B} = \rho_m \qquad (c)$$

$$\nabla \times \mathbf{E} = -\mathbf{M} - j\omega \mathbf{B} \qquad (b) \qquad \nabla \cdot \mathbf{D} = \rho \qquad (d) \qquad (1.1)$$

For linear, isotropic media D is related to E and B to H by the constitutive equations

$$\mathbf{D} = \varepsilon \mathbf{E} \quad (a) \qquad \mathbf{B} = \mu \mathbf{H} \quad (b) \tag{1.2}$$

and (1.1) becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\varepsilon \mathbf{E} \qquad (a) \qquad \nabla \cdot \mathbf{H} = \frac{\rho_m}{\mu} \qquad (c)$$

$$\nabla \times \mathbf{E} = -\mathbf{M} - j\omega\mu \mathbf{H} \qquad (b) \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \qquad (d)$$

In (1.1) the magnetic charge density ρ_m and the magnetic current density **M** are fictitious. Physical quantities corresponding to ρ_m and **M** do not exist. It is convenient, however, when considering some antenna problems to replace the actual sources by equivalent sources on a surface. The magnetic current density and charge density account for discontinuities in field components across the boundary surface [2, p. 31; 3, p. 3].

Since the Maxwell equations are linear, we may use superposition to account for the two sets of sources and solve separately the equations for electric and magnetic sources:

ELECTRIC SOURCES

$$\nabla \times \mathbf{H}_{J} = \mathbf{J} + j\omega \mathbf{D}_{J} = \mathbf{J} + j\omega\varepsilon \mathbf{E}_{J} \quad (a)$$

$$\nabla \times \mathbf{E}_{J} = -j\omega \mathbf{B}_{J} = -j\omega\mu \mathbf{H}_{J} \quad (b)$$

$$\nabla \cdot \mathbf{B}_{J} = \nabla \cdot \mathbf{H}_{J} = 0 \quad (c)$$

$$\nabla \cdot \mathbf{D}_{J} = \varepsilon \nabla \cdot \mathbf{E}_{J} = \rho \quad (d)$$

where subscript J refers to the partial fields produced by electric current density J and electric charge density ρ .

MAGNETIC SOURCES

$$\nabla \times \mathbf{H}_{M} = j\omega \mathbf{D}_{M} = j\omega \varepsilon \mathbf{E}_{M} \tag{a}$$

$$\nabla \times \mathbf{E}_{M} = -M - j\omega \mathbf{B}_{M} = -\mathbf{M} - j\omega \mu \mathbf{H}_{M} \quad (b)$$

 $\nabla \cdot \mathbf{B}_M = \mu \nabla \cdot \mathbf{H}_M = \rho_M \tag{c}$

$$\nabla \cdot \mathbf{D}_{M} = \nabla \cdot \mathbf{E}_{M} = 0 \tag{d}$$

(1.5)

where subscript M refers to the partial fields produced by magnetic current and charge densities M and ρ_M .

We consider first the Maxwell equations with electric sources (1.4). Since $\nabla \cdot \mathbf{B}_{j} = 0$, **B** can be represented as the curl of a vector potential **A**, commonly called the magnetic vector potential [4]

$$\mathbf{B}_{I} = \boldsymbol{\mu} \mathbf{H}_{I} = \nabla \times \mathbf{A} \tag{1.6}$$

Substituting (1.6) in (1.4b) leads to

$$\nabla \times (\mathbf{E}_J + j\omega \mathbf{A}) = 0 \tag{1.7}$$

Since the curl of the gradient of a scalar function is identically zero, we may set

$$\mathbf{E}_{J} + j\omega \mathbf{A} = -\nabla \Phi_{J} \tag{1.8}$$

where Φ_j is a scalar potential. It is commonly called the electric scalar potential.

Equations may be developed for A and Φ_j , and from their solutions \mathbf{D}_j and \mathbf{B}_j may be found. If we take the curl of (1.6) and substitute in (1.4a), we obtain

$$\nabla \times \nabla \times \mathbf{A} = \mu (\mathbf{J} + j\omega \varepsilon \mathbf{E}_J) \tag{1.9}$$

Use of a widely used vector identity for $\nabla \times \nabla \times A$ and the substitution of (1.8) in (1.9) gives

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - j \omega \mu \varepsilon \, \nabla \Phi_J + \omega^2 \mu \varepsilon \mathbf{A} \tag{1.10}$$

At this point only $\nabla \times \mathbf{A}$ has been constrained $(=\mathbf{B}_J)$. We are free to choose $\nabla \cdot \mathbf{A}$ according to Harrington [5], Sommerfeld [6], and Panofsky and Phillips [7]:

$$\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi,\tag{1.11}$$

With this choice, and with the definition

$$k^2 = \omega^2 \mu \varepsilon \tag{1.12}$$

(1.10) becomes

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} \tag{1.13}$$

The introduction of the relationship (1.11) between A and Φ_j makes it

THE VECTOR POTENTIALS

unnecessary for us to find Φ_j . The magnetic flux density \mathbf{B}_j may be found from the vector potential **A** once (1.13) is solved. Then we may find \mathbf{E}_j at points away from the sources by means of (1.4a), or alternatively we may find \mathbf{E}_j from (1.8) and (1.11), thus

$$\mathbf{E}_{J} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\,\nabla(\nabla\cdot\,\mathbf{A}) \tag{1.14}$$

Potentials for use when only magnetic sources **M** and ρ_M are present may be developed from the equation set (1.5) by analogy with the process used with electric sources. Using (1.5d) allows us to define an electric vector potential **F** by

$$\mathbf{D}_M = \varepsilon \mathbf{E}_M = -\nabla \times \mathbf{F} \tag{1.15}$$

where we note that the negative sign is arbitrary.

Substituting (1.15) into (1.5a) leads to

$$\nabla \times (\mathbf{H}_{M} + j\omega \mathbf{F}) = 0 \tag{1.16}$$

and the relationship to a magnetic scalar potential Φ_M ,

$$\mathbf{H}_{M} + j\omega\mathbf{F} = -\nabla\Phi_{M} \tag{1.17}$$

With the substitution of the curl of (1.15) into (1.5b) we obtain

$$\nabla \times \nabla \times \mathbf{F} = \varepsilon (\mathbf{M} + j\omega\mu \mathbf{H}_M) \tag{1.18}$$

and if we follow a process like that used previously and specify the divergence of **F** as

$$\nabla \cdot \mathbf{F} = -j\omega\mu\varepsilon \,\Phi_M \tag{1.19}$$

we arrive at an equation for F,

$$\nabla^2 \mathbf{F} + k^2 \mathbf{F} = -\varepsilon \mathbf{M} \tag{1.20}$$

Once **F** is found from (1.20) we may find \mathbf{D}_M from (1.15) and \mathbf{H}_M either from the appropriate Maxwell equation or from

$$\mathbf{H}_{M} = -j\omega\mathbf{F} - \frac{j}{\omega\mu\varepsilon}\,\nabla(\nabla\cdot\mathbf{F}) \tag{1.21}$$

As a final step we superimpose the solutions and find for the total fields due to electric and magnetic sources

$$\mathbf{E} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \mathbf{A}) - \frac{1}{\varepsilon} \nabla \times \mathbf{F}$$
(1.22)

$$\mathbf{H} = -j\omega\mathbf{F} - \frac{j}{\omega\mu\varepsilon}\,\nabla(\nabla\cdot\mathbf{F}) + \frac{1}{\mu}\,\nabla\times\mathbf{A} \tag{1.23}$$

1.3. INTEGRAL SOLUTIONS FOR THE VECTOR POTENTIALS

An integral solution for the vector potential equation (1.13) for A can be constructed by considering an infinitesimal current element at the origin of a spherical coordinate system (Fig. 1.2). For a z-directed current, (1.13) reduces to

$$\nabla^2 A_z + k^2 A_z = -\mu J_z \tag{1.24}$$

(1.25)

If we consider a point source (with source length infinitesimal), A_z must be spherically symmetric. In addition, everywhere except at the origin, $J_z = 0$, and (1.24) becomes



FIGURE 1.2. Current source at origin.

This equation has two independent solutions

$$\frac{1}{r}e^{-jkr}$$
 and $\frac{1}{r}e^{+jkr}$

which represent, respectively, outward- and inward-traveling spherical waves. From physical considerations we reject the inward-traveling wave and choose

$$A_z = \frac{C}{r} e^{-jkr} \tag{1.26}$$

where C is a constant. The constant may be quickly determined by noting that as $k \rightarrow 0$, (1.24) reduces to Poisson's equation

$$\nabla^2 A_z = -\mu J_z \tag{1.27}$$

with the well-known solution [5, 8]

$$A_{z} = \frac{\mu}{4\pi} \int \int \int \frac{J_{z}}{r} dv' \qquad (1.28)$$

where the prime denotes integration around the source point.

If we replace $J_z dv'$ by I dz' and integrate, the solution to (1.28) is

$$A_z = \frac{\mu I \ell}{4\pi r} \tag{1.29}$$

and if this is equated to (1.26) with k zero, we find for the constant

$$C = \frac{\mu I \ell}{4\pi} \tag{1.30}$$

The solution we then use for the magnetic vector potential of an infinitesimal current element at the coordinate origin is

$$A_z = \frac{\mu I \ell}{4\pi r} e^{-jkr} \tag{1.31}$$

This solution is readily generalized to the case of the infinitesimal element located at vector distance \mathbf{r}' from the origin and oriented along a line parallel to a general unit vector \mathbf{u} . It is

$$\mathbf{A}(\mathbf{r}) = \mathbf{u} \; \frac{\mu I \ell}{4\pi |\mathbf{r} - \mathbf{r}'|} \; e^{-jk|\mathbf{r} - \mathbf{r}'|} \tag{1.32}$$

Finally, if we have a linear current distribution, a current on a relatively thin wire for example, we obtain a solution to (1.13) in the form of an

outward-traveling wave

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \frac{I(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-jk|\mathbf{r} - \mathbf{r}'|} d\ell'$$
(1.33)

If we wish to return to our original formulation in terms of current density, the appropriate form is found by replacing $I d\ell' by J dv'$, giving

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \int \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-jk|\mathbf{r} - \mathbf{r}'|} dv'$$
(1.34)

By analogy we immediately find an integral form for the electric vector potential \mathbf{F} in (1.20). It is

$$\mathbf{F}(\mathbf{r}) = \frac{\varepsilon}{4\pi} \int \int \int \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-jk|\mathbf{r} - \mathbf{r}'|} dv'$$
(1.35)

or if we consider a magnetic current K,

$$\mathbf{F}(\mathbf{r}) = \frac{\varepsilon}{4\pi} \int \frac{K(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-jk|\mathbf{r} - \mathbf{r}'|} d\ell'$$
(1.36)

1.4. APPROXIMATIONS TO THE POTENTIALS

The evaluation of the vector potential integrals can be quite difficult to carry out in the general case, and approximations are desirable in the factor $|\mathbf{r} - \mathbf{r}'|$. We apply the binomial expansion

$$|\mathbf{r} - \mathbf{r}'| = [r^2 - 2\mathbf{r} \cdot \mathbf{r}' + {r'}^2]^{1/2}$$
$$= r - \mathbf{u}_r \cdot \mathbf{r}' + \frac{1}{2r} [r'^2 - (\mathbf{u}_r \cdot \mathbf{r}')^2] + \cdots$$
(1.37)

for r > r', where $\mathbf{u}_r = \mathbf{r}/r$, and terms in r^{-2} , r^{-3} , and so on, have been dropped.

Fresnel Zone

At distances from the sources where

$$r \ge r' \quad kr \ge 1$$

we may approximate $|\mathbf{r} - \mathbf{r}'|$ by r in the amplitude of the potential integrals and by (1.37) in the phase term. Then (1.34), for example, simplifies to FAR-ZONE FIELDS

$$\mathbf{A}(\mathbf{r}) = \frac{\mu e^{-jkr}}{4\pi r} \int \int \int \mathbf{J}(\mathbf{r}') \exp\left\{jk\left[\mathbf{u}_r \cdot \mathbf{r}' + \frac{(\mathbf{u}_r \cdot \mathbf{r}')^2}{2r} - \frac{r'^2}{2r}\right]\right\} dv' \quad (1.38)$$

Fraunhofer Zone (Far Zone)

At still greater distances from the sources we can drop the r^{-1} term of (1.37) in the phase of the potential integrals, and the equation for $A(\mathbf{r})$, for example, becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu e^{-jkr}}{4\pi r} \int \int \int \mathbf{J}(\mathbf{r}') e^{jk\mathbf{u}_r \cdot \mathbf{r}'} dv'$$
(1.39)

where it may be noted that

$$\mathbf{u}_r \cdot \mathbf{r}' = r' \cos \psi \tag{1.40}$$

with ψ the angle between **r** and **r**'.

The boundaries between the zones are not easily chosen and in fact depend on the distributions J and M. A commonly used dividing line between the Fresnel and far or Fraunhofer zones for an antenna with greatest linear dimension L is

$$r = \frac{2L^2}{\lambda} \tag{1.41}$$

With this choice the greatest value of the last two terms in (1.37), which we dropped in going from Fresnel to far zone, is

$$\frac{kr'^2}{2r} = \frac{2\pi}{\lambda} L^2 \frac{\lambda}{4L^2} = \frac{\pi}{2}$$

1.5. FAR-ZONE FIELDS

We can find expressions for the fields from the integral forms for the potentials. We first use the general forms for the potentials. From (1.6) and (1.34) we find

$$\mathbf{H}_{J} = \frac{1}{\mu} \nabla \times \mathbf{A} = \frac{1}{4\pi} \int \int \int \nabla \times \left[\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-jk|\mathbf{r} - \mathbf{r}'|} \right] d\upsilon'$$
$$= \frac{-1}{4\pi} \int \int \int \mathbf{J}(\mathbf{r}') \times \nabla \left[\frac{e^{-jk|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \right] d\upsilon'$$
(1.42)

We then find \mathbf{E}_{I} from

$$\mathbf{E}_{j} = \frac{1}{j\omega\varepsilon} \,\nabla \times \mathbf{H}_{j} = -\frac{1}{j4\pi\omega\varepsilon} \int \int \int \nabla \times \left[\mathbf{J} \times \nabla \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) \right] d\upsilon' \quad (1.43)$$

From the identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}$$
(1.44)

and the fact that J is a constant vector in the differentiation, E_J becomes

$$\mathbf{E}_{J} = \frac{-1}{j4\pi\omega\varepsilon} \int \int \int \left[\mathbf{J}\nabla^{2} \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) - (\mathbf{J}\cdot\nabla)\nabla \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) \right] d\upsilon' \quad (1.45)$$

Now, the function $e^{-jk|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r}-\mathbf{r}'|$ is a solution of the scalar Helmholtz equation [3, p. 38]

$$\nabla^2 \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) + k^2 \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) = 0$$
(1.46)

and, with this, we find for \mathbf{E}_{j}

$$\mathbf{E}_{J} = \frac{1}{j4\pi\omega\varepsilon} \int \int \int \left[\mathbf{J}k^{2} \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) + (\mathbf{J}\cdot\nabla)\nabla \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) \right] d\upsilon' \quad (1.47)$$

If we repeat the above procedure using the electric vector potential \mathbf{F} for magnetic source distribution \mathbf{M} , we obtain

$$\mathbf{E}_{M} = -\frac{1}{\varepsilon} \nabla \times \mathbf{F} = \frac{1}{4\pi} \int \int \int \mathbf{M}(\mathbf{r}') \times \nabla \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right) d\upsilon' \qquad (1.48)$$
$$\mathbf{H}_{M} = \frac{-1}{j\omega\mu} \nabla \times \mathbf{E}_{M}$$
$$= \frac{1}{j4\pi\omega\mu} \int \int \int \left[\mathbf{M}k^{2} \left(\frac{e^{-jk|\mathbf{k}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right) + (\mathbf{M}\cdot\nabla)\nabla \left(\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right)\right] d\upsilon' \qquad (1.49)$$

We now use the far-field approximation

$$\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \cong \frac{e^{-jk(r-\mathbf{u}_r\cdot\mathbf{r}')}}{r}$$
(1.50)

If only the terms of order 1/r are retained, the gradient may be shown to be

$$\nabla\left(\frac{e^{-jk(r-\mathbf{u}_r\cdot\mathbf{r}')}}{r}\right) \cong \frac{-jk}{r} e^{-jkr} e^{jk\mathbf{u}_r\cdot\mathbf{r}'}\mathbf{u}_r \tag{1.51}$$

and \mathbf{H}_{J} is immediately seen to be

FAR-ZONE FIELDS 11

$$\mathbf{H}_{J} = \frac{jk}{4\pi r} e^{-jkr} \int \int \int \mathbf{J}(\mathbf{r}') \times \mathbf{u}_{r} e^{jk\mathbf{u}_{r}\cdot\mathbf{r}'} d\upsilon'$$
(1.52)

which may be written as

$$\mathbf{H}_{J} = \frac{jk}{4\pi r} e^{-jkr} \int \int \int (J_{\phi} \mathbf{u}_{\theta} - J_{\theta} \mathbf{u}_{\phi}) e^{jk\mathbf{u}_{r} \cdot \mathbf{r}'} d\upsilon'$$
(1.53)

The corresponding value of the electric field for an electric source distribution may be found by substituting (1.50) and (1.51) into (1.47). The result is [3]

$$\mathbf{E}_{J} = \frac{-jkZ_{0}}{4\pi r} e^{-jkr} \int \int \int (J_{\theta}\mathbf{u}_{\theta} + J_{\phi}\mathbf{u}_{\phi}) e^{jk\mathbf{u}_{r}\cdot\mathbf{r}'} d\upsilon'$$
(1.54)

In the same manner we find the fields for a magnetic source distribution

$$\mathbf{E}_{M} = -\frac{jk}{4\pi r} e^{-jkr} \int \int \int (M_{\phi} \mathbf{u}_{\theta} - M_{\theta} \mathbf{u}_{\phi}) e^{jk\mathbf{u}_{r} \cdot \mathbf{r}'} d\nu' \qquad (1.55)$$

$$\mathbf{H}_{M} = -\frac{jk}{4\pi Z_{0}r} e^{-jkr} \int \int \int (M_{\theta}\mathbf{u}_{\theta} + M_{\phi}\mathbf{u}_{\phi}) e^{jk\mathbf{u}_{r}\cdot\mathbf{r}'} dv' \qquad (1.56)$$

where

$$Z_0 = \sqrt{\frac{\mu}{\varepsilon}} \tag{1.57}$$

is the intrinsic impedance of the medium of interest.

We can summarize these far fields:

MAGNETIC SOURCES ELECTRIC SOURCES $E_{r} = 0$ (a) $H_r = 0$ (g) $E_{\theta} = -j\omega A_{\theta}$ (b) $H_{\theta} = -j\omega F_{\theta}$ (h) (c) $H_{\phi} = -j\omega F_{\phi}$ $E_{\phi} = -j\omega A_{\phi}$ (i) (1.58)(d) $E_r = 0$ $H_{r} = 0$ (j) $H_{\theta} = \frac{j\omega A_{\phi}}{Z_0} = -\frac{E_{\phi}}{Z_0} \quad \text{(e)} \qquad E_{\theta} = -j\omega Z_0 F_{\phi} = Z_0 H_{\phi}$ (k) $H_{\phi} = \frac{-j\omega A_{\theta}}{Z_0} = \frac{E_{\theta}}{Z_0} \quad (f) \qquad E_{\phi} = j\omega Z_0 F_{\theta} = -Z_0 H_{\theta}$ (1)

We may see from these equations for the field components that in the far zone the \mathbf{E} and \mathbf{H} fields are orthogonal to each other and to \mathbf{r} and thus are TEM-to-r fields. This is true whether the sources are electric or magnetic or a combination of both.

1.6. USE OF THE POTENTIAL INTEGRALS FOR PHYSICAL STRUCTURES

The potential integrals for electric and magnetic sources were developed by considering an infinitesimal current element, with the implication that it was a current existing in a homogeneous medium throughout the region of interest to us. Antennas, however, have currents flowing in metallic conductors, and we must justify the use of the potential integrals for inhomogeneous media. To do this we use two of the Maxwell equations

$$\nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E} + \mathbf{J}^{s} = \hat{y}\mathbf{E} + \mathbf{J}^{s} \quad (a)$$

$$-\nabla \times \mathbf{E} = j\omega\mu\mathbf{H} + \mathbf{M}^{s} = \hat{z}\mathbf{H} + \mathbf{M}^{s} \quad (b)$$

(1.59)

where the superscript denotes a source, and the definitions of \hat{y} and \hat{z} are obvious.

The greater part of the region of interest is free space with parameters ε_0 and μ_0 , with a small region (the antenna) having different parameters, ε and μ . We may then rewrite the equations above as

$$\nabla \times \mathbf{H} = j\omega\varepsilon_0 \mathbf{E} + (\hat{y} - j\omega\varepsilon_0)\mathbf{E} + \mathbf{J}^s = j\omega\varepsilon_0 \mathbf{E} + \mathbf{J} \qquad (a)$$

$$-\nabla \times \mathbf{E} = j\omega\mu_0 \mathbf{H} + (\hat{z} - j\omega\mu_0)\mathbf{H} + \mathbf{M}^s = j\omega\mu_0 \mathbf{H} + \mathbf{M} \qquad (b)$$

where we have defined

$$\mathbf{J} = \mathbf{J}^{s} + (\hat{y} - j\omega\varepsilon_{0})\mathbf{E} \qquad (a)$$
$$\mathbf{M} = \mathbf{M}^{s} + (\hat{z} - j\omega\mu_{0})\mathbf{H} \qquad (b)$$

Formally, Eqs. (1.60) are the free-space Maxwell equations, and we may use the potential integrals with the sources **J** and **M** as if the region were homogeneous. The sources are now unknown, but we will see how this problem can be handled for a linear antenna.

Assume that a linear metallic antenna is fed by a current source \mathbf{J}^s . Further, for most conductors we can assume that $\mu = \mu_0$ and $\varepsilon \cong \varepsilon_0$ (the assumption about ε is unnecessary if we recognize that $\sigma \ge \omega \varepsilon$). Then the densities of (1.61) become

$$\mathbf{J} = \mathbf{J}^s + \sigma \mathbf{E} \quad (a) \qquad \mathbf{M} = 0 \quad (b) \tag{1.62}$$

Under these circumstances for a wire antenna we can write

$$\mathbf{A} = \frac{\mu}{4\pi} \int \int \int \frac{\mathbf{J}^s + \sigma \mathbf{E}}{|\mathbf{r} - \mathbf{r}'|} e^{-jk|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}v' \tag{1.63}$$

In (1.63) we have a valid potential integral that is applicable to our antenna problem even though we are dealing with an inhomogeneous region of space. The price that we must pay for obtaining this form is that the integrand is unknown.

Now the current source J^s is applied to the feed gap of a linear antenna. Also σE is the physical current density in the conducting structure of the antenna. We may then replace the integrand of (1.63) by measured currents on the antenna and write, for a thin antenna,

$$\mathbf{A}(\mathbf{r}) = \frac{u}{4\pi} \int \frac{I(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-jk|\mathbf{r} - \mathbf{r}'|} d\ell'$$
(1.64)

which agrees with (1.33). We have thus justified the use of the potential integrals to obtain the potentials produced by currents flowing in metallic antennas [9].

1.7. RADIATION PATTERN

We are now in a position to define the radiation pattern of an antenna and some related parameters. We will use IEEE Standard 145-1983 definitions for these and other quantities unless otherwise indicated [10].

In Section 1.3 we found the magnetic vector potential of an infinitesimal current element at the coordinate origin in order to develop the general equations for electric and magnetic vector potentials for more general source distributions. We now let the infinitesimal current element do double duty by using it as a basis for the definition of radiation pattern.

The magnetic vector potential for the current element of Fig. 1.2, which we obtained earlier, is

$$A_{z} = \frac{\mu I \ell}{4\pi r} e^{-jkr} \tag{1.31}$$

It is desirable to use spherical coordinates, and it is obvious that

$$A_{r} = A_{z} \cos \theta = \frac{\mu I \ell}{4\pi r} \cos \theta e^{-jkr} \qquad (a)$$
$$A_{\theta} = -A_{z} \sin \theta = -\frac{\mu I \ell}{4\pi r} \sin \theta e^{-jkr} \qquad (b) \qquad (1.65)$$

In order to define pattern and gain we need only the far-field components, which we may obtain immediately from (1.58)

$$E_r = 0 \tag{a}$$

$$E_{\theta} = -j\omega A_{\theta} = \frac{j\omega\mu I\ell}{4\pi r} \sin\theta e^{-jkr} \quad (b)$$

$$E_{\phi} = -j\omega A_{\phi} = 0 \tag{c}$$

$$H_r = 0$$
 (d)

$$H_{\theta} = -\frac{E_{\phi}}{Z_0} = 0 \tag{e}$$

$$H_{\phi} = \frac{E_{\theta}}{Z_0} = \frac{j\omega\mu I\ell}{4\pi Z_0 r} \sin\theta e^{-jkr} \qquad (f)$$

where

$$k = \frac{2\pi}{\lambda}$$

The time-average Poynting vector in the far field becomes

$$\mathbf{S} = \frac{1}{2} \operatorname{Re} \left(\mathbf{E} \times \mathbf{H}^* \right) = \frac{1}{2} \operatorname{Re} \left(E_{\theta} H_{\phi}^* \right) \mathbf{u}_r$$
$$= \frac{Z_0 |I|^2 \ell^2}{8\lambda^2 r^2} \sin^2 \theta \, \mathbf{u}_r \tag{1.67}$$

We found here the time-average Poynting vector in the far field. If we had used the more general relationships (1.22) and (1.23) for the fields in terms of **A**, we would have found the time-average Poynting vector for fields that vary as $1/r^2$ and $1/r^3$ to be zero, so that (1.67) is correct everywhere in the field. Of course, energy can be transferred to conducting objects by field terms other than the far fields (consider a transformer, for example). The difference is that, in the absence of nearby lossy objects, the energy in the 1/r field terms is lost to the antenna system, whereas energy represented by near-field terms is stored.

Radiation Intensity

Radiation intensity in a given direction is the power radiated from the antenna per unit solid angle. A bundle of rays, not all lying in a common plane, and intersecting at a common point, forms a solid angle, measured in steradians (dimensionless). If a sphere of radius r is constructed as in Fig. 1.3(a), with center at the ray intersection, the rays subtend area A on the sphere surface. The ratio



FIGURE 1.3. Illustration of solid angles: (a) solid angle and subtended area on a sphere; (b) elementary solid angle with general surface.

$$\Omega = \frac{A}{r^2} \tag{1.68}$$

is independent of the sphere radius and defines the solid angle formed by the rays. We may also write the infinitesimal element of solid angle in terms of the elementary area dA on any surface, as in Fig. 1.3(b). The projection of the surface area element onto a sphere centered at the ray intersection is

$\mathbf{u}_r \cdot \mathbf{n} \, dA$

where **n** is the unit normal vector to the surface, and **u**, the unit vector in the direction of **r**, the vector drawn from the ray intersection to the surface element. With the use of this surface area element projection, the element of solid angle is

$$d\Omega = \frac{dA}{r^2} \mathbf{u}_r \cdot \mathbf{n} \tag{1.69}$$

The radiation intensity U is then related to power radiated within solid angle Ω by

$$U = \frac{W}{\Omega} \tag{1.70}$$

if Ω is small enough that U is a constant. More generally

$$W = \int U \, d\Omega = \int \int U \, \sin \theta \, d\theta \, d\phi \tag{1.71}$$

The radiation intensity is readily related to S, the magnitude of the time-average Poynting vector, since

$$W = U\Omega = SA = Sr^2\Omega$$

or

$$U(\theta, \phi) = r^2 S(\theta, \phi) \tag{1.72}$$

in which we stress that both U and S are dependent on direction.

Radiation Pattern

The *radiation pattern* (antenna pattern) represents the spatial distribution of a quantity that characterizes the electromagnetic field generated by an antenna. In the usual case the radiation pattern is determined in the far-field region and is represented as a function of directional coordinates. Radiation properties

include power flux density, radiation intensity, phase, polarization, and field strength. (Note: Power flux density is not defined in the IEEE Standards, but in context it is used as the magnitude of the time-average Poynting vector. Normally we will use *power density*.)

The radiation pattern is measured as a function of direction at a constant radius from the antenna. Absolute or relative measurements may be made, although it is more common to see plots of relative power density or field magnitude. It is usual to present the three-dimensional information as a series of two-dimensional plots, in either polar or rectangular form. Fig. 1.4 shows the relative field magnitude and power density in constant-azimuth planes for the z-directed infinitesimal current element, or *elementary antenna* in our usage.

One two-dimensional plot is sufficient to describe the radiation pattern of the infinitesimal current element since the fields and power density are not functions of azimuth angle ϕ . For more general patterns, plots may be shown for variable polar angle θ in planes of constant-azimuth ϕ , or vice versa. Two



(c)

FIGURE 1.4. Radiation patterns of infinitesimal current element: (a) field strength, $d = |\sin \theta|$; (b) power density, $d = \sin^2 \theta$; (c) power density.



FIGURE 1.5 H-plane pattern for z-directed current element.

patterns of considerable importance for many antennas are the principal *E-plane* and *H-plane* patterns. The principal *E* plane is a plane containing the electric field vector and the direction of maximum radiation, and the principal *H* plane is one containing the magnetic field vector and the direction of maximum radiation [8]. Jointly, these are called the *principal planes*. The definitions apply only to antennas whose radiated wave is linearly polarized in the direction of maximum power density. Any constant-azimuth plane is an *E* plane for the *z*-directed current element, so Fig. 1.4 is an *E*-plane pattern. The *H* plane for the element is the plane $\theta = \pi/2$, or the *xy* plane. The *H*-plane pattern for the current element is shown in Fig. 1.5.

Beamwidth

The *half-power beamwidth*, in a plane containing the direction of the maximum of a beam, is the angle between the two directions in which the radiation intensity is one-half the maximum value of the beam. Half-power beamwidths in the E plane are shown in Figs. 1.4(a) and 1.4(b) for the current element. In the H plane it is inappropriate to speak of a beamwidth since the radiation intensity is constant. The current element is an *omnidirectional antenna*, one having a nondirectional pattern in a given plane (azimuth) and a directional pattern in any orthogonal plane.

Radiation Lobe

This is a portion of the radiation pattern bounded by regions of relatively weak radiated intensity [11]. Our current element fails us as an illustration, and we must go to a more general antenna that may have a principal plane pattern like that of Fig. 1.6.



FIGURE 1.6. Radiation lobes of general antenna.

1.8. GAIN AND DIRECTIVITY

The radiation patterns of the previous section show that the radiation intensity from an antenna is greater in some directions than in others, and this is of benefit to the user in many applications. This characteristic is usefully described by the directivity of the antenna. It is in essence a comparison of the radiation intensity of the antenna in a specified direction to the intensity that would exist if the antenna were to radiate the same total power equally in all directions, and can be greater than unity in some directions if it is less than unity in others. The *directivity* is the ratio of the radiation intensity in a given directions. The average radiation intensity is equal to the total power radiated by the antenna divided by 4π (the solid angle measure of a sphere). If the direction is not specified, the direction of maximum radiation intensity is implied [10].[†]

We may easily determine the directivity of the infinitesimal current source of Section 1.7. From (1.67) we find the radiation intensity to be

$$U(\theta, \phi) = \frac{Z_0 |I|^2 \ell^2}{8\lambda^2} \sin^2 \theta \qquad (1.73)$$

This represents a change from previous IEEE Standard definitions [11, 12] that used *directive* gain for this function and *directivity* for the maximum value of the directive gain. Earlier, Kraus (in 1950) used *directivity* as a function of θ , ϕ [1, Eq. 2-52a]. The author prefers *directive gain* (now deprecated by the Standard), but the usage in this text will conform to the current Standard.
Then the total radiated power is determined by integrating over the surface of a sphere of very large radius.

$$W_{\rm rad} = \int U(\theta, \phi) \, d\Omega = \frac{Z_0 |I|^2 \ell^2}{8\lambda^2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin^3 \theta \, d\theta \, d\phi$$
$$= \frac{\pi Z_0 |I|^2 \ell^2}{3\lambda^2} \tag{1.74}$$

The average radiation intensity is

$$U_{\rm av} = \frac{W_{\rm rad}}{4\pi} = \frac{Z_0 |I|^2 \ell^2}{12\lambda^2}$$
(1.75)

and the directivity is

$$D(\theta, \phi) = \frac{U(\theta, \phi)}{U_{\rm av}} = \frac{3}{2} \sin^2 \theta \qquad (1.76)$$

Radiation Resistance

The gain of an antenna is closely related to its directivity. Before considering antenna gain, however, it is useful to consider power losses in the antenna and also to anticipate a later discussion of antenna impedance by discussing *radiation resistance*. Power radiated is lost to the generator-transmission line-antenna system, and to the generator the loss is indistinguishable from heat loss in a resistance of appropriate value. We therefore define an equivalent resistance called the *radiation resistance* which is the ratio of power radiated by the antenna to the square of the rms current referred to a specified point.

The radiation resistance of the elementary antenna (infinitesimal current source) whose radiated power is given by (1.74) is obviously

$$R_{r} = \frac{2\pi Z_{0}\ell^{2}}{3\lambda^{2}} = 80\pi^{2} \left(\frac{\ell}{\lambda}\right)^{2}$$
(1.77)

where we use 120π for the characteristic impedance of free space. See Section 1.12 for a further discussion of radiation resistance.

The current is the same at all points of the elementary antenna, and it is clear that radiated power is divided by the square of that constant current to obtain the radiation resistance. A widely used antenna that does not have a constant current throughout it is a circular cylindrical center-fed dipole, shown in Fig. 1.7. If it is made of a wire whose radius is much smaller than a wavelength and much smaller than the dipole length, measurements show that the current, to a good approximation, is sinusoidal [1, p. 139].



FIGURE 1.7. Center-fed dipole antenna and associated current distribution.

$$I(z') = I_m \sin \left[k(\frac{1}{2}\ell - |z'|) \right] \quad -\frac{1}{2}\ell \le z' \le \frac{1}{2}\ell \tag{1.78}$$

At the antenna feed point

$$I_{in} = I(0) = I_m \sin\left(\frac{1}{2}k\ell\right)$$
(1.79)

The last phrase of the definition of radiation resistance is now clear. We may define the radiation resistance of the dipole referred to a current maximum

$$R_r = \frac{W_{\rm rad}}{|I_m|^2} \tag{1.80}$$

or we may define the resistance referred to the feed point

$$R_{\rm in} = \frac{W_{\rm rad}}{|I_{\rm in}|^2} \tag{1.81}$$

and the relationship between the two definitions is obviously

$$R_r = R_{\rm in} \sin^2\left(\frac{1}{2}k\ell\right) \tag{1.82}$$

Both definitions are used in the literature.

Antenna Losses

Antennas are constructed of conductors, with finite conductivity, and lossy dielectric materials. It is clear that not all of the power accepted by the antenna from its feed system is radiated—some is lost as heat. The determination of the losses is normally quite tedious and requires a knowledge of tangential magnetic fields (or surface current densities) at conducting surfaces and the electric fields in lossy dielectrics. We will consider here only one of the simplest cases, the losses in a circular cylindrical antenna made of a wire with conductivity σ and carrying a known current distribution I(z').

For a wire of radius a, we may treat an axial high-frequency current as though it flows with constant density to a depth δ at the wire surface, where δ is the *skin depth*, given by

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \tag{1.83}$$

Then the high-frequency resistance per unit length of such a wire is[†]

$$R_{\rm hf} = \frac{1}{2\pi a \delta \sigma} \tag{1.84}$$

Power loss per unit length then becomes

 $\frac{1}{2}I^2R_{\rm hf}$

and for a finite-length antenna

$$W_{\rm loss} = \frac{1}{2} \int_{-\ell/2}^{\ell/2} |I(z')|^2 R_{\rm hf} \, dz' \tag{1.85}$$

When applied to the elementary antenna we are using for most of our illustrations,

$$W_{\rm loss} = \frac{1}{2} |I|^2 R_{\rm hf} \,\ell \tag{1.86}$$

An equivalent loss resistance for this antenna can be defined as

$$R_{\rm loss} = \frac{2W_{\rm loss}}{\left|I\right|^2} = R_{\rm hf}\ell \tag{1.87}$$

More generally, a loss resistance for a dipole antenna, referred to the

Strictly, this is not quite correct, and the equation for a conductor of circular cross section is more complicated. Typically, for an antenna, $\delta \ll a$ and the approximation is excellent.

input, is

$$R_{\rm loss} = \frac{1}{|I_{\rm in}|^2} \int_{-\ell/2}^{\ell/2} |I(z')|^2 R_{\rm hf} \, dz' \tag{1.88}$$

Radiation Efficiency

In general the transmitter will deliver an incident power to the antenna. Part will be reflected because of an impedance mismatch and part will be *accepted* by the antenna. Of the power accepted, some will be radiated and some lost as heat in lossy conductors and dielectrics. A *radiation efficiency* is defined as the ratio of the total power radiated by the antenna to the net power accepted by the antenna from the connected transmitter.

We may write the efficiency as

$$e = \frac{W_{\rm rad}}{W_{\rm rad} + W_{\rm loss}} = \frac{W_{\rm rad}}{W_{\rm acc}}$$
(1.89)

and if we consider radiation resistance and loss resistance, referred to the same point,

$$e = \frac{R_r}{R_r + R_{\rm loss}} \tag{1.90}$$

Gain

We are now in a position to define the antenna gain (sometimes called *power* gain). It is the ratio of the radiation intensity, in a given direction, to the radiation intensity that would be obtained if the power accepted by the antenna were radiated isotropically (equally in all directions). Thus

$$G(\theta, \phi) = \frac{U(\theta, \phi)}{(1/4\pi)W_{\text{acc}}}$$
(1.91)

Gain does not include losses arising from impedance and polarization mismatches. If the direction is not specified, the direction of maximum radiation intensity is implied.

We may now readily use the relationship between radiated and accepted power to relate gain and directivity.

$$G(\theta, \phi) = \frac{U(\theta, \phi)}{(1/4\pi)W_{\rm rad}/e} = eD(\theta, \phi)$$
(1.92)

1.9. THE DIPOLE ANTENNA: FIELDS

In order to discuss the input impedance of an antenna, as we shall do in a later section, it is desirable to obtain the fields of a more complex antenna than the

23

elementary current source. We will use the center-fed dipole antenna of Fig. 1.8 for this purpose because it is a highly useful antenna and the equations describing it are readily available. We need both the near- and far-field terms, and we will save time by finding the complete fields and specializing as desired.

It was noted earlier that the current distribution of a center-fed dipole is

$$I(z') = I_m \sin \left[k \left(\frac{1}{2}\ell - |z'| \right) \right] \qquad -\frac{1}{2}\ell \le z' \le \frac{1}{2}\ell \qquad (1.78)$$

We may write for the magnetic vector potential

$$A_{z} = \frac{\mu I_{m}}{4\pi} \left\{ \int_{-\ell/2}^{0} \sin \left[k(\frac{1}{2}\ell + z')\right] \frac{e^{-jkR}}{R} dz' + \int_{0}^{\ell/2} \sin \left[k(\frac{1}{2}\ell - z')\right] \frac{e^{-jkR}}{R} dz' \right\}$$
(1.93)

FIGURE 1.8. Geometry for determining dipole fields.

If the sine terms of the integrands are replaced by their exponential equivalents, A_z becomes

$$A_{z} = \frac{\mu I_{m}}{8\pi j} \left[e^{jk(\ell/2)} \int_{-\ell/2}^{0} \frac{e^{-jk(R-z')}}{R} dz' - e^{-jk(\ell/2)} \int_{-\ell/2}^{0} \frac{e^{-jk(R+z')}}{R} dz' + e^{jk(\ell/2)} \int_{0}^{\ell/2} \frac{e^{-jk(R+z')}}{R} dz' - e^{-jk(\ell/2)} \int_{0}^{\ell/2} \frac{e^{-jk(R-z')}}{R} dz' \right]$$
(1.94)

If we use circular cylindrical coordinates, R may be written as

$$R = \sqrt{(z - z')^2 + \rho^2}$$
(1.95)

where ρ is the radial distance shown in Fig. 1.8. The magnetic field intensity is

$$H_{\phi} = -\frac{1}{\mu} \frac{\partial A_z}{\partial \rho} \tag{1.96}$$

and if we differentiate in (1.94) we obtain

$$H_{\phi} = -\frac{I_m}{8\pi j} \left\{ e^{jk(\ell/2)} \int_{-\ell/2}^{0} \frac{\partial}{\partial\rho} \frac{e^{-jk(R-z')}}{R} dz' - e^{-jk(\ell/2)} \int_{-\ell/2}^{0} \frac{\partial}{\partial\rho} \frac{e^{-jk(R+z')}}{R} dz' + e^{jk(\ell/2)} \int_{0}^{\ell/2} \frac{\partial}{\partial\rho} \frac{e^{-jk(R+z')}}{R} dz' - e^{-jk(\ell/2)} \int_{0}^{\ell/2} \frac{\partial}{\partial\rho} \frac{e^{-jk(R-z')}}{R} dz' \right\}$$
(1.97)

Carrying out the differentiation in the integrands gives us

$$\frac{\partial}{\partial \rho} \frac{e^{-jk(R\pm z')}}{R} = -\rho \left(\frac{jk}{R^2} + \frac{1}{R^3}\right) e^{-jk(R\pm z')}$$
(1.98)

These terms are exact differentials, and it may be verified in a straightforward manner that

$$\frac{d}{dz'} \frac{e^{-jk(R-z')}}{R(R-z'+z)} = \left(\frac{jk}{R^2} + \frac{1}{R^3}\right)e^{-jk(R-z')}$$
(a) (1.99)

$$\frac{d}{dz'} \frac{e^{-jk(R+z')}}{R(R+z'-z)} = -\left(\frac{jk}{R^2} + \frac{1}{R^3}\right)e^{-jk(R+z')}$$
(b)

The first integral in Eq. (1.97) therefore becomes

$$-\rho \left[\frac{e^{-jk(R-z')}}{R(R-z'+z)} \right]_{-\ell/2}^{0} = -\rho \left[\frac{e^{-jkr}}{r(r+z)} - \frac{e^{-jk(R_{2}+\ell/2)}}{R_{2}(R_{2}+\ell/2+z)} \right]$$
$$= -\rho \left[\frac{(r-z)e^{-jkr}}{r(r^{2}-z^{2})} - \frac{(R_{2}-\ell/2-z)e^{-jk(R_{2}+\ell/2)}}{R_{2}[R_{2}^{2}-(\ell/2+z)^{2}]} \right]$$

But

$$r^{2} = z^{2} + \rho^{2} \qquad (a)$$

$$R_{1}^{2} = (z - \frac{1}{2}\ell)^{2} + \rho^{2} \qquad (b) \qquad (1.100)$$

$$R_{2}^{2} = (z + \frac{1}{2}\ell)^{2} + \rho^{2} \qquad (c)$$

and the integral may be put into the form

$$-\frac{1}{\rho}\left[\left(1-\frac{z}{r}\right)e^{-jkr}-\left(1-\frac{\ell/2+z}{R_2}\right)e^{-jk(R_2+\ell/2)}\right]$$

The remaining integrals in (1.97) for H_{ϕ} give in order

$$\frac{1}{\rho} \left[\left(1 + \frac{z}{r} \right) e^{-jkr} - \left(1 + \frac{\ell/2 + z}{R_2} \right) e^{-jk(R_2 - \ell/2)} \right]$$
$$\frac{1}{\rho} \left[\left(1 - \frac{\ell/2 - z}{R_1} \right) e^{-jk(R_1 + \ell/2)} - \left(1 + \frac{z}{r} \right) e^{-jkr} \right]$$
$$- \frac{1}{\rho} \left[\left(1 + \frac{\ell/2 - z}{R_1} \right) e^{-jk(R_1 - \ell/2)} - \left(1 - \frac{z}{r} \right) e^{-jkr} \right]$$

If these four terms are substituted into the equation for H_{ϕ} , it becomes

$$H_{\phi} = -\frac{I_m}{4\pi j\rho} \left[e^{-jkR_1} + e^{-jkR_2} - 2\cos\frac{k\ell}{2} e^{-jkr} \right]$$
(1.101)

which is a remarkably simple equation for the field close to an antenna [13].

The electric field components are readily found from one of the Maxwell equations in circular cylindrical coordinates

$$E_{\rho} = -\frac{1}{j\omega\varepsilon} \frac{\partial H_{\phi}}{\partial z} \tag{1.102}$$

$$= \frac{jZ_0 I_m}{4\pi\rho} \left[\left(z - \frac{\ell}{2} \right) \frac{e^{-jkR_1}}{R_1} + \left(z + \frac{\ell}{2} \right) \frac{e^{-jkR_2}}{R_2} - 2z\cos\frac{k\ell}{2} \frac{e^{-jkr}}{r} \right]$$
(a)

$$E_{\phi} = 0 \tag{b}$$

$$E_{z} = -\frac{1}{j\omega\epsilon\rho} \frac{\partial(\rho H_{\phi})}{\partial\rho} = \frac{-jZ_{0}I_{m}}{4\pi} \left[\frac{e^{-jkR_{1}}}{R_{1}} + \frac{e^{-jkR_{2}}}{R_{2}} - 2\cos\frac{kl}{2}\frac{e^{-jkr}}{r}\right]$$
(c)

These fields may be readily specialized to the far zone. In (1.101) we make the substitutions

$$\rho = r \sin \theta \qquad (a)$$

$$R_1 = r - \frac{1}{2}\ell \cos \theta \qquad (b) \qquad (1.103)$$

$$R_2 = r + \frac{1}{2}\ell \cos \theta \qquad (c)$$

and find for H_{ϕ} and E_{θ} in the far zone.

$$H_{\phi} = \frac{jI_m e^{-jkr}}{2\pi r} \frac{\cos\left(\frac{1}{2}k\ell\cos\theta\right) - \cos\left(\frac{1}{2}k\ell\right)}{\sin\theta} \quad (a)$$

$$E_{\theta} = Z_0 H_{\phi} = \frac{jZ_0 I_m e^{-jkr}}{2\pi r} \frac{\cos\left(\frac{1}{2}k\ell\cos\theta\right) - \cos\left(\frac{1}{2}k\ell\right)}{\sin\theta} \quad (b)$$

The power density is readily formed from these far-zone equations for E_{θ} and H_{ϕ} . It is true for this antenna, just as for the infinitesimal current source of Section 1.7, that only the far fields contribute to the time-average Poynting vector. The power density thus formed may be integrated over a sphere of large radius to find the total power radiated and the radiation resistance found from the power. The integration is not easily done. Kraus gives the process for the half-wave antenna [1, p. 143] and Balanis gives the results for an antenna of general length [8, p. 120]. The resistance will not be given now but deferred to a later section when the input impedance will be developed.

1.10. RECIPROCITY THEOREM

In order to complete the center-fed dipole description by finding its input impedance, we need two theorems, one on reciprocity and one on equivalent sources. The reciprocity theorem will also be of use when we consider the receiving pattern of an antenna. We therefore pause to develop these theorems.

Consider two sets of sources, J^1 , M^1 and J^2 , M^2 in a linear isotropic medium. Then the Maxwell curl equations are

$$\nabla \times \mathbf{H}^{1} = \mathbf{J}^{1} + j\omega\varepsilon\mathbf{E}^{1} \qquad (a) \qquad \nabla \times \mathbf{H}^{2} = \mathbf{J}^{2} + j\omega\varepsilon\mathbf{E}^{2} \qquad (c)$$
$$\nabla \times \mathbf{E}^{1} = -\mathbf{M}^{1} - j\omega\varepsilon\mathbf{H}^{1} \qquad (b) \qquad \nabla \times \mathbf{E}^{2} = -\mathbf{M}^{2} - j\omega\varepsilon\mathbf{H}^{2} \qquad (d)$$

where \mathbf{E}^1 , \mathbf{H}^1 and \mathbf{E}^2 , \mathbf{H}^2 are the fields produced by sources 1 and 2, respectively. Multiplying (a) by \mathbf{E}^2 and (d) by \mathbf{H}^1 , adding, and using the identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$, and repeating (except with the multiplication of (b) by \mathbf{H}^2 and (c) by \mathbf{E}^1) leads to [5]

 $-\nabla \cdot (\mathbf{E}^1 \times \mathbf{H}^2 - \mathbf{E}^2 \times \mathbf{H}^1) = \mathbf{E}^1 \cdot \mathbf{J}^2 - \mathbf{E}^2 \cdot \mathbf{J}^1 + \mathbf{H}^2 \cdot \mathbf{M}^1 - \mathbf{H}^1 \cdot \mathbf{M}^2 \quad (1.106)$

Integrating over a volume and using the divergence theorem on the left side yields

$$-\oint (\mathbf{E}^{1} \times \mathbf{H}^{2} - \mathbf{E}^{2} \times \mathbf{H}^{1}) \cdot d\mathbf{A}$$
$$= \int \int \int (\mathbf{E}^{1} \cdot \mathbf{J}^{2} - \mathbf{E}^{2} \cdot \mathbf{J}^{1} + \mathbf{H}^{2} \cdot \mathbf{M}^{1} - \mathbf{H}^{1} \cdot \mathbf{M}^{2}) dv \qquad (1.107)$$

This equation represents the *Lorentz reciprocity theorem*. In a source-free region it reduces to

$$\oint (\mathbf{E}^1 \times \mathbf{H}^2 - \mathbf{E}^2 \times \mathbf{H}^1) \cdot d\mathbf{A} = 0 \qquad (1.108)$$

1.11. AN EQUIVALENCE THEOREM

The electromagnetic fields produced by a given source distribution are unique, but the reverse is not true. A given field within a region can be produced by more than one source distribution. An electric current above an infinite conducting plane produces the same field above the plane as the current and its image acting in free space, for example. Two sources that produce the same fields within a region are *equivalent* in that region [5].

Consider sources J and M within a region bounded by surface S, as shown in Fig. 1.9(a), producing fields E and H internal and external to S. The internal region may contain matter (conceivably nonlinear and nonisotropic), but external to S we assume free space without sources. Field values at the surface are E^s and H^s . Now consider a second case, shown in Fig. 1.9(b), with the same surface S the boundary between internal and external regions. We require that the fields external to S remain the same as for our first case, but



FIGURE 1.9. Fields of a source distribution and equivalent sources.

that the internal fields be zero. Further, we assume free space both internal and external to S for the second case. This field configuration will be established if surface currents

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H}^s \quad \text{(a)} \qquad \mathbf{M}_s = -\mathbf{n} \times \mathbf{E}^s \quad \text{(b)} \tag{1.109}$$

flow on surface S. The surface currents J_s and M_s are equivalent, for the fields external to S, to the original current distribution J and M. The equivalence expressed by (1.109), and its interpretation, are commonly called the *Love equivalence principle* [3]. It may be expressed more generally, with nonzero fields internal to S in the equivalent formulation [3, 5], but the form given here is the one most used.

Note that the fields \mathbf{E}^s and \mathbf{H}^s must be found from the original problem before the equivalent surface currents can be found. In some cases, such as apertures in conducting planes, good approximations can be made to the fields in the aperture. Once the fields over S are known the potential integrals may be used, since free space exists everywhere in the equivalent problem, to find the fields at all points external to S.

Two useful variations to the equivalence principle are possible. Since the fields internal to S are zero, we can place a perfectly conducting (for electric currents) surface just inside S or fill the entire internal region with a perfect electric conductor. The internal fields remain zero, and the external fields are unchanged. The conducting surface short circuits the electric current $J_s = n \times H^s$ and leaves only the magnetic surface current to radiate, as shown in Fig. 1.10(a). We may instead fill the internal region with a perfect magnetic conductor that short circuits the magnetic surface current $M_s = -n \times E^s$, leaving only the electric surface current to radiate, as in Fig. 1.10(b). In the general situation the potential integrals cannot be used to determine the fields produced by the equivalent surface current distributions of Fig. 1.10 because the currents do not radiate into a homogeneous medium. If the fields of the original problem are insignificant over the greater part of surface S, as in radiation from an aperture in a conducting plane, for example, and if the radii



FIGURE 1.10. Equivalent sources in the presence of conductors.

of curvature of S are large enough, in that part where \mathbf{E}^{s} and \mathbf{H}^{s} are significant, to use image theory, then one of the latter formulations of the equivalence principle can be useful. We will use this concept in a later section to determine the fields of an aperture antenna.

1.12. THE DIPOLE ANTENNA: INPUT IMPEDANCE

We shall develop in this section the input impedance of the linear center-fed antenna using the *induced emf method* commonly associated with the name of Carter [14], although Elliott [2] points out that it was introduced by Brillouin. The treatment here is similar to that of Elliott, with results in the form used by Balanis [8].

The dipole has a circular cross-section of radius a and length ℓ , as shown in Fig. 1.11. It is fed by an ideal source in an infinitesimally thin gap at the center. Because of skin effect, the current will flow in a thin layer at the conductor surface. We approximate this by assuming the layer to be of infinitesimal thickness with a surface current density $J_{sz}^{a}(z')$. We also assume a surface current density in the feed gap given by the same function. Now, in accordance with the development in Section 1.6, we can remove the conductors and leave the surface current distribution in free space without altering the external fields. If we surround the current distribution by a cylindrical surface S infinitesimally greater in length and radius than the original antenna, then the tangential electric field along the surface, $E_z^a(a, z')$, must be zero except at the feed gap. In the original problem, currents flow on the end caps, $z' = \pm \ell/2$, for a finite radius antenna, but if we assume the dipole to be thin, with $a \ll \lambda$ and $a \ll \ell$, we can ignore the contributions of the end caps.

Consider next a line current $I^{b}(z')$ in free space along the z axis and apply the reciprocity theorem (1.108) for a source-free region.

$$\oint (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) \cdot d\mathbf{A} = 0 \qquad (1.110)$$

where \mathbf{E}^{a} , \mathbf{H}^{a} are the fields produced by the surface current J_{sz}^{a} , and \mathbf{E}^{b} , \mathbf{H}^{b} are those produced by I^{b} . Then on surface S, if we note that E_{ϕ} and H_{z} are zero whether produced by the generator in the infinitesimal gap or by the line current on the axis, we get

$$\int_{-\ell/2}^{\ell/2} \int_{0}^{2\pi} \left[E_{z}^{a}(a, z') H_{\phi}^{b}(a, z') - E_{z}^{b}(a, z') H_{\phi}^{a}(a, z') \right] a \, d\phi \, dz' = 0 \quad (1.111)$$

Now we have noted that E_z^a is zero on S except at the infinitesimal feed gap. Further, we let

$$\int_{gap} E_z^a(a, z') \, dz' = -1 \tag{1.112}$$



FIGURE 1.11. Dipole antenna of finite radius.

so that E_z^a is a Dirac delta function (and the source voltage is 1 V). In addition we note that because of symmetry, all scalar quantities in our problem are independent of ϕ . We therefore carry out the integration in (1.111) and obtain

$$H^{b}_{\phi}(a,0) = -\int_{-\ell/2}^{\ell/2} E^{b}_{z}(a,z') H^{a}_{\phi}(a,z') dz'$$
(1.113)

Now we note that

$$H^{a}_{\phi}(a, z') = J^{a}_{sz}(z') \tag{1.114}$$

and

$$I^{a}(z') = 2\pi a J^{a}_{sz}(z') \tag{1.115}$$

With these substitutions, (1.113) becomes

$$H^{b}_{\phi}(a,0) = -\frac{1}{2\pi a} \int_{-\ell/2}^{\ell/2} E^{b}_{z}(a,z') I^{a}(z') dz' \qquad (1.116)$$

We can develop a second expression for the magnetic field $H_{\phi}^{b}(a, 0)$. We consider the thin disc lying between the arms of the dipole (Fig. 1.12) and integrate the Maxwell equation

$$\nabla \times \mathbf{H}^{b} = \mathbf{J}^{b} + j\omega\varepsilon\mathbf{E}^{b} \tag{1.117}$$

over one of its flat surfaces. This leads to

$$\oint \mathbf{H}^{b} \cdot d\ell = \iint \mathbf{J}^{b} \cdot d\mathbf{A} + j\omega\varepsilon \iint \mathbf{E}^{b} \cdot d\mathbf{A}$$
(1.118)

which becomes

$$2\pi a H^{b}_{\phi}(a,0) = I^{b}(0) + j2\pi\omega\varepsilon \int_{0}^{a} E^{b}_{z}(\rho,0)\rho \ d\rho \qquad (1.119)$$

where we have treated the top surface of the infinitesimally thin disc as lying at z = 0.

Now we have assumed $a \ll \ell$ and $a \ll \lambda$. The integral in (1.119) is therefore negligible in comparison to $I^b(0)$. This may be readily verified for specified fields, and it is left as an exercise to do so for the field of a sinusoidal current distribution on the axis. If the integral of (1.119) is neglected, comparison of the resulting equation with (1.116) leads to



FIGURE 1.12. Disc containing source distribution.

THE DIPOLE ANTENNA: INPUT IMPEDANCE

$$I^{b}(0) = -\int_{-\ell/2}^{\ell/2} E_{z}^{b}(a, z') I^{a}(z') dz'$$
(1.120)

Now $I^{b}(z')$ has not been constrained in any way (in contrast to I^{a} , which is the unknown current in a physical problem). We are therefore free to choose I^{b} arbitrarily, and we may choose it equal to I^{a} and drop the superscript. Then we obtain

$$I(0) = -\int_{-\ell/2}^{\ell/2} E_z(a, z') I(z') dz'$$
(1.121)

where we must remember that E_z is established by the axial current I^b and not by the surface current distribution of the physical problem.

We may now find the input impedance of the dipole antenna from

$$Z_{\rm in} = \frac{V}{I(0)} = \frac{VI(0)}{I^2(0)} = -\frac{1}{I^2(0)} \int_{-\ell/2}^{\ell/2} E_z(a, z')I(z') \, dz' \qquad (1.122)$$

since we took V earlier as 1 V. We must bear in mind that I(z') is not known. It may be measured or assumptions made about its form. Once I(z') is obtained, the field $E_z(a, z')$ may be determined in a straightforward manner since I(z') is a filamentary current in free space and the potential integral may be used.

We assumed in Section 1.9 a sinusoidal distribution of current on the z axis

$$I(z') = I_m \sin \left[k \left(\frac{1}{2}\ell - |z'| \right) \right] \quad -\frac{1}{2}\ell \le z' \le \frac{1}{2}\ell \quad (1.78)$$

and found E_z to be

$$E_{z} = -j \frac{Z_{0}I_{m}}{4\pi} \left[\frac{e^{-jkR_{1}}}{R_{1}} + \frac{e^{-jkR_{2}}}{R_{2}} - 2\cos\left(\frac{k\ell}{2}\right) \frac{e^{-jkr}}{r} \right]$$
(1.102c)

If we substitute these functions into (1.122), we can find the input impedance of the dipole antenna for the assumed sinusoidal current distribution. In E_z we use values for surface S,

$$R_{1} = \sqrt{\left(\frac{1}{2}\ell - z'\right)^{2} + a^{2}} \quad (a)$$

$$R_{2} = \sqrt{\left(\frac{1}{2}\ell + z'\right)^{2} + a^{2}} \quad (b) \quad (1.123)$$

$$r = \sqrt{z'^{2} + a^{2}} \quad (c)$$

In the integration for the real part of Z_{in} , the approximation a = 0 can be made, but not for the imaginary part since it causes the imaginary part to become infinite (except for special antenna lengths).

The results of the integrations are [8]:

$$R_{in} = \operatorname{Re} \left(Z_{in} \right) = \frac{Z_0}{2\pi \sin^2 (k\ell/2)} \\ \times \left\{ C + \ln (k\ell) - \operatorname{Ci} (k\ell) + \frac{1}{2} \sin (k\ell) [\operatorname{Si} (2k\ell) - 2 \operatorname{Si} (k\ell)] \right. \\ \left. + \frac{1}{2} \cos (k\ell) \left[C + \ln \frac{k\ell}{2} + \operatorname{Ci} (2k\ell) - 2 \operatorname{Ci} (k\ell) \right] \right\}$$
(1.124)
$$X_{in} = \operatorname{Im} \left(Z_{in} \right) = \frac{Z_0}{4\pi \sin^2 (k\ell/2)} \left\{ 2 \operatorname{Si} (k\ell) + \cos (k\ell) [2 \operatorname{Si} (k\ell) - \operatorname{Si} (2k\ell)] \right. \\ \left. - \sin (k\ell) \left[2 \operatorname{Ci} (k\ell) - \operatorname{Ci} (2k\ell) - \operatorname{Ci} \frac{2ka^2}{\ell} \right] \right\}$$
(1.125)

where

$$C = \text{Euler's constant} \cong 0.5772$$

Ci (x) = cosine integral =
$$-\int_{x}^{\infty} \frac{\cos u}{u} du$$
 (1.126)
Si (x) = sine integral = $\int_{0}^{x} \frac{\sin u}{u} du$

A commonly used antenna is the *half-wave dipole* for which the impedance, for a very thin dipole, is

$$Z_{\rm in} = 73.1 + j42.5 \,\Omega$$

For a sinusoidal current distribution it is common to reference the impedance (particularly the resistance) not to the input but to the position of maximum current (even if the antenna is so short that maximum current I_m is not reached at any point on the antenna). This is done by using I_m^2 rather than $I^2(0)$ in (1.122). The relationship between impedance referred to maximum current and input impedance is

$$Z_{\rm in} = \frac{I_m^2}{I^2(0)} \ Z_m = \frac{Z_m}{\sin^2\left(k\ell/2\right)}$$
(1.127)

The input resistance R_{in} for the dipole obtained by the induced emf method is the radiation resistance of the dipole, as defined in Section 1.8. We could have obtained it by forming the power density from the dipole far fields, (1.104), and integrating over a sphere to determine the total power radiated. The result would have been the same as that found by the induced emf method. We will more often use the symbol R_r , used in Section 1.8, than $R_{\rm in}$ for radiation resistance.

Equivalent Circuit for Antenna

We now recognize that *at one frequency* an antenna is seen by a generator and associated transmission line as an impedance, and the equivalent circuit of Fig. 1.13 may be used to determine power accepted by the antenna, power radiated, and so on. Even though we developed the impedance by considering a linear wire antenna in which it is reasonable to think of an ohmic loss resistance in series with a radiation resistance, we must keep in mind that losses may come from dielectrics or from induced ground currents if the antenna is near a lossy ground. In such cases a parallel admittance representation of the antenna (Norton equivalent) may have elements that vary less with frequency than does the Thevenin circuit of Fig. 1.13. In general, however, any equivalent circuit representation of an antenna is valid only over a narrow frequency range.

In considering the dipole antenna it is reasonable to consider feeding it with a two-wire transmission line with TEM mode fields. Now the concept of an antenna impedance is clearly dependent on our defining a driving point, or input port, for the antenna. Silver [15] points out that the current distribution in the line must be that characteristic of a transmission line up to the assigned driving point. At higher frequencies, interaction between the radiating system and the line may disturb the line currents back over a considerable distance, and there is no definite transition between transmission line currents and antenna currents. In such a case the concept of "antenna impedance" is ambiguous. Some antennas are fed by waveguides that do not propagate the TEM mode. If the waveguide propagates a single mode, as most waveguides are designed to do, it is equivalent to a two-wire line, and a mode impedance



FIGURE 1.13. Equivalent circuit of transmitting antenna.

may be defined. The antenna impedance can be expressed in terms of this mode impedance, but as before the validity of the impedance concept depends on our ability to define an antenna driving point with only a single waveguide mode on one side of this driving point. (A single mode on the other side is not precluded.) We will assume in our work that we have a situation in which the antenna impedance is clearly defined, and that it does not matter if the feeding transmission system is a two-wire or coaxial line carrying the TEM mode or a waveguide propagating some other single mode. We note that many antenna structures are so complex that their impedances have not yet been developed to a satisfactory extent, and we must rely on measurements to determine their impedances.

Finally, it should be noted that matching networks are commonly inserted between the antenna and the transmission line, and between the generator and transmission line. In our developments, we will not in general deal with the transmission line and matching networks since their principles are outside the scope of this book.



FIGURE 1.14. Waveguide opening into infinite plane.

1.13. WAVEGUIDE OPENING INTO INFINITE GROUND PLANE

In this section we will examine one of the many possible aperture antennas to see how the equivalence theorem of Section 1.11 may be used to determine the fields. Figure 1.14 shows a rectangular waveguide opening into an infinite ground plane. We assume that the waveguide allows only the dominant TE_{10} mode to propagate, with electric field

$$E_{y}^{i} = E_{0}^{\prime} \cos \frac{\pi x}{a} e^{-jk_{g}z}$$
(1.128)

where k_g is the propagation constant in the guide. At the guide opening a portion of the incident wave is reflected, and we take this to be the TE₁₀ mode also,

$$E'_{y} = \Gamma E'_{0} \cos \frac{\pi x}{a} e^{jk_{g}z}$$
(1.129)

At the guide opening, which we take as the z = 0 plane, the aperture field is the sum of incident and reflected waves

$$E_{y} = (1+\Gamma)E_{0}'\cos\frac{\pi x}{a} = E_{0}\cos\frac{\pi x}{a} \qquad \begin{cases} -\frac{1}{2}a \le x \le \frac{1}{2}a\\ -\frac{1}{2}b \le y \le \frac{1}{2}b \end{cases}$$
(1.130)

Along the ground plane the tangential electric field is zero.

We now apply the equivalence principle of Section 1.11, as illustrated by Fig. 1.15, to find the fields produced.

Figure 1.15(a) shows the waveguide opening into the ground plane with the tangential component of E (and H) nonzero in the aperture. In Fig. 1.15(b) the equivalence principle is used to establish a mathematical surface (an infinite plane) with equivalent surface currents J_s and M_s found from

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H}^s \quad \text{(a)} \qquad \mathbf{M}_s = -\mathbf{n} \times \mathbf{E}^s \quad \text{(b)} \qquad (1.109)$$

where $\mathbf{n} = \mathbf{u}_z$. Since we will make no use of \mathbf{J}_s , we do not obtain its value. In Fig. 1.15(c) we use a variation of the equivalence principle discussed in Section 1.11 to fill the region to the left of the surface (the "internal" region) with a perfect electric conductor. Finally, we apply image theory since we have an infinite conducting plane. The surface electric current that results from the application is zero since the image of a surface current density vector is an oppositely directed vector lying just inside the conductor, and the two add to zero. In the same way the surface magnetic current density is doubled, as shown in Fig. 1.15(d).

The magnetic surface current density for use in the potential integral is

$$2\mathbf{M}_{s} = -2\mathbf{u}_{z} \times \mathbf{u}_{y} E_{0} \cos \frac{\pi x}{a} = \mathbf{u}_{x} 2E_{0} \cos \frac{\pi x}{a} \qquad \begin{cases} |x| \leq \frac{1}{2}a\\ |y| \leq \frac{1}{2}b \end{cases}$$
(1.131)



FIGURE 1.15. Equivalent surface currents for waveguide opening into plane.

and if this is substituted into the electric vector potential specialized to the far zone,

$$\mathbf{F}(\mathbf{r}) = \frac{\varepsilon e^{-jkr}}{4\pi r} \int \int \mathbf{M}_s(\mathbf{r}') e^{jk\mathbf{u}_r \cdot \mathbf{r}'} dA'$$
(1.132)

we obtain

$$\mathbf{F}(\mathbf{r}) = \frac{\varepsilon e^{-jkr}}{4\pi r} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \mathbf{u}_x 2E_0 \cos \frac{\pi x'}{a} e^{jk\mathbf{u}_r \cdot \mathbf{r}'} dx' dy' \qquad (1.133)$$

which becomes in spherical coordinates

$$F_{\theta}(\mathbf{r}) = \frac{2\varepsilon E_0}{4\pi r} e^{-jkr} \cos\theta \cos\phi \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \cos\frac{\pi x'}{a}$$

$$\times \exp\left[jk\sin\theta \left(x'\cos\phi + y'\sin\phi\right)\right] dx' dy' \quad (a)$$

$$F_{\theta}(\mathbf{r}) = -\frac{2\varepsilon E_0}{4\pi r} e^{-jkr} \sin\phi \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \cos\frac{\pi x'}{a}$$

$$\times \exp\left[jk\sin\theta \left(x'\cos\phi + y'\sin\phi\right)\right] dx' dy' \quad (b)$$

where θ and ϕ are the polar and azimuth angles of Fig. 1.14.

