COLLECTED TRANSMISSION-LINE IDENTITIES

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(rev. 1.2)

Introduction

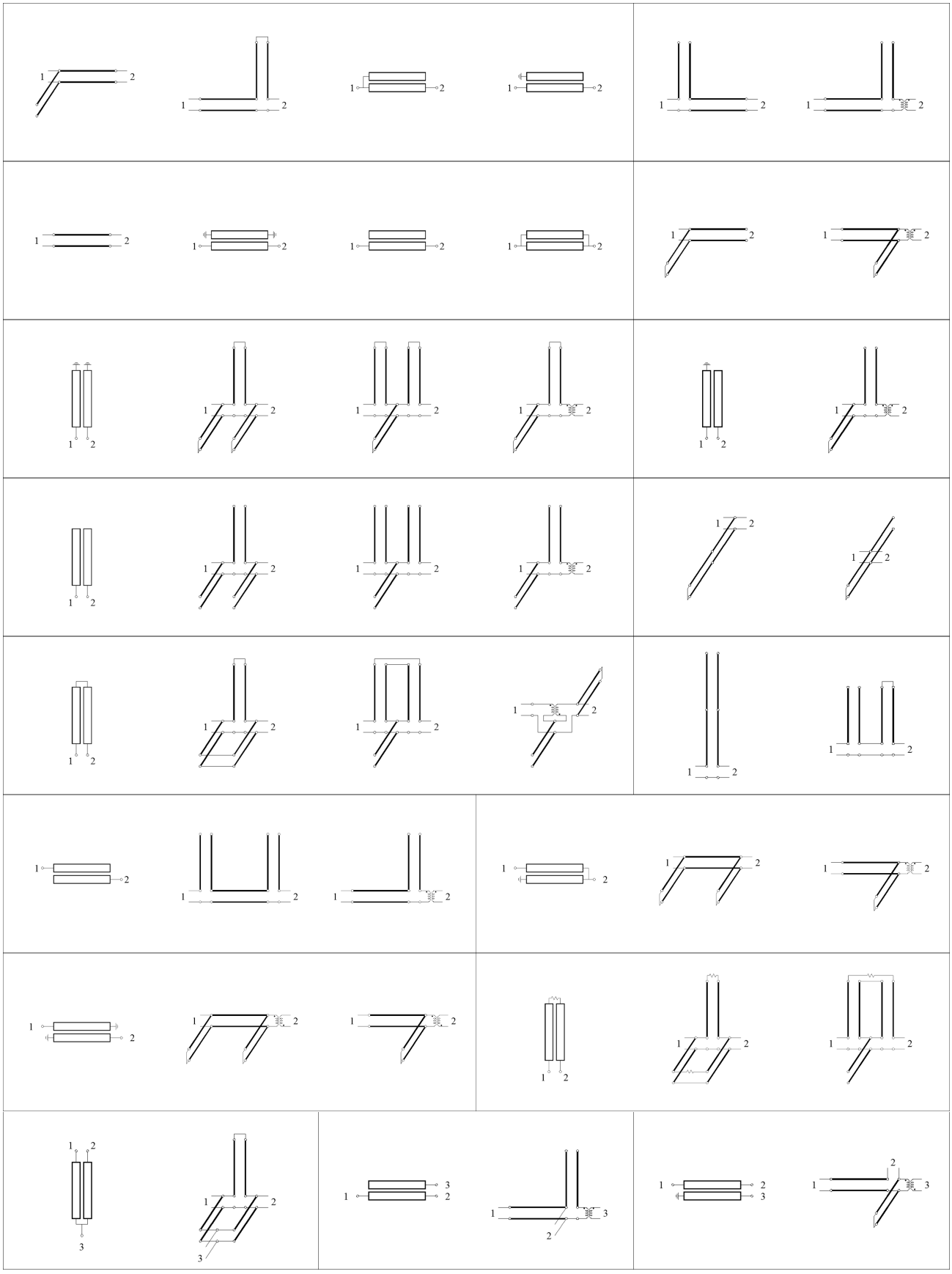
Transmission-line identities are a useful tool in the early stages of passive network design, and many such identities are well known and reported in the literature. To the author's knowledge, however, there is no single source a researcher can go to for a comprehensive list of all known identities, including those less commonly used, for easy reference. What follows is an attempt to collect all known transmission-line identities into one place to serve that purpose.

