research are enormous. Therefore, the limits to professional communication cannot be determined with the same degree of precision that corresponding limits in physics, chemistry, or other hard sciences can. This situation is likely to remain for many decades (if not centuries) to come.

It is also likely that for many decades and centuries to come, poets and other humanists will "see the human soul take wing in any shape, in any mood" to *unlimited* heights.

And, certainly, there is more good than harm in doing so.

## ACKNOWLEDGMENT

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## REFERENCES

[1] C. H. Perelman and L. Olbrechts-Tyteca, *The New Rhetoric: A Treatise on Argumentation*. London, England: Univ. Notre Dame Press, 1971, p. 2.
[2] C. Harkins, "Technical writer/visual apprentice," in *Proc. 25th Int. Technical Communication Conf.* (Society for Technical Communication), p. 129, 1978.
[3] G. A. Miller, "The magical number seven, plus or minus two," *The Psychol. Rev.*, vol. 63, no. 2, pp. 81-97, Mar. 1956.
[4] G. A. Barnett and C. A. Hughes, "The effects of media exposure on uncertainty reduction and voting," Int. Commun. Assoc., Chicago, IL, Apr. 1978.
[5] B. S. Greenberg, "Person-to-person communication in the diffusion of news events," *J. Quart.*, Autumn 1965.
[6] J. P. Robinson, "Mass communication and information diffusion," in *Current Perspectives in Mass Communication Research*, E. G. Kline and P. J. Tichemor, Eds. Beverly Hills, CA: Sage Publications, 1972, p. 87.
[7] J. Saltiel and J. Woelfel, "Inertia in cognitive processes: The role of accumulated information in attitude change," *Human Commun. Res.*, vol. 1, 1975.
[8] C. Shannon and W. Weaver, *The Mathematical Theory of Communication*. Urbana, IL: Univ. Illinois Press, 1949, pp. 5, 105.
[9] D. K. Berlo, *The Process of Communication*. New York: Holt, Rhinehart, and Winston, 1960, p. 72.
[10] E. M. Rogers with F. F. Shoemaker, *Communication of Innovations: A Cross-Cultural Approach*. New York: The Free Press, pp. 18-19.
[11] E. Katz, "The diffusion of new ideas and practices," in *The Science of Human Communication*, W. Schramm, Ed. New York: Basic Books, 1963, p. 80.
[12] P. F. Lazarfeld, B. Berelson, and H. Gaudet, *The People's Choice*. New York: Columbia Univ. Press, 1948, p. 152.
[13] G. A. Barnett, "An associational model for the diffusion of complex innovations," Int. Commun. Assoc., Chicago, IL. p. 15, 1978.
[14] Z. Griliches, "Hybrid corn: An exploration in the economics of technological change," *Econometrica*, vol. 25, 1957.
[15] G. Tarde, *The Laws of Imitation*, E. Eleos, Transl. New York: Holt, Rinehart, and Winston, 1903.
[16] F. S. Chapin, *Cultural Change*. New York: Century Company, 1928.
[17] H. E. Pemberton, "The curve of cultural diffusion rates," *Amer. Sociol. Rev.*, vol. 4, 1936.
[18] —, "The spatial order of diffusion," *Sociol. Social Res.*, vol. 4, 1938.
[19] R. L. Hamblin, R. B. Jacobsen, and J. L. L. Miller, *A Mathematical Theory of Social Change*. New York: Wiley, 1973, p. 48.
[20] S. C. Dodd, "Testing message diffusion in controlled experiments: Charting the distance and time factors in the interactance hypothesis," *Amer. Sociol. Rev.*, vol. 18, 1953.
[21] J. S. Coleman, *Introduction to Mathematical Sociology*. New York: Free Press, 1964, p. 494.
[22] E. M. Rogers with J. A. Danowski, R. Jara, and R. Wigand, "Interpersonal communication networks for family planning innovations," research proposal, Michigan State Univ., East Lansing, 1972.

# Fundamental Limitations in Antennas

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*Invited Paper*

*Abstract*—Four fundamental limitations in antennas have been identified in the areas of: electrically small antennas, superdirective antennas, superresolution antennas, and high-gain antennas. All exhibit roughly exponential increase in cost factors with performance increase beyond the robust range. This paper reviews these limitations.

Electrically small antennas are analyzed via spherical mode theory, with the antenna enclosed in a virtual sphere. Minimum $Q$ varies inversely as the cube of sphere radius in radian wavelengths when the radius is much less than the latter. This limits the achievable bandwidth.

Superdirective apertures require a constraint; the optimization is generally intractable except for line sources. Superdirective arrays have spacing below half-wavelength, and for small spacings a constraint is desirable to limit $Q$, tolerances, efficiency, sidelobes, etc. This is accomplished by expressing constrained directivity as a ratio of two Hermitian quadratic forms, for which a solution exists. Array $Q$ varies exponentially with directivity so only modest increases are practical.

Superresolution arrays use maximum entropy processing to improve spatial frequency resolution for short samples (short arrays), analogous to spectral analysis processing. An amplitude-tapered autocorrelation function is extended by linear least square prediction and autoregression; the latter contributes filter poles. This extension is with minimum added information, hence maximum entropy. In contrast to superdirective arrays which are all zero functions, superresolution maximum entropy uses an all pole function. Results are dependent upon the sampling subarray size and upon signal/noise ($S/N$). Required $S/N$ increases exponentially with inverse angular resolution.

Achievable gain of high-gain reflector antennas is limited by cost of the structure. For random surface errors maximum gain is proportional

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**to the mechanical tolerance ratio (antenna diameter/1σ tolerance) squared. Since cost increases rapidly with diameter and with tolerance ratio this comprises a gain limitation. Current best reflectors have maximum gain in the range of 90 to 100 dB.**

## I. Introduction

The ANTENNA field and, in fact, the parent discipline of electromagnetics has achieved a degree of maturity now that the benefits of the computer have been largely assimilated. Of the four areas in which fundamental limits apply to antennas, three are well known dating back to the period around World War II, while the fourth, although relatively new to antennas, has an older history in other disciplines. Thus it is likely that new developments in antennas will not include a new category of fundamental limitations.

It is important to carefully define the nature of a fundamental limitation. All engineering fields are full of principles which relate performance indices to generalized cost parameters such as size, weight, complexity, etc. Many of these performance indices have a comfortable range, i.e., a range of parameters in which the performance is robust. When a much higher performance index is required some of the cost parameters may increase at a rate roughly exponential. When this happens that performance index has a "fundamental limitation." In antennas there do not appear to be any rigid absolute upper limits that are meaningful. Of course, any system with a finite number of degrees of freedom, e.g., an $N$ element array with fixed spacing, has an absolute upper limit on directivity and a lower limit on bandwidth all at a single operating frequency. But the concern here is with situations where there is in principle no limit; the interesting questions then are how does the cost function vary, and how to choose an optimum operating point.

The four categories of fundamental limitations in antennas are: electrically small antennas where the bandwidth or $Q$ varies with size; superdirective apertures and arrays where directivity and robustness vary with complexity; superresolution arrays where angular resolution and robustness vary with complexity and noise including errors; and finally large antennas where gain and resolution vary with size and mechanical structure. All these categories exhibit a steeply rising cost with performance, and in all it is often desirable to design for operation somewhat above the robust region.

## II. Electrically Small Antennas

With the miniaturization of components endemic in almost all parts of electronics today, it is important to recognize the limits upon size reduction of antenna elements. These are related to the basic fact that the element's purpose is to couple to a free space wave, and the free space wavelength has not yet been miniaturized! A basic approach was taken by Chu [1] and subsequently by Harrington [2]. Since any radiating field can be written as a sum of spherical modes, the antenna, of whatever type it happens to be, is enclosed in a sphere. The radiated power can be calculated from the propagating modes within the sphere. All modes contribute to the reactive power. When the sphere is sufficiently large to support several propagating modes, this approach is of little value as the modal coefficients are difficult to calculate. With only one propagating mode, the radiated power arises primarily from that mode, analogous to the unit cell analysis developed by Oliner and Malech [3] for an electronic scanning (phased) array antenna. The utility of the Chu work becomes apparent when the sphere is too small to allow a propagating mode; all modes are then evanescent (below cutoff) and the $Q$ becomes large, as the evanescent modes contribute little real power. Note that unlike a closed waveguide there is a real part of each evanescent mode. Each mode has a $Q_n$ based on the ratio of stored energy to radiated energy, and the $Q_n$ rises rapidly when $kr$ drops below the mode number.