THE RECEIVING ANTENNA

The integral common to both equations of (1.134) becomes [2]

$$2\pi ab \ \frac{\cos\left(\pi X\right)}{\pi^2 - 4(\pi X)^2} \ \frac{\sin\left(\pi Y\right)}{\pi Y}$$

where

$$X = \frac{a}{\lambda} \sin \theta \cos \phi \quad (a)$$

$$Y = \frac{b}{\lambda} \sin \theta \sin \phi \quad (b)$$
(1.135)

If these values are substituted into (1.58), we obtain for the far fields of the waveguide carrying the TE_{10} mode opening into an infinite ground plane

$$E_{\theta} = \frac{\omega abE_{0}}{cr} e^{-jkr} \sin \phi \frac{\cos \left[(\pi a/\lambda) \sin \theta \cos \phi \right]}{\pi^{2} - 4 \left[(\pi a/\lambda) \sin \theta \cos \phi \right]^{2}} \times \frac{\sin \left[(\pi b/\lambda) \sin \theta \sin \phi \right]}{(\pi b/\lambda) \sin \theta \sin \phi}$$
(a)

$$E_{\phi} = \frac{\omega abE_{0}}{cr} e^{-jkr} \cos \theta \cos \phi \frac{\cos \left[(\pi a/\lambda) \sin \theta \cos \phi \right]}{\pi^{2} - 4 \left[(\pi a/\lambda) \sin \theta \cos \phi \right]^{2}} \times \frac{\sin \left[(\pi b/\lambda) \sin \theta \sin \phi \right]}{(\pi b/\lambda) \sin \theta \sin \phi}$$
(b)

1.14. THE RECEIVING ANTENNA

Impedance

If we postulate that all sources and matter are of finite extent, we have, far from the sources and matter,

$$E_{\theta} = Z_0 H_{\phi}$$
 (a) $E_{\phi} = -Z_0 H_{\theta}$ (b) (1.137)

and the left side of (1.107), representing the Lorentz reciprocity theorem, becomes, if we integrate over an infinitely large sphere,

$$-Z_{0} \oint \left(H_{\theta}^{1}H_{\theta}^{2} + H_{\phi}^{1}H_{\phi}^{2} - H_{\theta}^{2}H_{\theta}^{1} - H_{\phi}^{2}H_{\phi}^{1}\right) dA = 0 \qquad (1.138)$$

Equation (1.107) then becomes

$$\int \int \int (\mathbf{E}^1 \cdot \mathbf{J}^2 - \mathbf{H}^1 \cdot \mathbf{M}^2) \, dv = \int \int \int (\mathbf{E}^2 \cdot \mathbf{J}^1 - \mathbf{H}^2 \cdot \mathbf{M}^1) \, dv \quad (1.139)$$

The integrals in (1.139) have been called *reaction* [16]. The reaction of field 1 on source 2 is

$$\langle 1,2\rangle = \int \int \int (\mathbf{E}^1 \cdot \mathbf{J}^2 - \mathbf{H}^1 \cdot \mathbf{M}^2) \, dv \qquad (1.140)$$

and in this notation (1.139) may be written as

$$\langle 1, 2 \rangle = \langle 2, 1 \rangle \tag{1.141}$$

showing that the reaction of field 1 on source set 2 is equal to the reaction of field 2 on source set 1.

Consider a current source I_2 with $M^2 = 0$. Then the reaction $\langle 1, 2 \rangle$ becomes

$$\langle 1,2\rangle = \int \int \int \mathbf{E}^1 \cdot \mathbf{J}^2 \, d\upsilon = \int \mathbf{E}^1 \cdot I_2 \, d\ell = I_2 \int \mathbf{E}^1 \cdot d\ell = -V_1 I_2 \quad (1.142)$$

where V_1 is the voltage across source 2 due to the fields produced by some source 1 (which may be a voltage or current source, or both).

A linear, two-port network with voltages and currents shown in Fig. 1.16 may be represented by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$
(1.143)

where the Z matrix is the impedance matrix. We apply current sources at ports 1 and 2. The *partial voltage* V_{12} at port 1 due to the current source at port 2 is

$$V_{12} = Z_{12}I_2 \tag{1.144}$$

But

$$V_{12} = -\frac{\langle 2, 1 \rangle}{I_1}$$
(1.145)



40

so we may find from these two equations,

$$Z_{12} = -\frac{\langle 2, 1 \rangle}{I_1 I_2} \tag{1.146}$$

In the same way, if we apply a current source to port 1 and consider the partial voltage at port 2, we find

$$Z_{21} = -\frac{\langle 1, 2 \rangle}{I_2 I_1} \tag{1.147}$$

It follows immediately from the equality of the reactions $\langle 1,2\rangle$ and $\langle 2,1\rangle$ that

$$Z_{12} = Z_{21} \tag{1.148}$$

The linear two-port network we are considering may be the two antennas in a transmit-receive configuration shown in Fig. 1.17(a) with the equivalent circuit of Fig. 1.17(b). It is important to note that (1.143) and the equivalent circuit of Fig. 1.17(b) hold no matter whether an antenna is used to transmit or receive. If the antennas are widely separated, Z_{12} will be small, and an equivalent circuit (which may be used for the two antennas) with one of them, say 1, transmitting and the other receiving is as shown in Fig. 1.18 [17]. We assume that the antennas in Fig. 1.17 are matched in polarization, so that there are no polarization losses, and defer to a later chapter a discussion of polarization matching.



FIGURE 1.17. Polarization-matched antennas in transmit-receive configuration: (a) the antennas; (b) equivalent circuit for two antennas.



FIGURE 1.18. Approximate equivalent circuit for transmitting and receiving antenna system.

We may consider Z_{11} the input impedance of the transmitting antenna, 1, neglecting the effects of the receiving antenna on the transmitting antenna, an excellent approximation for widely separated antennas. Likewise Z_{22} would be the input impedance of antenna 2 if it were transmitting. If we call Z_{11} and Z_{22} the self-impedances of the antennas, we see that the self-impedance of an antenna is the same with the antenna transmitting and receiving. An interesting aspect of this equality is that it must hold for lossless antennas and for antennas with losses. If we think of the self-impedance as consisting of a radiation resistance in series with a loss resistance, then the total must be the same with the antenna transmitting and receiving, and the radiation resistance must also be the same with the antenna transmitting and receiving. (This would be the lossless case.) It follows that the loss resistance and the antenna efficiency must remain the same with the antenna receiving and for the transmitting case. We saw, however, in Section 1.8 that the loss resistance of a wire antenna is a function of the current distribution in the wire, and our developments therefore imply that the current distribution is the same in magnitude if the antenna is transmitting or receiving. Slater [18, p. 250] states that any difference in current distribution between the transmitting and receiving case is small and the effect may be noticed only at short distances.

While the loss resistance computation of Section 1.8 was done only for a wire antenna, it is obvious that these concepts can be extended to antennas in general if we consider efficiency rather than loss resistance. We state then, in general, that the self-impedance and efficiency of an antenna are the same when the antenna is receiving a signal as they are when it is transmitting.

Receiving Pattern

It is obvious that an antenna whose radiation pattern is directional in nature will also receive a wave in a manner that is directional. In other words, an antenna has a *receiving pattern* as well as a radiation pattern. We will define the receiving pattern here as the spatial distribution of the received power when a polarization-matched plane wave is incident on the antenna. In some cases the received voltage may be measured rather than the power. Consider two different positions for antenna 2 of the transmit–receive antenna configuration we have used previously, as shown in Fig. 1.19. The movement



FIGURE 1.19. Measurement of antenna patterns.

of antenna 2 from position a to position b is such that the distance from antenna 1 to antenna 2 is the same, and the orientation of antenna 2 with respect to a line drawn between the antennas is the same. Antenna 2, in other words, is moved along the surface of a sphere centered at antenna 1. If the sphere radius is large and if the absolute phase of the signal at antenna 2 is not important, the exact location of the sphere center is unimportant. We assume, as we did earlier, that the antennas are polarization matched, but we defer a discussion of what that means. We also assume that the antennas are far enough apart that a wave transmitted by one is for all practical purposes a plane wave at the other.

We first let antenna 1 serve as the transmitting antenna and use the equivalent circuit of Fig. 1.18. We note that the mutual impedance term Z_{21} in Fig. 1.18 is a function of θ and ϕ . The ratio of powers received (power to Z_L) in positions *a* and *b* is, from Fig. 1.18,

$$\frac{W_{2b}}{W_{2a}} = \frac{|Z_{21}(\theta_b, \phi_b)|^2}{|Z_{21}(\theta_a, \phi_a)|^2}$$
(1.149)

If we let position a serve as a reference position (it may be the position for maximum received power if we choose), then (1.149) describes the relative radiation pattern of antenna 1 as θ_b , ϕ_b take on arbitrary values.

In addition to the above measurement of power to the load on antenna 2, at each position of antenna 2 we connect the generator to antenna 2 and the load to antenna 1 and measure the load power. An equivalent circuit similar to Fig. 1.18, with generator $Z_{12}I_2$, leads to a ratio of load powers in position *a* and *b*,

AN INTRODUCTION TO ANTENNAS

$$\frac{W_{1b}}{W_{1a}} = \frac{|Z_{12}(\theta_b, \phi_b)|^2}{|Z_{12}(\theta_a, \phi_a)|^2}$$
(1.150)

This equation is the relative receiving pattern of antenna 1.

Now we showed earlier that $Z_{12}(\theta_b, \phi_b) = Z_{21}(\theta_b, \phi_b)$ and $Z_{12}(\theta_a, \phi_a) = Z_{21}(\theta_a, \phi_a)$. We conclude that the relative radiation pattern and receiving pattern of an antenna are equal.

Effective Area

The *effective area* of a receiving antenna in a given direction is the ratio of the available power at the terminals of the antenna to the power density of a polarization-matched plane wave incident on the antenna from that direction. By "available power" is meant the power that would be supplied to an impedance-matched load on the antenna terminals. In Fig. 1.18, impedance matching means that

$$Z_L = Z_{22}^* \tag{1.151}$$

where Z_{22} includes radiation resistance and loss resistance of the antenna. The effective area of an antenna is normally a more useful concept than the transmitter current and mutual impedance of Fig. 1.18 because it is independent of the transmitter parameters and the distance between the antennas. In addition, for aperture antennas it appears to be a natural characteristic. For wire antennas the effective area seems somewhat artificial since it does not correspond to any physical area of the antenna; nonetheless, it is a dimensionally correct and highly useful way to describe even a wire antenna.

We saw in previous work that the relative radiation and receiving patterns of an antenna are the same. It follows that for an antenna the gain and effective area are related by a constant, that is,

$$G(\theta, \phi) = CA_{e}(\theta, \phi) \tag{1.152}$$

We will in fact find that C is a universal constant for antennas.

Consider two antennas in a transmit-receive configuration, with local coordinate systems, as shown in Fig. 1.20. The antennas may be of arbitrary type and may be oriented arbitrarily with respect to their coordinate systems and to each other. We assume first that antenna 1 is transmitting and 2 is receiving with a matched load. They are separated by a sufficient distance R to cause the wave from 1 to be effectively a plane wave at 2 and to make the equivalent circuit of Fig. 1.18 valid. The receiving antenna and load are impedance matched. We note specifically that the antennas need not be lossless.

If antenna 1 accepts power W_{a_1} from the generator (and radiates a portion of it) and has gain G_1 , the power density at 2 is



FIGURE 1.20. Transmission between two antennas.

$$S = \frac{W_{a_1}G_1(\theta_1, \phi_1)}{4\pi R^2}$$
(1.153)

and the power to the impedance-matched load is

$$W_{L_2} = SA_{e_2}(\theta_2, \phi_2) \tag{1.154}$$

where A_{e_2} is the effective area of antenna 2. We find immediately from these equations that

$$G_1(\theta_1, \phi_1) A_{e_2}(\theta_2, \phi_2) = \frac{W_{L_2}}{W_{a_1}} (4\pi R^2)$$
(1.155)

If we reverse the transmitting and receiving roles of the antennas by connecting a generator to antenna 2 and causing it to accept power W_{a_2} , part of which is radiated, the power to a load that is impedance matched to antenna 1 is

$$W_{L_1} = \frac{W_{a_2}G_2(\theta_2, \phi_2)A_{e_1}(\theta_1, \phi_1)}{4\pi R^2}$$
(1.156)

which gives us

$$G_2(\theta_2, \phi_2) A_{e_1}(\theta_1, \phi_1) = \frac{W_{L_1}}{W_{a_2}} (4\pi R^2)$$
(1.157)

Let us make use once more of the valuable equivalent circuit of Fig. 1.18. With antenna 1 transmitting and power W_{a_1} supplied to the impedance Z_{11} , we find the ratio of load power (to load $Z_L = Z_{22}^*$) to the power W_{a_1} accepted by Z_{11} to be

$$\frac{W_{L_2}}{W_{a_1}} = \frac{|Z_{21}|^2}{4\operatorname{Re}\left(Z_{11}\right)\operatorname{Re}\left(Z_{22}\right)}$$
(1.158)

If we reverse the roles of transmitter and receiver, we find from the equivalent circuit

$$\frac{W_{L_1}}{W_{a_2}} = \frac{|Z_{12}|^2}{4 \operatorname{Re}(Z_{11}) \operatorname{Re}(Z_{22})}$$
(1.159)

and from the equality of Z_{12} and Z_{21} it follows that

$$\frac{W_{L_2}}{W_{a_1}} = \frac{W_{L_1}}{W_{a_2}} \tag{1.160}$$

We therefore obtain from a comparison of (1.155) and (1.157)

$$G_1(\theta_1, \phi_1) A_{e_2}(\theta_2, \phi_2) = G_2(\theta_2, \phi_2) A_{e_1}(\theta_1, \phi_1)$$
(1.161)

or

$$\frac{A_{e_1}(\theta_1, \phi_1)}{G_1(\theta_1, \phi_1)} = \frac{A_{e_2}(\theta_2, \phi_2)}{G_2(\theta_2, \phi_2)}$$
(1.162)

In (1.162) the angles are arbitrary and have been carried to show for one antenna that the effective area in a particular direction is being compared to the gain in the same direction. We note also that the equation holds for lossy antennas, and will hold also for lossless antennas if $G(\theta, \phi)$ is replaced by directivity $D(\theta, \phi)$ and $A_e(\theta, \phi)$ is determined for the lossless case. Antenna types were not specified, and it follows that if we can find the ratio A_e/G for one antenna, lossless or lossy, we know it for all.

We found in Section 1.8 that the directivity of the infinitesimal z-directed current source of Fig. 1.21 is

$$D = \frac{3}{2}\sin^2\theta \tag{1.76}$$

and the radiation resistance is

$$R_r = 80 \,\pi^2 \left(\frac{\ell}{\lambda}\right)^2 \tag{1.77}$$



FIGURE 1.21. Infinitesimal current source.

(Note: In Fig. 1.2, also representing the infinitesimal current element, the center feed point was not shown, but the difference is obviously irrelevant since we consider the current to be constant throughout the element for both Figs. 1.2 and 1.21.)

Consider now a wave incident on the antenna of Fig. 1.21, considering it as a receiving antenna. The open-circuit voltage induced at the antenna terminals is

$$V_{\rm oc} = E\ell \sin\theta \tag{1.163}$$

where V_{oc} and E are taken as peak values. Then the power to a matched load is

$$W = \frac{|V_{\rm oc}|^2}{8R_r} = \frac{|E|^2 \ell^2 \sin^2 \theta}{8R_r}$$
(1.164)

The power density at the antenna is

$$S = \frac{1}{2Z_0} |E|^2 \tag{1.165}$$

and therefore

$$W = \frac{Z_0 S \ell^2 \sin^2 \theta}{4R_r} \tag{1.166}$$

which gives an effective area for the lossless infinitesimal antenna of

47

$$A_e = \frac{W}{S} = \frac{3\lambda^2 \sin^2 \theta}{8\pi}$$
(1.167)

and a ratio of effective area to directivity of

$$\frac{\lambda^2}{4\pi}$$

This ratio was obtained for a lossless example, but as we saw earlier it also holds for the lossy case. We state therefore as a general rule that the effective area and gain of an antenna are related by

$$\frac{A_e(\theta, \phi)}{G(\theta, \phi)} = \frac{\lambda^2}{4\pi}$$
(1.168)

1.15. TRANSMISSION BETWEEN ANTENNAS

We have at this point defined the necessary terms and developed the equations that allow us to determine the power in a receiver load if the power accepted by the transmitting antenna is known. Let the power accepted by antenna 1 in Fig. 1.20 be W_{at} . If antenna 1 radiated isotropically, the power density at 2 would be

$$\frac{W_{ai}}{4\pi R^2}$$

Since antenna 1 does not radiate isotropically, but has gain G_i , the actual power density at 2 is

$$\frac{W_{at}G_t(\theta_t,\,\phi_t)}{4\pi R^2}$$

where we use t to indicate a transmitting antenna. The load power W_r in the load on the receiving antenna then is

$$W_r = \frac{W_{at}G_t(\theta_t, \phi_t)A_{cr}(\theta_r, \phi_r)}{4\pi R^2}$$
(1.169)

where subscript r indicates the receiving antenna.

Many variations on (1.169) are possible, but we will not write all of the possible equations. Instead we list here some of the more common alterations:

1. G_i may be replaced by $e_i D_i$, where e_i is the efficiency of the transmitting antenna and D_i its directivity.

48

2. W_{at} may be replaced by $(1 - |\Gamma_t|^2)W_{it}$ where W_{it} is the power incident on the (mismatched) transmitting antenna and Γ_t is the reflection coefficient obtained by treating the transmitting antenna as a load on the feeding transmission line.

3. A_{er} may be replaced by the gain G_r of the receiving antenna, according to $A_{er} = G_r \lambda^2 / 4\pi$.

4. If the receiving antenna is not terminated in a matched load, (1.169) must be multiplied by an impedance match factor, ranging between 0 and 1, to account for the mismatch loss. If the receiving antenna is represented by the series combination of R_a , including both radiation and loss resistances, and X_a , the antenna reactance, and the load impedance is $R_L + jX_L$, it is easy to show that this impedance match factor is

$$M_{z} = \frac{4R_{L}^{2}}{\left(R_{a} + R_{L}\right)^{2} + \left(X_{a} + X_{L}\right)^{2}}$$
(1.170)

5. If the antennas are not polarization matched, (1.169) must be multiplied by a polarization match factor. A discussion of this will be deferred to a later chapter.

1.16. THE RADAR EQUATION

Figure 1.22 shows a bistatic radar and target. The power density at the target is given by

$$S_{i} = W_{t} \frac{G_{t}(\theta_{t}, \phi_{t})}{4\pi R_{1}^{2}}$$
(1.171)

where W_i is the power accepted by the transmitting antenna. The transmitted signal may be pulsed or continuous wave, but its characteristics do not affect the development here.



FIGURE 1.22. Bistatic radar and target.

The wave striking the target is reradiated in a directional manner, and a portion of the reradiated, or scattered, power is intercepted by the receiving antenna. The power received depends on the transmitted power, the antenna gains, and the *scattering cross section* of the target. For transmitting and receiving antennas located apart, the radar is called *bistatic*, and we are concerned with the *bistatic cross section*.

The bistatic or scattering cross section of a target is the (fictional) area that, when multiplied by the power density of the incident wave, yields a power that would produce by isotropic radiation the same radiation intensity as that measured by the receiving antenna. If the transmitting and receiving antennas are located close to each other, as compared to the antenna-target distance, the cross section is referred to as monostatic. An alternate form of the definition is that the bistatic cross section is 4π times the radiation intensity of the scattered wave in a specified direction divided by the power density of the incident plane wave. The utility of this definition and the development of it may not be obvious. The following reasoning makes it clearer: An observer at the receiver can determine the power density of the scattered wave at the receiver and (multiplying by R_2^2) the radiation intensity of the scattered wave in the direction of the receiver. The observer does not know how the target scatters the incident wave (without much more information than can be obtained by one measurement), and yet to describe the target a commonly agreed on assumption is necessary. This assumption is that of isotropic scattering. With this assumption the observer determines that the total scattered power is 4π times the radiation intensity in his direction. It is then reasonable to say that this total power is scattered as a result of the target with a cross-section area σ intercepting an incident plane wave with power density established at the target by the radar transmitter. This should be clearer as we develop the radar equation.

The power density at the target is given by (1.171). Then for a target with cross-section σ , the intercepted power is

$$\sigma S_i = \frac{W_i G_i(\theta_i, \phi_i)\sigma}{4\pi R_1^2}$$
(1.172)

If this power is scattered isotropically as we assumed in defining σ (and as we must continue to assume), the power density at the receiving antenna is

$$S_{r} = \frac{\sigma S_{i}}{4\pi R_{2}^{2}} = \frac{W_{i}G_{i}(\theta_{i}, \phi_{i})\sigma}{(4\pi R_{1}R_{2})^{2}}$$
(1.173)

The power available to a matched load at the receiver is then

$$W_{r} = \frac{W_{t}G_{t}(\theta_{t}, \phi_{t})A_{er}(\theta_{r}, \phi_{r})\sigma}{(4\pi R_{1}R_{2})^{2}}$$
(1.174)

REFERENCES

Some characteristics of the cross-section σ are clear. The first is that the cross section depends on the direction of the transmitter and receiver from the target, not merely on the difference in their directions. For a target as complex as an aircraft a change of transmitter or receiver direction of as little as a degree can change the cross section by many decibels. A second characteristic is that the cross section is independent of the distances R_1 and R_2 if they are sufficiently large to cause a wave from either antenna to be plane at the target.

A third characteristic is that the received power is dependent on the polarizations of the two antennas, the geometry of the situation, and on the target itself. The *radar cross section* is that portion of the scattering cross section corresponding to a specified polarization component of the scattered wave [10]. We will defer discussion of polarization in radar to a later chapter.

If transmitter and receiver for a radar are at the same site, (1.174) simplifies to

$$W_r = \frac{W_t G_t(\theta, \phi) A_{er}(\theta, \phi) \sigma}{(4\pi R^2)^2}$$
(1.175)

where σ is the monostatic cross section of the target. In many cases the same antenna is used for transmitting and receiving, and the received power is

$$W_{r} = \frac{W_{l}G(\theta, \phi)A_{e}(\theta, \phi)\sigma}{(4\pi R^{2})^{2}} = \frac{W_{l}G^{2}(\theta, \phi)\lambda^{2}\sigma}{(4\pi)^{3}R^{4}}$$
(1.176)

REFERENCES

- 1. J. D. Kraus, Antennas, McGraw-Hill, New York, 1950.
- 2. R. S. Elliott, Antenna Theory and Design, Prentice-Hall, Englewood Cliffs, NJ, 1981.
- 3. R. E. Collin and F. J. Zucker, Antenna Theory, McGraw-Hill, New York, 1969.
- 4. W. H. Hayt, Engineering Electromagnetics, 4th ed., McGraw-Hill, New York, 1981.
- 5. R. F. Harrington, Time-Harmonic Electromagnetic Fields, McGraw-Hill, New York, 1961.
- 6. A. Sommerfeld, Electrodynamics, Academic Press, New York, 1952.
- 7. W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed., Addison-Wesley, Reading, MA, 1962.
- 8. C. A. Balanis, Antenna Theory, Harper & Row, New York, 1982.
- H. Mott and D. N. McQuiddy, "On the Use of the Potential Integral for Determining the Fields of Physical Radiating Structures," *IEEE Transactions on Education*, Vol. E-10, No. 4, pp. 237–239, December 1967.
- 10. IEEE Transactions on Antennas and Propagation, Vol. AP-31, No. 6, November 1983.
- 11. IEEE Transactions on Antennas and Propagation, Vol. AP-22, No. 1, January 1974.
- 12. IEEE Transactions on Antennas and Propagation, Vol. AP-17, No. 3, May 1969.
- 13. E. C. Jordan, *Electromagnetic Waves and Radiating Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1950.

- P. S. Carter, "Circuit Relations in Radiating Systems and Applications in Antenna Problems," Proc. IRE, Vol. 20, No. 6, pp. 1004–1041, June 1932.
- S. Silver, *Microwave Antenna Theory and Design*, Boston Technical Lithographers, Lexington, MA, 1963. (MIT Radiation Lab. Series, Vol. 12, McGraw-Hill, New York, 1949.)
- V. H. Rumsey, "The Reaction Concept in Electromagnetic Theory," Phys. Rev, Ser. 2, Vol. 94, No. 6, pp. 1483–1491, June 15, 1954.
- 17. S. Ramo, J. R. Whinnery, and T. Van Duzer, Fields and Waves in Communication Electronics, Wiley, New York, 1984.
- 18. J. C. Slater, Microwave Transmission, McGraw-Hill, New York, 1942.

PROBLEMS

- 1.1. Show that the integral of (1.28) is a solution to Poisson's equation (1.27).
- **1.2.** Prove that (1.51) is correct.
- 1.3. Develop (1.54) by substituting (1.50) and (1.51) into (1.47), keeping only terms that vary as 1/r.
- 1.4. Find the general E and H fields of the infinitesimal electric current element from the magnetic vector potential of (1.31). Show that the time-average Poynting vector formed from these fields is the same as that given by (1.67).
- 1.5. A practical antenna is the very short center-fed dipole shown in Fig. P1.5. If length $\ell \ll \lambda$, the current distribution on the antenna is, to a good approximation,

$$I = I_0 \left(1 - \frac{2}{\ell} z' \right) \qquad 0 \le z' \le \frac{\ell}{2}$$
$$I = I_0 \left(1 + \frac{2}{\ell} z' \right) \qquad -\frac{\ell}{2} \le z' \le 0$$



FIGURE P1.5. Short dipole and associated current distribution.

PROBLEMS

Assume that, in (1.33), $|\mathbf{r} - \mathbf{r}'|$ may be approximated by r in both the phase and amplitude of the integrand. Find the vector potential and far fields of the dipole.

- 1.6. Find the directivity of the short dipole of Problem 1.5.
- 1.7. Find the radiation resistance of the short dipole of Problem 1.5.
- **1.8.** Find the loss resistance of the short dipole of Problem 1.5. It is made of copper with length 10 cm and diameter 1 mm. The frequency is 100 MHz.
- 1.9. Determine the radiation efficiency of the short dipole of Problem 1.5.

REPRESENTATION OF WAVE POLARIZATION

2.1. INTRODUCTION

The electric vector of a harmonic plane wave traces an ellipse in the transverse plane with time, as is well known. In this chapter we develop the equation of the ellipse for a general, nonplane wave and consider the ellipse and the behavior of the field vectors in detail for a plane wave. The parameters commonly used to describe wave polarization, namely, the linear and circular polarization ratios, the ellipse axial ratio, tilt angle, rotation sense, and the Stokes parameters, are introduced and related to each other. A polarization chart based on the familiar Smith chart of transmission line theory, first discussed by Rumsey, is used, and contours for some common polarization parameters are shown on the chart. The Poincaré sphere is utilized, and mapping from the sphere onto several complex planes is described. This process, which results in standard and nonstandard polarization charts, is illustrated.

2.2. THE GENERAL HARMONIC WAVE

In this section we will show that a general (nonplanar) harmonic wave is elliptically polarized and find the equation of the polarization ellipse [1]. A nonplanar single-frequency wave with components

$$\mathscr{E}_{i}(\mathbf{r}, t) = a_{i}(\mathbf{r}) \cos\left[\omega t - g_{i}(\mathbf{r})\right] \qquad i = 1, 2, 3 \tag{2.1}$$

where a_i and g_i are real, may be written as

$$\mathscr{E}(\mathbf{r}, t) = \sum_{1}^{3} \mathbf{u}_{i} a_{i}(\mathbf{r}) \cos \left[\omega t - g_{i}(\mathbf{r})\right]$$
$$= \sum_{1}^{3} \mathbf{u}_{i} a_{i}(\mathbf{r}) \operatorname{Re}\left[e^{i\left[\omega t - g_{i}(\mathbf{r})\right]}\right]$$
(2.2)

and if we let

$$E_i(\mathbf{r}) = a_i(\mathbf{r})e^{-jg_i(\mathbf{r})}$$
(2.3)

be the complex, time-invariant term associated with each real, time-varying electric field component, then

$$\mathscr{E}(\mathbf{r}, t) = \operatorname{Re}\left[\sum_{1}^{3} \mathbf{u}_{i} E_{i}(\mathbf{r}) e^{j\omega t}\right]$$
(2.4)

where the \mathbf{u}_i are real orthogonal unit vectors. For a plane wave the phase term is given by

$$g_i(\mathbf{r}) = \mathbf{k} \cdot \mathbf{r} - \delta_i \tag{2.5}$$

but at this point we will not restrict ourselves to plane waves.

If we define the complex vector

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'(\mathbf{r}) + j\mathbf{E}''(\mathbf{r}) = \sum_{1}^{3} \mathbf{u}_{i}E_{i}(\mathbf{r})$$
(2.6)

then the harmonic vector field may be written as

$$\mathscr{E}(\mathbf{r}, t) = \operatorname{Re}\left[\mathbf{E}(\mathbf{r})e^{j\omega t}\right]$$
(2.7)

Let us assume that E may be transformed to a new set of axes defined by the orthogonal real vectors m and n, using the relation

$$\mathbf{E} = \mathbf{E}' + j\mathbf{E}'' = (\mathbf{m} + j\mathbf{n})e^{j\theta}$$
(2.8)

Equating real and imaginary parts of this equation yields

$$\mathbf{E}' = \mathbf{m} \cos \theta - \mathbf{n} \sin \theta$$
$$\mathbf{E}'' = \mathbf{m} \sin \theta + \mathbf{n} \cos \theta$$
(2.9)

and solving for **m** and **n** leads to

$$\mathbf{m} = \mathbf{E}' \cos \theta + \mathbf{E}'' \sin \theta$$
(2.10)
$$\mathbf{n} = -\mathbf{E}' \sin \theta + \mathbf{E}'' \cos \theta$$
If we assume, without loss of generality, that $|\mathbf{m}| \ge |\mathbf{n}|$, and require the orthogonality condition $\mathbf{m} \cdot \mathbf{n} = 0$, we find from (2.10) that

$$\tan 2\theta = \frac{2\mathbf{E}' \cdot \mathbf{E}''}{|\mathbf{E}'|^2 - |\mathbf{E}''|^2}$$
(2.11)

Since $\tan 2\theta$ as given by (2.11) is real, our assumed transformation may be carried out.

Next we substitute (2.8) into (2.7) to find the real field components. We obtain

$$\mathscr{E} = \operatorname{Re}\left[\operatorname{E} e^{j\omega t}\right] = \operatorname{Re}\left[(\mathbf{m} + j\mathbf{n})e^{j\theta}e^{j\omega t}\right]$$
(2.12)

and since m and n are real,

$$\mathscr{E} = \mathbf{m}\cos\left(\omega t + \theta\right) - \mathbf{n}\sin\left(\omega t + \theta\right) \tag{2.13}$$

If at each field point we now set up a local coordinate system with two of the axes directed along m and n, the field components are

$$\mathscr{C}_m = m \cos(\omega t + \theta)$$
 (a)
 $\mathscr{C}_n = -n \sin(\omega t + \theta)$ (b) (2.14)
 $\mathscr{C}_3 = 0$ (c)

where $m = |\mathbf{m}|$, $n = |\mathbf{n}|$. In (2.14) subscript 3 refers to the third of the three coordinates.

From (2.14) we see that

$$\frac{\mathscr{C}_m^2}{m^2} + \frac{\mathscr{C}_n^2}{n^2} = \cos^2\left(\omega t + \theta\right) + \sin^2\left(\omega t + \theta\right) = 1$$
(2.15)

This is the equation of an ellipse, in the plane defined by **m** and **n**, with semimajor and semiminor axes m and n. The field intensity ellipse is shown in Fig. 2.1. The field vector \mathscr{E} terminates on the ellipse, since its components \mathscr{E}_m and \mathscr{E}_n are not independent but obey the ellipse equation, (2.15). The direction of \mathscr{E} changes with time as its tip moves around the ellipse with a direction and velocity we will determine in a later section.

We see then that any harmonic wave, planar or nonplanar, is elliptically polarized. The plane of the ellipse and its shape and orientation in that plane are functions of the coordinates of the field point, but not of time.

This development has been concerned with the time-varying electric field, but it is clear that the magnetic field is also elliptically polarized. See problem 2.7 at the end of this chapter.



FIGURE 2.1. The polarization ellipse: (a) field intensity coordinates; (b) space coordinates.

2.3. POLARIZATION ELLIPSE FOR PLANE WAVES

A plane wave traveling in the z direction

$$\mathbf{E} = (E_x \mathbf{u}_x + E_y \mathbf{u}_y) e^{-jkz}$$
(2.16)

results if we use the phase term (2.5) and let the propagation constant be

$$\mathbf{k} = \mathbf{u}_z k \tag{2.17}$$

In (2.16) both E_x and E_y are complex and may be written as

$$E_x = |E_x|e^{j\phi_x}$$
 $E_y = |E_y|e^{j\phi_y}$ (2.18)

so that

$$\mathbf{E} = (\mathbf{u}_x | E_x | e^{j\phi_x} + \mathbf{u}_y | E_y | e^{j\phi_y}) e^{-jkz}$$
(2.19)

and the time-varying field is

$$\mathscr{E} = \operatorname{Re}\left(\operatorname{E}e^{j\omega t}\right) = \mathbf{u}_{x}|E_{x}|\cos\left(\omega t - kz + \phi_{x}\right) + u_{y}|E_{y}|\cos\left(\omega t - kz + \phi_{y}\right)$$
(2.20)

and if we set

$$\beta = \omega t - kz \tag{2.21}$$

the components of & become

$$\frac{\mathscr{E}_x}{|E_x|} = \cos\beta\cos\phi_x - \sin\beta\sin\phi_x \quad (a)$$

$$\frac{\mathscr{E}_y}{|E_y|} = \cos\beta\cos\phi_y - \sin\beta\sin\phi_y \quad (b)$$

Multiplying and subtracting as indicated leads to

$$\frac{\mathscr{E}_{x}}{|E_{x}|}\sin\phi_{y} - \frac{\mathscr{E}_{y}}{|E_{y}|}\sin\phi_{x} = \cos\beta\sin(\phi_{y} - \phi_{x}) \quad (a)$$

$$\frac{\mathscr{E}_{x}}{|E_{x}|}\cos\phi_{y} - \frac{\mathscr{E}_{y}}{|E_{y}|}\cos\phi_{x} = \sin\beta\sin(\phi_{y} - \phi_{x}) \quad (b)$$
(2.23)

Squaring and adding (2.23a) and (2.23b) gives

$$\frac{\mathscr{C}_{x}^{2}}{|E_{x}|^{2}} - 2 \frac{\mathscr{C}_{x}}{|E_{x}|} \frac{\mathscr{C}_{y}}{|E_{y}|} \cos(\phi_{y} - \phi_{x}) + \frac{\mathscr{C}_{y}^{2}}{|E_{y}|^{2}} = \sin^{2}(\phi_{y} - \phi_{x}) \quad (2.24)$$

This is the equation of a conic, and we have already seen in a more general case that it represents an ellipse. In (2.24) we set

$$\phi = \phi_y - \phi_x \tag{2.25}$$

and the equation becomes

$$\frac{\mathscr{C}_x^2}{|E_x|^2} - 2 \frac{\mathscr{C}_x}{|E_x|} \frac{\mathscr{C}_y}{|E_y|} \cos \phi + \frac{\mathscr{C}_y^2}{|E_y|^2} = \sin^2 \phi \qquad (2.26)$$

We may see from (2.22), which can be rewritten as

$$\frac{\mathscr{C}_x}{|E_x|} = \cos\left(\beta + \phi_x\right) \quad \text{(a)} \qquad \frac{\mathscr{C}_y}{|E_y|} = \cos\left(\beta + \phi_y\right) \quad \text{(b)} \qquad (2.27)$$

that the greatest values of \mathscr{C}_x and \mathscr{C}_y are, respectively, $|E_x|$ and $|E_y|$. Then the ellipse of (2.26) can be inscribed in a rectangle with sides parallel to the x and y axes and dimensions $2|E_x|$ and $2|E_y|$ as shown in Fig. 2.2.

From (2.27) we see that for \mathscr{E}_x to be maximum

$$\beta + \phi_x = 0$$
$$\beta + \phi_y = \beta + \phi_x + (\phi_y - \phi_x) = \phi$$

and for E, maximum

$$\beta + \phi_y = 0$$
 $\beta + \phi_x = -\phi$



FIGURE 2.2. Tilted polarization ellipse.

and we find that the ellipse of Fig. 2.2 intersects the sides of the rectangle at $\pm |E_x|$, $\pm |E_y| \cos \phi$ and $\pm |E_x| \cos \phi$, $\pm |E_y|$.

The angle τ of Fig. 2.2, measured from the x axis, is called the *tilt angle* of the polarization ellipse. We define it between the limits

$$0 \le \tau \le \pi \tag{2.28}$$

Let us find τ . From Fig. 2.3 we easily see that



FIGURE 2.3. Coordinate transformations.

$$\xi = x \cos \tau + y \sin \tau \quad (a)$$

$$\eta = y \cos \tau - x \sin \tau \quad (b)$$
(2.29)

and the field components transform as

$$\begin{aligned} \mathscr{E}_{\xi} &= \mathscr{E}_{x} \cos \tau + \mathscr{E}_{y} \sin \tau \quad \text{(a)} \\ \mathscr{E}_{\eta} &= -\mathscr{E}_{x} \sin \tau + \mathscr{E}_{y} \cos \tau \quad \text{(b)} \end{aligned}$$
(2.30)

Now the components $\mathscr{C}_{\varepsilon}$ and \mathscr{C}_{η} are also given by

$$\begin{aligned} \mathscr{E}_{\xi} &= m \cos \left(\beta + \phi_0\right) \quad \text{(a)} \\ \mathscr{E}_{\eta} &= \pm n \sin \left(\beta + \phi_0\right) \quad \text{(b)} \end{aligned} \tag{2.31}$$

where *m* and *n* are the positive semiaxes of Fig. 2.2, and ϕ_0 is some phase angle. That (2.31) is correct is easily seen by noting that it satisfies

$$\frac{\mathscr{C}_{\xi}^2}{m^2} + \frac{\mathscr{C}_{\eta}^2}{n^2} = 1$$

In (2.31) \mathscr{E}_{η} carries the \pm sign since we have not yet determined the rotation sense of \mathscr{E} . If we consider $\beta + \phi_0 = 0$, $\mathscr{E}_{\xi} = m$, and $\mathscr{E}_{\eta} = 0$ in (2.31), and then allow $\beta(=\omega t - kz)$ to increase infinitesimally, we see that the + sign corresponds to counterclockwise rotation of \mathscr{E} (as we look at Fig. 2.2) as β (or time) increases, and the - sign to clockwise rotation.

We equate (2.31) to (2.30).

$$\begin{aligned} & \mathscr{E}_{\xi} = m\cos\left(\beta + \phi_{0}\right) = \mathscr{E}_{x}\cos\tau + \mathscr{E}_{y}\sin\tau \quad \text{(a)} \\ & \mathscr{E}_{\eta} = \pm n\sin\left(\beta + \phi_{0}\right) = -\mathscr{E}_{x}\sin\tau + \mathscr{E}_{y}\cos\tau \quad \text{(b)} \end{aligned}$$

Expanding the left side and using on the right the wave components as given by (2.22), we get

$$m(\cos \beta \cos \phi_0 - \sin \beta \sin \phi_0)$$

= $|E_x|(\cos \beta \cos \phi_x - \sin \beta \sin \phi_x) \cos \tau$
+ $|E_y|(\cos \beta \cos \phi_y - \sin \beta \sin \phi_y) \sin \tau$ (a)
 $\pm n(\sin \beta \cos \phi_0 + \cos \beta \sin \phi_0)$
= $-|E_x|(\cos \beta \cos \phi_x - \sin \beta \sin \phi_x) \sin \tau$
+ $|E_y|(\cos \beta \cos \phi_y - \sin \beta \sin \phi_y) \cos \tau$ (b)
(2.33)

POLARIZATION ELLIPSE FOR PLANE WAVES

Equating the coefficients of $\cos \beta$ and $\sin \beta$ in (2.33) leads to

$$m\cos\phi_0 = |E_x|\cos\phi_x\cos\tau + |E_y|\cos\phi_y\sin\tau \qquad (a)$$

$$m\sin\phi_0 = |E_x|\sin\phi_x\cos\tau + |E_y|\sin\phi_y\sin\tau \qquad (b)$$
(2.34)

$$\pm n \cos \phi_0 = |E_x| \sin \phi_x \sin \tau - |E_y| \sin \phi_y \cos \tau \qquad (c)$$

$$\pm n \sin \phi_0 = -|E_x| \cos \phi_x \sin \tau + |E_y| \cos \phi_y \cos \tau \quad (d)$$

Squaring and adding the four equations of (2.34) results in

$$m^{2} + n^{2} = |E_{x}|^{2} + |E_{y}|^{2}$$
(2.35)

Next we multiply the first and third equations of (2.34) and also the second and fourth, and add the products, obtaining

$$\pm mn = -|E_x||E_y|\sin\phi \qquad (2.36)$$

Dividing the third equation of (2.34) by the first, and the fourth by the second gives

$$\pm \frac{n}{m} = \frac{|E_x|\sin\phi_x\sin\tau - |E_y|\sin\phi_y\cos\tau}{|E_x|\cos\phi_x\cos\tau + |E_y|\cos\phi_y\sin\tau}$$
$$= \frac{-|E_x|\cos\phi_x\sin\tau + |E_y|\cos\phi_y\cos\tau}{|E_x|\sin\phi_x\cos\tau + |E_y|\sin\phi_y\sin\tau}$$
(2.37)

Cross multiplying and collecting terms in (2.37) gives

$$(|E_x|^2 - |E_y|^2)\sin 2\tau = 2|E_x||E_y|\cos 2\tau\cos\phi$$
(2.38)

If we define the auxiliary angle α by

$$\tan \alpha = \frac{|E_y|}{|E_x|} \qquad 0 \le \alpha \le \frac{\pi}{2} \tag{2.39}$$

then (2.38) becomes

$$\tan 2\tau = \tan 2\alpha \cos \phi \tag{2.40}$$

We have thus obtained the ellipse tilt angle in terms of the field component magnitudes and phase difference.

In order to find the axial ratio of the ellipse and the rotation sense of the \mathscr{C} vector let us define another auxiliary angle δ by

$$\tan \delta = \mp \frac{n}{m} \qquad -\frac{\pi}{4} \le \delta \le \frac{\pi}{4} \tag{2.41}$$

From (2.41) we may obtain

$$\sin 2\delta = \mp \frac{2mn}{m^2 + n^2} \tag{2.42}$$

and the use of (2.35) and (2.36) leads to

$$\sin 2\delta = \frac{2|E_x||E_y|}{|E_x|^2 + |E_y|^2} \sin \phi$$
(2.43)

which will give us the axial ratio, n/m, from the field component magnitudes and phase difference.

Let us next determine the rotation sense of \mathcal{C} . The time-varying angle of \mathcal{C} , measured from the x axis, is

$$\psi = \tan^{-1} \frac{\mathscr{E}_{y}}{\mathscr{E}_{x}} = \tan^{-1} \frac{|E_{y}| \cos\left(\beta + \phi_{y}\right)}{|E_{x}| \cos\left(\beta + \phi_{x}\right)}$$
(2.44)

where $\beta = \omega t - kz$. Then

$$\frac{\partial \psi}{\partial \beta} = \frac{(|E_y|/|E_x|)[-\cos(\beta + \phi_x)\sin(\beta + \phi_y) + \sin(\beta + \phi_x)\cos(\beta + \phi_y)]}{[1 + |E_y|^2\cos^2(\beta + \phi_y)/|E_x|^2\cos^2(\beta + \phi_x)]\cos^2(\beta + \phi_x)}$$
(2.45)

and at some particular β , say $\beta = 0$,

$$\frac{\partial \psi}{\partial \beta} = -\frac{|E_x| |E_y| \sin \phi}{|E_x|^2 \cos^2 \phi_x + |E_y|^2 \cos^2 \phi_y}$$
(2.46)

Thus we see that

$$\frac{\partial \psi}{\partial \beta} < 0, \quad 0 < \phi < \pi$$

$$> 0, \quad \pi < \phi < 2\pi$$
(2.47)

If we look in the direction of wave propagation, in this case the +z direction, $\partial \psi / \partial \beta > 0$ corresponds to clockwise rotation of the \mathscr{E} vector as β (or time) increases. By definition we call this right-handed rotation of the vector. Conversely, $\partial \psi / \partial \beta < 0$ corresponds to counterclockwise or left-handed rotation. We may see from (2.43) and (2.47) that

63

(2.48)

 $\sin 2\delta < 0$, right-handed rotation

$$>0$$
, left-handed rotation

From (2.39) we can get

$$\sin 2\alpha = \frac{2|E_x||E_y|}{|E_x|^2 + |E_y|^2}$$
(2.49)

and, if this is used in (2.43), we get a simpler equation

$$\sin 2\delta = \sin 2\alpha \sin \phi \tag{2.50}$$

To summarize, from a knowledge of the field component amplitudes $|E_x|$ and $|E_y|$ and their phase difference $\phi = \phi_y - \phi_x$, we first find the auxiliary angle α from (2.39). Angle δ is next found from (2.50). The tilt angle of the polarization ellipse is then determined from (2.40) and the axial ratio and rotation sense from (2.41), where positive δ corresponds to right-handed rotation.

2.4. LINEAR AND CIRCULAR POLARIZATION

In the special cases of $|E_x| = 0$, or $|E_y| = 0$, or $\phi = 0$, the polarization ellipse degenerates to a straight line, and the wave is said to be linearly polarized. The axial ratio will of course be zero, and (2.39) and (2.40) may still be used to obtain the tilt angle.

If $|E_x| = |E_y|$ and $\phi = \pm \frac{1}{2}\pi$, the axial ratio as given by (2.41) becomes equal to one, the polarization ellipse degenerates to a circle, and the wave is said to be circularly polarized—right circular if $\phi = -\frac{1}{2}\pi$, and left circular if $\phi = +\frac{1}{2}\pi$.

2.5. POWER DENSITY

From

$$\mathbf{E} = (E_x \mathbf{u}_x + E_y \mathbf{u}_y) e^{-jkz}$$
(2.16)

and the Maxwell equations, we can find the magnetic field

$$\mathbf{H} = \frac{1}{Z_0} \left(-E_y \mathbf{u}_x + E_x \mathbf{u}_y \right) e^{-jkz}$$
(2.51)

where Z_0 is the characteristic impedance of the medium defined by

$$Z_0 = \sqrt{\frac{\mu}{\varepsilon}}$$

The complex Poynting vector is then

$$\mathbf{S}_{c} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^{*} = \frac{|E_{x}|^{2} + |E_{y}|^{2}}{2Z_{0}^{*}} \mathbf{u}_{z}$$
(2.52)

with the time-average Poynting vector given by

$$\mathbf{S} = \operatorname{Re}\left[\mathbf{S}_{c}\right] \tag{2.53}$$

2.6. ROTATION RATE OF THE FIELD VECTOR

In the z = 0 plane, the field given by (2.20) reduces to

$$\mathscr{E} = \mathbf{u}_x |E_x| \cos\left(\omega t + \phi_x\right) + \mathbf{u}_y |E_y| \cos\left(\omega t + \phi_y\right)$$
(2.54)

Then the angle ψ between \mathscr{C} and the positive x axis is given as a function of time by

$$\psi = \tan^{-1} \left[\frac{|E_y| \cos (\omega t + \phi_y)}{|E_x| \cos (\omega t + \phi_x)} \right]$$
(2.55)

and the rate of increase of ψ with time is

$$\frac{\partial \psi}{\partial t} = \frac{-\omega |E_x| |E_y| \sin \phi}{|E_x|^2 \cos^2 (\omega t + \phi_x) + |E_y|^2 \cos^2 (\omega t + \phi_y)}$$
(2.56)

where, as before,

$$\phi = \phi_y - \phi_x \tag{2.25}$$

We see that in general the rotation rate of the field vector is not constant. If we take the special case of circular polarization,

$$|E_x| = |E_y| \qquad \phi = \pm \frac{\pi}{2}$$

where the upper sign corresponds to left circular rotation and the lower to right circular, (2.56) reduces to

$$\frac{\partial \psi}{\partial t} = \mp \omega \tag{2.57}$$

Figure 2.4 indicates that $-\omega$ is consistent with left circular polarization.



FIGURE 2.4. Rotation relationships for the polarization ellipse.

We can simplify (2.56) if we note that (2.54) gives, at z = 0,

$$|\mathscr{C}|^{2} = \mathscr{C} \cdot \mathscr{C} = |E_{x}|^{2} \cos^{2} \left(\omega t + \phi_{x}\right) + |E_{y}|^{2} \cos^{2} \left(\omega t + \phi_{y}\right)$$
(2.58)

Using this, the rotation rate of the & vector becomes

$$\frac{\partial \psi}{\partial t} = \frac{-\omega |E_x| |E_y| \sin \phi}{|\mathcal{E}|^2}$$
(2.59)

On the major axis of the polarization ellipse, $|\mathcal{C}|$ is a maximum, given by m. Thus the rotation rate is a minimum, given by

$$\frac{\partial \psi}{\partial t}\Big|_{\min} = \frac{-\omega |E_x| |E_y| \sin \phi}{m^2}$$
(2.60)

On the minor axis, $|\mathscr{C}|$ is minimum (=n), and therefore the maximum rotation rate occurs on the minor axis and is

$$\frac{\partial \psi}{\partial t}\Big|_{\max} = \frac{-\omega |E_x| |E_y| \sin \phi}{n^2}$$
(2.61)

2.7. AREA SWEEP RATE

The area swept by the \mathscr{E} vector in a time dt as it moves through an angle $d\psi$ may be seen from Fig. 2.5 to be

$$dA = \frac{1}{2} \left| \mathcal{E} \right|^2 d\psi \tag{2.62}$$



FIGURE 2.5. Area sweep of the & vector.

Then the rate of increase of swept area is

$$\frac{\partial A}{\partial t} = \frac{1}{2} |\mathscr{E}|^2 \frac{d\psi}{\partial t}$$
(2.63)

and the use of (2.59) gives

$$\frac{\partial A}{\partial t} = -\frac{1}{2}\omega |E_x| |E_y| \sin \phi \qquad (2.64)$$

A negative value for the rate of area sweep is quite valid and indicates only that for right-handed rotation the rate of area sweep is positive, and for left-handed rotation it is negative.

Equation (2.64) shows that the rate of area sweep is not a function of time or of position of the tip of the electric field vector on the polarization ellipse. This may be considered a kind of Kepler's second law for electromagnetics. The laws are not precisely the same for electromagnetics and planetary motion, however, since the Kepler laws state that the planets move around the sun in ellipses with the sun at one focus, and the radius vector *from the sun* to a planet sweeps out equal areas in equal time intervals [2]. The electric field vector drawn from the ellipse origin, not a focus, sweeps out equal areas in equal intervals of time.

We note also that the & vector completes one rotation in the time

$$T = \frac{2\pi}{\omega} \tag{2.65}$$

2.8. ROTATION OF & WITH DISTANCE

If we set t = 0 in (2.20), we can find the position angle of \mathscr{C} as a function of distance z,

ROTATION OF & WITH DISTANCE

$$\psi = \tan^{-1} \frac{|E_y| \cos(-kz + \phi_y)}{|E_x| \cos(-kz + \phi_x)}$$
(2.66)

A comparison of (2.66) and (2.55) shows that we can find the rotation rate of \mathscr{C} with distance at a fixed time if we replace ω in (2.59) by -k and t by z. Then the rotation rate is

$$\frac{\partial \psi}{\partial z} = \frac{k|E_x||E_y|\sin\phi}{|E_x|^2\cos^2(-kz+\phi_x)+|E_y|^2\cos^2(-kz+\phi_y)}$$
(2.67)

and since the denominator is obviously $|\mathscr{C}|^2$ at t = 0,

$$\frac{\partial \psi}{\partial z} = \frac{k|E_x||E_y|\sin\phi}{|\mathcal{E}|^2} \tag{2.68}$$

This indicates that the rotation rate with z is minimum at the major axis of the polarization ellipse and maximum at the minor axis, just as it was with the time rotation rate.

If the rotation of \mathscr{C} with increasing time in a fixed plane is clockwise, the fact that (2.59) and (2.68) have different signs shows that the rotation with increasing distance at a fixed time is counterclockwise. We may think of a right-handed circular wave at fixed time in space as looking like a *left-handed* screw. With increasing time the screw rotates in a clockwise direction as we look in the direction of wave motion. This is shown in Fig. 2.6.



FIGURE 2.6. Rotation of & with time and distance for a right-handed circular wave.

We may see from (2.66) that the distance between two points of the wave having parallel field vectors at constant time is

$$\Delta z = \frac{2\pi}{k} = \lambda \tag{2.69}$$

2.9. THE POLARIZATION RATIOS

A description of the elliptically polarized wave in terms of tilt angle, axial ratio, and rotation sense leads to a good physical understanding of the wave, but it is not convenient mathematically. In this and the following sections the wave will be characterized by more tractable mathematical terms.

The time-invariant E field of (2.16) may also be written as

$$\mathbf{E} = E_0 (\mathbf{u}_x a + \mathbf{u}_y b e^{j\phi}) e^{-jkz}$$
(2.70)

if we extract a common complex term E_0 . For convenience we drop the distance phase term and write

$$\mathbf{E} = E_0(\mathbf{u}_x a + \mathbf{u}_y b e^{j\phi}) \tag{2.71}$$

Without loss of generality, we can choose E_0 and ϕ so that a and b are real and

$$a^2 + b^2 = 1 \tag{2.72}$$

Then E_0 has the same phase as E_x , and ϕ has the same meaning as in previous developments, the phase lead of E_y over E_x .

The value of E_0 does not affect the wave polarization in any way, and except in questions concerned with power, we will neglect it. We define a *polarization ratio P*, which alone carries all necessary polarization information, by

$$P = \frac{E_y}{E_x} = \frac{b}{a} e^{j\phi}$$
(2.73)

This is a commonly used definition, although we will see shortly that a slightly different definition is sometimes useful. Some special values of the polarization ratio are:

Characteristics	P
$E_x = a = 0$	8
$E_{y} = b = 0$	0
$a=b, \ \phi=-\pi/2$	-j
$a=b,\;\phi=+\pi/2$	+j
	Characteristics $E_x = a = 0$ $E_y = b = 0$ $a = b, \ \phi = -\pi/2$ $a = b, \ \phi = +\pi/2$

In the following sections we will determine ranges of P for other polarizations.

2.10. CIRCULAR WAVE COMPONENTS

Consider the complex vectors [3]

$$\boldsymbol{\omega}_{L} = \mathbf{u}_{x} + j\mathbf{u}_{y} \quad \text{(a)}$$

$$\boldsymbol{\omega}_{R} = \mathbf{u}_{x} - j\mathbf{u}_{y} \quad \text{(b)}$$
(2.74)

If we write these in the form of (2.71)

$$\omega_{L} = \sqrt{2} \left(\mathbf{u}_{x} \ \frac{1}{\sqrt{2}} + \mathbf{u}_{y} \ \frac{1}{\sqrt{2}} \ e^{j\pi/2} \right) \quad (a)$$

$$\omega_{R} = \sqrt{2} \left(\mathbf{u}_{x} \ \frac{1}{\sqrt{2}} + \mathbf{u}_{y} \ \frac{1}{\sqrt{2}} \ e^{-j\pi/2} \right) \quad (b)$$
(2.75)

it is clear that if we think of ω_L and ω_R as fields propagating in the z direction, then ω_L is a left circular wave $(a = b, \phi = \frac{1}{2}\pi)$, and ω_R is a right circular wave. To put this another way, if we find the real time-varying field associated with the complex time-invariant fields (2.74) we find

$$\operatorname{Re}\left[\boldsymbol{\omega}_{L}e^{j\omega t}e^{-jkz}\right] = \mathbf{u}_{x}\cos\left(\omega t - kz\right) + \mathbf{u}_{y}\cos\left(\omega t - kz + \frac{1}{2}\pi\right) \quad (a)$$

$$(2.76)$$

$$\operatorname{Re}\left[\boldsymbol{\omega}_{R}e^{j\omega t}e^{-jkz}\right] = \mathbf{u}_{x}\cos\left(\omega t - kz\right) + \mathbf{u}_{y}\cos\left(\omega t - kz - \frac{1}{2}\pi\right) \quad (b)$$

and these may be recognized as real time-varying vectors of constant amplitude rotating, in order, in a left- and right-handed sense.

The field **E** may be expanded in terms of ω_L and ω_R , giving

$$\mathbf{E} = E_0(\mathbf{u}_x a + \mathbf{u}_y b e^{j\phi}) = E_0(L\boldsymbol{\omega}_L + R e^{j\theta} \boldsymbol{\omega}_R)$$
(2.77)

where θ is the phase difference between left and right circular components, and if we use (2.74),

$$\mathbf{u}_x a + \mathbf{u}_y b e^{j\phi} = L(\mathbf{u}_x + j\mathbf{u}_y) + R e^{j\theta} (\mathbf{u}_x - j\mathbf{u}_y)$$
(2.78)

Equating coefficients of like unit vectors we obtain

$$a = L + Re^{j\theta} \quad (a)$$

$$be^{j\phi} = jL - jRe^{j\theta} \quad (b)$$
(2.79)

and solving for L and $Re^{i\theta}$ gives

$$L = \frac{1}{2}(a - jbe^{j\phi}) \quad (a)$$

$$Re^{j\theta} = \frac{1}{2}(a + jbe^{j\phi}) \quad (b)$$
(2.80)

Note that L and R, unlike a and b, are complex. The ratio

$$\frac{L}{Re^{j\theta}} = \frac{L}{R} e^{-j\theta} = \frac{a - jbe^{j\phi}}{a + jbe^{j\phi}}$$
(2.81)

is also in general complex. However, we can absorb the phase angle of $(a - jbe^{j\phi})/(a + jbe^{j\phi})$ into angle θ , so that L/R is real. We could also in (2.77), by extracting a term, E'_0 , with a different phase, make L and R real, but there is little reason to do so.

From (2.80) we may easily obtain

$$|L|^2 + |R|^2 = \frac{1}{2} \tag{2.82}$$

Just as a and $be^{j\phi}$ in (2.71) represent the x and y rectangular wave components, L and $Re^{j\theta}$ represent the left circular and right circular components of the general elliptical wave.

Let us define a ratio of the circular wave components, in analogy to the definition of P in (2.73). The complex quantity q is

$$q = \frac{L}{R} e^{-j\theta} \tag{2.83}$$

If E were written as

$$\mathbf{E} = (\boldsymbol{\omega}_{I} \boldsymbol{E}_{I} + \boldsymbol{\omega}_{R} \boldsymbol{E}_{R}) \boldsymbol{e}^{-jkz}$$
(2.84)

just as we wrote

$$\mathbf{E} = (\mathbf{u}_x E_x + \mathbf{u}_y E_y) e^{-jkz}$$
(2.16)

then q would also be

$$q = \frac{E_L}{E_R} \tag{2.85}$$

We will see later that all of the characteristics of the polarization ellipse may be determined from a knowledge of P or q. We previously used the name polarization ratio for P. It seems appropriate to call q the *circular polarization ratio*.

THE MODIFIED POLARIZATION RATIO

2.11 RELATIONSHIP BETWEEN P AND q, AND THE MODIFIED POLARIZATION RATIO

From (2.83) and (2.81) we see that

$$q = \frac{L}{R} e^{-j\theta} = \frac{a - jbe^{j\phi}}{a + jbe^{j\phi}} = \frac{1 - j(b/a)e^{j\phi}}{1 + j(b/a)e^{j\phi}}$$
(2.86)

and if we use (2.73),

$$q = \frac{1 - jP}{1 + jP} \tag{2.87}$$

Solving for P in terms of q gives

$$P = -j \,\frac{1-q}{1+q}$$
(2.88)

In both (2.87) and (2.88) P is multiplied by j. To remove this we define [4]

$$p = jP \tag{2.89}$$

and substitute into (2.87) and (2.88), obtaining

$$q = \frac{1 - p}{1 + p}$$
(2.90)

$$p = \frac{1-q}{1+q} \tag{2.91}$$

The symmetry is pleasing, but more importantly we will see later that the form of (2.90) and (2.91) allows polarizations to be plotted on the common Smith transmission line chart. Much of our later work will be carried out using p, which will be referred to as the *modified polarization ratio*. From (2.89) and (2.73) we have

$$p = j \frac{E_y}{E_x} = j \frac{b}{a} e^{j\phi}$$
(2.92)

Some special values of P, p, and q are:

Wave	Characteristics	Р	р	q	
Linear vertical	$E_x = a = 0$	8	j∞	-1	
Linear horizontal	$E_{y} = b = 0$	0	0	+1	
Right circular	$a = b, \ \phi = -\frac{1}{2}\pi;$	-j	+1	0	
Left circular	$L = \frac{1}{2}(a - Jbe^{-1}) = 0$ $a = b, \ \phi = \frac{1}{2}\pi;$	+j	-1	8	
	$Re^{j\theta} = \frac{1}{2}(a+jbe^{j\phi}) = 0$				

REPRESENTATION OF WAVE POLARIZATION

2.12. ELLIPSE CHARACTERISTICS IN TERMS OF q

From (2.77) and (2.83) we can obtain

$$\mathbf{E} = E_0 R e^{j\theta} (q \boldsymbol{\omega}_L + \boldsymbol{\omega}_R)$$
(2.93)

Let

$$\delta = \arg\left(E_0 R\right) \tag{2.94}$$

so that

$$\mathbf{E} = |E_0 R| e^{j\delta} e^{j\theta} (q \boldsymbol{\omega}_L + \boldsymbol{\omega}_R)$$
(2.95)

and

$$\mathscr{E} = \operatorname{Re}\left[|E_0 R| e^{j(\omega t + \delta - kz + \theta)} (q \omega_L + \omega_R)\right]$$
(2.96)

If we set

$$\beta = \omega t + \delta - kz \tag{2.97}$$

we obtain

$$\frac{\mathscr{E}}{|E_0 R|} = \operatorname{Re}\left[e^{i(\beta+\theta)}(q\omega_L + \omega_R)\right]$$
(2.98)

and

$$\frac{2\mathscr{E}}{|E_0R|} = e^{j(\beta+\theta)}(q\omega_L + \omega_R) + e^{-j(\beta+\theta)}(q^*\omega_L^* + \omega_R^*)$$
(2.99)

We see from (2.74) that

$$\omega_{L} \cdot \omega_{L} = \omega_{R} \cdot \omega_{R} = 0 \quad (a)$$

$$\omega_{L} \cdot \omega_{L}^{*} = \omega_{R} \cdot \omega_{R}^{*} = 2 \quad (b)$$

$$\omega_{L} \cdot \omega_{R} = \omega_{R} \cdot \omega_{L} = 2 \quad (c)$$

$$\omega_{L} \cdot \omega_{R}^{*} = \omega_{R} \cdot \omega_{L}^{*} = 0 \quad (d)$$
(2.100)

If we multiply (2.99) by its conjugate and use (2.100) we obtain

$$\frac{|\mathscr{C}|^2}{|E_0 R|^2} = 1 + |q|^2 + q e^{j2(\beta+\theta)} + q^* e^{-j2(\beta+\theta)}$$
(2.101)

and making use of the definitions for β and q, we get

$$\frac{|\mathscr{C}|^2}{|E_0 R|^2} = 1 + |q|^2 + |q|e^{j(2\beta+\theta)} + |q|e^{-j(2\beta+\theta)}$$
$$= 1 + |q|^2 + 2|q|\cos(2\omega t + 2\delta - 2kz + \theta)$$
(2.102)

We easily see from this equation that the maximum value of $|\mathscr{C}|$ is

$$|\mathscr{E}|_{\max} = |E_0 R| (1 + |q|) \tag{2.103}$$

which occurs at

$$2(\omega t + \delta - kz) + \theta = 0 \qquad 2\pi, \dots \qquad (2.104)$$

and the minimum value is

$$|\mathscr{E}|_{\min} = |E_0 R| |1 - |q| |$$
(2.105)

which occurs at

$$2(\omega t + \delta - kz) + \theta = \pi \qquad 3\pi, \dots \qquad (2.106)$$

The axial ratio of the polarization ellipse is then

$$\mathbf{AR} = \left| \frac{1 + |q|}{1 - |q|} \right| \tag{2.107}$$

Note that this definition of axial ratio is the inverse of the relation n/m used previously. This should not cause any confusion to the reader. Since we assumed m > n, an axial ratio greater than one is clearly m/n.

In order to find the tilt angle of the ellipse, we return to (2.99) and substitute in it the values for q, β , ω_L , and ω_R . This leads to

$$\frac{\mathscr{C}}{|E_0 R|} = \mathbf{u}_x \left[\cos\left(\omega t + \delta - kz + \theta\right) + \frac{L}{R} \cos\left(\omega t + \delta - kz\right) \right] + \mathbf{u}_y \left[\sin\left(\omega t + \delta - kz + \theta\right) - \frac{L}{R} \sin\left(\omega t + \delta - kz\right) \right]$$
(2.108)

Now we recall from (2.104) that $|\mathcal{E}|$ is maximum for

$$\omega t + \delta - kz = -\frac{1}{2}\theta, \ \pi - \frac{1}{2}\theta, \ldots$$

and if we substitute the first of these values into (2.108) we obtain

$$\frac{\mathscr{C}}{|E_0R|} = \mathbf{u}_x \left(\cos\frac{\theta}{2} + \frac{L}{R}\cos\frac{\theta}{2} \right) + \mathbf{u}_y \left(\sin\frac{\theta}{2} + \frac{L}{R}\sin\frac{\theta}{2} \right)$$
(2.109)

The rotation angle of \mathscr{C} , which is the tilt angle of the ellipse, since $|\mathscr{C}|$ is maximum, is given by

$$\tau = \tan^{-1} \frac{\mathscr{C}_{y}}{\mathscr{C}_{x}} = \tan^{-1} \frac{(1 + L/R)\sin(\theta/2)}{(1 + L/R)\cos(\theta/2)}$$
(2.110)

Solutions to this equation are

$$\tau = \frac{1}{2}\theta \qquad \pi + \frac{1}{2}\theta \tag{2.111}$$

We wish to keep τ in the range $0-\pi$, so the first form can be used for θ positive and the second for θ negative.

2.13. ELLIPSE CHARACTERISTICS IN TERMS OF p AND P

Combining (2.107) and (2.90) gives the axial ratio in terms of p. It is

$$AR = \left| \frac{|1+p| + |1-p|}{|1+p| - |1-p|} \right|$$
(2.112)

The tilt angle can be found from (2.111), (2.86), and (2.90). The result is

$$e^{-j2\tau} = e^{-j\theta} = \frac{q}{|q|} = \frac{(1-p)/(1+p)}{|(1-p)/(1+p)|}$$
(2.113)

These equations are not as convenient as those giving ellipse characteristics in terms of the circular polarization ratio q. In terms of the common polarization ratio P, using (2.89) quickly leads to

$$AR = \left| \frac{|1+jP| + |1-jP|}{|1+jP| - |1-jP|} \right|$$
(2.114)

and

$$e^{-j2\tau} = \frac{(1-jP)/(1+jP)}{|(1-jP)/(1+jP)|}$$
(2.115)

which are also less easy to use than the equations in terms of q.

2.14. POLARIZATION CHARACTERISTICS FOR RANGES OF p AND q

The complex parameters p and q each contain all the information about the polarization of a wave. It is not immediately clear, however, from a knowledge of p or q just what the polarization ellipse characteristics (or more generally the physical polarization characteristics) are. For example, what are the tilt angle, axial ratio, and rotation sense of a wave if $p = p_1 = 2e^{j\pi/6}$? For a second wave, if $p_2 = 2e^{-j\pi/6}$, are the two waves somewhat similar in polarization characteristics or do they differ greatly? If we have two antennas that will transmit, respectively, waves with these polarizations, can they be used satisfactorily in a transmit–receive configuration? The answer to the last question is reserved for a later chapter, but we will begin here to examine the first two.

From (2.59)

$$\frac{\partial \psi}{\partial t} = \frac{-\omega |E_x| |E_y| \sin \phi}{|\mathcal{E}|^2}$$
(2.59)

which gives the rate of increase of ψ , the angle of \mathscr{C} measured from the x axis, we see that $\partial \psi / \partial t$ is negative, corresponding to left elliptic rotation (LER), for

$$0 < \phi < \pi$$
 LER

and that $\partial \psi / \partial t$ is positive, corresponding to right elliptic rotation (RER), for

$$\pi < \phi < 2\pi$$
 RER

The end points of these ranges correspond to linear polarizations. Now

$$p = jP = j \frac{b}{a} e^{j\phi} = \frac{b}{a} (j \cos \phi - \sin \phi)$$
(2.116)

and we see that for left elliptic rotation, $0 < \phi < \pi$

$$\begin{array}{c} \operatorname{Re}(p) < 0\\ \operatorname{Im}(P) > 0 \end{array} \right\} \quad \text{LER}$$
 (2.117)

and for right elliptic rotation

$$\begin{array}{c} \operatorname{Re}(p) > 0 \\ \operatorname{Im}(P) < 0 \end{array} \right\} \quad \operatorname{RER}$$
 (2.118)

From (2.116) and the definition of q, it quickly follows that

$$|q|^{2} = \frac{1-p}{1+p} \frac{1-p^{*}}{1+p^{*}} = \frac{1+2(b/a)\sin\phi + (b/a)^{2}}{1-2(b/a)\sin\phi + (b/a)^{2}}$$
(2.119)

and we see immediately that for left elliptic rotation, $0 < \phi < \pi$, and right elliptic rotation, $\pi < \phi < 2\pi$, respectively,

$$|q| > 1$$
 LER (a) $|q| < 1$ RER (b) (2.120)

We might have expected this from the defining relation

$$q = \frac{L}{R} e^{-j\theta} \tag{2.83}$$

Thus for a general elliptical wave, which we know can be separated into left and right circular components, |q| < 1 corresponds to |L| < |R|, which results in a right-handed rotation of \mathscr{C} .

2.15. THE TRANSFORMATIONS p(q) AND q(p)

Before we consider the transformations

$$q = \frac{1-p}{1+p}$$
(2.90)
$$p = \frac{1-q}{1+q}$$
(2.91)

$$V = \frac{I_R}{2} \left[(Z_R + Z_0) e^{\gamma d} + (Z_R - Z_0) e^{-\gamma d} \right]$$
(a)

$$I = \frac{I_R}{2Z_0} \left[(Z_R + Z_0) e^{\gamma d} - (Z_R - Z_0) e^{-\gamma d} \right]$$
(b)



FIGURE 2.7. The transmission line.

where Z_R is the load impedance, Z_0 the characteristic line impedance, and γ the propagation constant, given in terms of the line parameters by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad (2.122)$$

The voltage reflection coefficient K is defined as the ratio of the reflected voltage wave [second term in (2.121a)] to the incident voltage wave [first term in (2.121a)]. Similarly, the current reflection coefficient is the ratio of the reflected current to the incident current. Mathematically, they are

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} e^{-2\gamma d}$$
(a)

$$K_I = -\frac{Z_R - Z_0}{Z_R + Z_0} e^{-2\gamma d} = -K$$
(b)
(2.123)

If the impedance of the line at any point is defined as

$$Z = \frac{V}{I} \tag{2.124}$$

its normalized value may be found from (2.121) and (2.123) to be

$$z = \frac{Z}{Z_0} = \frac{1+K}{1-K} = \frac{1-K_I}{1+K_I}$$
(2.125)

Solving for K and K_I leads to

$$K = \frac{z-1}{z+1}$$
 (a) $K_I = \frac{1-z}{1+z}$ (b) (2.126)

and we see that the transformation between z and K_I (not K) is the same as that between the polarization parameters p and q.