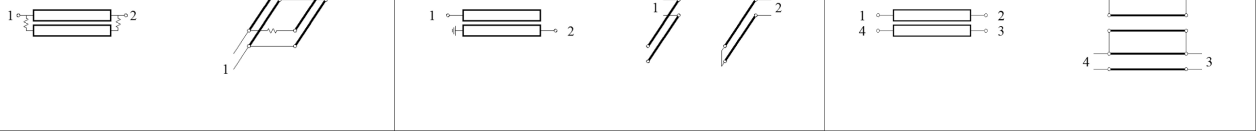
Section I contains a master chart with all identity sets shown topologically. All networks within a single box on this chart may be considered equivalent with the proper selection of values. The formulas for these values are given in the following sections. Two-port identity sets are covered in Section II, three-port sets in Section III, and four-ports in Section IV. Both impedance and admittance expressions have been used where appropriate to keep the formulas as compact as possible.

The intent is for this to be a living document, and to make it as complete as possible. To that end, the author welcomes feedback regarding any identities that may be missing, corrections to formulas, or references for identities that are already included if none is provided or an earlier one is found. Such contributions will be gratefully noted in the Acknowledgements section.

A few notes about what constitutes an "identity": An identity is a pair of circuits or networks for which all port parameters are mathematically identical at all frequencies. Approximate matches are not considered identities, nor are networks which match only at discrete frequencies. For brevity, this document will not include trivial combinations or repetition of known identities unless they are particularly revealing. Nor will it include networks which depend on hypothetical elements with no practical realization, such as all-frequency quadrature hybrids, ideal impedance/admittance inverters, or negative-length transmission lines, unless the identity enables the elimination of such elements to make the networks realizable.

I. Identity Sets

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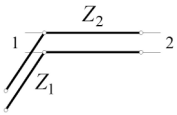
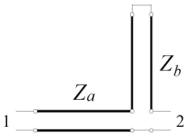
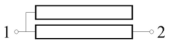
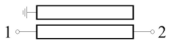
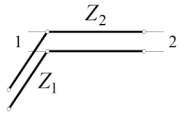
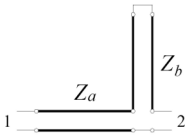
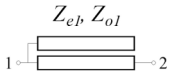
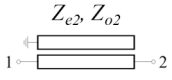
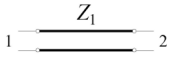
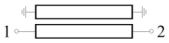
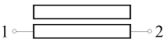
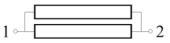
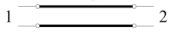
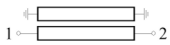
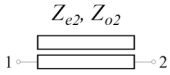
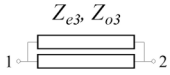
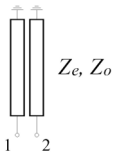
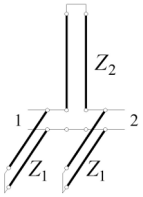
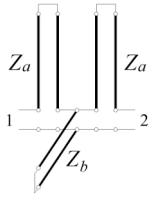
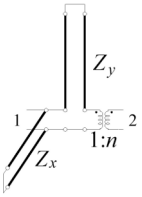
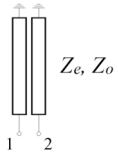
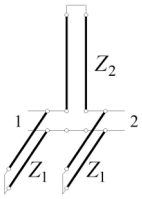
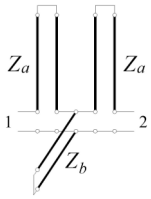
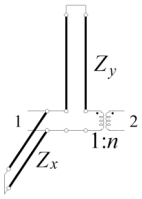
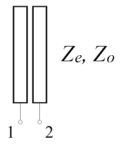
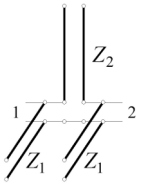
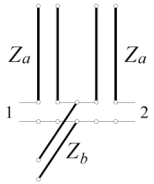
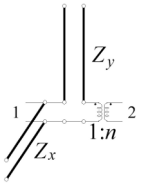
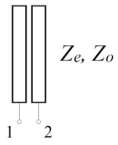
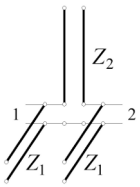
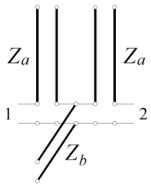
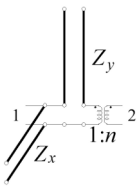
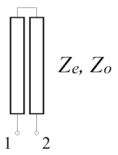
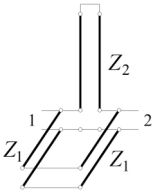
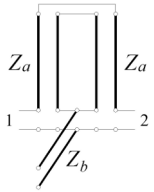
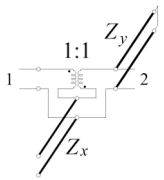
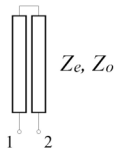
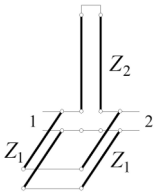
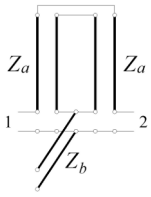
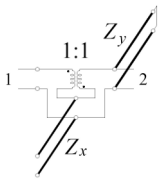
II. Two-Port Identities

