circuits of the second amplifier stage the condenser C_2 can be made to introduce a voltage E_1 across the cathode resistor of this stage that exactly balances out the voltage E_2 applied to the grid of the same tube because of the presence of voltage across the common plate impedance Z_c.

Let E_0 = voltage across the common plate impedance

E_1 = voltage introduced across the cathode of second amplifier stage due to the presence of E_0

E_2 = voltage existing on the grid of second amplifier stage due to the presence of E_0

$\omega = 2\pi \times$ frequency.

Then,

$$E_1 = E_0 \frac{1}{1 + \frac{C_3}{C_2} + \frac{1}{j\omega C_2 R_3}} \quad (1)$$

$$E_2 = E_0 \frac{1}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_p} + \frac{1 + \frac{R_1}{R_p}}{j\omega C_1 R_2}} \quad (2)$$

where,

R_p = plate resistance of the input amplifier tube

For perfect neutralization, the voltages E_1 and E_2 should be equal in magnitude and in phase, which gives

$$\frac{C_3}{C_2} = \frac{R_1}{R_2}\left(1 + \frac{R_2}{R_p}\right) \quad (3)$$

$$\frac{C_1}{C_2} = \frac{R_3}{R_2}\left(1 + \frac{R_1}{R_p}\right). \quad (4)$$

Equations (3) and (4) are independent and both must be satisfied. It will be noted that the conditions for balance are independent of frequency.

This neutralizing circuit reduces power-supply hum to the same extent as it does regeneration. This is because such hum is caused by a voltage fed back from the power supply to the input stages, just as regeneration is caused by a voltage fed back in the same way. The only difference in the two cases is that the voltage across the output of the power supply arises from different causes.

Formulas for the Skin Effect*

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Summary—*At radio frequencies, the penetration of currents and magnetic fields into the surface of conductors is governed by the skin effect. Many formulas are simplified if expressed in terms of the "depth of penetration," which has merely the dimension of length but involves the frequency and the conductivity and permeability of the conductive material. Another useful parameter is the "surface resistivity" determined by the skin effect, which has simply the dimension of resistance. These parameters are given for representative metals by a convenient chart covering a wide range of frequency. The "incremental-inductance rule" is given for determining not only the effective resistance of a circuit but also the added resistance caused by conductors in the neighborhood of the circuit. Simple formulas are given for the resistance of wires, transmission lines, and coils; for the shielding effect of sheet metal; for the resistance caused by a plane or cylindrical shield near a coil; and for the properties of a transformer with a laminated iron core.*

THE "skin effect" is the tendency for high-frequency alternating currents and magnetic flux to penetrate into the surface of a conductor only to a limited depth. The "depth of penetration" is a useful dimention, depending on the frequency and also on the properties of the conductive material, its conductivity or resistivity and its permeability. If the thickness of a conductor is much greater than the depth of penetration, its behavior toward high-frequency alternating currents becomes a surface phenomenon rather than a volume phenomenon. Its "surface resistivity" is the resistance of a conducting surface of equal length and width, and has simply the dimension of resistance. In the case of a straight wire, the width is the circumference of the wire.

Maxwell[1] discovered that the voltage required to force a varying current through a wire increases more than could be explained by inductive reactance. He explained this as caused by a departure from uniform current density. This discovery was followed up by Heaviside, Rayleigh, and Kelvin. It came to be called the "skin effect," because the current is concentrated in the outer surface of the conductor. The ratio of high-frequency resistance to direct-current resistance for a straight wire was computed in terms of Bessel functions and was reduced to tables.[2–7]

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[1] J. C. Maxwell, "Electricity and Magnetism," on page 385. 1873/1937, vol. 2, section 690, p. 322.

[2] Lord Rayleigh, *Phil. Mag.*, vol. 21, p. 381; 1886.

[3] C. P. Steinmetz, "Transient Electric Phenomena and Oscillations," pp. 361–393, 1909/1920.

[4] S. G. Starling, "Electricity and Magnetism," 1912/1914, pp. 364–369.

[5] E. B. Rosa and F. W. Grover, "Formulas and tables for the calculation of mutual and self-inductance (revised)," Bureau of Standards, S-169, pp. 172–182, 1916.

[6] "Radio Instruments and Measurements," Bureau of Standards, C-74, pp. 299–311, 1918/1924.

[7] J. H. Morecroft, "Principles of Radio Communication," 1921, pp. 114–136.

Steinmetz defined the "depth of penetration" without restriction as to the shape of the conductor. He applied this concept to laminated iron cores, as well as to conductors. Unfortunately, he gave two definitions which differ slightly, one for iron cores and another for conductors. The latter definition has been generally adopted, as in the Steinmetz tables on page 385.

More recent writers have reduced the treatment of the skin effect to simple terms and have generalized its application.[8–16] Schelkunoff and Stratton have given the most comprehensive treatment of the subject, including the depth of penetration in all kinds of problems involving conductors. They have introduced the concept of surface impedance, from which the surface resistivity is a by-product.

In spite of this active history of the skin effect, there is still a need for a simple and direct summary which will facilitate its appreciation and its application to simple problems. That is the purpose of this presentation.

Following Harnwell and Stratton, the mks rationalized system of units is employed for all relations, except where inches are specified. The properties of materials are taken for room temperature (20 degrees centigrade or 293 degrees absolute). The following list gives the principal symbols used herein.

d = depth of penetration (meters)
R_1 = surface resistivity (ohms)
σ = conductivity (mhos per meter)
$\rho = 1/\sigma$ = resistivity (ohm-meters)
μ = permeability (henrys per meter)
$\mu_0 = 4\pi \cdot 10^{-7}$ = permeability of space
f = frequency (cycles per second)
$\omega = 2\pi f$ = radian frequency (radians per second)
$j = \sqrt{-1}$
$e = 2.72$ = base of logarithms
$\exp x = e^x$ = exponential function
z = depth from the surface into the conductive medium (meters)
w = width (meters)
l = length (meters)
a = thickness or radius (meters)
b = distance, length or width (meters)
c = distance (meters)
r = radius (meters)
A = area (square meters)
I = current (amperes)
i = current density at a depth z (amperes per square meter)
i_0 = current density at the surface ($z=0$)
H = magnetic field intensity at a depth z (amperes per meter)
H_0 = magnetic field intensity at the surface ($z=0$)
E = electromotive force (volts)
P = power (watts)
P_1 = power dissipation per unit area (watts per square meter)
$Z = R + jX$ = impedance (ohms)
X = reactance (ohms)
R = resistance (ohms)
G = conductance (mhos)
L = inductance (henries)
L_0 = inductance in space outside of conductive medium
m = number of laminations
n = number of turns
r = ratio of resistivity
x = ratio of radii
Q = ratio of reactance to resistance

Fig. 1 is a chart[17] giving the surface resistivity R_1 and the depth of penetration d for various metals, over a wide range of frequency f. The depth is plotted in parts of an inch, since this aids in practical application and introduces no confusion with the mks electrical units. Each sloping line represents one metal, depending on its resistivity ρ or conductivity σ and its permeability μ at room temperature (20 degrees centigrade or 293 degrees absolute). The heavy lines are for copper, which is the logical standard of comparison. Additional lines can be drawn to meet special requirements, shifting them from the copper line in accordance with the properties of the metal.

Fig. 2 shows a slab of conductive material to be used in describing the skin effect. The current I is concentrated in the upper surface. From Harnwell, the alternating-current density i in the surface of a conductor decreases with depth z according to the formula

$$\frac{i}{i_0} = \exp - z\sqrt{j\omega\mu\sigma} = \exp - (1+j)z\sqrt{\frac{\omega\mu\sigma}{2}} = \exp - \frac{z}{d} \exp - j\frac{z}{d} \cdot \qquad (1)$$

[8] E. J. Sterba and C. B. Feldman, "Transmission lines for short-wave radio systems," Proc. I.R.E., vol. 20, pp. 1163–1202; July, 1932; *Bell Sys. Tech. Jour.*, vol. 11, pp. 411–450; July, 1932. (Convenient formulas.)