Now it is well known that if curves of constant Re (z) and constant Im (z) are plotted on the complex K plane, a widely available transmission line chart, the Smith chart, results. While the common usage of the Smith chart employs z and K (not K_I), nevertheless, it can also be used with the K_I coefficient.

We see here the advantage resulting from the use of the modified polarization ratio p = jP, rather than P itself. The transformation between p and q is such that it can be made with the commercially available Smith chart.

We must make a choice at this time. The Smith chart is plotted on the complex K plane. Shall we consider q analogous to K and plot our curves on the complex q plane, or shall we plot on the p plane? Rumsey [4] has pointed out that the simplest type of impedance transformation is that due to adding a uniform section of lossless transmission line, which causes a phase change at

REPRESENTATION OF WAVE POLARIZATION

constant amplitude of K. The simplest type of polarization transformation is that caused by rotation of the antenna producing an elliptically polarized wave that causes a phase change at constant amplitude of q. (It is obvious that an antenna producing a known pair of circular waves—to produce an elliptical resulting wave—must still produce the same pair if it is rotated; therefore |q|is constant.) Then it is convenient to consider the analogs q, K_l , and p, z.

Even more important, on the q plane all right elliptical rotations will fall within the unit circle |q| < 1. This unit circle is important since the common Smith chart is restricted (for convenience) to the unit circle. If the p plane were used for the plot, the left-half plane would contain all LER points and the right-half plane all RER points [from (2.117) and (2.118)]. Then the p plane would have to be infinite to contain all polarizations.

We will therefore use the q plane for our plots. Rather than use the equations for plotting curves on the Smith chart, we will develop new equations in terms of p and q. We may thus avoid the confusion inherent in translating, for example, from Re(z) to Re(p) and from $K_1(=-K)$ to q. In (2.90) and (2.91), let

$$p = u + jv$$
 (a) $q = s + jt$ (b) (2.127)

Then

$$p = u + jv = \frac{1 - q}{1 + q} = \frac{1 - s - jt}{1 + s + jt} = \frac{1 - s^2 - t^2 - j2t}{(1 + s)^2 + t^2}$$
(2.128)

Equating real and imaginary parts of this expression, collecting terms, and completing the squares gives, if $u \neq -1$ and $v \neq 0$,

$$\left(s + \frac{u}{u+1}\right)^2 + t^2 = \left(\frac{1}{u+1}\right)^2 \quad \text{(a)}$$

$$(s+1)^2 + \left(t + \frac{1}{v}\right)^2 = \left(\frac{1}{v}\right)^2 \quad \text{(b)}$$

The equations (2.129) describe two families of circles on the complex q plane. One family has centers at

$$s_c, t_c = -\frac{u}{u+1}, 0$$
 (2.130)

and radii

$$r = \left| \frac{1}{u+1} \right| \tag{2.131}$$

The second has centers at

$$s_c, t_c = -1, -\frac{1}{v}$$
 (2.132)

and radii

 $r = \left|\frac{1}{v}\right| \tag{2.133}$

Some of these circles are shown in Fig. 2.8. It is obvious that the portion of the complex plane within the circle of unit radius, |q| < 1, is identical to the common Smith chart.



FIGURE 2.8. The q plane with curves of constant u and v.

In developing the previous equations for the polarization plot, we excluded the values u = -1 and v = 0. We see, however, that as $u \rightarrow -1$, the first family of circles with centers and radii given by (2.130) and (2.131) merely degenerates to a straight line, the vertical line s = -1. As $v \rightarrow 0$, the second family of circles yields the horizontal axis. We are therefore not required to exclude these values for u and v.

2.16. THE TRANSFORMATION FOR u < 0

We see from (2.129) and Fig. 2.8 that for 0 > u > -1, the circle center for the family of circles u = constant is on the +s axis. Further, all the circles pass through the point (s = -1, t = 0) since the radius exceeds the location of the center by

$$\frac{1}{u+1} - \frac{|u|}{u+1} = \frac{1+u}{u+1} = 1$$

All the circles for 0 > u > -1 therefore lie outside the unit circle.

For u < -1, the circle center is on the -s axis. Again the circles pass through s = -1, t = 0, since the magnitude of the distance from the origin to the circle center exceeds the circle radius by

$$\frac{|u|}{|u+1|} - \frac{1}{|u+1|} = 1$$

We see therefore that all values of $u = \operatorname{Re}(p) < 0$ transform to points outside the unit radius circle |q| = 1. This might have been expected, since |q| < 1 corresponds to right elliptic rotation and $\operatorname{Re}(p) > 0$ also corresponds to right elliptic rotation. This appears to leave us in the unsatisfactory situation of being limited to right elliptic polarizations, at least inside the unit circle, which is normally all that is considered for the Smith chart.[†] In order to eliminate this restriction, for $u = \operatorname{Re}(p) < 0$ we define a new circular polarization ratio in the following way:

First we define a new phase term γ (not to be confused with the transmission line propagation constant discussed earlier).

$$\gamma = -\theta \tag{2.134}$$

Then the circular polarization ratio q may be written as

$$q = \frac{L}{R} e^{-j\theta} = Q e^{j\gamma}$$
(2.135)

We have set up a correspondence between p and z, and only positive values of Re (z) are needed for most transmission line studies. It is then not surprising to find ourselves restricted to positive Re (p) or right elliptic polarizations.

THE TRANSFORMATION FOR u < 0

where

$$Q = |q| = \frac{L}{R} \tag{2.136}$$

The phase term γ is defined in this manner in order to have a symbol for the phase of q without the confusion of the negative sign. Now if Q > 1, our polarization point will be outside the unit circle. It then seems appropriate to define a new polarization parameter involving 1/Q for use when Q > 1. Two reasonable definitions, 1/q or $1/q^*$, are possible. Let us use $1/q^*$, which has the same phase angle, γ , as q, and in fact is a reflection of q in the unit circle. Therefore we define

$$w = \frac{1}{q^*} = \frac{1}{Q} e^{-j\theta} = W e^{j\gamma}$$
(2.137)

and consider the transformation from p to w. Substituting (2.137) into (2.91) gives

$$p = \frac{1-q}{1+q} = \frac{1-1/w^*}{1+1/w^*} = \frac{w^*-1}{w^*+1}$$
(2.138)

and solving for w gives

$$w = \frac{1+p^*}{1-p^*} \tag{2.139}$$

We no longer have symmetric transformations as we did between p and q, but nevertheless the forms (2.138) and (2.139) suit our purpose.[†] We next let

$$w = e + jf \tag{2.140}$$

and repeat the steps leading to (2.129) for the families of circles on the q plane, setting

$$p = u + jv = \frac{w^* - 1}{w^* + 1} = \frac{e - jf - 1}{e - jf + 1} = \frac{e^2 - 1 + f^2 - 2jf}{(e + 1)^2 + f^2}$$
(2.141)

and equating real and imaginary terms and completing squares. The result is

$$\left(e + \frac{u}{u-1}\right)^{2} + f^{2} = \left(\frac{1}{u-1}\right)^{2} \quad \text{(a)}$$
$$(e+1)^{2} + \left(f + \frac{1}{v}\right)^{2} = \left(\frac{1}{v}\right)^{2} \quad \text{(b)}$$

^tThe use of w = 1/q, which was mentioned as a possibility, also would not have resulted in symmetric transforms between p and w.

These equations represent families of circles on the complex w plane. The first family, for constant u, has centers at

$$e_c, f_c = -\frac{u}{u-1}, 0$$
 (2.143)

and radii

$$r = \left| \frac{1}{u - 1} \right| \tag{2.144}$$

and the second family, for constant v, has centers at

$$e_c, f_c = -1, -\frac{1}{v}$$
 (2.145)

and radii

$$r = \left|\frac{1}{\nu}\right| \tag{2.146}$$

Let us compare the q- and w-plane plots

$$q = s + jt \text{ Plane} \qquad w = e + jf \text{ Plane}$$

$$q = \frac{1-p}{1+p} \qquad p = u + jv \qquad w = \frac{1+p^*}{1-p^*}$$

$$\left(s + \frac{u}{u+1}\right)^2 + t^2 = \left(\frac{1}{u+1}\right)^2 \qquad \left(e + \frac{u}{u-1}\right)^2 + f^2 = \left(\frac{1}{u-1}\right)^2$$

$$(s+1)^2 + \left(t + \frac{1}{v}\right)^2 = \left(\frac{1}{v}\right)^2 \qquad (e+1)^2 + \left(f + \frac{1}{v}\right)^2 = \left(\frac{1}{v}\right)^2$$

Family of Circles, u = Constant

Center at
$$\frac{-u}{u+1}$$
, 0
Radius = $\left|\frac{1}{u+1}\right|$
Center at $\frac{-u}{u-1}$, 0
Radius = $\left|\frac{1}{u-1}\right|$

Family of Circles, v = Constant

Center at $-1, -\frac{1}{v}$ Radius = $\left|\frac{1}{v}\right|$ Radius = $\left|\frac{1}{v}\right|$ Center at $-1, -\frac{1}{v}$ Radius = $\left|\frac{1}{v}\right|$

The comparison of the circle centers and radii for the two planes clearly shows two facts.

- 1. The circles on the q plane for some u, say $u = u_0$, are identical to the circles on the w plane for $u = -u_0$.
- 2. The circles on the q plane for some v, say $v = v_0$, are identical to the circles on the w plane for $v = v_0$.

It follows, therefore, that Fig. 2.8 may be considered the q plane with all curves u = constant inside the unit circle corresponding to positive values of u. All polarization points inside the unit circle on the q plane correspond to right elliptical polarization. With equal justification, Fig. 2.8 may be considered the w plane with all curves inside the unit circle corresponding to negative values of u. All polarization points inside the unit circle corresponding to negative values of u. All polarization points inside the unit circle on the w plane correspond to left elliptical polarization. A particular circle $u = u_0$ on the q plane would be labeled $-u_0$ on the w plane. For the two planes there is no difference in the constant v circles. In fig. 2.8 all values relating to the w plane are in parentheses.

We can find the polarization ellipse characteristics in terms of w by use of the equations for the ellipse characteristics in terms of q and the transformation between q and w. They are:

$$AR = \left| \frac{1+|q|}{1-|q|} \right| = \left| \frac{1+|1/w^*|}{1-|1/w^*|} \right| = \left| \frac{1+|w|}{1-|w|} \right|$$
(2.147)

and since by definition the phase angle of w is the same as that of q, that is, γ , the tilt angle is still

$$\tau = \frac{1}{2}\theta = -\frac{1}{2}\gamma \tag{2.148}$$

The rotation sense of the wave is left handed if

$$|q| > 1$$
 $|w| < 1$

and right handed if

$$|q| < 1$$
 $|w| > 1$

2.17. POLARIZATION CHART AS THE *p* PLANE

Since p(q) and q(p) have the same form, it is obvious that we can consider the polarization chart, Fig. 2.8, as the p plane, with the circles being curves of constant Re (q) and constant Im (q). This use has limited value, however, since all polarizations of interest do not fall within the unit circle.

2.18. COINCIDENT POINTS ON THE q AND w PLANES

Figure 2.8 represents the q plane for right elliptic polarizations and the w plane for left elliptic polarizations. Any wave polarization is represented by a point on this chart. On the other hand, a point on the chart represents two polarizations, one left handed and the other right handed. Consider the polarization described by a point q_0 and that described by the coincident point w_0 (not the transformed point $w = 1/q^*$). Then

$$w_0 = q_0$$

From (2.147) we see that

$$AR|_{q_0} = \left|\frac{1+|q_0|}{1-|q_0|}\right| = \left|\frac{1+|w_0|}{1-|w_0|}\right| = AR|_{w_0}$$

and

$$\tau|_{q_0} = \frac{1}{2}\theta = \tau|_{w_0}$$

We see then that coincident points on the q and w planes represent waves having the same axial ratios and tilt angles, but opposite rotation senses, since the q plane represents all right elliptic rotations and the w plane all left.

2.19 CONTOURS OF CONSTANT AXIAL RATIO AND TILT ANGLE

It is often useful to consider contours of constant axial ratio and tilt angle on the polarization chart. To obtain the curves of constant axial ratio, we note from (2.147) that contours of constant axial ratio are also contours of constant |q| or |w|. These are circles on the q or w planes with centers at the origins. On the q plane inside the unit circle where |q| < 1, the circle radius is found from (2.147) to be

Radius =
$$|q| = \frac{AR - 1}{AR + 1}$$
 (2.149)

and on the w plane, (2.147) likewise yields

Radius =
$$|w| = \frac{AR - 1}{AR + 1}$$
 (2.150)

We see that for a particular axial ratio, the circles on the q and w planes coincide. This also follows from Section 2.18 where we saw that coincident points on the q and w planes represent polarizations with the same axial ratio and tilt angle.



FIGURE 2.9. Curves of constant axial ratio and tilt angle.

Contours of constant tilt angle, for either the q or w plane, are found from

$$\gamma = \arg q = \arg w = -2\tau \tag{2.151}$$

Since we have constrained τ to the range $0 \le \tau \le \pi$, the range of γ is $0 \ge \gamma \ge -2\pi$.

Figure 2.9 shows curves of constant axial ratio and tilt angle on the q and w plane. Note that left and right circular waves, AR = 1, are represented by points at the origin, |q| = |w| = 0, and linear polarizations, $AR \rightarrow \infty$, by points on the unit circle, |q| = |w| = 1.

2.20. CONTOURS OF CONSTANT |p|

In Fig. 2.8 we have curves of constant Re (p) and Im (p). Let us now obtain curves of constant |p|. We take first the RER case and use

$$p = \frac{1-q}{1+q} \tag{2.91}$$

which becomes

$$|p|e^{j(\phi+\pi/2)} = \frac{1-Q\varepsilon^{j\gamma}}{1+Q\varepsilon^{j\gamma}}$$
(2.152)

Multiplying both sides by the complex conjugate and rearranging leads to

$$Q^{2} - 2 \frac{1 + |p|^{2}}{1 - |p|^{2}} Q \cos \gamma + \left(\frac{1 + |p|^{2}}{1 - |p|^{2}}\right)^{2} = \left(\frac{1 + |p|^{2}}{1 - |p|^{2}}\right)^{2} - 1 \quad (2.153)$$

Now a circle in plane polar coordinates (Q, γ) of radius r and center at (ℓ, α) is given by

$$Q^{2} - 2\ell Q \cos(\gamma - \alpha) + \ell^{2} = r^{2}$$
(2.154)

where $0 < (\gamma - \alpha) < 2\pi$. This is readily seen by applying the law of cosines to Fig. 2.10.

Therefore (2.153) represents a family of circles on the q plane with center at

$$Q_c, \gamma_c = \frac{1+|p|^2}{1-|p|^2}, 0 \qquad |p| < 1$$
 (2.155)

and with radius

$$r = \left[\left(\frac{1+|p|^2}{1-|p|^2} \right)^2 - 1 \right]^{1/2} \qquad |p| < 1$$
(2.156)





CONTOURS OF CONSTANT |p|

If |p| > 1, we note that (2.153) is not in the correct form (2.154) to represent a circle. For this case, we rewrite it, dividing numerators and denominators by $|p|^2$, to obtain

$$Q^{2} - 2\cos\left(\gamma \pm \pi\right) \frac{1 + 1/|p|^{2}}{1 - 1/|p|^{2}} Q + \left[\frac{1 + 1/|p|^{2}}{1 - 1/|p|^{2}}\right]^{2} = \left[\frac{1 + 1/|p|^{2}}{1 - 1/|p|^{2}}\right]^{2} - 1$$
(2.157)

Comparison to the standard equation shows that this represents a circle on the q plane with center at

$$Q_c, \gamma_c = \frac{1+1/|p|^2}{1-1/|p|^2}, \pi |p| > 1$$
 (2.158)

and radius

$$r = \left[\left(\frac{1+1/|p|^2}{1-1/|p|^2} \right)^2 - 1 \right]^{1/2} \qquad |p| > 1$$
 (2.159)

We consider next the left elliptical case for which

$$p = \frac{w^* - 1}{w^* + 1} = \frac{We^{-j\gamma} - 1}{We^{-j\gamma} + 1}$$
(2.160)

If we multiply both sides by the complex conjugate, just as we did with the RER case, we get

$$|p|^{2} = \frac{W^{2} - 2W\cos\gamma + 1}{W^{2} + 2W\cos\gamma + 1}$$
(2.161)

This equation may be put into the form of (2.153). Therefore (2.161) represents circles on the *w* plane with centers and radii that may be found immediately from the preceding equations. The results are

$$W_{c}, \gamma_{c} = \frac{1+|p|^{2}}{1-|p|^{2}}, 0 \qquad (a)$$

$$r = \left[\left(\frac{1+|p|^{2}}{1-|p|^{2}} \right)^{2} - 1 \right]^{1/2} \qquad (b)$$

$$W_{c}, \gamma_{c} = \frac{1+1/|p|^{2}}{1-1/|p|^{2}}, \pi \qquad (a)$$

$$r = \left[\left(\frac{1+1/|p|^{2}}{1-1/|p|^{2}} \right)^{2} - 1 \right]^{1/2} \qquad (b)$$

REPRESENTATION OF WAVE POLARIZATION

It is obvious from these results that the circles of constant |p| coincide for the q and w planes.

2.21. CONTOURS OF CONSTANT ϕ

Let us now examine contours of constant ϕ , the phase difference between x and y components of the propagating wave. Again we take the RER case first. If the conjugate of

$$p = |p|e^{i(\phi + \pi/2)} = \frac{1 - Qe^{i\gamma}}{1 + Qe^{i\gamma}}$$
(2.152)

is first added to p and then subtracted, and if the difference is divided by the sum, the result is

$$\tan\left(\phi + \frac{\pi}{2}\right) = \frac{-2Q\sin\gamma}{1-Q^2} \tag{2.164}$$

If we write

$$C = \cot \phi = -\tan\left(\phi + \frac{\pi}{2}\right) \tag{2.165}$$

then (2.164) becomes

$$Q^{2} - \frac{2Q}{C}\cos\left(\gamma + \frac{\pi}{2}\right) + \frac{1}{C^{2}} = 1 + \frac{1}{C^{2}}$$
(2.166)

This is the standard form for a circle, so it represents a family of circles on the q plane with centers at

$$Q_{c} = \frac{1}{C} = \tan \phi \quad (a)$$

$$\gamma_{c} = -\frac{\pi}{2} \qquad (b)$$

$$(2.167)$$

and radii

$$r = \left(1 + \frac{1}{C^2}\right)^{1/2} = \left(1 + \tan^2 \phi\right)^{1/2} = |\sec \phi| \qquad C > 0 \qquad (2.168)$$

These equations are restricted to C > 0 since (2.166) is in the standard form for a circle only if Q/C > 0, and Q > 0 by definition. The condition C > 0requires

$$C = \cot \phi > 0 \tag{2.169}$$

and this in turn leads to

$$0 < \phi < \frac{\pi}{2}$$
 LER $\pi < \phi < \frac{3\pi}{2}$ RER

We need not consider the range of ϕ giving left elliptic rotation, since for LER we do not use the q plane.

For C < 0, we may rewrite (2.166) as

$$Q^{2} - \frac{2Q}{|C|} \cos\left(\gamma - \frac{\pi}{2}\right) + \frac{1}{C^{2}} = 1 + \frac{1}{C^{2}}$$
(2.170)

Now

$$C = \cot \phi < 0 \tag{2.171}$$

gives ranges of ϕ that are

$$\frac{\pi}{2} < \phi < \pi \quad \text{LER} \quad \frac{3\pi}{2} < \phi < 2\pi \quad \text{RER}$$

Again we consider only the right elliptic case. From (2.170) the circle centers are

$$Q_{c} = \frac{1}{|C|} = |\tan \phi| \quad (a)$$

$$\gamma_{c} = \frac{\pi}{2} \qquad (b)$$

and the radii are

$$r = \sec \phi \tag{2.173}$$

Next, we consider the LER case, for which

$$p = |p|e^{j(\phi + \pi/2)} = \frac{w^* - 1}{w^* + 1} = \frac{We^{-j\gamma} - 1}{We^{-j\gamma} + 1}$$
(2.174)

Adding and subtracting the conjugate equations and dividing the difference by the sum leads to

$$\tan\left(\phi + \frac{\pi}{2}\right) = \frac{-2W\sin\gamma}{W^2 - 1} \tag{2.175}$$

Again we use

$$C = \cot \phi = -\tan\left(\phi + \frac{\pi}{2}\right) \tag{2.165}$$

and this leads to

REPRESENTATION OF WAVE POLARIZATION

$$W^{2} - \frac{2W}{C}\cos\left(\gamma - \frac{\pi}{2}\right) + \frac{1}{C^{2}} = 1 + \frac{1}{C^{2}}$$
(2.176)

For C > 0, this time we exclude the RER case, and ϕ has the range

$$0 < \phi < \frac{\pi}{2}$$
 LER

Then (2.176) represents circles on the w plane with centers

$$W_c = \frac{1}{C} = \tan \phi \quad (a)$$

$$\gamma_c = \frac{\pi}{2} \qquad (b)$$
(2.177)

and radii

$$r = \left(1 + \frac{1}{C^2}\right)^{1/2} = (1 + \tan^2 \phi)^{1/2} = \sec \phi \qquad (2.178)$$

For C < 0, we once more exclude the RER case and let the range of ϕ be

$$\frac{\pi}{2} < \phi < \pi$$
 LER

We rewrite (2.176) to be

$$W^{2} - \frac{2W}{|C|} \cos\left(\gamma + \frac{\pi}{2}\right) + \frac{1}{C^{2}} = 1 + \frac{1}{C^{2}}$$
(2.179)

and the circle centers and radii on the w plane turn out to be

$$W_{c} = \frac{1}{|C|} = |\tan \phi| \quad (a)$$

$$\gamma_{c} = -\frac{\pi}{2} \qquad (b)$$

$$r = |\sec \phi| \qquad (c)$$

At this point a summary would be useful, and we show here the centers, radii, and range of ϕ for the two polarization states.

RER LER

$$Q_c = \tan \phi$$
 $W_c = |\tan \phi|$
 $\gamma_c = -\frac{\pi}{2}$ $\gamma_c = -\frac{\pi}{2}$
 $r = |\sec \phi|$ $r = |\sec \phi|$ (2.181)
 $\pi < \phi < \frac{3\pi}{2}$ $\frac{\pi}{2} < \phi < \pi$

RER LER

$$Q_c = |\tan \phi|$$
 $W_c = \tan \phi$
 $\gamma_c = \frac{\pi}{2}$ $\gamma_c = \frac{\pi}{2}$
 $r = \sec \phi$ $r = \sec \phi$
 $\frac{3\pi}{2} < \phi < 2\pi$ $0 < \phi < \frac{\pi}{2}$
(2.182)

We can condense this summary further to



FIGURE 2.11. Curves of constant |p| and ϕ .
RER LER

$$Q_c = \tan \phi$$
 $W_c = \tan \phi$
 $\gamma_c = -\frac{\pi}{2}$ $\gamma_c = \frac{\pi}{2}$
 $r = |\sec \phi|$ $r = |\sec \phi|$
 $\pi < \phi < 2\pi$ $0 < \phi < \pi$
(2.183)

if we adopt the convention that Q_c and W_c can take on negative values. Then γ_c is simply increased by π whenever tan ϕ becomes negative.

We see from this summary that on the combined q and w plane the q-plane circles for $\pi < \phi < \frac{3}{2}\pi$ coincide with the w-plane circles for $\frac{1}{2}\pi < \phi < \pi$. Likewise the q-plane circles for $\frac{3}{2}\pi < \phi < 2\pi$ coincide with the w-plane circles for $0 < \phi < \frac{1}{2}\pi$.

Figure 2.11 shows the circles of constant |p| and constant phase angle ϕ on the q and w planes. The constant |p| circles are the same for both planes. The constant ϕ circles are labeled with the value of ϕ and either R or L, meaning, respectively, right elliptical polarization, in which case the chart is considered the q plane, and left elliptical polarization, in which case it is the w plane.

We have not plotted all of the contours of constant Re (p), Im (p), AR, τ , |p|, and ϕ on the same polarization chart because on a small chart this would be confusing. On a large chart, however, these plots allow one to obtain quickly any polarization parameter from a knowledge of others.

2.22. STOKES PARAMETERS

In his studies of partially polarized (quasi-monochromatic) light, Stokes introduced four quantities to characterize the amplitude and polarization of a wave. For strictly monochromatic waves these Stokes parameters are

$$S_{0} = |E_{x}|^{2} + |E_{y}|^{2}$$
(a)

$$S_{1} = |E_{x}|^{2} - |E_{y}|^{2}$$
(b)

$$S_{2} = 2|E_{x}| |E_{y}| \cos \phi$$
(c)

$$S_{3} = 2|E_{x}| |E_{y}| \sin \phi$$
(d)

where $|E_x|$, $|E_y|$, and ϕ are, as defined previously, the component amplitudes and phase difference of the wave.

It is obvious that the parameters are sufficient to describe both amplitude and polarization. Parameter S_0 gives the amplitude directly, while $|E_r|$ and $|E_y|$ can be found from S_0 and S_1 . Then ϕ may be determined from either S_2 or S_3 . Only three of the equations are independent since it is easily seen from (2.184) that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{2.185}$$

It is not difficult to relate the Stokes parameters to the terms used previously for describing the polarization ellipse. From (2.43) and the definitions of the Stokes parameters, we get

$$\sin 2\delta = \frac{2|E_x| |E_y| \sin \phi}{|E_x|^2 + |E_y|^2} = \frac{S_3}{S_0}$$
$$S_3 = S_0 \sin 2\delta$$
(2.186)

or

where δ is the auxiliary angle from which we found the axial ratio of the polarization ellipse.

From (2.40) for the tilt angle of the ellipse and (2.39) we find

$$\tan 2\tau = \tan 2\alpha \cos \phi = \frac{2 \tan \alpha \cos \phi}{1 - \tan^2 \alpha}$$
$$= \frac{2|E_x| |E_y| \cos \phi}{|E_x|^2 - |E_y|^2} = \frac{S_2}{S_1}$$
(2.187)

If (2.186) and (2.187) are substituted into (2.185) we are led to

$$S_1 = S_0 \cos 2\delta \cos 2\tau \tag{2.188}$$

Substitution of this equation into (2.187) in turn gives

$$S_2 = S_0 \cos 2\delta \sin 2\tau \tag{2.189}$$

2.23. THE POINCARÉ SPHERE

Collecting some of the previous equations, namely,

 $S_0 = |E_x|^2 + |E_y|^2$ (2.184a)

$$S_1 = S_0 \cos 2\delta \cos 2\tau \qquad (2.188)$$

$$S_2 = S_0 \cos 2\delta \sin 2\tau \tag{2.189}$$

$$S_3 = S_0 \sin 2\delta \tag{2.186}$$



FIGURE 2.12. The Poincaré sphere.

suggests a geometrical interpretation of the Stokes parameters. S_1 , S_2 , and S_3 are the Cartesian coordinates of a point on a sphere of radius S_0 . Then 2δ and 2τ are the latitude and azimuth angles measured to the point. This interpretation was introduced by Poincaré, and the sphere is called the Poincaré sphere. Such a sphere, with the Stokes parameters and the angles 2δ and 2τ , is shown in Fig. 2.12.

Since we can describe the amplitude of a wave by S_0 and its polarization by S_1 , S_2 , and S_3 , it is clear that any wave can be described by a point on the Poincaré sphere. To every state of polarization there corresponds one point on the sphere and vice versa.

2.24. SPECIAL POINTS ON THE POINCARÉ SPHERE

Left Circular

For this case

$$|E_x| = |E_y| \qquad \phi = \frac{1}{2}\pi$$

Then the Stokes parameters become

SPECIAL POINTS ON THE POINCARÉ SPHERE

$$S_{0} = |E_{x}|^{2} + |E_{y}|^{2} = 2|E_{x}|^{2}$$

$$S_{1} = |E_{x}|^{2} - |E_{y}|^{2} = 0$$

$$S_{2} = 2|E_{x}||E_{y}|\cos\phi = 0$$

$$S_{3} = 2|E_{x}||E_{y}|\sin\phi = 2|E_{x}|^{2} = S_{0}$$

and the point representing the polarization of this wave is the north pole (the +z axis) of the Poincaré sphere.

Right Circular

For this case

$$|E_x| = |E_y| \qquad \phi = -\frac{1}{2}\pi$$
$$S_0 = 2|E_x|^2$$
$$S_1 = S_2 = 0$$
$$S_3 = -S_0$$

which is the south pole of the sphere.

Left Elliptic

For this we have

 $0 < \phi < \pi$

and it follows from (2.184d) that

 $S_{3} > 0$

and all points for left elliptic polarizations are plotted on the upper hemisphere.

Right Elliptic

For this,

$$\pi < \phi < 2\pi$$
 $S_3 < 0$

and right elliptic polarization points are in the lower hemisphere.

Linear

For linear polarizations, if $|E_x|$ and $|E_y|$ are nonzero, then

 $\phi = 0, \pi$

and from (2.184d)

 $S_3 = 0$

and all linear polarization points are at the equator.

For linear vertical polarization the polarization point is at the -x-axis intersection with the sphere and for linear horizontal it is at the +x-axis intersection. The +y-axis intersection corresponds to linear polarization with a tilt angle of $\frac{1}{4}\pi$, and the -y-axis intersection to a tilt angle of $\frac{3}{4}\pi$.

Conjugate Point

If a wave with polarization p is represented by S_1 , S_2 , and S_3 on the Poincaré sphere, what point, S'_1 , S'_2 , S'_3 , represents p^* ? From

$$p = j \; \frac{|E_y|}{|E_x|} \; e^{j\phi}$$

we note that

$$p^* = -j \frac{|E_y|}{|E_x|} e^{-j\phi} = j \frac{|E_y|}{|E_x|} e^{-j(\phi+\pi)}$$

and the primed Stokes parameters are

$$S'_{0} = |E_{x}|^{2} + |E_{y}|^{2} = S_{0}$$

$$S'_{1} = |E_{x}|^{2} - |E_{y}|^{2} = S_{1}$$

$$S'_{2} = 2|E_{x}||E_{y}|\cos(-\phi + \pi) = -S_{2}$$

$$S'_{3} = 2|E_{x}||E_{y}|\sin(-\phi + \pi) = S_{2}$$

We see from Fig. 2.12 that this is a reflection of the first point in the xz plane.

Cross-Polarized Point

We will see later that the polarization ratio p' = -1/p has a special significance. From

OTHER RELATIONSHIPS BETWEEN THE VARIABLES

$$p' = -\frac{1}{p} = j \left| \frac{E_x}{E_y} \right| e^{-j\phi}$$

we find the Stokes parameters of the transformed point to be

$$S'_{0} = |E_{x}|^{2} + |E_{y}|^{2} = S_{0}$$

$$S'_{1} = |E_{y}|^{2} - |E_{x}|^{2} = -S_{1}$$

$$S'_{2} = 2|E_{x}| |E_{y}| \cos(-\phi) = S_{2}$$

$$S'_{3} = 2|E_{x}| |E_{y}| \sin(-\phi) = -S_{3}$$

2.25. OTHER RELATIONSHIPS BETWEEN THE VARIABLES

From (2.184c) and (2.184d) it is noted that

$$\phi = \tan^{-1} \frac{S_3}{S_2} \tag{2.190}$$

and from (2.184a) and (2.184b) it follows that

$$|E_x| = \sqrt{\frac{1}{2}(S_0 + S_1)}$$
 (a)
 $|E_y| = \sqrt{\frac{1}{2}(S_0 - S_1)}$ (b) (2.191)

From these equations we easily find that

$$p = \frac{-S_3 + jS_2}{S_0 + S_1} \quad (a)$$

$$q = \frac{S_1 - jS_2}{S_0 - S_3} \quad (b)$$
(2.192)

We may also find p and q in terms of the angles 2δ and 2τ on the Poincaré sphere. Substituting (2.186), (2.188), and (2.189) into (2.192) leads quickly to

$$p = \frac{-\sin 2\delta + j \cos 2\delta \sin 2\tau}{1 + \cos 2\delta \cos 2\tau}$$
(a)

$$q = \frac{\cos 2\delta \cos 2\tau - j \cos 2\delta \sin 2\tau}{1 - \sin 2\delta}$$
(b)

Finally, we note that the Poincaré sphere angles in terms of the Stokes parameters are

$$2\delta = \sin^{-1} \frac{S_3}{S_0} \quad (a)$$

$$2\tau = \tan^{-1} \frac{S_2}{S_1} \quad (b)$$
(2.194)

2.26. MAPPING THE POINCARÉ SPHERE ONTO A PLANE

Since the state of wave polarization can be represented by a point on the Poincaré sphere as well as by a point on the p, q, or w planes, it is not surprising that the Poincaré sphere may be mapped onto these complex planes. Before carrying out the mapping, however, let us find the Stokes parameters in terms of p. From (2.192a) we get

$$|p|^{2} = pp^{*} = \frac{S_{2}^{2} + S_{3}^{2}}{(S_{0} + S_{1})^{2}}$$

and using (2.185)

$$|p| = \sqrt{\frac{S_0 - S_1}{S_0 + S_1}} \tag{2.195}$$

Solving for S_1/S_0 gives

$$\frac{S_1}{S_0} = \frac{1 - |p|^2}{1 + |p|^2} = \frac{1 - |P|^2}{1 + |P|^2}$$
(2.196)

From

$$p = jP = j \; \frac{|E_y|}{|E_x|} \; e^{j\phi}$$

and the relations

$$S_2 = 2|E_x| |E_y| \cos \phi$$
 (c)
 $S_3 = 2|E_x| |E_y| \sin \phi$ (d)
(2.184)

we see that

$$S_{2} = 2|E_{x}|^{2} \operatorname{Re}(P) = 2|E_{x}|^{2} \operatorname{Im}(p) \quad (a)$$

$$S_{3} = 2|E_{x}|^{2} \operatorname{Im}(P) = -2|E_{x}|^{2} \operatorname{Re}(p) \quad (b)$$
(2.197)

Substituting (2.197) into

MAPPING THE POINCARÉ SPHERE ONTO A PLANE

$$\frac{S_2^2}{S_0^2} + \frac{S_3^2}{S_0^2} = 1 - \frac{S_1^2}{S_0^2} = \frac{4|p|^2}{(1+|p|^2)^2}$$
(2.198)

which results from (2.185) and (2.196), we obtain

$$|E_x|^2 = \frac{S_0}{1+|p|^2} \tag{2.199}$$

If (2.199) is substituted back into (2.197), there results

$$\frac{S_2}{S_0} = \frac{2 \operatorname{Im}(p)}{1 + |p|^2} = \frac{2 \operatorname{Re}(P)}{1 + |P|^2} \quad (a)$$

$$\frac{S_3}{S_0} = \frac{-2 \operatorname{Re}(p)}{1 + |p|^2} = \frac{2 \operatorname{Im}(P)}{1 + |P|^2} \quad (b)$$

We assert now that the Poincaré sphere may be mapped onto the p and P planes by a stereographic projection. The proof will be deferred until the projection is described. First, as an aid in orienting the Poincaré sphere, we tabulate the parameters for various special polarizations:

Polarization	р	Р	q	w	S_1	S_2	S_3
Right circular	1	-j	0	8	0	0	$-S_0$
Left circular	$^{-1}$	j	00	0	0	0	S_{0}
Linear vertical $(\tau = \frac{1}{2}\pi)$	j∞	00	-1	$^{-1}$	$-S_0$	0	0
Linear horizontal $(\tau = 0)$	0	0	1	1	So	0	0
Linear $(\tau = \frac{1}{4}\pi)$	j	1	-j	-j	0	So	0
Linear $(\tau = \frac{3}{4}\pi)$	-j	$^{-1}$	j	j	0	$-S_0$	0

The stereographic projection of a point on the Poincaré sphere onto the p or P plane is shown in Fig. 2.13. The sphere is oriented so that its north pole-south pole axis is parallel to the real axis of the p plane (imaginary axis of the P plane). Points are projected onto the plane by a ray from the sphere point farthest from the plane. Note that this projection point itself projects to ∞ on the polarization plane. At this time it might be wise to refer back to Fig. 2.12 to note that S_1 , S_2 , and S_3 are rectangular coordinates of a point on the sphere measured, respectively, along the x, y, and z axes.

Now the Stokes parameters and the Poincaré sphere give both amplitude and polarization information about the wave while points on the p plane describe only the wave polarization, so we expect something to be lost in the mapping, and this expectation is justified. Since the projection of the south pole ($S_1 = S_2 = 0$, $S_3 = -S_0$) onto the plane gives the point p = 1, the sphere radius must be



FIGURE 2.13. Stereographic projection of the Poincaré sphere onto the p and P planes.

$$S_0 = \frac{1}{2}$$
 (2.201)

and we give up amplitude information, commonly not of great interest in polarization problems, in going from the sphere to the polarization plane.

It is easy to see from the preceding table and Fig. 2.13 that all of the special points on the sphere project to the correct values of p and P on the polarization planes, and thus if the stereographic projection is valid, we have oriented the sphere properly on the plane. It has not yet been established, however, that the stereographic projection itself is valid. To show this we will find from Fig. 2.13 a graphical relationship between the Stokes parameters and their projection on the p plane. We will then compare these results with (2.195), (2.196), and (2.200) to see if the mapping is valid for the general case.

From Fig. 2.13

$$\sin \theta = \sqrt{\frac{S_2^2 + S_3^2}{(S_1 + S_0)^2 + S_2^2 + S_3^2}} = \sqrt{\frac{S_0^2 - S_1^2}{2S_0^2 + 2S_0S_1}} = \sqrt{\frac{S_0 - S_1}{2S_0}} \quad (2.202)$$

Then

$$|p| = 2S_0 \tan \theta = 2S_0 \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = 2S_0 \sqrt{\frac{S_0 - S_1}{S_0 + S_1}}$$
(2.203)

If we recall that $S_0 = \frac{1}{2}$, we see that (2.203) is identical to (2.195). Since (2.196) was derived from (2.195), it also must be satisfied by the mapping. From Fig. 2.13 we note that for the point to be projected,

$$S_1 > 0$$
, $S_2 > 0$, $S_3 < 0$

and for the corresponding polarization plane point

$$\operatorname{Re}(p) > 0, \quad \operatorname{Im}(p) > 0$$

Then it follows that

$$S_{2} = (\text{positive constant}) \operatorname{Im} (p) = C \operatorname{Im} (p) \quad (a)$$

$$S_{3} = (\text{negative constant}) \operatorname{Re} (p) = -C \operatorname{Re} (p) \quad (b)$$
(2.204)

where the two constants are equal in magnitude. We see that (2.204) agrees with (2.200) except for our lack of knowledge of the constant C.

To determine this constant, we can derive from (2.203) the relation

$$\frac{S_1}{S_0} = \frac{1 - |p|^2}{1 + |p|^2}$$

which is, of course, (2.196), as we know it must be. Substituting (2.196) and (2.204) into

$$S_1^2 + S_2^2 + S_3^2 = S_0^2 \tag{2.185}$$

gives

$$S_0^2 \left[\frac{1 - |p|^2}{1 + |p|^2} \right]^2 + C^2 [\operatorname{Im}(p)]^2 + C^2 [\operatorname{Re}(p)]^2 = S_0^2$$

from which it follows that

$$C = \frac{2S_0}{1+|p|^2} \tag{2.205}$$

and, if we use this constant in (2.204), we get

$$\frac{S_2}{S_0} = \frac{2 \operatorname{Im}(p)}{1 + |p|^2} \quad (a)$$

$$\frac{S_3}{S_0} = \frac{-2 \operatorname{Re}(p)}{1 + |p|^2} \quad (b)$$
(2.206)



FIGURE 2.14. Poincaré sphere projection from below.

which are the same as (2.200). We conclude then that the stereographic projection of the Poincaré sphere onto the *p* plane, with the sphere oriented as in Fig. 2.13, gives the correct correspondence between a polarization point on the *p* plane and a polarization point on the Poincaré sphere.

We have discussed the p plane with little mention of the P plane. Figure 2.13 also shows this plane which is, of course, related to the p plane by

$$p = jP \tag{2.89}$$

The p and P planes of Fig. 2.13 have an uncommon orientation as we look down on the planes. If Re (p) is plotted to the right, then Im (p) is plotted downward. This is not of great consequence, but if desired it can be changed by using left-handed coordinates for the Poincaré sphere (not generally desirable) or by projecting the Poincaré sphere onto the plane from below as shown in Fig. 2.14.

2.27. MAPPING ONTO THE q AND w PLANES

Since the transformation between p and q is linear, we might expect that the Poincaré sphere can be mapped onto the q plane, and this is correct. To obtain the transformation we substitute

$$p = p' + jp'' \qquad q = q' + jq''$$
 (2.207)

into (2.91), obtaining

$$p' = \frac{1 - |q|^2}{1 + 2q' + |q|^2} \quad (a)$$

$$p'' = \frac{-2q''}{1 + 2q' + |q|^2} \quad (b)$$
(2.208)

If (2.208) is substituted into (2.196) and (2.200), which give the Stokes parameters in terms of p, and if we note from (2.91) that

$$|p|^{2} = \frac{1 - q' + |q|^{2}}{1 + 2q' + |q|^{2}}$$
(2.209)

we find

$$\frac{S_1}{S_0} = \frac{2q'}{1+|q|^2} \tag{2.210}$$

$$\frac{S_2}{S_0} = \frac{-2q''}{1+|q|^2} \tag{2.211}$$

$$\frac{S_3}{S_0} = -\frac{1-|q|^2}{1+|q|^2} \tag{2.212}$$

Before discussing this mapping, we might as well develop the mapping equations for the w plane, since this is the analog of the q plane for left-handed polarizations. Substitution of $w = 1/q^*$ into (2.210), (2.111), and (2.212) leads to

$$\frac{S_1}{S_0} = \frac{2w'}{|w|^2 + 1} \qquad (a)$$

$$\frac{S_2}{S_0} = \frac{-2w''}{|w|^2 + 1} \qquad (b) \qquad (2.213)$$

$$\frac{S_3}{S_0} = -\frac{|w|^2 - 1}{|w|^2 + 1} \qquad (c)$$

If we tabulate the equations for the Stokes parameters in terms of p, q, and 2, we have

$$\frac{S_1}{S_0} = \frac{1 - |p|^2}{1 + |p|^2} = \frac{2 \operatorname{Re}(q)}{1 + |q|^2} = \frac{2 \operatorname{Re}(w)}{1 + |w|^2} \quad (a)$$

$$\frac{S_2}{S_0} = \frac{2 \operatorname{Im}(p)}{1 + |p|^2} = \frac{-2 \operatorname{Im}(q)}{1 + |q|^2} = \frac{-2 \operatorname{Im}(w)}{1 + |w|^2} \quad (b) \quad (2.214)$$

$$\frac{S_3}{S_0} = \frac{-2 \operatorname{Re}(p)}{1 + |p|^2} = -\frac{1 - |q|^2}{1 + |q|^2} = \frac{1 - |w|^2}{1 + |w|^2} \quad (c)$$





We see from this summary that the S_1 (or x) coordinate in the p-plane transformation is analogous to the $-S_3$ (or -z) coordinate in the q-plane transformation, and so on. It is then obvious that we can project the Poincaré sphere onto the q or w plane by appropriately interchanging the axes of the Poincaré sphere of Fig. 2.13. The appropriate interchanges are:

p Plane	q Plane	w Plane
x	-z	-z or $+z$
У	-y	-y or $+y$
Z	-x	x or $-x$

The first coordinate interchange for the w plane gives a left-handed coordinate system for the sphere. Therefore, following a hint given at the end of the preceding section, we reverse the coordinates to give a right-handed system, and to compensate we reverse the direction in which Im (w) is plotted.

The projections of the Poincaré sphere onto the q and w planes are shown in Fig. 2.15. We may see from Fig. 2.15(a) that the lower hemisphere (z < 0) maps into the unit circle on the q plane. This is expected since the lower hemisphere contains all right elliptically polarized points, and so does the unit circle on the q plane. From Fig. 2.15(b) we see that the hemisphere, z > 0, which contains all left elliptically polarized points, maps into the unit circle on the w plane.



FIGURE 2.16. Combined projections onto the q and w planes.

The two figures of 2.15 may be combined into one drawing, and such a combination is shown in Fig. 2.16. In this figure the q plane is above the sphere and the w plane below, but the reverse is also possible. The origin of the projection ray for the q plane is $S_3 = +S_0$ and for the w plane the ray origin is $S_3 = -S_0$.

In Fig. 2.17 the Poincaré sphere and the three projection planes p, q, and w are shown together. This drawing suggests that three other planes could be used in a simple fashion to describe polarization states since Fig. 2.17 shows only three of a possible six planes. The transformations from S_1 , S_2 , and S_3 to the remaining three parameters may be found from Fig. 2.17.

Carrying out the transformation to one of the possible planes, the r plane of Fig. 2.18, with a stereographic projection from the point $S_2 = -S_0$ on the Poincaré sphere yields the equations



FIGURE 2.17. Poincaré sphere with three projection planes.



FIGURE 2.18. Mapping onto the r plane.

$$\frac{S_1}{S_0} = \frac{2 \operatorname{Im} (r)}{1 + |r|^2} \quad (a)$$

$$\frac{S_2}{S_0} = \frac{1 - |r|^2}{1 + |r|^2} \quad (b)$$

$$\frac{S_3}{S_0} = \frac{-2 \operatorname{Re} (r)}{1 + |r|^2} \quad (c)$$

$$r = \frac{-S_3 + jS_1}{S_0 + S_2} \quad (d)$$

It is recommended that the reader develop these transformations.

In Fig. 2.17 we note that the p and q planes are each stereographic projections of polarization points on the Poincaré sphere, and we recall that p and q are related by a bilinear transform. The r plane is another projection of the Poincaré sphere, and it would not surprise us to find that r is a bilinear transform of p or q. To check this point, substitute (2.214) into (2.215d). Then we have

$$r = \frac{1 - |q|^2 + j2q'}{1 + |q|^2 - 2q''} = \frac{(1 + jq)(1 + jq^*)}{(1 + jq)(1 - jq^*)}$$
$$= \frac{1 + jq^*}{1 - jq^*} = \frac{w + j1}{w - j1}$$
(2.216)

REPRESENTATION OF WAVE POLARIZATION

We could obtain other polarization parameters either by interchanging the real and imaginary axes of one of the polarization planes (as in the relationship p = jP), a process which is not very significant, or by choosing one of the remaining projection planes of the Poincaré sphere. It appears, however, that the p and q parameters (together with w for left-handed polarizations) are the most useful, p because it is so easily obtained in terms of rectangular wave components. Figure 2.17 shows the utility of q and w. Since all left-handed polarizations are plotted on the hemisphere z > 0 and all right-handed polarizations be plotted in a finite region, and similarly for left-handed polarization on the w plane.

REFERENCES

- 1. M. Born and E. Wolf, Principles of Optics, 3rd ed., Pergamon Press, Oxford, 1965.
- R. P. Feynman, et al., *The Feynman Lectures on Physics*, Addison-Wesley, Reading, MA, 1973.
- 3. J. D. Kraus, Antennas, McGraw-Hill, New York, 1950.
- V. H. Rumsey, "Transmission Between Elliptically Polarized Antennas," Proc. IRE, Vol. 39, No. 5, pp. 535-540, May, 1951.

PROBLEMS

- **2.1.** A left-handed elliptically polarized wave has a tilt angle of 30° and an axial ratio of 2. Find the polarization ratios p and q (or w).
- 2.2. An antenna radiates a wave in the z direction with a polarization ratio $P = 2e^{j\pi/4}$. The antenna is rotated in the xy plane by 30° from the x axis toward the y axis. Find the new value of P.
- **2.3.** Find the points on the q (or w) plane Smith chart representing the two polarization states of problem 2.2. Can you draw a general conclusion about the change in polarization state caused by rotating the antenna?
- 2.4. The polarization parameters P (or p) and q (or w) developed in this chapter have physical significance, as the ratios, respectively, of rectangular and circular field components. Try to find a physical meaning for the parameter r as defined in this chapter.
- **2.5.** Consider the stereographic projection of the Poincaré sphere from the point $S_1 = +S_0$ onto a complex plane. Find the equation, corresponding to (2.215), for the complex variable on this plane in terms of the Stokes parameters.

PROBLEMS

- 2.6. The q-plane representation of polarization comes from a stereographic projection from the $S_3 = S_0$ point on the Poincaré sphere onto a plane parallel to the xy plane, where S_3 is measured on the z axis. Consider an orthographic projection, rather than stereographic, onto the same plane. Describe the resulting chart. Is it useful in working with polarization states?
- 2.7. Carry out the suggested work of Section 2.2 to prove that the polarization ellipse for the magnetic field is identical to that of the electric field except that it is rotated by angle $\frac{1}{2}\pi$ around the z axis.
- **2.8.** Make the substitution outlined immediately after Eq. (2.199) to obtain (2.200).
- **2.9.** Use (2.192a) and the relation between p and q to obtain (2.192b).

POLARIZATION MATCHING OF ANTENNAS

3.1. INTRODUCTION

It is obvious that when two antennas are used in a communication system, they should be matched in polarization so that the available power at the receiving antenna can be fully utilized. In this chapter a polarization match factor is developed and is given in terms of the standard polarization parameters. The relationship between the effective length of a receiving antenna and the field components of an incident wave necessary to yield maximum power is developed. In the final section a step-by-step process is outlined for obtaining the power received when two antennas are mismatched in polarization and do not have their main beam axes pointing at each other. It is interesting that this topic is not treated in most of the standard texts on antenna theory.

3.2. EFFECTIVE LENGTH OF AN ANTENNA

The electric field in the radiation zone of a dipole antenna, which is short compared to a free-space wavelength, as shown in Fig. 3.1, is given by

$$E_{\theta}(r,\,\theta,\,\phi) = \frac{jZ_0 I\ell}{2\lambda r} \,e^{-jkr}\sin\theta \tag{3.1}$$

where Z_0 is the intrinsic impedance of free space, k the free space propagation constant, λ the wavelength, and I the current into the antenna terminals of Fig. 3.1.



Equation (3.1) may be generalized to give the transmitted field of any antenna; thus [1]

$$\mathbf{E}'(r,\,\theta,\,\phi) = \frac{jZ_0I}{2\lambda r}\,e^{-jkr}\mathbf{h}(\theta,\,\phi) \tag{3.2}$$

where θ is the colatitude angle of Fig. 3.1 and ϕ is the azimuth angle. The current *I* is an input current at an arbitrary pair of terminals. Equation (3.2) describes a general antenna in terms of its *effective length* $\mathbf{h}(\theta, \phi)$. The effective length does not necessarily correspond to a physical length of the antenna, although there is a correspondence for the dipole. In fact, comparison of (3.1) and (3.2) shows that the effective length of the short dipole antenna is

$$\mathbf{h} = \mathbf{u}_{\theta} h_{\theta} = \mathbf{u}_{\theta} \,\ell \,\sin\theta \tag{3.3}$$

We see from this that **h** is not fixed for an antenna but depends on the angle θ (and more generally on ϕ) at which we measure the radiated field.

As mentioned, I is the current at an arbitrary pair of terminals, and it follows that the effective length **h** depends on the choice of terminal pair. Further, we note that if **E**' is to describe an elliptically polarized field, it must be complex, and therefore **h** is a complex vector. With a proper choice of coordinate system, **E**' and **h** will have only two components since in the radiation zone **E**' has no radial component.

3.3. RECEIVED VOLTAGE

We defined the effective length of an antenna in terms of the radiation field produced by it. We will show in this section that the open-circuit voltage induced in the antenna by an externally produced field is proportional to this effective length; in fact, some authors *define* effective length in terms of the open-circuit voltage produced when the antenna is receiving a wave.

By the principle of reciprocity, if two antennas are fed by equal current

sources, the open-circuit voltage produced across the terminals of antenna 1 by the current source feeding antenna 2 is the same as the open-circuit voltage produced across the terminals of antenna 2 by the current source feeding antenna 1.

We apply this principle to determine the open-circuit voltage across the terminals of our general antenna, whose transmitted field is given by (3.2), when it receives an incident wave. The general antenna is assumed to interact with a short dipole, as shown in Fig. 3.2, together with the coordinate system to be used and the assumed current directions and voltage polarities. Note that the same rectangular coordinate system is used for both antennas, although θ' , ϕ' are not equal to θ , ϕ .

First, we let the general antenna fed by a current source of 1 A transmit a wave toward the short dipole. Its field at the dipole is



FIGURE 3.2. General antenna and short dipole: (a) antennas; (b) coordinates and unit vectors.

RECEIVED VOLTAGE

and the open-circuit voltage across the dipole terminals, with the polarity shown in Fig. 3.2, is

$$V_2 = \mathbf{E}' \cdot \boldsymbol{\ell} \tag{3.5}$$

where ℓ is the vector length of the dipole. Here we are considering that the dipole has infinitesimal length so that the incident field E' is constant over the dipole length. The dipole may be arbitrarily oriented, but since E' has no radial component, a radial component of the dipole length will not contribute to the received voltage. Then, still using the coordinate system of Fig. 3.2,

$$V_2 = E'_{\theta}\ell_{\theta} + E'_{\phi}\ell_{\phi} \tag{3.6}$$

where the dipole components ℓ_{θ} and ℓ_{ϕ} are given by

$$\ell_{\theta} = \mathbf{u}_{\theta} \cdot \boldsymbol{\ell} \qquad \ell_{\phi} = \mathbf{u}_{\phi} \cdot \boldsymbol{\ell} \tag{3.7}$$

Combining (3.4) and (3.5) gives the voltage induced across the open dipole terminals by the incident wave from the general antenna:

$$V_2 = \frac{jZ_0}{2\lambda r} e^{-jkr} \mathbf{h} \cdot \boldsymbol{\ell}$$
(3.8)

Next, suppose the dipole fed by a 1-A current source is transmitting, and the general antenna, with open terminals, is receiving. The field produced at the general antenna (1) is given by

$$E_{\theta'}^{i} = \frac{jZ_{0}}{2\lambda r} e^{-jkr} \ell_{\theta'} \quad (a)$$

$$E_{\phi'}^{i} = \frac{jZ_{0}}{2\lambda r} e^{-jkr} \ell_{\phi'} \quad (b)$$
(3.9)

where we continue to use the same coordinate system but note that θ' , ϕ' differ from θ , ϕ . We note from Fig. 3.2 that although the angles just mentioned are different, we have

$$\mathbf{u}_{\theta} = \mathbf{u}_{\theta'}, \qquad \mathbf{u}_{\phi} = -\mathbf{u}_{\phi'}, \tag{3.10}$$

It follows that

$$E^{i}_{\theta} = E^{i}_{\theta} \quad (a) \qquad E^{i}_{\phi} = -E^{i}_{\phi} \quad (c)$$

$$\ell_{\theta} = \ell_{\theta} \quad (b) \qquad \ell_{\phi} = -\ell_{\phi} \quad (d)$$

and therefore the wave incident on antenna 1 is

$$E_{\theta}^{i} = \frac{jZ_{0}}{2\lambda r} e^{-jkr} \ell_{\theta} \qquad (a)$$

$$-E_{\phi}^{i} = \frac{jZ_{0}}{2\lambda r} e^{-jkr} (-\ell_{\phi}) \qquad (b)$$

or

$$\mathbf{E}^{i} = \frac{jZ_{0}}{2\lambda r} e^{-jkr} \ell \tag{3.13}$$

The open-circuit voltage induced in the general antenna (1) is V_1 , which by the reciprocity theorem is equal to V_2 , as given by (3.8). Then, from (3.8) and the reciprocity theorem, we get

$$V_1 = V_2 = \frac{jZ_0}{2\lambda r} e^{-jkr} \ell \cdot \mathbf{h}$$
(3.14)

and if we recognize that the first part of this expression is the incident wave of (3.13), we get an expression for the received voltage across the open terminals of antenna 1 in terms of the incident field, \mathbf{E}^{i} , and its effective length **h**. It is

$$V_1 = \mathbf{E}^i \cdot \mathbf{h} \tag{3.15}$$

It should be noted that in (3.15) both \mathbf{E}^{i} and \mathbf{h} are measured in the same coordinate system (in contrast to a situation to be discussed later). The voltage V_{1} is in general a complex phasor voltage, since both \mathbf{E}^{i} and \mathbf{h} are complex. Finally, in specifying the effective length \mathbf{h} of an antenna, a terminal pair at which input current is to be measured must be specified. Then V_{1} is the open-circuit voltage measured across those terminals.

3.4. MAXIMUM RECEIVED POWER

It is reasonable to believe from looking at (3.15) that by proper selection of the effective length **h** of a receiving antenna, we can increase the open-circuit voltage and hence the received power. If we neglect the extraneous problems of, for example, impedance mismatch, the power received by the general antenna is proportional to the square of the magnitude of the open-circuit voltage; thus using an equality rather than a proportional symbol (an inconsequential action since we will later consider a power ratio), we have

$$W = VV^* = |\mathbf{E}^i \cdot \mathbf{h}|^2 = |h_\theta E^i_\theta + h_\phi E^i_\phi|^2$$
(3.16)

where an appropriate coordinate system is used so that **h** has only two components and so only two are needed for \mathbf{E}^{i} .

Let us define

$$h_{\theta} = |h_{\theta}|e^{j\alpha} \quad (a) \quad \frac{h_{\phi}}{|h_{\phi}|} = \frac{h_{\theta}}{|h_{\theta}|}e^{j\delta_{1}} \quad (b)$$

$$\frac{E_{\theta}^{i}}{|E_{\theta}^{i}|} = \frac{h_{\theta}}{|h_{\theta}|}e^{j\beta} \quad (c) \quad \frac{E_{\phi}^{i}}{|E_{\phi}^{i}|} = \frac{E_{\theta}^{i}}{|E_{\theta}^{i}|}e^{j\delta_{2}} \quad (d)$$
(3.17)

with δ_1 the phase angle by which h_{ϕ} leads h_{θ} , β the angle by which E_{θ}^i leads h_{θ} , and δ_2 the angle by which E_{ϕ}^i leads E_{θ}^i . Using these equations, the received power, (3.16), becomes

$$W = ||h_{\theta}||E_{\theta}^{i}| + |h_{\phi}||E_{\phi}^{i}|e^{j(\delta_{1}+\delta_{2})}|^{2}$$
(3.18)

where the angle $2\alpha + \beta$ has been removed as common to both terms in the sum.

Clearly W is a maximum, from (3.18), if

$$\delta_1 + \delta_2 = 0 \tag{3.19}$$

and has value

$$W_m = [|h_{\theta}| |E_{\theta}^i| + |h_{\phi}| |E_{\phi}^i|]^2$$
(3.20)

Now W_m can be maximized further, for a fixed incident wave, \mathbf{E}' , by varying $|h_{\theta}|$ or $|h_{\phi}|$. Certainly, however, there must be some constraint on $|h_{\theta}|$ and $|h_{\phi}|$; otherwise, W_m could be made as great as we please by increasing $|h_{\theta}|$ and $|h_{\phi}|$ arbitrarily. To determine this constraint, return to (3.2), which gives the transmitted field of an antenna in terms of its effective length. The transmitted Poynting vector, from (3.2), is obviously proportional to $\mathbf{h} \cdot \mathbf{h}^*$. Then a reasonable constraint on an antenna is that this Poynting vector remain constant as we vary \mathbf{h} . Therefore, we vary \mathbf{h} to maximize W_m in (3.20) with the constraint

$$\mathbf{h} \cdot \mathbf{h}^* = |h_{\theta}|^2 + |h_{\phi}|^2 = C$$
(3.21)

Substituting (3.21) in (3.20), we get

$$W_m = [|h_{\theta}| |E_{\theta}^i| + (C - |h_{\theta}|^2)^{1/2} |E_{\phi}^i|]^2$$
(3.22)

and differentiating with respect to $|h_{\theta}|$ in order to maximize W_m gives

$$\frac{\partial W_m}{\partial |h_\theta|} = 2[|h_\theta| |E_\theta^i| + (C - |h_\theta|^2)^{1/2} |E_\phi^i|]^2 \left[|E_\theta^i| - \frac{|h_\theta| |E_\phi^i|}{(C - |h_\theta|^2)^{1/2}} \right] = 0$$
(3.23)

from which it is clear that

$$|E_{\theta}^{i}| - \frac{|h_{\theta}| |E_{\phi}^{i}|}{(C - |h_{\theta}|^{2})^{1/2}} = |E_{\theta}^{i}| - \frac{|h_{\theta}|}{|h_{\phi}|} |E_{\phi}^{i}| = 0$$
(3.24)

or

$$\frac{|h_{\theta}|}{|h_{\phi}|} = \frac{|E_{\theta}'|}{|E_{\phi}'|}$$
(3.25)

It seems quite reasonable that (3.25) will give maximum received power, rather than minimum, since if \mathbf{E}^{i} has a large θ component, we would expect a large h_{θ} to give best reception. However, we will substitute (3.25) into (3.20) to see if the received power is maximum. We rewrite (3.20) as

$$W_{m} = |h_{\theta}|^{2} |E_{\theta}^{i}|^{2} + |h_{\phi}|^{2} |E_{\phi}^{i}|^{2} + |h_{\theta}| |E_{\theta}^{i}| |h_{\phi}| |E_{\phi}^{i}| + |h_{\theta}| |E_{\theta}^{i}| |h_{\phi}| |E_{\phi}^{i}|$$
(3.26)

In the third term of W_m , we make the substitution from (3.25) that

$$\left|E_{\theta}^{i}\right|\left|h_{\phi}\right| = \left|h_{\theta}\right|\left|E_{\phi}^{i}\right| \tag{3.27}$$

and in the fourth term of W_m we make the substitution in reverse. Thus

$$W_{mm} = |h_{\theta}|^{2} |E_{\theta}^{i}|^{2} + |h_{\phi}|^{2} |E_{\phi}^{i}|^{2} + |h_{\theta}|^{2} |E_{\phi}^{i}|^{2} + |h_{\phi}|^{2} |E_{\theta}^{i}|^{2}$$
$$= (|h_{\theta}|^{2} + |h_{\phi}|^{2})(|E_{\theta}^{i}|^{2} + |E_{\phi}^{i}|^{2})$$
(3.28)

This is quite obviously maximum power rather than minimum. Finally, (3.28) may be written as

$$W_{mm} = (\mathbf{h} \cdot \mathbf{h}^*)(\mathbf{E}^i \cdot \mathbf{E}^{i*}) = |\mathbf{h}|^2 |\mathbf{E}^i|^2$$
(3.29)

There may be some concern on the part of the reader that (3.21) is a legitimate constraint. While we vary **h** for maximum *received* power, why should we apply a constraint that is meaningful only for the *transmitting* case? For this reason we return to (3.20), assume that **h** is fixed, and vary \mathbf{E}^{i} in order to maximize the power. Now in this situation it is quite clear that we can cause only a fixed power density at the receiving antenna. Therefore

$$\mathbf{E}^i \cdot \mathbf{E}^{i*} = C \tag{3.30}$$

Using this equation makes W_m become

POLARIZATION MATCH FACTOR 117

$$W_m = [|h_{\theta}| |E_{\theta}^i| + |h_{\phi}|(C - |E_{\theta}^i|^2)^{1/2}]^2$$
(3.31)

and differentiation with respect to $|E_{\theta}^{i}|$ gives

$$\frac{\partial W_m}{\partial |E_{\theta}^i|} = 2[|h_{\theta}||E_{\theta}^i| + |h_{\phi}|(C - |E_{\theta}^i|^2)^{1/2}] \Big[|h_{\theta}| - \frac{|h_{\phi}||E_{\theta}^i|}{(C - |E_{\theta}^i|^2)^{1/2}} \Big] = 0$$
(3.32)

from which it follows that

$$\frac{|h_{\theta}|}{|h_{\phi}|} = \frac{|E_{\theta}^{i}|}{|E_{\phi}^{i}|}$$
(3.33)

which is the same condition we arrived at previously.

Equation (3.33) gives one condition on h for maximum power reception. The other is given by

$$\delta_1 + \delta_2 = 0 \tag{3.19}$$

where δ_1 is the angle by which h_{ϕ} leads h_{θ} and δ_2 is the angle by which E_{ϕ}^i leads E_{θ}^i .

We rewrite (3.17b) as

$$\frac{h_{\phi}}{h_{\theta}} = \frac{|h_{\phi}|}{|h_{\theta}|} e^{j\delta_1}$$
(3.34)

and substitute (3.19) and (3.33) into it, obtaining

$$\frac{h_{\phi}}{h_{\theta}} = \frac{|E_{\phi}^{i}|}{|E_{\theta}^{i}|} e^{-j\delta_{2}}$$
(3.35)

From (3.17d) we recognize that the last term is $E_{\phi}^{i*}/E_{\theta}^{i*}$, so that the relationship between **h** and **E**^{*i*} for maximum received power is

$$\frac{h_{\phi}}{h_{\theta}} = \left(\frac{E_{\phi}^{i}}{E_{\theta}^{i}}\right)^{*} \tag{3.36}$$

3.5. POLARIZATION MATCH FACTOR

If we maintain the same degree of impedance matching for an antenna as we vary its polarization properties, then the ratio of actual power received to that received under the most favorable circumstances of matched polarization is, from (3.16) and (3.29),

$$\rho = \frac{|\mathbf{E}^{i} \cdot \mathbf{h}|^{2}}{|\mathbf{E}^{i}|^{2} |\mathbf{h}|^{2}}$$
(3.37)

We will refer to ρ as the *polarization match factor*, although it is sometimes called the polarization efficiency. Its range is obviously

$$0 \le \rho \le 1$$

The polarization match factor shows how well a receiving antenna of effective length \mathbf{h} is matched in polarization to an incoming wave. Now let us recognize that the incoming wave was transmitted by another antenna and so introduce the polarization properties of that antenna into the problem.

Figure 3.3 shows two antennas in a transmit-receive configuration. The transmitting antenna (1) will be described in its polarization properties by the right-handed coordinate system x, y, z adjacent to antenna 1 since the polarization of a wave is normally based on a right-handed coordinate system with one axis pointing in the direction of wave travel. The receiving antenna will be described by the right-handed ξ , η , ζ system. The antennas need not have their main beams pointed at each other, but the z and ζ axes are parallel and each points at the other antenna.

The incident wave from antenna 1 may be written as



FIGURE 3.3. Antennas and coordinate systems used in development of polarization match factor: (a) antennas; (b) coordinate systems.

POLARIZATION MATCH FACTOR

where p_1 is the modified polarization ratio [2] of the incident wave produced by antenna 1.

The *polarization ratio of an antenna* is defined as the polarization ratio of the field it transmits (far field). Therefore, p_1 is the modified polarization ratio of antenna 1. It is a function of θ and ϕ , the colatitude and azimuth angles measured for the transmission direction.

If antenna 2 were transmitting, its radiated wave could be written as

$$\mathbf{E}_{2} = E_{02}(a_{2}\mathbf{u}_{\xi} + b_{2}e^{j\phi_{2}}\mathbf{u}_{\eta})$$
(3.39)

where we use appropriate coordinates ξ , η , ζ for the wave propagating in the ζ direction, toward the first antenna. Equation (3.39) may be written in terms of the modified polarization ratio of antenna 2 as

$$\mathbf{E}_{2} = E_{02}a_{2} \Big(\mathbf{u}_{\xi} + j \, \frac{b_{2}}{a_{2}} \, e^{j\phi_{2}} e^{-j\pi/2} \mathbf{u}_{\eta} \Big) = E_{02}a_{2} \big(\mathbf{u}_{\xi} + p_{2} e^{-j\pi/2} \mathbf{u}_{\eta} \big) \quad (3.40)$$

where p_2 is the modified polarization ratio of antenna 2 in the ζ direction, using the appropriate right-handed coordinates at antenna 2.

Now the transmitted field (3.40) is related to the vector length of antenna 2 by

$$\mathbf{E}_2 = \frac{jZ_0 I_2}{2\lambda r} \, e^{-jkr} \mathbf{h}_2 \tag{3.41}$$

Therefore, \mathbf{h}_2 becomes, using (3.40) and (3.41),

$$\mathbf{h}_{2} = \frac{2\lambda r}{jZ_{0}I_{2}} e^{jkr} \mathbf{E}_{2} = \frac{2\lambda r}{jZ_{0}I_{2}} e^{jkr} E_{02} a_{2} (\mathbf{u}_{\xi} + p_{2}e^{-j\pi/2}\mathbf{u}_{\eta})$$

= $h_{02} (\mathbf{u}_{\xi} + p_{2}e^{-j\pi/2}\mathbf{u}_{\eta})$ (3.42)

where

$$h_{02} = \frac{2\lambda r}{jZ_0 I_2} e^{jkr} E_{02} a_2 \tag{3.43}$$

Let us return now to the situation where antenna 1 transmits and antenna 2 receives. The open-circuit voltage across the appropriate terminals of 2 is

$$V_2 = \mathbf{E}_1^i \cdot \mathbf{h}_2 = E_{1\xi}^i h_{2\xi} + E_{1\eta}^i h_{2\eta}$$
(3.44)

If we note from Fig. 3.3 that

$$E_{1\xi}^{i} = -E_{1x}^{i}$$
 (a) $E_{1\eta}^{i} = E_{1y}^{i}$ (b) (3.45)

and use the field and effective length components from (3.38) and (3.42), we

get for the open-circuit voltage

$$V_{2} = -E_{01}a_{1}h_{02} + E_{01}a_{1}p_{1}e^{-j\pi/2}h_{02}p_{2}e^{-j\pi/2}$$
$$= -E_{01}a_{1}h_{02}(1+p_{1}p_{2})$$
(3.46)

We find also, from (3.38) and (3.42), that

$$|\mathbf{E}^{i}|^{2} = |E_{01}a_{1}|^{2}(\mathbf{u}_{x} + p_{1}e^{-j\pi/2}\mathbf{u}_{y}) \cdot (\mathbf{u}_{x} + p_{1}^{*}e^{j\pi/2}\mathbf{u}_{y})$$
$$= |E_{01}a_{1}|^{2}(1 + p_{1}p_{1}^{*})$$
(3.47)

and

$$|\mathbf{h}_{2}|^{2} = |h_{02}|^{2} (\mathbf{u}_{\xi} + p_{2}e^{-j\pi/2}\mathbf{u}_{\eta}) \cdot (\mathbf{u}_{x} + p_{2}^{*}e^{j\pi/2}\mathbf{u}_{\eta})$$
$$= |h_{02}|^{2} (1 + p_{2}p_{2}^{*})$$
(3.48)

If we substitute (3.44), (3.46), (3.47), and (3.48) into the polarization match factor (3.37), we obtain

$$\rho = \frac{|E_{01}a_1h_{02}|^2|1+p_1p_2|^2}{|E_{01}a_1|^2(1+p_1p_1^*)|h_{02}|^2(1+p_2p_2^*)} = \frac{(1+p_1p_2)(1+p_1^*p_2^*)}{(1+p_1p_1^*)(1+p_2p_2^*)} \quad (3.49)$$

It is worthwhile to repeat that the definition of p_1 uses wave components measured in a right-handed system with the z axis pointing away from antenna 1 and toward 2. In defining p_2 , we used a right-handed system with the η axis parallel to and in the same direction as the y axis and with the ζ axis pointing toward antenna 1.