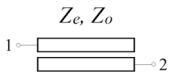
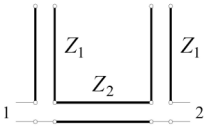
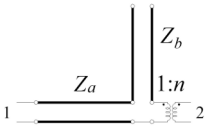
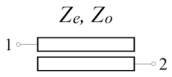
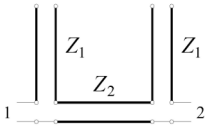
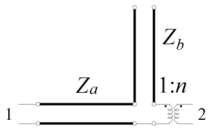
				
		$Z_1 = Z_a\left(1+\frac{Z_a}{Z_b}\right)$ $Z_2 = Z_a + Z_b$ [1], [3], [4]	$Z_1 = \frac{1}{2}Z_{e1}\left(1+\frac{Z_{e1}}{Z_{o1}}\right)$ $Z_2 = \frac{1}{2}\left(Z_{e1}+Z_{o1}\right)$	$Y_1 = \frac{1}{2}\frac{\left(Y_{o2}-Y_{e2}\right)^2}{Y_{o2}+Y_{e2}}$ $Z_2 = \frac{1}{2}\left(Z_{e2}+Z_{o2}\right)$ [1], [3], [4]
	$Z_a = \frac{Z_1 Z_2}{Z_1+Z_2}$ $Z_b = \frac{Z_2^2}{Z_1+Z_2}$ [1], [3], [4]		$Z_a = \frac{1}{2}Z_{e1}$ $Z_b = \frac{1}{2}Z_{o1}$	$Y_a = \frac{1}{2}\left(Y_{o2}+Y_{e2}\right)$ $Z_b = \frac{1}{2}\frac{\left(Z_{e2}-Z_{o2}\right)^2}{Z_{e2}+Z_{o2}}$ [1], [3], [4]
	$Z_{e1} = 2\frac{Z_1 Z_2}{Z_1+Z_2}$ $Z_{o1} = 2\frac{Z_2^2}{Z_1+Z_2}$	$Z_{e1} = 2Z_a$ $Z_{o1} = 2Z_b$		$Y_{e1} = \frac{1}{4}\left(Y_{o2}+Y_{e2}\right)$ $Z_{o1} = \frac{\left(Z_{e2}-Z_{o2}\right)^2}{Z_{e2}+Z_{o2}}$
	$Z_{e2} = Z_2\left(1+\sqrt{\frac{Z_2}{Z_1+Z_2}}\right)$ $Z_{o2} = Z_2\left(1-\sqrt{\frac{Z_2}{Z_1+Z_2}}\right)$ [1], [3], [4]	$Z_{e2} = Z_a + Z_b + \sqrt{Z_b\left(Z_a+Z_b\right)}$ $Z_{o2} = Z_a + Z_b - \sqrt{Z_b\left(Z_a+Z_b\right)}$ [1], [3], [4]	$Z_{e2} = \frac{1}{2}\left(Z_{e1}+Z_{o1}\right) + \frac{1}{2}\sqrt{Z_{o1}\left(Z_{e1}+Z_{o1}\right)}$ $Z_{o2} = \frac{1}{2}\left(Z_{e1}+Z_{o1}\right) - \frac{1}{2}\sqrt{Z_{o1}\left(Z_{e1}+Z_{o1}\right)}$	