[9] S. A. Schelkunoff, "The electromagnetic theory of coaxial transmission lines and cylindrical shields," *Bell Sys. Tech. Jour.*, vol. 8, pp. 532–579; October, 1934. (The most complete theoretical treatment.)

[10] S. A. Schelkunoff, "Coaxial communication transmission lines," *Elec. Eng.*, vol. 53, pp. 1592–1593; December, 1934. (A brief description of the physical behavior.)

[11] E. I. Green, F. A. Leibe, and H. E. Curtis, "The proportioning of shielding circuits for minimum high-frequency attenuation," *Bell Sys. Tech. Jour.*, vol. 15, pp. 248–283; April, 1936.

[12] August Hund, "Phenomena in High-Frequency Systems," 1936, pp. 333–338.

[13] S. A. Schelkunoff, "The impedance concept and its application to problems of reflection, refraction, shielding and power absorption," *Bell Sys. Tech. Jour.*, vol. 17, pp. 17–48; January, 1938.

[14] G. P. Harnwell, "Principles of Electricity and Electromagnetism," pp. 313–317, 1938.

[15] W. R. Smythe, "Static and Dynamic Electricity," 1939, pp. 388–417.

[16] J. A. Stratton, "Electromagnetic Theory," pp. 273–278, 500–511, and 520–554, 1941. (mks units.)

[17] This chart has been reprinted in the report of the Rochester Fall Meeting in *Electronics*, December, 1941. More recently, a similar chart has appeared in the following reference, together with other valuable formulas and curves: J. R. Whinnery, "Skin effect formulas," *Electronics*, vol. 15, pp. 44–48; February, 1942.

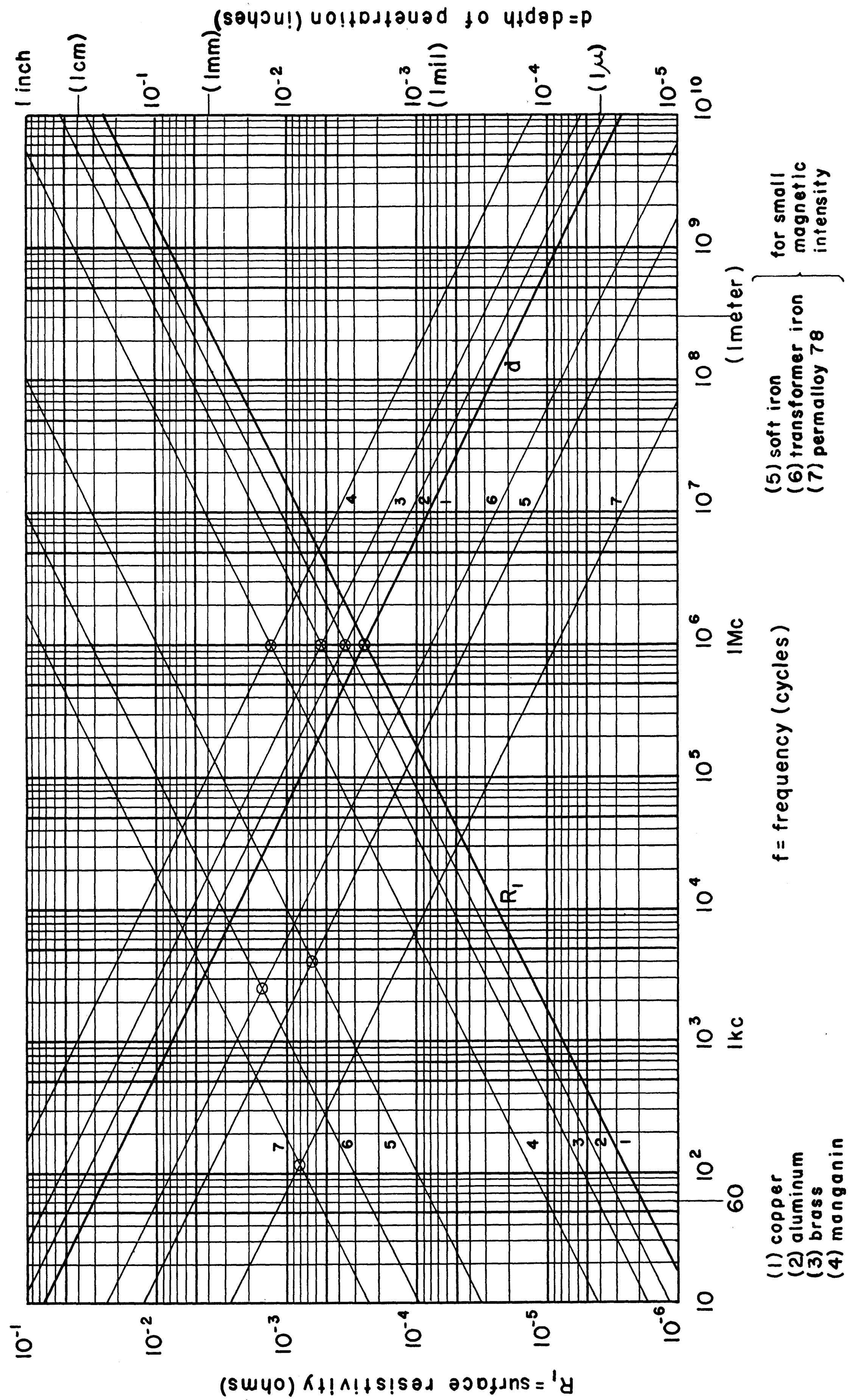


Fig. 1—Surface Resistivity and Depth of Penetration.

This decay of current density is shown by the shaded area plotted on the side of the slab.

The depth of penetration is defined by the last formula, as the depth at which the current density (or magnetic flux) is attenuated by 1 napier (in the ratio $1/e = 1/2.72$, or -8.7 decibels). At the same depth, its phase lags by 1 radian, so d is $1/2\pi$ wavelength or 1 radian length in terms of the wave propagation in the conductor.

The depth of penetration, by this definition, is

$$d = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu \sigma}} \qquad \text{meters.} \quad (2)$$

It is noted that the $\sqrt{2}$ factor arises when the $\sqrt{j}$ is resolved into its real and imaginary components in the exponent in (1).

The total current is the integral of the current density in the conductive medium. This integral from the surface into the medium is a decaying spiral in the complex plane, which rapidly approaches its limit if the thickness is much greater than the depth of penetration. The total current is therefore given by the integral for infinite depth, over the width w:

$$\begin{aligned} I &= w \int_0^\infty i \cdot dz \\ &= i_0 w \int_0^\infty \exp - (1 + j) \frac{z}{d} \, dz \\ &= \frac{i_0 w d}{1 + j} \qquad \text{amperes.} \quad (3) \end{aligned}$$

The voltage E on the surface along the length of the conductor is obtained from the current density and

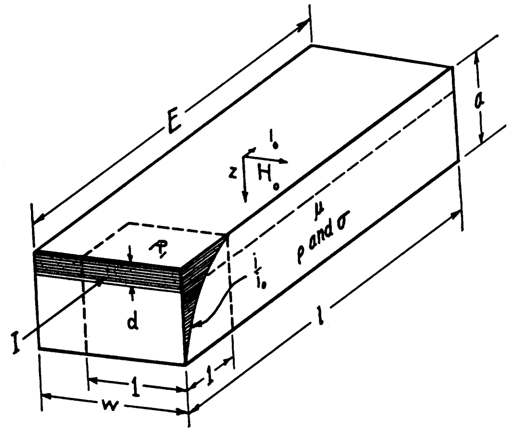


Fig. 2—The skin effect on the surface of a conductor.

the volume resistivity.

$$E = i_0 l \rho \qquad \text{volts.} \quad (4)$$

If this voltage were to be measured, the return circuit would have to be adjacent to the surface so as not to include any of the magnetic flux in the near-by space.