3.6. POLARIZATION MATCH FACTOR: SPECIAL CASES

Polarization-Matched Antennas

If we have two polarization-matched antennas in a transmit-receive system, the polarization match factor of (3.49) is equal to 1. (Note that it may change if the orientation of one of the antennas is changed.) Thus

$$\rho = \frac{(1+p_1p_2)(1+p_1^*p_2^*)}{(1+p_1p_1^*)(1+p_2p_2^*)} = 1$$
(3.50)

Cross multiplying, expanding, and canceling terms gives

$$(p_1^* - p_2)(p_2^* - p_1) = 0$$

which has a solution

$$p_1 = p_2^*$$
 (3.51)

In terms of the circular polarization ratio q,

$$q_1 = \frac{1 - p_1}{1 + p_1} = \frac{1 - p_2^*}{1 + p_2^*} = q_2^*$$
(3.52)

and the axial ratios and tilt angles of the polarization ellipse of the two antennas are related by

$$AR_{1} = \left| \frac{1 + |q_{1}|}{1 - |q_{1}|} \right| = \left| \frac{1 + |q_{2}|}{1 - |q_{2}|} \right| = AR_{2} \quad (a)$$

$$\tau_{1} = \frac{\theta_{1}}{2} = -\frac{\gamma_{1}}{2} = +\frac{\gamma_{2}}{2} = -\tau_{2} \qquad (b)$$

Now, in (3.53), τ_1 and τ_2 are described in different coordinate systems, as shown in Fig. 3.3. It is obvious from Fig. 3.3 that the condition $\tau_1 = -\tau_2$ means that the major axes of the two polarization axes coincide. Equation (3.52) also shows that the rotation senses of the two polarization ellipses are the same when described in the appropriate coordinate systems. Having the same rotation sense, using the coordinate systems of Fig. 3.3, means that if we think of both antennas transmitting a right elliptic wave, for example, the two waves will appear to rotate in opposite directions at a point in space at which they "meet."

Cross-Polarized Antennas

Two antennas in a transmit-receive configuration that are so polarized that no signal is received are said to be cross-polarized. For this situation

$$\rho = 0 = \frac{(1+p_1p_2)(1+p_1^*p_2^*)}{(1+p_1p_1^*)(1+p_2p_2^*)}$$
(3.54)

from which it follows that

$$p_1 = -\frac{1}{p_2} \tag{3.55}$$

Solving for q, we obtain

$$q_1 = \frac{1 - p_1}{1 + p_2} = \frac{p_2 + 1}{p_2 - 1} = -\frac{1}{q_2}$$
(3.56)

We see immediately that the rotation senses of the polarization ellipses of the

antennas are opposite (so that if both antennas transmitted simultaneously, their field vectors would appear to rotate in the same direction).

The axial ratios are, from (3.56),

$$AR_{1} = \left| \frac{1 + |q_{1}|}{1 - |q_{1}|} \right| = \left| \frac{|q_{2}| + 1}{|q_{2}| - 1} \right| = AR_{2}$$
(3.57)

Also from (3.56)

$$Q_1 e^{j\gamma_1} = -\frac{1}{Q_2 \varepsilon^{j\gamma_2}} = \frac{1}{Q_2} e^{-j(\gamma_2 + \pi)}$$

so that

 $\gamma_1 = -\gamma_2 + \pi$

and

$$\tau_1 = -\frac{1}{2}\gamma_1 = \frac{1}{2}\gamma_2 \mp \frac{1}{2}\pi = -\tau_2 \mp \frac{1}{2}\pi$$
(3.58)

Bearing in mind that τ_1 is measured from the x axis toward the y axis in Fig. 3.3, and τ_2 is measured from the ξ axis toward the η axis in Fig. 3.3, we see that (3.58) means that the major axis of one polarization ellipse coincides with the minor axis of the other.

Identical, Polarization-Matched Antennas

It would seem to be quite easy to define identical antennas, but surprisingly there is a degree of arbitrariness involved. When placed side by side and oriented similarly, identical antennas are indistinguishable except by position. Although not overly precise, this definition is quite clear. Now we make the assumption that they are placed into a transmit-receive configuration by rotating one of them by π radians around a *vertical* axis (the y axis of Fig. 3.3). We might also consider a rotation about a horizontal axis or the major or minor axis of the polarization ellipse—hence the arbitrariness mentioned but we will rotate first about the vertical axis.

For identical antennas, before one is rotated into a receiving position,

$$h_{x1} = h'_{x2} \qquad h_{y1} = h'_{y2}$$

$$p_1 = j \frac{h_{y1}}{h_{x1}} = p'_2 = j \frac{h'_{y2}}{h'_{x2}}$$
(3.59)

where the primes are used with the parameters of the antenna to be rotated.

After antenna 2 is rotated 180° about the y axis of Fig. 3.3, its new length components are

$$h_{x2} = -h'_{x2} \qquad h_{y2} = h'_{y2} \tag{3.60}$$

Changing these components to the ξ , η , ζ coordinates, which are now appropriate for antenna 2, we have

$$h_{\xi 2} = -h_{x2} = h'_{x2}$$
 (a)
 $h_{n2} = h_{y2} = h'_{y2}$ (b) (3.61)

Then the new value for the polarization ratio p_2 is

$$p_2 = j \frac{h_{\eta 2}}{h_{\xi 2}} = j \frac{h'_{y 2}}{h'_{x 2}} = p'_2$$
(3.62)

and thus p_2 is unchanged by rotation about a vertical axis. A little thought will show that, in general, the major axes of the ellipses no longer coincide.

Let the antennas be not only identical but polarization matched. Then they must satisfy (3.51), (3.59), and (3.62), or

$$p_1 = p_2^*$$
 (a) $p_1 = p_2$ (b) (3.63)

and also

$$q_1 = q_2^*$$
 (a) $q_1 = q_2$ (b) (3.64)

We conclude then that identical, polarization-matched antennas must have

$$p_1 = p_2 = \text{real quantity}$$
 $q_1 = q_2 = \text{real quantity}$ (3.65)

from which it follows that

$$\gamma_1 = \gamma_2 = 0, \ \pi \qquad \tau_1 = \tau_2 = 0, \ -\frac{1}{2}\pi$$
 (3.66)

We see that the major axis of the polarization ellipse must be either vertical or horizontal if the antennas are to be identical (in our sense of rotation about a vertical axis) and matched. This does not exclude circularly polarized antennas for which the concept of major axis is not meaningful.

Antennas that are identical and cross-polarized must satisfy

$$p_1 = -\frac{1}{p_2}$$
 $q_1 = -\frac{1}{q_2}$ (3.67)

which gives

$$p_1 = q_1 = \pm j1 \tag{3.68}$$

from which we find that

$$AR \to \infty$$

$$\tau = \frac{1}{4}\pi \qquad \frac{3}{4}\pi$$
(3.69)

which describe linearly polarized antennas with tilt angles of 45° or 135°.

When we rotated one of the antennas about a vertical axis, we found that the two antennas would be matched if their major axes were vertical. Perhaps then if we started with identical, side-by-side antennas and rotated one of them about its major axis, we would obtain polarization matching.

Let the polarization parameters *before* rotation be p_1, p'_2, \ldots , where the primes are used with the parameters of the antenna to be rotated. Then for identical antennas

$$p_1 = p'_2 \qquad q_1 = q'_2$$

 $AR_1 = AR'_2 \qquad \tau_1 = \tau'_2$
(3.70)

After antenna 2 is rotated about its major axis, we recognize that the axial ratio and rotation sense are unchanged, that is,

$$AR_2 = AR_2'$$

$$|q_2| = |q_2'|$$
(3.71)

Since the rotation takes place about the major axis, obviously the major axis does not change, but as Fig. 3.3 shows, the tilt angle in the ξ , η , ζ system is measured oppositely from that in x, y, z. Therefore, after rotation the new tilt angle is given by

$$\tau_2 = -\tau_2' \tag{3.72}$$

Equations (3.71) and (3.72) lead to

$$q_2 = q_2^{\prime *} \tag{3.73}$$

and from (3.70)

$$q_2 = q_1^* \tag{3.74}$$

MATCH FACTOR IN OTHER FORMS

Now, from (3.52), this is the condition for polarization matching. We conclude then that identical antennas (indistinguishable when placed side by side and similarly oriented) will be matched in polarization if one of them is rotated 180° about its major polarization axis to bring it to a receive position.

3.7. MATCH FACTOR IN OTHER FORMS

Since the modified polarization ratio p is not always the most convenient parameter for an antenna, we need the equation for ρ in terms of other parameters. If we make the substitution

$$p = \frac{1-q}{1+q} \tag{2.91}$$

in (3.49), the match factor is found in terms of q to be

$$\rho = \frac{(1+q_1q_2)(1+q_1^*q_2^*)}{(1+q_1q_1^*)(1+q_2q_2^*)}$$
(3.75)

It is not surprising that ρ has the same form in q as in p, since the form for q in terms of p is the same as for p in terms of q.

Now (3.75) is valid for any value of q, but nonetheless if we treat left elliptic polarizations by means of the parameter w(|q| > 1, |w| < 1), we might wish ρ in terms of w. Substituting

$$p = \frac{w^* - 1}{w^* + 1} \tag{2.138}$$

into (3.49) leads to

$$\rho = \frac{(1+w_1w_2)(1+w_1^*w_2^*)}{(1+w_1w_1^*)(1+w_2w_2^*)}$$
(3.76)

an equation that also has the same form as (3.49).

A mixed form in terms of q_1 and w_2 or q_2 and w_1 might also be useful. Replacing q_2 in (3.75) by $1/w_2^*$ leads to

$$\rho = \frac{(w_2^* + q_1)(w_2 + q_1^*)}{(1 + q_1q_1^*)(1 + w_2w_2^*)}$$
(3.77)

and interchanging subscripts in (3.77) gives

$$\rho = \frac{(w_1^* + q_2)(w_1 + q_2^*)}{(1 + q_2 q_2^*)(1 + w_1 w_1^*)}$$
(3.78)

All four forms (3.75)–(3.78) are valid for any value of q and w, but it

would be natural to use (3.75) for both antennas right handed, (3.76) for both left handed, and (3.77) or (3.78) for one left and the other right handed.

We may find ρ in terms of axial ratios and tilt angles of the polarization ellipses, but here we must be careful about the rotation sense of the ellipses, since axial ratio and tilt alone are not sufficient to describe the antenna polarization. Consider first that both antennas are right handed. We have, from (2.107), if |q| < 1,

$$AR = \frac{1 + |q|}{1 - |q|}$$
(3.79)

In (3.75) we write

$$q = |q|e^{-j2\tau} (3.80)$$

and ρ becomes

$$\rho = \frac{1+2|q_1q_2|\cos 2(\tau_1+\tau_2)+|q_1q_2|^2}{(1+|q_1|^2)(1+|q_2|^2)}$$
(3.81)

and if |q| from (3.79) is substituted into (3.81), there results, after some manipulation,

$$\rho = \frac{(AR_1AR_2 + 1)^2 + (AR_1 + AR_2)^2 + (AR_1^2 - 1)(AR_2^2 - 1)\cos 2(\tau_1 + \tau_2)}{2(AR_1^2 + 1)(AR_2^2 + 1)}$$
(3.82)

If both antennas are left handed, |w| < 1, we find from (2.147) that

$$AR = \frac{1+|w|}{1-|w|}$$
(3.83)

We also have

$$w = |w|e^{-j2\tau}$$
(3.84)

If we substitute (3.83) and (3.84), which have the same forms as (3.79) and (3.80), into (3.76), which has the same form as (3.75), it is obvious that (3.82) will result. Therefore, (3.82) holds if both antennas are right handed or if both are left handed.

If antenna 1 is right handed and 2 is left handed, we substitute

$$AR_{1} = \frac{1 + |q_{1}|}{1 - |q_{1}|} \qquad q_{1} = |q_{1}|e^{-j2\tau_{1}} \quad (a)$$

$$AR_{2} = \frac{1 + |w_{2}|}{1 - |w_{2}|} \qquad w_{2} = |w_{2}|e^{-j2\tau_{2}} \quad (b)$$
(3.85)

into the mixed form (3.77) and obtain

$$\rho = \frac{(AR_1AR_2 - 1)^2 + (AR_1 - AR_2)^2 + (AR_1^2 - 1)(AR_2^2 - 1)\cos 2(\tau_1 + \tau_2)}{2(AR_1^2 + 1)(AR_2^2 + 1)}$$
(3.86)

for the match factor in terms of axial ratios and tilt angles.

If antenna 1 is left handed and 2 is right handed, we could make the appropriate substitutions in (3.78), and (3.86) would again result.

In terms of axial ratios and tilt angles, the polarization match factor may be found from (3.82) if both antennas have the same polarization rotation sense and from (3.86) if they are of opposite sense.

Finally, we note that ρ may be written in terms of a, b, and ϕ of (2.70) as

$$\rho = \frac{1 - 2(b_1/a_1)(b_2/a_2)\cos(\phi_1 + \phi_2) + [(b_1/a_1)(b_2/a_2)]^2}{[1 + (b_1/a_1)^2][1 + (b_2/a_2)^2]}$$
(3.87)

in terms of left and right circular components as

$$\rho = \frac{1 + 2(L_1/R_1)(L_2/R_2)\cos(\theta_1 + \theta_2) + [(L_1/R_1)(L_2/R_2)]^2}{[1 + (L_1/R_1)^2][1 + (L_2/R_2)^2]} \quad (3.88)$$

and in terms of the common polarization ratio P(=-jp) as

$$\rho = \frac{(1 - P_1 P_2)(1 - P_1^* P_2^*)}{(1 + P_1 P_1^*)(1 + P_2 P_2^*)}$$
(3.89)

3.8. CONTOURS OF CONSTANT MATCH FACTOR

Examination of one of the equations for polarization match factor, say (3.75), leads one to suspect that, for a given value of q_1 , a range of q_2 values might give the same polarization match factor ρ . We consider this point further, holding ρ constant and using

$$q = Q e^{j\gamma} \tag{2.135}$$

With the substitution (2.135), (3.75) may be put into the form

$$Q_{2}^{2} - \frac{2Q_{1}\cos\left(\gamma_{1} + \gamma_{2}\right)}{(Q_{1}^{2} + 1)\rho - Q_{1}^{2}} Q_{2} + \frac{Q_{1}^{2}}{\left[(Q_{1}^{2} + 1)\rho - Q_{1}^{2}\right]^{2}} = \frac{(1 - \rho)\rho}{\left[\rho - Q_{1}^{2}/(1 + Q_{1}^{2})\right]^{2}}$$
(3.90)

Comparison to the standard form (2.154) shows (3.90) to represent a family
of circles on the q plane (actually the q_2 plane), with center and radius

$$Q_{2c}, \gamma_{2c} = \frac{Q_1}{(Q_1^2 + 1)\rho - Q_1^2}, -\gamma_1 \quad (a)$$

$$r = \left| \frac{[(1 - \rho)\rho]^{1/2}}{\rho - Q_1^2/(1 + Q_1^2)} \right| \quad (b)$$
(3.91)

Now, Eq. (3.90) is in the correct form (2.154) only if

$$(Q_1^2+1)\rho - Q_1^2 > 0$$

or

$$\rho > \frac{Q_1^2}{1 + Q_1^2} \tag{3.92}$$

If this condition on ρ is not met, (3.90) may be rewritten as

$$Q_{2}^{2} - \frac{2Q_{1}\cos\left(\gamma_{1} + \gamma_{2} + \pi\right)}{\left|(1 + Q_{1}^{2})\rho - Q_{1}^{2}\right|} Q_{2} + \frac{Q_{1}^{2}}{\left[(1 + Q_{1}^{2})\rho - Q_{1}^{2}\right]^{2}} = \frac{(1 - \rho)\rho}{\left[\rho - Q_{1}^{2}/(1 + Q_{1}^{2})\right]^{2}}$$
(3.93)

which represents a family of circles with center and radius

$$Q_{2c}, \gamma_{2c} = \frac{Q_1}{\left|(1+Q_1^2)\rho - Q_1^2\right|}, -\gamma_1 + \pi \quad \text{(a)}$$

$$r = \left|\frac{\left[(1-\rho)\rho\right]^{1/2}}{\rho - Q_1^2/(1+Q_1^2)}\right| \qquad \text{(b)}$$

if

$$\rho < \frac{Q_1^2}{1+Q_1^2} \tag{3.95}$$

These equations, (3.91) and (3.94), represent contours of constant match factor on the q plane representing antenna 2 in terms of given polarization characteristics for antenna 1, the other part of a communications system. Given a transmitting antenna with polarization q_1 , this family of circles on the q_2 plane allows us to determine quickly the effect of varying the receiving antenna polarization. The words *transmitting* and *receiving* were used above for clarity. Of course, it makes no difference which antenna transmits and which receives. It is tempting at this point to draw these constant ρ contours on the q plane, but the temptation should be resisted, since we used an equation, (3.75), in which both antennas are represented by q, which prevents our considering a left-handed antenna 2 if we wish to remain in the unit circle on the q_2 plane. We therefore will go on to consider, before drawing the contours, antennas described by w_1 and w_2 , q_1 and w_2 , and q_2 and w_1 .

For antennas described by w_1 and w_2 we should substitute

$$w = W e^{j\gamma} \tag{3.96}$$

into (3.76), but (3.96) is identical in form to (2.135) and (3.76) to (3.75). The result is that contours of constant ρ on the w_2 plane are circles with centers and radii given by (3.91) and (3.94) using W instead of Q.

$$W_{2c}, \gamma_{2c} = \frac{W_1}{(1+W_1^2)\rho - W_1^2}, -\gamma_1 \quad (a)$$

$$r = \left| \frac{[(1-\rho)\rho]^{1/2}}{\rho - W_1^2/(1+W_1^2)} \right| \quad (b)$$
(3.97)

if

$$\rho > \frac{W_1^2}{1 + W_1^2} \tag{3.98}$$

and

$$W_{2c}, \gamma_{2c} = \frac{W_1}{|(1+W_1^2)\rho - W_1^2|}, -\gamma_1 + \pi \quad (a)$$

$$r = \left| \frac{[(1-\rho)\rho]^{1/2}}{\rho - W_1^2/(1+W_1^2)} \right| \qquad (b)$$

if

$$\rho < \frac{W_1^2}{1 + W_1^2} \tag{3.100}$$

Now we take the situation of antenna 1 described by q_1 and antenna 2 by w_2 . We substitute (2.135) and (3.96) into (3.77) and obtain

٩

$$\rho = \frac{W_2^2 + 2Q_1W_2\cos\left(\gamma_1 + \gamma_2\right) + Q_1^2}{(1 + Q_1^2)(1 + W_2^2)}$$
(3.101)

This equation may be put into the standard circle form

POLARIZATION MATCHING OF ANTENNAS

$$W_{2}^{2} - \frac{2Q_{1}\cos\left(\gamma_{1} + \gamma_{2}\right)}{(1+Q_{1}^{2})\rho - 1} W_{2} + \frac{Q_{1}^{2}}{\left[(1+Q_{1}^{2})\rho - 1\right]^{2}} = \frac{(1-\rho)\rho}{\left[\rho - 1/(1+Q_{1}^{2})\right]^{2}}$$
(3.102)

if the denominator of the second term is positive. If the denominator is negative, the correct form is

$$W_{2}^{2} - \frac{2Q_{1}\cos\left(\gamma_{1} + \gamma_{2} + \pi\right)}{\left|(1 + Q_{1}^{2})\rho - 1\right|} W_{2} + \frac{Q_{1}^{2}}{\left[(1 + Q_{1}^{2})\rho - 1\right]^{2}} = \frac{(1 - \rho)\rho}{\left[\rho - 1/(1 + Q_{1}^{2})\right]^{2}}$$
(3.103)

These equations represent circles on the W_2 plane with centers and radii

$$W_{2c}, \gamma_{2c} = \frac{Q_1}{(1+Q_1^2)\rho - 1}, -\gamma_1 \quad (a)$$

$$r = \left| \frac{[(1-\rho)\rho]^{1/2}}{\rho - 1/(1+Q_1^2)} \right| \quad (b)$$

if

$$\rho > \frac{1}{1 + Q_1^2} \tag{3.105}$$

and

$$W_{2c}, \gamma_{2c} = \frac{Q_1}{|(1+Q_1^2)\rho - 1|}, -\gamma_1 + \pi \quad (a)$$

$$r = \left| \frac{[(1-\rho)\rho]^{1/2}}{\rho - 1/(1+Q_1^2)} \right| \qquad (b)$$

if

$$\rho < \frac{1}{1 + Q_1^2} \tag{3.107}$$

Finally we take the sense of antenna 1 being described by w_1 and antenna 2 by q_2 . Substitution of (2.135) and (3.96) into (3.76) gives quickly

$$\rho = \frac{W_1^2 + 2Q_2W_1\cos\left(\gamma_1 + \gamma_2\right) + Q_2^2}{(1 + W_1^2)(1 + Q_2^2)}$$
(3.108)

an equation that is the same as (3.101) with Q_1 replaced by W_1 and W_2 replaced by Q_2 . We may therefore use the circle equations (3.104) and (3.106) with the same replacements, giving

$$Q_{2c}, \gamma_{2c} = \frac{W_1}{(1+W_1^2)\rho - 1}, -\gamma_1 \quad (a)$$

$$r = \left| \frac{[(1-\rho)\rho]^{1/2}}{\rho - 1/(1+W_1^2)} \right| \quad (b)$$
(3.109)

if

$$\rho > \frac{1}{1 + W_1^2} \tag{3.110}$$

and

$$Q_{2c}, \gamma_{2c} = \frac{W_1}{|(1+W_1^2)\rho - 1|}, -\gamma_1 + \pi \quad (a)$$

$$r = \left| \frac{[(1-\rho)\rho]^{1/2}}{\rho - 1/(1+W_1^2)} \right| \quad (b)$$

if

$$\rho < \frac{1}{1 + W_1^2} \tag{3.112}$$

All of the preceding equations for centers and radii of the constant ρ circles were given in terms of circles on the q_2 and w_2 planes. Obviously, the designations 1 and 2 are arbitrary, so in any equation the subscripts may be interchanged. Also, it clearly makes no difference which antenna transmits and which receives.

We have not used any restriction that an antenna must have a particular rotation sense. However, to stay in the unit circle, if antenna 2 is right handed, we would normally use circle equations (3.91) and (3.94), or (3.109) and (3.111), which represents circles on the q plane, on which all polarizations fall within the unit circle.

We can combine these eight equation sets into two if we note first that (3.91) and (3.94) differ only by the magnitude sign in the form for Q_{2c} and in the value of γ_{2c} (and similarly for the other pairs of equations) and, second, if we recognize, for example, that if we use $W_1 = 1/Q_1$ in (3.91), it becomes

$$Q_{2c} = \frac{Q_1}{(1+Q_1^2)\rho - Q_1^2} = \frac{1/W_1}{(1+1/W_1^2)\rho - 1/W_1^2} = \frac{W_1}{(1+W_1^2)\rho - 1} \quad (3.113)$$

which is the same equation as (3.109a).

Combining the equations for the constant ρ circles appropriately leads to

$$Q_{2c} = \frac{Q_1}{|(1+Q_1^2)\rho - Q_1^2|} = \frac{W_1}{|(1+W_1^2)\rho - 1|}$$
(a)

$$\gamma_{2c} = -\gamma_1 \qquad \rho > \frac{Q_1^2}{1+Q_1^2} = \frac{1}{1+W_1^2}$$
(b)

$$\gamma_{2c} = -\gamma_1 + \pi \qquad \rho < \frac{Q_1^2}{1+Q_1^2} = \frac{1}{1+W_1^2}$$
(c)

$$r = \left| \frac{\left[(1-\rho)\rho \right]^{1/2}}{\rho - Q_1^2 / (1+Q_1^2)} \right| = \left| \frac{\left[(1-\rho)\rho \right]^{1/2}}{\rho - 1 / (1+W_1^2)} \right| \quad (d)$$

and

$$W_{2c} = \frac{W_1}{|(1+W_1^2)\rho - W_1^2|} = \frac{Q_1}{|(1+Q_1^2)\rho - 1|}$$
(a)

$$\gamma_{2c} = -\gamma_1 \qquad \rho > \frac{W_1^2}{1+W_1^2} = \frac{1}{1+Q_1^2}$$
(b)

$$W^2 \qquad (3.115)$$

$$\gamma_{2c} = -\gamma_1 + \pi \qquad \rho < \frac{W_1^2}{1 + W_1^2} = \frac{1}{1 + Q_1^2} \qquad (c)$$
$$r = \left| \frac{\left[(1 - \rho)\rho \right]^{1/2}}{\rho - W_1^2 / (1 + W_1^2)} \right| = \left| \frac{\left[(1 - \rho)\rho \right]^{1/2}}{\rho - 1 / (1 + Q_1^2)} \right| \qquad (d)$$

Example

As an aid in understanding these equations and the constant match factor curves, consider an example of antenna 1 right handed with

 $q_1 = \frac{1}{2}$

Substituting in (3.114) gives

$$Q_{2c} = \frac{2}{|5\rho - 1|}$$
$$\gamma_{2c} = \begin{cases} 0, & \rho > 0.2\\ \pi, & \rho < 0.2 \end{cases}$$
$$r = \frac{[(1 - \rho)\rho]^{1/2}}{|\rho - 0.2|}$$

and (3.115) yields

$$W_{2c} = \frac{2}{|5\rho - 4|}$$
$$\gamma_{2c} = \begin{cases} 0, & \rho > 0.8\\ \pi, & \rho < 0.8 \end{cases}$$
$$r = \frac{[(1-\rho)\rho]^{1/2}}{|\rho - 0.8|}$$

Figure 3.4 shows the circles of constant ρ determined from these equations. The constant ρ contours are labeled in decibels, with the negative sign omitted on the plot.

If we know that antenna 2 is right handed, it is only necessary to show the first set of circles on the q_2 plane. However, it is possible that we might wish to select a left elliptic antenna for antenna 2, so from the second set of equations the constant ρ circles are also drawn on the w_2 plane. When considered as the q_2 plane (antenna 2 right handed), the constant ρ contours are solid in Fig. 3.4. When considered as the w_2 plane, the constant ρ curves are dashed in Fig. 3.4.



FIGURE 3.4. The q_2 (or w_2) plane with curves of constant polarization match factor: antenna 1 right elliptic, $q_1 = 0.5$; antenna 2, right elliptic (solid curves) and left elliptic (dashed curves).

A study of this example and the resulting Fig. 3.4 illustrates various points worth considering:

1. The chart, Fig. 3.4, applies for only one value of Q_1 or W_1 .

2. The angle γ_1 of q_1 or w_1 may change, and the chart will remain valid. The circles of Fig. 3.4 are drawn with centers on the line $-\gamma_1, -\gamma_1 + \pi$. Then for an angle $\gamma_1 \neq 0$ the line of circle centers is simply rotated to $-\gamma_1$ by rotating the chart.

3. A wide range of antenna polarizations will result in the same polarization match, since in general one of the constant ρ circles spans a wide range of values of q_2 .

4. Some of the constant ρ circles lie completely in the unit circle, and some intersect it.

5. For those constant ρ circles that intersect the unit circle, we have right elliptic and left elliptic antennas with the same polarization match. This is shown in Fig. 3.4 by the intersection at the unit circle of solid (right-handed antenna) and dashed (left-handed antenna) constant ρ circles having the same value of ρ . On the unit circle itself, sense of rotation is meaningless, of course.

6. In the example used, with antenna 1 right elliptic and $Q_1 = \frac{1}{2}$, the greatest polarization loss (smallest ρ value) for any right elliptic receiving antenna (2) is 10 dB, and the smallest loss for any right-handed antenna is 0 dB. Greatest and smallest losses for a left-handed receiving antenna are ∞ and 0.46 dB. We see from this that the polarization match *may* be better between right and left elliptic antennas than between two right-handed antennas.

7. The circles for different ρ values never intersect if antenna 2 is right handed (or left handed), but right elliptic and left elliptic curves do intersect. For example, if antenna 2 is represented by the point at the center of the unit circle (circularly polarized), the ρ loss is 0.97 dB for antenna 2 right handed or 7 dB for antenna 2 left handed.

8. The two curves of 7 which intersect at the origin have "complementary" curves (opposite rotation sense and equal polarization loss) that are straight lines.

9. Curves outside the straight lines mentioned in 8 have reverse curvature.

10. The 3-dB-loss curves intersect at the unit circle bisector 90° from the center line of the circles.

Special Points

The process of constructing the constant ρ circles can be shortened by considering special ranges on the q_2 or w_2 plane.

First we note that we need consider only real values of q_1 since for q_1 having a general angle, we need only rotate the constant ρ curves for q_1 real.



On the unit circle of the q_2 or w_2 planes, the equations for ρ reduce to

$$\rho = \frac{1}{2} + \frac{Q_1 \cos \gamma_2}{1 + Q_1^2} = \frac{1}{2} + \frac{W_1 \cos \gamma_2}{1 + W_1^2}$$
(3.116)

Since values of ρ are the same on the real axis for both the q_2 and w_2 planes, the same equation holds for both planes.

On the real axis of either the q_2 or w_2 planes ρ values are given by

$$\rho = \begin{cases}
\frac{(1 \pm Q_1 Q_2)^2}{(1 + Q_1^2)(1 + Q_2^2)} = \frac{(Q_2 \pm W_1)^2}{(1 + Q_2^2)(1 + W_1^2)}, & q_2 = \pm Q_2 \\
\frac{(1 \pm W_1 W_2)^2}{(1 + W_1^2)(1 + W_2^2)} = \frac{(Q_1 \pm W_2)^2}{(1 + Q_1^2)(1 + W_2^2)}, & w_2 = \pm W_2
\end{cases}$$
(3.117)

where the upper signs are used for points to the right of the q_2 or w_2 plane origin and the lower for points to the left.

For given Q_1 or W_1 values (3.116) and (3.117) can be used to determine ρ for values of γ_2 on the unit circle and Q_2 or W_2 on the real axis. Circles may then be drawn through points of equal ρ values (since the circle center is known to be on the real axis). Alternatively, for known Q_1 , (3.116) and (3.117) may be equated to give γ_2 on the unit circle in terms of the equal ρ point, Q_2 , on the real axis.

Contours for Varying Q_1

Figure 3.5 shows a set of contours for ρ values of 0.1, 0.25, and 0.5 as Q_1 varies from 0 to 1. For clarity the q_2 and w_2 planes are shown separately. Contour labels are given only for the $Q_1 = 1$ pair, but they are obvious for all Q_1 values.

The choice of the range of Q_1 means that antenna 1 is right elliptic, ranging from right circular to linear.

We may see from the figure that for antenna 2 right handed, any value of q_2 leads to a match for which $\rho \ge 0.5$, if the transmitting antenna (1) is right circular. As the axial ratio of antenna 1 increases, the allowable region of q_2 , for a match $\rho \ge 0.5$, shrinks.

For antenna 2 left handed and antenna 1 right circular, no value of w_2 gives a match $\rho > 0.5$. As the axial ratio of antenna 1 increases, the allowable region of w_2 for a good polarization match increases.

3.9. THE POINCARÉ SPHERE AND POLARIZATION MATCH FACTOR

Since we have determined contours of constant polarization match factor ρ on the q and w planes, and since the planes are appropriate stereographic

projections of polarization points on the Poincaré sphere, we might expect to find constant polarization match curves on the Poincaré sphere itself, and we are not disappointed.

Two antennas in a transmit-receive configuration are polarization matched ($\rho = 1$) if $p_1 = p_2^*$. Thus if the points corresponding to p_1 and p_2^* (or p_1^* and p_2) coincide when plotted on the Poincaré sphere, complete polarization matching exists between the two antennas.

As a matter of notation, let us use p_a and p_b for antennas A and B to avoid confusion with the Stokes parameters notation S_1 , S_2 , S_3 .

The polarization match between antennas A and B is

$$\rho = \frac{(1+p_a p_b)(1+p_a^* p_b^*)}{(1+|p_a|^2)(1+|p_b|^2)}$$
(3.118)

where

$$p_{a} = j \frac{|E_{y}^{a}|}{|E_{x}^{a}|} e^{j\phi_{a}} \quad (a)$$

$$p_{b} = j \frac{|E_{\eta}^{b}|}{|E_{\xi}^{b}|} e^{j\phi_{b}} \quad (b)$$
(3.119)

with all quantities measured in the x, y, z or ξ , η , ζ coordinate systems of Fig. 3.3.

Substitution of (3.119) into (3.118) gives

 $\rho =$

$$\frac{|E_x^a|^2 |E_\xi^b|^2 + |E_y^a|^2 |E_\eta^b|^2 - 2|E_x^a| |E_y^a| |E_\xi^b| |E_\eta^b| (\cos \phi_a \cos \phi_b - \sin \phi_a \sin \phi_b)}{(|E_x^a|^2 + |E_y^a|^2)(|E_\xi^b|^2 + |E_\eta^b|^2)}$$
(3.120)

The Stokes parameters of the waves are

$$S_{0}^{a} = |E_{x}^{a}|^{2} + |E_{y}^{a}|^{2} \qquad S_{0}^{b} = |E_{\xi}^{b}|^{2} + |E_{\eta}^{b}|^{2}$$

$$S_{1}^{a} = |E_{x}^{a}|^{2} - |E_{y}^{a}|^{2} \qquad S_{1}^{b} = |E_{\xi}^{b}|^{2} - |E_{\eta}^{b}|^{2}$$

$$S_{2}^{a} = 2|E_{x}^{a}| |E_{y}^{a}| \cos \phi_{a} \qquad S_{2}^{b} = 2|E_{\xi}^{b}| |E_{\eta}^{b}| \cos \phi_{b}$$

$$S_{3}^{a} = 2|E_{x}^{a}| |E_{y}^{a}| \sin \phi_{a} \qquad S_{3}^{b} = 2|E_{\xi}^{b}| |E_{\eta}^{b}| \sin \phi_{b}$$
(3.121)

and if we use these relationships in (3.120), the result is

$$\rho = \frac{1}{2} \left(1 + \frac{S_1^a}{S_0^a} \frac{S_1^b}{S_0^b} - \frac{S_2^a}{S_0^a} \frac{S_2^b}{S_0^b} + \frac{S_3^a}{S_0^a} \frac{S_3^b}{S_0^b} \right)$$
(3.122)

Now if the Stokes parameters corresponding to polarizations p_a and p_b are

$$S_1^a, S_2^a, S_3^a$$
 and S_1^b, S_2^b, S_3^b

then the Stokes parameters corresponding to p_b^* are

$$S_1^b, -S_2^b, S_3^b$$

If the two points corresponding to p_a and p_b^* with the Stokes parameters S_1^a , S_2^a , S_3^a and S_1^b , $-S_2^b$, S_3^b are plotted on the Poincaré sphere and two rays drawn from the origin to these points, the angle between the rays is given by

$$\cos \beta = \frac{S_1^a}{S_0^a} \frac{S_1^b}{S_0^b} - \frac{S_2^a}{S_0^a} \frac{S_2^b}{S_0^b} + \frac{S_3^a}{S_0^a} \frac{S_3^b}{S_0^b}$$
(3.123)

Then

$$\cos\frac{\beta}{2} = \left(\frac{1+\cos\beta}{2}\right)^{1/2} = \frac{1}{\sqrt{2}} \left(1 + \frac{S_1^a}{S_0^a} \frac{S_1^b}{S_0^b} - \frac{S_2^a}{S_0^a} \frac{S_2^b}{S_0^b} + \frac{S_3^a}{S_0^a} \frac{S_3^b}{S_0^b}\right)^{1/2}$$
(3.124)

Comparing this equation to (3.122) shows that

$$\rho = \cos^2 \frac{1}{2}\beta \tag{3.125}$$

We see from this equation that if we consider two antennas arranged to transmit and receive, plot the polarization point (modified polarization ratio) of one antenna and the conjugate polarization point of the other on the Poincaré sphere using (2.196) and (2.200) to determine the Stokes parameters for the plot, then the polarization match factor ρ is the square of the cosine of half the angle measured at the sphere center defined by the points.

It is evident that we could draw on the Poincaré sphere contours of constant-polarization match factor for a two-antenna system. These contours are circles with center at the plotted polarization point of one of the antennas.

3.10. MATCH FACTOR USING ONE COORDINATE SYSTEM

Throughout this chapter we have consistently defined the polarization ratio of an antenna in terms of a right-handed coordinate system with the z axis pointing away from the antenna. When two antennas in a transmit-receive system were considered, as in Fig. 3.3, two right-handed coordinate systems, one for each antenna, were used, with the z and ζ axes pointing at each other. We defined polarization ratios using the two coordinate systems as POLARIZATION MATCH FACTOR: MISALIGNED ANTENNAS

$$p_1 = j \frac{E_y}{E_x}$$
 $p_2 = j \frac{E_\eta}{E_\xi}$ (3.126)

and obtained a polarization match factor

$$\rho = \frac{(1+p_1p_2)(1+p_1^*p_2^*)}{(1+p_1p_1^*)(1+p_2p_2^*)}$$
(3.49)

Now, some workers prefer to use only one coordinate system, the x, y, z system of Fig. 3.3, and define both antenna polarizations in this one system. The change is relatively simple. Let p'_2 be the modified polarization ratio of antenna 2 defined in the x, y, z coordinate system of Fig. 3.3. Then

$$p_2' = j \; \frac{E_y}{E_x}$$

But, from Fig. 3.3,

$$E_y = E_\eta \qquad E_x = -E_\xi \tag{3.127}$$

and

$$p_{2}' = j \, \frac{E_{y}}{E_{x}} = -j \, \frac{E_{\eta}}{E_{\xi}} = -p_{2} \tag{3.128}$$

Substitution into the equation for ρ gives

$$\rho = \frac{(1 - p_1 p_2')(1 - p_1^* p_2'^*)}{(1 + p_1 p_1^*)(1 + p_2' p_2'^*)}$$
(3.129)

We may note from this that for matched antennas, $p_1 = -p_2^{\prime*}$, and for cross-polarized antennas, $p_1 = 1/p_2^{\prime}$.

Similarly, we find, using the common polarization ratio P in (3.89), that if P'_2 is the polarization ratio of antenna 2 defined in the same x, y, z coordinate system used for antenna 1, the polarization match factor becomes

$$\rho = \frac{(1 + P_1 P_2')(1 + P_1^* P_2'^*)}{(1 + P_1 P_1^*)(1 + P_2' P_2'^*)}$$
(3.130)

3.11. POLARIZATION MATCH FACTOR: MISALIGNED ANTENNAS

In Section 3.5 and subsequent sections we considered two antennas in a transmit-receive configuration with aligned axes (see Fig. 3.3). Orientation of the antennas was arbitrary, but the fields and effective length components were known in the particular coordinate systems. In the general case the

radiated field components and effective length components will be known in some appropriate coordinate system for the antennas, and these coordinate systems will not be aligned. We must then transform antenna locations and field components to other coordinate systems before determining the polarization match factor.

Figure 3.6 shows the coordinate systems we will consider. The system without subscripts is a ground or reference system. The *a* system is appropriate to the transmitting antenna, with the radiated fields known in that system. The *b* system is rotated so that its *z* axis points toward the receiving antenna. Likewise, the *c* system is the natural one for the receiving antenna, the one in which its radiated field (or equivalently its effective length **h**) is known. The *d* system is rotated so that its *z* axis points to the transmitting antenna.

Economy and conciseness of notation are essential to clarity when the number of coordinate systems is considered. We will use E to represent the field of the transmitting antenna and h the effective length of the receiving antenna. A letter superscript refers to the coordinate system in which a quantity is measured. Vector fields will be represented by the usual boldface letters and will be treated in this section as column matrices; thus

$$\mathbf{E} = \operatorname{col}\left(E_x, E_y, E_z\right) \tag{3.131}$$



FIGURE 3.6. Coordinate systems for misaligned antennas.

In addition, we define a column matrix to represent the coordinates of a point; thus

$$\mathbf{X} = \operatorname{col}\left(x, \, y, \, z\right) \tag{3.132}$$

A 3×3 matrix will be represented by a square bracket; thus

$$[A] = \begin{bmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix}$$
(3.133)

The transformation, by rotations, of a point from coordinate system 1 to system 2, having the same origin, is carried out by the Euler angle matrix,

$$[E] = \begin{bmatrix} \cos \beta \cos \gamma & \cos \beta \sin \gamma & -\sin \beta \\ \sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma & \sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \cos \beta \\ \cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma & \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \cos \beta \end{bmatrix}$$

The angles α , β , γ are measured from an axis in the old system (1) toward the corresponding axis in the new (2). The rotations are taken in order:

- 1. γ around the z axis in the direction $x \rightarrow y$.
- 2. β around the y axis in the direction $z \rightarrow x$.
- 3. α around the x axis in the direction $y \rightarrow z$.

Not all authors define the Euler angle transformations in the same manner [3].

The location of point X^2 in the new system is related to its location X^1 in the old system by

$$\mathbf{X}^2 = [E]\mathbf{X}^1 \tag{3.135}$$

where

$$\mathbf{X} = \operatorname{col}\left(x, \, y, \, z\right) \tag{3.136}$$

Transformation of vector functions is carried out by the same matrix; thus

$$\mathbf{F}^2 = [E]\mathbf{F}^1 \tag{3.137}$$

where

(3.134)

$$\mathbf{F} = \operatorname{col}\left(F_x, F_y, F_z\right) \tag{3.138}$$

The transformation from system 2 to 1 is carried out with the inverse matrix

$$\mathbf{X}^{1} = [E]^{-1} \mathbf{X}^{2} \tag{3.139}$$

But the inverse of the Euler angle matrix is its transpose $[E]^{T}$, so that

$$\mathbf{X}^1 = [E]^{\mathrm{T}} \mathbf{X}^2 \tag{3.140}$$

We will use these Euler angle matrices for the coordinate systems of Fig. 3.6.

- [A] from the ground/reference system (translated to x_i, y_i, z_i) to system a.
- [B] from the ground/reference system to system b.
- [C] from the ground/reference system to system c.
- [D] from the ground/reference system to system d.

Note that in many cases the geometry is simpler than this general case. The transmitter, for example, may also be the reference system, and z^a may already point to the receiver, making two transformations unnecessary.

Let us now consider the polarization matching problem for the two antennas. More generally, we will obtain the received power at the receiving antenna and separate the effects of polarization mismatch from the antenna gains. Let us assume that we know the orientations of the transmitting and receiving antenna systems with respect to the reference system, that is, we know the matrices [A] and [C]. Further, we know the far fields of the transmitting antenna in its natural coordinate system, E_{θ}^{a} and E_{ϕ}^{a} . We also know the effective length components of the receiving antenna in its natural coordinate system, h_{θ}^{c} and h_{ϕ}^{c} . We may proceed using one of two methods, both of which will be given here, step by step.

Method 1

STEP 1. Translate the reference system to the transmitter position. Obtain the receiver position X'_r in this translated system.

$$\mathbf{X}_{r}^{\prime} = \mathbf{X}_{r} - \mathbf{X}_{t} \tag{3.141}$$

STEP 2. Use the Euler angle matrix [A] to find the receiver position in the natural system (system a) of the transmitter:

POLARIZATION MATCH FACTOR: MISALIGNED ANTENNAS

$$\mathbf{X}_r^a = [A]\mathbf{X}_r' \tag{3.142}$$

143

Determine the colatitude and azimuth angles of the receiver in system a.

STEP 3. From the known properties of the transmitter, find E_{θ}^{a} and E_{ϕ}^{a} at the receiver. The absolute values of E_{θ}^{a} and E_{ϕ}^{a} must be found if the receiver power is needed. This requires a knowledge of transmitted power, and the distance from transmitter to receiver (easily found from $|\mathbf{X}_{r} - \mathbf{X}_{t}|$). If relative values are sufficient, the transmitter–receiver distance and the transmitter power may be neglected. In fact, only the effective length $\mathbf{h}(\theta, \phi)$ of the transmitting antenna is needed. We will continue to use \mathbf{E} , however, since it is more general and so that we may distinguish it easily from the \mathbf{h} value used for the receiving antenna.

STEP 4. Convert E^a_{θ} and E^a_{ϕ} at the receiver to rectangular form.

$$E_x^a = E_\theta^a \cos \theta \cos \phi - E_\phi^a \sin \phi \quad (a)$$

$$E_y^a = E_\theta^a \cos \theta \sin \phi + E_\phi^a \cos \phi \quad (b) \quad (3.143)$$

$$E_z^a = -E_\theta^a \sin \theta \quad (c)$$

where θ and ϕ are the known values at the receiver found in step 2. The subscripts refer to axes in the *a* system, specifically x^a , y^a , z^a .

STEP 5. Transform the field components to the receiving antenna system, going to the reference system as an intermediate step and then to the receiving antenna system (system c) using the known matrix [C].

$$\mathbf{E} = [A]^{\mathrm{T}} \mathbf{E}^{a} \tag{3.144}$$

$$\mathbf{E}^{c} = [C]\mathbf{E} = [C][A]^{\mathrm{T}}\mathbf{E}^{a}$$
(3.145)

STEP 6. If the absolute value of \mathbf{E}^{c} is known, find the receiver open-circuit voltage using the known receiving antenna value of **h** in system c:

$$V = \mathbf{E}^c \cdot \mathbf{h}^c \tag{3.146}$$

The received power is found easily if antenna and load impedances are known.

If the power is not needed, or if only a relative value of \mathbf{E}^{c} has been obtained, find the polarization match factor from

$$\rho = \frac{|\mathbf{E}^c \cdot \mathbf{h}^c|^2}{|\mathbf{E}^c|^2 |\mathbf{h}^c|^2}$$
(3.147)

Method 2

It may have been noted that method 1 did not utilize the polarization ratios of the antennas, nor did it use the coordinate systems b and d. An alternate method to obtain the polarization match factor between the antennas does lead naturally to the use of the polarization ratios.

STEPS 1-4. Same as steps 1-4 of method 1.

STEP 5. Create two new coordinate systems, b and d of Fig. 3.6, and obtain the Euler angle matrices [B] and [D]. The z axes are to be antiparallel and so are the x axes. The y axes are parallel. These systems will then correspond to those of Fig. 3.3, and equations developed using that figure will be valid. The requirements on systems b and d are not yet sufficient to yield unique coordinates. It is convenient to further require that the axes x^b and x^d lie in the xz plane of the reference system. In the Euler angle matrices this leads to the requirement that $\gamma = 0$. The Euler angle matrices then become

$$[B] = \begin{bmatrix} \cos \beta_b & 0 & -\sin \beta_b \\ \sin \alpha_b \sin \beta_b & \cos \alpha_b & \sin \alpha_b \cos \beta_b \\ \cos \alpha_b \sin \beta_b & -\sin \alpha_b & \cos \alpha_b \cos \beta_b \end{bmatrix}$$
(3.148)
$$[D] = \begin{bmatrix} \cos \beta_d & 0 & -\sin \beta_d \\ \sin \alpha_d \sin \beta_d & \cos \alpha_d & \sin \alpha_d \cos \beta_d \\ \cos \alpha_d \sin \beta_d & -\sin \alpha_d & \cos \alpha_d \cos \beta_d \end{bmatrix}$$
(3.149)

Consider again the reference system translated to the transmitter position so that the receiver position in the translated system is

$$\mathbf{X}_r' = \mathbf{X}_r - \mathbf{X}_t \tag{3.141}$$

The receiver position in system b is then

$$\mathbf{X}_{r}^{b} = [B]\mathbf{X}_{r}^{\prime} \tag{3.150}$$

or

$$\begin{bmatrix} x_r^b \\ y_r^b \\ z_r^b \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} x_r' \\ y_r' \\ z_r' \end{bmatrix}$$
(3.151)

But in system b, the x and y coordinates, x_r^b and y_r^b , of the receiver are zero. Then POLARIZATION MATCH FACTOR: MISALIGNED ANTENNAS

$$\cos \beta_b x'_r - \sin \beta_b z'_r = 0 \tag{a}$$

 $\sin \alpha_b \sin \beta_b x'_r + \cos \alpha_b y'_r + \sin \alpha_b \cos \beta_b z'_r = 0 \quad (b)$

These two equations may be solved to give α_b and β_b in terms of the known quantities x'_r , y'_r , and z'_r . The solutions are

$$\tan \beta_{b} = \frac{x'_{r}}{z'_{r}}$$
(a)
$$\tan \alpha_{b} = -\frac{1}{\cos \beta_{b}} \frac{y'_{r}z'_{r}}{(x'_{r})^{2} + (z'_{r})^{2}}$$
(b)

The matrix [B] is thus completely specified.

To find the [D] matrix, we proceed in the manner that led to the [B] matrix, by first translating the reference system to the position of the receiving antenna. The location of the transmitter in this translated system is

$$\mathbf{X}_{t}^{\prime\prime} = \mathbf{X}_{t} - \mathbf{X}_{r} \tag{3.154}$$

and the transmitter in system d is at

$$\mathbf{X}_{t}^{d} = [D]\mathbf{X}_{t}^{"} \tag{3.155}$$

The x_t^d and y_t^d coordinates are zero, which leads to the equations

$$\cos \beta_d x_i'' - \sin \beta_d z_i'' = 0 \quad (a)$$

$$\sin \alpha_d \sin \beta_d x_i'' + \cos \alpha_d y_i'' + \sin \alpha_d \cos \beta_d z_i'' = 0 \quad (b)$$
(3.156)

with solutions

$$\tan \beta_{d} = \frac{x_{t}''}{z_{t}''}$$
(a)
$$\tan \alpha_{d} = -\frac{1}{\cos \beta_{d}} \frac{y_{t}'' z_{t}''}{(x_{t}'')^{2} + (z_{t}'')^{2}}$$
(b)

STEP 6. Transform the field components obtained in step 4 to the b system using the reference system as an intermediate step:

$$\mathbf{E}^{b} = [B][A]^{\mathrm{T}} \mathbf{E}^{a} \tag{3.158}$$

It is to be noted that \mathbf{E}^{b} will not have a z^{b} component but only transverse components.

STEP 7. Use the Euler angle matrix [C] to find the transmitter position in the natural system (system c) of the receiving antenna:

$$\mathbf{X}_{t}^{c} = [C]\mathbf{X}_{t}^{"} \tag{3.159}$$

Determine the colatitude and azimuth angles of the transmitter in system c.

STEP 8. From knowledge of the receiver effective length $\mathbf{h}^{c}(\theta_{c}, \phi_{c})$ in its natural coordinate system, find the effective length in the direction of the transmitter.

STEP 9. Transform the effective length to system d using the reference system as an intermediate step:

$$\mathbf{h}^{d} = [D][C]^{\mathrm{T}} \mathbf{h}^{c} \tag{3.160}$$

Note that in system d, \mathbf{h}^{d} will have only transverse components.

STEP 10. Define polarization ratios P_r and P_r (or modified polarization ratios p_r and p_r) for the transmitting and receiving antennas

$$P_{t} = \frac{E_{y}^{b}}{E_{x}^{b}}$$
 (a) $P_{r} = \frac{h_{y}^{d}}{h_{x}^{d}}$ (b) (3.161)

The subscripts refer to axes in the proper coordinate system. Thus in defining P_i , the components are those along the y^b and x^b axes, while in defining P_r , the components are along the y^d and x^d axes.

STEP 11. Find the polarization match factor from either

$$\rho = \frac{(1 - P_t P_r)(1 - P_t^* P_r^*)}{(1 + P_t P_t^*)(1 + P_r P_r^*)}$$
(3.162)

or

$$\rho = \frac{(1+p_t p_r)(1+p_t^* p_r^*)}{(1+p_t p_t^*)(1+p_r p_r^*)}$$
(3.163)

It is obvious that the second method laid out here is more cumbersome than the first. It has the advantage, however, that the polarization ratios are obtained, and the aids developed for understanding polarization problems, such as the Poincaré sphere and the complex plane charts, can be readily applied.

REFERENCES

 G. Sinclair, "The Transmission and Reception of Elliptically Polarized Waves," Proc. IRE, Vol. 38, No. 2, pp. 148–151, February 1950.

PROBLEMS

- V. H. Rumsey, "Transmission between Elliptically Polarized Antennas," Proc. IRE, Vol. 39, No. 5, pp. 535–540, May 1951.
- 3. H. Goldstein, Classical Mechanics, Addison-Wesley, Reading, MA, 1950.

PROBLEMS

- 3.1. In a reference coordinate system located at the ground (see Fig. 3.6) a transmitting antenna is located at 500, 1000, 2000 and a receiving antenna is located at 0, 500, 5000. Develop a coordinate system at the transmitting antenna with its z axis pointing at the receiving antenna and its x axis parallel to the xz plane of the reference system. Find the Euler angle matrix of this system.
- **3.2.** Two short dipoles are used as transmitter and receiver in a communications link. The first dipole lies along the y axis of the reference coordinate system of Fig. 3.6. The xz plane of the reference system is parallel to the ground. The z axis of the reference system points to the east. The second dipole is located at 400, 400, 2000. It leans toward the northeast and makes an angle of 75° with the ground. Find the polarization match factor between the antennas.
- **3.3.** Show that the polarization match factor between two antennas can be written as

$$\rho = \frac{|\mathbf{h}_1 \cdot \mathbf{h}_2|^2}{|\mathbf{h}_1|^2 |\mathbf{h}_2|^2}$$

where \mathbf{h}_1 and \mathbf{h}_2 are the effective lengths of the two antennas measured in the same coordinate system.

- **3.4.** Show that, for two antennas in a communication link, if the axial ratio of one antenna is much greater than that of the other, it is of little concern that the antennas have the same or opposite rotation sense.
- **3.5.** Verify the statement following Eq. (3.62) that if two identical antennas are first placed side by side and then moved into a transmit-receive configuration by rotating one of them 180° around a vertical axis, the major axes of their polarization ellipses no longer coincide.



POLARIZATION CHARACTERISTICS OF SOME ANTENNAS

4.1. INTRODUCTION

In this chapter we shall obtain the polarization parameters of several common antennas. We shall also obtain polarization match factors when these antennas are paired with standard antennas in a transmit–receive configuration. In this way we can compare the off-axis performance of the antennas of interest to their polarization performance on-axis (or in some design direction).

In the previous chapters we defined the polarization parameters in the context of a wave traveling in the z direction as in Fig. 4.1(a). Specifically, we defined the polarization ratio P as

$$P = \frac{E_y}{E_x} \tag{2.73}$$

The coordinates of Fig. 4.1(a) are common in a discussion of polarization [1]. The value of P clearly depends on the coordinates used. For example, if we use the ratio $P = E_y/E_x$ for the rotated system of Fig. 4.1(b), it will not be the same as (2.73). A somewhat more general definition of the polarization ratio is

$$P = \frac{E_{\text{vertical}}}{E_{\text{horizontal}}} \tag{4.1}$$

with unit vectors chosen so that

$$\mathbf{u}_{\text{horizontal}} \times \mathbf{u}_{\text{vertical}} = \mathbf{u}_{\text{propagation direction}}$$
(4.2)



FIGURE 4.1. Coordinate systems for defining P.

Note that the first coordinate system of Fig. 4.1 satisfies the requirement (4.2) if x is taken as the "horizontal" component and y the "vertical," but the coordinates of Fig. 4.1(b) do not satisfy (4.2) if y is treated as horizontal.

Near the earth's surface we may define horizontal somewhat loosely as parallel to the surface. More precisely, if a line is drawn from the coordinate origin to the earth's center and if a tangent plane is drawn at the intersection of this line and the earth's surface, then the horizontal axis of the coordinate system is parallel to the tangent plane. The direction of wave propagation

may or may not be parallel to this plane, but in either case, vertical is defined as perpendicular to the horizontal axis and to the direction of propagation.

At points far from the earth the coordinate system for defining the polarization ratio is essentially arbitrary.

The electric field radiated by an antenna is commonly defined by a spherical coordinate system as in Fig. 4.1(c). The wave, having only E_{θ} and E_{ϕ} components (in the far field), travels in the radial direction. If the xy plane is parallel to the earth's surface, then E_{ϕ} is the horizontal component of the wave (it is always parallel to the xy plane and hence to the earth's surface), and $-E_{\theta}$ is the vertical component. [To establish this, either lay the coordinates of Fig. 4.1(a) over those of Fig. 4.1(c) with the z axis coincident with **r** or take the cross product $\mathbf{u}_{\phi} \times (-\mathbf{u}_{\theta})$.] The appropriate definition of the polarization ratio is then

$$P = \frac{-E_{\theta}}{E_{\phi}} \tag{4.3}$$

Some difficulty arises if we establish a coordinate system whose xy plane is not parallel to the earth's surface. Neither E_{θ} nor E_{ϕ} is in general horizontal (parallel to the earth's surface), and we cannot define P in terms of vertical and horizontal components. In this text we will therefore use coordinate systems, if possible, with the z axis perpendicular to the earth's surface. If that is inappropriate, we will continue to use (4.3) to define P and consider horizontal to mean parallel to the xy plane.

4.2. TEST ANTENNAS FOR DETERMINING EFFECT OF POLARIZATION

The purpose of obtaining polarization parameters of an antenna is to find the polarization match factor between that antenna used, for example, as a transmitter and some other antenna used as a receiver. What antenna should we use as a receiver, and how should it be oriented? It is reasonable that if the antenna being examined is intended to produce a circularly polarized wave, for example, we should see how faithfully it does so by using a circularly polarized receiving antenna. Since most antennas are meant to produce either linear or circular polarizations (and we will consider only these in this chapter), we will use a linearly or circularly polarized antenna, as appropriate, to receive the wave.

Now we redefined the polarization ratio in Section 4.1, and we must determine the effect on the equations for the polarization match factor developed in Chapter 3. To do this, consider the coordinate systems of Fig. 4.1(c). Antenna 1 is the transmitter and antenna 2 the receiver. The rectangular coordinate system at antenna 2 is translated from antenna 1. The

polarization match factor developed in Chapter 3 is

$$\rho = \frac{|\mathbf{E}^{i} \cdot \mathbf{h}|^{2}}{|\mathbf{E}^{i}|^{2} |\mathbf{h}|^{2}}$$
(3.37)

where \mathbf{E}^{i} is the field at the receiving antenna caused by the transmitter and **h** is the effective length of the receiving antenna. Both \mathbf{E}^{i} and **h** are measured in the same coordinate system. Let us arbitrarily choose to do so in the receiver system of Fig. 4.1(c). Then

$$\mathbf{E}^{i} \cdot \mathbf{h} = E^{i}_{\theta} \cdot h_{\theta'} + E^{i}_{\phi} \cdot h_{\phi'}$$
(4.4)

From Fig. 4.1(c) it may be seen that

$$E^{i}_{\theta} = E^{i}_{\theta'}, \qquad E^{i}_{\phi} = -E^{i}_{\phi'}.$$
 (4.5)

and therefore

$$\mathbf{E}^{i} \cdot \mathbf{h} = E^{i}_{\theta} h_{\theta'} - E^{i}_{\phi} h_{\phi'} = E^{i}_{\phi} h_{\phi'} \left(\frac{E^{i}_{\theta}}{E^{i}_{\phi}} \frac{h_{\theta'}}{h_{\phi'}} - 1 \right)$$

Now, it is clear that for the transmitting antenna (1)

$$P_1 = \frac{-E'_{\theta}}{E'_{\phi}} \tag{4.6}$$

and if we think for a moment of antenna 2 as transmitting a wave E toward 1, its field would be proportional to its effective length. Then

$$P_{2} = \frac{-E_{\theta'}}{E_{\phi'}} = -\frac{h_{\theta'}}{h_{\phi'}}$$
(4.7)

If these substitutions are made above,

$$\mathbf{E}^{i} \cdot \mathbf{h} = E^{i}_{\phi} h_{\phi'} (P_1 P_2 - 1) \quad (a)$$

$$|\mathbf{E}^{i}|^2 = |E^{i}_{\phi'}|^2 (1 + P_1 P_1^*) \quad (b) \quad (4.8)$$

$$|\mathbf{h}|^2 = |h_{\phi'}|^2 (1 + P_2 P_2^*) \quad (c)$$

and substitution into (3.37) gives for the polarization match factor

$$\rho = \frac{(1 - P_1 P_2)(1 - P_1^* P_2^*)}{(1 + P_1 P_1^*)(1 + P_2 P_2^*)}$$
(4.9)

This equation is identical to (3.89), which we developed using the definition (2.73) for polarization ratio. We may therefore use (3.89) or any equation for ρ involving polarization parameters p, q, w, \ldots , developed in Chapter 3.

It may not always be convenient to define P_2 in the translated coordinate system of Fig. 4.1(c). If we have, for example, two identical antennas pointing at each other, a reversed coordinate system would be appropriate for one of them. Two such systems are shown in Fig. 4.1(d) and (e). If P_2 is defined in the manner shown, (4.9) remains valid. While we will define P_2 in one of the coordinate systems of Fig. 4.1(c), (d), or (e), it is nonetheless convenient to express P_2 in terms of θ and ϕ , the direction of the receiving antenna from the transmitter. This will be clearer as we discuss the polarization of the receiving antenna necessary to give a polarization match with the transmitter.

In order to select the receiving antenna and orient it correctly, we start by recognizing that a dipole antenna produces a wave that is everywhere linearly polarized. If we use it as a transmitter, then we should use another linearly polarized antenna (perhaps another dipole) as a receiver. If the receiver is correctly oriented, there is no polarization mismatch in any direction from the transmitter. Figure 4.2 shows the correct orientation for the receiving antenna if the antenna undergoing examination is a z-directed dipole. The receiving dipole must lie in the plane containing the transmitting dipole, and the line from transmitter to receiver must be perpendicular to the receiving dipole. We anticipate the results of Section 4.3 and note that the z-directed transmitter has a polarization ratio

$$P_{z} = \infty \tag{4.24}$$

except at $\theta = 0$ where the field is zero. From Fig. 4.2 it is clear that on a line perpendicular to the receiving dipole $h_{\phi'} = 0$ and the receiver has a polarization ratio



FIGURE 4.2. Orientation of receiving dipole for testing z-directed transmitting dipole antenna.

Substitution in (4.9) shows that the polarization matching requirement is met for these two antennas.

Let us now suppose that the transmitting antenna is a dipole lying on the x axis. Again we anticipate the results of Section 4.3 and note that the polarization ratio in the direction θ , ϕ is

$$P_x = \cos\theta \cot\phi \tag{4.22}$$

For polarization matching the receiving antenna should be a dipole lying in a plane that contains the x axis, and a perpendicular from the receiving antenna should point to the transmitter. Since we know that these antennas will be polarization matched, then the polarization ratio of the receiving antenna must be

$$P_r = -P_x^* = -\cos\theta \cot\phi \tag{4.11}$$

An earlier statement may now be clearer. In this expression, P_r is defined as $-h_{\theta'}/h_{\phi'}$, using the translated coordinate system of Fig. 4.1(c) [or the translated and reversed systems of Fig. 4.1(d) or (e)]. Nonetheless, it is convenient to give P_r not in terms of the angles θ' , ϕ' of Fig. 4.1(c) but in terms of the direction θ , ϕ of the receiving antenna from the transmitting antenna.

In Section 4.3 we will find the polarization ratio of a y-directed dipole to be

$$P_{v} = -\cos\theta \tan\phi \tag{4.23}$$

For test purposes we will use a dipole receiver with polarization ratio

$$P_r = -P_v^* = \cos\theta \tan\phi \tag{4.12}$$

Now suppose that our transmitting antenna is intended to produce a wave with polarization characteristics similar to one of these x-, y-, or z-oriented dipoles. The open waveguide antenna of Section 4.8 is an example, with a polarization similar to that of the y-oriented dipole. For such an antenna we will continue to use as a test (receiving) antenna a dipole with polarization ratio given by (4.12), even though there will be a polarization mismatch in some directions. The antenna is intended to produce a linearly polarized wave, and it is appropriate to examine how well it does so. We are not measuring the polarization of the antenna, although we could certainly do so; rather we are comparing its polarization to a standard, and the dipole is the standard.

It should be noted that the receiving antenna changes only in orientation as it is moved from one point to another. The change in polarization ratio is caused by this change in orientation.

Right and left circularly polarized antennas have polarization ratios of -j

and +j, respectively. If we have an antenna intended to produce a right circular wave, whether it does so in all directions or not, we will test it by obtaining its polarization match factor with a receiving antenna having

$$P_r = -(-j)^* = -j \tag{4.13}$$

Similarly, to test a left circular antenna, we will use a receiver with

$$P_r = +j \tag{4.14}$$

In this chapter we will have occasion to consider relative radiation intensity

$$|E_{\theta}(\theta, \phi)|^2 + |E_{\phi}(\theta, \phi)|^2$$

and polarization match factor

 $\rho(\theta, \phi)$

We will refer to a radiation intensity pattern and to the *radiation intensity* beamwidth as the angle between the two directions in which the radiation intensity drops to one-half (-3 dB) of its maximum value.

In the same way the *polarization beamwidth* is the angle between two directions for which ρ is one-half.

If both radiation intensity and polarization effects are used in determining the 3-dB points for an antenna, we will use *overall beamwidth* as the angle between two directions for which received power in an appropriate receiving antenna drops to one-half the possible received power.

4.3. THE SHORT DIPOLE

We will obtain the far fields produced by short dipoles oriented along the coordinate axes. The fields of a dipole with any orientation may then be written as the sum of the fields produced by these.

The magnetic vector potential of a short dipole directed along one of the coordinate axes is given by (1.31) and similar forms,

$$A_{x,y,z} = \frac{\mu I \ell}{4\pi r} e^{-jkr}$$
(4.15)

where I is the value of the current at all points in the dipole length ℓ . A more realistic model for a short antenna is a center-fed dipole having a triangular current distribution with maximum current I_0 at the center and zero current at the ends. The vector potential for this antenna is (see problem 1.5)

THE SHORT DIPOLE

$$A_{x,y,z} = \frac{\mu I_0 \ell}{8\pi r} e^{-jkr}$$
(4.16)

There is no essential difference between these equations, and we will continue to use (4.15).

We can find the fields of the dipoles by the transformations

$$A_{r} = A_{x} \sin \theta \cos \phi + A_{y} \sin \theta \sin \phi + A_{z} \cos \theta$$
$$A_{\theta} = A_{x} \cos \theta \cos \phi + A_{y} \cos \theta \sin \phi - A_{z} \sin \theta \qquad (4.17)$$
$$A_{\phi} = -A_{x} \sin \phi + A_{y} \cos \phi$$

and the far-field equations of (1.58), repeated here for convenience,

$$E_r = 0 \qquad E_\theta = -j\omega A_\theta \qquad E_\phi = -j\omega A_\phi \qquad (4.18)$$

Fields

The fields that result from these equations are:

x-DIRECTED DIPOLE

$$E_r = 0 \tag{a}$$

$$E_{\theta} = -\frac{j\omega\mu I\ell}{4\pi r}\cos\theta\cos\phi e^{-jkr} \quad (b)$$
(4.19)

$$E_{\phi} = \frac{j\omega\mu I\ell}{4\pi r} \sin \phi e^{-jkr}$$
(c)

y-DIRECTED DIPOLE

$$E_r = 0 \tag{a}$$

$$E_{\theta} = -\frac{j\omega\mu I\ell}{4\pi r} \cos\theta \sin\phi e^{-jkr} \quad (b)$$
(4.20)

$$E_{\phi} = -\frac{j\omega\mu I\ell}{4\pi r}\cos\phi e^{-jkr} \qquad (c)$$

z-DIRECTED DIPOLE

$$E_r = 0 \tag{a}$$

$$E_{\theta} = \frac{j\omega\mu I\ell}{4\pi r} \sin\theta e^{-jkr} \quad (b) \tag{4.21}$$

$$E_{\phi} = 0 \tag{c}$$

Polarization Ratios

x-DIRECTED DIPOLE

$$P_x = \frac{-E_\theta}{E_\phi} = \cos\theta \cot\phi \tag{4.22}$$

y-DIRECTED DIPOLE

$$P_{\nu} = -\cos\theta\,\tan\phi\tag{4.23}$$

z-DIRECTED DIPOLE $P_z = \infty$ (4.24)

Polarization Match Factors

This is essentially trivial since we have already postulated that the dipole in question is to be compared to a correctly oriented receiving antenna, which is also a dipole with

$$P_r = -P_{x,y,z}^*$$
(4.25)

Nevertheless, let us see the process for the x-directed dipole. We substitute

$$P_r = \cos\theta \cot\phi \tag{4.22}$$

and

$$P_r = -\cos\theta \cot\phi \tag{4.11}$$

into the match factor equation

$$\rho = \frac{(1 - P_x P_r)(1 - P_x^* P_r^*)}{(1 + P_x P_x^*)(1 + P_r P_r^*)}$$
(4.26)

and immediately obtain $\rho = 1$. We note that of course the test antenna does not receive the same power at all points, but the variation is due to the dipole directive gain, not to its polarization properties.