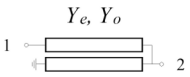
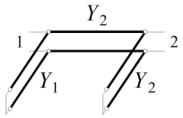
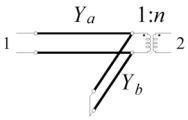
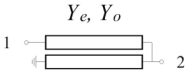
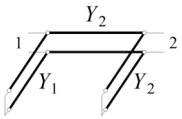
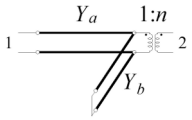
	Z_1 	Y_{e1}, Y_{o1} 	Z_{e2}, Z_{o2} 	Z_{e3}, Z_{o3} 
Z_1 		$Y_1 = \frac{1}{2}\left(Y_{e1} + Y_{o1}\right)$ [1], [3], [4]	$Z_1 = \frac{1}{2}\left(Z_{e2} + Z_{o2}\right)$ [1], [3], [4]	$Z_1 = \frac{1}{2} Z_{e3}$
Y_{e1}, Y_{o1} 	$Y_{e1} = Y_1 \frac{2}{1+\rho}$ $Y_{o1} = Y_1 \frac{2\rho}{1+\rho}$ ρ = free [1], [3], [4]		$Z_{e1} = \left(Z_{e2} + Z_{o2}\right)\frac{1+\rho}{4}$ $Z_{o1} = \left(Z_{e2} + Z_{o2}\right)\frac{1+\rho}{4\rho}$ ρ = free [1], [3], [4]	$Z_{e1} = Z_{e3} \frac{1+\rho}{4}$ $Z_{o1} = Z_{e3} \frac{1+\rho}{4\rho}$ ρ = free
Z_{e2}, Z_{o2} 	$Z_{e2} = Z_1 \frac{2\rho}{1+\rho}$ $Z_{o2} = Z_1 \frac{2}{1+\rho}$ ρ = free [1], [3], [4]	$Y_{e2} = \left(Y_{e1} + Y_{o1}\right)\frac{1+\rho}{4\rho}$ $Y_{o2} = \left(Y_{e1} + Y_{o1}\right)\frac{1+\rho}{4}$ ρ = free [1], [3], [4]		$Z_{e2} = Z_{e3} \frac{\rho}{1+\rho}$ $Z_{o2} = Z_{e3} \frac{1}{1+\rho}$ ρ = free
Z_{e3}, Z_{o3} 	$Z_{e3} = 2Z_1$ Z_{o3} = free	$Z_{e3} = \frac{4}{Y_{e1} + Y_{o1}}$ Z_{o3} = free	$Z_{e3} = Z_{e2} + Z_{o2}$ Z_{o3} = free	

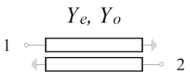
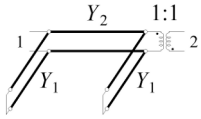
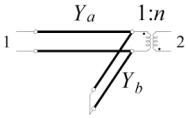
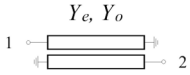
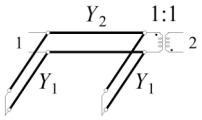
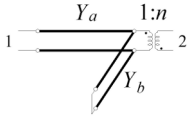
				