The "internal impedance" or "surface impedance" is computed from this voltage E and the current I.

$$\begin{aligned} Z &= \frac{E}{I} = (1 + j) \frac{\rho l}{wd} \\ &= (1 + j) \frac{l}{w} \sqrt{\pi f \mu \rho} = \frac{l}{w} \sqrt{j \omega \mu \rho} \qquad \text{ohms.} \quad (5) \end{aligned}$$

Its real and imaginary components are the resistance

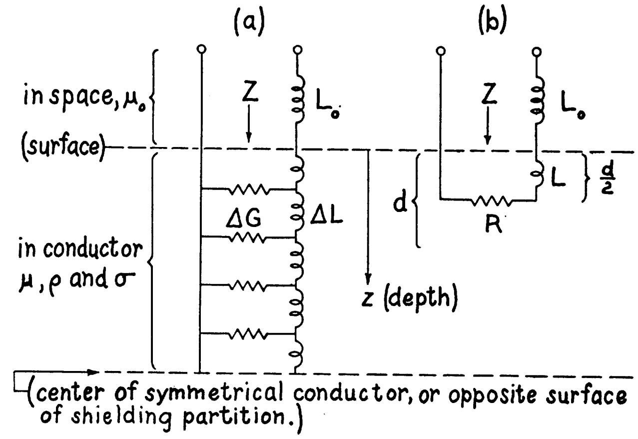


Fig. 3—The internal impedance of a conductor, in terms of distributed circuit parameters (a) and equivalent lumped parameters (b).

and the internal reactance, which are equal.

$$\begin{aligned} Z &= R + jX, \\ R = X &= \frac{l}{w} \sqrt{\pi f \mu \rho} \qquad \text{ohms.} \quad (6) \end{aligned}$$

The surface resistivity R_1, given in the chart, is defined as the resistance of a surface of equal length and width.

$$\begin{aligned} R_1 &= \frac{\rho}{d} = \sqrt{\pi f \mu \rho} \\ R = X &= \frac{l}{w} R_1 \qquad \text{ohms.} \quad (7) \end{aligned}$$

For example, R_1 is the resistance of the unit square surface in Fig. 2.[18]

The internal inductance is the part of the total inductance which is caused by the magnetic flux in the conductive medium. It is computed from the internal reactance.

$$L = \frac{X}{\omega} = \frac{l}{w}\left(\mu \frac{d}{2}\right) \qquad \text{henries.} \quad (8)$$

This is the inductance of a layer of the conductive material having a thickness of $d/2$, one half the depth of penetration. This merely means that the mean depth of the current is one-half the thickness of the conducting layer.

Fig. 3 illustrates the concept of internal impedance in terms of electric circuit elements. In the diagram (a), the inductance L_0 is that caused by the magnetic flux in the space adjacent to the conductor. Each part

[18] Schelkunoff (footnote 9, p. 550) calls R_1 the "intrinsic resistance" of the material.

of the current meets additional inductance in proportion to its depth from the surface of the conductor. This inductance is ΔL per element of depth. The conductance of the material is ΔG for the same element of depth. The conductive slab behaves as a transmission line with paths of shunt conductance in layers parallel to the surface, and series inductance between layers. This hypothetical line presents the internal impedance Z in series with the external inductance L_0. If the thickness of the conductor is much greater than the depth of penetration, the impedance is unaffected by conditions at the far end of the line, or beyond the other surface of the conductor.

The internal impedance of the hypothetical transmission line in Fig. 3(a) is computed from its distributed inductance and conductance, by circuit theory. Since the magnetic flux path has an area $l\Delta z$ and a length w

$$\Delta L = \frac{\mu l \Delta z}{w} \qquad \text{henries. (9)}$$

Since the current path has an area $w\Delta z$ and a length l,

$$\Delta G = \frac{\sigma w \Delta z}{l} = \frac{w \Delta z}{\rho l} \qquad \text{mhos. (10)}$$

The impedance of a long line with these properties is

$$Z = \sqrt{j\omega \Delta L/\Delta G} = \frac{l}{w}\sqrt{j\omega\mu\rho} \qquad \text{ohms. (11)}$$

This is an independent complete derivation of (5), without recourse to electromagnetic-wave equations.

The components of internal impedance are shown in Fig. 3(b) as R and L. The resistance R is that of a layer whose thickness is equal to the depth of penetration d. The internal inductance is that of a layer whose thickness is $d/2$, one half the depth of penetration.

Some inductance formulas carry the assumption that the current travels in a thin sheet on the surface of the conductor, as if the resistivity were zero. Such assumptions are usual for transmission lines, wave guides, cavity resonators, and piston attenuators. Such formulas can be corrected for the depth of penetration by assuming that the current sheet is at a depth $d/2$ from the surface. This is the same as assuming that the surface of the conductor recedes by the amount

$$\frac{d}{2} \cdot \frac{\mu}{\mu_0} \cdot \qquad (12)$$

The second factor has an effect only if the conductive material has a permeability μ differing from that of space μ_0. The same correction is applicable to shielding partitions, regarding their effect on the inductance of near-by circuits.

There is sometimes a question which surface of a conductor will carry the current. The rule is, that the current follows the path of least impedance. Since the impedance is mainly inductive reactance, in the common cases, the current tends to follow the path of least inductance. In a ring, for example, the current density is greater on the inner surface. In a coaxial line, the current flows one way on the outer surface of the inner conductor and returns on the inner surface of the outer conductor.

In determining whether the thickness is much greater than the depth of penetration, the effective thickness corresponds to the length of the hypothetical line in Fig. 3(a). In a symmetrical conductor with penetration from both sides, as in a strip or a wire, the effective thickness is the depth to the center of the conductor. In a shielding partition with penetration into the surface on one side and with open space on the other side, the effective thickness is the actual thickness. If the effective thickness exceeds twice the depth of penetration, the accuracy of the above impedance formulas is sufficient for most purposes, within two per cent for a plane surface.

The shielding effect of a conductive partition depends not only on the material and thickness of the partition, but also on its location. For example, two layers of metal have more shielding effect if they are separated by a layer of free space than if they are close together. If a shielding partition carries current on one surface ($z=0$) and is exposed to free space at the other surface ($z=a$) the current density has a definite ratio between one surface and the the other. For the thickness a, much greater than the depth of penetration, as in Fig. 2, this ratio is

$$\frac{i_a}{i_0} = 2 \exp - \frac{a}{d} \qquad (a \gg d)$$

$$= 0.69 - \frac{a}{d} \qquad \text{napiers}$$

$$= 6 - 8.7\frac{a}{d} \qquad \text{decibels. (13)}$$

The factor 2 is caused by reflection at the far surface. The space on either side of a shield usually adds to the attenuation indicated by this formula.

The shielding ability of a given metal at a given frequency is best expressed as the attenuation for a convenient unit of thickness, disregarding the reflection factor. The unit of thickness may be 1 millimeter (10^{-3} meter) or 1 mil ($2.54 \cdot 10^{-5}$ meter). In copper at 1 megacycle, for example, it is 132 decibels per millimeter or 3.3 decibels per mil. In iron, it is much greater and depends also on the magnetic flux density, since that affects the permeability.