Received Power

The received power density is quickly found from the fields to be:

x-DIRECTED DIPOLE

$$S_x = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \left(|E_\theta|^2 + |E_\phi|^2 \right) = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \left(\frac{\omega \mu I \ell}{4\pi r} \right)^2 (1 - \sin^2 \theta \cos^2 \phi) \quad (4.27)$$

y-DIRECTED DIPOLE

$$S_{y} = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \left(\frac{\omega \mu I \ell}{4\pi r}\right)^{2} (1 - \sin^{2} \theta \sin^{2} \phi)$$
(4.28)

z-DIRECTED DIPOLE

$$S_z = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \left(\frac{\omega \mu I \ell}{4\pi r}\right)^2 \sin^2 \theta \tag{4.29}$$

4.4. CROSSED DIPOLES (TURNSTILE ANTENNA)

An antenna used to produce a circularly polarized wave is shown in Fig. 4.3. If the vertical (y-directed) and horizontal (x-directed) dipoles are identical and are fed with currents having the same amplitudes and $\frac{1}{2}\pi$ phase difference, the radiated wave is circularly polarized on the z axis.

Let us take the feed current or voltage to the x-directed dipole as a reference, and let the feed to the y dipole lead it by $\frac{1}{2}\pi$. The resulting fields are the sum of (4.19) and (4.20) multiplied by j. The result is

$$E_{\theta} = -\frac{j\omega\mu I\ell}{4\pi r} (\cos\theta\cos\phi + j\cos\theta\sin\phi)e^{-jkr} \quad (a)$$

$$E_{\phi} = \frac{j\omega\mu I\ell}{4\pi r} (\sin\phi - j\cos\phi)e^{-jkr} \quad (b)$$
(4.30)

$$L_{\phi} = \frac{4\pi r}{4\pi r} (\sin \phi - f \cos \phi) e^{-\frac{\pi}{2}}$$

The polarization ratio is

$$P_{i} = \frac{\cos\theta\cos\phi + j\cos\theta\sin\phi}{\sin\phi - j\cos\phi} = j\cos\theta \qquad (4.31)$$

On the z axis, $\theta = 0$ and P = j1, which corresponds to a left circular wave propagating in the z direction. Had the y dipole feed lagged in phase by $\frac{1}{2}\pi$, the wave would have been right circular along the z axis.

In Section 4.2 we saw that it is appropriate to examine the polarization loss as a function of propagation direction by allowing the antenna to radiate



toward a left circular receiving antenna with $P_r = +j$. Then for any angle θ the polarization match factor for this antenna pair is

$$\rho(\theta, \phi) = \frac{(1 - P_r P_r)(1 - P_r^* P_r^*)}{(1 + P_r P_r^*)(1 + P_r P_r^*)} = \frac{(1 + \cos\theta)^2}{2(1 + \cos^2\theta)} = \frac{1}{2} + \frac{\cos\theta}{1 + \cos^2\theta} \quad (4.32)$$

We note from (4.32) that the power received drops to one-half its maximum value at $\theta = \frac{1}{2}\pi$.

The antenna gain, neglecting polarization effects, is proportional to

$$|E_{\theta}|^{2} + |E_{\phi}|^{2} = |\cos\theta\cos\phi + j\cos\theta\sin\phi|^{2} + |\sin\phi - j\cos\phi|^{2}$$
$$= 1 + \cos^{2}\theta$$

and the gain relative to the maximum gain (in the direction $\theta = 0$) is

$$G_r = \frac{1}{2}(1 + \cos^2 \theta)$$
 (4.33)

Note that the half-power angle for polarization is $\theta = \frac{1}{2}\pi$, and the half-power beamwidth for polarization is π . The half-power beamwidth, neglecting polarization, is also π , from (4.33). When ρ and G_r are combined to determine the actual power received by a circularly polarized antenna, we obtain

$$G_{r}\rho = \frac{1}{4}(1 + \cos\theta)^{2}$$
(4.34)

Setting $G_{,\rho}$ to $\frac{1}{2}$ gives

 $\theta_{3dB} = 65.5^{\circ}$

Half-power beamwidth = $2\theta_{3dB} = 131^{\circ}$

A note of caution is in order. The polarization ratio for the crossed dipoles was found to be

$$P_i = j \cos \theta \tag{4.31}$$

In the xz plane, for large θ (measured from the z axis), the electric field is primarily y directed, whereas in the yz plane, for large θ , it is primarily x directed. The reader incautiously thinking of y as the "vertical" axis and x as "horizontal" in Fig. 4.3, will be concerned that the polarization ratio is the same in both planes. It was pointed out in Section 4.1, and is repeated here, that the xy plane must be treated as the horizontal plane in defining polarization ratio in terms of E_{θ} and E_{ϕ} , even though it need not be parallel to the earth. In that context, both the x and y axes in Fig. 4.3 are horizontal, and it is reasonable that the polarization ratio has the same form in the two cases described above.

4.5. CROSSED DIPOLES WITH GROUND PLANE

The radiation intensity pattern of the crossed dipoles can be sharpened by placing the dipoles in front of an infinite conducting plane, as in Fig. 4.4. By image theory the fields in front of the plane remain the same if the screen is removed and image dipoles, fed by currents differing in phase from the real dipoles by π radians, are placed at a distance 2a from the real dipoles.

From the pattern multiplication principle of array theory the far-zone field of a uniform array of identical elements is the product of the field of a single element and the *array factor*. The array factor is a function of geometry and the excitation phases of the elements and is essentially the pattern of an array of isotropic radiators located at the real antennas [2]. The array factor of two



FIGURE 4.4. Crossed dipoles near infinite conducting plane.

elements on the z axis separated by distance d is [2]

$$AF = 2\cos\left[\frac{1}{2}(kd\cos\theta + \beta)\right]$$
(4.35)

where β is the phase of the excitation of the element at the greater z value compared to that of the element at the lesser value of z.

If we let the dipoles be a quarter wavelength from the plane, and note that $\beta = \pi$ and $d = 2a = \frac{1}{2}\lambda$, the array factor for the crossed dipoles in front of the conducting screen becomes

$$AF = 2\cos\left[\frac{1}{2}\pi(\cos\theta + 1)\right]$$
(4.36)

Both E_{θ} and E_{ϕ} of (4.30) are multiplied by this array factor to give the new fields, and since both are altered by the same factor, it is clear that P_t , as given by (4.31), and ρ , as given by (4.32), are unchanged. On the other hand, the radiation intensity is multiplied by the square of the array factor, and the relative gain becomes

$$G_r = \frac{1}{2} (1 + \cos^2 \theta) \cos^2 \left[\frac{1}{2} \pi (\cos \theta + 1) \right]$$
(4.37)

The product of G_r and ρ is now

$$G_r \rho = \frac{1}{4} [(1 + \cos \theta)^2] \cos^2 \left[\frac{1}{2}\pi(\cos \theta + 1)\right]$$
(4.38)

and if this is set equal to $\frac{1}{2}$, we find for the half-power beamwidth, considering both radiation intensity and polarization match, that

Beamwidth =
$$2\theta_{3dB} = 98.4^{\circ}$$

It should be noted that the value of ρ , the polarization match factor, is 0.854 at the overall 3-dB angle for the crossed dipoles without a screen and 0.958 for the crossed dipoles with a screen. Use of the screen produced a narrower beam and one that is still almost circularly polarized at the beam edge.

The array principle used here can obviously be extended. The array factor has no polarization properties, since it is the pattern of an array of isotropic radiators. By the pattern multiplication principle, both E_{θ} and E_{ϕ} produced by one element are multiplied by the same factor. Then the polarization ratio of an array of identical elements whose fields are not altered by the presence of other elements in the array is the same as that of one of the elements. This may be advantageous in some applications since the radiation intensity pattern can be primarily controlled by the array geometry, whereas the polarization of the radiated wave is completely established by the choice of array element.

THE LOOP ANTENNA

4.6. THE LOOP ANTENNA

The far electric field of a small circular loop antenna with uniform in-phase current lying in the xy plane as shown in Fig. 4.5 is [2]

$$E_r = E_\theta = 0 (a) (4.39)$$
$$E_\phi = \frac{\omega\mu k a^2 I \sin \theta}{4r} e^{-jkr} (b)$$

It is obvious that the field is everywhere linearly polarized and horizontal, as discussed in Section 4.1. We may then use either another circular loop antenna or a horizontal dipole to receive a field radiated by the transmitting loop without polarization loss. The receiving loop is oriented so that the radial line from the transmitting loop center is in the receiving loop plane. This will keep the receiving loop gain constant as it is moved from one location to another. A horizontal dipole receiving antenna must be tangent to a circle drawn with the z axis as center. With either receiving antenna, $\rho = 1$. We now have two additional test antennas to use with an antenna intended to produce a field linearly polarized in a horizontal direction.

The loop may be considered small, and (4.39) is valid, if $a \le \lambda$. If that is not the case, more general equations for the loop fields are [1, p. 161]

$$E_r = E_{\theta} = 0 \qquad (a)$$

$$E_{\phi} = \frac{\omega \mu a I_L J_1(ka \sin \theta)}{2r} e^{-jkr} \qquad (b)$$

where J_1 is the Bessel function of the first kind and first order. For loops with



FIGURE 4.5. Circular loop antenna and test antennas.

circumference $\frac{1}{4}\lambda$ or greater, phase shifters must be inserted at intervals around the loop to maintain a uniform in-phase current in the loop [1].

It should be noted that the radiated field of the large loop is still linearly polarized in the azimuth direction.

4.7. LOOP AND DIPOLE

We saw in the previous section that the field of a small loop is azimuthal and varies as $\sin \theta$. In Section 4.3 we noted that the field of a z-directed short dipole is directed wholly in the θ direction and also varies as $\sin \theta$. It is obvious then that a combination of the two antennas with proper current amplitudes and phases can produce a wave that is everywhere circular. Figure 4.6 shows such a combination.

From a comparison of the E_{θ} field of the short dipole and the E_{ϕ} field of a small loop,

$$E_{\theta} = \frac{j\omega\mu I_{d}\,\ell\,\sin\theta}{4\pi r} \,e^{-jkr} \quad (a)$$

$$E_{\phi} = \frac{\omega\mu ka^{2}I_{L}\,\sin\theta}{4r} \,e^{-jkr} \quad (b)$$
(4.41)

it is obvious that if the wave is to be, for example, right circular, so that

$$P = -\frac{E_{\theta}}{E_{\phi}} = -j$$

then we must require that

$$\frac{I_d}{I_L} = \frac{\pi k a^2}{\ell} \tag{4.42}$$

Reversing either current will give a left circular wave.



FIGURE 4.6. Loop and dipole.

LOOP AND DIPOLE

The use of the loop and dipole antenna with a circularly polarized receiving antenna, of the correct sense, obviously gives a polarization match factor of unity. The relative gain is also clearly

$$G_r = \sin^2 \theta \tag{4.43}$$

and the pattern is omnidirectional in azimuth with a half-power beamwidth of $\frac{1}{2}\pi$ in a constant-azimuth plane.

Let us consider now a longer dipole with a field

$$E_{\theta} = \frac{jZ_0 I_m}{2\pi r} \frac{\cos\left[(k\ell/2)\cos\theta\right] - \cos\left(k\ell/2\right)}{\sin\theta} e^{-jkr} \quad (1.104)$$

and the larger loop of Section 4.6 with field

$$E_{\phi} = \frac{\omega \mu a I_L J_1(ka \sin \theta)}{2r} e^{-jkr}$$
(4.40)

It will be seen that this combination no longer is circularly polarized for all values of θ . It is clear that by a choice of the relative feed currents of loop and dipole, the antenna can be made to radiate a circularly polarized wave in one direction, θ . Let us choose the wave to be right circular at $\theta = \frac{1}{2}\pi$ (in the *xy* plane).

The polarization ratio is, from the equations for E_{θ} and E_{ϕ} ,

$$P_{t} = C \frac{\cos\left[(k\ell/2)\cos\theta\right] - \cos\left(k\ell/2\right)}{\sin\theta J_{1}(ka\sin\theta)}$$
(4.44)

where C is a constant. If the wave is to be right circular at $\theta = \frac{1}{2}\pi$,

$$C \ \frac{1 - \cos\left(k\ell/2\right)}{J_1(ka)} = -j$$

which gives

$$C = \frac{-jJ_1(ka)}{1 - \cos(k\ell/2)}$$
(4.45)

and for general θ

$$P_{t} = \frac{-jJ_{1}(ka)}{1 - \cos\left(k\ell/2\right)} \frac{\cos\left[(k\ell/2)\cos\theta\right] - \cos\left(k\ell/2\right)}{\sin\theta J_{1}(ka\sin\theta)}$$
(4.46)

If we match this transmitting antenna with a right circular antenna having $P_r = -j$, the match factor can be written as
$$\rho = \frac{1}{2} \times$$

$$\frac{(\sin\theta[1-\cos(k\ell/2)]J_1(ka\sin\theta) + J_1(ka)\{\cos[(k\ell/2)\cos\theta] - \cos(k\ell/2)\})^2}{\sin^2\theta[1-\cos(k\ell/2)]^2J_1^2(ka\sin\theta) + J_1^2(ka)\{\cos[(k\ell/2)\cos\theta] - \cos(k\ell/2)\}^2}$$
(4.47)

This is a rather complicated equation, but matters can be simplified if we consider the product of ρ and the relative gain G_r . Using (1.104) and (4.40), we note that

$$|E_{\theta}|^{2} + |E_{\phi}|^{2} = \left(\frac{Z_{0}I_{m}}{2\pi r}\right)^{2} \frac{\left\{\cos\left[(k\ell/2)\cos\theta\right] - \cos\left(k\ell/2\right)\right\}^{2}}{\sin^{2}\theta} + \left(\frac{\omega\mu aI_{L}}{2r}\right)^{2}J_{1}^{2}(ka\sin\theta)$$
(4.48)

At $\theta = \frac{1}{2}\pi$, the two terms above are equal from our choice of circular polarization at $\theta = \frac{1}{2}\pi$. In addition, we will use the intensity there to normalize the intensity at any angle. It follows from these two facts that

$$G_{r} = \frac{\left[\cos\left[(k\ell/2)\cos\theta\right] - \cos\left(k\ell/2\right)\right]^{2}}{2\sin^{2}\theta\left[1 - \cos\left(k\ell/2\right)\right]^{2}} + \frac{J_{1}^{2}(ka\sin\theta)}{2J_{1}^{2}(ka)}$$
(4.49)

If we take the product of G_r and ρ , the result is

$$G_{r}\rho = \frac{1}{4} \times \frac{(\sin\theta[1 - \cos(k\ell/2)]J_{1}(ka\sin\theta) + J_{1}(ka)\{\cos[(k\ell/2)\cos\theta] - \cos(k\ell/2)\})^{2}}{\sin^{2}\theta[1 - \cos(k\ell/2)]^{2}J_{1}^{2}(ka)}$$
(4.50)

It may be verified that this is unity in the plane $\theta = \frac{1}{2}\pi$. For a half-wave dipole the product is simpler, namely

$$G_r \rho|_{\ell=\lambda/2} = \frac{1}{4} \left[\frac{J_1(ka\sin\theta)}{J_1(ka)} + \frac{\cos\left[(\pi/2)\cos\theta\right]}{\sin\theta} \right]^2$$
(4.51)

4.8. WAVEGUIDE OPENING INTO INFINITE GROUND PLANE

In Section 1.13 we developed the equations for the far fields of a rectangular waveguide carrying the TE_{10} mode and opening into an infinite ground plane. With the ground plane taken as the *xy* plane and with the long dimension of

164

the waveguide directed along x, as shown in Fig. 4.7, the far fields are

$$E_{\theta} = \frac{\omega abE_{0}}{cr} \sin \phi \frac{\cos \left[(\pi a/\lambda) \sin \theta \cos \phi \right]}{\pi^{2} - 4 \left[(\pi a/\lambda) \sin \theta \cos \phi \right]^{2}} \frac{\sin \left[(\pi b/\lambda) \sin \theta \sin \phi \right]}{(\pi b/\lambda) \sin \theta \sin \phi} \quad (a)$$

$$E_{\phi} = \frac{\omega abE_{0}}{cr} \cos \theta \cos \phi \frac{\cos \left[(\pi a/\lambda) \sin \theta \cos \phi \right]}{\pi^{2} - 4 \left[(\pi a/\lambda) \sin \theta \cos \phi \right]^{2}} \times \frac{\sin \left[(\pi b/\lambda) \sin \theta \sin \phi \right]}{(\pi b/\lambda) \sin \theta \sin \phi} \quad (b)$$

where a and b are the waveguide dimensions in the x and y directions, respectively.



FIGURE 4.7. Waveguide opening into plane.

In spite of the complexity of the field components, the polarization ratio for this antenna is quite simple. It is

$$P_{t} = -\frac{\tan\phi}{\cos\theta} \tag{4.52}$$

On the z axis the wave is polarized in the y direction. (We avoid the use of the word *vertical* as ambiguous.)

We earlier considered an antenna that also produces a y-polarized wave on the z axis, namely, the y-directed dipole of Section 4.3, with a polarization ratio

$$P_{v} = -\cos\theta\,\tan\phi\tag{4.23}$$

Since the polarization is the same in at least one direction, an important direction at that, it is interesting to compare polarizations in other directions.

It should be noted first that both antennas radiate a wave that is everywhere linearly polarized and that may be received everywhere without polarization loss by a correctly oriented, linearly polarized antenna. Since the polarization ratios are different, we might expect, correctly, that the receiving antenna orientation will be different for the dipole and the waveguide.

Let us consider first the principal E and H planes and the xy plane. The polarization ratios and the field components are compared here:

Plane	Polarization		Fields	
	Dipole	Waveguide	Dipole	Waveguide
Principal E plane, $\phi = \frac{1}{2}\pi$	œ	œ	$E_{\theta}(E_y, E_z)$	$E_{\theta}(E_y, E_z)$
Principal H plane, $\phi = 0$	0	0	$E_{\phi}(E_{y})$	$E_{\phi}(E_{y})$
xy Plane, $\theta = \frac{1}{2}\pi$	0	∞	$E_{\phi}(E_x, E_y)$	$E_{\theta}(E_z)$

We see from this table that the polarization behavior of the two antennas is the same in the principal E and H planes but differs markedly in the xy plane (which is generally of little interest for the waveguide opening).

Let us now consider the reception of the wave transmitted by the waveguide antenna. It is worthwhile to repeat that the radiated wave is everywhere linear and can be received without polarization loss by any linearly polarized antenna that is correctly oriented. We saw in Section 4.2 and Fig. 4.2 that if the transmitter is a dipole, then the receiver can be a dipole that lies in the same plane as the transmitter and is perpendicular to a line drawn from it. That natural orientation will not do for the waveguide antenna, however. In theory, the field components of the waveguide antenna

HORNS

can be calculated and a receiving antenna (dipole) oriented parallel to the field. This means nothing, however, since, in theory, $\rho = 1$. In making gain measurements of the waveguide antenna, a receiving dipole can be oriented perpendicular to a line from the waveguide antenna and rotated around that axis to maximize received power. This eliminates any polarization mismatch and allows a correct measurement of the gain.

If we wish specifically to consider the polarization behavior of the waveguide opening into a plane, it is appropriate to use as a receiving antenna a dipole oriented as it would be if the transmitting antenna itself were a *y*-directed dipole. The polarization ratio of the receiver is then to be taken as

$$P_r = \cos\theta \tan\phi \tag{4.12}$$

and the polarization match factor between this dipole and the waveguide antenna is

$$\rho = \frac{\cos^2 \theta (1 + \tan^2 \phi)^2}{(\cos^2 \theta + \tan^2 \phi)(1 + \cos^2 \theta \tan^2 \phi)}$$
(4.53)

It is quickly noted that $\rho = 1$ in the principal *E* and *H* planes. It may also be determined without difficulty that the maximum rate of change of ρ with angle θ , near $\theta = 0$, occurs for $\phi = \frac{1}{4}\pi, \frac{3}{4}\pi, \ldots$ (unsurprising since ρ is independent of θ for $\phi = 0, \frac{1}{2}\pi, \ldots$).

Along a line giving maximum rate of change of ρ with θ (tan $\phi = 1$), the value of ρ drops 3 dB where $\theta_{3dB} = 65.53^{\circ}$. Then the minimum 3-dB beamwidth is

$$2\theta_{3dB} = 131.1^{\circ}$$

for polarization effects alone.

The radiation intensity beamwidth can be found from the fields of (1.136). The 3-dB beamwidths in the principal *E* and *H* planes are given by Balanis [2, p. 469] as $50.6\lambda/b$ and $68.8\lambda/a$ respectively. These beamwidths are on the order of the polarization beamwidth for standard rectangular waveguides used in their designed frequency ranges. Polarization effects are therefore important within the radiation intensity beamwidth, and they decrease the overall beamwidth of this antenna significantly.

4.9. HORNS

Horns are among the most widely used microwave antennas. In this section we will consider primarily the polarization properties of a pyramidal horn fed by a rectangular waveguide carrying the TE_{10} mode with a y-directed electric field. The geometry is shown in Fig. 4.8. The radiation fields are relatively



FIGURE 4.8. Pyramidal horn antenna.

complex, and their development is widely available in the literature and thus is not repeated here. The notation used here is that of Balanis [2, Chapter 12].

The far fields of the pyramidal horn of Fig. 4.8 are given by Balanis as

$$E_{\theta} = \frac{jkE_0 e^{-jkr}}{4\pi r} \sin \phi (1 + \cos \theta) I_1 I_2 \quad (a)$$

$$E_{\phi} = \frac{jkE_0 e^{-jkr}}{4\pi r} \cos \phi (1 + \cos \theta) I_1 I_2 \quad (b)$$
(4.54)

where I_1 and I_2 are given by the rather complicated expressions

$$\begin{split} I_1 &= \frac{1}{2} \, \sqrt{\frac{\pi \rho_2}{k}} \, \left(e^{jk_1^{\prime 2} \rho_2 / 2k} \{ \left[C(t_2') - C(t_1') \right] - j \left[S(t_2') - S(t_1') \right] \} \\ &+ e^{jk_1^{\prime 2} \rho_2 / 2k} \{ \left[C(t_2'') - C(t_1'') \right] - j \left[S(t_2'') - S(t_1'') \right] \} \right) \end{split}$$
(a) (4.55)
$$I_2 &= \sqrt{\frac{\pi \rho_1}{k}} \, e^{jk_2^2 \rho_1 / 2k} \{ \left[C(t_2) - C(t_1) \right] - j \left[S(t_2) - S(t_1) \right] \}$$
(b)

where

$$k = \frac{2\pi}{\lambda} \tag{a}$$

$$k'_x = k \sin \theta \cos \phi + \frac{\pi}{a_1}$$
 (b)

$$k''_x = k \sin \theta \cos \phi - \frac{\pi}{a_1}$$
 (c)

$$k_{v} = k \sin \theta \sin \phi \tag{d}$$

168

HORNS

$$t'_{1} = \sqrt{\frac{1}{\pi k \rho_{2}}} \left(-\frac{k a_{1}}{2} - k'_{x} \rho_{2} \right)$$
 (e)

$$t_2' = \sqrt{\frac{1}{\pi k \rho_2}} \left(\frac{k a_1}{2} - k_x' \rho_2 \right) \qquad (f)$$

$$t_1'' = \sqrt{\frac{1}{\pi k \rho_2}} \left(-\frac{k a_1}{2} - k_x'' \rho_2 \right) \quad (g) \tag{4.56}$$

$$t_2'' = \sqrt{\frac{1}{\pi k \rho_2}} \left(\frac{k a_1}{2} - k_x'' \rho_2 \right)$$
 (h)

$$t_{1} = \sqrt{\frac{1}{\pi k \rho_{1}}} \left(-\frac{k b_{1}}{2} - k_{y} \rho_{1} \right)$$
(i)

$$t_{2} = \sqrt{\frac{1}{\pi k \rho_{1}}} \left(\frac{k b_{1}}{2} - k_{y} \rho_{1} \right) \qquad (j)$$

$$C(x) = \int_0^x \cos\left(\frac{1}{2}\pi t^2\right) dt$$
 (k)

$$S(x) = \int_0^x \sin(\frac{1}{2}\pi t^2) dt$$
 (1)

If one looks at the horn in the x direction of Fig. 4.8, the upper and lower horn surfaces, if extended, meet inside the waveguide. The distance from this line to the aperture plane z = 0 is ρ_1 . Similarly, the two side surfaces of the horn, if extended, meet in a line, and the distance to the aperture plane is ρ_2 . These values occur in (4.56).

It is evident from the field equations that numerical computation of the radiation pattern of the pyramidal horn is necessary for the best understanding of its radiation characteristics. The reader is referred to Balanis for three-dimensional patterns [2, pp. 565–576].

In contrast to the radiation intensity, the polarization ratio of the pyramidal horn is quite simple. From (4.54) it is

$$P_t = -\frac{E_0}{E_\phi} = -\tan\phi \tag{4.57}$$

It is noteworthy that the *E*-plane sectoral horn (flared in the *E* plane, the *y* direction, but not in the *x* direction) and the *H*-plane sectoral horn (flared in the *H* plane, the *x* direction, but not in the *y* direction) have fields that are quite different from those of the pyramidal horn [2, Chapter 12] and yet have the same polarization ratio. Note that the fields are everywhere linearly polarized, but not in the same direction.

Since these horn antennas are intended to produce a y-polarized linear

field in the main beam, it is appropriate to use as a test receiving antenna the dipole that was also used for the waveguide opening into a plane. The polarization ratio of that test dipole is

$$P_r = \cos\theta \tan\phi \tag{4.12}$$

Then the polarization match factor between this dipole and any one of the horn antennas is

$$\rho = \frac{(1 + \cos\theta \tan^2 \phi)^2}{(1 + \tan^2 \phi)(1 + \cos^2\theta \tan^2 \phi)}$$
(4.58)

Let us look first at the principal E and H planes. In the principal E plane $\phi = \frac{1}{2}\pi$, and in the principal H plane $\phi = 0$; it is immediately seen from (4.58) that $\rho = 1$. In those planes the field radiated from a horn is indistinguishable from that of a dipole in its polarization characteristics.

By differentiating (4.58) with respect to $\tan^2 \phi$, it can be determined that the greatest rate of change of ρ with θ near the direction $\theta = 0$ occurs, as it did for the waveguide opening into a plane, where $\tan^2 \phi = 1$ or $\phi = \frac{1}{4}\pi, \frac{3}{4}\pi, \ldots$. If this value is substituted into the polarization match factor equation, it becomes

$$\rho = \frac{(1 + \cos \theta)^2}{2(1 + \cos^2 \theta)} \qquad \phi = \frac{\pi}{4}$$
(4.59)

It is quickly ascertained from this equation that the polarization beamwidth in a plane tilted at 45° with respect to the principal *E* and *H* planes is

$$2\theta_{3dB} = 120^{\circ}$$

Furthermore, this is the minimum polarization beamwidth.

Typically, a pyramidal horn antenna will have E- and H-plane beamwidths determined by radiation intensity that are much smaller than the minimum polarization beamwidth. In many situations we may therefore neglect polarization in the main beam. It is a different matter for horns other than pyramidal, however; the E-plane sectoral horn generally has a large H-plane radiation intensity beamwidth, and the H-plane sectoral horn has a large E-plane radiation intensity beamwidth. It is necessary therefore to consider polarization effects in the main beam of the E- and H-plane sectoral horns.

4.10. PARABOLOIDAL REFLECTOR

The surface formed by rotating a parabola about its axis is the most frequently used reflector antenna. Geometric optics, valid for vanishingly small wavelengths, shows that rays from the focal point are reflected in a beam parallel to the axis. Analyses using finite wavelengths show that the beam diverges, but the beamwidth is small for a paraboloid whose dimensions are large compared to a wavelength.

In this section we will carry out an analysis to find the fields produced by a



FIGURE 4.9. Paraboloidal reflector and aperture plane.

172 POLARIZATION CHARACTERISTICS OF SOME ANTENNAS

source at the focal point using the aperture distribution technique. The field reflected by the paraboloid is found over a plane passing through the focal point and perpendicular to the paraboloid axis using geometric optics techniques. Equivalent sources are formed over that plane, and the far fields are obtained by integration over these sources using the procedures of Section 1.13. This method and others for finding the fields of a paraboloid are discussed in standard texts [2-4].

The geometry of the reflection problem is shown in Fig. 4.9, and a cross-sectional view of the reflector in a plane $\phi' = \text{constant}$ appears as Fig. 4.10. From the equation for the paraboloid surfaces the unit normal **n** may be found [2],

$$\mathbf{n} = -\mathbf{u}_{r'} \cos\left(\frac{1}{2}\theta'\right) + \mathbf{u}_{\theta'} \sin\left(\frac{1}{2}\theta'\right) \tag{4.60}$$

and from **n** the angles α and β are easily shown to be equal to each other, and

$$\alpha = \beta = \frac{1}{2}\theta' \tag{4.61}$$

Since α is the angle from the surface normal measured to an incident ray from the focal point, and β is the angle from **n** to a ray parallel to the paraboloid axis, it is clear that all rays from the focal point are reflected parallel to each other.

A source at the origin produces a wave that is incident on the reflector



FIGURE 4.10. Cross section of reflector.

surface, inducing a surface current with density

$$\mathbf{J}_{s} = \mathbf{n} \times \mathbf{H} = \mathbf{n} \times (\mathbf{H}_{r} + \mathbf{H}_{r}) \tag{4.62}$$

where \mathbf{H}_i and \mathbf{H}_i are incident and reflected fields. If we can approximate the surface in the vicinity of the reflection point by an infinite plane conductor, the tangential components of incident and reflected magnetic fields are equal, and

$$\mathbf{n} \times \mathbf{H}_{i} = \mathbf{n} \times \mathbf{H}_{i} \tag{4.63}$$

which allows J_{c} to be written as

$$\mathbf{J}_{s} = 2\mathbf{n} \times \mathbf{H}_{i} = 2\mathbf{n} \times \mathbf{H}_{r} \tag{4.64}$$

We next require all points on the reflecting surface to be in the far field of the source so that the electric field of the incident wave is transverse to the magnetic field and the direction of propagation and is related to \mathbf{H}_i by the intrinsic impedance of free space, Z_0 . Then

$$\mathbf{J}_{s} = \frac{2}{Z_{0}} \left[\mathbf{n} \times (\mathbf{u}_{r} \times \mathbf{E}_{i}) \right]$$
(4.65)

where $\mathbf{u}_{r'}$ is a unit vector directed from the focal point to the point of reflection. In the same way the electric field of the reflected wave is related to the reflected magnetic field, and we have at the reflector surface

$$\mathbf{J}_s = \frac{2}{Z_0} \left[\mathbf{n} \times (-\mathbf{u}_z \times \mathbf{E}_r) \right]$$
(4.66)

with $-\mathbf{u}_z$ a unit vector from the reflection point and antiparallel to the z axis.

Let a source at the focal point with gain G accept power W, from a generator. The power density is then

$$S(r', \theta', \phi') = \frac{W_{t}G(\theta', \phi')}{4\pi {r'}^{2}}$$
(4.67)

From the relationship

$$S = \frac{1}{2Z_0} |\mathbf{E}|^2 \tag{4.68}$$

we can find the field of the incident wave at the reflector surface.

$$\mathbf{E}_{i}(r',\,\theta',\,\phi') = \mathbf{e}_{i} \left[Z_{0} \,\frac{W_{i}}{2\pi} \,G(\theta',\,\phi') \right]^{1/2} \,\frac{e^{-jkr'}}{r'} \tag{4.69}$$

where \mathbf{e}_i is a unit vector perpendicular to \mathbf{u}_r . We extract the constants from this equation and define

$$C = \sqrt{Z_0 W_t / 2\pi} \tag{4.70}$$

We may then write

$$\mathbf{E}_{i}(r',\,\theta',\,\phi') = \mathbf{e}_{i}C\sqrt{G(\theta',\,\phi')}\,\frac{e^{-jkr'}}{r'} \tag{4.71}$$

If this expression for \mathbf{E}_i is substituted into (4.65), we may write the surface current density in terms of the source parameters as

$$\mathbf{J}_{s} = \frac{2}{Z_{0}} C \sqrt{G(\theta', \phi')} \frac{e^{-j\kappa r}}{r'} \mathbf{a}$$
(4.72)

where

$$\mathbf{a} = \mathbf{n} \times (\mathbf{u}_{i} \times \mathbf{e}_{i}) \tag{4.73}$$

We may also write the second expression for J_s , (4.66), as

$$\mathbf{J}_{s} = \frac{2}{Z_{0}} E_{r} [\mathbf{n} \times (-\mathbf{u}_{z} \times \mathbf{e}_{r})]$$
(4.74)

where E_r is the reflected field value at the surface and e_r is a unit vector expressing its polarization (considered here a real vector). Since both

 $\mathbf{n} \times (\mathbf{u}_r, \times \mathbf{e}_i)$

and

$$\mathbf{n} \times (-\mathbf{u}, \times \mathbf{e}_r)$$

are unit vectors, and since both give the direction of \mathbf{J}_s , it is apparent that they are equal and that at the surface

$$\mathbf{E}_{r} = C\sqrt{G(\theta', \phi')} \, \frac{e^{-jkr'}}{r'} \, \mathbf{e}_{r}$$
(4.75)

Source Polarized in the y Direction

At this point it is difficult to proceed further without assuming a polarization for the source. If we take it to be in the y direction, we may write \mathbf{e}_i as PARABOLOIDAL REFLECTOR 175

$$\mathbf{e}_{i} = \frac{\mathbf{u}_{r} \times (\mathbf{u}_{y} \times \mathbf{u}_{r})}{|\mathbf{u}_{r} \times (\mathbf{u}_{y} \times \mathbf{u}_{r})|}$$
(4.76)

and if \mathbf{u}_y , the unit vector in the y direction, is expanded as

$$\mathbf{u}_{y} = \sin \theta' \sin \phi' \mathbf{u}_{r'} + \cos \theta' \sin \phi' \mathbf{u}_{\theta'} + \cos \phi' \mathbf{u}_{\phi'}$$
(4.77)

it is straightforward to show that

$$\mathbf{e}_{i} = \frac{\cos \theta' \sin \phi' \mathbf{u}_{\theta'} + \cos \phi' \mathbf{u}_{\phi'}}{\sqrt{1 - \sin^{2} \theta' \sin^{2} \phi'}}$$
(4.78)

It is tedious, but not difficult, to obtain **a** in rectangular coordinates by substituting \mathbf{e}_i into (4.73) and transforming all vectors to rectangular coordinates. The result is

$$\mathbf{a} = \frac{-\sin\theta'\sin(\theta'/2)\sin\phi'\cos\phi'}{\sqrt{1-\sin^2\theta'\sin^2\phi'}} \mathbf{u}_x + \frac{\cos(\theta'/2)(\cos\theta'\sin^2\phi'+\cos^2\phi')}{\sqrt{1-\sin^2\theta'\sin^2\phi'}} \mathbf{u}_y - \frac{\sin(\theta'/2)\cos\theta'\sin\phi'}{\sqrt{1-\sin^2\theta'\sin^2\phi'}} \mathbf{u}_z$$
(4.79)

We may next find e, by writing a as

$$\mathbf{a} = \mathbf{n} \times (-\mathbf{u}_z \times \mathbf{e}_r) = -(\mathbf{n} \cdot \mathbf{e}_r)\mathbf{u}_z + (\mathbf{n} \cdot \mathbf{u}_z)\mathbf{e}_r$$
$$= -(\mathbf{n} \cdot \mathbf{e}_r)\mathbf{u}_z - \cos\left(\frac{1}{2}\theta'\right)\mathbf{e}_r \qquad (4.80)$$

If these expressions for a are equated, we may quickly obtain the components of \mathbf{e} , transverse to z. (We need not be concerned with ascertaining if \mathbf{e} , has a z component since it will not contribute to equivalent surface currents over the aperture plane.) The result is

$$\mathbf{e}_{r} = \frac{\sin \phi' \cos \phi' (1 - \cos \theta') \mathbf{u}_{x} - (\cos \theta' \sin^{2} \phi' + \cos^{2} \phi') \mathbf{u}_{y}}{\sqrt{1 - \sin^{2} \theta' \sin^{2} \phi'}} \quad (4.81)$$

We now make the assumption that the electric field intensity at a point on the aperture plane is given by the field intensity transverse to z at a corresponding point (same x and y coordinates) on the reflector, except for the phase retardation

$$kr'\cos\theta'$$

caused by the path from reflector to aperture [3]. The aperture field is then given by

$$\mathbf{E}_{ap} = C\sqrt{G_y(\theta', \phi')} \frac{e^{-jkr'(1+\cos\theta')}}{r'} \mathbf{e}_r$$
(4.82)

where we use G_y as a reminder that the source is y directed.

If we write E_{ap} as

$$\mathbf{E}_{ap} = \mathbf{u}_{x} E_{ax} + \mathbf{u}_{y} E_{ay} \tag{4.83}$$

where

$$E_{ax} = C\sqrt{G_y(\theta', \phi')} \frac{e^{-jkr'(1+\cos\theta')}}{r'} \frac{\sin\phi'\cos\phi'(1-\cos\theta')}{\sqrt{1-\sin^2\theta'\sin^2\phi'}}$$
(a)

$$E_{ay} = -C\sqrt{G_y(\theta', \phi')} \frac{e^{-jkr'(1+\cos\theta')}}{r'} \frac{\cos\theta'\sin^2\phi' + \cos^2\phi'}{\sqrt{1-\sin^2\theta'\sin^2\phi'}}$$
(b)

we can find equivalent surface currents on the aperture plane by

$$\mathbf{M}_{sa} = -\mathbf{u}_{z} \times \mathbf{E}_{ap} = \mathbf{u}_{x} E_{ay} - \mathbf{u}_{y} E_{ax}$$
(a)

$$\mathbf{J}_{sa} = \mathbf{u}_{z} \times \mathbf{H}_{ap} = -\mathbf{u}_{x}H_{ay} + \mathbf{u}_{y}H_{ax} = -\mathbf{u}_{x}\frac{E_{ax}}{Z_{0}} - \mathbf{u}_{y}\frac{E_{ay}}{Z_{0}} \quad (b)$$
(4.85)

We fill the region on the reflector side of the aperture plane with a perfect electric conductor, as discussed in Section 1.11, and apply image theory, as in Section 1.13. The result is that we double M_{sa} over the aperture plane and let J_{sa} be zero. Further, we assume that M_{sa} has value only over the circle on the aperture plane, which is the projection of the paraboloidal reflector on the plane. Since we are concerned only with the far fields, we can use the approximation

$$\mathbf{F}(\mathbf{r}) = \frac{\varepsilon e^{-jkr}}{4\pi r} \int \int 2\mathbf{M}_{sa}(\mathbf{r}') e^{jk\mathbf{u}_r \cdot \mathbf{r}'} dA'$$
(4.86)

where

$$\mathbf{u}_r \cdot \mathbf{r}' = x' \sin \theta \cos \phi + y' \sin \theta \sin \phi \tag{4.87}$$

Substitution of the magnetic surface current value of (4.85) into the equation for $F(\mathbf{r})$ gives

$$F_{x} = \frac{\varepsilon e^{-jkr}}{2\pi r} \int \int E_{ay} e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)} dx' dy' \quad (a)$$

$$F_{y} = -\frac{\varepsilon e^{-jkr}}{2\pi r} \int \int E_{ax} e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)} dx' dy' \quad (b)$$
(4.88)

PARABOLOIDAL REFLECTOR

where the integration is carried out over the projection of the reflector onto the aperture plane.

The electric field components may be found from

$$E_{\theta} = -j\omega Z_0 F_{\phi} = -j\omega Z_0 (-F_x \sin \phi + F_y \cos \phi)$$
(a)

$$E_{\phi} = j\omega Z_0 F_{\theta} = j\omega Z_0 (F_x \cos \theta \cos \phi + F_y \cos \theta \sin \phi)$$
(b)
(4.89)

Substitution of the components of F leads to

$$E_{\theta} = \frac{j\omega Z_{0}\varepsilon e^{-jkr}}{2\pi r} \int \int (E_{ax}\cos\phi + E_{ay}\sin\phi) \times e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)} dx' dy'$$
(a)

$$E_{\phi} = \frac{j\omega Z_0 \varepsilon e^{-jkr}}{2\pi r} \cos\theta \int \int (-E_{ax} \sin\phi + E_{ay} \cos\phi)$$
(4.90)
 $\times e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)} dx' dy'$ (b)

Using the geometry of the reflector, we may change variables in the integrals and achieve forms that may be easier to evaluate. We may express r' in Fig. 4.10 as [2]

$$r' = \frac{2f}{1 + \cos\theta'} \tag{4.91}$$

and also

$$x' = r' \sin \theta' \cos \phi' \quad (a)$$

$$y' = r' \sin \theta' \sin \phi' \quad (b) \quad (4.92)$$

$$z' = r' \cos \theta' \quad (c)$$

With these changes, the aperture fields become

$$E_{ax} = C\sqrt{G_y(\theta', \phi')} \frac{(1 + \cos \theta')e^{-j2kf}}{2f} \frac{\sin \phi' \cos \phi'(1 - \cos \theta')}{\sqrt{1 - \sin^2 \theta' \sin^2 \phi'}} \quad (a)$$

$$E_{ay} = -C\sqrt{G_y(\theta', \phi')} \frac{(1 + \cos \theta')e^{-j2kf}}{2f} \frac{\cos \theta' \sin^2 \phi' + \cos^2 \phi'}{\sqrt{1 - \sin^2 \theta' \sin^2 \phi'}} \quad (b)$$

and the radiated fields are

POLARIZATION CHARACTERISTICS OF SOME ANTENNAS

$$E_{\theta} = \frac{j\omega Z_{0}\varepsilon e^{-jkr}}{2\pi r} \int \int (E_{ax}\cos\phi + E_{ay}\sin\phi)$$

$$\times \exp\left(j2kf\,\frac{\sin\theta'\sin\theta\cos(\phi'-\phi)}{1+\cos\theta'}\right) \frac{(2f)^{2}\sin\theta'}{(1+\cos\theta')^{2}}\,d\theta'\,d\phi' \quad (a)$$

$$E_{\phi} = \frac{j\omega Z_{0}\varepsilon e^{-jkr}}{2\pi r}\,\cos\theta \int \int (-E_{ax}\sin\phi + E_{ay}\cos\phi) \quad (4.94)$$

$$\times \exp\left(j2kf\,\frac{\sin\theta'\sin\theta\cos(\phi'-\phi)}{1+\cos\theta'}\right) \frac{(2f)^{2}\sin\theta'}{(1+\cos\theta')^{2}}\,d\theta'\,d\phi' \quad (b)$$

Principal Plane Fields: y-Polarized Source

The expressions for the fields in general require numerical integration, but they can be simplified if we consider the principal *E* and *H* planes only, given, respectively, for the *y*-polarized source by $\phi = \frac{1}{2}\pi$ and $\phi = 0$.

In the principal E plane E_{ϕ} reduces to

$$E_{\phi} = \frac{j\omega Z_0 \varepsilon e^{-jkr}}{2\pi r} \cos\theta \int \int E_{ax} e^{jky'\sin\theta} dx' dy'$$
(4.95)

where E_{ax} may be written from (4.84) and (4.92) as

$$E_{ax} = C\sqrt{G_{y}(\theta', \phi')} \frac{e^{-jkr'(1+\cos\theta')}}{r'} \frac{x'y'(1-\cos\theta')}{r'^{2}\sin^{2}\theta'\sqrt{1-\sin^{2}\theta'\sin^{2}\phi'}}$$
(4.96)

We have previously restricted our development to a y-directed primary source. Let us now make the assumption that the gain $G_y(\theta', \phi')$ is symmetric so that the x-directed fields over the aperture plane have the same magnitude at symmetrically located points, as shown in Fig. 4.11. Then

$$G_{y}(x', y') = G_{y}(-x', y') = G_{y}(x', -y') = G_{y}(-x', -y')$$
(4.97)

It follows that

$$E_{ax}(x', y') = -E_{ax}(-x', y') = -E_{ax}(x', -y') = E_{ax}(-x', -y') \quad (4.98)$$

and the contributions to the integral for E_{ϕ} are

$$E_{ax}(x', y')e^{jk|y'|\sin\theta} - E_{ax}(x', y')e^{-jk|y'|\sin\theta}$$
$$-E_{ax}(x', y')e^{jk|y'|\sin\theta} E_{ax}(x', y')e^{-jk|y'|\sin\theta}$$

178



FIGURE 4.11. Aperture plane for y-directed source.

We see that these contributions cancel in pairs. It follows that in the principal E plane $E_{\phi} = 0$, and the radiated field has only an E_{θ} component.

In the principal H plane, $\phi = 0$, E_{θ} becomes

$$E_{\theta} = j\omega Z_0 \frac{\varepsilon e^{-jkr}}{2\pi r} \int \int E_{ax} e^{jkx'\sin\theta} dx' dy'$$
(4.99)

where E_{ax} is given by (4.96).

If the analysis carried out for the principal E plane is repeated, we will find that $E_{\theta} = 0$ in the principal H plane.

If the fields in the principal planes are converted to rectangular coordinates, it may be seen that $E_x = 0$ in both planes, and in the *H* plane the field has only an E_y component. We recognize that both statements would be correct for a y-directed dipole, for example, without the parabolic reflector, and it is interesting to note that the conditions carry over to the reflector antenna if the required symmetry conditions are met.

Source Polarized in the x Direction

If the source is x polarized instead of y, the vector (4.76) for the field incident on the reflector should be altered to

$$\mathbf{e}'_{i} = \frac{\mathbf{u}_{r'} \times (\mathbf{u}_{x} \times \mathbf{u}_{r'})}{|\mathbf{u}_{r'} \times (\mathbf{u}_{x} \times \mathbf{u}_{r'})|}$$
(4.100)

which becomes

$$\mathbf{e}'_{i} = \frac{\cos \theta' \cos \phi' \mathbf{u}_{\theta'} - \sin \phi' \mathbf{u}_{\phi'}}{\sqrt{1 - \sin^{2} \theta' \cos^{2} \phi'}}$$
(4.101)

The vector \mathbf{a} , given by (4.79) for the y-directed source, becomes, for the x-polarized case,

$$\mathbf{a}' = \frac{\cos\left(\theta'/2\right)(\sin^2 \phi' + \cos \theta' \cos^2 \phi')}{\sqrt{1 - \sin^2 \theta' \cos^2 \phi'}} \mathbf{u}_x - \frac{\sin \theta' \sin\left(\theta'/2\right) \sin \phi' \cos \phi'}{\sqrt{1 - \sin^2 \theta' \cos^2 \phi'}} \mathbf{u}_y - \frac{\cos \theta' \sin\left(\theta'/2\right) \cos \phi'}{\sqrt{1 - \sin^2 \theta' \cos^2 \phi'}} \mathbf{u}_z$$
(4.102)

The vector \mathbf{e}'_r representing the polarization of the reflected field is found by the process used earlier and is

$$\mathbf{e}'_{r} = -\frac{\sin^{2} \phi' + \cos \theta' \cos^{2} \phi'}{\sqrt{1 - \sin^{2} \theta' \cos^{2} \phi'}} \mathbf{u}_{x} + \frac{\sin \phi' \cos \phi' (1 - \cos \theta')}{\sqrt{1 - \sin^{2} \theta' \cos^{2} \phi'}} \mathbf{u}_{y} \quad (4.103)$$

This equation can be verified, or in fact could have been derived, by noting that the x-polarized source is the y-polarized source rotated in azimuth by -90° . Then we should have

$$e'_{rx}(\phi' + \frac{1}{2}\pi) = e_{ry}(\phi')$$
 (a)
 $e'_{ry}(\phi' + \frac{1}{2}\pi) = -e_{rx}(\phi')$ (b)
(4.104)

These equalities are easily verified.

The aperture fields may now be found from

$$E'_{ax} = -C\sqrt{G_x(\theta', \phi')} \frac{e^{-jkr'(1+\cos\theta')}}{r'} \frac{\sin^2\phi' + \cos\theta'\cos^2\phi'}{\sqrt{1-\sin^2\theta'\cos^2\phi'}} \quad (a)$$

$$(4.105)$$

$$E'_{ay} = C\sqrt{G_x(\theta', \phi')} \frac{e}{r'} \frac{\sin \phi \cos \phi \left(1 - \cos \phi\right)}{\sqrt{1 - \sin^2 \theta' \cos^2 \phi'}} \quad (b)$$

and the radiated fields found from (4.90) as before.

Principal Plane Fields: x-Polarized Source

Earlier we considered the fields in the principal planes for a y-polarized source with gain symmetry and found E_{ϕ} to be zero in the E plane, $\phi = \frac{1}{2}\pi$, and E_{θ} to be zero in the H plane, $\phi = 0$. If this development is repeated for an x-polarized source with symmetric gain

$$G_x(x', y') = G_x(-x', y') = G_x(x', -y') = G_x(-x', -y')$$
(4.106)

a result is that in the E plane, $\phi = 0$, of the x-polarized source, $E_{\phi} = 0$. In the

H plane, $\phi = \frac{1}{2}\pi$, the component $E_{\theta} = 0$. In both planes the rectangular component E_y is zero, an unsurprising result for an *x*-polarized source. Note that the *E* plane for an *x*-polarized source is the *H* plane for a *y*-polarized source, and vice versa.

Dipole Feed Antenna: y Directed

A commonly used feed antenna is a center-fed dipole. We found in Section 4.3 that the field components of a y-directed short dipole are

$$E_{r} = 0 \qquad (a)$$

$$E_{\theta} = -\frac{j\omega\mu I\ell}{4\pi r'} e^{-jkr'} \cos \theta' \sin \phi' \qquad (b) \qquad (4.107)$$

$$E_{\phi} = \frac{j\omega\mu I\ell}{4\pi r'} e^{-jkr'} \cos \phi' \qquad (c)$$

where the factor 4 in the denominators is doubled if a triangular current distribution is assumed. With this feed the incident field magnitude at the reflector surface is

$$|\mathbf{E}_i|^2 = \left(\frac{\omega\mu I\ell}{4\pi r'}\right)^2 (1 - \sin^2\theta' \sin^2\phi') \tag{4.108}$$

Comparison of this equation with that for the incident field, (4.71), shows that for the y-directed dipole, we should use

$$C\sqrt{G_y(\theta',\phi')} = \frac{j\omega\mu I\ell}{4\pi} \sqrt{1-\sin^2\theta'\sin^2\phi'}$$
(4.109)

Substitution into the aperture fields simplifies them to

$$E_{ax} = \frac{j\omega\mu I\ell}{4\pi r'} \sin \phi' \cos \phi' (1 - \cos \theta') e^{-jkr'(1 + \cos \theta')}$$
(a)

$$E_{ay} = \frac{-j\omega\mu I\ell}{4\pi r} (\cos \theta' \sin^2 \phi' + \cos^2 \phi') e^{-jkr'(1 + \cos \theta')}$$
(b)

Principal Plane Polarization: y-Directed Dipole

We saw earlier that for a y-directed source with symmetric gain, $E_{\phi} = 0$ in the principal E plane and $E_{\theta} = 0$ in the principal H plane. Using this finding, the fields produced by a y-directed dipole feed become

$$E \text{ PLANE, } \phi = \frac{1}{2}\pi$$

$$E_{\theta} = j\omega Z_{0} \frac{\varepsilon e^{-jkr}}{2\pi r} \int \int E_{ay} e^{jky'\sin\theta} dx' dy'$$

$$= j\omega Z_{0} \frac{\varepsilon e^{-jkr}}{2\pi r} \int \int \left(-\frac{j\omega\mu I\ell}{4\pi r'}\right) (\cos\theta'\sin^{2}\phi' + \cos^{2}\phi')$$

$$\times e^{-jkr'(1+\cos\theta')} e^{jky'\sin\theta} dx' dy'$$

$$E_{\phi} = 0$$
(b)

The polarization ratio is

$$P = \infty \tag{4.112}$$

$$H \text{ PLANE, } \phi = 0 \tag{a}$$

$$E_{\phi} = j\omega Z_{0} \frac{\varepsilon e^{-jkr}}{2\pi r} \cos\theta \int \int E_{ay} e^{jkx'\sin\theta} dx' dy'$$

$$= j\omega Z_{0} \frac{\varepsilon e^{-jkr}}{2\pi r} \cos\theta \int \int \left(\frac{-j\omega\mu I\ell}{4\pi r'}\right) (\cos\theta'\sin^{2}\phi' + \cos^{2}\phi')$$

$$\times e^{-jkr'(1+\cos\theta')} e^{jkx'\sin\theta} dx' dy'$$
(4.113)

The polarization ratio is

$$P = 0 \tag{4.114}$$

Dipole Feed Antenna: x Directed

We found in Section 4.3 the field components of a short x-directed dipole to be

$$E_{\theta} = \frac{-j\omega\mu I\ell}{4\pi r'} e^{-jkr'} \cos \theta' \cos \phi' \quad (a)$$

$$E_{\phi} = \frac{j\omega\mu I\ell}{4\pi r'} e^{-jkr'} \sin \phi' \quad (b)$$

It follows from the same reasoning used previously that we should let in (4.105)

$$\dot{C}\sqrt{G_x(\theta',\phi')} = \frac{j\omega\mu I\ell}{4\pi}\sqrt{1-\sin^2\theta'\cos^2\phi'}$$
(4.116)

The aperture fields for the x-directed dipole reduce to

$$E'_{ax} = -\frac{j\omega\mu I\ell}{4\pi r'} (\sin^2 \phi' + \cos \theta' \cos^2 \phi') e^{-jkr'(1+\cos \theta')}$$
(a)

$$E'_{ay} = \frac{j\omega\mu I\ell}{4\pi r'} \sin \phi' \cos \phi'(1-\cos \theta') e^{-jkr'(1+\cos \theta')}$$
(b)
(4.117)

Principal Plane Polarization: x-Directed Dipole

We saw earlier that for an x-polarized source, $E_{\phi} = 0$ in the principal E plane, $\phi = 0$, and $E_{\theta} = 0$ in the H plane, $\phi = \frac{1}{2}\pi$. Using this information and the fields of the x-directed dipole in (4.90), we find in the principal planes.

E PLANE,
$$\phi = 0$$

$$E'_{\theta} = j\omega Z_0 \frac{\varepsilon e^{-jkr}}{2\pi r} \int \int E'_{ax} e^{jkx' \sin \theta} dx' dy'$$

= $j\omega Z_0 \frac{\varepsilon e^{-jkr}}{2\pi r} \int \int \left(-\frac{j\omega\mu I\ell}{4\pi r'} \right) (\sin^2 \phi' + \cos \theta' \cos^2 \phi')$
 $\times e^{-jkr'(1+\cos \theta')} e^{jkx' \sin \theta} dx' dy'$ (a)
 $E'_{\phi} = 0$ (b)

$$P = \infty$$
 (c)

 $H \text{ PLANE, } \phi = \frac{1}{2}\pi$ $E'_{\theta} = 0 \tag{a}$

$$E'_{\phi} = j\omega Z_0 \frac{\varepsilon e^{-jkr}}{2\pi r} \cos\theta \int \int -E'_{ax} e^{jky'\sin\theta} dx' dy'$$

$$= j\omega Z_0 \frac{\varepsilon e^{-jkr}}{2\pi r} \cos\theta \int \int \frac{j\omega\mu I\ell}{4\pi r'} (\sin^2\phi' + \cos\theta'\cos^2\phi')$$

$$\times e^{-jkr'(1+\cos\theta')} e^{jky'\sin\theta} dx' dy' \qquad (b)$$

$$P = 0$$
 (c)

Crossed-Dipole Feed

A circularly polarized wave can be produced on the paraboloid axis if crossed dipoles with a $\frac{1}{2}\pi$ phase difference are used as the feed antenna. The fields in a general direction must be computed numerically, but the polarization ratio

in two planes is relatively simple. We will therefore restrict our consideration to the planes $\phi = 0$ and $\phi = \frac{1}{2}\pi$.

In the equations for the dipole feeds previously developed we let the y-directed dipole lead in phase by $\frac{1}{2}\pi$ and use currents I and jI, respectively, in the x- and y-directed dipoles. Then, in general, the fields produced by the crossed-dipole feed are

$$\hat{E}_{\theta} = jE_{\theta} + E'_{\theta} \quad (a)$$

$$\hat{E}_{\phi} = jE_{\phi} + E'_{\phi} \quad (b)$$
(4.120)

where E_{θ} , E_{ϕ} , E'_{θ} , and E'_{ϕ} are given by (4.111), (4.113), (4.118), and (4.119).

In the plane $\phi = 0$ these equations give

$$\hat{E}_{\theta} = E'_{\theta} \quad (a)$$

$$\hat{E}_{\phi} = jE_{\phi} \quad (b)$$
(4.121)

where E'_{θ} and E_{ϕ} are given by (4.118) and (4.113). From these equations the polarization ratio may be written as

$$P = -\frac{\hat{E}_{\theta}}{\hat{E}_{\phi}} = \frac{jE'_{\theta}}{E_{\phi}}$$
$$= \frac{j}{\cos\theta} \frac{\int \int (1/r')(\sin^2\phi' + \cos\theta'\cos^2\phi')e^{-jkr'(1+\cos\theta')}e^{jkx'\sin\theta}dx'dy'}{\int \int (1/r')(\cos\theta'\sin^2\phi' + \cos^2\phi')e^{-jkr'(1+\cos\theta')}e^{jkx'\sin\theta}dx'dy'}$$
(4.122)

Now if we use (4.92), portions of the integrands in the numerator and denominator of this expression can be written as

$$\sin^{2} \phi' + \cos \theta' \cos^{2} \phi' = \frac{1}{r'^{2} \sin^{2} \theta'} \left(y'^{2} + \frac{z'}{r'} x'^{2} \right) \quad \text{(a)}$$

$$\cos \theta' \sin^{2} \phi' + \cos^{2} \phi' = \frac{1}{r'^{2} \sin^{2} \theta'} \left(\frac{z'}{r'} y'^{2} + x'^{2} \right) \quad \text{(b)}$$
(4.123)

We may interchange the variables x' and y' without having any effect on other terms in the integrands, and we note that the limits on x' and y' are the same. Therefore, the numerator and denominator integrals in (4.122) are equal, and in the plane $\phi = 0$ the paraboloidal reflector with crossed-dipole feed has polarization ratio

184

PARABOLOIDAL REFLECTOR 185

$$P = \frac{j}{\cos \theta} \qquad \frac{\pi}{2} \le \theta \le \pi \tag{4.124}$$

In the plane $\phi = \frac{1}{2}\pi$ the fields are

$$\hat{E}_{\theta} = jE_{\theta}$$
 (a)
 $\hat{E}_{\phi} = E'_{\phi}$ (b) (4.125)

and the polarization ratio becomes

$$P = \frac{-jE_{\theta}}{E_{\phi}'}$$

$$= \frac{j}{\cos\theta} \frac{\int \int (1/r')(\cos\theta'\sin^2\phi' + \cos^2\phi')e^{-jkr'(1+\cos\theta')}e^{jky'\sin\theta}dx'dy'}{\int \int (1/r')(\sin^2\phi' + \cos\theta'\cos^2\phi')e^{-jkr'(1+\cos\theta')}e^{jky'\sin\theta}dx'dy'}$$
(4.126)

As before, it is easily shown that the integrands are equal, and the polarization ratio in the plane $\phi = \frac{1}{2}\pi$ becomes

$$P = \frac{j}{\cos \theta} \qquad \frac{\pi}{2} \le \theta \le \pi \tag{4.127}$$

which is what it was also for the $\phi = 0$ plane.

The polarization match factor in either plane is obtained by using a circularly polarized receiver with

$$P_r = -j \tag{4.128}$$

since in the region of interest $\cos \theta$ is negative. The polarization match factor then becomes

$$\rho = \frac{1}{2} - \frac{\cos\theta}{1 + \cos^2\theta} \tag{4.129}$$

If this value of ρ is compared to that of (4.32) for the crossed dipoles without the parabolic reflector, it may be seen that they are the same if the different reference for the measurement of polar angle θ is considered. Another difference is that for the reflector antenna, the equation (4.129) for ρ is valid only in the planes $\phi = 0$ and $\phi = \frac{1}{2}\pi$, whereas for the crossed dipoles alone (4.32) is valid everywhere.

It is obvious from (4.129) that the 3-dB polarization beamwidth for the

crossed-dipole feed is π in the planes $\phi = 0$ and $\phi = \frac{1}{2}\pi$. On the other hand, the 3-dB radiation intensity beamwidth of a parabolic reflector is approximately λ/D for a uniformly illuminated aperture [1]. For a tapered illumination, which would occur with a crossed-dipole feed, the beamwidth will be somewhat greater. In addition, the polarization beamwidth obtained here is valid only in two planes. Nevertheless, it is clear that for large aperture diameter, polarization effects will be small in the main beam of the parabolic reflector if received power is the quantity of interest.

4.11. NARROW-POLARIZATION-BEAMWIDTH ARRAY

In our examination of the polarization characteristics of various antennas we have not yet encountered one with a small polarization beamwidth, even though some of them have small radiation intensity beamwidths. In this section we will examine an array, shown in Fig. 4.12, that can produce narrow beams in both radiation intensity and polarization. The array elements will be treated as short dipoles, although other linearly polarized elements could be used. The array is intended to produce a circularly polarized wave in the main beam, so we assume that the phases of the elements along the y axis lead those of the x axis elements by $\frac{1}{2}\pi$ when the beam is broadside. An even number of elements is shown for each linear array, but an odd number can be used without changing the equations of this section.



FIGURE 4.12. Narrow-polarization-beamwidth array.

NARROW-POLARIZATION-BEAMWIDTH ARRAY

For simplicity, a uniform array, with equal feed amplitudes for all elements and a constant difference between the phases of adjacent elements, is assumed on both x and y axes. Furthermore, the same number of elements, with the same spacing, is assumed for the x- and y-axis arrays.

The array factor for a linear array of isotropic elements along the axes is [2]

$$AF = \frac{1}{N} \frac{\sin [(N/2)\psi]}{\sin [(1/2)\psi]}$$
(4.130)

where

$$\psi = \psi_x = kd \sin \theta \cos \phi + \beta_x \qquad x \text{ axis} \quad (a)$$

$$\psi = \psi_y = kd \sin \theta \sin \phi + \beta_y \qquad y \text{ axis} \quad (b)$$
(4.131)

with β_x and β_y the feed phase differences.

If we use the pattern multiplication principle of array theory and the fields of x- and y-directed short dipoles of (4.19) and (4.20), the fields produced by the x-directed dipoles along the x axis are

$$E_{r} = 0$$
(a)

$$E_{\theta} = -\frac{j\omega\mu I\ell}{4\pi r} \cos\theta \cos\phi \frac{\sin\left[(N/2)(kd\sin\theta\cos\phi + \beta_{x})\right]}{N\sin\left[(1/2)(kd\sin\theta\cos\phi + \beta_{x})\right]} e^{-jkr}$$
(b)
(4.132)

$$E_{\phi} = \frac{j\omega\mu I\ell}{4\pi r} \sin\phi \, \frac{\sin\left[(N/2)(kd\sin\theta\cos\phi + \beta_x)\right]}{N\sin\left[(1/2)(kd\sin\theta\cos\phi + \beta_x)\right]} \, e^{-jkr} \tag{c}$$

and the fields produced by the array of y-directed dipoles along the y axis are

$$E_r = 0 \tag{a}$$

$$E_{\theta} = -\frac{j\omega\mu I\ell}{4\pi r} \cos\theta \sin\phi \frac{\sin\left[(N/2)(kd\sin\theta\sin\phi+\beta_{y})\right]}{N\sin\left[(1/2)(kd\sin\theta\sin\phi+\beta_{y})\right]} e^{-jkr}$$
(b)

$$E_{\phi} = -\frac{j\omega\mu I\ell}{4\pi r} \cos\phi \frac{\sin\left[(N/2)(kd\sin\theta\sin\phi+\beta_{y})\right]}{N\sin\left[(1/2)(kd\sin\theta\sin\phi+\beta_{y})\right]} e^{-jkr}$$
(c)
(4.133)

For convenience, we consider only the broadside case, $\beta_x = \beta_y = 0$, and group several factors in the field equations as constant C. If the dipoles along the y axis lead those along the x axis by phase difference $\frac{1}{2}\pi$, the total fields are

POLARIZATION CHARACTERISTICS OF SOME ANTENNAS

$$E_{\theta} = -C \cos \theta \cos \phi \frac{\sin \left[(N/2)kd \sin \theta \cos \phi \right]}{N \sin \left[(1/2)kd \sin \theta \cos \phi \right]}$$

$$-jC \cos \theta \sin \phi \frac{\sin \left[(N/2)kd \sin \theta \sin \phi \right]}{N \sin \left[(1/2)kd \sin \theta \sin \phi \right]} \quad (a)$$

$$E_{\phi} = C \sin \phi \frac{\sin \left[(N/2)kd \sin \theta \cos \phi \right]}{N \sin \left[(1/2)kd \sin \theta \cos \phi \right]}$$

$$-jC \cos \phi \frac{\sin \left[(N/2)kd \sin \theta \sin \phi \right]}{N \sin \left[(1/2)kd \sin \theta \sin \phi \right]} \quad (b)$$

Radiation intensity and polarization ratio are readily found from (4.134). It is sufficient here to consider these quantities only in the yz plane, $\phi = \frac{1}{2}\pi$. The xz plane is obviously like the yz plane. In other planes the beamwidths are more difficult to determine but still may be found from (4.134).

In the yz plane the radiation intensity, normalized to its maximum value at $\theta = 0$, is

$$G_r = \frac{U}{U_{\text{max}}} = \frac{1}{2} \left(\cos^2 \theta \, \frac{\sin^2 \left[(N/2) k d \sin \theta \right]}{N^2 \sin^2 \left[(1/2) k d \sin \theta \right]} + 1 \right)$$
(4.135)

and if we obtain the radiation intensity half-power beamwidth by setting this equal to $\frac{1}{2}$, we find

$$\sin^2\left(\frac{1}{2}Nkd\sin\theta\right) = 0\tag{4.136}$$

The value of θ for which this holds is readily recognized as the first array factor null of the linear array on the y axis.

Still in the yz plane, let us determine the polarization ratio of the wave. It is

$$P = -\frac{E_{\theta}}{E_{\phi}} = j \cos \theta \, \frac{\sin \left[(N/2)kd \sin \theta \right]}{N \sin \left[(1/2)kd \sin \theta \right]} \tag{4.137}$$

from which it is readily seen that the wave is circularly polarized on the z axis.

If we use a circularly polarized receiving antenna in conjunction with this array, the polarization match factor in the yz plane is readily seen to be

$$\rho = \frac{1}{2} + \cos\theta \frac{\sin\left[(N/2)kd\sin\theta\right]}{N\sin\left[(1/2)kd\sin\theta\right]} \Big/ \Big(1 + \cos^2\theta \frac{\sin^2\left[(N/2)kd\sin\theta\right]}{N^2\sin^2\left[(1/2)kd\sin\theta\right]}\Big)$$
(4.138)

If this expression is set equal to $\frac{1}{2}$ to obtain the polarization beamwidth, we again obtain (4.136) and conclude that the polarization beamwidth is the same as the radiation intensity beamwidth and may be quite small for a large number of array elements.

PROBLEMS

The product of G_r and ρ in the yz plane is simpler than either factor alone. It is

$$G_r \rho = \frac{1}{4} \left(1 + \cos \theta \, \frac{\sin \left[(N/2)kd \sin \theta \right]}{N \sin \left[(1/2)kd \sin \theta \right]} \right)^2 \tag{4.139}$$

If we set this to $\frac{1}{2}$ to find the overall beamwidth in the yz plane, we find, not unexpectedy, that the overall beamwidth is smaller than that for the radiation intensity or polarization alone.

It is instructive to study the radiation intensity and polarization if all of the dipole antenna elements are rotated by $\frac{1}{2}\pi$, so that the dipoles on the y axis are oriented in the x direction, and vice versa. On the z axis the wave is still circularly polarized, but the off-axis behavior is different. This is left to the reader as an exercise. One of the problems at the end of this chapter also asks for the polarization behavior if each element in Fig. 4.12 is replaced by crossed dipoles, with each element producing a circularly polarized wave on the z axis.

REFERENCES

- 1. J. D. Kraus, Antennas, McGraw-Hill, New York, 1950.
- 2. C. A. Balanis, Antenna Theory, Harper & Row, New York, 1982.
- S. Silver, Microwave Antenna Theory and Design, MIT Radiation Laboratory Series, Vol. 12, McGraw-Hill, New York, 1949.
- 4. R. E. Collin and F. J. Zucker, Antenna Theory, Vol. 2, McGraw-Hill, New York, 1969.
- R. E. Ziemer and W. H. Tranter, Principles of Communications, Houghton Mifflin, Boston, 1976.

PROBLEMS

- **4.1.** Find the fields of a small circular loop antenna, with uniform current, lying in the xz plane. *Hint*: Compare the fields of y-oriented and z-oriented short dipoles.
- 4.2. Verify the text statement in Section 4.8 that the maximum rate of change of ρ with angle θ occurs, for example, for $\phi = \frac{1}{4}\pi$.
- **4.3.** Find the 3-dB polarization beamwidth in the plane $\phi = \frac{1}{8}\pi$ of the waveguide opening into a plane (Section 4.8).
- **4.4.** Plot the relative radiation intensity as a function of θ in the plane $\phi = \frac{1}{8}\pi$ for the waveguide opening into a plane (Section 4.8). Find the 3-dB beamwidth and compare to the polarization beamwidth of problem 4.3. Assume standard x-band waveguide (a = 0.9 in., b = 0.4 in.) and a frequency of 10 GHz.

