Preface

This Instructors' Manual provides solutions to most of the problems in ANTENNAS: FOR ALL APPLICATIONS, THIRD EDITION. All problems are solved for which answers appear in Appendix F of the text, and in addition, solutions are given for a large fraction of the other problems. Including multiple parts, there are 600 problems in the text and solutions are presented here for the majority of them.

Many of the problem titles are supplemented by key words or phrases alluding to the solution procedure. Answers are indicated. Many tips on solutions are included which can be passed on to students.

Although an objective of problem solving is to obtain an answer, we have endeavored to also provide insights as to how many of the problems are related to engineering situations in the real world.

The Manual includes an index to assist in finding problems by topic or principle and to facilitate finding closely-related problems.

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Chapter 2. Antenna Basics

2-7-1. Directivity.

Show that the directivity D of an antenna may be written

$$D = \frac{\dfrac{E(\theta,\phi)_{\max} E^*(\theta,\phi)_{\max}}{Z} r^2}{\dfrac{1}{4\pi}\iint_{4\pi} \dfrac{E(\theta,\phi)E^*(\theta,\phi)}{Z} r^2 d\Omega}$$

Solution:

, $\quad U(\theta,\phi)_{\max} = S(\theta,\phi)_{\max} r^2, \quad U_{av} = \dfrac{1}{4\pi}\iint_{4\pi} U(\theta,\phi) d\Omega$

$$U(\theta,\phi) = S(\theta,\phi) r^2, \quad S(\theta,\phi) = \frac{E(\theta,\phi)E^*(\theta,\phi)}{Z}$$

Therefore

$$D = \frac{\dfrac{E(\theta,\phi)_{\max} E^*(\theta,\phi)_{\max}}{Z} r^2}{\dfrac{1}{4\pi}\iint_{4\pi} \dfrac{E(\theta,\phi)E^*(\theta,\phi)}{Z} r^2 d\Omega} \qquad \text{q.e.d.}$$

Note that $r^2 =$ area/steradian, so $U = Sr^2$ or (watts/steradian) = (watts/meter^2) $\times$ meter^2

2-7-2. Approximate directivities.

Calculate the approximate directivity from the half-power beam widths of a unidirectional antenna if the normalized power pattern is given by: (a) $P_n = \cos\theta$, (b) $P_n = \cos^2\theta$, (c) $P_n = \cos^3\theta$, and (d) $P_n = \cos^n\theta$. In all cases these patterns are unidirectional (+z direction) with P_n having a value only for zenith angles $0° \le \theta \le 90°$ and $P_n = 0$ for $90° \le \theta \le 180°$. The patterns are independent of the azimuth angle ϕ.

Solution:

(a) $\theta_{HP} = 2\cos^{-1}(0.5) = 2 \times 60° = 120°$, $\qquad D = \dfrac{40{,}000}{(120)^2} = 278$ (*ans.*)

(b) $\theta_{HP} = 2\cos^{-1}(\sqrt{0.5}) = 2 \times 45° = 90°$, $\qquad D = \dfrac{40{,}000}{(90)^2} = 4.94$ (*ans.*)

(c) $\theta_{\text{HP}} = 2\cos^{-1}(\sqrt[3]{0.5}) = 2\times 37.47^{\text{o}} = 74.93^{\text{o}}$, $\qquad D = \dfrac{40{,}000}{(75)^2} = 7.3$ (*ans.*)

2-7-2. continued

(d) $\theta_{\text{HP}} = 2\cos^{-1}(\sqrt[n]{0.5})$, $\qquad D = \dfrac{10{,}000}{(\cos^{-1}(\sqrt[n]{0.5}))^2}$ (*ans.*)

***2-7-3. Approximate directivities.**

Calculate the approximate directivities from the half-power beam widths of the three unidirectional antennas having power patterns as follows:

$$P(\theta,\phi) = P_m \sin\theta \sin^2\phi$$

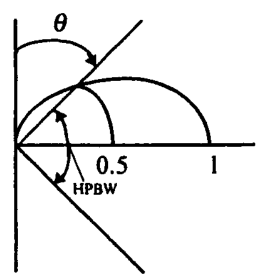
$$P(\theta,\phi) = P_m \sin\theta \sin^3\phi$$

$$P(\theta,\phi) = P_m \sin^2\theta \sin^3\phi$$

$P(\theta,\phi)$ has a value only for $0 \le \theta \le \pi$ and $0 \le \phi \le \pi$ and is zero elsewhere.