The power dissipation in the surface of a shield is determined by the magnetic field intensity at its surface. The same is true of current conductors or iron cores but in those cases there are more direct methods of computation in terms of current and effective resistance. Since the magnetic flux path has a length equal to the width w of the conductor, and since the magnetomotive force is equal to the current I, the magnetic intensity at the surface is

$$H_0 = \frac{I}{w} \qquad \text{amperes per meter.} \quad (14)$$

The power dissipation is

$$P = I^2R = (wH_0)^2 \frac{l}{w} R_1$$
$$= lwH_0{}^2R_1 = lwP_1 \qquad \text{watts} \quad (15)$$

in which the power dissipation per unit area of surface is

$$P_1 = H_0{}^2R_1 \qquad \text{watts per square meter.} \quad (16)$$

For most purposes, the power dissipation is more readily computed by the following method, in terms of effective resistance in a circuit.

The "incremental-inductance rule" is a formula which gives the effective resistance caused by the skin effect, but is based entirely on inductance computations. Its great value lies in its general validity for all metal objects in which the current and magnetic intensity are governed by the skin effect. In other words, the thickness and the radius of curvature of exposed metal surfaces must be much greater than the depth of penetration, say at least twice as great. It is equally applicable to current conductors, shields, and iron cores.

This rule is a generalization of (7) which states that the surface resistance R is equal to the internal reactance X as governed by the skin effect. The internal reactance is the reactance of the internal inductance L in (8). This inductance is the increment of the total inductance which is caused by the penetration of magnetic flux under the conductive surface. This change of inductance is the same as would be caused by the surface receding to the depth given in (12). Starting with a knowledge of this depth, the reverse process of computation gives the increment of inductance caused by the penetration, and from that the effective resistance as governed by the skin effect.

The incremental-inductance rule is stated, that the effective resistance in a circuit is equal to the change of reactance caused by the penetration of magnetic flux into metal objects. It is valid for all exposed metal surfaces which have thickness and radius of curvature much greater than the depth of penetration, say at least twice as great.

The application of the incremental-inductance rule involves the following steps:

(a) Select the circuit in which the effective resistance is to be evaluated, and identify the exposed metal surfaces in which the skin effect is prevalent.

(b) Compute the rate of change of inductance of this circuit with recession of each of the metal surfaces, $\sigma L_0/\sigma z$, assuming zero depth of penetration.[19]

(c) Note that the increment of inductance caused by penetration into each surface is

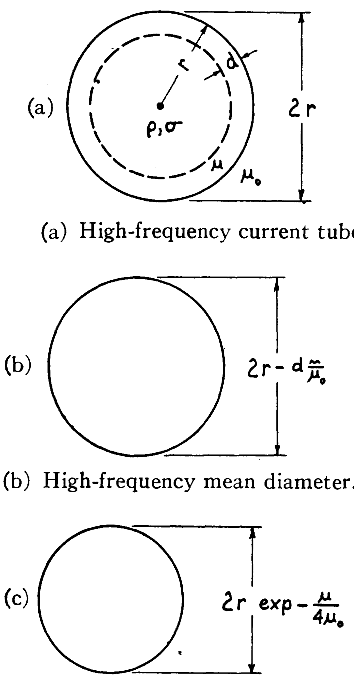
$$L = \frac{\mu}{\mu_0} \cdot \frac{d}{2} \cdot \frac{\partial L_0}{\partial z} \qquad \text{henries.} \quad (17)$$

(d) Compute the effective resistance contributed by each surface,

$$R = \omega L = \frac{1}{\mu_0} \cdot \frac{\partial L_0}{\partial z} R_1 \qquad \text{ohms.} \quad (18)$$

For a surface carrying the current of the circuit, this is identical with (7). For the effect of near-by metal objects, such as shields, this formula is easily applied in many practical cases. It is most useful in cases of nonuniform current distribution, which otherwise would require special integrations.

A straight wire has its current concentrated in a tubular surface layer as shown in Fig. 4(a). The depth



(a) High-frequency current tube.

(b) High-frequency mean diameter.

(c) Low-frequency mean diameter.

Fig. 4—The current distribution in a straight wire.

of this layer is d. The radius of the wire is r but the mean radius of the current tube is $r-d/2$. The resistance ratio of the wire is the ratio of the alternating-current resistance R of the direct-current resistance R_0. It is the inverse ratio of the effective cross-sectional areas,

$$\frac{R}{R_0} = \frac{\pi r^2}{\pi(2r - d)d}$$
$$= \frac{r}{2d} + \frac{1}{4} + \cdots \qquad (r > 2d). \quad (19)$$

[19] A second-order approximation is secured if $\delta L_0/\delta z$ is computed assuming that the surface is below the actual surface by the amount given in (12).

Since the assumptions are an approximation at best, only the first two terms of this series deserve attention. They give a close approximation if the radius exceeds twice the depth of penetration, or if the resistance ratio exceeds 5/4.[20–26]

The inductance of a straight wire is determined by the mean diameter of the current path. Fig. 4(b) shows the equivalent current sheet for the case in which the radius is very much greater than the depth of penetration. A perfect conductor, to have the same inductance with zero depth of penetration, has a radius which is less by the amount given in (12). This rule is reliable only if the equivalent radius is greater than 7/8 the actual radius.

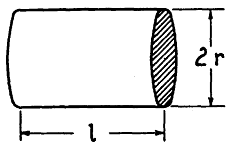


Fig. 5—Straight wire.

The low-frequency inductance of a straight wire, with uniform current distribution, is its maximum inductance. As shown in Fig. 4(b), the equivalent current sheet has the radius

$$r \exp - \frac{\mu}{4\mu_0} \qquad (20)$$

in which the factor exp-1/4 is the "geometric-mean distance" of a circular area.[27]

The straight wire of Fig. 5, assuming a depth of penetration very much less than the radius, has its resistance expressed by the simple formula

$$R = \frac{l}{2\pi r} R_1 \qquad \text{ohms.} \quad (21)$$

This neglects the second term in the series of (19). It is on this simple basis that the following cases are described.[28,8,11]

The coaxial line of Fig. 6 has its current flowing one way on the lesser radius r_1 and returning on the greater radius r_2. The total resistance is

$$R = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\frac{l}{2\pi} R_1 \qquad \text{ohms.} \quad (22)$$

[20] Morecroft, (footnote 7, p. 116), curves of resistance ratio.
[21] E. Jahnke and F. Emde, "Tables of Functions," B. G. Teubner, Berlin, Germany, 1933, chapter 18, p. 314, Fig. 165, curve $rb_0/2b_1$.
[22] August Hund, "High-Frequency Measurements," 1933. pp. 263–266. Series expansions.
[23] Schelkunoff, footnote 9, pp. 551–553, formulas and curves for resistance and reactance ratio.
[24] August Hund, "Phenomena in High Frequency Systems," 1936. p. 338. Series expansions.
[25] J. H. Miller, "R−F resistance of copper wire," *Electronics*, vol. 9, no. 2, p. 338; February, 1936. Curves and formula.
[26] Stratton, footnote 16, p. 537, series expansions.
[27] Rosa and Grover, footnote 5, p. 167.
[28] Alexander Russell, "The effective resistance and inductance of a concentric main," *Phil. Mag.*, sixth series, vol. 17, pp. 524–552; April, 1909.

The inductance in the space between the conductors is[29]

$$L_0 = \frac{\mu_0 l}{2\pi} \log \frac{r_2}{r_1} \qquad \text{henries.} \quad (23)$$

For a given value of the greater radius r_2, minimum attenuation in this line requires minimum R/L_0, and this is obtained with $r_2/r_1 = 3.59$, approximately.[30,31] With

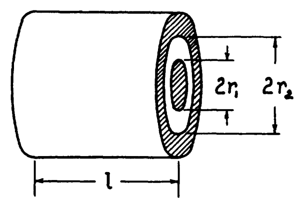


Fig. 6—Coaxial conductors.

this shape, the resistance of inner and outer conductors is divided as 78 per cent and 22 per cent of the total. Since the optimum ratio satisfies the equation

$$\log \frac{r_2}{r_1} = 1 + \frac{r_1}{r_2}, \qquad (24)$$

the ratio of reactance to resistance for this shape is, for a nonmagnetic conductor,[32]

$$Q = \frac{2r_1}{d} = \frac{r_2}{1.8\,d} \qquad (\mu = \mu_0). \quad (25)$$

This is the ratio of the diameter of the inner conductor to the depth of penetration. In general,

$$Q = \frac{2r_1}{d} \cdot \frac{\log \frac{r_2}{r_1}}{1 + \frac{r_1}{r_2}} \cdot \qquad (26)$$

This value is reduced slightly by end effects.