Turning to the derivation, the electric and magnetic fields are written in a spherical harmonic series with no azimuthal variation, e.g., the radiation is omnidirectional. Each term in such a series [4] contains Legendre functions and spherical Bessel functions. Chu [1] postulated an equivalent circuit for each spherical mode, and showed that the modal impedance (ratio of mode voltage to mode current in the equivalent circuit) is:

$$Z_n = j\,\frac{(krh_n^{(2)})'}{krh_n^{(2)}}$$

where $k = 2\pi/\lambda$, $r$ is the sphere radius and $h_n^{(2)}$ is the spherical Hankel function. The prime indicates a derivative with respect to the argument. Through the use of the Hankel function recurrence relations a continued fraction expansion of $Z_n$ can be made. From this the structure of the equivalent circuit emerges: each term in the expansion is a series $C$, shunt $L$ section, with the circuit being a ladder network. For the first mode ($n = 1$) there is only one section in the network; each integer increase in $n$ adds an $L$–$C$ section. In all cases there is a final shunt resistive load. From the equivalent circuit the input resistance and reactance are found by equating the resistance, reactance, and frequency derivative of reactance of $Z_n$ to those of the circuit. Results are

$$R_n = \frac{1}{FF^*}, \qquad F = krh_n^{(2)}(kr)$$

$$X_n = \operatorname{Re}\frac{F'}{F}.$$

The $Q$ of the $n$th mode is found from the ratio of stored to radiated energy

$$Q_n = \frac{2\omega W_n}{P_n} = \frac{\omega FF^*}{2}\left(\frac{\partial X_n}{\partial\omega} - \frac{X_n}{\omega}\right)$$

$$Q_n = \frac{krFF^*}{2}\,[krX_n' - X_n] = \frac{krFF^*}{2}\,(krX_n)'.$$

Now define $H_1 = \operatorname{Re}(h_n^{(2)*}\,h_{n+1}^{(2)})$ and $H = h_n^{(2)}h_n^{(2)*}$. After considerable manipulation:

$$Q_n = \frac{k^3r^3(H'H_1 - HH_1') - 2kr\,H^2}{H}.$$

When several modes can be supported, the overall $Q$ is

$$Q = \frac{\displaystyle\sum_{n=1}^{N}\frac{a_na_n^*Q_n}{(2n+1)}}{\displaystyle\sum_{n=1}^{N}\frac{a_na_n^*}{(2n+1)}}$$

with $a_n$ the excitation coefficient of the $n$th mode. Higher modes become evanescent for roughly $kr < 1$; the $Q$ becomes:

$$Q = \frac{1 + 3k^2r^2}{k^3r^3\,[1 + k^2r^2]}.$$

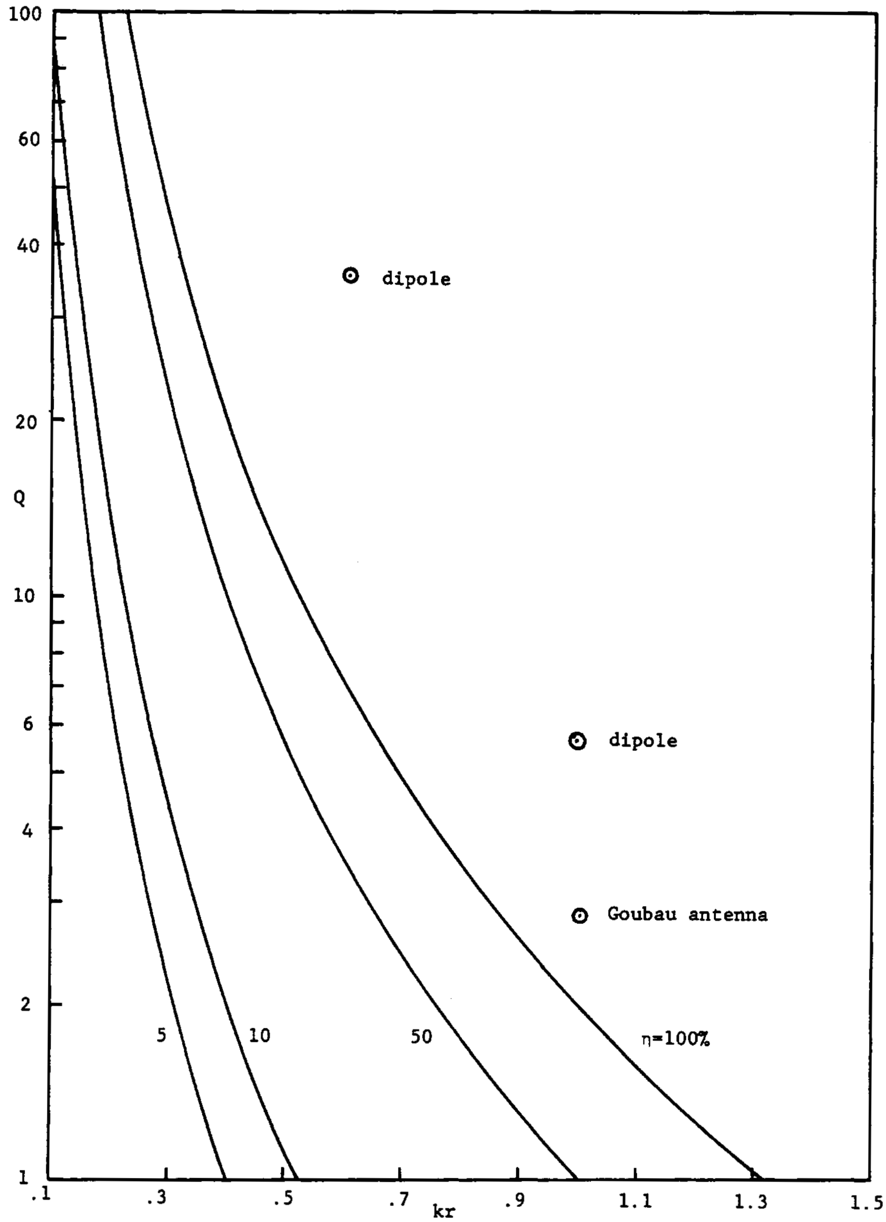


Fig. 1. Chu-Harrington fundamental limitations for single-mode antenna versus efficiency.

The efficiency $\eta$ will be discussed later. This $Q$ is for the lowest TM mode. When both a TM mode and a TE mode are excited, the value of $Q$ is halved. Note that for $kr \ll 1$ the $Q$ varies inversely as the cube of sphere radius in radian wavelengths. The importance of the Chu result is that it relates the lowest achievable $Q$ to the maximum dimension of an electrically small antenna, and this result is independent of the art that is used to construct the antenna within the hypothetical sphere, except in determining whether a pure TE or pure TM mode, or both, is excited. Since the $Q$ grows rapidly (inverse cube) as size decreases, this indeed represents a fundamental limit which has only been approached but never even equaled, much less exceeded.

The foregoing derivation assumed a lossless antenna, except for radiation resistance. If the antenna is lossy the effect is to insert a loss resistance in series with the radiation resistance, so its effect on $Q$ is apparent. Fig. 1 plots single mode $Q$ for various efficiencies. Bandwidth is derived from $Q$ by assuming that the antenna equivalent is a resonant circuit with fixed values. Then the fractional bandwidth which is defined as the normalized spread between the half-power frequencies is:

$$\text{Bandwidth} = \frac{f_{\text{upper}} - f_{\text{lower}}}{f_{\text{center}}} = \frac{1}{Q}.$$

For $Q \gg 1$ this relationship is meaningful as the fixed resonant circuit is a good approximation to the antenna. But for $Q < 2$, the respresentation is no longer accurate. However, the curves are still useful for low $Q$ even though imprecise. An octave bandwidth for example requires $Q = \sqrt{2}$ and with no losses this requires a minimum antenna length of $0.365\lambda$. Since most small antennas are loops or dipoles, which do not use the spherical volume efficiently, an actual octave antenna is significantly larger, often larger than $\lambda/2$. From this it is clear that improving bandwidth for an electrically small antenna is only possible by fully utilizing the volume in establishing a TM and TE mode, or by reducing efficiency. The latter is typified by the ELF SQUID which can be extremely small in wavelengths yet possess a sizeable bandwidth [5]. However, the efficiency is also extremely low.

It is useful to evaluate simple electrically small antennas against the minimum $Q$ curve. A short dipole has impedance of

$$Z \simeq 20k^2h^2 - j\,\frac{120\,(\ln h/a - 1)}{\tan kh}$$

where $h$ is the half-length and $a$ is the radius. $Q$ is then:

$$Q \simeq \frac{6\,(\ln h/a - 1)}{k^2h^2\,\tan kh}.$$

The antenna has minimum $Q$ for low $h/a$, but the approximate formulas above are not accurate for fat dipoles. Using $h/a = 50$, short dipole points are shown in Fig. 1. To obtain performance closer to the minimum $Q$ curve the spherical volume must be used more effectively; a dipole is essentially one dimensional. A more effective design using three dimensions is a clover leaf dipole with coupling loops over a ground plane (or double clover leaf dipole without ground plane) developed by Goubau [6]. This antenna in symmetric form requires $ka = 1.04$ and gives an octave bandwidth, or $Q = \sqrt{2}$. However, if the performance is attributed equally to electric and magnetic modes, the single mode $Q$ is $2\sqrt{2}$. Fig. 1 includes the effect of a matched resistive load.

Electrically small antennas are also superdirective, although their normal directivity is 1.5. However, using the definitions in Section III, if the electrically small antenna were uniformly excited its directivity would be $4r/\lambda$ where $r$ is the radius as before. Since $r \ll \lambda$ the directivity has increased to 1.5. Thus it is not surprising that electrically small antennas also suffer from the classic superdirective ailments: low radiation resistance, sensitive tolerances, and narrow bandwidth. And so electrically small antennas are superdirective in all respects.

## III. Superdirectivity

### A. Historical Notes

A useful operational definition of antenna superdirectivity (formerly called supergain) is directivity[1] higher than that obtained with the same antenna configuration uniformly excited (constant amplitude and linear phase). Excessive array superdirectivity inflicts major problems in low radiation resistance (hence low efficiency), sensitive excitation and position tolerances, and narrow bandwidth. Superdirectivity applies in principle to arrays of isotropic elements although, of course, actual antenna arrays are composed of nonisotropic elements.