		$Y_e = Y_1$ $Y_o = Y_1 + 2Y_2$ [1], [3], [4]	$Z_e = Z_a + 2Z_b$ $Z_o = Z_a$ [3]	if $n^2 = \frac{Z_x}{Z_x + Z_y}$, then : $Z_e = Z_x(n+1)$ $Z_o = Z_y \frac{n^2}{n+1}$
	$Y_1 = Y_e$ $Y_2 = \frac{1}{2}(Y_o - Y_e)$ [1], [3], [4]		$Z_1 = Z_a + 2Z_b$ $Z_2 = (Z_a + 2Z_b)\frac{Z_a}{Z_b}$ [3]	if $n^2 = \frac{Z_x}{Z_x + Z_y}$, then : $Z_1 = Z_x(n+1)$ $Z_2 = Z_y n$
	$Z_a = Z_o$ $Z_b = \frac{1}{2}(Z_e - Z_o)$ [3]	$Y_a = Y_1 + 2Y_2$ $Y_b = (Y_1 + 2Y_2)\frac{Y_1}{Y_2}$ [3]		if $n^2 = \frac{Z_x}{Z_x + Z_y}$, then : $Z_a = Z_y \frac{n^2}{n+1}$ $Z_b = Z_x n$
	$Z_x = \frac{1}{2}(Z_e + Z_o)$ $Y_y = \frac{1}{2}\frac{(Y_o - Y_e)^2}{Y_o + Y_e}$ $n = \frac{Z_e - Z_o}{Z_e + Z_o}$	$Z_x = \frac{Z_1}{1+n}$ $Z_y = \frac{Z_2}{n}$ $n = \frac{Z_1}{Z_1 + Z_2}$	$Z_x = \frac{Z_b}{n}$ $Z_y = Z_a \frac{n+1}{n^2}$ $n = \frac{Z_b}{Z_a + Z_b}$	

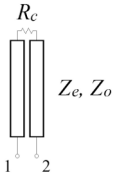
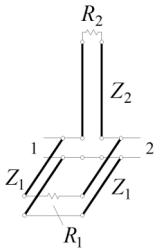
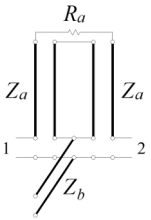
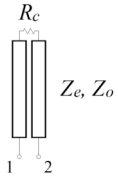
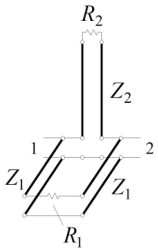
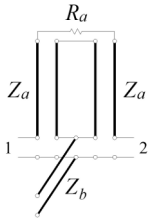
				
		$Y_e = Y_1$ $Y_o = Y_1 + 2Y_2$	$Z_e = Z_a + 2Z_b$ $Z_o = Z_a$ [1], [3], [4]	if $n^2 = \frac{Z_x}{Z_x + Z_y}$, then : $Z_e = Z_x(n+1)$ $Z_o = Z_y \frac{n^2}{n+1}$
	$Y_1 = Y_e$ $Y_2 = \frac{1}{2}(Y_o - Y_e)$		$Z_1 = Z_a + 2Z_b$ $Z_2 = (Z_a + 2Z_b)\frac{Z_a}{Z_b}$	if $n^2 = \frac{Z_x}{Z_x + Z_y}$, then : $Z_1 = Z_x(n+1)$ $Z_2 = Z_y n$
	$Z_a = Z_o$ $Z_b = \frac{1}{2}(Z_e - Z_o)$ [1], [3], [4]	$Y_a = Y_1 + 2Y_2$ $Y_b = (Y_1 + 2Y_2)\frac{Y_1}{Y_2}$		if $n^2 = \frac{Z_x}{Z_x + Z_y}$, then : $Z_a = Z_y \frac{n^2}{n+1}$ $Z_b = Z_x n$
	$Z_x = \frac{1}{2}(Z_e + Z_o)$ $Y_y = \frac{1}{2}\frac{(Y_o - Y_e)^2}{Y_o + Y_e}$ $n = \frac{Z_e - Z_o}{Z_e + Z_o}$	$Z_x = \frac{Z_1}{1+n}$ $Z_y = \frac{Z_2}{n}$ $n = \frac{Z_1}{Z_1 + Z_2}$	$Z_x = \frac{Z_b}{n}$ $Z_y = Z_a \frac{n+1}{n^2}$ $n = \frac{Z_b}{Z_a + Z_b}$	