If a coaxial line is used as the inductance of a resonant circuit, maximum impedance at parallel resonance may be desired. This is obtained with maximum L_0^2/R, which determines the condition

$$\frac{1}{2} \log \frac{r_2}{r_1} = 1 + \frac{r_1}{r_2} \cdot \qquad (27)$$

The required shape is $r_2/r_1 = 9.2$, approximately.[33] If the length of the line is much less than one-quarter wavelength, so its shunt capacitance is negligible, this optimum shape has the following resistance at parallel resonance: $(\mu = \mu_0)$.

$$R' = Q^2 R = 0.307 \frac{l r_2}{d^2} R_1 \qquad \text{ohms.} \quad (28)$$

For given frequency and material, this resistance is proportional to the area of the conducting surfaces.

[29] Harnwell, footnote 14, p. 304.
[30] Sterba and Feldman, footnote 8, p. 419.
[31] Green, Leibe, and Curtis, footnote 11, p. 253.
[32] In all cases, Q is expressed on the assumption of a nonmagnetic conductor.
[33] F. E. Terman, "Resonant lines in radio circuits," *Elec. Eng.*, vol. 53, pp. 1046–1053; July, 1934.

A pair of straight parallel wires is shown in cross section in Fig. 7. The same current flows in opposite directions in the two wires, and is concentrated on the surface. If the wire diameter 2a is much less than the center-to-center separation 2b, the resistance of each wire is given by (21) for Fig. 5. As the diameter in Fig. 7 approaches equality with the separation, the proximity of wires causes greater current density on the inner sides.[34,35] This effect is easily evaluated by the incremental-inductance rule. The approximate and

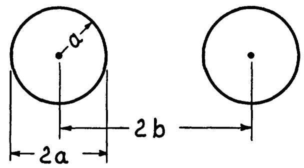


Fig. 7—Parallel wires.

exact formulas for the external inductance of this pair of wires, of length l, are

$$L_0 = \frac{\mu_0 l}{\pi}\log\frac{2b}{a} \quad (a \ll b)$$
$$= \frac{\mu_0 l}{\pi}\,\text{anticosh}\,\frac{b}{a}$$
$$= \frac{\mu_0 l}{\pi}\log\left[\frac{b}{a}\left(1+\sqrt{1-\left(\frac{a}{b}\right)^2}\right)\right]. \quad (29)$$

This first formula neglects the proximity of the inner sides of the wires. The third formula shows in the parenthetical factor, by the amount the factor departs from 2, the reduction of inductance by the extra concentration of current on the inner sides. Since the penetration ∂z corresponds to $-\partial a$, the effect of surface recession is

$$\frac{\partial L_0}{\partial z} = -\frac{\partial L_0}{\partial a} = \frac{\mu_0 l}{\pi a}\cdot\frac{1}{\sqrt{1-(a/b)^2}} \quad (30)$$

in which the last factor is the proximity factor. From this formula and (18), the resistance is

$$R = \frac{l}{\pi a\sqrt{1-(a/b)^2}}R_1 \quad \text{ohms.} \quad (31)$$

The proximity factor appears as a reduction of the effective circumference of the wire, because the current is concentrated toward one side of each wire. Otherwise, this formula is the same as for a single wire of length $2l$.

A ring of wire is shown in Fig. 8, with r_2 as the radius of the ring and r_1 as the much smaller radius of the wire, both being much greater than the depth of penetration d. The resistance is

$$R = \frac{2\pi r_2}{2\pi r_1}R_1 = \frac{r_2}{r_1}R_1 \quad \text{ohms.} \quad (32)$$

On the same assumption, $r_1 \ll r_2$, the inductance is[36]

$$L_0 = \mu_0 r_2\left(\log\frac{8r_2}{r_1} - 2\right)$$
$$= \mu_0 r_2 \log\frac{8r_2}{e^2 r_1} \quad \text{henries.} \quad (33)$$

For a given ring diameter $2r_2$, the maximum ratio of reactance to resistance is obtained with approximately

$$\frac{r_2}{r_1} = \frac{e^3}{8} = 2.5 \quad (34)$$

in which case the inductance and the ratio of reactance to resistance are

$$L = \mu_0 r_2 \quad \text{henries.}$$
$$Q = \frac{2r_1}{d} \quad (35)$$

This ratio is the same for the ring as for the coaxial line. Only the simple approximate formulas are given for the ring because no exact formula is known. In the absence of an exact inductance formula, it is also impossible to find easily the effect of current concentra-

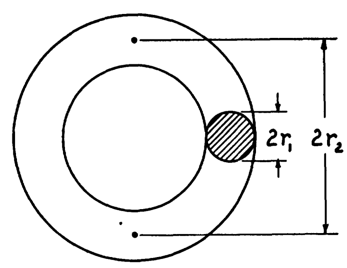


Fig. 8—Circular ring.

tion on the inner side of the ring conductor. The added resistance of the ring caused by radiation and near-by objects is neglected.

A shielding wall near the ring of Fig. 8 is shown in Fig. 9, the wall being a metal sheet parallel to the ring at a distance c. The added resistance caused by this shield is computed by the incremental-inductance rule. Assuming first that the shield is a perfect conductor, the effective inductance of the coil is reduced by an amount equal to the mutual inductance with its image (shown in dotted lines) at a distance $2c$. Therefore, the change of inductance is[37]

$$L_0' = -\frac{\pi\mu_0 r_2^4}{16c^3} \quad \text{henries.} \quad (36)$$

To obtain the effect of penetration in the shild, ∂c corresponds to ∂z, so

$$\frac{\partial L_0'}{\partial z} = \frac{\partial L_0'}{\partial c} = \frac{3\pi\mu_0 r_2^4}{16c^4}. \quad (37)$$

[34] J. R. Carson, "Wave propagation over parallel wires: The proximity effect," *Phil. Mag.*, vol. 41, p. 627; April, 1921.
[35] Green, Leibe, and Curtis, footnote 11, pp. 267–268.
[36] Harnwell, footnote 14, p. 305.
[37] Harnwell, footnote 14, pp. 304–305.

From this formula and (18), the added resistance is

$$R' = \frac{3\pi r_2^4}{16c^4} R_1' \text{ ohms} \tag{38}$$

in which R_1' is the surface resistivity of the shield. This is equal to the change of reactance which would be caused if the shield were moved further back by the displacement $(d/2)(\mu/\mu_0)$ as shown in Fig. 9. Comparing the added resistance with the resistance R of the ring alone, formula (32), the relative change of resistance is

$$\frac{R'}{R} = \frac{3\pi r_1 r_2^3 R_1'}{16c^4 R_1}\cdot \tag{39}$$

This ratio is independent of the frequency, so long as the depth of penetration is the controlling factor. As an example, a copper ring with the optimum shape

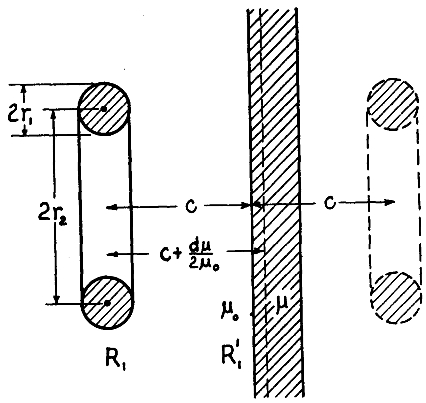


Fig. 9—A ring near a shielding wall.