Probably the earliest work[2] on the possibility of superdirectivity was by Oseen [7]. A limited endfire superdirectivity using a monotonic phase function was accomplished by Hansen and Woodyard [8]. Another early contributor was Franz [9]. Schelkunoff [10] in a classic paper on linear arrays discussed, among other topics, array spacings less than $\lambda/2$, showing how equal spacing of the array polynomial zeros over that portion of the unit circle represented by the spacing gives superdirectivity. The field received wide publicity when La Paz and Miller [11] purported to show that a given aperture would allow a maximum directivity, and when Bouwkamp and De Bruijn [12] showed that they had made an error and that there was no limit on theoretical directivity. Thus the important theorem: a fixed aperture size can achieve (in theory) any desired directivity value. This theorem is now widely recognized, but the practical implications are less well known. Bloch *et al.* [13] say that the theorem has been rediscovered several times; the practical limitations of superdirectivity occur as a surprise to systems engineers and others year after year! In 1946, a burst of war-time research reporting occurred. Reid [15] generalized the Hansen–Woodyard endfire superdirectivity to include an element pattern. Uzkov [16] derived the endfire directivity as $d \to 0$. And Dolph [17] invented the widely used Dolph–Chebyshev array distribution wherein the equal level oscillations of a Chebyshev polynomial are used to produce an array pattern with equal level sidelobes. To follow this last development, Riblet [18] developed Dolph–Chebyshev arrays for spacing below $\lambda/2$, i.e., superdirective. DuHamel [19] and Stegen [20] developed complimentary advances in the computation of Dolph–Chebyshev coefficients and directivity. Maximum directivity for an array with fixed spacings was derived, for acoustic arrays, by Pritchard [21].

Superdirective aperture design thus requires a constraint, and will be discussed later. Arrays with fixed number of elements and spacing due to the finite number of variables do not, as clearly there is an excitation that provides maximum directivity. The Lagrangian process for determining this maximum will be discussed, followed by a discussion of Chebyshev array design.

### B. Maximum Directivity for Fixed Spacing

Consider an array with a given number $N$ of elements and a fixed spacing. The array coefficients that produce this maximum directivity are found by the Lagrange multiplier method [22]. Since the set of coefficients can be scaled by any common factor, it is convenient to make the sum unity. Then for an array of an even number of elements

$$\sum_{n=1}^{N/2} A_n = 1.$$

This allows the inverse directivity of a broadside array (which will be minimized) to be expressed as

$$\frac{1}{G} = 2\sum_{n=1}^{N/2}\sum_{m=1}^{N/2} A_nA_m[f_{n+m-1} + f_{n-m}]$$

where

$$f_n = \operatorname{sinc} nkd.$$

An odd number requires a change of limits and introduction of the Neumann number.

Applying the variational method gives a set of equations, where the Lagrangian multiplier is $\mu$

$$\sum_{n=1}^{N/2} A_n[f_{n+m-1} + f_{n-m}] + \mu = 0, \qquad m = 1, 2, \cdots, N/2.$$

The set of equations has $N/2 + 1$ unknowns and $N/2$ equa-

[1] Directivity is the ratio of peak field intensity at any far field radius from the antenna to the integral of field intensity over a sphere of that radius.

[2] See Bloch *et al.* [14] for a list of early references.

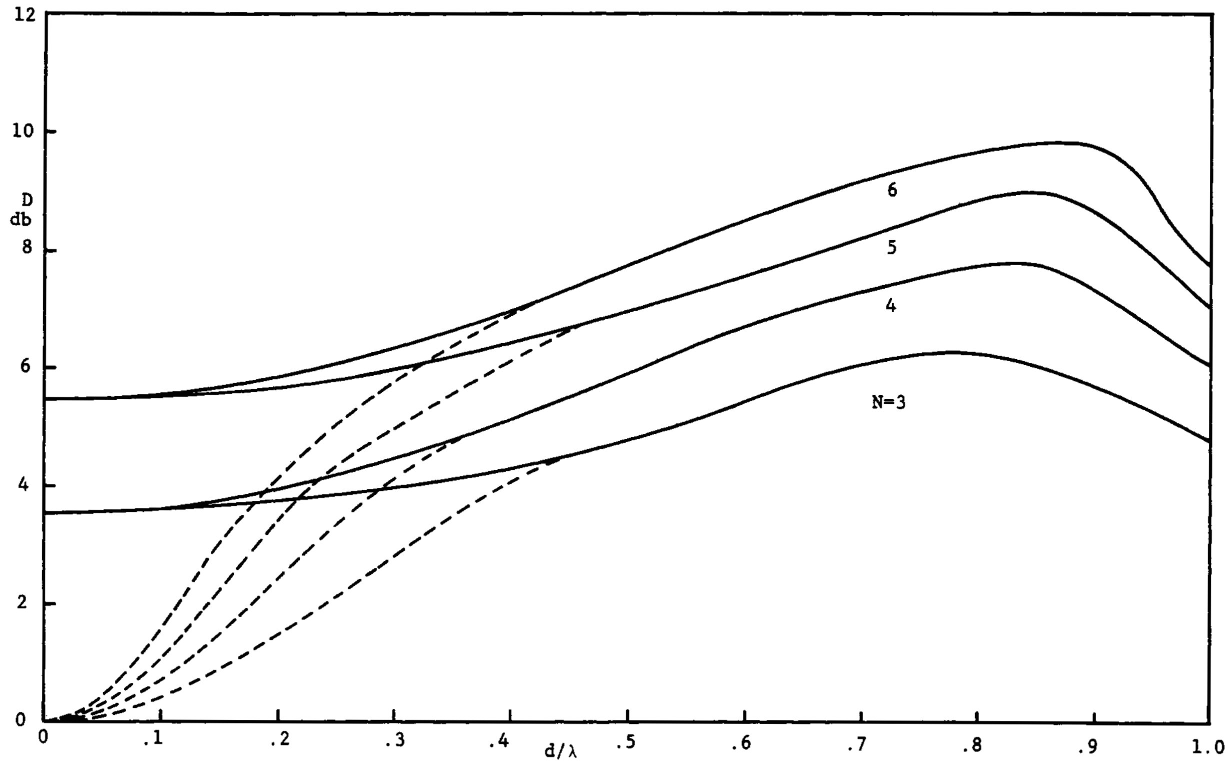


**Fig. 2. Maximum directivity for fixed spacing (dashed curves for uniform amplitude).**

tions; again the sum equation completes the set. The multiplier can be eliminated by substituting from the $m = 1$ equation

$$\mu = -\sum_{n=1}^{N/2} A_n [f_n + f_{n-1}].$$

This gives a set of $N/2$ simultaneous equations in the array coefficients. These were solved by hand for $N = 3, 5, 7$ by Pritchard [21] and can readily be solved by computer for any reasonable $N$. Writing the set of equations:

$$\sum_{n=1}^{N/2} A_n = 1$$

$$\sum_{n=1}^{N/2} A_n [f_{n+m-1} + f_{n-m} - f_{n-1} - f_n] = 0, \qquad m = 2, 3, 4, \cdots, M.$$

The right-hand side vector is $(1, 0, \cdots)$. Although matrix inversion can be used to solve for the array coefficient vector, it is much faster to use an equation solver such as the Crout reduction [23]. For small spacing (in wavelengths) multiple precision calculations may be needed due to the subtraction of large numbers. Calculations of maximum broadside directivity were made with the results shown in Fig. 2. Also shown is the directivity for uniform excitation. Above $d/\lambda = .5$, the two are very close. Also some minor oscillations in the directivity curves have been smoothed out, as they are not important here. The coalescing of pairs of curves at zero spacing occurs because arrays of $2N$ and $2N - 1$ elements have the same number of degrees of freedom. For endfire beams, in the directivity expression, $f_n$ is replaced by $f_{2n}$ and this carries over to the simultaneous eqns. The element coefficients will now be complex, due to the Hansen–Woodyard endfire aperture. For progressive phasing (phasing matches space wave phasing), the directivity is $N$ for spacing of both $\lambda/4$ and $\lambda/2$. However, at $\lambda/4$ spacing, the optimum excitation in amplitude and phase yields a directivity of $N^2$.

### *C. Dolph–Chebyshev Arrays*

A symmetrically tapered (amplitude) distribution over the array or aperture is associated with a pattern having lower sidelobes than those of the uniform (amplitude) array. Lowering the sidelobes broadens the beamwidth and lowers the excitation efficiency. The latter is the ratio of directivity to directivity for uniform excitation. Some improvement in both beamwidth and efficiency is obtained by raising the farther out sidelobes. Intuitively one might expect equal level sidelobes to be optimum for a given sidelobe level. A method of accomplishing this for a half wave spaced broadside array was invented by Dolph [17] who recognized that the Chebyshev polynomials were ideally suited: in the range $\pm 1$ there are oscillations of unit amplitude, while outside this range the polynomial becomes monotonically large.