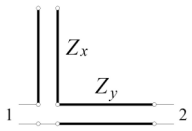
				
		$Z_e = Z_1$ $Y_o = Y_1 + 2Y_2$	$Z_e = Z_a + 2Z_b$ $Z_o = Z_a$	$Z_e = 2Z_x$ $Z_o = 2Z_y$ [1], [4]
	$Z_1 = Z_e$ $Y_2 = \frac{1}{2}(Y_o - Y_e)$		$Z_1 = Z_a + 2Z_b$ $Z_2 = \frac{Z_a}{Z_b}(Z_a + 2Z_b)$	$Z_1 = 2Z_x$ $Y_2 = \frac{1}{4}(Y_y - Y_x)$
	$Z_a = Z_o$ $Z_b = \frac{1}{2}(Z_e - Z_o)$	$Y_a = Y_1 + 2Y_2$ $Y_b = \frac{Y_1}{Y_2}(Y_1 + 2Y_2)$		$Z_a = 2Z_x$ $Z_b = Z_x - Z_y$
	$Z_x = \frac{1}{2}Z_e$ $Z_y = \frac{1}{2}Z_o$ [1], [4]	$Z_x = \frac{1}{2}Z_1$ $Y_y = 2(Y_1 + 2Y_2)$	$Z_x = \frac{1}{2}(Z_a + 2Z_b)$ $Z_y = \frac{1}{2}Z_a$	

			
		$Z_e = Z_1 + 2Z_2$ $Z_o = Z_1$ [1], [3], [4]	if $n^2 = \frac{Z_a}{Z_a + Z_b}$, then : $Z_e = Z_a(1+n)$ $Z_o = Z_a(1-n)$
	$Z_1 = Z_o$ $Z_2 = \frac{1}{2}(Z_e - Z_o)$ [1], [3], [4]		if $n^2 = \frac{Z_a}{Z_a + Z_b}$, then : $Z_1 = Z_a(1-n)$ $Z_2 = Z_a n$
	$Z_a = \frac{1}{2}(Z_e + Z_o)$ $Y_b = \frac{1}{2}\frac{(Y_o - Y_e)^2}{(Y_o + Y_e)}$ $n = \frac{Z_e - Z_o}{Z_e + Z_o}$	$Z_a = \frac{Z_1}{1-n}$ $Z_b = Z_1 \frac{1+n}{n^2}$ $n = \frac{Z_2}{Z_1 + Z_2}$	

			
		$Y_e = Y_2$ $Y_o = 2Y_1 + Y_2$	if $n = \frac{1}{2}\left(1 + \frac{Y_b}{Y_a}\right)$, then : $Y_e = Y_a \frac{1}{n}$ $Y_o = Y_b \frac{1}{n}$
	$Y_1 = \frac{1}{2}\left(Y_o - Y_e\right)$ $Y_2 = Y_e$		if $n = \frac{1}{2}\left(1 + \frac{Y_b}{Y_a}\right)$, then : $Y_1 = \frac{1}{2n}\left(Y_b - Y_a\right)$ $Y_2 = Y_a \frac{1}{n}$
	$Y_a = Y_e n$ $Y_b = Y_o n$ $n = \frac{1}{2}\left(1 + \frac{Y_o}{Y_e}\right)$	$Y_a = Y_2 n$ $Y_b = \left(2Y_1 + Y_2\right)n$ $n = \frac{Y_1 + Y_2}{Y_2}$	

			
		$Y_e = Y_1$ $Y_o = Y_1 + 2Y_2$ [1], [3], [4]	if $n^2 = \frac{Y_b}{Y_a} + 1$, then : $Y_e = Y_a\left(1 - \frac{1}{n}\right)$ $Y_o = Y_a\left(1 + \frac{1}{n}\right)$
	$Y_1 = Y_e$ $Y_2 = \frac{1}{2}\left(Y_o - Y_e\right)$ [1], [3], [4]		if $n^2 = \frac{Y_b}{Y_a} + 1$, then : $Y_1 = Y_a\left(1 - \frac{1}{n}\right)$ $Y_2 = Y_a\,\frac{1}{n}$
	$Y_a = \frac{1}{2}\left(Y_o + Y_e\right)$ $Z_b = \frac{1}{2}\frac{\left(Z_e - Z_o\right)^2}{Z_e + Z_o}$ $n = \frac{Y_o + Y_e}{Y_o - Y_e}$	$Y_a = Y_1 + Y_2$ $Y_b = Y_1 n\left(1 + n\right)$ $n = 1 + \frac{Y_1}{Y_2}$	