Solution:

To find D using approximate relations,



we first must find the half-power beamwidths.

$$\frac{\text{HPBW}}{2} = 90 - \theta \quad \text{or} \quad \theta = 90 - \frac{\text{HPBW}}{2}$$

For $\sin\theta$ pattern, $\sin\theta = \sin\left(90 - \dfrac{\text{HPBW}}{2}\right) = \dfrac{1}{2}$,

$$90 - \frac{\text{HPBW}}{2}\sin^{-1}\left(\frac{1}{2}\right), \quad -\frac{\text{HPBW}}{2}\sin^{-1}\left(\frac{1}{2}\right) - 90, \quad \therefore \text{HPBW} = 120^{\text{o}}$$

For $\sin^2\theta$ pattern, $\quad \sin^2\theta = \sin^2\left(90 - \dfrac{\text{HPBW}}{2}\right) = \dfrac{1}{2}$,

$$\sin\left(90 - \frac{\text{HPBW}}{2}\right) = \frac{1}{\sqrt{2}}, \quad \therefore \text{HPBW} = 90^{\text{o}}$$

For $\sin^3\theta$ pattern, $\quad \sin^3\theta = \sin^3\left(90 - \dfrac{\text{HPBW}}{2}\right) = \dfrac{1}{2}$,

$$\sin\left(90 - \frac{\text{HPBW}}{2}\right) = \frac{1}{\sqrt[3]{2}}, \quad \therefore \text{HPBW} = 74.9^{\circ}$$

***2-7-3. continued**

Thus,

$$D = \frac{41{,}253\,\text{sq. deg.}}{\theta_{\text{HP}}\phi_{\text{HP}}} = \frac{41{,}253}{(120)(90)} = 3.82 \cong \frac{40{,}000}{(120)(90)} = 3.70 \quad (ans.)$$

for $P(\theta,\phi) = \sin\,\theta \sin^2\phi$

$$= \frac{41{,}253}{(120)(74.9)} = 4.59 \cong \frac{40{,}000}{(120)(74.9)} = 4.45 \quad (ans.)$$

for $P(\theta,\phi) = \sin\,\theta \sin^3\phi$

$$= \frac{41{,}253}{(90)(74.9)} = 6.12 \cong \frac{40{,}000}{(90)(74.9)} = 5.93 \quad (ans.)$$

for $P(\theta,\phi) = \sin^2\,\theta \sin^3\phi$

***2-7-4. Directivity and gain.**

(a) Estimate the directivity of an antenna with $\theta_{\text{HP}} = 2°$, $\phi_{\text{HP}} = 1°$, and (b) find the gain of this antenna if efficiency $k = 0.5$.

Solution:

(a) $$D = \frac{40{,}000}{\theta_{\text{HP}}\phi_{\text{HP}}} = \frac{40{,}000}{(2)(1)} = 2.0 \times 10^4 \quad \text{or} \quad 43.0\text{ dB} \quad (ans.)$$

(b) $$G = kD = 0.5(2.0 \times 10^4) = 1.0 \times 10^4 \quad \text{or} \quad 40.0\text{ dB} \quad (ans.)$$

2-9-1. Directivity and apertures.

Show that the directivity of an antenna may be expressed as

$$D = \frac{4\pi}{\lambda^2} \frac{\iint_{Ap} E(x,y)dxdy \iint_{Ap} E^*(x,y)dxdy}{\iint_{Ap} E(x,y)E^*(x,y)dxdy}$$

where $E(x, y)$ is the aperture field distribution.

Solution: If the field over the aperture is uniform, the directivity is a maximum (= D_{m}) and the power radiated is P'. For an actual aperture distribution, the directivity is D and the power radiated is P. Equating effective powers

$$D_{\mathrm{m}}\,P' = D\,P\,, \qquad D = D_{\mathrm{m}}\frac{P'}{P} = \frac{4\pi}{\lambda^2}A_p\frac{\dfrac{E_{\mathrm{av}}E_{\mathrm{av}}^*}{Z}A_p}{\iint_{Ap}\dfrac{E(x,y)E^*(x,y)}{Z}dxdy}$$