However, a direct correspondence between this polynomial and the array polynomial is not feasible, becasue the main beam must be symmetric and have zero slope at its center. The $N$ element array has $N - 1$ zeros while the $N$th-order Chebyshev polynomial has $N$ zeros. Thus an $N - 1$ degree Chebyshev is used. Part of the $x > 1$ region is mapped onto one side of the main beam while the oscillatory portion of the Chebyshev is mapped out once onto the sidelobes on one side of the main beam. The transformation from $T_{N-1}(z)$ to space factor $F(u)$, with $u = \frac{1}{2} kd \sin\theta$, is $z = z_0 \cos u$. Element spacing is $d$. Sidelobe ratio (SLR)[3] is given by:

$$\mathrm{SLR} = T_{N-1}(z_0)$$

or inversely,

$$z_0 = \cosh \frac{\operatorname{arc\,cosh} \mathrm{SLR}}{N-1}.$$

Dolph's derivation and the formulas of Stegen are limited to $d \geqslant \lambda/2$. Riblet [18] showed that this restriction could be removed, but only for $N$ odd. For spacing below half wave, the space factor is formed by starting at a point near the end of the Chebyshev $\pm 1$ region,[4] tracing the oscillatory region to the other end, then retracing back to the start end and up the monotonic portion to form the main beam half. Since the $M$th order Chebyshev has $M - 1$ oscillations, which are traced twice, and since the trace from 0 to 1 and back forms the center sidelobe (in between the trace out and back), the space factor always has an odd number of sidelobes each side, or an even number of zeros. Hence only an odd number of elements can be formed into a Chebyshev array for $d < \lambda/2$. The pattern is given by

$$T_M(a \cos \psi + b)$$

$$a = \frac{z_0 + 1}{1 - \cos kd}$$

$$b = \frac{z_0 \cos kd + 1}{\cos kd - 1}$$

where as before $\psi = kd \sin \theta$. The value of $z_0$ is different:

$$z_0 = \cosh \frac{\operatorname{arc\,cosh} \mathrm{SLR}}{M}$$

$$\mathrm{SLR} = T_M(z_0).$$

Formulas have been developed by DuHamel [19], Brown [25], [26] Salzer [27], and Drane [28], [29]. Those of Drane will be used here as they are suitable for computer calculation of superdirective arrays. The array amplitudes are

$$A_n = \frac{\epsilon_n}{4M} \sum_{m=0}^{M_1} \epsilon_m \epsilon_{M_2-m} T_n(x_n) \cdot [T_M(ax_n + b) + (-1)^n T_M(b - ax_n)]$$

where $\epsilon_i = 1$ for $i = 0$ and equal to 2 for $i > 0$; $x_n = \cos n\pi/M$. The integers $M_1$ and $M_2$ are, respectively, the integer parts of $M/2$ and $(M+1)/2$. This result is valid for $d \leqslant \lambda/2$. Small spacings (highly superdirective arrays) may require multiple precision due to the subtraction of terms. Many arrays are half wave spaced; for these the $a$ and $b$ reduce to

$$a = \frac{z_0 + 1}{2}$$

$$b = \frac{z_0 - 1}{2}.$$

For half-wave spacing the two approaches give identical results! In fact, due to the properties of the Chebyshev polynomial, the two space factors, in precursor form, are equal

$$T_{N-1}\left(z_0^{N-1} \cos \frac{\psi}{2}\right) \equiv T_M\left(\frac{z_0^M (\cos \psi + 1) + \cos \psi - 1}{2}\right), \quad N - 1 = 2m$$

where the superscripts on $z_0$ indicate that each must be chosen for the proper form. Since many computers have no inverse hyperbolic functions, it is convenient to rewrite the $z_0$ as:

$$z_0 = \tfrac{1}{2}\left[\mathrm{SLR} + \sqrt{\mathrm{SLR}^2 - 1}\right]^{1/M} + \tfrac{1}{2}\left[\mathrm{SLR} - \sqrt{\mathrm{SLR}^2 - 1}\right]^{1/M}$$

Du Hamel [19] extended the Chebyshev design principle to endfire arrays, but only for $d < \lambda/2$. In fact, to avoid a back lobe spacing is customarily made $\leqslant \lambda/4$. To do this for any scan angle, $\psi$ is modified as usual to

$$\psi = kd (\sin \theta - \sin \theta_0)$$

where $\theta_0$ is the scan (main beam) angle, and the interelement phase shift is $kd \sin \theta_0$. Coefficients $a$ and $b$ become

$$a = -\frac{3 + z_0 + 2\sqrt{2(z_0 + 1)} \cos kd}{2 \sin^2 kd}$$

$$b = \frac{(\sqrt{z_0 + 1} + \sqrt{2} \cos kd)^2}{2 \sin^2 kd}.$$

Returning to the broadside array, $Q$ is given by Lo *et al.* [24] as

$$Q = \frac{\sum_{n=1}^{N} A_n^2}{\sum_{n=1}^{N} \sum_{m=1}^{N} A_n A_m \operatorname{sinc}(n - m) kd}.$$

$Q$, which is the inverse of fractional bandwidth, is plotted for Dolph–Chebyshev arrays versus $d/\lambda$ in Fig. 3. The $N = 3$ array has a log–log slope of $Q$ versus spacing of 4:1, while the $N = 5$ array has a slope of 8:1. The $N = 7$ and 9 arrays have even higher slopes. For small values of $Q$, the curve is not too accurate.

Bandwidth appears to be more restrictive than tolerances; the $N = 3$ array to be practical requires a spacing of the order of $0.1\lambda$ or larger. And the $N > 3$ arrays are even less forgiving.

Directivity is given by

$$G = \frac{\left(\sum_{n=1}^{N} A_n\right)^2}{\sum_{n=1}^{N} \sum_{m=1}^{N} A_n A_m \operatorname{sinc}(n - m) kd}.$$

Note the similarity to the result for $Q$. Two cases of Chebyshev superdirective arrays have been calculated to illustrate the variation of $Q$ with directivity increase. First is a two wavelength array of isotropic elements, which with half-wave spacing has a minimum directivity of 5. This two wavelength aperture is occupied by 7, 9, 11 or 13 elements. Table I shows the element spacing, directivity and $Q$, for a design SLR of 20 dB. The $Q$ values are shown in Fig. 4, where the straight line has been fitted to the data, and is

$$\log Q = 3.3175 (G - 5).$$

The form of the equation, with the normal directivity ($G_0 = 5$)

[3] Ratio of main beam peak to first sidelobe peak.
[4] The exact starting point depends on $N$ and $kd$.

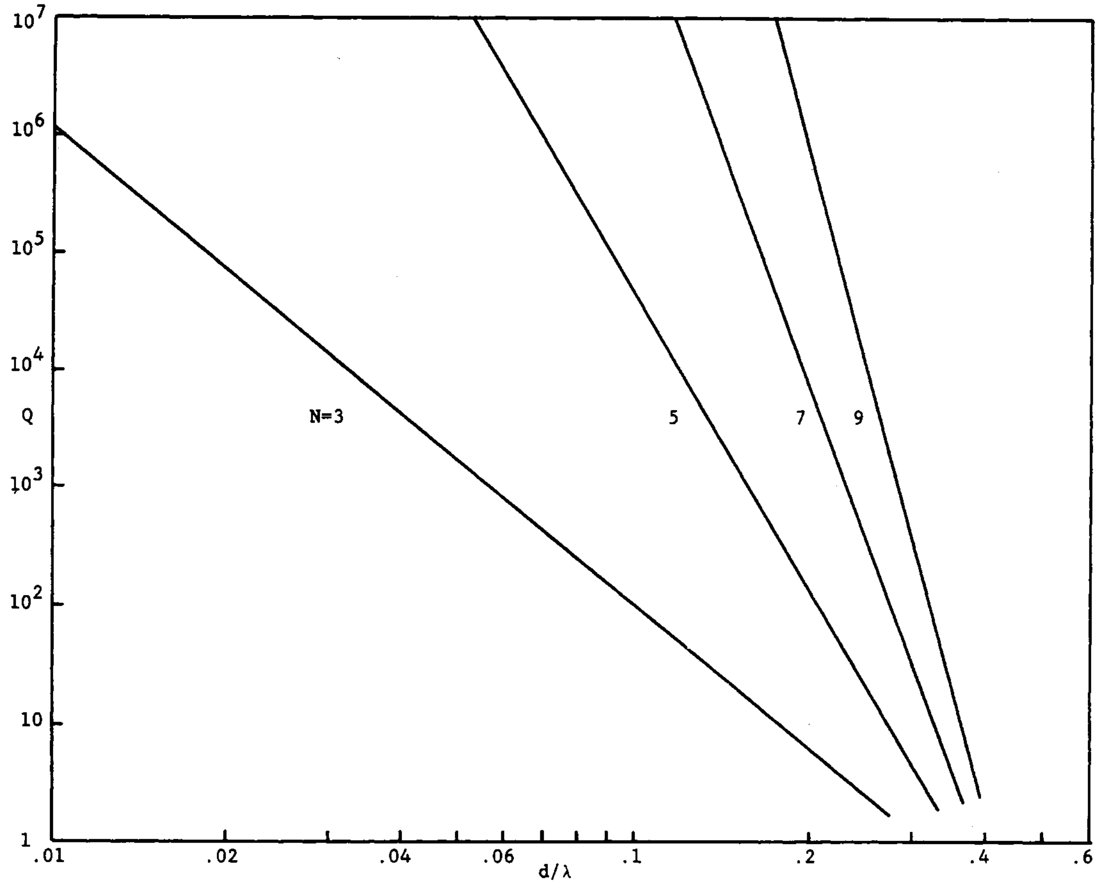


Fig. 3. $Q$ of broadside arrays.