190 POLARIZATION CHARACTERISTICS OF SOME ANTENNAS

- **4.5.** Find the 3-dB polarization beamwidth of the pyramidal horn antenna as a function of the azimuth angle ϕ .
- **4.6.** If each dipole element in Fig. 4.12 is rotated by 90° so that the y axis array consists of x-directed dipoles, and vice versa, find the normalized radiation intensity in the yz plane. The other conditions of Section 4.11 remain the same. The number of elements on each axis and the element spacings are the same. All feed amplitudes are equal and the phases of all element feeds in each linear array are the same. The feed phases of the elements on the y axis lead those on the x axis by $\frac{1}{2}\pi$. Compare radiation intensity beamwidth and polarization beamwidth in the yz plane. The receiving antenna is to be circularly polarized.
- 4.7. If each element in Fig. 4.12 is replaced by crossed dipoles, with the y-directed dipole leading the x-directed dipole in phase by $\frac{1}{2}\pi$, and if no phase difference exists between crossed dipoles on the x axis and those on the y axis, compare the radiation intensity and polarization beamwidths.
- **4.8.** Suppose the turnstile antenna of Section 4.4 is used to transmit from an unstabilized satellite so that it rolls and tumbles. Let the earth-based receiving antenna be circularly polarized and always pointing at the transmitter. If all values of angle θ in (4.34) are equally probable, find the expected value of $G_r \rho$ [5, p. 188].



GENERATION OF GENERAL POLARIZATIONS

5.1. INTRODUCTION

It is desirable, when working with elliptically polarized waves, to have an antenna system that can generate any desired polarization and to know, from attenuator and phase shifter settings, what this polarization is. Conversely, we need a receiving antenna that can measure the polarization of an incoming wave. In this chapter, three such antenna systems are described. All have been constructed and found to perform satisfactorily.

5.2. SIMPLE WAVEGUIDE SYSTEM FOR ELLIPTICAL POLARIZATION

Figure 5.1 shows a waveguide and antenna system capable of transmitting a wave with any desired polarization [1]. It consists of a circular horn fed by a circular guide loaded with a quarter-wave plate, a circular waveguide to





circular waveguide rotary joint, a rectangular-to-circular waveguide (TE_{11} mode) transducer, and a rectangular-to-rectangular waveguide collinear rotary joint.

In operation the TE_{11} mode is established in the first circular guide section with a plane of symmetry dependent on the orientation of the input rectangular guide. The quarter-wave plate, whose orientation is independent of the symmetry plane of the mode, then establishes a phase shift for one component of this mode with respect to the orthogonal component.

For reference purposes the broadwall of the input waveguide is taken parallel to the horizontal plane and considered the x axis of the fixed coordinate system. The broadwall of the rotatable rectangular waveguide serves as a reference for the angular displaced axis x' with the angle of displacement β . The angle between the y' axis and the plane of the quarter-wave plate (plane in which the linear component is delayed in phase by $\frac{1}{2}\pi$) is denoted by δ . The unit vectors \mathbf{u}_{\parallel} and \mathbf{u}_{\perp} are, respectively, in the plane and perpendicular to the plane of the quarter-wave plate, and both are transverse to the axis of revolution of the circular horn. Figure 5.2 shows these coordinates.

It is obvious that the first rectangular guide section, apart from serving a transmission function, merely establishes a reference frame. It may be omitted and the rectangular-to-rectangular rotary joint replaced by, for example, a movable coaxial-to-rectangular transducer. Also, any antenna with circular symmetry, such as a cylindrical polyrod, may be used in place of the horn.

The far field transmitted by the antenna is

$$\mathbf{E}(\boldsymbol{\beta}, \boldsymbol{\delta}) = E_{\parallel} \mathbf{u}_{\parallel} + j E_{\perp} \mathbf{u}_{\perp}$$
(5.1)

where E_{\parallel} and E_{\perp} are the relative field strengths, in the plane and perpendicular to the plane of the quarter-wave plate, respectively, on the axis of the horn at the far-field point. From Fig. 5.2 it is seen that



FIGURE 5.2. Coordinates for waveguide system.

$$\mathbf{u}_{\parallel} = \cos\left(\beta + \delta\right)\mathbf{u}_{y} - \sin\left(\beta + \delta\right)\mathbf{u}_{x} \quad (a)$$

$$\mathbf{u}_{\perp} = \sin\left(\beta + \delta\right)\mathbf{u}_{\perp} + \cos\left(\beta + \delta\right)\mathbf{u}_{\perp} \quad (b)$$
(5.2)

and that

$$E_{\parallel} = E_{y} \cos \delta \quad (a)$$

$$E_{\perp} = E_{y} \sin \delta \quad (b)$$
(5.3)

For convenience, (5.1) can be normalized by requiring $|E_{y'}| = 1$. Using (5.2) and (5.3), (5.1) becomes

$$\mathbf{E}(\beta, \delta) = \cos \delta [\cos (\beta + \delta) \mathbf{u}_{y} - \sin (\beta + \delta) \mathbf{u}_{x}] + e^{j\pi/2} \sin \delta [\sin (\beta + \delta) \mathbf{u}_{y} + \cos (\beta + \delta) \mathbf{u}_{x}]$$
(5.4)

which by simple trigonometric manipulation becomes

$$\mathbf{E}(\beta, \delta) = \frac{1}{2} \{ [\mathbf{u}_{y} \cos \beta - \mathbf{u}_{x} \sin \beta] + [\mathbf{u}_{y} \cos (\beta + 2\delta) - \mathbf{u}_{x} \sin (\beta + 2\delta)] + e^{j\pi/2} [\mathbf{u}_{y} \cos \beta - \mathbf{u}_{x} \sin \beta] + e^{-j\pi/2} [\mathbf{u}_{y} \cos (\beta + 2\delta)] - \mathbf{u}_{x} \sin (\beta + 2\delta)] \}$$

$$(5.5)$$

The first and third bracketed terms in (5.5) are identified as unit vectors in the y' direction $(\mathbf{u}_{y'})$. The second and fourth terms describe a unit vector leading $\mathbf{u}_{y'}$ by angle $2\delta(\mathbf{u}_{2\delta})$ as shown in Fig. 5.2.

The linearly polarized field components of (5.5) may be expressed in terms of right circular and left circular rotating components.

$$\mathbf{u}_{L} = e^{-j(\pi/2 + \beta)} \boldsymbol{\omega}_{L} \quad \text{(a)}$$
$$\mathbf{u}_{R} = e^{+j(\pi/2 + \beta)} \boldsymbol{\omega}_{R} \quad \text{(b)}$$

where $\boldsymbol{\omega}_L$ and $\boldsymbol{\omega}_R$ are defined by (2.74). The phase angles of \mathbf{u}_L and \mathbf{u}_R are chosen so that at the time origin they coincide with $\mathbf{u}_{v'}$. Then

$$\mathbf{u}_{y'} = \frac{1}{2}(\mathbf{u}_L + \mathbf{u}_R) \tag{5.7}$$

Making similar substitutions for $\mathbf{u}_{2\delta}$ causes (5.5) to become

$$\mathbf{E}(\beta,\delta) = \frac{1}{2}\sqrt{2}e^{j\pi/4} \left[e^{-j(\beta+\delta+3\pi/4)}\cos\left(\frac{1}{4}\pi+\delta\right)\boldsymbol{\omega}_L + e^{j(\beta+\delta+\pi/4)}\cos\left(\delta-\frac{1}{4}\pi\right)\boldsymbol{\omega}_R\right]$$
(5.8)

From (5.8) we obtain the circular polarization ratio

$$q = e^{-j(2\beta + 2\delta + \pi)} \cot(\delta + \frac{1}{4}\pi)$$
(5.9)

A plot of (5.9) is shown in Fig. 5.3 for $\beta = 0$. It is evident that all possible axial ratios are included as δ increases from $-\frac{1}{4}\pi$ to 0 (left circular to linear vertical polarization), with the field vector rotating in the left-hand sense. The polarization changes from linear vertical to right circular as δ increases from 0 to $+\frac{1}{4}\pi$, with all possible axial ratios included. From (5.9) axial ratio and tilt angle are

$$AR = \frac{\cos(\delta + \pi/4) + \sin(\delta + \pi/4)}{\sin(\delta + \pi/4) - \cos(\delta + \pi/4)}$$
(a)
$$\tau = \frac{1}{2}\pi + \beta + \delta$$
(b)

where the axial ratio is negative for left elliptical polarization.



FIGURE 5.3. Polarization contour on Smith chart used as q and w planes.

ANOTHER WAVEGUIDE SYSTEM

A logical change in this system is the replacement of the quarter-wave plate with a "variable-wave" plate (such as a ferrite slab biased transverse to the direction of propagation and parallel to the slab) which introduces a phase delay ψ into the field component parallel to the plate. Physical considerations fix the angle of polarization inclination δ at $\frac{1}{4}\pi$ radians. This configuration is amenable to the same analysis applied to the quarter-wave plate, with the result

$$q = e^{-j(\pi + 2\beta)} \cot(\frac{1}{2}\psi + \frac{1}{4}\pi)$$
(5.11)

leading to

$$\begin{aligned} & \operatorname{AR} = \operatorname{cot} \left(\frac{1}{2} \psi \right) & (a) \\ & \tau = \frac{1}{2} \pi + \beta & (b) \end{aligned} \right\}^{-\frac{1}{2}} \pi \leq \psi \leq \frac{1}{2} \pi \\ & \operatorname{AR} = \tan \left(\frac{1}{2} \psi \right) & (c) \\ & \tau = \beta & (d) \end{aligned} \right\}^{-\frac{1}{2}} \pi \leq \psi \leq \frac{3}{2} \pi \end{aligned}$$

$$(5.12)$$

This second arrangement allows the replacement of a mechanical rotation by a bias current.

While this discussion has been concerned with the transmission of an elliptically polarized wave, it is obvious that the system can also be used to measure the polarization of an incoming wave. It is left as an exercise to develop the required equations.

5.3. ANOTHER WAVEGUIDE SYSTEM

Figure 5.4 shows a second waveguide system for radiating an elliptical wave with any desired polarization ratio. It has been constructed and found to perform well. The two inputs are fed from a common source using a power splitter. Placing an attenuator before each input and a phase shifter before



FIGURE 5.4. Two-port waveguide system for elliptical waves.

one input allows two orthogonal waveguide modes to be established with relative amplitude and phase controllable over any desired ranges.

The input signal at port 1 establishes a TE_{11} circular guide mode that travels toward the dielectric rod radiating element. The electric field of this mode is horizontal (in the plane of the paper) on the axis of the guide. The input at port 2 establishes the TE_{10} mode in the rectangular guide with a vertical electric field. At the junction of the rectangular and circular guides, this TE_{10} rectangular mode excites a TE_{11} mode in the circular guide with a vertical electric field on the guide axis. A vertical post placed in the circular guide serves to prevent this vertical TE_{11} mode from traveling to the left, toward port 1. A grid of horizontal wires at the junction of the guides.

We then have two orthogonal TE_{11} modes in the circular guide with relative amplitudes and phase difference independently controlled. Off-axis the fields produced by a dielectric rod antenna excited with a TE_{11} mode are complex and will not be discussed here, but on-axis, because of the symmetry of the TE_{11} mode, the vertical TE_{11} mode will produce a vertical linearly polarized wave in the far field. The orthogonal TE_{11} mode will produce a horizontal far field. Since the amplitudes and phase difference can be set at will, it is clear that on the axis a wave of any desired polarization can be radiated.

It is quite clear that this system can also be used to measure the polarization ratio of an incoming wave. The two paths from the rod antenna to ports 1 and 2 are not equivalent, however, so the system must be calibrated in order to measure polarization by comparing outputs at ports 1 and 2. In this respect it is not as convenient as the waveguide system described in Section 5.2 or the one to be discussed in Section 5.4.

5.4. LOSSLESS POWER COMBINER AND DIVIDER SYSTEM

Figure 5.5 shows a lossless power combiner and divider system that is well suited for generating a wave with arbitrary polarization or measuring the polarization of an incident elliptically polarized wave [2]. It is based on the variable-ratio power divider of Teeter and Bushore [3, 4]. The system may be set up either in waveguide or transmission line. The hybrid tees may be replaced by circulators, and in fact other variations are possible [3, 5]. One antenna is linear horizontal and the other is linear vertical. They are placed adjacent to each other and pointed in the same direction. Either antenna may be the vertically polarized one, but for definiteness we let this be the one marked 4. Then 2 is horizontally polarized.

We will utilize scattering matrices [6] in examining the microwave networks, with the ports of the hybrid tee numbered as in Fig. 5.6. The scattering



FIGURE 5.5. Power combiner and divider for transmitting and receiving arbitrarily polarized waves.

matrix is given by

$$[b] = [S][a] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & -1 & 0 & 1\\ 1 & 0 & 1 & 0 \end{bmatrix} [a]$$
(5.13)

Let us consider first the system as used in receiving, Fig. 5.5(a). We assume that there is at least one direction in which the effective lengths of the two antennas used will be equal. For example, crossed dipoles with equal length transmission paths to the ports of the upper tee of Fig. 5.5(a) have equal effective lengths along a direction perpendicular to the dipoles. For simplicity in the initial development we will take this direction to be the z axis of Fig.



FIGURE 5.6. Port designations for the hybrid tee.

5.7. We will relax this requirement later. The incident wave travels in the ζ direction in Fig. 5.7.

Let the incident wave, in xyz coordinates, be

T.I

$$\mathbf{E}^{i} = E_{0}(\mathbf{u}_{x}a + \mathbf{u}_{y}be^{j\phi})$$
(5.14)



FIGURE 5.7. Antennas and coordinates for the power combiner system.

Without loss of generality we drop any common amplitude coefficients and neglect phase shifts common to both input arms of the upper tee of Fig. 5.5(a). Then the inputs to the upper tee are

$$a_2 = ae^{-j\alpha}$$
 (a)
 $a_4 = be^{j\phi}$ (b) (5.15)

where we have made use of the assumption that the effective lengths of the two antennas are equal in magnitude.

The outputs from the top tee, using the scattering matrix of (5.13), are

$$b_{1} = \frac{1}{\sqrt{2}} (a_{2} + a_{4}) = \frac{1}{\sqrt{2}} (ae^{-j\alpha} + be^{j\phi})$$
(a)
$$b_{3} = \frac{1}{\sqrt{2}} (-a_{2} + a_{4}) = \frac{1}{\sqrt{2}} (-ae^{-j\alpha} + be^{j\phi})$$
(b) (5.16)

Phase shifts common to both lines or waveguides connecting top and bottom tees can be neglected. Then if β is the differential phase shift, the inputs to the bottom tee are

$$a_{1} = b_{1} = \frac{1}{\sqrt{2}} (ae^{-j\alpha} + be^{j\phi})$$
(a)
$$a_{3} = b_{3}e^{-j\beta} = \frac{1}{\sqrt{2}} (-ae^{-j\alpha} + be^{j\phi})e^{-j\beta}$$
(b)

and the outputs from the bottom tee are

$$b_{2} = \frac{1}{\sqrt{2}} (a_{1} - a_{3}) = \frac{1}{2} [ae^{-j\alpha} + be^{j\phi} - e^{-j\beta}(-ae^{-j\alpha} + be^{j\phi})]$$
(a)
(5.18)
$$b_{4} = \frac{1}{\sqrt{2}} (a_{1} + a_{3}) = \frac{1}{2} [ae^{-j\alpha} + be^{j\phi} + e^{-j\beta}(-ae^{-j\alpha} + be^{j\phi})]$$
(b)

Removing $e^{-j\alpha}$ from each expression leaves

$$b_{2} = \frac{1}{2}e^{-j\alpha}[a + be^{j(\phi + \alpha)} + e^{-j\beta}(a - be^{j(\phi + \alpha)})] \quad (a)$$

$$b_{4} = \frac{1}{2}e^{-j\alpha}[a + be^{j(\phi + \alpha)} - e^{-j\beta}(a - be^{j(\phi + \alpha)})] \quad (b)$$
(5.19)

Now we let

$$\phi + \alpha = \pm \frac{1}{2}\pi \tag{5.20}$$

and require
GENERATION OF GENERAL POLARIZATIONS

$$b_2 = 0$$
 (5.21)

It follows from (5.19) that

$$a \pm jb + e^{-j\beta}(a \mp jb) = 0$$

or

$$e^{-j\beta} = -\frac{a\pm jb}{a\mp jb} \tag{5.22}$$

Using the upper and lower signs, respectively, in (5.22) gives

$$\beta = \pi \mp 2 \tan^{-1} \frac{b}{a} \tag{5.23}$$

For either of these values, which make $b_2 = 0$, we get for b_4

$$b_4 = \frac{1}{2}e^{-j\alpha} \left[a \pm jb + \frac{a \pm jb}{a \mp jb} \left(a \mp jb \right) \right] = e^{-j\alpha} (a \pm jb)$$
(5.24)

and it follows that

$$|b_4|^2 = a^2 + b^2 = 1 \tag{5.25}$$

We see then that all of the incident energy is directed to port 4 of the bottom tee by our choices of phase delays

$$\phi + \alpha = \pm \frac{1}{2}\pi \tag{5.20}$$

$$\beta = \pi \mp 2 \tan^{-1} \frac{b}{a} \tag{5.23}$$

We could just as well direct all of the energy to port 2 of the lower tee by the choices

$$\phi + \alpha = \pm \frac{1}{2}\pi \tag{5.26}$$

$$e^{-j\beta} = \frac{a \pm jb}{a \mp jb} \tag{5.27}$$

$$\beta = \mp 2 \tan^{-1} \frac{b}{a} \tag{5.28}$$

Then

$$b_4 = 0$$
 (a)
 $b_2 = e^{-j\alpha}(a \pm jb)$ (b) (5.29)
 $|b_2|^2 = 1$ (c)

This system has several uses. First, it can extract maximum power from an incident wave of any polarization by appropriate choice of the phase shifts α and β . Second, it may be used to measure the polarization of an incident wave, using procedures outlined in Section 8.5. Finally, we will see here that it allows the formation of a polarization-adaptive two-way communication system.

Let us consider that the system is set up for maximum output at port 4, with

$$\phi + \alpha = \pm \frac{1}{2}\pi \tag{5.20}$$

$$e^{-j\beta} = -\frac{a \pm jb}{a \mp jb} \tag{5.22}$$

$$\beta = \pi \mp 2 \tan^{-1} \frac{b}{a} \tag{5.23}$$

and is used for transmission, with an input to arm 4 of the lower tee and a matched load at port 2, as in Fig. 5.5(b). Then at the bottom tee

$$a_2 = 0$$
 (a) $a_4 = 1$ (b) (5.30)

The outputs from the bottom tee, using the scattering matrix of (5.13), are

$$b_{1} = \frac{1}{\sqrt{2}} (a_{2} + a_{4}) = \frac{1}{\sqrt{2}}$$
(a)
$$b_{3} = \frac{1}{\sqrt{2}} (-a_{2} + a_{4}) = \frac{1}{\sqrt{2}}$$
(b) (5.31)

and the inputs to the top tee are

$$a_1 = \frac{1}{\sqrt{2}}$$
 (a)
 $a_3 = b_3 e^{-j\beta} = -\frac{1}{\sqrt{2}} \frac{a \pm jb}{a \mp jb}$ (b) (5.32)

where the special value of (5.22) is used for β .

The top tee outputs are then

$$b_{2} = \frac{1}{\sqrt{2}} (a_{1} - a_{3}) = \frac{1}{2} \left(1 + \frac{a \pm jb}{a \mp jb} \right) = \frac{a}{a \mp jb} \quad (a)$$

$$b_{2} = \frac{1}{\sqrt{2}} (a_{1} - a_{3}) = \frac{1}{2} \left(1 + \frac{a \pm jb}{a \mp jb} \right) = \frac{\pi jb}{a \mp jb} \quad (b)$$

 $b_4 = \frac{1}{\sqrt{2}} (a_1 + a_3) = \frac{1}{2} \left(1 - \frac{a - y_2}{a \mp jb} \right) = \frac{1}{a \mp jb}$ (b)

The waves transmitted from the antennas are

$$E'_{x} = b_{2}e^{-j\alpha} = \frac{a}{a \mp jb} e^{j(\phi \mp \pi/2)}$$
 (a)
 $\mp ib$ (5.34)

$$E'_{y} = b_{4} = \frac{+jb}{a \mp jb}$$
(b)

The total power radiated is of course

$$E'_{x}E'_{x} + E'_{y}E'_{y} = 1 (5.35)$$

and the polarization ratio of the transmitted wave, in the directions for which $|h_2| = |h_4|$, is

$$p' = j \, \frac{E'_y}{E'_x} = j \, \frac{\mp jb}{ae^{j(\phi \mp \pi/2)}} = j \, \frac{b}{a} \, e^{-j\phi} \tag{5.36}$$

Now the polarization of the incoming wave is

$$p^{i} = j \, \frac{E_{\eta}}{E_{\xi}} = -j \, \frac{E_{y}}{E_{x}} = -j \, \frac{b}{a} \, e^{j\phi}$$
(5.37)

and we have

$$p' = p' \tag{5.38}$$

The conclusion is that if we set the phase shifters of our power combiner to give maximum output at port 4 of the bottom tee of Fig. 5.5 on reception, and then use the system to transmit by applying a signal to port 4, the polarization ratio of the transmitted wave is the conjugate of the polarization ratio of the received wave. This would be true also if we maximized the power output of arm 2 of the lower tee and then applied the generator to the same arm.

Suppose we leave our phase shifters set so that on reception $b_2 = 0$ and b_4 is maximum, with phase shifts given by (5.20) and (5.23). Now, however, instead of connecting a generator to port 4 for transmission, we connect it to port 2, so that at the bottom tee inputs are

$$a_2 = 1$$
 (a) $a_4 = 0$ (b) (5.39)

An analysis similar to the preceding one gives the transmitted signals as

$$E'_{x} = -\frac{b}{a \mp jb} e^{j\phi} \quad (a)$$

$$E'_{y} = \frac{a}{a \mp jb} \quad (b)$$

leading to a polarization ratio

$$p' = -j \frac{a}{b} e^{-j\phi} = -\frac{1}{p^i}$$
(5.41)

Suppose now that we have a communication configuration with this variable polarization system at one end, as in Fig. 5.8, and a fixed arbitrarily polarized antenna at the other. The fixed polarization antenna transmits a wave with polarization p^i toward the variable polarization system, which is then set to receive maximum power. In turn, when used to transmit from the port at which maximum power is received, the power combiner system transmits a wave with polarization $p' = p^i$. But this is the polarization that the fixed polarization antenna receives best. Thus adjustment of the variable polarization antenna until it receives maximum signal causes it, on transmission, to transmit a wave from which the fixed polarization antenna receives maximum power. This offers the opportunity for an automatically adaptive (in polarization) two-way communication system.

The condition (5.41) may be recognized as the cross-polarization condition in a communication link. Thus, in the link of Fig. 5.8, if the variable polarization system on reception is set for maximum power out at port 4,



FIGURE 5.8. Power-maximized communications link.

203

GENERATION OF GENERAL POLARIZATIONS

connecting the generator to arm 2 on transmission would cause the fixed polarization antenna in the link to receive *no* power.

It was mentioned earlier that for simplicity, we would assume that in the z direction the effective lengths of the two antennas are equal. Now, in general, the effective lengths are complex functions of direction, and if the reference points for the effective lengths are taken at the inputs to the upper tee of Fig. 5.5, the effective lengths also depend on the lengths of the transmission paths to the tee. We therefore use

$$h_{2}(\theta, \phi) = |h_{2}|e^{j\delta_{2}}$$
(a)
$$h_{4}(\theta, \phi) = |h_{4}|e^{j\delta_{4}} = |h_{4}|e^{j(\delta_{2}+\delta)}$$
(b)
(5.42)

Again, neglecting the common phase shift δ_2 , the inputs to the top tee of Fig. 5.5 change from (5.15) to

$$a_2 = |h_2|ae^{-j\alpha}$$
 (a)
 $a_4 = |h_4|e^{j\delta}be^{j\phi}$ (b)
(5.43)

We can find the outputs of the bottom tee replacing *a* by $|h_2|a$ and *b* by $|h_4|e^{i\delta}b$ in (5.19). The result is

$$b_{2} = \frac{1}{2}e^{-j\alpha}[|h_{2}|a + |h_{4}|be^{j(\phi + \alpha + \delta)} + e^{-j\beta}(|h_{2}|a - |h_{4}|be^{j(\phi + \alpha + \delta)})]$$
(a)

$$b_{4} = \frac{1}{2}e^{-j\alpha}[|h_{2}|a + |h_{4}|be^{j(\phi + \alpha + \delta)} - e^{-j\beta}(|h_{2}|a - |h_{4}|be^{j(\phi + \alpha + \delta)})]$$
(b)
(5.44)

If now we set

$$\phi + \alpha + \delta = \pm \frac{1}{2}\pi \tag{5.45}$$

and require

$$b_2 = 0$$
 (5.46)

it follows from (5.44) that

$$|h_2|a \pm j|h_4|b + e^{-j\beta}(|h_2|a \mp j|h_4|b) = 0$$

or

$$e^{-j\beta} = -\frac{|h_2|a \pm j|h_4|b}{|h_2|a \mp j|h_4|b}$$
(5.47)

which gives

LOSSLESS POWER COMBINER AND DIVIDER SYSTEM

$$\beta = \pi \mp 2 \tan^{-1} \frac{|h_4|b}{|h_2|a}$$
(5.48)

Then

$$b_{4} = \frac{1}{2}e^{-j\alpha} \left[|h_{2}|a \pm j|h_{4}|b + \frac{|h_{2}|a \pm j|h_{4}|b}{|h_{2}|a \mp j|h_{4}|b} (|h_{2}|a \mp j|h_{4}|b) \right]$$

= $e^{-j\alpha} (|h_{2}|a \pm j|h_{4}|b)$ (5.49)

which leads to the equation

$$|b_4|^2 = |h_2|^2 a^2 + |h_4|^2 b^2$$
(5.50)

This again is all of the incident power, since $b_2 = 0$.

If we now use this system for transmission by connecting a generator to arm 4 of the lower hybrid tee and a matched load to arm 2, while leaving phase shifts α and β set as in (5.45) and (5.48), we have lower tee inputs

$$a_2 = 0$$
 (a) $a_4 = 1$ (b) (5.51)

The inputs to the top tee may be found from $a_3 = b_3 \exp(-j\beta)$, using (5.31) for b_3 and (5.47) for $\exp(-j\beta)$. They are

$$a_{1} = \frac{1}{\sqrt{2}}$$
 (a)

$$a_{3} = -\frac{1}{\sqrt{2}} \frac{|h_{2}|a \pm j|h_{4}|b}{|h_{2}|a \mp j|h_{4}|b}$$
 (b)
(5.52)

The outputs from the top tee then become

$$b_{2} = \frac{1}{\sqrt{2}} (a_{1} - a_{3}) = \frac{|h_{2}|a}{|h_{2}|a \mp j|h_{4}|b} \quad (a)$$

$$b_{4} = \frac{1}{\sqrt{2}} (a_{1} + a_{3}) = \frac{\mp j|h_{4}|b}{|h_{2}|a \mp j|h_{4}|b} \quad (b)$$
(5.53)

The transmitted wave then has components

$$E'_{x} = b_{2}e^{-j\alpha}|h_{2}| = \frac{|h_{2}|^{2}a}{|h_{2}|a \mp j|h_{4}|b} e^{j(\phi + \delta \mp \pi/2)}$$
(a)

$$E'_{y} = b_{4}|h_{4}|e^{j\delta} = \frac{\mp j|h_{4}|^{2}be^{j\delta}}{|h_{2}|a \mp j|h_{4}|b}$$
(b)

205

and polarization

$$p' = j \frac{E'_{y}}{E'_{x}} = j \frac{|h_{4}|^{2}}{|h_{2}|^{2}} \frac{b}{a} e^{-j\phi} = \frac{|h_{4}|^{2}}{|h_{2}|^{2}} p''$$
(5.55)

We conclude that in this more general case for which $h_2 \neq h_4$ in the direction of the incoming wave (due perhaps to the use of nonidentical antennas, to improper orientation of the antennas, or to unequal transmission path lengths between antennas and hybrid tee inputs), if the system is set up for maximum power reception and then used for transmitting, the transmitted signal *in the direction from which the original signal was received* is modified in its polarization characteristics by the ratio $|h_4|^2/|h_2|^2$.

REFERENCES

- H. Mott and D. N. McQuiddy, "A Simple Waveguide System for Radiating Elliptically Polarized Waves," *IEEE Trans. on Antennas and Propagation*, Vol. AP-16, No. 1, pp. 134–135, January 1968.
- H. Mott and D. N. McQuiddy, "A Polarization-Adaptive Antenna System," IEEE Region 3 Convention Record, pp. 27.4.1–27.4.5, April 1968.
- W. L. Teeter and K. R. Bushore, "A Variable-Ratio Microwave Power Divider and Multiplexer," *IRE Trans. on Microwave Theory and Techniques*, Vol. MTT-5, No. 4, pp. 227–229, October 1957.
- R. M. Vaillancourt, "Analysis of the Variable-Ratio Microwave Power Divider," *IRE Trans.* on Microwave Theory and Techniques, Vol. MTT-6, No. 2, pp. 238–239, April 1958.
- H. J. Riblet, "The Short-Slot Hybrid Junction," Proc. IRE, Vol. 40, No. 2, pp. 180–184, February 1952.
- 6. R. N. Ghose, Microwave Circuit Theory and Analysis, McGraw-Hill, New York, 1963.

6.1. LINEAR POLARIZATION

In this section we shall consider the reflection and transmission of a linearly polarized plane wave at the plane interface between two media. It is convenient for this problem to use a rectangular coordinate system with two axes lying in the plane of the interface, but since the incident wave strikes the interface at some angle other than perpendicular, we must change its components to the appropriate coordinates for the interface. The transformations are readily apparent from Fig. 6.1. The plane wave being considered travels in the ζ direction. The x and ξ axes coincide and are into the plane of the page. From the figure, we see that

$$\zeta = z \cos \theta + y \sin \theta$$

$$\eta = y \cos \theta - z \sin \theta$$
 (6.1)

$$\xi = x$$

and since space vector components transform like the coordinates of a point,

$$\mathbf{u}_{\zeta} = \mathbf{u}_{z} \cos \theta + \mathbf{u}_{y} \sin \theta$$
$$\mathbf{u}_{\eta} = \mathbf{u}_{y} \cos \theta - \mathbf{u}_{z} \sin \theta$$
$$\mathbf{u}_{\xi} = \mathbf{u}_{x}$$
(6.2)



FIGURE 6.1. Coordinate transformations.

We define a plane of incidence as that plane containing a vector in the direction of wave travel, \mathbf{u}_{ς} , and a vector normal to the interface, \mathbf{u}_{z} . It is then convenient to consider linearly polarized waves by two cases, E lying in the plane of incidence (**H** perpendicular to the plane) and E perpendicular to the plane of incidence (**H** in the plane).

Fields: Polarization Normal to Plane of Incidence

For this case

$$E_{\xi} = E_0 e^{-jk\xi} \quad (a)$$

$$H_{\eta} = \frac{E_0}{Z} e^{-jk\xi} \quad (b)$$
(6.3)

where Z is the characteristic impedance of the medium.

In the general case, for lossy media, k and Z are given by

$$k = \omega \sqrt{\mu \varepsilon} \sqrt{1 - j \frac{\sigma}{\omega \varepsilon}}$$
(6.4)

LINEAR POLARIZATION

$$Z = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \tag{6.5}$$

and are specialized to the lossless case by setting conductivity $\sigma = 0$.

Transforming to x, y, and z coordinates gives

$$\mathbf{E} = \mathbf{u}_{\xi} E_{\xi} = \mathbf{u}_{x} E_{x} + \mathbf{u}_{y} E_{y} + \mathbf{u}_{z} E_{z}$$

$$\mathbf{H} = \mathbf{u}_{\eta} H_{\eta} = \mathbf{u}_{x} H_{x} + \mathbf{u}_{y} H_{y} + \mathbf{u}_{z} H_{z}$$
(6.6)

and if we substitute (6.2) into (6.6) and equate coefficients of like unit vectors, then

$$E_x = E_{\xi}$$

$$H_y = \cos \theta H_{\eta}$$

$$H_z = -\sin \theta H_{\eta}$$
(6.7)

If (6.1) and (6.3) are substituted into (6.7) the result is

$$E_{x} = E_{0}e^{-jk(z\cos\theta + y\sin\theta)}$$

$$H_{y} = \cos\theta \frac{E_{0}}{Z} e^{-jk(z\cos\theta + y\sin\theta)}$$

$$H_{z} = -\sin\theta \frac{E_{0}}{Z} e^{-jk(z\cos\theta + y\sin\theta)}$$
(6.8)

Fields: Polarization in the Plane of Incidence

For this case, **H** has only a ξ component and it is appropriate to write

$$H_{\xi} = H_0 e^{-jk\zeta} \qquad E_{\eta} = -ZH_0 e^{-jk\zeta}$$
(6.9)

where the negative sign for E_{η} is necessary to give wave travel in the ζ direction. Obviously, the transformation to new coordinates is the same as for the electric field normal to the plane of incidence, with the roles of E_{ξ} and H_{ξ} interchanged. The resulting fields are

$$H_{x} = H_{0}e^{-jk(z\cos\theta + y\sin\theta)}$$

$$E_{y} = -\cos\theta Z H_{0}e^{-jk(z\cos\theta + y\sin\theta)}$$

$$E_{z} = \sin\theta Z H_{0}e^{-jk(z\cos\theta + y\sin\theta)}$$
(6.10)

209



FIGURE 6.2. Coordinate systems for wave reflection.

Figure 6.2 is a superposition of all of the rotated coordinate systems we need to study the reflection problem. The incident wave strikes the plane boundary at angle θ_i , a part is reflected at angle θ_r , and a part transmitted at angle θ_i . We restrict our examination to θ_i real so that (6.8) and (6.10) represent uniform plane incident waves.

Snell's Laws

Boundary conditions at the interface require the sum of two tangential fields to be equal to a third. It is obvious from the form of the waves [Eqs. (6.8) and (6.10) may represent incident, reflected, or transmitted waves with a proper choice of k and Z] that the boundary conditions can be met only if the phase variation with y in both media is the same for all field terms. These phase terms are [using the appropriate angles from Fig. 6.2 in (6.8) and (6.10)]:

Incident wave	$k_1 y \sin \theta_i$
Reflected wave	$k_1 y \sin(\pi - \theta_r) = k_1 y \sin \theta_r$
Transmitted wave	$k_2 y \sin \theta_t$

It follows that

$$k_1 \sin \theta_i = k_1 \sin \theta_r = k_2 \sin \theta_r \tag{6.11}$$

LINEAR POLARIZATION

and from this equation we obtain Snell's laws,

$$\theta_r = \theta_i$$
 (a)
 $\frac{\sin \theta_i}{\sin \theta_i} = \frac{k_1}{k_2}$ (b)

Our equations to this point are valid for lossy media, for which k and Z are complex. In a commonly encountered situation the first medium is air and the second the lossy earth. Equation (6.12) holds, but the transmitted wave is nonuniform. In this situation we are normally most interested in the reflected wave and, for lossy media, will restrict our attention to it.

Reflection and Transmission Coefficients: Perpendicular Polarization

We consider the waves to be composed of incident, reflected, and transmitted terms with appropriate superscripts. We define reflection and transmission coefficients as

$$\Gamma_{\perp} = \frac{E_{x}'}{E_{x}'}\Big|_{z=0} \qquad T_{\perp} = \frac{E_{x}'}{E_{x}'}\Big|_{z=0}$$
(6.13)

The wave components of the incident wave are

$$E_x^{i} = E_0 e^{-jk_1(z\cos\theta_i + y\sin\theta_i)}$$

$$H_{y'}^{i} = \frac{E_0}{Z_1} e^{-jk_1(z\cos\theta_i + y\sin\theta_i)}$$

$$H_y^{i} = \cos\theta_i \frac{E_0}{Z_1} e^{-jk_1(z\cos\theta_i + y\sin\theta_i)}$$

$$H_z^{i} = -\sin\theta_i \frac{E_0}{Z_1} e^{-jk_1(z\cos\theta_i + y\sin\theta_i)}$$
(6.14)

Noting that for the reflected wave the appropriate angle to use in (6.8) is $\pi - \theta_r$ and using the first of Snell's laws, we get from (6.8) and (6.13)

$$E_{x}^{r} = \Gamma_{\perp} E_{0} e^{-jk_{1}(-z\cos\theta_{i} + y\sin\theta_{i})}$$

$$H_{y^{*}}^{r} = \frac{E_{x}^{r}}{Z_{1}} = \Gamma_{\perp} \frac{E_{0}}{Z_{1}} e^{-jk_{1}(-z\cos\theta_{i} + y\sin\theta_{i})}$$

$$H_{y}^{r} = -\cos\theta_{i} H_{y^{*}}^{r} = -\cos\theta_{i} \Gamma_{\perp} \frac{E_{0}}{Z_{1}} e^{-jk_{1}(-z\cos\theta_{i} + y\sin\theta_{i})}$$

$$H_{z}^{r} = -\sin\theta_{i} H_{y^{*}}^{r} = -\sin\theta_{i} \Gamma_{\perp} \frac{E_{0}}{Z_{1}} e^{-jk_{1}(-z\cos\theta_{i} + y\sin\theta_{i})}$$
(6.15)

For the transmitted wave,

$$E_{x}^{t} = T_{\perp} E_{0} e^{-jk_{2}(z \cos \theta_{t} + y \sin \theta_{t})}$$

$$H_{y^{m}}^{t} = \frac{E_{x}^{t}}{Z_{2}} = T_{\perp} \frac{E_{0}}{Z_{2}} e^{-jk_{2}(z \cos \theta_{t} + y \sin \theta_{t})}$$

$$H_{y}^{t} = \cos \theta_{t} H_{y^{m}}^{t} = \cos \theta_{t} T_{\perp} \frac{E_{0}}{Z_{2}} e^{-jk_{2}(z \cos \theta_{t} + y \sin \theta_{t})}$$

$$H_{z}^{t} = -\sin \theta_{t} H_{y^{m}}^{t} = -\sin \theta_{t} T_{\perp} \frac{E_{0}}{Z_{2}} e^{-jk_{2}(z \cos \theta_{t} + y \sin \theta_{t})}$$
(6.16)

The boundary condition on the tangential electric field components that must be met is

$$E'_{x}|_{z=0} + E'_{x}|_{z=0} = E'_{x}|_{z=0}$$
(6.17)

Use of the appropriate components from (6.14), (6.15), and (6.16) and noting that the phase terms are equal because of Snell's laws leads to

$$1 + \Gamma_{\perp} = T_{\perp} \tag{6.18}$$

Since the magnetic field components are also necessarily continuous across the boundary, we have

$$H_{y}^{i}|_{z=0} + H_{y}^{r}|_{z=0} = H_{y}^{i}|_{z=0}$$
(6.19)

which becomes, using the field components and Snell's laws,

$$\frac{\cos \theta_i}{Z_1} \left(1 - \Gamma_\perp \right) = \frac{\cos \theta_i}{Z_2} T_\perp$$
(6.20)

If (6.18) is utilized, we find for the reflection coefficient

$$\Gamma_{\perp} = \frac{Z_2 \sec \theta_i - Z_1 \sec \theta_i}{Z_2 \sec \theta_i + Z_1 \sec \theta_i}$$
(6.21)

Reflection and Transmission Coefficients: Parallel Polarization

For this case we define reflection and transmission coefficients as

$$\Gamma_{\parallel} = \frac{H_{x}'}{H_{x}^{i}}\Big|_{z=0} \qquad T_{\parallel} = \frac{H_{x}'}{H_{x}^{i}}\Big|_{z=0}$$
(6.22)

Some authors define Γ_{\parallel} as E'_{ν}/E'_{ν} , rather than as above. This definition does

LINEAR POLARIZATION

not utilize the symmetry of the Maxwell equations, and it chooses one of two electric field components in preference to using the only magnetic field component. Our choice here agrees with that of Stratton [1].

From (6.10) and (6.22) the wave components are

$$\begin{aligned} H_{x}^{i} &= H_{0}e^{-jk_{1}(z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{y}^{i} &= -Z_{1}H_{x}^{i} = -Z_{1}H_{0}e^{-jk_{1}(z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{y}^{i} &= \cos\theta_{i}E_{y}^{i} = -\cos\theta_{i}Z_{1}H_{0}e^{-jk_{1}(z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{z}^{i} &= -\sin\theta_{i}E_{y}^{i} = \sin\theta_{i}Z_{1}H_{0}e^{-jk_{1}(z\cos\theta_{i}+y\sin\theta_{i})} \\ H_{x}^{r} &= \Gamma_{\parallel}H_{0}e^{-jk_{1}(-z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{y}^{r} &= -Z_{1}H_{x}^{r} = -Z_{1}\Gamma_{\parallel}H_{0}e^{-jk_{1}(-z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{y}^{r} &= -\cos\theta_{i}E_{y}^{r} = \cos\theta_{i}Z_{1}\Gamma_{\parallel}H_{0}e^{-jk_{1}(-z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{z}^{r} &= -\sin\theta_{i}E_{y}^{r} = \sin\theta_{i}Z_{1}\Gamma_{\parallel}H_{0}e^{-jk_{1}(-z\cos\theta_{i}+y\sin\theta_{i})} \\ H_{x}^{i} &= T_{\parallel}H_{0}e^{-jk_{2}(z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{y}^{r} &= -Z_{2}H_{x}^{t} = -Z_{2}T_{\parallel}H_{0}e^{-jk_{2}(z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{y}^{r} &= \cos\theta_{i}E_{y}^{r} = -\cos\theta_{i}Z_{2}T_{\parallel}H_{0}e^{-jk_{2}(z\cos\theta_{i}+y\sin\theta_{i})} \\ E_{z}^{i} &= -\sin\theta_{i}E_{y}^{r} = \sin\theta_{i}Z_{2}T_{\parallel}H_{0}e^{-jk_{2}(z\cos\theta_{i}+y\sin\theta_{i})} \end{aligned}$$
(6.25)

Using these fields, the boundary condition

$$H_x^i|_{z=0} + H_x^r|_{z=0} = H_x^i|_{z=0}$$
(6.26)

and Snell's laws give immediately

$$1 + \Gamma_{\parallel} = T_{\parallel} \tag{6.27}$$

The boundary condition

$$E_{y}^{i}|_{z=0} + E_{y}^{r}|_{z=0} = E_{y}^{\prime}|_{z=0}$$
(6.28)

and Snell's laws lead to

$$Z_1 \cos \theta_i (\Gamma_{\parallel} - 1) = -Z_2 T_{\parallel} \cos \theta_i \tag{6.29}$$

Then, use of (6.27) gives for the reflection coefficient

$$\Gamma_{\parallel} = -\frac{Z_2 \cos \theta_t - Z_1 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i}$$
(6.30)

Alternate Forms for the Fresnel Coefficients

It is obvious that Snell's laws can be used to remove θ_i from Eqs. (6.21) and (6.30). At the same time we will use the grazing angle

$$\alpha = \frac{1}{2}\pi - \theta_i \tag{6.31}$$

and specialize to the interface between air and a lossy medium, replacing μ_1 and ε_1 by μ_0 and ε_0 and using for the lossy medium $\mu_2 = \mu_0$, $\varepsilon_0 = \varepsilon$, and $\sigma_2 = \sigma$.

Since ε always occurs in the combination $\sigma + j\omega\varepsilon$, we can define a complex dielectric constant

$$\sigma + j\omega\varepsilon = j\omega\varepsilon \left(1 - j \frac{\sigma}{\omega\varepsilon}\right) = j\omega\hat{\varepsilon}$$
(6.32)

where

$$\hat{\varepsilon} = \varepsilon \left(1 - j \; \frac{\sigma}{\omega \varepsilon} \right) \tag{6.33}$$

We can also write

$$k_2 = \omega \sqrt{\mu_0 \varepsilon} \sqrt{1 - j\sigma/\omega\varepsilon} = \omega \sqrt{\mu_0 \varepsilon}$$
(6.34)

$$Z_2 = \sqrt{\frac{j\omega\mu_0}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\mu_0}{\hat{\varepsilon}}}$$
(6.35)

Substitution into (6.21) and (6.30) causes them to reduce to

$$\Gamma_{\perp} = \frac{\cos\theta_i - (\hat{\varepsilon}/\varepsilon_0 - \sin^2\theta_i)^{1/2}}{\cos\theta_i + (\hat{\varepsilon}/\varepsilon_0 - \sin^2\theta_i)^{1/2}} = \frac{\sin\alpha - (\hat{\varepsilon}/\varepsilon_0 - \cos^2\alpha)^{1/2}}{\sin\alpha + (\hat{\varepsilon}/\varepsilon_0 - \cos^2\alpha)^{1/2}}$$
(6.36)

$$\Gamma_{\parallel} = \frac{\left(\hat{\varepsilon}/\varepsilon_{0}\right)\cos\theta_{i} - \left(\hat{\varepsilon}/\varepsilon_{0} - \sin^{2}\theta_{i}\right)^{1/2}}{\left(\hat{\varepsilon}/\varepsilon_{0}\right)\cos\theta_{i} + \left(\hat{\varepsilon}/\varepsilon_{0} - \sin^{2}\theta_{i}\right)^{1/2}} = \frac{\left(\hat{\varepsilon}/\varepsilon_{0}\right)\sin\alpha - \left(\hat{\varepsilon}/\varepsilon_{0} - \cos^{2}\alpha\right)^{1/2}}{\left(\hat{\varepsilon}/\varepsilon_{0}\right)\sin\alpha + \left(\hat{\varepsilon}/\varepsilon_{0} - \cos^{2}\alpha\right)^{1/2}}$$

$$(6.37)$$

Power: Perpendicular Polarization

We restrict this discussion to lossless media, although the extension to lossy media is simple. We also will distinguish between Poynting vectors and the

proportion of incident power that is reflected and transmitted, a distinction not often made in texts.

The Poynting vectors, taken from the appropriate fields, are

$$S_{i} = \frac{1}{2} \frac{E_{x}^{'} E_{x}^{'}}{Z_{1}} = \frac{1}{2} \frac{|E_{0}|^{2}}{Z_{1}}$$

$$S_{r} = \frac{1}{2} \frac{E_{x}^{r} E_{x}^{r}}{Z_{1}} = \frac{1}{2} \frac{|E_{0}|^{2}}{Z_{1}} |\Gamma_{\perp}|^{2}$$

$$S_{t} = \frac{1}{2} \frac{E_{x}^{'} E_{x}^{'}}{Z_{2}} = \frac{1}{2} \frac{|E_{0}|^{2}}{Z_{2}} |T_{\perp}|^{2}$$
(6.38)

The ratio of reflected to incident power is the same as the ratio of their Poynting vectors, thus

$$\frac{W_r}{W_i} = \frac{S_r}{S_i} = |\Gamma_\perp|^2 \tag{6.39}$$

However, the ratio of transmitted to incident power is not equal to the Poynting vector ratio, as may be seen from Fig. 6.3. The power incident on the interface within the confines of some arbitrary channel is partly reflected in a channel of equal cross section, and partly transmitted in a channel of different cross section. Equation (6.39) follows immediately, and we can get the ratio of transmitted to incident power as

$$\frac{W_i}{W_i} = 1 - \frac{W_r}{W_i} = 1 - \frac{S_r}{S_i} = 1 - |\Gamma_{\perp}|^2$$
(6.40)



FIGURE 6.3. Channel widths for reflection and transmission.

Power: Parallel Polarization

The Poynting vectors for this case are

$$S_{i} = \frac{1}{2}Z_{1}H_{x}^{i}H_{x}^{i^{*}} = \frac{1}{2}Z_{1}|H_{0}|^{2}$$

$$S_{r} = \frac{1}{2}Z_{1}H_{x}^{r}H_{x}^{r^{*}} = \frac{1}{2}Z_{1}|H_{0}|^{2}|\Gamma_{\parallel}|^{2}$$

$$S_{t} = \frac{1}{2}Z_{2}H_{x}^{t}H_{x}^{t^{*}} = \frac{1}{2}Z_{2}|H_{0}|^{2}|T_{\parallel}|^{2}$$
(6.41)

The proportions of reflected and transmitted power are

$$\frac{W_r}{W_i} = \frac{S_r}{S_i} = |\Gamma_{\parallel}|^2$$

$$\frac{W_i}{W_i} = 1 - \frac{S_r}{S_i} = 1 - |\Gamma_{\parallel}|^2$$
(6.42)

Total Transmission

For parallel polarization an incidence angle, called the Brewster angle, can be found for which all of the incident power is transmitted across the interface into the second medium. From (6.30), $\Gamma_{\parallel} = 0$ if

$$Z_2 \cos \theta_i = Z_1 \cos \theta_i \tag{6.43}$$

Considering lossless dielectrics, with equal permeabilities, for which

$$\frac{Z_2}{Z_1} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \qquad (a)$$

$$\frac{\sin \theta_i}{\sin \theta_i} = \frac{k_1}{k_2} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \qquad (b)$$

we obtain a solution to (6.43),

$$\tan \theta_i = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \tag{6.45}$$

At this angle of incidence, all of the wave is transmitted and none reflected. There is no comparable solution for perpendicular polarization, as may be seen by setting $\Gamma_{\perp} = 0$ in (6.21). This phenomenon allows the production of a linearly polarized wave by reflection of a wave with general polarization.

ELLIPTICAL WAVES

Total Reflection

For lossless media with equal permeabilities, Snell's law for transmission is

$$\frac{\sin \theta_i}{\sin \theta_i} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \tag{6.46}$$

If $\sqrt{\varepsilon_1/\varepsilon_2} > 1$ then $\sin \theta_i > 1$ for a range of incidence angles θ_i . Then the exponential term for the transmitted fields, with either perpendicular or parallel polarization, from (6.16) or (6.25), becomes

$$e^{-jk_2(z\cos\theta_l+y\sin\theta_l)} = e^{-jk_2(z\sqrt{1-\sin^2\theta_l}+y\sin\theta_l)} = e^{-k_2z\sqrt{\sin^2\theta_l-1}}e^{-jk_2y\sin\theta_l}$$
(6.47)

which no longer represents a uniform plane wave in region 2. Examination of the fields shows that no power propagates in the z direction in region 2, and therefore no wave propagates across the interface. It follows that for this situation, the magnitude of the reflection coefficients is unity,

$$|\Gamma_{\perp}| = |\Gamma_{\parallel}| = 1$$

and all incident power is reflected. The angle

$$\theta_i = \sin^{-1} \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \tag{6.48}$$

which gives $\sin \theta_i = 1$, is called the "critical angle." All greater angles of incidence lead to complete reflection at a boundary.

Note from (6.47) that in region 2 a nonuniform wave is set up which appears to propagate in the y direction and falls off in amplitude with z. Obviously, this field could have been set up only by waves propagating across the interface, but this is not predicted by our steady-state solution.

6.2. ELLIPTICAL WAVES

A linearly polarized wave that is neither perpendicular to nor parallel to the plane of incidence can be broken into perpendicular and parallel components and each component multiplied by the appropriate reflection and transmission coefficients in order to obtain the complete reflected and transmitted fields. As a matter of fact, this procedure can be applied to a generally polarized incident wave if the phase difference between perpendicular and parallel components is accounted for.

An incident wave of general polarization may be written, using the primed coordinate system of Fig. 6.2, as

$$\mathbb{E}^{i} = E_{0}(\mathbf{u}_{x'}a + \mathbf{u}_{y'}be^{j\phi})e^{-jk_{1}z'}$$
(6.49)

where E_0 is taken as real without loss of generality. To assist us in determining the reflected and transmitted wave components, we find the magnetic field components corresponding to the electric field.

$$E_{x'} = E_0 a e^{-jk_1 z'}$$
 (a) $H_{y'} = \frac{E_0}{Z_1} a e^{-jk_1 z'}$ (b) (6.50)

-

$$E_{y'} = E_0 b e^{j\phi} e^{-jk_1 z'}$$
 (a) $H_{x'} = -\frac{E_0 b}{Z_1} e^{j\phi} e^{-jk_1 z'}$ (b) (6.51)

From (6.13) and (6.50) the reflected and transmitted fields arising from the incident $E_{x'}$ are

$$E'_{x''} = \Gamma_{\perp} E_0 a e^{-jk_1 z''}$$
 (a)
 $E'_{x'''} = T_{\perp} E_0 a e^{-jk_2 z'''}$ (b) (6.52)

From (6.22) and (6.51) those fields arising from the incident $H_{x'}$ are

$$H'_{x''} = -\Gamma_{\parallel} \frac{E_0 b}{Z_1} e^{j\phi} e^{-jk_1 z''} \quad (a)$$

$$H'_{x'''} = -T_{\parallel} \frac{E_0 b}{Z_1} e^{j\phi} e^{-jk_2 z''} \quad (b)$$

with associated electric fields

$$E_{y''}^{r} = -Z_{1}H_{x''}^{r} = \Gamma_{\parallel}E_{0}be^{j\phi}e^{-jk_{1}z''}$$
(a)

$$E_{y'''}^{r} = -Z_{2}H_{x'''}^{r} = \frac{Z_{2}}{Z_{1}}T_{\parallel}E_{0}be^{j\phi}e^{-jk_{2}z'''}$$
(b)

The total fields, incident, reflected, and transmitted, are then

$$\mathbf{E}^{i} = E_{0}(\mathbf{u}_{x'}a + \mathbf{u}_{y'}be^{j\phi})e^{-jk_{1}z'}$$
(6.49)

$$\mathbb{E}^{r} = E_{0}(\mathbf{u}_{x''}\Gamma_{\perp}a + \mathbf{u}_{y''}\Gamma_{\parallel}be^{j\phi})e^{-jk_{1}z''}$$
(6.55)

$$\mathbf{E}' = E_0(\mathbf{u}_{x''}T_{\perp}a + \mathbf{u}_{y''}\frac{Z_2}{Z_1}T_{\parallel}be^{j\phi})e^{-jk_2z''}$$
(6.56)

Polarization ratios are easily obtained from these fields. They are

ELLIPTICAL WAVES

$$p^{i} = j \frac{b}{a} e^{j\phi}$$
 (a)

$$p' = j \frac{\Gamma_{\parallel} b}{\Gamma_{\perp} a} e^{j\phi} = \frac{\Gamma_{\parallel}}{\Gamma_{\perp}} p^{i}$$
 (b) (6.57)

$$p' = j \frac{Z_2 T_{\parallel} b}{Z_1 T_{\perp} a} e^{j\phi} = \frac{Z_2}{Z_1} \frac{T_{\parallel}}{T_{\perp}} p^i$$
 (c)

Special Cases

Let us look at a few special cases as a means of verifying (6.57) and perhaps discovering some physical facts about the reflection of elliptical waves.

1. Let θ_i , the angle of incidence, be 0. Then $\theta_i = 0$, and the equations for reflection and transmission coefficients simplify to

$$\begin{split} \Gamma_{\perp} &= \frac{Z_2 - Z_1}{Z_2 + Z_1} & \Gamma_{\parallel} = -\frac{Z_2 - Z_1}{Z_2 + Z_1} = -\Gamma_{\perp} \\ T_{\perp} &= 1 + \Gamma_{\perp} = \frac{2Z_2}{Z_1 + Z_2} & T_{\parallel} = 1 + \Gamma_{\parallel} = \frac{2Z_1}{Z_1 + Z_2} \end{split}$$

Then, from (6.57)

$$p^r = -p^i \qquad p^t = p^i$$

and we see that the transmitted wave has the same polarization as the incident wave, but in general the reflected wave is neither matched to the incident wave nor cross-polarized.

At this point we must consider our coordinate systems. In Chapter 3, when we developed the equations for polarization match of two antennas, we used coordinate systems with coincident vertical axes as shown in Fig. 6.4(a). In our study of reflection at an interface we have used coordinate systems with coincident horizontal axes, as in Fig. 6.4(b). Examination of Fig. 6.4 shows that since



FIGURE 6.4. Coordinate systems for (a) transmission between antennas and (b) reflection at normal incidence.

$$\frac{E_{\eta}}{E_{\xi}} = \frac{-E_{y''}}{-E_{x''}} = \frac{E_{y''}}{E_{x''}}$$

the polarization parameter p is the same in the ξ , η , ζ system and the x", y", z" system. Therefore, our previously developed equations for polarization match factor apply to reflection at normal incidence.

2. Let $\theta_i = 0$ and $p^i = \pm 1$, where the upper sign corresponds to right circular polarization and the lower to left circular. Then

$$p' = \mp 1$$
 $p' = \pm 1$

The reflected wave is circularly polarized, but in the opposite sense to the incident wave. The incident and reflected waves are related by

$$p' = -\frac{1}{p'}$$

which is the condition for cross-polarization. Thus, if a circularly polarized wave is transmitted normally toward a plane interface, the reflected wave cannot be received by the transmitting antenna.

3. Let the incident wave be linearly polarized. Then $\phi = 0$ and

$$p^{i} = j \frac{b}{a}$$
 $p' = j \frac{\Gamma_{\parallel}}{\Gamma_{\perp}} \frac{b}{a}$ $p' = j \frac{Z_{2}}{Z_{1}} \frac{T_{\parallel}}{T_{\perp}} \frac{b}{a}$

and we see that the reflected and transmitted waves are elliptically polarized unless both media are lossless or the wave incidence is normal.

4. Let the incident wave be linearly polarized and "vertical." Note that vertical here means in the plane of incidence. Then

$$p^i \rightarrow \infty$$
 $p' \rightarrow \infty$ $p' \rightarrow \infty$

so the reflected and transmitted waves are also linear vertical.

5. Let the incident wave be linearly polarized and horizontal. Then

$$p^i = p^r = p^t = 0$$

and all waves are linear horizontal.

6. At the Brewster angle, $\Gamma_{\parallel} = 0$, and from (6.57),

$$p' = 0$$
 $p' = \frac{Z_2}{Z_1} \frac{1}{T_{\perp}} p^i$

We see that the reflected wave is linear horizontal, no matter what the polarization of the incident wave (except for a linear vertical incident wave,

ELLIPTICAL WAVES

which would not be reflected at all). The transmitted wave is in general different from the incident wave in polarization. This characteristic can be used to produce a linearly polarized wave. Its use is rare at the lower frequencies, but more frequent for light.

Reflections from a Conductor

Let a wave in air be reflected from a plane surface that is a good conductor. From (6.35) we see that $Z_2 = 0$, and from (6.21) and (6.30) we obtain

$$\Gamma_{\perp} = -1 \qquad \Gamma_{\parallel} = +1 \tag{6.58}$$

independent of the angle of incidence. Then (6.57) gives immediately

$$p^r = -p^i \tag{6.59}$$

or in terms of the common polarization ratio P,

$$P' = -P^i \tag{6.60}$$

Again we stress that this is independent of the angle of incidence.

The Flat Plate

We consider an infinite (so that edge effects are unimportant) flat plate, Fig. 6.5. Let the incident wave be linearly polarized perpendicular to the plane of



FIGURE 6.5. Polarization changes by reflection from flat conducting plate.

incidence, so that $E_y^i = 0$, $E_x^i \neq 0$, $p^i = jE_y^i / E_x^i = 0$. This is a linear horizontal wave, according to the arbitrary definitions of Chapter 2. Then, from (6.59),

p' = 0

which is also a linear horizontal wave.

For a linear vertical wave (polarized in the plane of incidence), $E_x^i = 0$, $E_y^i \neq 0$, $p^i = j\infty$. Then

$$p' = -j\infty$$

which also represents a linear vertical wave.

Let the incident wave be right circular so that

$$p' = +1$$

Then from (6.59) we see that

p' = -1

which represents a left circular wave. The reverse is also true; a left circular wave will be reflected as a right circular wave. This leads to the well-known result that a monostatic radar transmitting a circular wave and receiving a wave of the same sense will be blind to a flat plate. It is clear that this is true also for a bistatic radar.

Dihedral Corner Reflector

A dihedral corner reflector that can be used for cross-section enhancement of a radar target is shown in Fig. 6.6. The plane conducting surfaces form a right



FIGURE 6.6. Dihedral corner reflector.

angle. It is readily apparent that a ray striking one of the surfaces from a direction perpendicular to the line of intersection of the two planes will be reflected in the direction from which it came. The scattering matrix for this reflector will be developed in a later section. We consider now only the polarization ratios.

Application of (6.59) twice for a dihedral corner oriented so that its fold line coincides with either the x or y axis gives

$$p' = -p^{i}$$

$$p' = -p' = p^{i}$$
(6.61)

and thus the reflected wave has the same polarization as the incident wave, provided that the plates are large and edge effects may be neglected.

We recall from Section 3.6 that for the reflected wave to be matched to the same antenna used to transmit the incident wave, it is necessary that

$$p' = p^{i^*} \tag{6.62}$$

It follows that a monostatic radar may be blind to a dihedral corner reflector for some polarizations of the radar. Obviously, linear vertical and linear horizontal (with respect to the line of intersection of the plates) represent the polarization-matched cases.

If the incident wave is right circular,

$$p' = +1$$

then the reflected wave is also right circular, with

$$p' = +1$$

which represents a polarization-matched case. These cases should not mislead one, however, into thinking that any wave reflected from the dihedral corner reflector is polarization matched to the transmitter. Consider an incident wave that is linear and tilted at 45°, so that $E'_y = E'_x$ and

$$p' = j1$$

Then

which represents a linear wave tilted at 45° in a reversed coordinate system appropriate to the reflected wave, Fig. 6.7. We see that the reflected wave is cross-polarized and the radar is blind to the dihedral corner.

p' = i1



FIGURE 6.7. Incident wave with 45° tilt and reflected wave for dihedral corner reflector.

Trihedral Corner Reflector

The dihedral corner reflector has a serious defect as a cross-sectional enhancement device. The incident ray must lie in a plane perpendicular to the line of intersection of the planes that form the corner if the reflected ray is to be directed back to the radar. This deficiency can be overcome by using a trihedral corner reflector. Figure 6.8 shows a triangular trihedral corner reflector, although other shapes are possible.

In general, rays that strike an interior surface of the trihedral corner will undergo three reflections as indicated by Fig. 6.8 and will be returned parallel to the incident ray. There are exceptions to this rule, however. If the incident ray is at a sufficiently large angle from the axis of symmetry of the corner reflector, it will undergo two reflections only. The ray may, for example, strike plane *AOB* and be reflected to plane *AOC*. If plane *BOC* is not sufficiently extended, the ray from *AOC* will not strike it and thus will not be returned parallel to the incident ray.



FIGURE 6.8. Triangular trihedral corner reflector.

ELLIPTICAL WAVES

It is also obvious that, if the incident ray is parallel to one of the reflecting planes forming the trihedral corner, it will be doubly reflected back to the source with polarization

$$p' = p'$$

If we apply (6.59) three times for the triply reflecting case, it is clear for the trihedral corner reflector that the polarizations of reflected and incident waves are related by

$$p^r = -p^i \tag{6.63}$$

It is quickly seen from this that linear vertical waves are reflected as linear vertical and linear horizontal as linear horizontal. In fact, any linearly polarized wave is reflected so that it is polarization matched to the transmitting antenna. To see this, consider Eq. (3.49) which gives the polarization match factor

$$\rho = \frac{(1+p_1p_2)(1+p_1^*p_2^*)}{(1+p_1p_1^*)(1+p_2p_2^*)}$$
(3.49)

We consider the incident wave to be wave 1 and the reflected wave to be 2. If the incident wave is linearly polarized, the phase angle in the modified polarization ratio is zero. Then

$$p^i = j \frac{b}{a}$$

with b/a real. For the trihedral reflector

$$p' = -j \frac{b}{a}$$

and substitution in (3.49) leads to

$$\rho = 1$$

This result is not unexpected because, although we used "vertical" polarization as an example, this reflector has no natural vertical axis for an incident ray directed along the symmetry axis of the trihedral corner. Note, however, that if the wave is reflected from only two planes of the trihedral corner, it may be significantly cross-polarized.

Finally, we note that if the incident wave is circularly polarized, the reflected wave is also circularly polarized with the opposite sense, and thus the trihedral corner reflector is invisible to a circularly polarized radar using the same antenna system for transmitting and receiving.

We can obviously extend the developments for the dihedral and trihedral corner reflectors to obtain the polarization of a wave reflected by n plane surfaces (large compared to a wavelength). It is

$$p' = (-1)^n p' \tag{6.64}$$

6.3. REFLECTION AND TRANSMISSION MATRICES

In reflection problems we sometimes need to know field magnitudes of the reflected and transmitted waves, in addition to their polarizations. In the field equations we can drop the distance exponentials since they do not affect either the power in a wave or its polarization, and write

$$\mathbf{E}' = \mathbf{u}_{x'}E'_{x'} + \mathbf{u}_{y'}E'_{y'} = E_0(\mathbf{u}_{x'}a + \mathbf{u}_{y'}be^{j\phi})$$

$$\mathbf{E}' = \mathbf{u}_{x''}E'_{x''} + \mathbf{u}_{y''}E'_{y''} = E_0(\mathbf{u}_{x''}\Gamma_{\perp}a + \mathbf{u}_{y''}\Gamma_{\parallel}be^{j\phi})$$

$$\mathbf{E}' = \mathbf{u}_{x'''}E'_{x'''} + \mathbf{u}_{y'''}E'_{y'''} = E_0\left(\mathbf{u}_{x'''}T_{\perp}a + \mathbf{u}_{y'''}\frac{Z_2}{Z_1}T_{\parallel}be^{j\phi}\right)$$
(6.65)

It is obvious from (6.65) that we can write the relationships between the field components as

$$\begin{bmatrix} E_{x''} \\ E_{y''} \end{bmatrix} = \begin{bmatrix} \Gamma_{\perp} & 0 \\ 0 & \Gamma_{\parallel} \end{bmatrix} \begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix}$$
(6.66)

and

$$\begin{bmatrix} E'_{x''} \\ E'_{y'''} \end{bmatrix} = \begin{bmatrix} T_{\perp} & 0 \\ 0 & \frac{Z_2}{Z_1} T_{\parallel} \end{bmatrix} \begin{bmatrix} E'_{x'} \\ E'_{y'} \end{bmatrix}$$
(6.67)

The coefficient matrices are called the reflection and transmission matrices. They are not the scattering matrices commonly encountered in radar (to be discussed in the next section) because they are concerned with reflections from an infinite plane, they involve bistatic reflections, and different coordinate systems are used for \mathbf{E}' and \mathbf{E}' .

We may write the fields in terms of left and right circular components using the relationships developed in Chapter 2. They are

$$\begin{split} \mathbf{E}^{i} &= E_{0}(\mathbf{u}_{x'}a + \mathbf{u}_{y'}be^{j\phi}) = E_{0}(L^{i}\boldsymbol{\omega}_{L}^{i} + R^{i}e^{-j\gamma_{i}}\boldsymbol{\omega}_{R}^{i}) \quad \text{(a)} \\ \mathbf{E}^{r} &= E_{0}(\mathbf{u}_{x''}\Gamma_{\perp}a + \mathbf{u}_{y''}\Gamma_{\parallel}be^{j\phi}) \\ &= E_{0}(L^{r}\boldsymbol{\omega}_{L}^{r} + R^{r}e^{-j\gamma_{r}}\boldsymbol{\omega}_{R}^{r}) \quad \text{(b)} \\ \mathbf{E}^{i} &= E_{0}\left(\mathbf{u}_{x'''}T_{\perp}a + \mathbf{u}_{y'''}\frac{Z_{2}}{Z_{1}}T_{\parallel}be^{j\phi}\right) \\ &= E_{0}(L^{i}\boldsymbol{\omega}_{L}^{i} + R^{i}e^{-j\gamma_{r}}\boldsymbol{\omega}_{R}^{i}) \quad \text{(c)} \end{split}$$

(c)

REFLECTION AND TRANSMISSION MATRICES

where we use $\gamma = -\theta$ to avoid confusion between the space angles θ_i , θ_r , θ_r , and the phase angles between left and right circular wave components. In (6.68) the vectors ω_L and ω_R are identified as to their corresponding coordinate system by their superscripts so that, for example, $\omega'_L \neq \omega'_L$.

From (6.68) we find the left and right circular wave components to be

$$L' = \frac{1}{2}(a - jbe^{j\phi})$$
 (a) (6.69)

$$R^{i}e^{-j\gamma_{h}} = \frac{1}{2}(a+jbe^{j\phi})$$
 (b)

$$L' = \frac{1}{2} \left(\Gamma_{\perp} a - j \Gamma_{\parallel} b e^{j \phi} \right) \tag{a}$$

$$R^{r}e^{-j\gamma_{r}} = \frac{1}{2}(\Gamma_{\perp}a + j\Gamma_{\parallel}be^{j\phi}) \qquad (b$$

$$L' = \frac{1}{2} \left(T_{\perp} a - j \frac{Z_2}{Z_1} T_{\parallel} b e^{j\phi} \right) \quad (a)$$

$$R' e^{-j\gamma_1} = \frac{1}{2} \left(T_{\perp} a + j \frac{Z_2}{Z_1} T_{\parallel} b e^{j\phi} \right) \quad (b)$$
(6.71)

These equations may also be put into matrix form without difficulty. The result is

$$\begin{bmatrix} L'\\ R'e^{-j\gamma_{t}} \end{bmatrix} = \begin{bmatrix} \Gamma_{LL} & \Gamma_{LR}\\ \Gamma_{RL} & \Gamma_{RR} \end{bmatrix} \begin{bmatrix} L^{i}\\ R^{i}e^{-j\gamma_{t}} \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} \Gamma_{\perp} + \Gamma_{\parallel} & \Gamma_{\perp} - \Gamma_{\parallel}\\ \Gamma_{\perp} - \Gamma_{\parallel} & \Gamma_{\perp} + \Gamma_{\parallel} \end{bmatrix} \begin{bmatrix} L^{i}\\ R^{i}e^{-j\gamma_{t}} \end{bmatrix}$$
$$\begin{bmatrix} L'\\ R^{i}e^{-j\gamma_{t}} \end{bmatrix} = \begin{bmatrix} T_{LL} & T_{LR}\\ T_{RL} & T_{RR} \end{bmatrix} \begin{bmatrix} L^{i}\\ R^{i}e^{-j\gamma_{t}} \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} T_{\perp} + \frac{Z_{2}}{Z_{1}} & T_{\parallel} & T_{\perp} - \frac{Z_{2}}{Z_{1}} & T_{\parallel}\\ T_{\perp} - \frac{Z_{2}}{Z_{1}} & T_{\parallel} & T_{\perp} + \frac{Z_{2}}{Z_{1}} & T_{\parallel} \end{bmatrix} \begin{bmatrix} L^{i}\\ R^{i}e^{-j\gamma_{t}} \end{bmatrix}$$
(6.73)

We note that since the off-diagonal terms of (6.66) and (6.67) are zero, an incident linear vertical wave cannot give rise to a reflected or transmitted horizontal wave. Since the off-diagonal terms of (6.72) and (6.73) are not zero in general, an incident circular wave of one sense can create a circular wave of the other sense, either reflected or transmitted.