2-9-1. continued

where $$E_{\mathrm{av}} = \frac{1}{A_p}\iint_{A_p}E(x,y)dxdy$$

therefore $$D = \frac{4\pi}{\lambda^2}\frac{\iint_{Ap}E(x,y)\,dxdy\iint_{Ap}E^*(x,y)\,dxdy}{\iint_{Ap}E(x,y)E^*(x,y)\,dxdy} \quad \text{q.e.d.}$$

where $$\frac{E_{\mathrm{av}}E_{\mathrm{av}}^*A_p}{\iint_{Ap}E(x,y)E^*(x,y)\,dxdy} = \frac{E_{\mathrm{av}}E_{\mathrm{av}}^*}{\dfrac{1}{A_p}\iint E(x,y)E^*(x,y)\,dxdy} = \frac{E_{\mathrm{av}}^*}{(E^2)_{\mathrm{av}}} = \varepsilon_{ap} = \frac{A_e}{A_p}$$

2-9-2. Effective aperture and beam area.

What is the maximum effective aperture (approximately) for a beam antenna having half-power widths of 30° and 35° in perpendicular planes intersecting in the beam axis? Minor lobes are small and may be neglected.

Solution:

$$\Omega_A \cong \theta_{\mathrm{HP}}\phi_{\mathrm{HP}} = 30^{\mathrm{o}}\times 35^{\mathrm{o}}, \qquad A_{em} = \frac{\lambda^2}{\Omega_A} \cong \frac{57.3^2}{30^{\mathrm{o}}\times 35^{\mathrm{o}}}\lambda^2 = 3.1\lambda^2 \quad (ans.)$$

***2-9-3. Effective aperture and directivity.**

What is the maximum effective aperture of a microwave antenna with a directivity of 900?

Solution: $$D = 4\pi\ A_{em}/\lambda^2, \qquad A_{em} = \frac{D\lambda^2}{4\pi} = \frac{900}{4\pi}\lambda^2 = 71.6\,\lambda^2 \quad (ans.)$$

2-11-1. Received power and the Friis formula.

What is the maximum power received at a distance of 0.5 km over a free-space 1 GHz circuit consisting of a transmitting antenna with a 25 dB gain and a receiving antenna with a 20 dB gain? The gain is with respect to a lossless isotropic source. The transmitting antenna input is 150 W.

Solution:

$$\lambda = c / f = 3\times10^8/10^9 = 0.3 \text{ m}, \qquad A_{et} = \frac{D_t\lambda^2}{4\pi}, \qquad A_{er} = \frac{D_r\lambda^2}{4\pi}$$

2-11-1. continued

$$P_r = P_t \frac{A_{et}A_{er}}{r^2\lambda^2} = P_t \frac{D_t\lambda^2 D_r\lambda^2}{(4\pi)^2 r^2\lambda^2} = 150\frac{316\times0.3^2\times100}{(4\pi)^2 500^2} = 0.0108 \text{ W} = 10.8 \text{ mW} \quad (\textit{ans.})$$

***2-11-2. Spacecraft link over 100 Mm.**

Two spacecraft are separated by 100 Mm. Each has an antenna with D = 1000 operating at 2.5 GHz. If craft A's receiver requires 20 dB over 1 pW, what transmitter power is required on craft B to achieve this signal level?

Solution:

$$\lambda = c / f = 3\times10^8/2.5\times10^9 = 0.12 \text{ m}, \qquad A_{et} = A_{er} = \frac{D\lambda^2}{4\pi}$$

$$P_r(\text{required}) = 100\times10^{-12} = 10^{-10} \text{ W}$$

$$P_t = P_r\frac{r^2\lambda^2}{A_{et}^2} = P_r\frac{(4\pi)^2 r^2\lambda^2}{D^2\lambda^4} = P_r\frac{r^2(4\pi)^2}{D^2\lambda^2} = 10^{-10}\frac{10^{16}(4\pi)^2}{10^6\ 0.12^2} = 10966 \text{ W} \cong 11 \text{ kW} \quad (\textit{ans.})$$

2-11-3. Spacecraft link over 3 Mm.

Two spacecraft are separated by 3 Mm. Each has an antenna with D = 200 operating at 2 GHz. If craft A's receiver requires 20 dB over 1 pW, what transmitter power is required on craft B to achieve this signal level?