			
		if $\frac{R_1}{R_2} = 2\frac{Z_1}{Z_2}$, then : $Z_e = Z_1$ $Y_o = Y_1 + 2Y_2$ $R_c = \frac{R_1 R_2}{R_1 + R_2}$	$Z_e = Z_a + 2Z_b$ $Z_o = Z_a$ $R_c = R_a$
	$Z_1 = Z_e$ $Y_2 = \frac{1}{2}(Y_o - Y_e)$ $R_1 = \frac{Z_e}{Z_o} R_c$ $R_2 = \frac{Z_e}{Z_e - Z_o} R_c$		$Z_1 = Z_a + 2Z_b$ $Z_2 = \frac{Z_a}{Z_b}(Z_a + 2Z_b)$ $R_1 = \left(1 + 2\frac{Z_b}{Z_a}\right) R_a$ $R_2 = \left(1 + \frac{Z_a}{2Z_b}\right) R_a$
	$Z_a = Z_o$ $Z_b = \frac{1}{2}(Z_e - Z_o)$ $R_a = R_c$	if $\frac{R_1}{R_2} = 2\frac{Z_1}{Z_2}$, then : $Y_a = Y_1 + 2Y_2$ $Y_b = \frac{Y_1}{Y_2}(Y_1 + 2Y_2)$ $R_a = \frac{R_1 R_2}{R_1 + R_2}$	

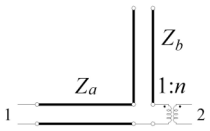


if $n = \dfrac{Z_a}{Z_a + Z_b}$, then :

$Z_x = Z_b n$

$Z_y = Z_a n$

[1], [3], [4]

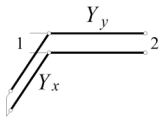


$Z_a = Z_y \frac{1}{n}$

$Z_b = Z_x \frac{1}{n}$

$n = \dfrac{Z_y}{Z_x + Z_y}$

[1], [3], [4]

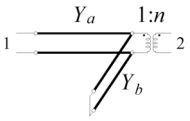


if $n = 1 + \dfrac{Y_b}{Y_a}$, then :

$Y_x = Y_b \frac{1}{n}$

$Y_y = Y_a \frac{1}{n}$

[1], [3], [4]

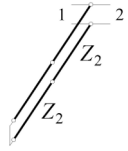


$Y_a = Y_y n$

$Y_b = Y_x n$

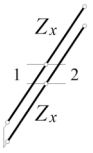
$n = 1 + \dfrac{Y_x}{Y_y}$

[1], [3], [4]



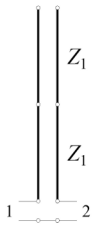
$Z_1 = \frac{1}{2} Z_x$

[2]



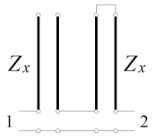
$Z_x = 2Z_1$

[2]



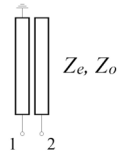
$Z_1 = 2Z_x$

[2]



$Z_x = \frac{1}{2} Z_1$

[2]

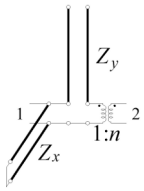


if $n^2 = \dfrac{Z_x}{Z_x + Z_y}$, then :

$Z_e = Z_x (n+1)$

$Z_o = Z_y \dfrac{n^2}{n+1}$

[1], [3], [4]

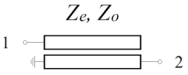


$Z_x = \frac{1}{2}(Z_e + Z_o)$

$Y_y = \frac{1}{2} \dfrac{(Y_o - Y_e)^2}{Y_o + Y_e}$

$n = \dfrac{Z_e - Z_o}{Z_e + Z_o}$

[1], [3], [4]