We may see also from (6.72) and (6.73) that an incident left circular wave gives the same reflected and transmitted powers as an incident right circular wave.

227

6.4. BACKSCATTERING FROM A TARGET: THE SCATTERING MATRIX

The backscattered power from a radar target is dependent on the polarization of the wave transmitted toward it. Further, many targets can reflect a wave that is significantly cross-polarized, that is, has a field component not present in the incident wave. For example, consider the wave to be linear vertical and the target to be a thin wire transverse to a line from radar to target and neither vertical nor horizontal in the transverse plane. It is obvious that a current induced in the wire by the incident vertical field will reradiate a wave with a substantial horizontal component.

In many cases the transmitting and receiving radar antennas are so close that one coordinate system may be used for both antennas, as illustrated in Fig. 6.9, and so that $r_1 = r_2 = r$. In many cases, also, one antenna is used both for transmitting and receiving so that we need consider only one effective length and polarization ratio.

The wave incident on the target from the transmitting antenna is

$$\mathbf{E}^{i} = j \, \frac{Z_0 I}{2\lambda r} \, \mathbf{h}(\theta, \, \phi) \tag{6.74}$$

where we have dropped the phase term e^{-jkr} .

The field components are transverse to the direction of propagation, and if the z axis of Fig. 6.9 is oriented toward the target, these are x and y components. We may then rewrite (6.74) in a matrix form, and it is desirable to do so. Then

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = j \, \frac{Z_0 I}{2\lambda r} \begin{bmatrix} h_x \\ h_y \end{bmatrix}$$
(6.75)

At the target an incident x component of the field gives rise to reflected x and y components, and similarly for an incident y component. Then we can write the reflected wave at the radar as

$$\begin{bmatrix} E'_{x} \\ E'_{y} \end{bmatrix} = \frac{1}{\sqrt{4\pi}r} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} E'_{x} \\ E'_{y} \end{bmatrix}$$
(6.76)



FIGURE 6.9. Backscattering from a target.

The [A] matrix is known as the *target scattering matrix*. Some discussion of the form chosen for it is in order. Some authors use subscripts \perp and \parallel rather than x and y, but in reflection problems it is sometimes desirable to work with reversed coordinates so that a right-handed system is available for both incident and reflected waves. This is more easily done when using x and y rather than with the less versatile \perp and \parallel . The use of $\sqrt{4\pi r}$ in the definition of the scattering matrix is not universal. Here we follow the usage of Sinclair [2] and Copeland [3]. On the other hand, Ruck et al. [4] use for the scattering matrix

$$\begin{bmatrix} E_1^s \\ E_2^s \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_1' \\ E_2' \end{bmatrix}$$

omitting the distance term. In their definition $[E^s]$ is the scattered field at the radar and $[E^i]$ the electric field of an incident plane wave. The reader is advised that this usage is widespread.

The scattering cross section of a target, when polarization is not taken into account, is defined as (see also Chapter 1)

$$\sigma = \lim_{r \to \infty} \frac{4\pi r^2 S^r}{S^i} \tag{6.77}$$

where S' is the magnitude of the Poynting vector of the reflected wave measured at the receiver, and S' is the Poynting vector magnitude of the transmitted wave measured at the target position. If the target depolarizes the incident wave, we must describe the scattering cross section by a matrix also.

Ruck et al. give the relationship between the scattering matrix elements and the scattering cross section as

$$\sigma_{ij} = 4\pi r^2 |a_{ij}|^2$$

However, in the notation of this book the distance term is avoided, which is desirable, and the relationship is

$$\sigma_{\alpha\beta} = |A_{\alpha\beta}|^2 \tag{6.78}$$

where α and β independently take on values x and y. Thus both σ and A are independent of the radar-target distance.

From (6.75) and (6.76) the reflected fields at the receiving antenna may be written as

$$\begin{bmatrix} E'_{x} \\ E'_{y} \end{bmatrix} = \frac{jZ_{0}I}{\sqrt{4\pi}(2\lambda r^{2})} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} h_{x} \\ h_{y} \end{bmatrix}$$
(6.79)

If the roles of transmitting and receiving antennas are interchanged, the

reciprocity theorem states that the same open-circuit voltage is induced in the new receiving antenna as in the old. This requires that the scattering matrix be symmetric, that is

$$A_{xy} = A_{yx} \tag{6.80}$$

To see this, we rewrite (6.79) to show explicitly that [h] shown there is for the transmitter

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \frac{jZ_0 I}{\sqrt{4\pi}(2\lambda r^2)} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} h'_x \\ h'_y \end{bmatrix}$$
(6.81)

Now the voltage induced in the receiver load is

$$V_{R} = \begin{bmatrix} h_{x}^{r} & h_{y}^{r} \end{bmatrix} \begin{bmatrix} E_{x}^{r} \\ E_{y}^{r} \end{bmatrix} = \frac{jZ_{0}I}{\sqrt{4\pi}(2\lambda r^{2})} \begin{bmatrix} h_{x}^{r} & h_{y}^{r} \end{bmatrix} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} h_{x}^{i} \\ h_{y}^{i} \end{bmatrix}$$
(6.82)

where superscript r denotes the effective length of the receiving antenna. If we interchange roles of transmitting and receiving antennas while keeping the same current I, we must have the same open-circuit voltage. Therefore

$$V_{R} = \frac{jZ_{0}I}{\sqrt{4\pi}(2\lambda r^{2})} \begin{bmatrix} h_{x}^{i} & h_{y}^{i} \end{bmatrix} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} h_{x}^{\prime} \\ h_{y}^{\prime} \end{bmatrix}$$
(6.83)

Equating these two forms for V_R gives

$$\begin{bmatrix} h_x^r & h_y^r \end{bmatrix} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} h_x^i \\ h_y^i \end{bmatrix} = \begin{bmatrix} h_x^i & h_y^i \end{bmatrix} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} h_x^r \\ h_y^r \end{bmatrix}$$
(6.84)

Multiplication of these matrices shows that the equation can be satisfied only if $A_{xy} = A_{yx}$.

Further, if the target is symmetric about a plane containing a ray from transmitting antenna to the target, the coordinate system may be chosen so that $A_{xy} = 0$. To see this, consider a simple target such as two diagonal wires, as shown in Fig. 6.10. A ray from the transmitter to the target lies in the symmetry plane and is perpendicular to the wires. We choose the coordinate system so that the x axis lies in the plane, and transmit a wave with only an x component, E_x^i in (6.76). At some instant of time the incident wave will set up currents as shown in Fig. 6.10. Obviously, when these currents reradiate, the vertical components will cancel. Then in (6.76), E_y^r must be zero, and therefore

$$A_{yx} = 0$$



FIGURE 6.10. Simple target with symmetry plane.

The incident power density at a target may be written, using (6.75), as

$$S^{i} = \frac{Z_0 I^2}{8\lambda^2 r^2} \left(|h_{1x}|^2 + |h_{1y}|^2 \right)$$
(6.85)

where Z_0 and I are taken as real, and I is the amplitude of the sinusoidal feed current rather than its rms value. If Z_0 and I are not real, their amplitudes should be used in (6.85). We use subscript 1 for the transmitting antenna.

We may write S^{i} in matrix notation as

$$S^{i} = \frac{Z_{0}I^{2}}{8\lambda^{2}r^{2}} [h_{1}]^{\mathrm{T}}[h_{1}]^{*}$$
(6.86)

where $[h_1]^T$ is the transpose of the column matrix $[h_1]$, and $[h_1]^*$ is the conjugate matrix.

The Poynting vector density of the reflected wave at the receiver is

$$S' = \frac{1}{2Z_0} \left(|E_x'|^2 + |E_y'|^2 \right) = \frac{1}{2Z_0} \left[E' \right]^{\mathrm{T}} \left[E' \right]^{\mathrm{T}}$$
(6.87)

and using (6.79), this becomes

$$S' = \frac{1}{2Z_0} \left(\frac{jZ_0 I}{2\lambda r} \right) \left(\frac{-jZ_0 I}{2\lambda r} \right) \frac{1}{4\pi r^2} \left[[A][h_1] \right]^{\mathrm{T}} [[A][h_1]]^*$$
(6.88)

Now the transpose of the product of two matrices is the product of their transposes in reverse order, so that

$$S^{r} = \frac{Z_{0}I^{2}}{32\pi\lambda^{2}r^{4}} \left[[h_{1}]^{\mathrm{T}}[A]^{\mathrm{T}}] [[A]^{*}[h_{1}]^{*} \right]$$
(6.89)

Using (6.85) and (6.89) in the backscattering cross section (6.77) we get

$$\sigma = \frac{[[h_1]^{\mathrm{T}}[A]^{\mathrm{T}}][[A]^*[h_1]^*]}{[h_1]^{\mathrm{T}}[h_1]^*}$$
(6.90)

or if the matrices are multiplied as indicated,

$$\sigma = \frac{|A_{xx}h_{1x} + A_{xy}h_{1y}|^2 + |A_{yx}h_{1x} + A_{yy}h_{1y}|^2}{|h_{1x}|^2 + |h_{1y}|^2}$$
(6.91)

This equation for σ makes it appear that the backscattering cross section of a target is a function of the effective length components of the transmitting antenna, but of course this is not the case. We write for the transmitting antenna the modified polarization ratio p_1 ,

$$p_1 = j \, \frac{h_{1y}}{h_{1x}} \tag{6.92}$$

and substitute it into (6.91), which then becomes

$$\sigma = \frac{|A_{xx} - jp_1 A_{xy}|^2 + |A_{yx} - jp_1 A_{yy}|^2}{1 + |p_1|^2}$$
(6.93)

We see that, as expected, the cross section depends on the polarization of the transmitting antenna but not on its effective length.

If we require in (6.91) that the denominator be constant while we vary h_{1x} and h_{1y} (a constraint discussed in Chapter 3), then (6.91) is the sum of two terms like that for received power, (3.16). We may maximize each term separately, and it follows from (3.28) and (3.36) that the first and second terms are maximum, respectively, if

$$\frac{h_{1y}}{h_{1x}} = \left(\frac{A_{xy}}{A_{xx}}\right)^* \tag{6.94}$$

and

$$\frac{h_{1y}}{h_{1x}} = \left(\frac{A_{yy}}{A_{xy}}\right)^* \tag{6.95}$$

The two terms may be maximized simultaneously if we set (6.94) and (6.95) equal to each other. This leads to

$$A_{xy} = \sqrt{A_{xx}A_{yy}} \tag{6.96}$$

and

$$\frac{h_{1y}}{h_{1x}} = \sqrt{\frac{A_{yy}^*}{A_{xx}^*}} \tag{6.97}$$

assuming that all parameters are nonzero, giving a maximum cross section

$$\sigma_m = (|A_{xx}| + |A_{yy}|)^2 \tag{6.98}$$

More often than not the target scattering parameters are not under our control, and (6.98) cannot be satisfied.

More generally, (6.91) may be maximized by differentiating with respect to magnitude and angle of $h_{1\nu}/h_{1x}$. The choices are

$$\arg (h_{1y}/h_{1x}) = \arg (A_{xx}A_{xy}^* + A_{yy}^*A_{xy})$$
(a)
$$\left| \frac{h_{1y}}{h_{1x}} \right| = \frac{|A_{yy}|^2 - |A_{xx}|^2}{2|A_{xx}A_{xy}^* + A_{xy}A_{yy}^*|} \pm \frac{(|A_{yy}|^2 - |A_{xx}|^2)^2}{(2|A_{xx}A_{xy}^* + A_{xy}A_{yy}^*|)^2}$$
(b)

where again we assume that all parameters are nonzero.

The open-circuit voltage induced in the receiving antenna by the reflected wave is

$$V = \mathbf{h}_2 \cdot \mathbf{E}' = [h_2]^{\mathrm{T}} [E'] = h_{2x} E'_x + h_{2y} E'_y$$
(6.100)

giving a received power density

$$S' = VV^* = |h_{2x}E'_x + h_{2y}E'_y|^2$$
(6.101)

where we use an equality rather than a proportional symbol since we will later take a power ratio.

Now (6.101) is the same as (3.16), and S' may be maximized in (6.101) just as it was in Chapter 3. The result is

$$S_m^r = |\mathbf{h}_2|^2 |\mathbf{E}^r|^2 = (|h_{2x}|^2 + |h_{2y}|^2)(|E_x^r|^2 + |E_y^r|)^2$$
(6.102)

if we choose

$$\frac{h_{2y}}{h_{2x}} = \left(\frac{E_y'}{E_x'}\right)^*$$
(6.103)

The polarization match factor between the receiving antenna and the wave backscattered from the target is

$$\rho = \frac{|\mathbf{h}_2 \cdot \mathbf{E}'|^2}{|\mathbf{h}_2|^2 |\mathbf{E}'|^2} = \frac{|h_{2x} E_x' + h_{2y} E_y'|^2}{(|h_{2x}|^2 + |h_{2y}|^2)(|E_x'|^2 + |E_y'|^2)}$$
(6.104)

where the components of [E'] are given in (6.79).

In terms of the target scattering parameters and the effective length components of the transmitting antenna, the choice of h_{2y}/h_{2x} for maximum received power becomes, from (6.103) and (6.79)

$$\frac{h_{2y}}{h_{2x}} = \left(\frac{A_{xy}h_{1x} + A_{yy}h_{1y}}{A_{xx}h_{1x} + A_{xy}h_{1y}}\right)^*$$
(6.105)

where we have used $A_{yx} = A_{xy}$. The match factor (6.104) becomes

$$\rho = \frac{|h_{2x}(A_{xx}h_{1x} + A_{xy}h_{1y}) + h_{2y}(A_{xy}h_{1x} + A_{yy}h_{1y})|^2}{(|h_{2x}|^2 + |h_{2y}|^2)(|A_{xx}h_{1x} + A_{xy}h_{1y}|^2 + |A_{xy}h_{1x} + A_{yy}h_{1y}|^2)}$$
(6.106)

If the same antenna is used for transmitting and receiving, so that

$$h_{1x} = h_{2x} = h_x$$
 $h_{1y} = h_{2y} = h_y$ (6.107)

then the relationship for maximum received power, (6.105), becomes

$$\frac{h_{y}}{h_{x}} = \left(\frac{A_{xy}h_{x} + A_{yy}h_{y}}{A_{xx}h_{x} + A_{xy}h_{y}}\right)^{*}$$
(6.108)

We have used primarily in this chapter the modified polarization ratio p, but it is convenient here to use the more common ratio P. The ratios are of course related by p = jP. From the definitions of antenna effective length and P, (6.108) becomes

$$P = \frac{h_y}{h_x} = \left(\frac{A_{xy} + A_{yy}P}{A_{xx} + A_{xy}P}\right)^*$$
(6.109)

Equation (6.109) can be solved for P with some ingenuity. We first write it as

$$P = \frac{A_{xy}^* + A_{yy}^* P^*}{A_{xx}^* + A_{xy}^* P^*}$$
(6.110)

and substitute it into the right side of (6.109) to eliminate the conjugate of *P*. The result is a quadratic:

$$P^2 + BP + C = 0 \tag{6.111}$$

with solution

$$P = -\frac{1}{2}B \pm \frac{1}{2}\sqrt{B^2 - 4C}$$
(6.112)

where

$$B = \frac{|A_{xx}|^2 - |A_{yy}|^2}{A_{xy}^* A_{yy} + A_{xy} A_{xx}^*}$$
(6.113)

and

$$C = -\frac{A_{xy}A_{yy}^{*} + A_{xy}^{*}A_{xx}}{A_{xy}^{*}A_{yy} + A_{xy}A_{xx}^{*}}$$
(6.114)

The polarization match factor, if the same antenna is used for transmitting and receiving (but not configured for maximum received power), becomes

$$\rho = \frac{|h_x^2 A_{xx} + 2h_x h_y A_{xy} + h_y^2 A_{yy}|^2}{(|h_x|^2 + |h_y|^2)(|A_{xx} h_x + A_{xy} h_y|^2 + |A_{xy} h_x + A_{yy} h_y|^2)}$$
(6.115)

In terms of the polarization ratio P of the antenna, this becomes

$$\rho = \frac{|A_{xx} + 2A_{xy}P + A_{yy}P^2|^2}{(1+|P|^2)(|A_{xx} + A_{xy}P|^2 + |A_{xy} + A_{yy}P|^2)}$$
(6.116)

If the polarization ratio P is found from (6.112) and used in (6.116), the largest polarization match factor (polarization efficiency) is obtained.

6.5. SCATTERING MATRIX FOR CIRCULAR WAVE COMPONENTS

It is often convenient to use the scattering parameters in left and right circular component form so that the scattered fields are given by

$$\begin{bmatrix} E_R^r \\ E_L^r \end{bmatrix} = \frac{1}{\sqrt{4\pi}r} \begin{bmatrix} A_{RR} & A_{RL} \\ A_{LR} & A_{LL} \end{bmatrix} \begin{bmatrix} E_R^i \\ E_L^i \end{bmatrix}$$
(6.117)

The elements of the circular scattering matrix may be determined by measurement or by transformation from the rectangular matrix elements. We will develop the transformation.

A plane wave may be written in right and left circular component form as

$$\mathbf{E} = E_R \boldsymbol{\omega}_R + E_L \boldsymbol{\omega}_L \tag{6.118}$$

where E_R and E_L are the complex right and left circular field components. The rotating vectors $\boldsymbol{\omega}_R$ and $\boldsymbol{\omega}_L$ are related to the rectangular unit vectors by

$$\boldsymbol{\omega}_{L} = \mathbf{u}_{x} + j\mathbf{u}_{y} \quad \text{(a)}$$

$$\boldsymbol{\omega}_{R} = \mathbf{u}_{x} - j\mathbf{u}_{y} \quad \text{(b)}$$
(2.74)

Equating circular and rectangular component forms for the fields leads to

$$E_R = \frac{1}{2}(E_x + jE_y)$$
 (a)
 $E_L = \frac{1}{2}(E_x - jE_y)$ (b)
(6.119)
236 POLARIZATION CHANGES BY REFLECTION AND TRANSMISSION

We substitute into (6.117) these expressions for E_R and E_L . For the incident wave E, the substitution is as written, but for the reflected wave we must go to a right-handed coordinate system with unit vectors $-\mathbf{u}_z$, \mathbf{u}_x , $-\mathbf{u}_y$ before transforming to circular form. Therefore, for the reflected wave we replace E_y by $-E_y$, and (6.117) becomes

$$\frac{1}{2} \begin{bmatrix} E_x' - jE_y' \\ E_x' + jE_y' \end{bmatrix} = \frac{1}{2} \frac{1}{\sqrt{4\pi r}} \begin{bmatrix} A_{RR} & A_{RL} \\ A_{LR} & A_{LL} \end{bmatrix} \begin{bmatrix} E_x' + jE_y' \\ E_x^i - jE_y^i \end{bmatrix}$$
(6.120)

If the matrices in (6.120) are multiplied and compared to (6.76), also multiplied out, we obtain the following relationships among the rectangular and circular scattering matrix elements

$$A_{RR} = \frac{1}{2} (A_{xx} - jA_{xy} - jA_{yx} - A_{yy})$$

$$A_{RL} = \frac{1}{2} (A_{xx} + jA_{xy} - jA_{yx} + A_{yy})$$

$$A_{LR} = \frac{1}{2} (A_{xx} - jA_{xy} + jA_{yx} + A_{yy})$$

$$A_{LL} = \frac{1}{2} (A_{xx} + jA_{xy} + jA_{yx} - A_{yy})$$

$$A_{xx} = \frac{1}{2} (A_{RR} + A_{RL} + A_{LR} + A_{LL})$$

$$A_{xy} = \frac{1}{2} j(A_{RR} - A_{RL} + A_{LR} - A_{LL})$$

$$A_{yx} = \frac{1}{2} j(A_{RR} + A_{RL} - A_{LR} - A_{LL})$$

$$A_{yy} = \frac{1}{2} (-A_{RR} + A_{RL} + A_{LR} - A_{LL})$$
(6.122)

Since $A_{xy} = A_{yx}$, it follows that $A_{RL} = A_{LR}$, and these equations simplify to

$$A_{RR} = \frac{1}{2}(A_{xx} - j2A_{xy} - A_{yy})$$

$$A_{RL} = \frac{1}{2}(A_{xx} + A_{yy})$$

$$A_{LR} = A_{RL}$$

$$A_{LL} = \frac{1}{2}(A_{xx} + j2A_{xy} - A_{yy})$$

$$A_{xx} = \frac{1}{2}(A_{RR} + 2A_{RL} + A_{LL})$$

$$A_{xy} = \frac{1}{2}j(A_{RR} - A_{LL})$$

$$A_{yx} = A_{xy}$$

$$A_{yy} = \frac{1}{2}(-A_{RR} + 2A_{RL} - A_{LL})$$
(6.124)

RELATIONSHIP OF THE SCATTERING MATRIX AND POLARIZATION RATIO 237

Let us specialize to a target with a plane of symmetry so that

$$A_{xy} = A_{yx} = 0 \tag{6.125}$$

Then

$$A_{RR} = \frac{1}{2} (A_{xx} - A_{yy})$$

$$A_{RL} = A_{LR} = \frac{1}{2} (A_{xx} + A_{yy})$$

$$A_{LL} = \frac{1}{2} (A_{xx} - A_{yy})$$
(6.126)

Note that a plane of symmetry is sufficient to cause the returns for left and right circular polarizations to be equivalent.

Finally, let us specialize to a spherical target (or to a target with a plane of symmetry that also appears to be unaltered by a rotation through 90°) as we did earlier when we were considering the scattering matrix in rectangular form. Since $A_{xx} = A_{yy}$, (6.126) reduces to

$$A_{RR} = A_{LL} = 0 \qquad A_{RL} = A_{LR} = A_{xx} \tag{6.127}$$

This shows clearly that the polarization sense is reversed for a circular wave incident on a sphere.

Note that we used only one set of coordinates for the scattering matrix in rectangular form, which is common, but in transforming to the scattering matrix for circular components we first established a right-handed coordinate set for the reflected wave because to refer to a wave as right circular in an improper coordinate system would be quite confusing.

6.6. RELATIONSHIP OF THE SCATTERING MATRIX AND POLARIZATION RATIO

The scattering matrix and knowledge of the incident wave are sufficient to describe the properties of the reflected wave, including polarization. Let us find the polarization ratio of the reflected wave in terms of the scattering matrix elements and the incident wave. We start with

$$\begin{bmatrix} E_x'\\ E_y' \end{bmatrix} = \frac{1}{\sqrt{4\pi}r} \begin{bmatrix} A_{xx} & A_{xy}\\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} E_x'\\ E_y' \end{bmatrix}$$
(6.76)

Now, substituting

$$p^{i} = \frac{jE_{y}^{i}}{E_{x}^{i}} \qquad p^{r} = -\frac{jE_{y}^{r}}{E_{x}^{r}}$$

into (6.76) leads to

$$\begin{bmatrix} 1\\ jp^r \end{bmatrix} = \frac{1}{\sqrt{4\pi}r} \frac{E'_x}{E'_x} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} 1\\ -jp^i \end{bmatrix}$$
(6.128)

which is easily solved for p',

$$p' = \frac{A_{yy}p' + jA_{yx}}{jA_{xy}p' - A_{xx}}$$
(6.129)

where we assume $E'_x \neq 0$. The equation simplifies slightly if we recognize that $A_{xy} = A_{yx}$.

If we take the case of a target with a plane of symmetry so that $A_{xy} = A_{yx} = 0$, (6.129) reduces to

$$p' = -\frac{A_{yy}}{A_{xx}} p^i$$

and if we go further to a target that is unaltered by a 90° rotation

$$p' = -p'$$

a result we found earlier for the infinite conducting plane.

We can also relate the circular polarization ratio q to the elements of a circular scattering matrix. Just as we wrote the reflected wave in terms of a scattering matrix for rectangular components by (6.76) we can relate the circular components of the reflected and incident waves by

$$\begin{bmatrix} E_{R}'\\ E_{L}' \end{bmatrix} = \frac{1}{\sqrt{4\pi}r} \begin{bmatrix} A_{RR} & A_{RL}\\ A_{LR} & A_{LL} \end{bmatrix} \begin{bmatrix} E_{R}^{i}\\ E_{L}^{i} \end{bmatrix}$$
(6.117)

We use the definition

$$q = \frac{E_L}{E_R} \tag{2.85}$$

noting that it applies both to incident and reflected waves, unlike the rectangular form p (or P), since in writing (6.76) we used the same x and y axes for the incident and reflected waves, but in writing (6.117) we used a right-handed system for both incident and reflected waves in order to define clockwise and counterclockwise rotations. Combining the last two equations gives

$$\begin{bmatrix} 1\\q' \end{bmatrix} = \frac{1}{\sqrt{4\pi}r} \frac{E_R'}{E_R'} \begin{bmatrix} A_{RR} & A_{RL} \\ A_{LR} & A_{LL} \end{bmatrix} \begin{bmatrix} 1\\q^i \end{bmatrix}$$
(6.130)

which leads to

$$q^{r} = \frac{A_{LL}q' + A_{LR}}{A_{RL}q^{i} + A_{RR}}$$
(6.131)

Again if we take the symmetric case $A_{LR} = A_{RL} = 0$, we get the simpler form

$$q' = \frac{A_{LL}}{A_{RR}} q'$$

6.7. SCATTERING MATRICES FOR SOME COMMON REFLECTORS

In Section 6.2 we developed the polarization ratios for the waves scattered from a flat plate and dihedral and trihedral corners. In this section we will find their scattering matrices.

Flat Plate

For an infinite flat conducting plate at normal incidence the reflected and incident electric fields are related by

$$E_{x}^{r} = -E_{x}^{i} \qquad E_{y}^{r} = -E_{y}^{i} \tag{6.132}$$

Then from the relationship

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} E'_x \\ E'_y \end{bmatrix}$$

where the $\sqrt{4\pi r}$ of (6.76) is omitted as inappropriate for a flat plate, it is obvious that the scattering matrix is

$$[A] = \begin{bmatrix} -1 & 0\\ 0 & -1 \end{bmatrix} \tag{6.133}$$

For a disc of area πR^2 , a physical optics approach gives a normal incidence cross section [4]

$$\sigma = \frac{4\pi}{\lambda^2} \left(\pi R^2\right)^2 \tag{6.134}$$

Since the cross section is related to the matrix elements by (6.78), it follows that the scattering matrix of the disc at normal incidence is

240 POLARIZATION CHANGES BY REFLECTION AND TRANSMISSION

$$[A] = \frac{2\sqrt{\pi}}{\lambda} (\pi R^2) \begin{bmatrix} -1 & 0\\ 0 & -1 \end{bmatrix}$$
(6.135)

If these values are substituted into (6.129), we find

$$p' = -p^i$$

which we know to be true for a flat plate.

Dihedral Corner

Consider first a dihedral corner oriented so that its fold line (intersection line between the planes forming the dihedral) is parallel to the y axis. Figure 6.11 shows this corner with incident E_y and E_x fields. (In the figure a dot represents an electric field out of the paper and a cross one into the paper.) It is easily seen from the figure that the E_y component is reflected unchanged (except for the phase change with distance which is not shown), while the E_x component is reversed. No cross-polarized component occurs. Then the scattering matrix for a corner constructed with semi-infinite plates is

$$[A] = \begin{bmatrix} -1 & 0\\ 0 & 1 \end{bmatrix} \tag{6.136}$$

Consider next a dihedral whose fold line is rotated in the xy plane by angle θ measured from the y axis toward the x axis, Fig. 6.12. The dihedral plates are taken to be so large that edge effects are negligible. If the incident field is y directed, E_y^i , then components parallel and perpendicular to the dihedral fold line are

$$E_{x'} = -E_{y}^{i} \sin \theta$$
 (a) $E_{y'} = E_{y}^{i} \cos \theta$ (b) (6.137)



FIGURE 6.11. Electric field components for dihedral corner.



FIGURE 6.12. Tilted dihedral corner.

After two reflections

$$E_{x'}^{r} = -E_{x'} = E_{y}^{i} \sin \theta \quad (a)$$

$$E_{y'}^{r} = E_{y'} = E_{y}^{i} \cos \theta \quad (b)$$
(6.138)

From the figure, the x and y components of the reflected field are

$$E'_{x} = E'_{x} \cos \theta + E'_{y} \sin \theta = E'_{y} \sin \theta \cos \theta + E'_{y} \sin \theta \cos \theta \quad (a)$$

$$E'_{y} = E'_{y} \cos \theta - E'_{x} \sin \theta = E'_{y} \cos^{2} \theta - E'_{y} \sin^{2} \theta \qquad (b)$$

From

$$\begin{bmatrix} E_x'\\ E_y' \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy}\\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} 0\\ E_y^i \end{bmatrix}$$
(6.140)

it follows that

242 POLARIZATION CHANGES BY REFLECTION AND TRANSMISSION

$$A_{xy} = 2\sin\theta\cos\theta = \sin 2\theta \quad (a)$$

$$A_{yy} = \cos^2\theta - \sin^2\theta = \cos 2\theta \quad (b)$$
(6.141)

If the incident field is x directed, then

$$E_{x'} = E_x^i \cos \theta$$
 (a) $E_{y'} = E_x^i \sin \theta$ (b) (6.142)

After two reflections

$$E_{x}^{r} = -E_{x}^{i}\cos\theta \quad (a) \qquad E_{y}^{r} = E_{x}^{i}\sin\theta \quad (b) \qquad (6.143)$$

and

$$E'_{x} = E'_{x} \cos \theta + E'_{y} \sin \theta = -E'_{x} \cos^{2} \theta + E'_{x} \sin^{2} \theta \qquad (a)$$

$$E'_{y} = E'_{y} \cos \theta - E'_{x} \sin \theta = E'_{x} \sin \theta \cos \theta + E'_{x} \sin \theta \cos \theta \qquad (b)$$

It is then obvious that

$$A_{xx} = \sin^2 \theta - \cos^2 \theta = -\cos 2\theta \quad (a)$$

$$A_{yx} = 2\sin \theta \cos \theta = \sin 2\theta \quad (b)$$

For a dihedral corner with plate dimensions and a and b, Fig. 6.12, Ruck et al. [4] give a scattering cross section

$$\sigma = \frac{16\pi a^2 b^2 \sin^2(\pi/4 + \phi)}{\lambda^2}$$
(6.146)

where it is assumed that the incident ray path is perpendicular to the dihedral fold line. It follows that the scattering matrix for a dihedral rotated by angle θ in the xy plane from the y axis is

$$[A] = \frac{4\sqrt{\pi}ab\sin\left(\frac{\pi}{4} + \phi\right)}{\lambda} \begin{bmatrix} -\cos 2\theta & \sin 2\theta\\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$
(6.147)

Trihedral Corner

In Section 6.2 we saw that a trihedral corner reflector constructed of very large plates has the same polarization as one flat plate at normal incidence, namely, $p^r = -p^i$. The scattering matrix of the trihedral corner of finite size can then be expected to approximate that for the flat plate, (6.133), modified to account for the finite cross section. Ruck et al. give for the maximum cross sections of the square and triangular trihedral corners of Fig. 6.13:







FIGURE 6.13. Trihedral corners.

Square:
$$\sigma = \frac{12\pi L^4}{\lambda^2}$$
(6.148)

Triangular:
$$\sigma = \frac{4\pi L^4}{\lambda^2}$$
 (6.149)

Then, if edge effects are neglected, the scattering matrices at angles giving maximum cross section are

Square:
$$[A] = \frac{2\sqrt{3\pi}L^2}{\lambda} \begin{bmatrix} -1 & 0\\ 0 & -1 \end{bmatrix}$$
 (6.150)

Triangular:
$$[A] = \frac{2\sqrt{\pi}L^2}{\lambda} \begin{bmatrix} -1 & 0\\ 0 & -1 \end{bmatrix}$$
(6.151)

Sphere

It was pointed out earlier that for a target that is symmetric about a plane containing a ray from antenna to target, we may choose a coordinate system so that $A_{xy} = 0$. To do so causes (6.115) to become

$$\rho = \frac{|h_x^2 A_{xx} + h_y^2 A_{yy}|^2}{(|h_x|^2 + |h_y|^2)(|A_{xx}h_x|^2 + |A_{yy}h_y|^2)}$$
(6.152)

It is obvious that a spherical target is symmetric about any plane containing a ray from antenna to target, and (6.152) is applicable to the sphere. Further, for a sphere it is clear that

$$A_{xx} = A_{yy}$$

no matter what coordinate system we choose. Therefore

244 POLARIZATION CHANGES BY REFLECTION AND TRANSMISSION

$$\rho = \frac{|h_x^2 + h_y^2|^2}{(|h_x|^2 + |h_y|^2)^2}$$
(6.153)

For a linearly polarized incident wave h_x and h_y are either in phase or 180° out of phase, except when one is zero. It follows in any case of a linearly polarized wave that $\rho = 1$.

For a circularly polarized incident wave we can take h_x as real and

$$h_y = \pm jh_x \tag{6.154}$$

Then (6.153) gives

 $\rho = 0$

and we see that a spherical target is invisible to a circularly polarized radar. For this reason it is common for radars to use circular polarization to suppress the return from raindrops, which to a first approximation are spherical. The return from a desired target in rain may be reduced also, but generally the ratio of desired signal to rain clutter is improved.

If the raindrops are significantly deformed from the spherical so that, for example, $A_{xx} > A_{yy}$, then (6.152) indicates that for maximum rejection of the rain signal, it would be appropriate to make $h_x > h_y$. Beckmann has pointed out that while circular polarization gives the greatest rejection of the return from a (spherical) raindrop, it does not necessarily allow the greatest degree of discrimination between rain and the desired target signal because that discrimination depends on the polarization characteristics of the target [5].

From the discussion given here, it is apparent that for the sphere the scattering matrix is

$$[A] = A_{xx} \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$$
(6.155)

The reader is referred to the extensive discussion by Barrick to find cross sections of the sphere [4, Ch. 3].

6.8. REFLECTIONS FROM ARBITRARILY ORIENTED PLANE

In this discussion it is convenient to consider the arbitrarily oriented plane as one that is tilted with respect to a "horizontal" reference surface and to consider an incident wave to have "vertical" and "horizontal" components. The surface in question may be a dielectric with appropriate Fresnel coefficients or it may be conducting, in which case the Fresnel coefficients are set to ± 1 . The material in this section was developed to describe the reflections from Earth that occur with propagation between antennas, or between a radar and target, when they are in the vicinity of the earth's surface. These reflections are commonly referred to as multipath reflections. The development follows in many respects the work of Beckmann [6, 7], although equations for some angles will be derived by vector methods rather than by the spherical trigonometry methods of Beckmann and Spizzichino [6]. In addition, Beckmann gives only polarization ratios, not fields.

It may be shown that a wave reflected in the plane of incidence is not depolarized if the incident wave is either horizontally or vertically polarized. If the wave is not horizontally or vertically polarized, it will be depolarized even by pure specular reflection in the plane of incidence from a smooth surface. This occurs because the incident wave must be split into horizontal and vertical components and the Fresnel coefficients applied separately to each component. Since the Fresnel coefficients are different for the two polarizations, the reflected components will be differently affected in amplitude and phase by the reflection, and the reflected wave will have polarization properties that differ from those of the incident wave.

If the earth is not flat, specular points may exist that do not lie on the intersection of the earth with a vertical plane containing radar and target. Thus, hilly terrain may have multiple specular points. In general, the reflected wave from such a specular point is depolarized since the incident wave, even if horizontally or vertically polarized with respect to the average terrain, is not horizontally or vertically polarized with respect to the local terrain at a specular point.

In this section we shall use a polarization ratio that is the complex ratio of vertical to horizontal field components

$$P = \frac{E_V}{E_H} \tag{6.156}$$

Figure 6.14 shows the geometry to be used in this discussion. In Fig. 6.14, the coordinate system is drawn for convenience with the origin at the specular point, but this does not detract from the generality of the development. We take the unit vectors in the cyclic order \mathbf{k} , \mathbf{u}_H , \mathbf{u}_V , which are, respectively, a vector in the direction of propagation, a unit vector parallel to the flat earth (horizontal), and a unit vector that is "vertical." (It would be perpendicular to the earth for a wave propagating parallel to the carth. Otherwise it is merely perpendicular to \mathbf{k} and \mathbf{u}_H .) When the final equations are given, the specular point will be taken at (x, y, 0). The incidence plane of Fig. 6.14 is the "main" plane of incidence determined by the propagation vector \mathbf{k}_1 (or \mathbf{r}_1) and the unit vector \mathbf{u}_2 normal to the average earth plane (the *xy* plane). Since in general the specular point lies on a tilted surface, the "local" plane of incidence at the specular point. The normal vector \mathbf{n} can be seen to lie in the plane defined by \mathbf{r}_1 and \mathbf{r}_2 and is given by



FIGURE 6.14. Depolarization by scattering out of incidence plane.

$$\mathbf{n} = \frac{\mathbf{r}_2 / r_2 - \mathbf{r}_1 / r_1}{|\mathbf{r}_2 / r_2 - \mathbf{r}_1 / r_1|}$$
(6.157)

The plane of scattering or reflection is similarly defined by the vectors \mathbf{k}_2 and \mathbf{u}_z , while the local scattering plane is defined by \mathbf{k}_2 and \mathbf{n} . Note that the local scattering plane is the same as the local incidence plane, since Snell's laws require that \mathbf{k}_1 , \mathbf{k}_2 , and \mathbf{n} lie in the same plane.

Figure 6.15 describes the geometry at the scattering point more completely. The incident wave is taken as linearly polarized without loss of generality since an elliptically polarized wave can be considered the sum of two orthogonal linearly polarized waves with an appropriate phase difference. The local plane of incidence is perpendicular to the reflecting plane surface and is established by **n** and \mathbf{k}_1 . We treat \mathbf{E}_1 in Fig. 6.15 as horizontally polarized and measure β as the included angle between it and the reflecting plane.

A general incident field is shown in Fig. 6.16. Angle β is taken positive as measured clockwise from \mathbf{u}_{H1} of Fig. 6.16 if θ_3 in Fig. 6.14 is positive as measured from \mathbf{u}_x toward \mathbf{u}_y . The range of β in Fig. 6.16 is $-\pi \leq \beta \leq \pi$, although in practice the range will be much smaller.

In Fig. 6.16, E_{1V} and E_{1H} may differ in phase so that E_1 may be a general elliptically polarized plane wave.

Beckmann points out [6, p. 170] that if the scattering plane of Fig. 6.16 is tilted simultaneously about both x and y axes of Fig. 6.14, it is necessary to consider an angle β_2 measured between the horizontal component of the



FIGURE 6.15. Scattering point geometry.



FIGURE 6.16. Decomposition of the incident field.

reflected wave and the intersection of the wave front with the reflecting plane. Angle β_2 is approximately equal to β and in much of Beckmann's work the difference is neglected.

Figure 6.17 shows the ground reflection point at x, y, 0 rather than at the coordinate origin as in Fig. 6.14. The reflecting surface is not shown and neither are the various unit vectors and angles. For those refer to Fig. 6.14.

From Fig. 6.17,

$$\mathbf{r}_1 = x\mathbf{u}_x + y\mathbf{u}_y - h_1\mathbf{u}_z \tag{6.158}$$

$$\mathbf{r}_2 = (d-x)\mathbf{u}_x - y\mathbf{u}_y + h_2\mathbf{u}_z \tag{6.159}$$

and from Figs. 6.14 and 6.17

$$\theta_1 = \frac{\pi}{2} - \tan^{-1} \frac{h_1}{\sqrt{x^2 + y^2}} \tag{6.160}$$

$$\theta_2 = \frac{\pi}{2} - \tan^{-1} \frac{h_2}{\sqrt{(d-x)^2 + y^2}}$$
(6.161)

Also, from Fig. 6.14 and 6.17, θ_3 is the angle between a vector drawn from the point immediately below the transmitter to the scattering point and a vector from the scattering point to a point immediately below the receiver. (Note that one of these may be a target.) Then θ_3 is given by

$$\cos \theta_{3} = \frac{x\mathbf{u}_{x} + y\mathbf{u}_{y}}{\sqrt{x^{2} + y^{2}}} \cdot \frac{(d - x)\mathbf{u}_{x} - y\mathbf{u}_{y}}{\sqrt{(d - x)^{2} + y^{2}}}$$
$$= \frac{x(d - x) - y^{2}}{\sqrt{x^{2} + y^{2}}\sqrt{(d - x)^{2} + y^{2}}}$$
(6.162)

We take θ_3 as positive for $y \leq 0$.



FIGURE 6.17. Reflection point at (x, y, 0).

In order to use the Fresnel coefficients, it is necessary to know the local grazing angle γ of Fig. 6.15. Now k_1 (or equivalently r_1) and n of Fig. 6.15 are coplanar, and γ is measured in that plane. Therefore we have

$$\frac{\mathbf{n} \cdot \mathbf{r}_1}{r_1} = \cos\left(\frac{\pi}{2} + \gamma\right) = -\sin\gamma \tag{6.163}$$

Using (6.157) for **n** gives

$$\sin \gamma = \frac{r_1 r_2 - \mathbf{r}_1 \cdot \mathbf{r}_2}{|r_1 \mathbf{r}_2 - r_2 \mathbf{r}_1|} \tag{6.164}$$

The equations for β and β_2 will be derived here using vectors rather than the spherical trigonometry methods of Beckmann [6]. In Fig. 6.15, β is defined as the angle between the incident electric field (assumed horizontal) and the intersection of the wave front with the reflecting plane. Vector t along this intersection is perpendicular both to **n**, which is perpendicular to the reflecting plane, and to \mathbf{r}_1 . Then

$$\mathbf{t} = \frac{\mathbf{n} \times \mathbf{r}_1 / r_1}{|\mathbf{n} \times \mathbf{r}_1 / r_1|} = \frac{\mathbf{n} \times \mathbf{r}_1}{|\mathbf{n} \times \mathbf{r}_1|}$$
(6.165)

Also from Fig. 6.14, the horizontal component of the incident electric field is directed along \mathbf{u}_{H1} , given by

$$\mathbf{u}_{H1} = \frac{\mathbf{u}_z \times \mathbf{r}_1}{|\mathbf{u}_z \times \mathbf{r}_1|} \tag{6.166}$$

Then we can write

$$\cos \beta = \mathbf{t} \cdot \mathbf{u}_{H1} = \frac{\mathbf{n} \times \mathbf{r}_1}{|\mathbf{n} \times \mathbf{r}_1|} \cdot \frac{\mathbf{u}_z \times \mathbf{r}_1}{|\mathbf{u}_z \times \mathbf{r}_1|}$$
(6.167)

If (6.157) for **n** is used, the first term of (6.167) can be shown to be

$$\frac{\mathbf{n} \times \mathbf{r}_1}{|\mathbf{n} \times \mathbf{r}_1|} = \frac{\mathbf{r}_2 \times \mathbf{r}_1}{|\mathbf{r}_2 \times \mathbf{r}_1|} \tag{6.168}$$

so that

$$\cos \beta = \frac{(\mathbf{r}_2 \times \mathbf{r}_1)}{|\mathbf{r}_2 \times \mathbf{r}_1|} \cdot \frac{(\mathbf{u}_z \times \mathbf{r}_1)}{|\mathbf{u}_z \times \mathbf{r}_1|}$$
(6.169)

Using the identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \tag{6.170}$$

250 POLARIZATION CHANGES BY REFLECTION AND TRANSMISSION

and introducing rectangular coordinates according to Fig. 6.17 yields

$$\cos \beta = \frac{r_1^2 h_2 + h_1 x (d - x) - h_1 y^2 - h_1^2 h_2}{\sqrt{y^2 (h_1 - h_2)^2 + [x(h_2 - h_1) + h_1 d]^2 + y^2 d^2} \sqrt{x^2 + y^2}}$$
(6.171)

We can find β_2 in a similar fashion. It is defined as the angle between the reflected electric field (horizontal component) and the intersection of the wave front with the reflecting surface. (Note that this intersection of the wave front with the reflecting surface is the same for both incident and reflected waves.) Then

$$\cos\beta_2 = \mathbf{t} \cdot \mathbf{u}_{H2} \tag{6.172}$$

Since \mathbf{u}_{H_2} is perpendicular to \mathbf{r}_2 and to \mathbf{u}_2 , we can express it as

$$\mathbf{u}_{H2} = \frac{\mathbf{u}_z \times \mathbf{r}_2}{|\mathbf{u}_z \times \mathbf{r}_2|} \tag{6.173}$$

and

$$\cos \beta_2 = \frac{\mathbf{r}_2 \times \mathbf{r}_1}{|\mathbf{r}_2 \times \mathbf{r}_1|} \cdot \frac{\mathbf{u}_z \times \mathbf{r}_2}{|\mathbf{u}_z \times \mathbf{r}_2|}$$
(6.174)

which becomes in our coordinate system

$$\cos \beta_2 = \frac{h_2 x (d-x) - h_2 y^2 - h_1 h_2^2 + r_2^2 h_1}{\sqrt{y^2 (h_1 - h_2)^2 + [x(h_2 - h_1) + h_1 d]^2 + y^2 d^2} \sqrt{(d-x)^2 + y^2}}$$
(6.175)

Before applying the Fresnel coefficients to the incident field of Fig. 6.16, it is necessary to find the locally vertical and horizontal field components. They may be found from

$$E_{1LH} = E_{1H} \cos \beta - E_{1V} \sin \beta \quad (a)$$

$$E_{1LV} = E_{1H} \sin \beta + E_{1V} \cos \beta \quad (b)$$
(6.176)

Note that

$$|E_{1LH}|^2 + |E_{1LV}|^2 = |E_{1H}|^2 + |E_{1V}|^2 = |E_1|^2$$
(6.177)

as required by conservation of energy.

Applying the Fresnel coefficients $\Gamma_{\nu}(=\Gamma_{\parallel})$ and $\Gamma_{H}(=\Gamma_{\perp})$ gives the locally horizontal and vertical reflected field components

REFLECTIONS FROM ARBITRARILY ORIENTED PLANE 251

$$E_{2LH} = \Gamma_H E_{1LH}$$
 (a) $E_{2LV} = \Gamma_V E_{1LV}$ (b) (6.178)

Note that E_{1LV} is not perpendicular to the reflecting surface; that is, it is not directed along **n**. Instead it is perpendicular to \mathbf{k}_1 and lies in the plane defined by **n** and \mathbf{k}_1 .

Now since β_2 is the angle between the horizontal component of the reflected wave and the intersection of the wave front with the reflecting plane, we can write equations for the locally horizontal and vertical field components of the reflected wave similar to (6.176) for the incident field

$$E_{2LH} = E_{2H} \cos \beta_2 - E_{2V} \sin \beta_2 \quad (a)$$

$$E_{2LV} = E_{2H} \sin \beta_2 + E_{2V} \cos \beta_2 \quad (b)$$
(6.179)

Inverting gives the horizontal and vertical components of the reflected wave \mathbf{E}_2 .

$$E_{2H} = E_{2LH} \cos \beta_2 + E_{2LV} \sin \beta_2 \quad (a)$$

$$E_{2V} = E_{2LV} \cos \beta_2 - E_{2LH} \sin \beta_2 \quad (b)$$
(6.180)

If (6.176) is first substituted into (6.178) and the resulting forms for E_{2LV} and E_{2LH} are inserted into (6.180), the result is

$$E_{2H} = \Gamma_H (E_{1H} \cos \beta - E_{1V} \sin \beta) \cos \beta_2$$

+ $\Gamma_V (E_{1H} \sin \beta + E_{1V} \cos \beta) \sin \beta_2$ (a)
$$E_{2V} = \Gamma_V (E_{1H} \sin \beta + E_{1V} \cos \beta) \cos \beta_2$$

- $\Gamma_H (E_{1H} \cos \beta - E_{1V} \sin \beta) \sin \beta_2$ (b)

All quantities in (6.181) are known: the incident field components E_{1H} and E_{1V} , the scattering angles θ_1 , θ_2 , and θ_3 , which allow β and β_2 to be found, and the Fresnel coefficients Γ_H and Γ_V . The scattered field components may then be determined.

The polarization ratio P_2 may be found as the ratio of E_{2V} to E_{2H} . Dividing (6.181b) by (6.181a) and removing $\cos \beta \cos \beta_2$ from numerator and denominator yields

$$P_{2} = \frac{\Gamma_{V}(E_{1H} \tan \beta + E_{1V}) - \Gamma_{H}(E_{1H} - E_{1V} \tan \beta) \tan \beta_{2}}{\Gamma_{H}(E_{1H} - E_{1V} \tan \beta) + \Gamma_{V}(E_{1H} \tan \beta + E_{1V}) \tan \beta_{2}}$$
(6.182)

Dividing numerator and denominator by E_{1H} and noting that

252 POLARIZATION CHANGES BY REFLECTION AND TRANSMISSION

$$P_1 = \frac{E_{1V}}{E_{1H}}$$
(6.183)

gives

$$P_{2} = \frac{P_{1}(\Gamma_{H} \tan \beta \tan \beta_{2} + \Gamma_{V}) - \Gamma_{H} \tan \beta_{2} + \Gamma_{V} \tan \beta}{\Gamma_{H} + \Gamma_{V} \tan \beta \tan \beta_{2} - P_{1}(\Gamma_{H} \tan \beta - \Gamma_{V} \tan \beta_{2})}$$
(6.184)

At the beginning of this section it was stated that the material was developed to describe waves reflected from the earth when antennas are in the vicinity of the earth's surface. Scattering of the type described here, with the reflected wave completely polarized (see Chapter 7) is referred to as *specular scattering*. In effect, the earth is considered to be made up of large tilted facets, and we have considered each facet to be an infinite smooth plane. In practice multiple specular points will occur for some terrain types. If so, the fields reflected from all the specular points add coherently with phases determined by path lengths. The fields must be weighted by antenna gains and, for targets, by appropriate cross sections.

Other applications of the developments in this section will certainly occur to the reader.

REFERENCES

- 1. J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941.
- G. Sinclair, "The Transmission and Reception of Elliptically Polarized Waves," Proc. IRE, Vol. 38, No. 2, pp. 148–151, February 1950.
- J. R. Copeland, "Radar Target Classification by Polarization Properties," Proc. IRE, Vol. 48, No. 7, pp. 1290–1296, July 1960.
- 4. G. T. Ruck, D. E. Barrick, W. D. Stuart, and C. K. Krichbaum, *Radar Cross Section Handbook*, Plenum Press, New York, 1970.
- P. Beckmann, "Optimum Polarization for Polarization Discrimination," Proc. IEEE, Vol. 56, No. 10, pp. 1755–1756, October 1968.
- P. Beckmann and A. Spizzichino, The Scattering of Electromagnetic Waves from Rough Surfaces, Pergamon Press, New York, 1963.
- 7. P. Beckmann, *The Depolarization of Electromagnetic Waves*, The Golem Press, Boulder, 1968.



PARTIAL POLARIZATION

7.1. INTRODUCTION

To this point we have considered only monochromatic waves. Such waves are completely polarized, with the end point of the field vector tracing an ellipse of constant eccentricity and tilt angle. A wave arising from some physical source is never completely monochromatic. The amplitude and phase change, with an irregular variation superimposed on a regular variation, and the tip of the field vector traces an ellipse whose shape and orientation change with time. Such a wave is said to be partially polarized. In the limit, as the amplitude and phase of the wave become more random, the wave is randomly polarized.

We will consider a quasi-monochromatic field variation, with a wave that has a finite band width that is small compared to the mean wave frequency. Such a wave obviously is partially polarized. *Note*: In this chapter timevarying fields are not represented by script but by italic letters. Confusion is unlikely, since partially polarized fields cannot be represented by timeinvariant quantities.

7.2. ANALYTIC SIGNALS

We can represent a field component of a partially polarized wave as

$$E'(t) = a(t)\cos\left[\omega t + \phi(t)\right] \tag{7.1}$$

where a(t) and $\phi(t)$ are the real amplitude and phase of the wave component. Now (7.1) is valid whether we apply the quasi-monochromatic constraint or not, but if the wave is quasi-monochromatic, a(t) and $\phi(t)$ vary slowly enough so that the wave approximates a cosine.

PARTIAL POLARIZATION

Rather than using (7.1) for our wave component, we will use instead its analytic signal representation

$$E(t) = a(t)e^{j[\omega t + \phi(t)]}$$
(7.2)

The analytic signal as developed by Gabor [1] is a complex function associated with the real field function E'(t). It is a generalization of the exponential function $e^{j\omega t}$ normally used for convenience to represent the real function $\cos \omega t$ in many areas of electrical engineering and physics. The reader is referred to Gabor, Born, and Wolf [2] or other references [3, 4] for a more extensive treatment of the analytic signal. Here we will assume that the functions we are concerned with possess Fourier transforms and note that we can form the analytic signal associated with any real function E'(t) by using the formulation

$$E(t) = E'(t) + jE'(t)$$
(7.3)

where the imaginary part, E'(t), of the analytic signal is the Hilbert transform of E'(t); thus

$$E'(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{E'(t')}{t'-t} dt'$$
(7.4)

The bar across the integral symbol in (7.4) signifies the Cauchy principal value of the integral, that is

$$\int_{-\infty}^{\infty} \frac{E'(t')}{t'-t} dt' = \lim_{T \to t^{-}} \int_{-\infty}^{T} \frac{E'(t')}{t'-t} dt' + \lim_{T \to t^{+}} \int_{T}^{\infty} \frac{E'(t')}{t'-t} dt'$$
(7.5)

7.3. COHERENCY MATRIX OF A QUASI-MONOCHROMATIC PLANE WAVE

Consider a quasi-monochromatic plane wave traveling in the z direction with components

$$E_{x}(t) = a_{x}(t)e^{j[\omega t + \phi_{x}(t)]}$$
(a)

$$E_{y}(t) = a_{y}(t)e^{j[\omega t + \phi_{y}(t)]}$$
(b)
(7.6)

where the phase term -kz is omitted from both components. The mean radian frequency is ω , and the a(t) and $\phi(t)$ are slowly varying time functions.

In analogy to monochromatic waves, we could define a complex vector

$$\mathbf{a} = \mathbf{u}_{x} a_{x}(t) e^{j\phi_{x}(t)} + \mathbf{u}_{y} a_{y}(t) e^{j\phi_{y}(t)}$$
(7.7)

to represent the wave, but in contrast to the monochromatic case, a varies with time.

We may write the field as a single component at angle θ to the x axis by

$$E(t, \theta) = E_x(t) \cos \theta + E_y(t) \sin \theta$$
(7.8)

It is convenient to take the time average

$$I = \langle E(t, \theta) E^*(t, \theta) \rangle \tag{7.9}$$

which is defined by

$$I = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E_T E_T^* dt$$
 (7.10)

where E_T is the truncated function

$$E_T = E(t, \phi) \qquad |t| \le T$$

= 0
$$|t| > T \qquad (7.11)$$

Substitution of the wave components into the desired time average gives

$$I = \langle E_x(t) E_x^*(t) \rangle \cos^2 \theta + \langle E_y(t) E_y^*(t) \rangle \sin^2 \theta + [\langle E_x(t) E_y^*(t) \rangle + \langle E_x^*(t) E_y(t) \rangle] \sin \theta \cos \theta$$
(7.12)

The presence of the four time averages in (7.12) makes it desirable to define a matrix, called the *coherency matrix* [2] of the wave. We do so by

$$[J] = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix} = \frac{1}{2Z_0} \begin{bmatrix} \langle E_x(t)E_x^*(t) \rangle & \langle E_x(t)E_y^*(t) \rangle \\ \langle E_x^*(t)E_y(t) \rangle & \langle E_y(t)E_y^*(t) \rangle \end{bmatrix}$$
(7.13)

which may also be written, using (7.6), as

$$[J] = \frac{1}{2Z_0} \begin{bmatrix} \langle a_x^2 \rangle & \langle a_x a_y e^{j(\phi_x - \phi_y)} \rangle \\ \langle a_x a_y e^{j(\phi_y - \phi_x)} \rangle & \langle a_y^2 \rangle \end{bmatrix}$$
(7.14)

If the amplitude and phase functions of (7.6) vary so slowly that the time derivatives in the Maxwell equations, for example, in

$$\nabla \times \mathbf{H} = \varepsilon \; \frac{\partial \mathbf{E}}{\partial t} \tag{7.15}$$

can be replaced by $j\omega$, as is customary in treating monochromatic waves, the

time-average Poynting vector of the field we are considering is proportional to the trace of the coherency matrix

$$S = \left(\left\langle a_x^2 \right\rangle + \left\langle a_y^2 \right\rangle \right) = \operatorname{Tr} \left[J \right]$$
(7.16)

The mixed terms of the coherency matrix may be normalized by setting

$$\mu_{xy} = |\mu_{xy}|e^{j\beta_{xy}} = \frac{J_{xy}}{\sqrt{J_{xx}}\sqrt{J_{yy}}}$$
(7.17)

It may be shown by the Schwarz inequality that

$$|\mu_{xy}| \le 1 \tag{7.18}$$

The term μ_{xy} is a measure of the degree of correlation between the x and y field components, similar to the degree of coherence relating values of the same wave component at different points as used in scalar diffraction theory.

From (7.14) we note that

$$J_{xy} = J_{yx}^*$$
(7.19)

Then the matrix determinant may be written as

$$||J|| = J_{xx}J_{yy} - |J_{xy}|^2$$
(7.20)

or, using (7.17), as

$$||J|| = J_{xx}J_{yy}(1 - |\mu_{xy}|^2)$$
(7.21)

Since J_{xx} and J_{yy} are positive real and $|\mu_{xy}| \leq 1$, we see that

$$|J|| \ge 0 \tag{7.22}$$

Unpolarized Waves

Waves that are unpolarized have the characteristic that the time average, given by I in (7.12), is independent of angle θ . In addition, if a fixed phase retardation is introduced into one of the field components, I is unchanged. This requires that

$$J_{xx} = J_{yy}$$
 (a) $J_{xy} = J_{yx} = 0$ (b) (7.23)

We see then that the components E_x and E_y are completely uncorrelated and the coherency matrix reduces to

COHERENCY MATRIX OF A QUASI-MONOCHROMATIC PLANE WAVE 257

$$\begin{bmatrix} J \end{bmatrix} = \begin{bmatrix} J_{xx} & 0\\ 0 & J_{yy} \end{bmatrix} = \frac{S}{2} \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$$
(7.24)

where S is the power density of the wave.

Complete Polarization

Let us first think about monochromatic waves. For these the terms a and ϕ of (7.14) are time independent, and the coherency matrix becomes

$$[J] = \frac{1}{2Z_0} \begin{bmatrix} a_x^2 & a_x a_y e^{-j\phi} \\ a_x a_y e^{j\phi} & a_y^2 \end{bmatrix}$$
(7.25)

where

$$\phi = \phi_y - \phi_x \tag{7.26}$$

From the coherency matrix elements we can get

$$\mu_{xy} = \frac{J_{xy}}{\sqrt{J_{xx}}\sqrt{J_{yy}}} = \frac{a_x a_y e^{-j\phi}}{a_x a_y} = e^{-j\phi}$$
(7.27)

and we have complete coherence, since

 $|\mu_{xy}| = 1$

The phase of μ_{xy} is the phase difference between the wave components.

We may also have complete polarization for nonmonochromatic waves. If a_x , a_y , ϕ_x , ϕ_y depend on time in such a way that the ratio of amplitudes and the difference in phase are independent of time, that is, if

$$\frac{a_{y}(t)}{a_{x}(t)} = C_{1} \qquad \phi = \phi_{y}(t) - \phi_{x}(t) = C_{2}$$
(7.28)

with C_1 and C_2 constants, then the coherency matrix, (7.14), becomes

$$[J] = \frac{1}{2Z_0} \begin{bmatrix} \langle a_x^2 \rangle & C_1 \langle a_x^2 \rangle e^{-jC_2} \\ C_1 \langle a_x^2 \rangle e^{jC_2} & C_1^2 \langle a_x^2 \rangle \end{bmatrix}$$
(7.29)

from which we obtain

$$\mu_{xy} = e^{-jC_2} \tag{7.30}$$

and the wave is completely polarized.

The coherency matrix for this case is the same as for the monochromatic wave with components equal to

$$E_x = \sqrt{\langle a_x^2 \rangle} e^{j(\omega t + \phi_x)} \tag{7.31}$$

$$E_{y} = C_{1} \sqrt{\langle a_{x}^{2} \rangle} e^{j(\omega t + \phi_{x} + C_{2})}$$
(7.32)

From (7.21) and (7.30) it is clear that for a completely polarized wave

||J|| = 0

Linear Polarization

For linear polarization the wave must, of course, satisfy the requirements for complete polarization, and in addition

$$\phi = 0, \pm \pi, \pm 2\pi, \dots \tag{7.33}$$

Then the coherency matrices for monochromatic and completely polarized polychromatic waves are, respectively,

$$[J] = \frac{1}{2Z_0} \begin{bmatrix} a_x^2 & (-1)^m a_x a_y \\ (-1)^m a_x a_y & a_y^2 \end{bmatrix} \qquad m = 0, 1, 2, \dots$$
(7.34)

and

$$[J] = \frac{1}{2Z_0} \begin{bmatrix} \langle a_x^2 \rangle & (-1)^m C_1 \langle a_x^2 \rangle \\ (-1)^m C_1 \langle a_x^2 \rangle & C_1^2 \langle a_x^2 \rangle \end{bmatrix}$$
(7.35)

More particularly, the matrices

$$[J] = S\begin{bmatrix} 1 & 0\\ 0 & 0 \end{bmatrix}, S\begin{bmatrix} 0 & 0\\ 0 & 1 \end{bmatrix}, \frac{S}{2}\begin{bmatrix} 1 & 1\\ 1 & 1 \end{bmatrix}, \frac{S}{2}\begin{bmatrix} 1 & -1\\ -1 & 1 \end{bmatrix}$$
(7.36)

represent linear polarizations that are, respectively, x directed, y directed, 45° from the x axis, and 135° from the x axis.

Circular Polarization

We saw previously that for circular polarization the component amplitudes are equal, and

$$\phi = \pm \frac{1}{2}\pi \tag{7.37}$$

for left and right circular polarization, respectively. Then the coherency

matrix becomes

$$[J] = \frac{S}{2} \begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}, \frac{S}{2} \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}$$
(7.38)

for left and right circular polarization, respectively.

7.4. DEGREE OF POLARIZATION

A plane wave may be considered as the sum of N independent plane waves traveling in the same direction. In particular, we will consider a quasimonochromatic wave to be the sum of a completely polarized wave and a completely unpolarized wave. We may show that this representation is unique by showing that any coherency matrix can be uniquely expressed in the form

$$[J] = [J^{(1)}] + [J^{(2)}]$$
(7.39)

where

$$[J^{(1)}] = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \qquad [J^{(2)}] = \begin{bmatrix} B & D \\ D^* & C \end{bmatrix}$$
(7.40)

with

$$A \ge 0 \qquad B \ge 0 \qquad C \ge 0 \qquad BC - DD^* = 0 \tag{7.41}$$

If we compare (7.40) to the special case (7.24), we see that $[J^{(1)}]$ is the coherency matrix for a completely unpolarized wave. If we use ||J|| = 0 as the criterion for a completely polarized wave, then $[J^{(2)}]$ is the coherency matrix for a completely polarized wave.

We must next show that the decomposition into completely polarized and completely unpolarized waves is unique. This we will do by obtaining the elements of $[J^{(1)}]$ and $[J^{(2)}]$ from the known elements of [J]. From (7.39) and (7.40) we may write

$$A + B = J_{xx}$$
 (a) $D^* = J_{yx}$ (c)
 $D = J_{xy}$ (b) $A + C = J_{yy}$ (d) (7.42)

Substituting (7.42) into the last equation of (7.41) gives

$$(J_{xx} - A)(J_{yy} - A) - J_{xy}J_{yx} = 0$$
(7.43)

which is a quadratic in A with solution

PARTIAL POLARIZATION

$$A = \frac{1}{2}(J_{xx} + J_{yy}) \pm \frac{1}{2}[(J_{xx} + J_{yy})^2 - 4||J||]^{1/2}$$
(7.44)

Substituting (7.44) in (7.42a) gives

$$B = \frac{1}{2} (J_{xx} - J_{yy}) \mp \frac{1}{2} [(J_{xx} + J_{yy})^2 - 4 ||J||]^{1/2}$$

= $\frac{1}{2} (J_{xx} - J_{yy}) \mp \frac{1}{2} [(J_{xx} - J_{yy})^2 + 4 J_{xy} J_{xy}^*]^{1/2}$ (7.45)

From the second form of (7.45) we see that the negative sign for the last term is not allowed since it would make *B* negative, contrary to our hypothesis. Then the *A*, *B*, *C*, *D* values of (7.42) are found uniquely from

$$A = \frac{1}{2}(J_{xx} + J_{yy}) - \frac{1}{2}[(J_{xx} + J_{yy})^{2} - 4||J||]^{1/2} \quad (a)$$

$$B = \frac{1}{2}(J_{xx} - J_{yy}) + \frac{1}{2}[(J_{xx} + J_{yy})^{2} - 4||J||]^{1/2} \quad (b)$$

$$C = \frac{1}{2}(J_{yy} - J_{xx}) + \frac{1}{2}[(J_{xx} + J_{yy})^{2} - 4||J||]^{1/2} \quad (c)$$
(7.46)

$$D = J_{xy} \tag{d}$$

$$D^* = J_{yx} \tag{e}$$

The Poynting vector magnitude of the total wave is

$$S_t = \text{Tr}[J] = J_{xx} + J_{yy}$$
 (7.47)

and that of the polarized part of the wave is

$$S_p = \operatorname{Tr} \left[J^{(2)} \right] = B + C = \left[\left(J_{xx} + J_{yy} \right)^2 - 4 \|J\| \right]^{1/2}$$
(7.48)

Quite reasonably, the ratio of the power densities of the polarized part and the total wave is called the *degree of polarization* of the wave. It is given by

$$R = \frac{S_p}{S_t} = \left[1 - \frac{4\|J\|}{(J_{xx} + J_{yy})^2}\right]^{1/2}$$
(7.49)

Now

$$||J|| \leq J_{xx}J_{yy} \leq \frac{1}{4}(J_{xx} + J_{yy})^2$$

and therefore

$$0 \le R \le 1 \tag{7.50}$$

Let us consider the two extreme values of R. For R = 1, (7.49) requires that

$$||J|| = 0$$

which is the condition for complete polarization. Then $|\mu_{xy}| = 1$, and the x and y wave components are mutually coherent. For R = 0, (7.49) requires that

$$(J_{xx} - J_{yy})^2 + 4|J_{xy}|^2 = 0$$

which can be satisfied only by

$$J_{xx} = J_{yy} \qquad J_{xy} = J_{yx} = 0$$

It follows that $|\mu_{xy}| = 0$ and E_x and E_y are mutually incoherent.

As we have just seen, R = 0 requires that E_x and E_y be mutually incoherent. The converse is not true. For mutual incoherence $J_{xy} = J_{yx} = 0$ and $|\mu_{xy}| = 0$. Then

$$R = \left[1 - \frac{4J_{xx}J_{yy}}{(J_{xx} + J_{yy})^2}\right]^{1/2} = \frac{|J_{xx} - J_{yy}|}{J_{xx} + J_{yy}}$$

We see that $|\mu_{xy}| = 0$ is not sufficient to give an unpolarized wave. To make it completely unpolarized, we must also have

$$J_{xx} = J_{yy}$$

We can separate the matrix [J] of (7.24), representing the unpolarized part of a wave, even further, as

$$[J] = \frac{S}{2} \begin{bmatrix} 1 & 0\\ 0 & 0 \end{bmatrix} + \frac{S}{2} \begin{bmatrix} 0 & 0\\ 0 & 1 \end{bmatrix}$$
(7.51)

which indicates that an unpolarized wave can be regarded as being composed of two independent linearly polarized waves orthogonal to each other, each of equal power density.

Just as readily, we could have written

$$[J] = \frac{S}{4} \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix} + \frac{S}{4} \begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}$$
(7.52)

showing that with equal validity we could consider an unpolarized wave to be made up of two independent circular waves of opposite rotation sense.