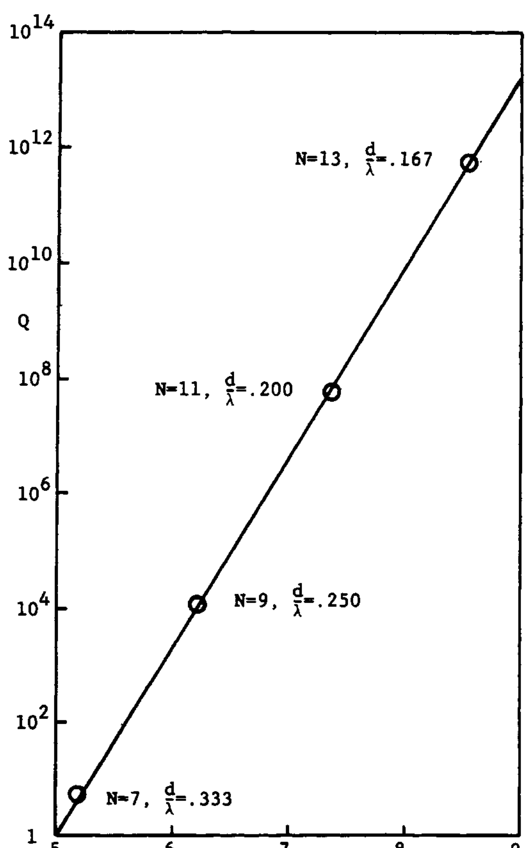


Fig. 4. $Q$ versus directivity, 20 dB SLR, array length 2λ.

TABLE I
Two Wavelength Array, SLR = 20 dB

| N | d/λ | G | Q |
|---|---|---|---|
| 5 | .500 | 4.69 | 1.7 |
| 7 | .333 | 5.18 | 7.0 |
| 9 | .250 | 6.21 | $1.2 \times 10^4$ |
| 11 | .200 | 7.36 | $5.8 \times 10^7$ |
| 13 | .167 | 8.54 | $5.5 \times 10^{11}$ |

subtracted, was suggested by calculations of Rhodes. To show the effects of SLR, Table II is a repeat for SLR = 10 dB. And Fig. 5 shows a straight line fit, with coefficient 4.1475. Higher sidelobes increase directivity and $Q$. Calculated points for $N = 13$ are inaccurate, double precision on a 32 bit machine is inadequate. Thus the $Q$ increases approximately exponentially with directivity above the normal value. For modest increases above normal directivity, it may be inferred from Rhodes that the $Q$ curve has minor oscillations. Note the very rapid increase of $Q$ with directivity: even a 10 percent increase produces a bandwidth limited to a few percent. However, the Chebyshev design may not give the lowest $Q$; a constrained synthesis is apparently necessary to produce the lowest $Q$ for a given directivity. Newman *et al.* [30] have done this for endfire arrays, but only for two cases.

The second case is a half-wavelength aperture, originally computed by Yaru [31] which with two isotropic elements gives a maximum directivity of 2. Into this small aperture is placed 3, 5, 7, or 9 elements. Table III gives the pertinent

TABLE II
TWO WAVELENGTH ARRAY, SLR = 10 dB

| N | d/λ | G | Q |
|---|---|---|---|
| 5 | .500 | 4.89 | 1.8 |
| 7 | .333 | 5.40 | 37 |
| 9 | .250 | 6.19 | $7.6 \times 10^4$ |
| 11 | .200 | 6.99 | $3.5 \times 10^8$ |
| 13 | .167 | 7.74 | $3.3 \times 10^{12}$ |

TABLE III
HALF-WAVELENGTH ARRAY, SLR = 20 dB

| N | d/λ | G | Q |
|---|---|---|---|
| 3 | .25 | 2.11 | 2.6 |
| 5 | .125 | 3.17 | $8.7 \times 10^3$ |
| 7 | .0833 | 4.40 | $5.7 \times 10^8$ |
| 9 | .0625 | 5.66 | $1.7 \times 10^{14}$ |

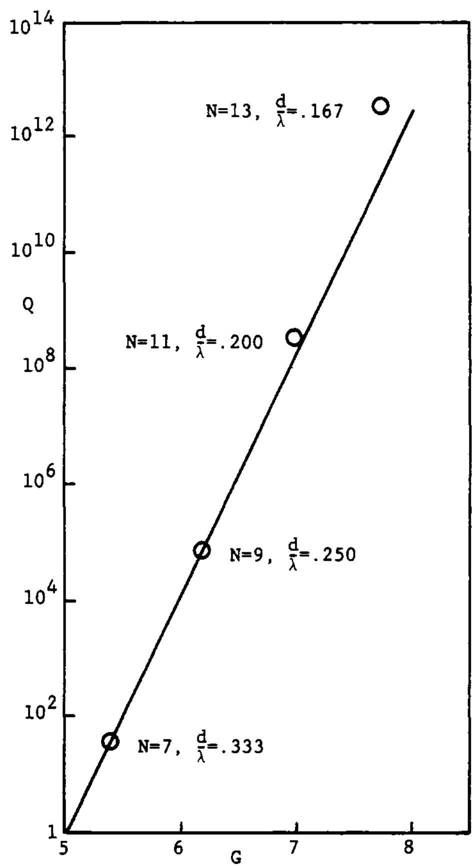


Fig. 5. $Q$ versus directivity, 10 dB SLR, array length $2\lambda$.

parameters, again for a 20-dB SLR. A straight line fit of the same type, $\log Q = a(G - 2)$, was tried but the fit is less good. It is not known whether this is due to computing errors or to the small size of the aperture in wavelengths.

### *D. Superdirective Apertures*

Much more difficult than the array is the superdirective aperture, as a constraint is always required. $Q$ was first used as a constraint. Taylor [32] defined a supergain ratio $\kappa$ which is the ratio of total (radiated plus stored) energy in the field of the antenna to radiated energy. It is related to $Q$ for simple sources by:

$$\kappa = 1 + Q.$$

Rhodes [33] also related $Q$ and $\kappa$ for various types of strip and line sources. Rhodes [34], [35] maximized directivity of a line source subject to a constraint on $\kappa$ using the eigenfunctions of the source: prolate spheroidal functions. These are particularly suited to allow a simple derivation as they are doubly orthogonal: over infinite limits, which fits the total energy integral over all angles, and over finite limits, which fits the radiated energy integral over real angles. He found a roughly linear relationship between $\log Q$ and directivity, with a coefficient of roughly 2. This substantiates the suspicion that the Chebyshev design may result in a higher $Q$ than necessary for a given directivity, since the coefficients from Tables I and II are in the 3–4 range.

A review of constrained synthesis is beyond the scope of this paper so only a brief mention will be made. Except for the work of Rhodes discussed above, constrained synthesis applies to either arrays, or to a sampled aperture so that a discrete vector of variables is used. The directivity to be maximized and its constraint are written as the ratio of two Hermitian quadratic forms. Then a solution exists and can be found in a direct manner. The basic framework was developed by Gilbert and Morgan [36] and Uzsoky and Solymar [37] and was extended by Lo *et al.* [24]. A review paper is by Cheng [38]. Directivity or $G/T$ can be directly minimized as mentioned earlier or maximized, subject to a constraint on $Q$, sidelobe structure, tolerances, or any combination of these.

## IV. SUPERRESOLUTION

Superresolution is the production of an array pattern with one or more main beams that are appreciably narrower than the nominal Rayleigh resolution $\theta = \lambda/L$. It differs from superdirectivity in that the latter produces a space factor with a narrow beamwidth (implied by the high directivity) through choice of array element coefficients; the spatial frequency function contains only zeros, which are those of the array polynomial. In contrast, superresolution in essence uses a filter function that is an all pole model as will be explained briefly below.

Superresolution was invented by Burg [39], [40] and Capon [41] for use in estimating frequency spectra of a set of measurements at intervals of time. The Burg technique, known as maximum entropy spectral analysis (ME for short) is advantageous when the data sample is short (in terms of one cycle of the lowest frequency component). A related technique developed by Capon is called maximum likelihood (ML for short) and is also superresolution but somewhat less so. Both of these techniques can be applied to spatial frequency resolution using an array. Conventional spectral analysis uses a window function on the data, then applies an FFT. Directly analogous is the conventional multiple beam array where the window function is now the array excitation taper, and the beam forming network provides the FFT. Indeed the Butler beam former is literally a microwave layout of the FFT as has been observed. ME is a salient modification of an older spectral estimation technique used before computers made direct FFT easy: autocorrelation, sometimes called moving average. In this early technique the autocorrelation function is calculated from the data and then windowed. For short data sets, zeros are added and then the FT is taken. This all zero model is limited to Rayleigh resolution. In the ME method, the windowed autocorrelation function is extended not by adding zeros but by adding information of which a minimum amount is new. As shown by van den Bos [42], the Burg technique is equivalent to a combination of least squares linear prediction and autoregression. The latter is a statistical technique of fitting an all pole filter to the data. The resulting prediction error filter is minimum phase and the corresponding autocorrelation matrix is positive definite and Toeplitz. The ML technique does not extrapolate the autocorrelation function, but applies a window shape that varies with the wave number so as to minimize interfering spectral components, i.e., minimum variance. Burg [43] has shown that these two are closely related when sampling times are equal and regular. Key to the improved performance of both is that they adapt to the characteristics of the data. Extensive reference sources on superresolution are a reprint book Childers [44] and a workshop on spectrum estimation RADC [45]. Other useful references are Barnard [46], Griffiths and Prieto-Diaz [47], and Landus and Lacoss [48].