$(r_2 = 2.5r_1)$ at a distance of 1 diameter from a soft-iron shield $(R_1' = 40R_1)$ would suffer about 59 per cent increase of resistance caused by the shield. In this location, a slightly smaller wire diameter would be optimum, because the inductance of the ring would increase in a greater ratio than the total resistance. The reduction of inductance (36) by the shield varies with the inverse cube of the distance, whereas the added resistance (38) varies with the inverse fourth power.

A ring perpendicular to the shield, instead of parallel as in Fig. 9, and with its center at the same distance, would suffer only one half as much change of inductance and resistance. This follows from the fact that the mutual inductance with its image would be one half as great. This is a striking example of the utility of the incremental-inductance rule, since the departure from axial symmetry would make this problem very difficult of solution by field-integration methods.

A coil of n turns near a shield has its inductance and resistance changed by n^2 times as much as the ring, that is, by n^2L_0' and n^2R', formulas (36) and (38).

The air-core toroidal coil of Fig. 10 has n turns on a coil radius of r_1 and a ring radius of r_2. The following simple formulas are based on the asumptions that the coil radius is much less than the ring radius $(r_1 \ll r_2)$ and that the current is concentrated on the inner surface with uniform distribution in a layer of depth very much less than the coil radius $(d \ll r_1)$.

$$R = \frac{2\pi r_1 n}{2\pi r_2/n} R_1 = \frac{r_1}{r_2} n^2 R_1 \quad \text{ohms} \tag{40}$$

$$L_0 = \mu_0 n^2 \frac{r_1^2}{2r_2} \quad \text{henries} \tag{41}$$

$$Q = \frac{r_1}{d}\cdot \tag{42}$$

These values are realized in a "one-turn" air-core toroid of a continuous metal sheet. They are closely approximated in a coil of round wire wound with a pitch only slightly greater than the wire diameter.

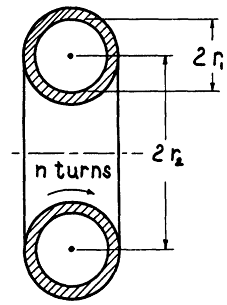


Fig. 10—Toroidal coil.

The preferable shapes of the air-core toroid of Fig. 10 involve a coil radius comparable with the ring radius, a departure from the above assumptions. This is of interest only in the "one-turn" case $(n=1)$ since a layer of wire cannot be wound with optimum pitch over the entire surface of the coil. The exact formula for the inductance is[38]

$$L_0 = \mu_0(r_2 - \sqrt{r_2^2 - r_1^2}) = \frac{\mu_0 r_1^2}{r_2 + \sqrt{r_2^2 - r_1^2}} \quad \text{henries.} \tag{43}$$

Since the penetration ∂z corresponds with ∂r_1,

$$\frac{\partial L_0}{\partial z} = \frac{\partial L_0}{\partial r_1} = \frac{\mu_0 r_1}{\sqrt{r_2^2 - r_1^2}}\cdot \tag{44}$$

From this formula and (18), the resistance is

$$R = \frac{r_1}{\sqrt{r_2^2 - r_1^2}} R_1 \tag{45}$$

and the ratio of reactance to resistance is

$$Q = \frac{r_1}{d}\cdot\frac{2\sqrt{r_2^2 - r_1^2}}{r_2 + \sqrt{r_2^2 - r_1^2}}\cdot \tag{46}$$

There is an optimum design for this case, with the coil diameter slightly less than the ring diameter, but the practical optimum is affected by so many factors that a theoretical optimum is of little value. If the ring diameter $2r_2$ is given, the optimum shape happens to be $r_1 = 0.78r_2$, in which case

$$Q = 0.60\frac{r_2}{d} = 0.77\frac{r_1}{d}\cdot \tag{47}$$

[38] Harnwell, footnote 14, p. 302.

There is another optimum design for a given outside diameter, $2(r_1+r_2)$:

$$\frac{r_1}{r_1+r_2} = 0.41; \quad \frac{r_2}{r_1+r_2} = 0.59$$

$$\frac{r_1}{r_2} = 0.70 \tag{48}$$

$$Q = 0.343\,\frac{r_1+r_2}{d} = 0.83\,\frac{r_1}{d}\cdot$$

The solenoidal coil of Fig. 11 has n turns wound on a radius a in an axial length b. If such a coil has a length much greater than its radius and is wound closely with rectangular wire of thickness much greater than the

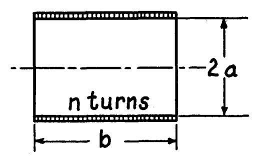


Fig. 11—Solenoidal coil.

depth of penetration, the current flows in a sheet on the inner surface of the wire, and the resistance is

$$R = \frac{2\pi a}{b}\,n^2R_1 \qquad \text{ohms.} \tag{49}$$

In a practical coil of many turns of round wire, there is an optimum diameter of wire slightly less than the pitch of winding. This formula is a rough approximation for practical coils with optimum wire diameter. It corresponds to a coil resistance slightly less than π times as great as the resistance of a straight wire of the same length and diameter. (The effect of distributed capacitance and dielectric resistance is omitted.) The inductance is approximately,[39] for $b>0.8a$

$$L_0 = \frac{\mu_0\pi a^2n^2}{b+0.9\,a} \qquad \text{henries.} \tag{50}$$

The corresponding ratio of reactance to resistance is approximately

$$Q = \frac{a}{d}\cdot\frac{1}{1+0.9\,a/b}\cdot \tag{51}$$

These simple formulas are applicable to coils in which the length is greater than the radius, the optimum wire diameter exceeds $4d$, and the number of turns exceeds about 4. In comparison with some recent measurements, these formulas check fairly well the component of resistance caused by the skin effect as distinguished from capacitance effects.[40]

A solenoidal coil in a coaxial tubular shield is shown in Fig. 12. The radius of coil and shield a_1 and a_2 determines the relative distribution of current on the inner and outer surfaces of the coil. The theoretical relations are based on the ideal long coil and shield, closely wound of rectangular wire, but the conclusions are approximately correct for practical coils. The magnetic intensity H_1 inside the coil and H_2 between coil

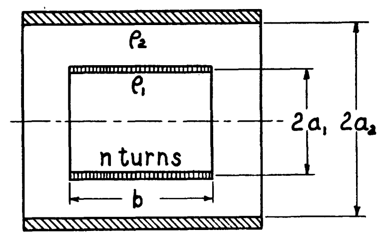


Fig. 12—Solenoidal coil in a coaxial tubular shield.

and shield are in the inverse ratio of the cross-sectional areas because all the flux inside the coil has to return in the space between coil and shield.

$$\frac{H_2}{H_1} = \frac{a_1^2}{a_2^2-a_1^2} = \frac{1}{a_2^2/a_1^2-1}\cdot \tag{52}$$

The power dissipation is divided among the inner and outer surfaces of the coil and the inner surface of the shield. By (15), the total is

$$\begin{aligned} P &= 2\pi a_1bH_1^2R_1 + 2\pi a_1bH_2^2R_1 + 2\pi a_2bH_2^2R_2 \\ &= 2\pi a_1bH_1^2R_1\left(1+\frac{H_2^2}{H_1^2}+\frac{a_2H_2^2R_2}{a_1H_1^2R_1}\right) \qquad \text{watts} \end{aligned} \tag{53}$$

in which R_1 and R_2 are the respective values of surface resistivity for the metals of coil and shield. By (14), the total current on both surfaces of the coil is

$$I = (H_1+H_2)b/n \qquad \text{amperes.} \tag{54}$$

Therefore, the effective resistance of the coil is

$$\begin{aligned} R &= \frac{P}{I^2} \\ &= \frac{2\pi a_1n^2R_1}{b}\cdot\frac{\left(1+\dfrac{H_2^2}{H_1^2}+\dfrac{a_2H_2^2R_2}{a_1H_1^2R_1}\right)}{\left(1+\dfrac{H_2}{H_1}\right)^2}\cdot \end{aligned} \tag{55}$$