PARTIAL POLARIZATION

7.5. STOKES PARAMETERS OF PARTIALLY POLARIZED WAVES

We previously defined the Stokes parameters of a monochromatic wave by the equations †

$$S_{0} = |E_{x}|^{2} + |E_{y}|^{2}$$
(a)

$$S_{1} = |E_{x}|^{2} - |E_{y}|^{2}$$
(b)

$$S_{2} = 2|E_{x}| |E_{y}| \cos \phi$$
(c)

$$S_{3} = 2|E_{x}| |E_{y}| \sin \phi$$
(d)

For quasi-monochromatic waves a more general definition, which reduces to (2.184) for time-independent amplitude and phase of the wave components, is

$$S_{0} = \langle a_{x}^{2} \rangle + \langle a_{y}^{2} \rangle \qquad (a)$$

$$S_{1} = \langle a_{x}^{2} \rangle - \langle a_{y}^{2} \rangle \qquad (b)$$

$$S_{2} = 2 \langle a_{x}a_{y} \cos \phi \rangle \qquad (c)$$

$$S_{3} = 2 \langle a_{x}a_{y} \sin \phi \rangle \qquad (d)$$

where

$$\phi = \phi_y - \phi_x \tag{7.54}$$

If we compare these parameters to the elements of the coherency matrix (7.14), we see that

$$S_{0} = 2Z_{0}(J_{xx} + J_{yy}) \quad (a)$$

$$S_{1} = 2Z_{0}(J_{xx} - J_{yy}) \quad (b)$$

$$S_{2} = 2Z_{0}(J_{xy} + J_{yx}) \quad (c)$$

$$S_{3} = 2Z_{0}j(J_{xy} - J_{yx}) \quad (d)$$
(7.55)

^{\dagger} The author regrets the conflict in notation where S is used for power density and the Stokes parameters. The Stokes parameters will have a numerical subscript and the power density will not.

or, solving for the coherency matrix elements,

$$J_{xx} = \frac{1}{4Z_0} (S_0 + S_1) \quad (a)$$

$$J_{yy} = \frac{1}{4Z_0} (S_0 - S_1) \quad (b)$$

$$J_{xy} = \frac{1}{4Z_0} (S_2 - jS_3) \quad (c)$$

$$J_{yx} = \frac{1}{4Z_0} (S_2 + jS_3) \quad (d)$$

In terms of the Stokes parameters, the statement

$$\|J\| \ge 0 \tag{7.22}$$

becomes

$$S_0^2 \ge S_1^2 + S_2^2 + S_3^2 \tag{7.57}$$

For a completely polarized wave the requirement

||J|| = 0

gives immediately

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

in accordance with (2.185).

Just as we separated the coherency matrix of a quasi-monochromatic wave into the sum of coherency matrices for a completely polarized wave and a completely unpolarized wave, we can decompose a wave in the same manner in terms of its Stokes parameters. We write for the general wave

$$S_0 = S_0^{(1)} + S_0^{(2)} \quad (a) \qquad S_2 = S_2^{(1)} + S_2^{(2)} \quad (c)$$

$$S_1 = S_1^{(1)} + S_1^{(2)} \quad (b) \qquad S_3 = S_3^{(1)} + S_3^{(2)} \quad (d)$$
(7.58)

where superscript (1) refers to a completely unpolarized wave and (2) to the polarized wave.

Unpolarized Waves

For a completely unpolarized wave we found earlier that

$$J_{xx} = J_{yy}$$
 (a) $J_{xy} = J_{yx} = 0$ (b) (7.23)

Then from (7.55) we have

$$S_1^{(1)} = S_2^{(1)} = S_3^{(1)} = 0 (7.59)$$

Complete Polarization

For this case we have, rewriting (2.185),

$$(S_0^{(2)})^2 = (S_1^{(2)})^2 + (S_2^{(2)})^2 + (S_3^{(2)})^2$$
(7.60)

Degree of Polarization

In light of (7.59) the general Stokes parameters of (7.58) simplify to

$$S_{0} = S_{0}^{(1)} + S_{0}^{(2)} \quad (a) \qquad S_{2} = S_{2}^{(2)} \quad (c)$$

$$S_{1} = S_{1}^{(2)} \quad (b) \qquad S_{3} = S_{3}^{(2)} \quad (d)$$
(7.61)

Equations (7.60) and (7.61) can be combined to give

$$S_0^{(1)} = S_0 - \sqrt{S_1^2 + S_2^2 + S_3^2}$$
(7.62)

and

$$S_0^{(2)} = \sqrt{S_1^2 + S_2^2 + S_3^2} \tag{7.63}$$

The degree of polarization was defined earlier as the ratio of power densities of the polarized part and the total wave. But $S_0^{(2)}$ measures the density of the polarized part and S_0 the density of the total wave. Then the degree of polarization of the wave in terms of its Stokes parameters is

$$R = \frac{S_0^{(2)}}{S_0} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}$$
(7.64)

7.6. POLARIZATION RATIO OF PARTIALLY POLARIZED WAVES

We can obtain the polarization ratio and the polarization ellipse characteristics of the polarized part of a wave just as we did for the completely polarized wave. From (2.192a) and the relation between p and P, we can write the polarization ratio in terms of the Stokes parameters for the polarized part of the wave as

$$P = \frac{S_2^{(2)} + jS_3^{(2)}}{S_0^{(2)} + S_1^{(2)}}$$
(7.65)

and if we use (7.63) and (7.61) to find P in terms of the parameters of the total wave, this becomes

$$P = \frac{S_2 + jS_3}{S_1 + \sqrt{S_1^2 + S_2^2 + S_3^2}}$$
(7.66)

In the same way the circular polarization ratio q, from (2.192b), is

$$q = \frac{S_1^{(2)} - jS_2^{(2)}}{S_0^{(2)} - S_3^{(2)}} = \frac{S_1 - jS_2}{\sqrt{S_1^2 + S_2^2 + S_3^2} - S_3}$$
(7.67)

In terms of the coherency matrix elements for the partially polarized wave, the polarization ratio becomes, substituting in (7.66) from (7.55) and making use of (7.64),

$$P = \frac{2J_{yx}}{(R+1)J_{xx} + (R-1)J_{yy}}$$
(7.68)

where R is the degree of polarization of the wave.

For complete polarization we have

$$R = 1 \qquad J_{xx} = \frac{1}{2Z_0} E_x E_x^* \qquad J_{yx} = \frac{1}{2Z_0} E_x^* E_y$$

and (7.68) reduces to

$$P = \frac{E_y}{E_x}$$

7.7. RECEPTION OF PARTIALLY POLARIZED WAVES

A wave with field intensity E falling on a receiving antenna with effective length h produces an open-circuit voltage at the antenna terminals,

$$V = \mathbf{E} \cdot \mathbf{h} \tag{3.15}$$

This holds whether E is coherent or not, but we are concerned here with partially polarized waves and will accordingly consider the power supplied to a matched load on the antenna to be [5]

$$W = \frac{\langle VV^* \rangle}{8R_a} \tag{7.69}$$

where R_a is the antenna resistance (radiation resistance plus loss resistance).

PARTIAL POLARIZATION

Using (A.5) and (A.8), R_a may be put into the form

$$R_a = \frac{Z_0 \mathbf{h} \cdot \mathbf{h}^*}{4A_e} \tag{7.70}$$

and if we use this and (3.15) in (7.69), we get

$$W = \frac{A_e}{2Z_0 \mathbf{h} \cdot \mathbf{h}^*} \left\langle (\mathbf{E} \cdot \mathbf{h}) (\mathbf{E} \cdot \mathbf{h})^* \right\rangle$$
(7.71)

If we note that time averaging is unnecessary for the receiving antenna, the received power becomes

$$W = \frac{A_e}{2Z_0 \mathbf{h} \cdot \mathbf{h}^*} \left(|h_x|^2 \langle E_x E_x^* \rangle + h_x h_y^* \langle E_x E_y^* \rangle + h_x^* h_y \langle E_x^* E_y \rangle + |h_y|^2 \langle E_y E_y^* \rangle \right)$$
(7.72)

which becomes, using the elements of the coherency matrix of the incident wave,

$$W = \frac{A_e}{\mathbf{h} \cdot \mathbf{h}^*} \left(|h_x|^2 J_{xx} + h_x h_y^* J_{xy} + h_x^* h_y J_{yx} + |h_y|^2 J_{yy} \right)$$
(7.73)

We saw earlier that a partially polarized plane wave may be considered the sum of a completely polarized wave and a completely unpolarized wave. The coherency matrix elements of the component waves are given by (7.40). Substituting into the equation for received power then gives

$$W = \frac{A_{e}}{\mathbf{h} \cdot \mathbf{h}^{*}} \left[|h_{x}|^{2} (A+B) + h_{x} h_{y}^{*}(D) + h_{x}^{*} h_{y}(D^{*}) + |h_{y}|^{2} (A+C) \right]$$
(7.74)

This form may be separated to give

$$W = W' + W'' = A_e A + \frac{A_e}{\mathbf{h} \cdot \mathbf{h}^*} \left(|h_x|^2 B + h_x h_y^* D + h_x^* h_y D^* + |h_y|^2 C \right)$$
(7.75)

where the first term,

$$W' = A_c A \tag{7.76}$$

which represents the power received from the unpolarized portion of the wave, is independent of the polarization characteristics of the receiving antenna. It is informative to express this power in terms of the degree of polarization of the wave. From (7.46a) and (7.49) we get

$$A = \frac{1}{2}(J_{xx} + J_{yy})(1 - R)$$
(7.77)

RECEPTION OF PARTIALLY POLARIZED WAVES

or, in terms of the power density of the wave,

$$W' = A_r \frac{1}{2} S_r (1 - R) \tag{7.78}$$

Note that if the wave is unpolarized (R = 0), the maximum power that can be extracted from the wave is one-half the power that could be utilized from a completely polarized wave polarization matched to the receiving antenna.

We need not be concerned further with W' since nothing we can do with the polarization of the receiving antenna will either increase or decrease it. We therefore turn our attention to the power received from the completely polarized part of the wave and attempt to maximize it. From (7.41) and (7.46) we note that B and C in

$$W'' = \frac{A_e}{\mathbf{h} \cdot \mathbf{h}^*} \left(|h_x|^2 B + h_x h_y^* D + h_x^* h_y D^* + |h_y|^2 C \right)$$
(7.79)

are positive real. We therefore first maximize the sum of the two middle terms of (7.79) by setting

$$h_x = |h_x|e^{j\beta_x}$$
 (a) $h_y = |h_y|e^{j\beta_y}$ (b) $D = |D|e^{j\delta}$ (c) (7.80)

It is at once obvious that the sum

$$h_{x}h_{y}^{*}D + h_{x}^{*}h_{y}D^{*}$$

is maximum if we choose

$$\beta_{y} - \beta_{x} = \delta \tag{7.81}$$

Then W" becomes

$$W''_{m} = \frac{A_{e}}{\mathbf{h} \cdot \mathbf{h}^{*}} \left(|h_{x}|^{2}B + 2|h_{x}| |h_{y}| |D| + |h_{y}|^{2}C \right)$$
(7.82)

We can maximize W''_m by varying $|h_x|$ and $|h_y|$ while holding $\mathbf{h} \cdot \mathbf{h}^*$ constant. This is an appropriate constraint and was discussed in Section 3.4. Differentiating W''_m with respect to $|h_x|$ given by

$$|h_x| = (\mathbf{h} \cdot \mathbf{h}^* - |h_y|^2)^{1/2}$$
(7.83)

and setting the derivative to zero gives

$$\frac{|h_y|^2 - |h_x|^2}{|h_x| |h_y|} = \frac{C - B}{|D|}$$
(7.84)

with solution

PARTIAL POLARIZATION

$$\frac{|h_y|}{|h_x|} = \frac{C}{|D|}$$
 (a) $\frac{|h_x|}{|h_y|} = \frac{B}{|D|}$ (b) (7.85)

Note that $|D| \neq 0$ unless the wave is completely unpolarized. Obviously one of these forms is the inverse of the other, and this leads to the requirement

 $BC - |D|^2 = 0$

which agrees with (7.41). Since $|D| \neq 0$, then $B \neq 0$ and $C \neq 0$.

Combining (7.85) and (7.81) leads to the relations

$$\frac{h_y}{h_x} = \frac{C}{D^*}$$
 (a) $\frac{h_x}{h_y} = \frac{B}{D}$ (b) (7.86)

If these values are substituted into the equation for W''_m , the power received from the polarized part of the wave becomes

$$W''_{mm} = A_c(B+C)$$
(7.87)

which is obviously maximum power rather than minimum.

From the equations for B and C, (7.46); the power densities (7.47) and (7.48); and the definition for degree of polarization, R; the maximum power that may be received from the polarized part of the wave is

$$W_{mm}'' = A_e S_p = A_e S_t R \tag{7.88}$$

where S_p is the power density of the polarized part of the wave and S_i is that of the total wave.

It may be shown that if the wave is completely polarized, the choices made for the receiving antenna effective lengths in (7.86) are the same as those made in (3.36). This is left as an exercise.

It was noted earlier that for the unpolarized part of the wave the maximum power that can be received is one-half the power that could be received from a polarized wave of the same power density using a matched polarization receiver. The received power is independent of the receiver polarization. Then in order to maximize total received power, we need only to match our receiver to the completely polarized portion of the wave using (7.86). The total received power is then the sum

$$W_m = W' + W''_{mm} = \frac{1}{2}A_c S_t (1+R)$$
(7.89)

REFERENCES

 D. Gabor, "Theory of Communication," Journal of the Institution of Electrical Engineers, Vol. 93, Part III, pp. 429–457, 1946.

PROBLEMS

- 2. M. Born and E. Wolf, Principles of Optics, Pergamon Press, New York, 1965.
- M. J. Beran and G. B. Parrent, Jr., Theory of Partial Coherence, Prentice-Hall, Englewood Cliffs, NJ, 1964.
- 4. C. H. Papas, Theory of Electromagnetic Wave Propagation, McGraw-Hill, New York, 1965.
- H. C. Ko, "The Interaction of Radio Antennas with Statistical Radiation," Notes for Short Course in Antennas, Ohio State University, 1965.

PROBLEMS

- 7.1. Consider a monochromatic wave so that, for example, in (7.79), $B = J_{xx}$, and the coherency matrix elements can be simplified. Show that the choices made for the receiving antenna in (7.86) reduce to the choices made for the monochromatic wave in (3.36).
- 7.2. Derive Eq. (7.87).
- **7.3.** Obtain the effective length components analogous to those in (7.86) if it is desired to minimize the power received from the polarized part of the wave.



POLARIZATION MEASUREMENTS

8.1. INTRODUCTION

There are several standard techniques for measuring the polarization characteristics of an antenna. It is common to use the antenna under test to transmit and to use certain standard antennas, or one antenna whose orientation is varied, as receivers. This is the point of view taken in this chapter. It is, of course, equally correct to measure the response of the antenna under test while transmitting toward it waves of known polarization. All of the techniques described here work better for some antennas than for others, and there is no "best" method for measuring polarization.

8.2. THE LINEAR COMPONENT METHOD

Since the polarization of a wave, and that of the antenna that transmits it, is completely defined by the polarization ratio

$$P = \frac{E_y}{E_x} = \frac{|E_y|}{|E_x|} e^{j\phi}$$
(8.1)

measurement of the amplitudes of the x- and y-directed fields and the phase difference between them using two linearly polarized receiving antennas is clearly an effective method for determining the polarization. It is, of course, power to a receiver load that is measured and not the field strength components, so it is essential that the two linear antennas have equal gains and impedances. A receiver can be switched from one antenna to the other so that no problem will arise from unequal receiver gains in the two channels. The phase difference can be measured by a slotted line method [1] or by use of a calibrated phase shifter.

At lower frequencies dipole antennas are satisfactory linearly polarized standards, while at higher frequencies standard gain horns can be used. Typically, on axis, their axial ratios are on the order of 40 dB, which is satisfactory for most measurements [2]. Gains of the standard antennas can be measured at the desired frequencies prior to their use in the polarization measurement system. Placement of the two receiving standards is critical when measuring the phase difference between the components, and for this reason the method is not attractive at high frequencies.

8.3. THE CIRCULAR COMPONENT METHOD

The polarization of a wave is also specified completely by its circular polarization ratio

$$q = \frac{E_L}{E_R} = \frac{|E_L|}{|E_R|} e^{-j\theta}$$
(8.2)

It follows that with two antennas having equal gains and impedances, one left circularly polarized and the other right circular, we can use the procedures outlined for the linear component method to measure an antenna's polarization. Kraus [1, p. 483] recommends the method and suggests the use of helices for the standard antennas, but Rubin [3] points out the difficulty of constructing identical (except for rotation sense) antennas, particularly when it is necessary to cover a wide frequency range. Another problem exists, also. Since the axial ratio of an *n*-turn helix is not 1 but is given by [1, p. 206]

$$AR = \frac{2n+1}{2n} \tag{8.3}$$

if we require that the helix polarization approach circular as closely as the standard gain horn approaches linear polarization (AR \rightarrow 40 dB), it is readily seen that the helix must have a very large number of turns. This may make the method impractical for very precise measurements.

An alternative to the measurement of θ is the use of a linearly polarized receiving antenna to measure the tilt angle of the polarization ellipse [3]. This eliminates the problem of antenna placement when measuring phase difference of the wave components. The tilt angle, taken together with the rotation sense, obtainable from |q|, and the axial ratio, which may be found from

$$AR = \left| \frac{1 + |q|}{1 - |q|} \right|$$
(2.107)

define the polarization completely.
POLARIZATION MEASUREMENTS

8.4. THE POLARIZATION PATTERN

Equation (3.82) for the polarization match factor of two antennas with the same rotation sense and (3.86) for antennas of opposite sense both reduce to

$$\rho = \frac{AR_1^2 \cos^2(\tau_1 + \tau_2) + \sin^2(\tau_1 + \tau_2)}{AR_1^2 + 1}$$
(8.4)

if antenna 2 is linearly polarized, with $AR_2 \rightarrow \infty$. This equation leads to a widely used method for obtaining the polarization ellipse of an antenna experimentally.

Antenna 2 is rotated around a line drawn between the two antennas, say the z axis of Fig. 3.3. Further, antenna 2 is so oriented that it cannot receive any z-directed wave components as it is rotated. For a dipole the rotation axis is perpendicular to the dipole.

At $\tau_2 = -\tau_1$, which corresponds to coincidence between the major axes of the ellipses for the two antennas,

$$\rho = \frac{AR_1^2}{AR_1^2 + 1}$$
(8.5)

which is a maximum. At $\tau_2 = -\tau_1 \pm \frac{1}{2}\pi$, which corresponds to the major axis of the linearly polarized antenna coinciding with the minor axis of the antenna being tested.

$$\rho = \frac{1}{AR_1^2 + 1}$$
(8.6)

which is a minimum.

The open-circuit voltage is proportional to the square root of ρ , so the ratio of maximum to minimum open-circuit voltage, in magnitude, is

$$\frac{|V_{\text{max}}|}{|V_{\text{min}}|} = AR_1 \tag{8.7}$$

We thus have the axial ratio of the antenna undergoing test, and of course we have its tilt angle from the known rotation angle of the linear antenna when maximum power is received (or better yet, from the angle for minimum power plus 90°, since the minimum power angle is more sharply defined than the maximum power angle).

A plot of the square root of ρ from (8.4) is called the *polarization pattern* of the antenna whose polarization is being measured. Figure 8.1 shows the polar form of the pattern for (a) an antenna with an axial ratio of 2 and (b) a linearly polarized antenna. Since the ratio of maximum to minimum values of the polarization pattern is the axial ratio of the antenna under test, the



FIGURE 8.1. Polarization patterns for measuring axial ratio and tilt angle of the polarization ellipse: (a) AR = 2, with inscribed polarization ellipse; (b) $AR \rightarrow \infty$.

polarization ellipse can be inscribed in the polarization pattern, and Fig. 8.1(a) shows this.

The polarization pattern method lends itself well to the rapid testing of an antenna's polarization properties as a function of angle from beam maximum. The linear sampling antenna is rotated rapidly while the antenna under test is scanned slowly. A recording of the received voltage shows the antenna pattern with a rapid cyclic variation on it caused by the spinning of the sampling antenna. The ratio of amplitudes of adjacent maxima and minima will yield the axial ratio of the antenna being tested if the antenna pattern does not change significantly while the sampling antenna rotates through one-half revolution [2]. This automated process will clearly be more effective for antennas almost circularly polarized than for linearly polarized antennas.

An obvious deficiency of the polarization pattern method is its failure to give the rotation sense of the antenna under test. This information sometimes may be inferred from the antenna construction. It may also be obtained by making additional measurements with two equal-gain, opposite-sense, circularly polarized antennas.

Since the sampling antenna is mechanically rotated, care must be taken that the received power is not affected by the motion. In particular, rotary joints must have a constant output or be calibrated. As a final remark about the polarization pattern method, if the antenna undergoing test is nominally linearly polarized, a measurement of its axial ratio will be inaccurate unless the axial ratio of the sampling antenna is much greater than that of the antenna being tested.

8.5. POWER COMBINER AND DIVIDER SYSTEM

The two-antenna (linear vertical and horizontal) lossless power combiner and divider system shown in Fig. 5.5 and described in Section 5.4 can be used to measure the polarization of an incoming wave, assuming that the α and β phase shifters are calibrated. We take first the case in which all incoming power is directed to port 4 of the lower hybrid of Fig. 5.5(a). From (5.20) and (5.23), using the upper signs, we can find the phase difference ϕ between the incoming linear wave components, and their magnitude ratio, b/a, by

$$\phi = \frac{1}{2}\pi - \alpha \qquad (a)$$

$$\frac{b}{a} = -\tan\frac{\beta + \pi}{2} \qquad (b)$$
(8.8)

These quantities are sufficient to describe the polarization of the incident wave. The modified polarization ratio of the incident wave may be written as

$$p^{i} = j \, \frac{E_{\eta}}{E_{\xi}} = -j \, \frac{E_{y}}{E_{x}} = -j \, \frac{b}{a} \, e^{j\phi} \tag{8.9}$$

and the use of (8.8) gives

$$p^{i} = +j \tan\left[\frac{1}{2}(\beta + \pi)\right] e^{j(\pi/2 - \alpha)} = e^{-j\alpha} \cot\left(\frac{1}{2}\beta\right)$$
(8.10)

The use of the lower sign in (5.20) and (5.23) gives

$$\phi = -\frac{1}{2}\pi - \alpha \qquad (a)$$

$$\frac{b}{a} = \tan \frac{\beta + \pi}{2} \qquad (b)$$

which when substituted into (8.9) gives

$$p^{i} = e^{-j\alpha} \cot\left(\frac{1}{2}\beta\right) \tag{8.12}$$

which is the same as (8.10).

In the same way the case in which all incoming power is directed to port 2 leads to, using (5.26) and (5.28),

POLARIZATION MEASUREMENT WITH UNEQUAL EFFECTIVE LENGTHS 275

$$p' = -e^{-j\alpha} \tan\left(\frac{1}{2}\beta\right) \tag{8.13}$$

In use, greater precision is obtained by nulling one of the outputs, say b_2 , rather than maximizing the other. Errors will occur if the antennas are not linearly polarized. The effect of system losses, such as attenuation in the phase shifters, is not significant unless very high accuracy is desired [4].

The two antennas of Fig. 5.5 may be replaced by a horn of square cross section with orthogonal feeds or by a circular waveguide supporting two orthogonal TE_{11} modes and terminated by a circular horn or polyrod antenna.

The α and β phase shifters of this measurement system may be calibrated in the system by transmitting waves of known polarization toward the measurement system antennas. For example, if the incoming wave is linear horizontal, so that $p^i = 0$, and if β is varied to null the power to port 4, while α is left general, (8.13) requires that $\beta = 0$, which establishes the zero for that phase shifter. If a left circular wave, with $p^i = -1$, is radiated toward the system and if β is set to π radians, then the output of port 4 can be nulled by setting $\alpha = 0$. Other calibration points can also be determined by rotating the linear calibration antenna.

This system may also be used to obtain the polarization pattern of an antenna without the mechanical rotations used in the standard polarization pattern procedure.

8.6. POLARIZATION MEASUREMENT WITH UNEQUAL EFFECTIVE LENGTHS

It is not difficult to ensure that the magnitudes of the effective lengths of the antennas used in the power combiner-divider system of the previous section are equal, since small pointing errors will change the magnitudes only slightly. It is more difficult to make certain that the effective lengths have the same phase, since phase is more sensitive to pointing errors and also depends on relative path lengths between the antennas and the upper tee of Fig. 5.5.

The system may be calibrated if a wave of known polarization is available. Once the system parameters are known, polarization of a general wave may be obtained from the equations

$$\phi = -\alpha - \delta \pm \frac{1}{2}\pi \qquad (a)$$

$$\frac{b}{a} = -\frac{|h_2|}{|h_4|} \tan \frac{\beta + \pi}{2} \qquad (b)$$

which come from (5.45) and (5.48), with maximum output from arm 4 of the lower tee of Fig. 5.5(a) assumed. From (8.14), we can find

POLARIZATION MEASUREMENTS

$$p^{i} = \frac{|h_{2}|}{|h_{4}|} e^{-j(\alpha+\delta)} \cot\left(\frac{1}{2}\beta\right)$$
(8.15)

If the system can be rotated about an axis pointing in the direction of the incident wave without changing the relative amplitudes and phases of h_2 and h_4 , it may be used for measuring the polarization of an incident wave without the necessity of calibration.

Let the polarization ratio of the incident wave, which is to be measured, be p_1 , and that of our measuring system be p_2 . The α and β phase shifts are varied until the output from port 4 of Fig. 5.5(a) is maximum. For this situation

$$p_2 = p_1^* \tag{8.16}$$

Next we decrease the setting of the α phase shifter by $\Delta \alpha$, which changes the polarization of the measuring system to

$$p_{2}' = p_{2}e^{j\Delta\alpha} = p_{1}^{*}e^{j\Delta\alpha}$$
(8.17)

For this new setting the polarization match factor, which is the ratio of output at port 4 to the output under polarization-matched conditions, and hence is known, is

$$\rho' = \frac{(1+p_1p_2')(1+p_1^*p_2'^*)}{(1+p_1p_1^*)(1+p_2'p_2'^*)}$$
(8.18)

and with the use of (8.17), this becomes

$$\rho' = 1 - \frac{2|p_1|^2(1 - \cos \Delta \alpha)}{(1 + |p_1|^2)^2}$$
(8.19)

and since ρ' is known, $|p_1|$ may be found.

We next rotate both receiving antennas through a convenient angle, say $\frac{1}{4}\pi$, about the z axis of Fig. 5.7, as shown in Fig. 8.2. The fields received at the polarization measuring system are E_x and E_y , which become, after the system is rotated,

$$E_{x'} = \frac{1}{2}\sqrt{2}(E_x + E_y)$$
 (a) $E_{y'} = \frac{1}{2}\sqrt{2}(-E_x + E_y)$ (b) (8.20)

and the polarizations in the original and rotated systems are

$$p_1 = -j \frac{E_y}{E_x} \qquad p'_1 = -j \frac{E_{y'}}{E_{x'}}$$
(8.21)



FIGURE 8.2. Rotation of polarization measurement system.

Using (8.20) the last equation becomes

$$p'_{1} = -j \; \frac{E_{y}}{E_{x'}} = j \; \frac{E_{x} - E_{y}}{E_{x} + E_{y}} = j \; \frac{1 - jp_{1}}{1 + jp_{1}} \tag{8.22}$$

Equation (8.22) may be put into the form

Im
$$(p_1) = \frac{1+|p_1|^2}{2} \frac{1-|p_1'|^2}{1+|p_1'|^2}$$
 (8.23)

Now we found $|p_1|$ by introducing a known change $\Delta \alpha$ into the α phase shifter. We could find $|p'_1|$ with the physically rotated system in the same way. Then (8.23) allows Im (p_1) to be found. Finally, from a knowledge of $|p_1|$ and Im (p_1) , we may find p_1 itself.

In summary, if we can introduce a known phase shift into the phase shifter, and if we can physically rotate the antennas of our polarization-measuring equipment about the z axis of Fig. 5.7 without changing the relative amplitudes and phases of h_2 and h_4 , we can measure the polarization of an incoming wave without being concerned with any field phase differences in our system.

8.7. POLARIZATION PROPERTIES FROM AMPLITUDE MEASUREMENTS

We found in Section 3.9 that if the modified polarization ratio of a transmitting antenna and the conjugate of the modified polarization ratio of a

receiving antenna (or vice versa) are plotted on a Poincaré sphere by means of the Stokes parameters, using (2.196) and (2.200), the polarization match factor between the antennas is given by

$$\rho = \cos^2\left(\frac{1}{2}\beta\right) \tag{3.125}$$

where β is the angle between the rays from the sphere center to the two plotted points. If we have a transmitting antenna with polarization unknown and a receiving antenna with known polarization, we are assured that a circle drawn on the Poincaré sphere, with radius compatible with (3.125) and center at the receiver conjugate polarization point, will pass through the sphere point defining the transmitter polarization.[†] If we take a second receiving antenna, a circle with its conjugate point as center will also pass through the transmitter polarization point. In general, the two circles will intersect in two points on the Poincaré sphere. A third receiving antenna can be used to remove the ambiguity. As a general rule, the three circles generated by using three receiving antennas paired with the transmitting antenna will intersect at one point on the Poincaré sphere, thus uniquely defining the polarization of the transmitting antenna. Note that amplitude measurements only are needed [5].

If the circles on the Poincaré sphere interact at small angles, it is obvious that small errors in the amplitude measurements can lead to significant uncertainty in the polarization. Prior knowledge of the antenna under test can be used to select the sampling antennas, and in fact, if the rotation sense of the antenna being tested is known, it may be possible to eliminate one measurement.

The polarization match factor ρ is not measured directly; rather, we measure power to a receiver load, and this is determined by polarization, antenna gains, transmitted power, and so on. It is then clear that additional measurements are needed to determine an antenna's polarization properties.

A convenient method of handling the requirement for additional information is by using pairs of receiving antennas that have the same gains but are orthogonally polarized, such as left and right circular antennas. Power ratios are then used to determine the polarization of the antenna tested. We illustrate the method by using three pairs of receiving antennas, linear horizontal (x directed) and vertical, linear at 45° and 135° from the x axis, and left and right circular.

Linear Vertical and Horizontal

In terms of the common polarization ratio, the polarization match factor between a transmitting antenna (1) and a receiving antenna (2) is

[†] We do not consider the case of transmitter and receiver orthogonal.

POLARIZATION PROPERTIES FROM AMPLITUDE MEASUREMENTS 279

$$\rho = \frac{(1 - P_1 P_2)(1 - P_1^* P_2^*)}{(1 + P_1 P_1^*)(1 + P_2 P_2^*)}$$
(3.89)

If a linear vertical receiving antenna is used, with $P_2 \rightarrow \infty$, the power received is

$$W_V = C_1 \rho_V = C_1 \frac{|P_1|^2}{1 + |P_1|^2}$$

where C_1 is a constant that includes the antenna gains, power transmitted, receiver gain, impedance match, antenna separation, and impedance match but not polarization.

If next we use a linear horizontal receiving antenna, with $P_2 = 0$, keeping all other factors the same, the power received is

$$W_H = C_1 \rho_H = \frac{C_1}{1 + |P_1|^2}$$

and the ratio of the two received powers is

$$\frac{W_V}{W_H} = \frac{\rho_V}{\rho_H} = |P_1|^2 \tag{8.24}$$

Linear 45° and 135°

If a linearly polarized antenna tilted at 45° ($P_2 = 1$) and one at 135° ($P_2 = -1$) are used successively with the antenna under test, the ratio of powers received is

$$\frac{W_{45}}{W_{135}} = \frac{1 + |P_1|^2 - 2\operatorname{Re}(P_1)}{1 + |P_1|^2 + 2\operatorname{Re}(P_1)}$$
(8.25)

if the gains of the two receiving antennas are equal.

Left and Right Circular

Using left circular $(P_2 = j)$ and right circular $(P_2 = -j)$ antennas leads to a power ratio

$$\frac{W_R}{W_L} = \frac{1 + |P_1|^2 + 2 \operatorname{Im}(P_1)}{1 + |P_1|^2 - 2 \operatorname{Im}(P_1)}$$
(8.26)

Equations (8.24), (8.25), and (8.26) are readily solved to give

POLARIZATION MEASUREMENTS

Re
$$(P_1) = \frac{(1 + W_V/W_H)(1 - W_{45}/W_{135})}{2(1 + W_{45}/W_{135})}$$
 (a)
Im $(P_1) = \frac{(1 + W_V/W_H)(W_R/W_L - 1)}{2(W_R/W_L + 1)}$ (b)

We see then that six amplitude measurements will lead to the polarization of a general antenna. In (8.27) the term W_V/W_H can be replaced by $|P_1|^2$, and since this is equal to the sums of the squares of (a) and (b) of (8.27), it appears that P_1 can be determined by two power ratios, W_{45}/W_{135} and W_R/W_L . Ambiguities in P_1 will appear, however, if this is done.

The choice of antenna pairs utilized to obtain (8.27) may not be optimum for the measurement of a general antenna, but the chosen antennas are easily obtained. One linearly polarized antenna may be used for four of the measurements. If helices are used for the circularly polarized receiving antennas, their small departure from the circular, indicated by (8.3), will not affect polarization measurements substantially and, in fact, (8.26) can be modified to account for the ellipticity. The problem remains, however, of constructing equal-gain helices of opposite rotation. Nevertheless, the freedom from measuring phase, for which accurate positioning of the receiving antennas is necessary, makes this method attractive.

REFERENCES

- 1. J. D. Kraus, Antennas, McGraw-Hill, New York, 1950.
- T. G. Hickman, et al., "Polarization Measurements," Chapter 10 in J. S. Hollis, et al., Microwave Antenna Measurements, Scientific-Atlanta, Atlanta, GA, 1970.
- R. Rubin, "Antenna Measurements," Chapter 34 in H. Jasik, ed., Antenna Engineering Handbook, McGraw-Hill, New York, 1961.
- 4. R. N. Ghose, Microwave Circuit Theory and Analysis, McGraw-Hill, New York, 1963.
- 5. G. H. Knittel, "The Polarization Sphere as a Graphical Aid in Determining the Polarization of an Antenna by Amplitude Measurements Only," *IEEE Trans. on Antennas and Propagation*, Vol. AP-15, No. 2, pp. 217–221, March 1967.

PROBLEMS

8.1. A helix is desired for polarization measurements with a quality as high as that of a good linearly polarized antenna (AR = 40 dB). Define equal quality to mean that $|E_R|/|E_L|$ for the helix is the same as $|E_y|/|E_x|$ for the linear antenna. Find the number of turns needed for the helix.

PROBLEMS

8.2. A dipole, a left circular, and a right circular antenna are used on an antenna range to determine the polarization ratio of a transmitting antenna. The two circularly polarized antennas have the same gain and impedance. With respect to a coordinate system at the receiving antenna having its z axis directed toward the transmitter, the dipole is successively oriented along the y axis (vertical), along the x axis (horizontal), and at 45° and 135° from the x axis. The same receiver load impedance is used for all antennas. The received powers are (in milliwatts)

Vertical:	3.82	135°:	4.04
Horizontal:	0.95	Right Circular:	7.80
45°:	0.73	Left Circular:	3.34

Find the polarization ratio of the transmitting antenna.

APPENDIX

RELATION BETWEEN EFFECTIVE LENGTH AND GAIN

The effective length of an antenna is defined in terms of its transmitted far field by

$$\mathbf{E}(r,\,\theta,\,\phi) = \frac{jZ_0I}{2\lambda r} \,e^{-jkr}\mathbf{h}(\theta,\,\phi) \tag{3.2}$$

where I is the input current at an arbitrary pair of terminals.

The directivity of the antenna is, from Section 1.8, the ratio of the radiation intensity in a specified direction to the radiation intensity averaged over all space. From

$$U(\theta, \phi) = r^2 S(r, \theta, \phi) = \frac{r^2}{2Z_0} \mathbf{E} \cdot \mathbf{E}^*$$
(A.1)

and

$$U_{\rm av} = \frac{1}{4\pi} \int \int_{4\pi} U \, d\Omega \tag{A.2}$$

we can obtain the directivity

$$D(\theta, \phi) = \frac{\mathbf{E} \cdot \mathbf{E}^*}{(1/4\pi) \iint_{4\pi} \mathbf{E} \cdot \mathbf{E}^* d\Omega} = \frac{\mathbf{h} \cdot \mathbf{h}^*}{(1/4\pi) \iint_{4\pi} \mathbf{h} \cdot \mathbf{h}^* d\Omega}$$
(A.3)

and gain

$$G(\theta, \phi) = \frac{e\mathbf{h} \cdot \mathbf{h}^*}{(1/4\pi) \int \int_{4\pi} \mathbf{h} \cdot \mathbf{h}^* d\Omega}$$
(A.4)

where e is antenna efficiency.

The gain and effective area of the antenna are related by $\lambda^2/4\pi$, so that

$$A_{e}(\theta, \phi) = \frac{\lambda^{2} e \mathbf{h} \cdot \mathbf{h}^{*}}{\int \int_{4\pi} \mathbf{h} \cdot \mathbf{h}^{*} d\Omega}$$
(A.5)

Note that (θ, ϕ) now refers to the direction from which the wave comes to strike the receiving antenna.

Radiation resistance of an antenna, from Section 1.8, is the ratio of the power radiated to the square of the rms current at arbitrarily chosen terminals. Then

$$R_r = \frac{2}{I^2} \frac{r^2}{2Z_0} \int \int_{4\pi} \mathbf{E} \cdot \mathbf{E}^* \, d\Omega \tag{A.6}$$

and if we use (3.2),

$$R_r = \frac{Z_0}{4\lambda^2} \int \int_{4\pi} \mathbf{h} \cdot \mathbf{h}^* \, d\Omega \tag{A.7}$$

In terms of a total antenna resistance $R_a (= R_r + R_{\text{loss}})$, we can use (1.90) and obtain

$$R_{a} = \frac{Z_{0}}{4e\lambda^{2}} \int \int_{4\pi} \mathbf{h} \cdot \mathbf{h}^{*} d\Omega \qquad (A.8)$$



ISOTROPIC ANTENNAS AND NULL-FREE ANTENNAS

B.1. AN ISOTROPIC ANTENNA

It was shown in Chapter 4 that the relative gain of a crossed-dipole (turnstile) antenna with the two dipoles lying in the xy plane and fed by equal-amplitude currents in quadrature phase is

$$G_r = \frac{1}{2}(1 + \cos^2 \theta)$$
 (4.33)

The gain is independent of azimuth angle and varies only 3 dB with the polar angle. Since the fields of an array of identical elements are those of the individual element multiplied by an array factor, it is apparent that a linear array of turnstiles on the z axis, with an array factor

$$AF = \frac{C}{\sqrt{1 + \cos^2 \theta}} \tag{B.1}$$

(where C is a constant), would produce a radiation intensity pattern independent of angles, an isotropic antenna.

Saunders [1] has shown that a continuous even distribution of turnstiles on the z axis, with feed function $K_0(kz)$, where $k = 2\pi/\lambda$ and K_0 is the modified Bessel function of the second kind and zero order, will produce such an array factor. Further, he shows that the distribution can be truncated to a reasonable length without disturbing the radiation pattern significantly. In fact, he found that with only two discrete turnstiles spaced a quarter of a wavelength apart, the radiation intensity varied less than 0.5 dB over the full range of polar angle. We may conclude that for practical purposes it is possible to construct an isotropic antenna insofar as radiated power density is concerned. Zaidi [2], using concentric ring radiators, has also shown that it is theoretically possible to radiate power isotropically.

Now we saw in Chapter 4 that the product $G_r\rho$, where ρ is the polarization match factor between two antennas, determines the power available to a receiving antenna and is therefore a more useful figure of merit for an antenna pair than the gain alone. We saw also in that chapter that the polarization characteristics of a linear array are those of an element of the array. If a circularly polarized antenna is used to receive the wave radiated either by the single turnstile of Section 4.4, with a polarization match factor

$$\rho = \frac{1}{2} + \frac{\cos\theta}{1 + \cos^2\theta} \tag{4.32}$$

or the turnstile array proposed by Saunders, the polarization loss is zero on the positive z axis, 3 dB in the xy plane and infinite on the negative z axis. Any other receiving antenna of fixed polarization would show a similar variation in polarization match, and it is apparent that polarization effects cannot be neglected in a consideration of the isotropicity of antennas.

Rather than consider further the isotropic antenna, we will examine the more general class of null-free antennas, those that do not have a zero in the radiated power density over the far-field sphere of the antenna. We exclude from consideration systems that create isotropic patterns by exciting different antennas either by different frequencies or in a time sequence [3].

B.2. BROUWER'S THEOREM

We begin with a theorem of Brouwer [4], slightly restated: A vector distribution everywhere single valued and continuous on (and tangent to) a singly connected, two-sided, closed surface must be zero or infinity in at least one point. The vector distribution we consider is the time-varying radiated electric far field of an antenna, which meets all the conditions of Brouwer's theorem and of course cannot be infinite anywhere, and the surface is a large sphere, centered at the antenna, over which only the far field exists. Brouwer's theorem has served as a basis for developments by Mathis [5, 6], Saunders [1], and Scott and Soo Hoo [7].

B.3. A THEOREM OF MATHIS

Mathis has shown that the radiation pattern must contain either a null or some point at which the field is linearly polarized or both. Saunders' proof of the theorem is simple. He assumes that the wave is everywhere circularly or elliptically polarized (excluding linear polarization as a special case of

APPENDIX B

elliptical). But the time-varying field of such a wave cannot vanish at a point unless it vanishes for all time, and we have a contradiction that proves Mathis's theorem.

B.4. SAUNDERS' THEOREM

Saunders went a step further by showing that a null-free radiation pattern cannot be linearly polarized everywhere. His proof parallels the proof of Mathis's theorem in that he assumes that the pattern is null free and linearly polarized. A linearly polarized wave can be written as the sum of two oppositely rotating, circularly polarized waves, each of which must satisfy Brouwer's theorem. This they can do only by vanishing, but if one circular component vanishes, the resultant field is circularly polarized, and if both vanish at the same point, the radiation pattern contains a null. Thus Saunders' theorem is proved.

B.5. ANOTHER PROOF OF MATHIS'S THEOREM

As an introduction to a more general theorem, Scott and Soo Hoo have presented another proof of the theorem of Mathis [7]. Since it provides insights into polarization behavior not given by other proofs, it will be given here.

The far field of the radiating antenna can be written in complex (timeinvariant) form as

$$\mathbf{E}(r,\theta,\phi) = \frac{e^{-jkr}}{r} \left[E_{\theta}(\theta,\phi) \mathbf{u}_{\theta} + E_{\phi}(\theta,\phi) \mathbf{u}_{\phi} \right]$$
(B.2)

If we write the real and imaginary parts of the complex fields as

$$E_{\theta} = f_1 + jf_2$$
 (a) $E_{\phi} = g_1 + jg_2$ (b) (B.3)

and transform to the time domain, we find the time-varying electric field to be

$$\mathscr{E}(r,\theta,\phi,t) = \frac{1}{r} \left\{ \left[f_1(\theta,\phi) \cos\left(\omega t - kr\right) - f_2(\theta,\phi) \sin\left(\omega t - kr\right) \right] \mathbf{u}_{\theta} + \left[g_1(\theta,\phi) \cos\left(\omega t - kr\right) - g_2(\theta,\phi) \sin\left(\omega t - kr\right) \right] \mathbf{u}_{\phi} \right\}$$
(B.4)

Since \mathscr{E} is single valued and continuous on, and tangent to, the far-field sphere, it conforms to Brouwer's theorem and must be zero at some point (θ_0, ϕ_0) on the sphere (we note again that it cannot be infinite). The zero value in \mathscr{E} can be achieved in several ways, and we examine them by case.

Case 1

$$f_1(\theta_0, \phi_0) = f_2(\theta_0, \phi_0) = g_1(\theta_0, \phi_0) = g_2(\theta_0, \phi_0) = 0$$
(B.5)

This case clearly makes \mathscr{E} zero for all time and represents a radiation pattern null in the direction (θ_0, ϕ_0) . More than one pattern null can exist, since (B.5) can have more than one solution.

Case 2

$$f_1(\theta_0, \phi_0) = f_2(\theta_0, \phi_0) = 0 \quad (a)$$

$$\tan(\omega t_0 - kr) = \frac{g_1(\theta_0, \phi_0)}{g_2(\theta_0, \phi_0)} \quad (b)$$

Equations (B.6b) and (B.4) show that an instantaneous zero in \mathscr{E}_{ϕ} occurs at time t_0 in the direction (θ_0, ϕ_0) . As t_0 takes on new values, the direction (θ_0, ϕ_0) given by (B.6b) also changes, and the point on the far-field sphere corresponding to (θ_0, ϕ_0) traces a path with time. On this path \mathscr{E}_{ϕ} is zero at specific instants (when the radially propagating sinusoidal field instantaneously becomes zero), but in general, \mathscr{E}_{ϕ} is not zero on the path. On the other hand, (B.6a) and (B.4) show that \mathscr{E}_{θ} is always zero along the path. It follows that on this path the radiation intensity is not zero, and the wave is linearly polarized.

Case 3

$$\tan (\omega t_0 - kr) = \frac{f_1(\theta_0, \phi_0)}{f_2(\theta_0, \phi_0)} \quad (a)$$

$$g_1(\theta_0, \phi_0) = g_2(\theta_0, \phi_0) = 0 \quad (b)$$
(B.7)

This case is the same as case 2 except that the roles of the functions f and g are interchanged. The wave represented here also has a nonzero radiation intensity and is linearly polarized.

Case 4

$$\tan\left(\omega t_0 - kr\right) = \frac{f_1(\theta_0, \phi_0)}{f_2(\theta_0, \phi_0)} = \frac{g_1(\theta_0, \phi_0)}{g_2(\theta_0, \phi_0)} \tag{B.8}$$

The point on the far-field sphere corresponding to (θ_0, ϕ_0) traces a path along which \mathcal{E}_{θ} and \mathcal{E}_{ϕ} take on instantaneous zero values together, and on which the radiation intensity is nonzero.

Now if (B.8) is substituted into (B.4), we find on the path defined by

 (θ_0, ϕ_0) that the general time-varying fields are

$$\mathcal{E}_{\theta} = f_2(\theta_0, \phi_0) [\tan(\omega t_0 - kr)\cos(\omega t - kr) - \sin(\omega t - kr)] \quad (a)$$

$$\mathcal{E}_{\phi} = g_2(\theta_0, \phi_0) [\tan(\omega t_0 - kr)\cos(\omega t - kr) - \sin(\omega t - kr)] \quad (b)$$
(B.9)

Equation (B.9) shows that \mathscr{E}_{θ} and \mathscr{E}_{ϕ} are in phase, and the wave along the path (θ_0, ϕ_0) is linearly polarized.

From these four cases we see from the requirement of Brouwer's theorem that the time-varying electric field must be zero at some point, it follows that the radiation intensity must have a null (case 1) or it must have at least one point at which the field is linearly polarized (cases 2, 3, and 4). This is an alternate proof of the theorem of Mathis.

B.6. A THEOREM OF SCOTT AND SOO HOO

A theorem by Scott and Soo Hoo contains the theorems of Mathis and Saunders as special cases. It may be stated as: Elliptical polarization of all axial ratios, ranging from circular polarization of purely one sense, through linear, to circular polarization of the opposite sense must exist in the far field of a null-free antenna. Two comments are in order: The theorem does *not* say that all possible polarizations exist. It does not apply to many standard antennas, the dipole for example, but it does apply to the turnstile antenna we have considered.

We may write the electric field (B.2) in terms of right and left circular components,

$$\mathbf{E}(r,\,\theta,\,\phi) = \frac{e^{-jkr}}{r} \left[E_L(\theta,\,\phi)\boldsymbol{\omega}_L + E_R(\theta,\,\phi)\boldsymbol{\omega}_R \right] \tag{B.10}$$

where the vectors $\boldsymbol{\omega}_L$ and $\boldsymbol{\omega}_R$, if converted to time-varying form by the usual convention, would represent constant-amplitude waves rotating in opposite directions. In (2.74), if we use $\mathbf{u}_x = \mathbf{u}_{\phi}$ and $\mathbf{u}_y = -\mathbf{u}_{\theta}$, they become

$$\boldsymbol{\omega}_L = \mathbf{u}_{\phi} - j\mathbf{u}_{\theta}$$
 (a) $\boldsymbol{\omega}_R = \mathbf{u}_{\phi} + j\mathbf{u}_{\theta}$ (b) (B.11)

With the use of (B.3) and (B.11) the circular waves that sum to give the general propagating wave may be written as

$$\mathbf{E}_{L}(r, \theta, \phi) = \frac{e^{-jkr}}{2r} \left[(g_{1} - f_{2}) + j(f_{1} + g_{2}) \right] \boldsymbol{\omega}_{L} \quad (a)$$

$$\mathbf{E}_{R}(r, \theta, \phi) = \frac{e^{-jkr}}{2r} \left[(g_{1} + f_{2}) - j(f_{1} - g_{2}) \right] \boldsymbol{\omega}_{R} \quad (b)$$

We note that ω_L and ω_R are complex, and in finding the time-varying circular wave components, it is desirable to revert to the \mathbf{u}_{θ} and \mathbf{u}_{ϕ} vectors of (B.11). The resulting time-varying circular fields are

$$\mathscr{E}_{L}(r, \theta, \phi, t) = \frac{1}{2r} \left\{ [f_{1} + g_{2}) \cos(\omega t - kr) + (g_{1} - f_{2}) \sin(\omega t - kr)] \mathbf{u}_{\theta} + [(g_{1} - f_{2}) \cos(\omega t - kr) - (f_{1} + g_{2}) \sin(\omega t - kr)] \mathbf{u}_{\phi} \right\}$$
(a)
(B.13)

$$\mathscr{C}_{R}(r,\,\theta,\,\phi,\,t) = \frac{1}{2r} \left\{ [f_{1} - g_{2})\cos(\omega t - kr) - (g_{1} + f_{2})\sin(\omega t - kr)]\mathbf{u}_{\theta} + [(g_{1} + f_{2})\cos(\omega t - kr) + (f_{1} - g_{2})\sin(\omega t - kr)]\mathbf{u}_{\phi} \right\}$$
(b)

The magnitudes of the circular components, from (B.12) or (B.13), are

$$\begin{aligned} |\mathscr{C}_{L}(r,\,\theta,\,\phi,\,t)| &= \frac{1}{2r} \left[\left(f_{1} + g_{2} \right)^{2} + \left(g_{1} - f_{2} \right)^{2} \right]^{1/2} \quad \text{(a)} \\ |\mathscr{C}_{R}(r,\,\theta,\,\phi,\,t)| &= \frac{1}{2r} \left[\left(f_{1} - g_{2} \right)^{2} + \left(g_{1} + f_{2} \right)^{2} \right]^{1/2} \quad \text{(b)} \end{aligned}$$
(B.14)

Now \mathscr{C}_L and \mathscr{C}_R must each satisfy Brouwer's theorem, since each is a single-valued, continuous vector function tangent to the far-field sphere. However, a circularly polarized wave cannot be zero at any time unless its magnitude is zero. It follows that for a null-free radiation pattern (see below) there are at least two points on the far-field sphere corresponding to the directions (θ_1, ϕ_1) and (θ_2, ϕ_2) for which

$$|\mathscr{C}_{L}(\theta_{1}, \phi_{1})| = 0$$
 (a) $|\mathscr{C}_{R}(\theta_{2}, \phi_{2})| = 0$ (b) (B.15)

Let us note first that if the wave is to be linearly polarized everywhere, then

$$|\mathscr{C}_L| = |\mathscr{C}_R|$$

everywhere. In turn, this requires that the two directions (θ_1, ϕ_1) and (θ_2, ϕ_2) coincide, and the result is a radiation pattern null in that direction. This explains the restriction to a null-free pattern in the sentence before (B.15). The development proves Saunders' theorem that a null-free pattern cannot be everywhere linearly polarized.

We can deduce another fact very quickly from (B.15). It is clear that for a null-free antenna $|\mathscr{C}_L| = 0$ at (θ_1, ϕ_1) and $|\mathscr{C}_R| \neq 0$. The radiated wave is then right circularly polarized in that direction. Further, it is left circular in the direction (θ_2, ϕ_2) . A null-free antenna then must have directions of both right and left circular polarization.

APPENDIX B

The intermediate-value theorem of calculus can be used to show that the locus of points of linear polarization on the far-field sphere will be one or more closed paths on the sphere. It may be stated as: Suppose S is a connected set and f is a function that is continuous at each point of S. Suppose f takes on two different values C_1 and C_2 at points P_1 and P_2 in S. Then, for every number K between C_1 and C_2 , there is some point of S at which f takes on the value K [7]. We let S be all points on the far-field sphere; P_1 and P_2 , respectively, correspond to (θ_1, ϕ_1) and (θ_2, ϕ_2) ; and f be

$$f(\theta, \phi) = |\mathscr{C}_L(\theta, \phi)| - |\mathscr{C}_R(\theta, \phi)|$$
(B.16)

Now, from (B.15) and (B.16), it is clear that

$$f(P_1) < 0$$
 and $f(P_2) > 0$ (B.17)

and it follows from the intermediate value theorem that a point P_0 corresponding to (θ_0, ϕ_0) exists such that

$$f(P_0) = 0$$
 (B.18)

or

$$|\mathscr{C}_L(\theta_0, \phi_0)| = |\mathscr{C}_R(\theta_0, \phi_0)| \tag{B.19}$$

But (B.19) is the requirement for linear polarization at (θ_0, ϕ_0) , and we have another proof of Mathis's theorem.

Next let N curves terminating on P_1 and P_2 be drawn on surface S, the far-field sphere (N > 1). If the curves do not cross, they divide S into N closed, connected subsets, each satisfying the intermediate-value theorem. Since P_1 and P_2 are members of each subset, there is at least one point in each subset for which f = 0, and the wave is linearly polarized. If we now let $N \rightarrow \infty$, the points f = 0 approach a continuous, closed curve. Then we see that the locus of points of linear polarization is one or more closed curves on the far-field sphere.

We need not restrict the function f to the form we have utilized so far. Instead, let f be the magnitude of the circular polarization ratio of Chapter 2:

$$f(\theta, \phi) = |q(\theta, \phi)| = \frac{|\mathscr{C}_L(\theta, \phi)|}{|\mathscr{C}_R(\theta, \phi)|}$$
(B.20)

Let S be a simply connected surface on the far-field sphere that includes the points corresponding to (θ_0, ϕ_0) of (B.19) and (θ_1, ϕ_1) but excludes the point corresponding to (θ_2, ϕ_2) , for which f is infinite. Since $f(\theta_1, \phi_1) = 0$ and $f(\theta_0, \phi_0) = 1$, we infer from the intermediate-value theorem that a point on S

can always be found for which

$$|q(\theta, \phi)| = k \qquad 0 \le k \le 1 \tag{B.21}$$

and we see that the wave takes on all values of |q| from 0 (corresponding to a right circular wave) to 1 (corresponding to linear polarization).

In the same way we can show that a point on S can always be found for which

$$|w(\theta, \phi)| = \frac{|\mathscr{C}_{R}(\theta, \phi)|}{|\mathscr{C}_{L}(\theta, \phi)|} = k \qquad 0 \le k \le 1$$
(B.22)

and we see that the wave takes on all values of |w|, from that corresponding to a left circular wave to the value corresponding to a linearly polarized wave. Since |q| = 1/|w|, we note that the full range of |q| on the far-field sphere of a null-free antenna is

$$0 \le |q(\theta, \phi)| < \infty \tag{B.23}$$

The axial ratio of the polarization ellipse is related to the circular polarization ratio by

$$AR = \left| \frac{1+|q|}{1-|q|} \right| \tag{2.107}$$

so it is clear that all axial ratios exist in the radiated field of a null-free antenna.

In the superb paper of Scott and Soo Hoo it is stated that for any (fixed) polarization of a receiving antenna and a null-free transmitting antenna that is rolling and tumbling (as in a satellite vehicle), there exists at least one orientation of that vehicle for which the antenna would receive no signal. The conclusion is unjustified. The cross-polarization condition between two antennas is

$$q_1 = -\frac{1}{q_2}$$
(3.56)

For the null-free antennas $|q_1|$ is constrained to take on all values, but q_1 itself is not so constrained. It is then possible to choose a fixed receiving antenna so that (3.56) is not met. As an exercise, it is suggested that the reader select a receiving antenna to pair with the turnstile antenna, for which

$$q = \frac{1 + \cos \theta}{1 - \cos \theta} \tag{B.24}$$

so that (3.56) cannot be satisfied by any value of θ .

APPENDIX B

REFERENCES

- 1. W. K. Saunders, "On the Unity Gain Antenna," *Electromagnetic Theory and Antennas*, Part 2, E. C. Jordan, ed., Pergamon Press, New York, 1963, pp. 1125–1130.
- S. H. R. Zaidi, "On Synthesis of Isotropic Patterns with Concentric Ring Circular Array," University of Tennessee Engineering Experiment Station, Knoxville, Rept. 4 (AD-293868), October 1962.
- F. F. Fulton, Jr., "The Combined Radiation Pattern of Three Orthogonal Dipoles," *IEEE Trans. on Antennas and Propagation*, Vol. AP-13, No. 2, pp. 323–324, March 1965.
- L. E. J. Brouwer, "On Continuous Vector Distributions on Surfaces," Proc. Royal Academy (Amsterdam), English ed., Vol. 11, pp. 850–858, 1909.
- 5. H. F. Mathis, "A Short Proof That an Isotropic Antenna is Impossible," *Proc. IRE*, Vol. 39, No. 8, p. 970, August 1951.
- 6. H. F. Mathis, "On Isotropic Antennas," Proc. IRE, Vol. 42, No. 12, p. 1810, December 1954.
- W. G. Scott and K. M. Soo Hoo, "A Theorem on the Polarization of Null-Free Antennas," *IEEE Trans. on Antennas and Propagation*, Vol. AP-14, No. 5, pp. 587–590, September 1966.

Analytic signal, 253-254 Antenna(s): aperture, 37 bandwidth, 2 crossed-dipole, 157-160, 183-186, 190, 284 - 288cross-polarized, 121 dipole, 20, 23, 27, 30-35, 154-157, 162, 181-182, 281, 288 efficiency, 23, 42, 53, 283 elementary, 17, 20 E-plane sectoral horn, 169-170 equivalent circuit of, 41-46 gain, 1, 14, 19-20, 23, 44, 282-284 helix, 271, 280 horn, 167-170, 190 H-plane sectoral horn, 169-170 identical cross-polarized, 123 identical polarization-matched, 122-125 impedance, 1, 2, 36, 39 impedance matrix of, 40 infinitesimal, 47 isotropic, 160, 284-285 linear, 12 loop, 161, 189 loop and dipole, 162-164 losses, 1, 22 misaligned, 139-146 null-free, 284-292 omnidirectional, 18 open waveguide, 37-38, 164 parabolic reflector, 170-186 pattern, 1, 13-17, 43-44, 289-291 polarization-matched, 120-125

polarization ratio, 119 pyramidal horn, 168-170, 190 receiving, 1, 39 receiving pattern, 42-44 transmitting, 1-2 turnstile, 157-160, 183-186, 190, 284-288 Aperture antenna, 37 Aperture plane, 175-176 Area, effective, 1, 44, 47, 283 Area sweep rate, 65-66 Array, narrow polarization beamwidth, 186-189 Array factor, 159, 284 Axial ratio, 54, 61-63, 68, 73, 75, 84-85, 109, 121-122, 126, 291 Balanis, C. A., 27, 30, 51, 167-169, 189 Bandwidth, antenna, 2 Barrick, D. E., 244, 252 Beckmann, P., 244-252 Beamwidth: half-power, see Beamwidth, radiation intensity overall, 154, 158, 167 polarization, 154, 158, 167, 186-189 radiation intensity, 154, 167, 170, 186 Beran, M. J., 269 Bistatic cross section, 50 Born, M., 108, 254, 269 Brewster angle, 216 Brillouin, L., 30 Brouwer, L. E. J., 285-286, 289, 292 Bushore, K. R., 196, 206

Carter, P. S., 30, 52 Cauchy principal value, 254 Circulator, 196 Coherence: complete, 257 degree of, 256 mutual, 262 Coherency matrix, 254-259, 262-266 Collin, R. E., 1, 51, 189 Communication system, polarization-adaptive, 201, 203 Complete polarization, 253, 257-259, 263-268 Copeland, J. R., 229, 252 Cosine integral, 34 Critical angle, 217 Crossed-dipole antenna, 157-160, 183-186, 190, 284-288 Cross polarization, 96, 121-123, 240 Cross section: backscattering, 231 bistatic, 50 monostatic, 50-51 radar, 51 scattering, 50, 229, 232-233 Current element, 6-7, 12-13, 17, 19-20, 47 Degree of coherence, 256 Degree of polarization, 259-261, 264, 268 Depolarization, 229 Dihedral corner reflector, 222-226, 240 Dipole antenna, 20, 23, 27, 30-35, 154-157, 162, 181-182, 281, 288 Directive gain, 20 Directivity, 19-20, 23, 46, 53, 282-283 Effective area, 1, 44, 47, 283 Effective length, 110-120, 146, 151, 233, 268-269. 275, 282-283 Efficiency, see Antenna(s), efficiency; Polarization, match factor Electric charge density, 3 Electric current density, 3 Electric source, 3-4, 11 Elementary antenna, 17, 20 Elliott, R. S., 1, 30, 51 Ellipse, polarization, 54, 57-67, 70, 73, 75, 83, 93 Elliptically polarized waves, 75, 80, 83-84, 89, 92, 95, 109 generation of, 191-206 reflection of, 217-221 E-plane, 18 Equivalence theorem, 28-30, 37

Equivalent circuit of antennas, 41, 46 Equivalent current, 37 Equivalent source, 3, 28, 172 Euler angles, 141-146 Euler's constant, 34 Far zone, 9, 14 Far-zone fields, 9-12 Feynman, R. P., 108 Field: complex time-invariant, 69 time-varying, 69 Flat plate, 221-222, 239-241 Focal point, 171-173 Fourier transform, 254 Fraunhofer zone, 9 Fresnel coefficients, 214, 244-245, 249-251. See also Reflection coefficients; Transmission coefficients Fresnel zone, 8-9 Friis transmission formula, 1, 48 Fulton, F. F., Jr., 292 Gabor, D., 254, 268 Gain, 1, 14, 19-20, 23, 44, 282-284 Geometric optics, 170-172 Ghose, R. N., 206, 280 Goldstein, H., 148 Harrington, R. F., 4, 51 Hayt, W. H., 51 Helical antenna, 271, 280 Helmholtz equation, 10 Hickman, T. G., 280 Hilbert transform, 254 Hollis, J. S., 280 Horn antenna, 167-170, 190 E-plane sectoral, 169-170 H-plane sectoral, 169-170 pyramidal, 168-170, 190 H-plane, 18 Hybrid tee, 196 Illumination, tapered, 186 Image theory, 37 Impedance: antenna, 1, 36, 39, 42 mode, 36 mutual, of antennas, 43-44 transmission line characteristic, 77 Impedance match, 23 Impedance match factor, 49 Impedance matrix, 40 Incidence plane, 208, 245

Induced emf, 30, 34-35 Infinitesimal antenna, 47 Interface: reflection from, 207-252 transmission through, 207-221 Intermediate-value theorem, 290 Isotropic antenna, 160, 284-285 Isotropic radiation, 23, 48 Jordan, E. C., 51 Kepler's laws, 66 Knittel, G. H., 280 Ko, H. C., 269 Kraus, J. D., 1, 27, 51, 108, 189, 271, 280 Krichbaum, C. K., 252 Length, see Effective length Linear antenna, 12 Lobe, radiation, 18 Loop antenna, 161, 189 Loop and dipole antenna, 162-164 Losses, antenna, 1, 22 Loss resistance, 22-23, 42, 53 Love equivalence principle, 29 Magnetic charge density, 3-4 Magnetic current, 37 Magnetic current density, 3-4 Magnetic source, 2-3, 11 Match factor: impedance, 49 polarization, 49, 117-147, 150-152, 156, 278 Matching network, 36 Mathis, H. F., 285-292 Maxwell equations, 2 McQuiddy, D. N., 51, 206 Monochromatic wave, 253 Monostatic cross section, 50 Mott, H., 51, 206 Multipath, 2 Mutual coherence, 261 Nonuniform wave, 217 North pole, Poincaré sphere, 95 Null, radiation pattern, 287, 289 Null-free antenna, 284-292 Omnidirectional antenna, 18 Open-waveguide antenna, 37-38, 164 Panofsky, W. K. H., 4, 51 Papas, C. H., 269

Parabolic reflector antenna, 170-186 Parrent, G. B., Jr., 269 Partial polarization, 92, 253-269 Pattern: antenna, 1, 13-17, 43-44, 289-291 polarization, 272-274 radiation, 13-17, 43-44, 289-291 receiving antenna, 42 Pattern multiplication, 159 Phillips, M., 4, 51 Plane of incidence, 208 local, 245 main, 245 Plane wave, 56-57 Poincaré sphere, 54, 93-109, 136-138, 146, 278 Poisson's equation, 7, 52 Polarization, 17 chart, 83-84, 92, 109, 146 circular, 63-64, 94-95, 99, 258, 261 complete, 253, 257-268 degree of, 259-261, 264, 268 efficiency, 118. See also Polarization, match factor ellipse, 54, 57-67, 70, 73-75, 83, 93 magnetic field, 109 elliptic, 75, 80, 83-84, 89, 92, 95, 109 linear, 63, 96, 99, 207-208, 258, 261 loss, 134. See also Polarization, match factor match, 23 match factor, 49, 117-147, 150-152, 156, 278 misaligned antennas, 139-146 matching, 110-147 measurement, 270-281 parallel to plane of incidence, 209, 212, 216, 222 partial, 92, 253-269 pattern, 272-274 perpendicular to plane of incidence, 208, 211, 214 random, 253 Polarization-adaptive communications system, 201, 203 Polarization ratio: circular, 54, 70, 74, 80, 109, 264-265, 271 common, 54, 68-69, 74, 109, 127, 138-139, 150, 153, 245, 264-265, 270, 281 modified, 71, 77, 109, 119, 125, 138, 237 Polarization ratio of antenna, 119 Potential: electric scalar, 4 electric vector, 5, 8, 38

Potential (Continued) magnetic scalar, 5 magnetic vector, 4, 7, 13 Potential integral, 12 Power, maximum received, 114-117 Power density, 17, 50 Power flux density, 17 Power gain, 23 Poynting vector, 14, 16, 115 complex, 64 time average, 64 P plane, 102 p plane, 102, 105 Principal plane, 18, 166, 178-183. See also E-plane; H-plane Projection: orthographic, 109 stereographic, 99-102, 109 Propagation constant, 57 transmission line, 77 q plane, 102-106 Quarter-wave plate, 195 Quasi-monochromatic wave, 253-254, 259, 262-263 Radar, 2 bistatic, 49, 222 monostatic, 222-223 Radar cross section, see Cross section Radar equation, 1, 49 Radar target, 2, 49-50 Radiation: efficiency, see Antenna(s), efficiency intensity, 14-20, 50, 282 lobe, 18 pattern, 13-17, 43-44 null-free, 289-291 resistance, 20-23, 34, 46, 53, 283 Raindrops, scattering from, 244 Ramo, S., 52 Random polarization, 253 Reaction, 40-41 Receiving antenna, 1, 39 Receiving pattern, antenna, 42 Reciprocity theorem, 27-28, 39, 111-114 Reflection: from arbitrarily-oriented plane, 244-252 from conductor, 221 of elliptically-polarized waves, 217-221 at interface, 207-252 total, 217 Reflection coefficients, 211-214 transmission line, 77

Reflection matrix, 226 Reflector: dihedral corner, 222-226, 240 flat plate, 221-222, 239-241 sphere, 243 trihedral corner, 224-226, 241 Resistance: high-frequency, 22 loss, 22-23, 42, 53 radiation, 20-23, 34, 46, 53, 283 Riblet, H. J., 206 Rotation rate: distance, 66 time, 64-65 Rotation sense, 54, 60-63, 68, 75-78, 84, 89, 121, 126 Rubin, R., 271, 280 Ruck, G. T., 229, 241, 252 Rumsey, V. H., 52, 54, 77, 108, 147 Satellite, 190 Saunders, W. K., 284-289, 292 Scattering: coefficients, 226-235, 239-244 circular, 235-238 cross section, 50, 229 matrix, 226-244 circular, 235-238 microwave network, 196 specular, 252 Schwarz inequality, 256 Scott, W. G., 285-288, 291-292 Silver, S., 35, 52, 189 Sinclair, G., 146, 229, 252 Sine integral, 34 Skin depth, 22 Slater, J. C., 42, 52 Smith chart, 54, 71, 77-80, 109 Snell's laws, 210-213, 246 Solid angle, 14 Sommerfeld, A., 4, 51 Soo Hoo, K. M., 285-288, 291-292 Source: electric, 3-4 equivalent, 3, 172 magnetic, 2-5 Specular scattering, 252 Sphere, 243 Spizzichino, A., 245-252 Stereographic projection, 99-102, 109 Stokes parameters, 54, 92-109, 137-138, 262-264, 278 Stratton, J. A., 213, 252 Stuart, W. D., 252

Tapered illumination, 186 Target, radar, 2, 49-50 Teeter, W. L., 196, 206 Tilt angle, 54, 59-63, 68, 73-75, 84-85, 93, 109, 121, 126 Total reflection, 217 Transmission: through interface, 207-221 line, 76 matrix, 226 Transmission coefficients, 211-212 Transmitting antenna, 1-2 Tranter, W. H., 189 Trihedral corner reflector, 224-226, 241 Turnstile antenna, 157-160, 190, 284 Unpolarized wave, 256, 259-263, 266-268 Vaillancourt, R. M., 206 Van Duzer, T., 52 Vector length, see Effective length

Wave: circular, 71 completely polarized, 257-259, 263-268

elliptic, 75, 80, 83-84, 89, 92, 95, 109, 191-206, 217-221 harmonic, 54-56 linear horizontal, 71, 149-150, 158 linear vertical, 71, 149-150, 158 monochromatic, 253-254, 257-258, 262, 269 nonmonochromatic, 257-258 nonplanar, 54, 56 nonuniform, 217 partially-polarized, 92, 253-269 plane, 56-57 polychromatic, 257-258 quasi-monochromatic, 92, 253-254, 259, 262-263 unpolarized, 256, 259-263, 266-268 Wave components: circular, 69-70 rectangular, 70 Whinnery, J. R., 52 Wolf, E., 108, 254, 269 w plane, 102-106 Zaidi, S. H. R., 285, 292 Ziemer, R. E., 189 Zucker, F. J., 1, 51, 189