Array superresolution is in general analogous to that involved in spectral analysis but there are several minor differences. A linear array provides a two-dimensional data set, in space and time. However, the time dimension can be effectively eliminated by narrow-band filtering, leaving spatial frequency as the analog to frequency. The signal from an array is usually of long duration, as opposed to the common case of "single snapshot" time data, e.g., earthquake or seismic data. Proceeding with ME applied to an $N$ element antenna array, let $v = (d/\lambda) \sin \theta$ where $d/\lambda$ is the element spacing in wavelengths, and let $w = \exp(j2\pi v)$. Linear prediction estimates $N$-$K$ values of $v$ based on previous values:

$$\hat{v}_n = \sum_{k=1}^{K} A_k v_{n-k}.$$

The error between true value and predicted value is

$$\epsilon_n = v_n - \hat{v}_n = -\sum_{k=0}^{K} A_k v_{n-k}.$$

When this error vector is minimized in a least squares sense, its spectrum is equivalent to white noise; the filter is maximum entropy. Taking $Z$ transforms

$$E = -\frac{\text{constant}}{\left|\prod_{k=1}^{K} (1 - B_k w)\right|^2}.$$

The denominator of this expression is the filter with $K$ poles. No convolution is incurred in this ME formulation so the fine structure is not smoothed. The autocorrelation matrix has elements $R_{i-k}$ where

$$R_n = \sum_{m=-\infty}^{\infty} P_m P_{m+n}$$

where $P_m$ is a signal at an angle in space. The matrix is symmetric Toeplitz. The equations are determinate and could be solved simultaneously. However, for large matrices, an iterative technique is advantageous. An iterative set of the excitation coefficients $A_k^K$ is calculated, where the superscript $K$ is for the $K$th iteration. In the Burg algorithm the set obeys

$$\sum_{k=1}^{K} A_k^K R_{j-k} = -R_j, \qquad j = 1, K$$

with errors

$$\sigma_K^2 = R_0 + \sum_{k=1}^{K} A_k^K R_k^*.$$

The iterative equation is

$$a_k^K = -\frac{1}{\sigma_{K-1}^2}\left[R_K + \sum_{k=1}^{K-1} A_k^{K-1} R_{K-1}\right].$$

The previous derivation can be viewed as an adaptive array [49] of the sidelobe canceller type. However, when a snapshot of discrete noise sources is taken, the array samples are coherent and the resulting autocorrelation matrix has one eigenvalue per discrete noise source and is not Toeplitz. To allow ME techniques to be used spatial incoherence can be introduced by a pseudo-Doppler synthetic movement of the $K$ data sample for example. As an example of the superresolution Gabriel [49] has calculated the equivalent pattern of an eight element half wave spaced array with equal sources of 30 dB $S/N$ at 18° and 22° (see Fig. 6). The beam peaks are shown sharp because they result from the inverse of a pole. A superdirective array in contrast (see Section III) with half-wave spacing has essentially the Rayleigh beamwidth. Other references on array superresolution are Borgiotti and Kaplan [50], McDonough [51], Evans [52], White [53], and Strickland [54].

The ME method is sensitive to noise and to the ratio of sampling subarray $K$ to the overall array $N$. Too small a value of $K$ broadens the resolution, while too large a value produces splitting of peaks and extraneous peaks. A value of $K = N/2$ is roughly acceptable King [55] although sophisticated methods have been developed for determining the optimum value Akaike [56]. Assuming that an acceptable value of $K/N$ is used, the angular resolution that can be realized is limited by $S/N$. Gabriel [49] has recently shown that for two equal incoherent sources the log resolution varies exponentially with $S/N$, see Fig. 7. And so the fundamental superresolution limit has appeared for the simplest case of two equal sources:

$$\frac{S}{N} \simeq \left(\frac{\theta_{3\ \text{Rayleigh}}}{\theta_{3\ \text{superresolution}}}\right)^{3.26}$$

where $S/N$ is a power ratio. Perhaps it will be possible to derive a general relationship for ME superresolution for several sources of variable relative levels and spacings. There is already a basic limitation for the resolution of two signals in noise using the Cramer–Rao bound; see Manasse [57], Sklar and Schweppe [58], and Pollon and Lank [59].

## V. High-Gain Antennas

The final category of fundamental limitations concerns large antennas which exhibit high gain not due to superdirectivity but due to large area in square wavelengths. Almost all high gain antennas are reflectors, since large arrays are usually more expensive. The discussion here is limited to reflectors steerable in both planes. Cost of a reflector is affected by size of course since a given diameter requires a volume of structural support members. However, the cost is also a function of the

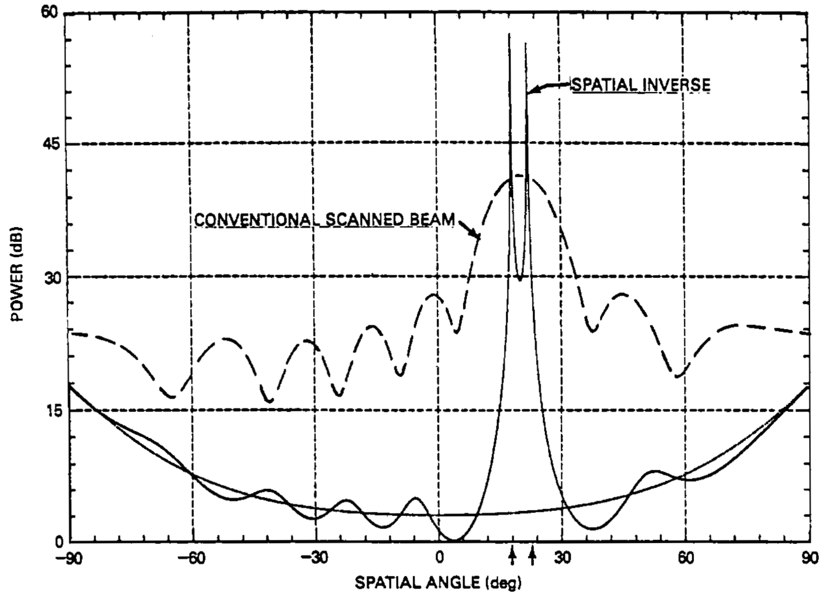


Fig. 6. Equivalent pattern of 8 element maximum entropy array with two sources (courtesy of W. F. Gabriel).

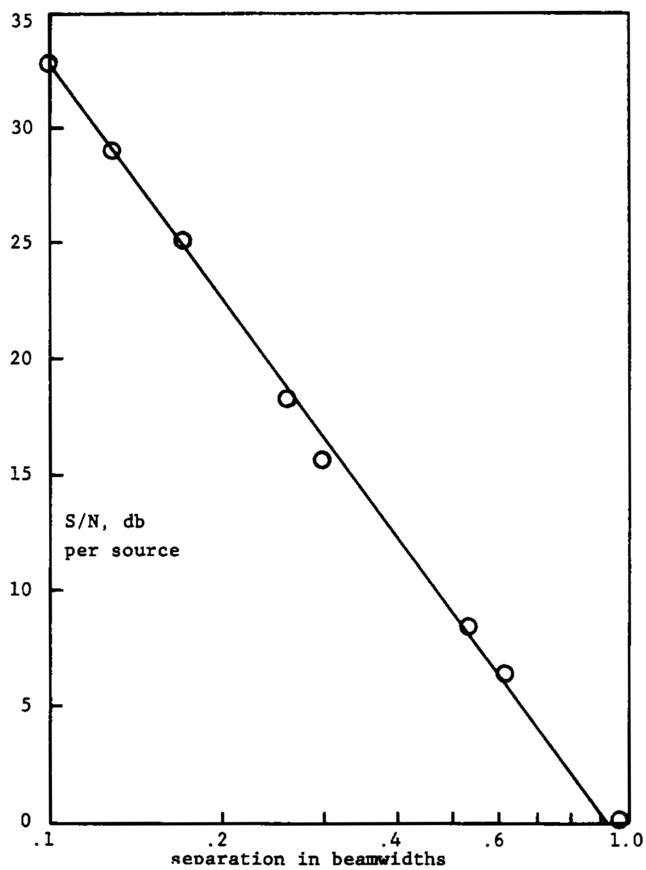


Fig. 7. Resolution limit for two sources with Gaussian noise, from simulations (courtesy of W. F. Gabriel).