Solution:

$$\lambda = c / f = 3\times10^8/2\times10^9 = 0.15 \text{ m} \qquad A_{et} = A_{er} = \frac{D\lambda^2}{4\pi}$$

$$P_r = 100\times10^{-12} = 10^{-10} \text{ W}$$

$$P_t = P_r\frac{r^2\lambda^2}{A_{et}A_{er}} = P_r\frac{(4\pi)^2 r^2\lambda^2}{D^2\lambda^2\lambda^2} = 10^{-10}\frac{(4\pi)^2\ 9\times10^{12}}{4\times10^4\times0.15^2} = 158 \text{ W} \quad (\textit{ans.})$$

2-11-4. Mars and Jupiter links.

(a) Design a two-way radio link to operate over earth-Mars distances for data and picture transmission with a Mars probe at 2.5 GHz with a 5 MHz bandwidth. A power of 10^{-19}

W Hz^{-1} is to be delivered to the earth receiver and 10^{-17} W Hz^{-1} to the Mars receiver. The Mars antenna must be no larger than 3 m in diameter. Specify effective aperture of Mars and earth antennas and transmitter power (total over entire bandwidth) at each end. Take earth-Mars distance as 6 light-minutes. (b) Repeat (a) for an earth-Jupiter link. Take the earth-Jupiter distance as 40 light-minutes.

2-11-4. continued

Solution:

(a) $\lambda = c/f = 3\times10^8/2.5\times10^9 = 0.12$ m

$P_r(\text{earth}) = 10^{-19}\times5\times10^6 = 5\times10^{-13}$ W

$P_r(\text{Mars}) = 10^{-17}\times5\times10^6 = 5\times10^{-11}$ W

Take $A_e(\text{Mars}) = (1/2)\pi\,1.5^2 = 3.5\,\text{m}^2 \quad (\varepsilon_{\text{ap}} = 0.5)$

Take $P_t(\text{Mars}) = 1\,\text{kW}$

Take $A_e(\text{earth}) = (1/2)\pi\,15^2 = 350\,\text{m}^2 \quad (\varepsilon_{\text{ap}} = 0.5)$

$$P_t(\text{earth}) = P_r(\text{Mars})\frac{r^2\lambda^2}{A_{et}(\text{earth})A_{et}(\text{Mars})}$$

$$P_t(\text{earth}) = 5\times10^{-11}\frac{(360\times3\times10^8)^2\,0.12^2}{3.5\times350} = 6.9\,\text{MW}$$

To reduce the required earth station power, take the earth station antenna

$$A_e = (1/2)\pi\ 50^2 = 3927\,\text{m}^2 \quad (\textit{ans.})$$

so

$$P_t(\text{earth}) = 6.9\times10^6(15/50)^2 = 620\ \text{kW} \quad (\textit{ans.})$$

$$P_r(\text{earth}) = P_t(\text{Mars})\frac{A_{et}(\text{Mars})A_{er}(\text{earth})}{r^2\lambda^2} = 10^3\frac{3.5\times3930}{(360\times3\times10^8)^2\,0.12^2} = 8\times10^{-14}\ \text{W}$$

which is about 16% of the required 5 x 10^{-13} W. The required 5 x 10^{-13} W could be obtained by increasing the Mars transmitter power by a factor of 6.3. Other alternatives would be (1) to reduce the bandwidth (and data rate) reducing the required value of P_r or (2) to employ a more sensitive receiver.

As discussed in Sec. 12-1, the *noise power* of a receiving system is a function of its system temperature T and bandwidth B as given by $P = kTB$, where k = Boltzmann's constant = 1.38×10^{-23} JK^{-1}.

For $B = 5 \times 10^6$ Hz (as given in this problem) and $T = 50$ K (an attainable value),

$$P(\text{noise}) = 1.38 \times 10^{-23} \times 50 \times 5 \times 10^6 = 3.5 \times 10^{-15} \text{ W}$$

2-11-4. continued

The received power (8×10^{-14} W) is about 20 times this noise power, which is probably sufficient for satisfactory communication. Accordingly, with a 50 K receiving system temperature at the earth station, a Mars transmitter power of 1 kW is adequate.

(b) The given Jupiter distance is 40/6 = 6.7 times that to Mars, which makes the required transmitter powers $6.7^2 = 45$ times as much *or* the required receiver powers 1/45 as much.

Neither appears feasible. But a practical solution would be to reduce the bandwidth for the Jupiter link by a factor of about 50, making $B = (5/50) \times 10^6 = 100$ kHz.