The last factor gives the effect of the shield. It may actually reduce the resistance, by redistribution of surface currents, but not as much as it reduces the inductance. The effective inductance of the coil in the shield is

$$\begin{aligned} L_0 &= \frac{\mu_0}{\dfrac{b}{\pi a_1^2}+\dfrac{b}{\pi(a_2^2-a_1^2)}} \\ &= \frac{\pi a_1^2\mu_0}{b}\left(1-\frac{a_1^2}{a_2^2}\right) \qquad \text{henries} \end{aligned} \tag{56}$$

[39] H. A. Wheeler, "Simple inductance formulas for radio coils," Proc. I.R.E., vol. 16, pp. 1398–1400; October, 1928.
[40] F. E. Terman, "Radio Engineering," 1932/1937, pp. 37–42.

in which the last factor gives the reduction of inductance by the shield. With these substitutions,

$$x = \frac{a_1}{a_2}, \qquad r = \frac{R_2}{R_1} \tag{57}$$

and the ratio of reactance to resistance is

$$Q = \frac{a_2}{d} \cdot \frac{x(1 - x^2)}{1 - (2 - r)x^2 + 2x^4} . \tag{58}$$

This is expressed in terms of the shield radius a_2 since that determines the space in which the coil is located. The maximum Q is obtained if x satisfies the equation

$$0 = 1 - (1 + r)x^2 - (4 + r)x^4 + x^6. \tag{59}$$

This is most easily solved by trial. The solutions for the optimum design in several cases are as follows:

(60)

$r = \frac{R_2}{R_1}$	x^2	$x = \frac{a_1}{a_2}$	Q
0	0.41	0.64	$0.72 \frac{a_2}{d} = 1.14 \frac{a_1}{d}$
1	0.30	0.55	$0.44 \frac{a_2}{d} = 0.80 \frac{a_1}{d}$
2	0.23	0.48	$0.33 \frac{a_2}{d} = 0.70 \frac{a_1}{d}$
∞	$\frac{1}{r}$	$\frac{1}{\sqrt{r}}$	$\frac{1}{2\sqrt{r}} \frac{a_2}{d} = 0.50 \frac{a_1}{d}$.

An approximate formula for the optimum ratio of radii is given by the relation,

$$x^2 = \frac{1}{2.3 + r}$$
$$\frac{a_1}{a_2} = \frac{1}{\sqrt{2.3 + R_2/R_1}} . \tag{61}$$

This formula is exact for $r = 1, 2, \infty$. In the first two rows of the table (60), the coefficient in the last column indicates that the reduction of inductance by the shield is not accompanied by a proportionate reduction of Q. Therefore, the shield reduces the effective resistance in these cases. In practice, the shield always decreases the Q.

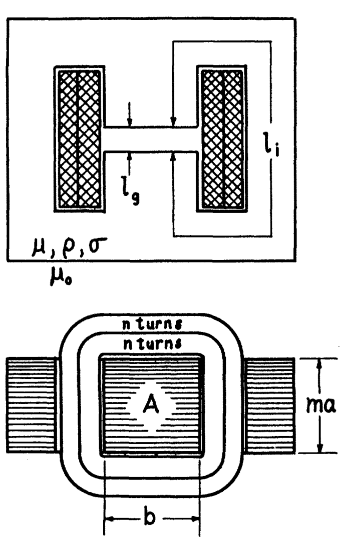


Fig. 13—Transformer with laminated iron core.

The transformer of Fig. 13 has a laminated iron core of cross-sectional area A. The flux path in the iron has a length l_i while that in the air gap has a length l_g. For simplicity of analysis, the two coils have the same number of turns n. If the actual number of turns is n_1 and n_2, the respective self-impedances and mutual impedance are obtained by letting

$$n^2 = n_1^2, \ n_2^2, \ n_1 n_2. \tag{62}$$

Fig. 14 shows the impedance network which is the equivalent of this transformer. The upper part represents the coil resistance and the part of the inductance caused by magnetic flux in the space outside the core, as if the core space had zero permeability. The lower

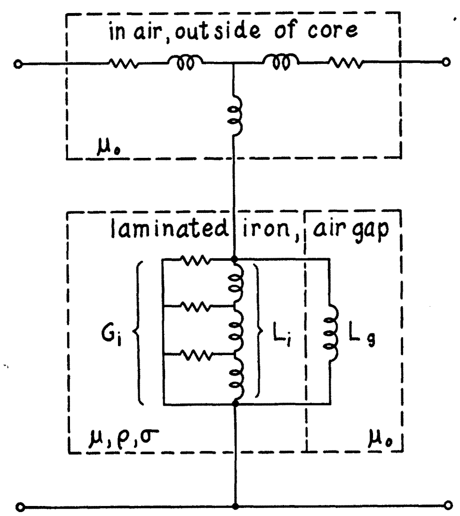


Fig. 14—The distributed-impedance network equivalent to the iron-core transformer.

part represents the impedance caused by the core, including the air gap. The inductance which would be caused by the flux in the iron core of permeability μ, with no air gap, is

$$L_i = \frac{\mu n^2 A}{l_i} \quad \text{henries.} \tag{63}$$

The inductance which would be caused by the flux in the air gap, if the iron core had infinite permeability, is

$$L_g = \frac{\mu_0 n^2 A}{l_g} \quad \text{henries.} \tag{64}$$

The inductance effective at low frequencies is that of L_i and L_g in parallel,

$$L_0 = \frac{L_i L_g}{L_i + L_g} = \frac{\mu_0 n^2 A}{l_g + l_i \mu_0/\mu} \quad \text{henries.} \tag{65}$$

The eddy currents and skin effect depend on the division of the core area into laminations,

$$A = mab \quad \text{square meters} \tag{66}$$

in which m is the number of laminations of thickness a and width b. The current paths in the laminations cause an apparent distributed conductance, associated with the iron inductance L_i, which has the value

$$G_i = \frac{\sigma a l_i}{4n^2 m b} \qquad \text{mhos} \quad (67)$$

in which σ is the conductivity of the iron. The effect of this distributed conductance is least at low frequencies and merges into the skin effect at high frequencies.

Fig. 15 shows a simplified equivalent network in which the shunt resistance R and inductance L have values depending on the frequency. These parallel components are used rather than series components, because the effective shunt resistance varies less with frequency than the effective series resistance.

At low frequencies, the apparent shunt conductance approaches the constant value

$$G = \tfrac{1}{3} G_i \qquad \text{mhos.} \quad (68)$$

The corresponding value of shunt resistance is

$$R = \frac{12n^2 mbd}{a l_i} R_1 \qquad \text{ohms.} \quad (69)$$

in which R_1 is the surface resistivity of the iron. The inductance L has its low-frequency value L_i. This is based on the assumption that the alternating flux within the lamination suffers only a small phase lag and no appreciable reduction in magnitude, which is true if the depth of penetration is greater than the thickness of laminations.[41] The corresponding ratio of shunt susceptance to conductance, is

$$Q = \frac{R}{\omega L} = 6 \frac{d^2}{a^2} \qquad (d \gg a). \quad (70)$$

At frequencies so high that the depth of penetration is less than 1/4 the thickness of laminations, the skin effect governs the impedance caused by the iron core. The effective impedance of R and L in parallel is the impedance of the line with distributed series L_i and shunt G_i:

$$Z = \frac{1}{1/R + 1/j\omega L} = \sqrt{\frac{j\omega L_i}{G_i}} = \frac{4n^2 mb}{(1-j) l_i} R_1 \qquad \text{ohms.} \quad (71)$$

The shunt components of this impedance have the value

$$R = \omega L = \frac{4n^2 mb}{l_i} R_1 \qquad \text{ohms.} \quad (72)$$

The apparent shunt inductance is

$$L = \frac{2d}{a} L_i \qquad \text{henries.} \quad (73)$$

This is the inductance based on twice the depth of penetration as the effective thickness of each lamination.