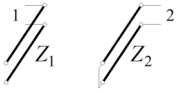


if $Z_2 > Z_1$, then :

$Z_e = Z_2 + \sqrt{Z_2 (Z_2 - Z_1)}$

$Z_o = Z_2 - \sqrt{Z_2 (Z_2 - Z_1)}$

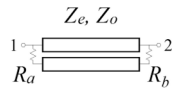
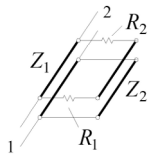
[1], [3], [4]



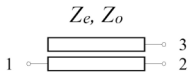
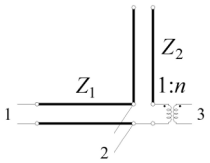
$Y_1 = \frac{1}{2}(Y_o + Y_e)$

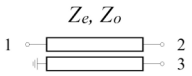
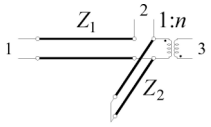
$Z_2 = \frac{1}{2}(Z_e + Z_o)$

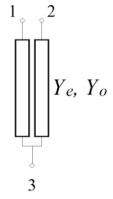
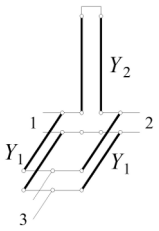
[1], [3], [4]

	
$Z_e = \dfrac{2Z_1Z_2}{Z_1 + Z_2}$ $Z_o = \dfrac{2Z_1^2}{Z_1 + Z_2}$ $R_a = \dfrac{4R_1Z_1^2}{(Z_1 + Z_2)^2}$ $R_b = \dfrac{4R_2Z_1^2}{(Z_1 + Z_2)^2}$	$Z_1 = \frac{1}{2}(Z_e + Z_o)$ $Z_2 = \frac{1}{2}\dfrac{Z_e}{Z_o}(Z_e + Z_o)$ $R_1 = \frac{1}{4}R_a\left(\dfrac{Z_e}{Z_o} + 1\right)^2$ $R_2 = \frac{1}{4}R_b\left(\dfrac{Z_e}{Z_o} + 1\right)^2$

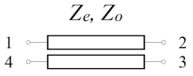
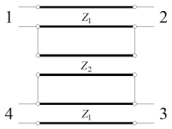
III. Three-Port Identities

	
if $n^2 = \frac{Z_1}{Z_1 + Z_2}$, then : $Z_e = Z_1(1+n)$ $Z_o = Z_1(1-n)$ [5]	$Z_1 = \frac{1}{2}(Z_e + Z_o)$ $Y_2 = \frac{1}{2}\frac{(Y_o - Y_e)^2}{(Y_o + Y_e)}$ $n = \frac{Z_e - Z_o}{Z_e + Z_o}$ [5]

	
if $n^2 = 1 + \frac{Z_1}{Z_2}$, then : $Z_e = Z_2 n(n+1)$ $Z_o = Z_2 n(n-1)$	$Y_1 = \frac{1}{2}(Y_o + Y_e)$ $Z_2 = \frac{1}{2}\frac{(Z_e - Z_o)^2}{(Z_e + Z_o)}$ $n = \frac{Z_e + Z_o}{Z_e - Z_o}$

	
$Y_e = Y_1$ $Y_o = Y_1 + 2_1 Y_2$	$Y_1 = Y_e$ $Y_2 = \frac{1}{2}(Y_o - Y_e)$

IV. Four-Port Identities

	
$Z_e = Z_1$ $Y_o = Y_1 + 2Y_2$	$Z_1 = Z_e$ $Y_2 = \frac{1}{2}(Y_o - Y_e)$

References

[1] H. Ozaki and J. Ishii, "Synthesis of a class of strip-line filters," IRE Trans. Circuit Theory, pp. 104-109, June 1958.
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[4] R. Wenzel, "Exact design of TEM microwave networks using quarter-wave lines," IEEE Trans. Microw. Theory Techn., pp. 94-11, January 1964.
[5] M. Morgan and T. Boyd, "Reflectionless filter structures," IEEE Trans. Microw. Theory Techn., vol. 63, no. 4, pp. 1263-1271, April 2015.