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*2-11-5. Moon link.

A radio link from the moon to the earth has a moon-based 5λ long right-handed monofilar axial-mode helical antenna (see Eq. (8-3-7)) and a 2 W transmitter operating at 1.5 GHz. What should the polarization state and effective aperture be for the earth-based antenna in order to deliver 10^{-14} W to the receiver? Take the earth-moon distance as 1.27 light-seconds.

Solution:

$$\lambda = c / f = 3 \times 10^8 / 1.5 \times 10^9 = 0.2 \text{ m},$$

From (8-3-7) the directivity of the moon helix is given by

$$D = 12 \times 5 = 60 \quad \text{and} \quad A_{et}(\text{moon}) = \frac{D\,\lambda^2}{4\pi}$$

From Friis formula

$$A_{er} = \frac{P_r r^2 \lambda^2}{P_t A_{et}} = \frac{P_r (4\pi)\; r^2 \lambda^2}{P_t D \lambda^2} = \frac{10^{-14}(3 \times 10^8 \times 1.27)^2 4\pi}{2 \times 60} = 152 \text{ m}^2 \text{ RCP or}$$

about 14 m diameter (*ans.*)

2-16-1. Spaceship near moon.

A spaceship at lunar distance from the earth transmits 2 GHz waves. If a power of 10 W is radiated isotropically, find (a) the average Poynting vector at the earth, (b) the rms electric field **E** at the earth and (c) the time it takes for the radio waves to travel from the spaceship to the earth. (Take the earth-moon distance as 380 Mm.) (d) How many photons per unit area per second fall on the earth from the spaceship transmitter?

2-16-1. continued

Solution:

(a) $\text{PV (at earth)} = \dfrac{P_t}{4\pi\, r^2} = \dfrac{10}{4\pi(380\times10^6)^2} = 5.5\times10^{-18}\ \text{Wm}^{-2} = 5.5\,\text{aWm}^{-2}$ (*ans.*)

(b) $\text{PV} = S = E^2/Z$ or $E = (SZ)^{1/2}$

or $E = (5.5\times10^{-18}\times377)^{1/2} = 45\times10^{-9} = 45\,\text{nVm}^{-1}$ (*ans.*)

(c) $t = r/c = 380\times10^6/3\times10^8 = 1.27\ \text{s}$ (*ans.*)

(d) Photon = $hf = 6.63\times10^{-34}\times2\times10^9 = 1.3\times10^{-24}\ \text{J}$, where $h = 6.63\times10^{-34}$ Js

This is the energy of a 2.5 MHz photon. From (a), $\text{PV} = 5.5\times10^{-18}\ \text{Js}^{-1}\text{m}^{-2}$

Therefore, number of photons $= \dfrac{5.5\times10^{-18}}{1.3\times10^{-24}} = 4.2\times10^6\ \text{m}^{-2}\text{s}^{-1}$ (*ans.*)

2-16-2. More power with CP.

Show that the average Poynting vector of a circularly polarized wave is twice that of a linearly polarized wave if the maximum electric field **E** is the same for both waves. This means that a medium can handle twice as much power before breakdown with circular polarization (CP) than with linear polarization (LP).

Solution:

From (2-16-3) we have for rms fields that $\text{PV} = S_{av} = \dfrac{E_1^2 + E_2^2}{Z_o}$

For LP, E_2 (or E_1) $= 0$, so $S_{av} = \dfrac{E_1^2}{Z_o}$

For CP, $E_1 = E_2$, so $S_{av} = \dfrac{2E_1^2}{Z_o}$

Therefore $S_{CP} = 2S_{LP}$ (*ans.*)

2-16-3. PV constant for CP.

Show that the instantaneous Poynting vector (PV) of a plane circularly polarized traveling wave is a constant.