The air gap sometimes increases the ratio of reactance to resistance in the impedance of an iron-core inductor. This question involves the series resistance R_c of each coil, while the inductance in the space outside the coil is usually negligible. Increasing the air gap

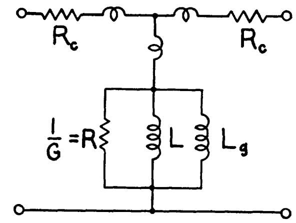


Fig. 15—The lumped-impedance network equivalent to the iron-core transformer.

decreases L_g, thereby causing more dissipation in the coils (R_c) and less in the core (R). The optimum length of air gap is approximately that which divides the dissipation equally between coil and core. The optimum condition is

$$\frac{1}{\omega L} + \frac{1}{\omega L_g} = \sqrt{\frac{1 + R_c/R}{R R_c}} \quad (74)$$

For this condition, the maximum ratio is[42]

$$Q = \frac{1}{2} \sqrt{\frac{R/R_c}{1 + R_c/R}} \cdot \quad (75)$$

This is nearly independent of the number of turns. Its value is expressed in terms of three properties of the coil; ρ_c is the resistivity of the copper wire, l_c is the average length of wire per turn, and A_c is the total cross-sectional area of the turns of wire on the winding in question. In a self-inductor of one coil, A_c is somewhat less than the area of each window. The following formulas are simplified on the assumption that $R \gg R_c$ so the optimum air gap gives $Q \gg 1$. At the higher frequencies, where the skin effect predominates, the optimum air gap gives

$$Q = \sqrt{\frac{A A_c \rho}{a d l_i l_c \rho_c}} \qquad (a > 4d). \quad (76)$$

At the lower frequencies, where eddy currents are induced by nearly uniform flux in the laminations, the optimum air gap gives[43]

$$Q = \sqrt{\frac{3 A A_c \rho}{a^2 l_i l_c \rho_c}} \qquad (a < d). \quad (77)$$

These maximum values cannot be realized if the optimum length of air gap is negative. This is true at very low frequencies where the eddy currents are negligible, in which case the air gap is reduced to zero, giving

$$Q = \frac{\omega L_i}{R_c} \qquad (\omega L_i \ll \sqrt{R R_c}) \\ = \frac{2 A A_c \rho}{d^2 l_i l_c \rho_c} = \frac{2 A A_c \mu}{d_c^2 l_i l_c \mu_0} \quad (78)$$

in which d and d_c are the depths of penetration in iron

[41] V. E. Legg, "Survey of magnetic materials and applications in the telephone system," *Bell. Sys. Tech. Jour.*, vol. 18, pp. 438–464, July 1939. (In Fig. 7, θ is the ratio of thickness to depth of penetration.)

[42] L. A. Arguimbau, "Losses in audio-frequency coils," General Radio Experimenter, vol. 11, no. 6, pp. 1–4, November, 1936.

[43] P. K. McElroy and R. F. Field, "How good is an iron-cored coil?" General Radio Experimenter, vol. 16, no. 10, pp. 1–12, March, 1942.

and copper, the materials of the core and the coil. In these formulas, the depth of penetration includes the frequency dimension, enabling the expression entirely in terms of ratios. The value of (78) is actually independent of the ρ of the iron and the μ_0 of the copper, those being involved also in the depth of penetration.

Since copper is the usual material for conductors, it is useful to remember the depth of penetration in copper at a certain frequency and room temperature (20 degrees centigrade):

$$\begin{aligned} At \quad f &= 10^6 \text{ cycles} = 1\ Mc, \\ d_c' &= 66\cdot 10^{-6} \text{ meter} = 0.066 \text{ mm} = 66 \text{ microns} \\ &= 2.6\cdot 10^{-3} \text{ inch} = 2.6 \text{ mils.} \end{aligned} \tag{79}$$

The values for copper and other materials (at a temperature of 0 degrees centigrade) are found in the Steinmetz[3] table, p. 385. The essential properties of copper are (at 20 degrees centigrade):

$$\begin{aligned} \mu_c = \mu_0 &= 4\pi\cdot 10^{-7} \text{ henry per meter} \\ &= 1.257 \text{ microhenrys per meter} \\ \sigma_c &= 5.80\cdot 10^7 \text{ mhos per meter} \\ &= 58 \text{ megamhos per meter} \\ \rho_c &= 1.724\cdot 10^{-8} \text{ ohm-meter} \\ &= 1/58 \text{ microhm-meter} \\ \mu_0\sigma_c &= 72.8 \text{ seconds per square meter} \\ \mu_0\rho_c &= 2.17\cdot 10^{-14} \text{ ohm}^2\text{-second} \end{aligned} \tag{80}$$

in which μ_0 is the permeability of space. The other important value for copper is the surface resistivity, still at 1 megacycle:

$$\begin{aligned} R_{1c}' &= 2.60\cdot 10^{-4} \text{ ohm} \\ &= 0.260 \text{ milohm.} \end{aligned} \tag{81}$$

In order to convert d_c and R_{1c} for other materials, it is necessary to know only their permeability and resistivity relative to copper:

$$\begin{aligned} d &= d_c' \sqrt{\frac{1 \text{ Mc}}{f}\,\frac{\mu_0}{\mu}\,\frac{\rho}{\rho_c}} \\ R_1 &= R_{1c}' \sqrt{\frac{f}{1 \text{ Mc}}\,\frac{\mu}{\mu_0}\,\frac{\rho}{\rho_c}}\,. \end{aligned} \tag{82}$$

The chart of Fig. 1 gives the depth of penetration d and the surface resistivity R_1 plotted against frequency. Each pair of crossed lines is for one material. Some of the materials shown are chosen for their extreme properties (at least, among the common materials). Copper has the least resistivity. The permalloy shown (78 per cent nickel) is used for loading submarine telegraph cables and for shielding against alternating magnetic fields; it has the least depth of penetration, by virtue of its high permeability and small resistivity:

$$\begin{aligned} \mu &= 9000\ \mu_0 \text{ (at small flux density)} \\ \rho &= 9.3\ \rho_c = 0.16 \text{ microhm-meter} \\ &At \text{ 1 Mc} \\ d' &= 2.1\cdot 10^{-6} \text{ meter} = 0.084 \text{ mil} \\ R_1' &= 75 \text{ milohms} \end{aligned} \tag{83}$$

Manganin is the material usually used in resistance standards; it has about the highest resistivity compatible with minimum permeability, and therefore the greatest depth of penetration:

$$\begin{aligned} \mu &= \mu_0 \\ \rho &= 25.5\ \rho_c = 0.44 \text{ microhm-meter} \\ &At \text{ 1 Mc} \\ d' &= 0.33 \text{ mm} = 13 \text{ mils} \\ R_1' &= 1.3 \text{ milohms.} \end{aligned} \tag{84}$$

Most of the ordinary materials fall within the limits of these three cases.

On the chart, the intersection of each pair of lines moves upward with increasing resistivity and toward the left with increasing permeability. (It is purely coincidental that the intersection is at 1 megacycle for nonmagnetic materials.)

In this collection of formulas, the properties of the conductive materials are usually expressed in terms of depth of penetration d and surface resistivity R_1, both of which involve also the frequency. The former appears in ratios with other "length" dimensions. The latter appears in impedance formulas, where it brings in the "resistance" dimension. Other quantities usually appear in ratios so they do not complicate the dimensions. The two parameters d and R_1 are most useful because they have not only dimensional simplicity but also obvious physical significance.