manufacturing tolerance ratio $R$, which is the ratio of the largest dimension to the $1\sigma$ error $\epsilon$

$$R = D/\epsilon.$$

Reflector gain is related to $R$ as a high-quality pattern (low sidelobes) requires roughly $\epsilon \leqslant \lambda/40$. Typical reflector efficiency varies from below 0.5 for large $\epsilon$ to 0.8 for a shaped high efficiency reflector. For broad band reflectors a value of 0.5 is representative and will be used here. Gain then is

$$G = \pi^2 D^2 / 2\lambda^2$$

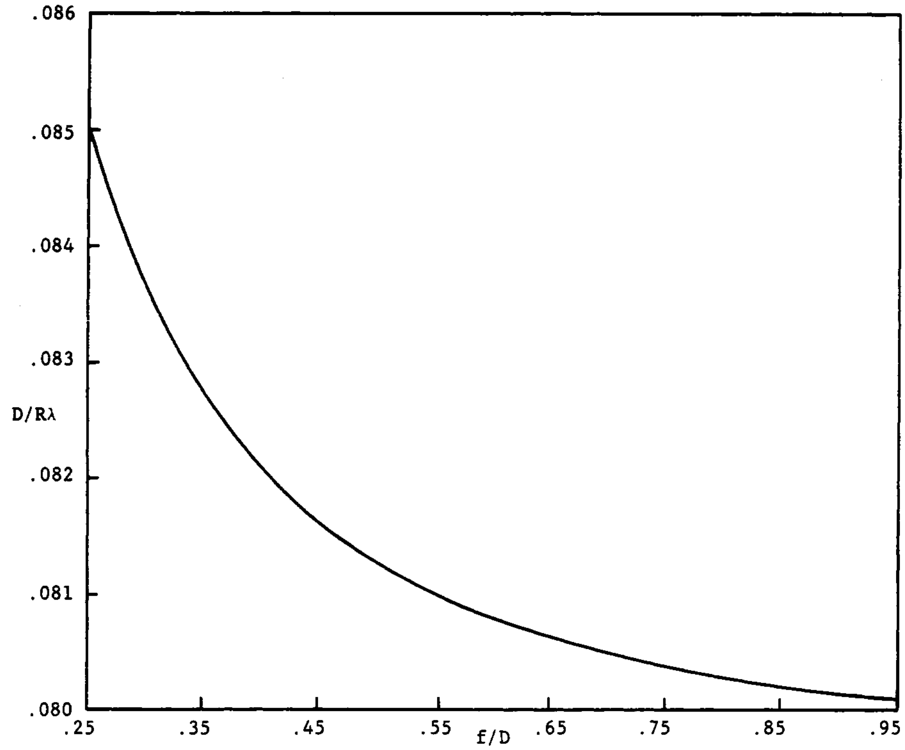

Fig. 8. Reflector diameter/tolerance ratio versus $f/D$.

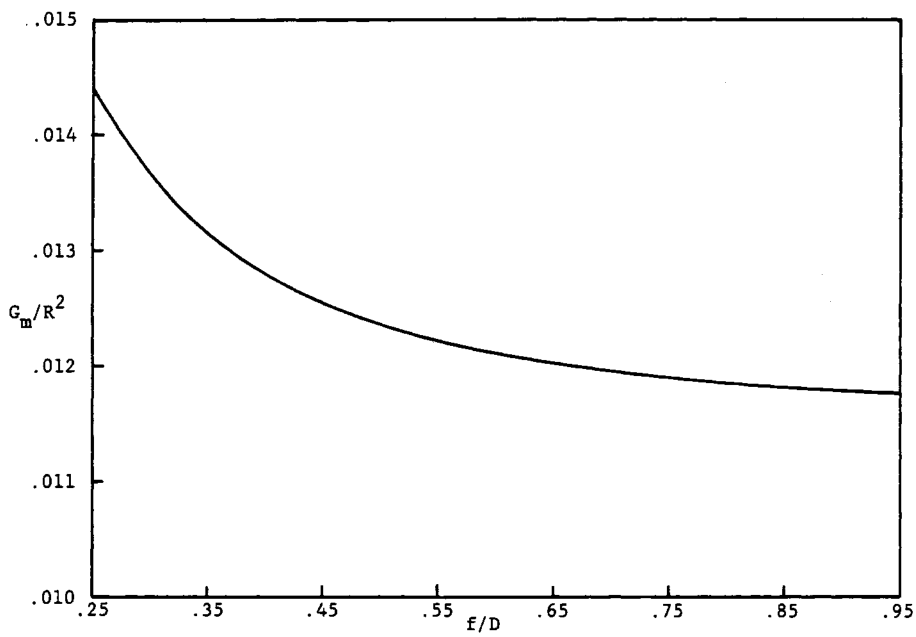

Fig. 9. Maximum gain/tolerance ratio squared versus reflector $f/D$.

which for the $\epsilon = \lambda/40$ value gives $G = \pi^2 R^2/3200$. This is not the maximum gain as the pattern can suffer some degradation yet realize higher gain. Assuming the reflector surface errors are random gain can be found from the formulas of Ruze [60] for high $f/D$ values. For deep dishes (perhaps $f/D < 1$) an improved formula has been developed by Wested [61]:

$$G = \frac{1}{2}\left(\frac{\pi D}{\lambda}\right)^2 \frac{A+1}{A + \exp(4\pi\epsilon/\lambda)^2}$$

where aperture efficiency of 0.5 is included, and the correction factor $A$ is

$$A = \frac{1}{(4f/D)^2 \ln(1 + 1/(4f/D)^2)} - 1.$$

In terms of $D/\lambda$ and $R$ the gain is:

$$G = \frac{1}{2}\left(\frac{\pi D}{\lambda}\right)^2 \frac{A+1}{A + \exp(4\pi D/R\lambda)^2}$$

which gives maximum gain for fixed $f/D$ and $R$, for the value

TABLE IV
REPRESENTATIVE HIGH-GAIN REFLECTOR ANTENNAS

| Institution | Location | D | $\epsilon$ | R | notes |
|---|---|---|---|---|---|
| Cal Tech | Owens Valley | 10.4m | 10μm | $10^6$ | 1980 completion |
| NRAO | Charlottesville | 25m | 70μm | $3.6 \times 10^5$ | |
| Max Planck Inst. | So. Spain | 30m | 90μm | $3.3 \times 10^5$ | 1982 completion, homologous |
| Max Planck Inst. | Effelsberg | 100m<br>80m * | 1mm<br>.5mm | $10^5$<br>$1.6 \times 10^5$ | homologous |
| Chalmers Univ. | Gothenburg | 20.1m | .21mm | $9.6 \times 10^4$ | |
| NRAO | Green Bank | 140 ft. | .9mm | $4.7 \times 10^4$ | |

* using inner 80 m only

of $4\pi D/R\lambda$ that is the solution to:

$$A = \left(\left(\frac{4\pi D}{R\lambda}\right)^2 - 1\right) \exp\left(\frac{4\pi D}{R\lambda}\right)^2.$$

The solution $D/R\lambda$ is shown in Fig. 8; for large $f/D$ the solution approaches $1/4\pi$. Maximum gain, normalized by $R^2$ is in Fig. 9; for large $f/D$, $G/R^2$ approaches $1/32e$, which is a factor of 3.7 larger than the $\epsilon = \lambda/40$ value. It is interesting to examine $R$ values for current large reflector antennas. Table IV gives diameter, best estimate of tolerance and manufacturing tolerance ratio.[5]

These antennas manufacturing tolerance ratios of $10^5$ to $10^6$ are to be compared with the $10^{-4}$ resolution achieved by the best imaging microphotolithography lenses, which corresponds to $R = 4\pi 10^4$. The art of telescope building has advanced appreciably in the last two decades; the best reflectors represent maximum gains in the range of 90–100 dB! Technology in current use includes homologous design in which a large reflector deforms into a new paraboloid with elevation angle change so that a feed refocus restores performance, and computer design of lightweight support structure.

Cost of large reflectors is highly variable, and a study of costs is beyond the scope of this paper. Previous studies include array cost, Cantafio [62], Provencher *et al.* [63], and reflector cost, Potter *et al.* [64]. The latter used two antennas, of 85 and 210 ft diameter, to determine cost variation which was as diameter to the 2.8 power. Both antennas were originally designed for $R = 10^4$ but it is suspected that the larger antenna has intrinsically a larger $R$ value. A determination of how reflector cost varies with $R$ and with $D$ is yet to be made. It can be said that cost increases roughly exponentially with $D$ (and probably with $R$ also), just as the cost of any structure increases exponentially for large values of $R$. Thus the fundamental limit is the rapidly increasing cost of large gain.

## REFERENCES


[1] L. J. Chu, "Physical limitations of omnidirectional antennas," *J. Appl. Phys.*, vol. 19, pp. 1163–1175, Dec. 1948.

[2] R. F. Harrington, "Effect of antenna size on gain, bandwidth, and efficiency," *J. Res. Nat. Bur. Stand.*, vol. 64D, pp. 1–12, Jan.–Feb. 1960.

[3] A. A. Oliner and R. G. Malech, "Mutual coupling in infinite scanning arrays," in *Microwave Scanning Antennas*, R. C. Hansen, Ed., vol. II. New York: Academic Press, 1966, ch. 3.

[4] J. A. Stratton, *Electromagnetic Theory.* New York: McGraw-Hill, 1941.

[5] S. A. Wolf *et al.*, "Superconducting extremely low frequency (ELF) magnetic field sensors for submarine communications," *IEEE Trans. Commun.*, vol. COM-22, pp. 549–554, 1974.

[6] G. Goubau, "Multi-element monopole antennas," in *Proc. Workshop on Electrically Small Antennas* ECOM, Ft. Monmouth, NJ, pp. 63–67, May 1976.

[7] C. W. Oseen, "Die Einsteinsche Nadelstichstrahlung und die Maxwellschen Gleichungen," *Ann. Phys.*, vol. 69, p. 202, 1922.

[8] W. W. Hansen and J. R. Woodyard, "A new principle in directional antenna design," *Proc. IRE*, vol. 26, pp. 333–345, Mar. 1938.

[9] K. Franz, "The gain and the (Rudenberg) 'absorption surfaces' of large directive arrays," *Hochfreq. Elektroakust.*, vol. 54, p. 198, 1939.