Solution:

$$E_{\mathrm{CP}} = E_x \cos\omega t + E_y \sin\omega t \qquad \text{where } E_x = E_y = E_{\mathrm{o}}$$

2-16-3. continued

$$\left|E_{\mathrm{CP}}\right| = (E_{\mathrm{o}}^2 \cos^2 \omega t + E_{\mathrm{o}}^2 \sin^2 \omega t)^{1/2} = E_{\mathrm{o}}(\cos^2 \omega t + \sin^2 \omega t)^{1/2} = E_{\mathrm{o}} \quad \text{(a constant)}$$

$$\text{Therefore PV or } S(\text{instantaneous}) = \frac{E_{\mathrm{o}}^2}{Z} \quad \text{(a constant)} \quad (\textit{ans.})$$

*2-16-4. EP wave power

An elliptically polarized wave in a medium with constants $\sigma = 0$, $\mu_r = 2$, $\varepsilon_r = 5$ has H-field components (normal to the direction of propagation and normal to each other) of amplitudes 3 and 4 A m^{-1}. Find the average power conveyed through an area of 5 m^2 normal to the direction of propagation.

Solution:

$$S_{av} = \frac{1}{2} Z(H_1^2 + H_2^2) = \frac{1}{2} 377(\mu_r / \varepsilon_r)^{1/2}(H_1^2 + H_2^2) = \frac{1}{2} 377(2/5)^{1/2}(3^2 + 4^2) = 2980 \text{ Wm}^{-2}$$

$$P = AS_{av} = 5 \times 2980 = 14902 \text{ W} = 14.9 \text{ kW} \quad (\textit{ans.})$$

2-17-1. Crossed dipoles for CP and other states.

Two $\lambda/2$ dipoles are crossed at 90°. If the two dipoles are fed with equal currents, what is the polarization of the radiation perpendicular to the plane of the dipoles if the currents are (a) in phase, (b) phase quadrature (90° difference in phase) and (c) phase octature (45° difference in phase)?

Solution:

(a) LP (*ans.*)

(b) CP (*ans.*)

(c) From (2-17-3) $\sin 2\varepsilon = \sin 2\gamma \sin \delta$

where $\gamma = \tan^{-1}(E_2 / E_1) = 45^\circ$

$\delta = 45^\circ$

$\varepsilon = 22\tfrac{1}{2}^\circ$

$\text{AR} = \cot\varepsilon = 1/\tan\varepsilon = 2.41$ (EP)...(*ans.*)

***2-17-2. Polarization of two LP waves.**

A wave traveling normally out of the page (toward the reader) has two linearly polarized components

$$E_x = 2\cos\omega t$$

$$E_y = 3\cos\left(\omega t + 90^\circ\right)$$

(a) What is the axial ratio of the resultant wave?
(b) What is the tilt angle τ of the major axis of the polarization ellipse?
(c) Does **E** rotate clockwise or counterclockwise?

Solution:

(a) From (2-15-8) , $\text{AR} = 3/2 = 1.5$ (*ans.*)

(b) $\tau = 90^\text{o}$ (*ans.*)

(c) At $t = 0$, $E = E_x$; at $t = T/4$, $E = -E_y$, therefore rotation is CW (*ans.*)

2-17-3. Superposition of two EP waves.

A wave traveling normally outward from the page (toward the reader) is the resultant of two elliptically polarized waves, one with components of **E** given by

$$E'_y = 2\cos\omega t \quad \text{and} \quad E'_x = 6\cos\left(\omega t + \tfrac{\pi}{2}\right)$$

and the other with components given by $E''_y = 1\cos\omega t$ and $E''_x = 3\cos\left(\omega t - \tfrac{\pi}{2}\right)$

(a) What is the axial ratio of the resultant wave?
(b) Does **E** rotate clockwise or counterclockwise?

Solution:

$$E_y = E'_y + E''_y = 2\cos\omega t + \cos\omega t = 3\cos\omega t$$

$$E_x = E'_x + E''_x = 6\cos(\omega t + \pi/2) + 3\cos(\omega t - \pi/2) = -6\sin\omega t + 3\sin\omega t = -3\sin\omega t$$

(a) E_x and E_y are in phase quadrature and $\mathrm{AR} = 3/3 = 1$ (CP) (*ans.*)

(b) At $t = 0$, $\mathbf{E} = \hat{\mathbf{y}}3$, at $t = T/4$, $\mathbf{E} = -\hat{\mathbf{x}}3$, therefore rotation is CCW (*ans.*)

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*2-17-4. Two LP components.

An elliptically polarized plane wave traveling normally out of the page (toward the reader) has linearly polarized components E_x and E_y. Given that $E_x = E_y = 1$ V m^{-1} and that E_y leads E_x by 72°,
(a) Calculate and sketch the polarization ellipse.
(b) What is the axial ratio?
(c) What is the angle τ between the major axis and the x-axis?