[10] S. A. Schelkunoff, "A mathematical theory of linear arrays," *Bell Syst. Tech. J.*, vol. 22, pp. 80–107, 1943.

[11] L. La Paz and G. A. Miller, "Optimum current distributions on vertical antennas," *Proc. IRE*, vol. 31, pp. 214–232, 1943.

[12] C. J. Bouwkamp and N. G. de Bruijn, "The problems of optimum antenna current distribution," *Philips Res. Rep.*, vol. 1, pp. 135–158, 1946.

[13] A. Bloch *et al.*, "Superdirectivity," *Proc. IRE*, vol. 48, p. 1164, 1960.

[14] A. Bloch *et al.*, "A new approach to the design of super-directive aerial arrays," *Proc. Inst. Elec. Eng.*, vol. 100, pp. 303–314, Sept. 1953.

[15] D. G. Reid, "The gain of an idealized Yagi array," *J. Inst. Elec. Eng.*, vol. 93, pt. IIIA, pp. 564–566, 1946.

[16] A. I. Uzkov, "An approach to the problem of optimum directive antennae design," *Compt. Rend. Acad. Sci. U.R.S.S.*, vol. 53, p. 35, 1946.

[17] C. L. Dolph, "A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level," *Proc. IRE*, vol. 34, pp. 335–348, June 1946.

[18] H. J. Riblet, "Discussion on 'a current distribution for broad-side arrays which optimizes the relationship between beam width and side-lobe level," *Proc. IRE*, vol. 35, pp. 489–492, 1947.

[19] R. H. DuHamel, "Optimum patterns for endfire arrays," *Proc. IRE*, vol. 41, pp. 652–659, 1953.

[20] R. J. Stegen, "Excitation coefficients and beamwidths of Tschebyscheff arrays," *Proc. IRE*, vol. 41, pp. 1671–1674, Nov. 1953.

[21] R. L. Pritchard, "Optimum directivity patterns for linear point arrays," *J. Acoust. Soc. Amer.*, vol. 25, pp. 879–891, Sept. 1953.

[22] R. Courant, *Differential and Integral Calculus, Vol. II.* New York: Interscience, 1936.

[23] J. R. Westlake, *A Handbook of Numerical Matrix Inversion and Solution of Linear Equations.* New York: Wiley, 1968, p. 32.

[24] Y. T. Lo *et al.*, "Optimization of directivity and signal-to-noise ratio of an arbitrary antenna array," *Proc. IEEE*, vol. 54, pp. 1033–1045, Aug. 1966.

[25] J. L. Brown, "A simplified derivation of the Fourier coefficients


[5] Assistance of Dr. N. Kern, Dr. J. W. M. Baars, Prof. R. B. Leighton, Dr. S. Hensel and Dr. J. W. Findlay in providing results of extensive and careful measurements is greatly appreciated.

for Chebyshev patterns," *Proc. Inst. Elec. Eng.*, vol. 105C, pp. 167-168, Nov. 1957.

[26] ——, "On the determination of excitation coefficients for a Tchebycheff pattern," *IRE Trans. Antenna Propagat.*, vol. AP-10, pp. 215-216, Mar. 1962.

[27] H. E. Salzer, "Calculating Fourier coefficients for Chebyshev patterns," *Proc. IEEE*, vol. 63, pp. 195-197, Jan. 1975.

[28] C. J. Drane, "Derivation of excitation coefficients for Chebyshev arrays," *Proc. Inst. Elec. Eng.*, vol. 110, pp. 1755-1758, Oct. 1963.

[29] C. J. Drane, "Dolph-Chebyshev excitation coefficient approximation," *IEEE Trans. Antenna Propagat.*, vol. AP-12, pp. 781-782, Nov. 1964.

[30] E. H. Newman *et al.*, "Superdirective receiving arrays, *IEEE Trans. Antennas Propagat.*, vol. AP-26, pp. 629-635, Sept. 1978.

[31] N. Yaru, "A note on supergain antenna arrays," *Proc. IRE*, vol. 39, pp. 1081-1085, Sept. 1951.

[32] T. L. Taylor, "Design of line-source antennas for narrow beamwidth and low sidelobes," *IRE Trans. Antennas Propagat.*, vol. AP-3, pp. 16-28, 1955.

[33] D. R. Rhodes, "On the quality factor of strip and line source antennas and its relationship to superdirectivity ratio," *IEEE Trans. Antennas Propagat.*, vol. AP-20, pp. 318-325, May 1972.

[34] ——, "On an optimum line source for maximum directivity," *IEEE Trans. Antennas Propagat.*, vol. AP-19, pp. 485-492, July 1971.

[35] ——, *Synthesis of Planar Antenna Sources.* Oxford, England: Clarendon Press, 1974.

[36] E. N. Gilbert and S. P. Morgan, "Optimum design of directive antenna arrays subject to random variations," *Bell Syst. Tech. J.*, vol. 34, pp. 637-663, May 1955.

[37] M. Uzsoky and L. Solymar, "Theory of super-directive linear arrays," *Acta Phys. Hung.* (Budapest), vol. 6, pp. 185-205, 1956.

[38] D. K. Cheng, "Optimization techniques for antenna arrays," *Proc. IEEE*, vol. 59, pp. 1664-1674, Dec. 1971.

[39] J. P. Burg, "Maximum entropy spectral analysis," in *Proc. 37th Meeting Society Exploration Geophysists*, 1967.

[40] ——, "A new analysis technique for time series data," NATO Adv. Study Inst. on Signal Processing With Emphasis on Underwater Acoustics, Aug. 1968.

[41] J. Capon, "High-resolution frequency-wavenumber spectrum analysis," *Proc. IEEE*, vol. 57, pp. 1408-1418, Aug. 1969.

[42] A. van den Bos, "Alternative interpretation of maximum entropy spectral analysis," *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 493-494, July 1971.

[43] J. P. Burg, "The relationship between maximum entropy spectra and maximum likelihood spectra," *Geophys.*, vol. 37, pp. 375-376, Apr. 1972.

[44] D. G. Childers, *Modern Spectrum Analysis.* New York: Wiley, 1978.

[45] *Proc. Spectrum Estimation Workshop*, RADC, Oct. 1979.

[46] T. E. Barnard, "The maximum entropy spectrum and the Burg technique," Rep. TR-75-01, Texas Instruments, June 1975.

[47] L. G. Griffiths and R. Prieto-Diaz, "Spectral analysis of natural seismic events using autoregressive techniques," *IEEE Trans. Geosci. Electron.*, vol. GE-15, pp. 13-25, Jan. 1977.

[48] T. E. Landers and R. T. Lacoss, "Geophysical applications of autoregressive spectral estimates," *IEEE Trans. Geosci. Electron.*, vol. GE-15, pp. 26-32, Jan. 1977.

[49] W. F. Gabriel, "Spectral analysis and adaptive array superresolution techniques," *Proc. IEEE*, vol. 68, pp. 654-666, June 1980.

[50] G. V. Borgiotti and L. J. Kaplan, "Superresolution of uncorrelated interference sources by using adaptive array techniques," *IEEE Trans. Antennas Propagat.*, vol. AP-27, Nov. 1979, pp. 842-845.

[51] R. N. McDonough, "Maximum-entropy spatial processing of array data," *Geophys.*, vol. 39, pp. 843-851, Dec. 1974.

[52] J. E. Evans, "Aperture sampling techniques for precision direction finding," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-15, pp. 891-894, Nov. 1979.

[53] W. D. White, "Angular spectra in radar applications," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-15, pp. 895-898, Nov. 1979.

[54] R. N. Strickland *et al.*, "Resolution of subwavelength-spaced scatterers by superdirective data processing simulating evanescent wave illumination," *Microwave, Optics Acoust.*, vol. 3, pp. 37-42, Mar. 1979.

[55] W. R. King, "Maximum entropy wavenumber analysis," NRL Rep. 8298, Mar. 1979.

[56] H. Akaike, "A new look at the statistical model identification," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 716-723, Dec. 1974.

[57] J. Ruze, "Antenna tolerance theory—A review," *Proc. IEEE*, vol. 54, pp. 633-640, Apr. 1966.

[58] J. H. Wested, "Effect of deviation from the ideal paraboloid shape of large antenna reflectors," in *Proc. IEE Conf. Large Steerable Antennas* (London, England), pp. 115-119, 1966.

[59] R. Manasse, "Maximum angular accuracy of tracking a radio star by lobe comparison," *IRE Trans. Antennas Propagat.*, vol. AP-8, pp. 50-56, Jan. 1960.

[60] J. R. Sklar and F. C. Schweppe, "On the angular resolution of multiple targets," *Proc. IEEE*, vol. 52, pp. 1044-1045, Sept. 1964.

[61] G. E. Pollon and G. W. Lank, "Angular tracking of two closely spaced radar targets," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-4, pp. 541-550, July 1968.

[62] L. J. Cantafio, "Prediction of the minimum investment cost of phased array radars," *IEEE Trans. Aerosp. Electron Syst.*, vol. AES-3, supplement, pp. 207-225, Nov. 1967.

[63] J. H. Provencher *et al.*, "Multifrequency arrays: Design and cost considerations," NELC TR-1956, July 1975.

[64] P. D. Potter *et al.*, "Large antenna apertures and arrays for deep space communications," JPL TR 32-848, Nov. 1965.