Solution:

(b) $\gamma = \tan^{-1}(E_2 / E_1) = 45^\circ$, $\delta = 72^\circ$

From (2-17-3), $\varepsilon = 36^\circ$, therefore $\mathrm{AR} = 1/\tan\varepsilon = 1.38$ (*ans.*)

(c) From (2-17-3), $\sin 2\tau = \tan 2\varepsilon / \tan\delta$ or $\tau = 45^\circ$ (*ans.*)

2-17-5. Two LP components and Poincaré sphere.

Answer the same questions as in Prob. 2-17-4 for the case where E_y leads E_x by 72° as before but $E_x = 2$ V m^{-1} and $E_y = 1$ V m^{-1}.

Solution:

(b) $\gamma = \tan^{-1} 2 = 63.4^\circ$

$\delta = 72^\circ$

$\varepsilon = 24.8^\circ$ and $\mathrm{AR} = 2.17$ (*ans.*)

(c) $\tau = 11.2^\circ$ (*ans.*)

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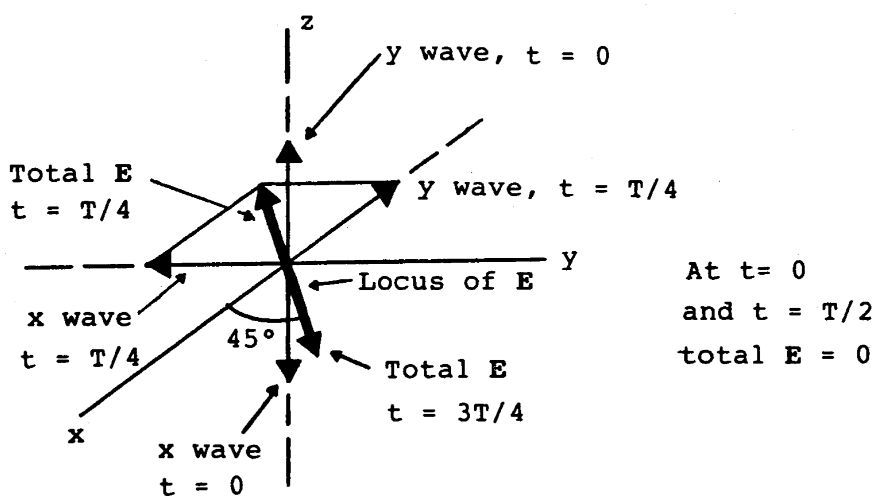
*2-17-6. Two CP waves.

Two circularly polarized waves intersect at the origin. One (y-wave) is traveling in the positive y direction with **E** rotating clockwise as observed from a point on the positive y-axis. The other (x-wave) is traveling in the positive x direction with **E** rotating clockwise as observed from a point on the positive x-axis. At the origin, **E** for the y-wave is in the positive z direction at the same instant that **E** for the x-wave is in the negative z direction. What is the locus of the resultant **E** vector at the origin?

Solution:

Resolve 2 waves into components or make sketch as shown. It is assumed that the waves have equal magnitude.

*2-17-6. continued



Locus of **E** is a straight line in *xy* plane at an angle of 45° with respect to *x* (or *y*) axis.

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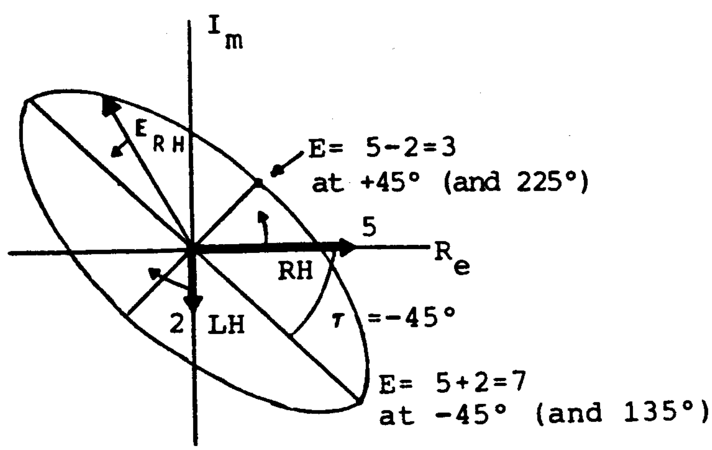
*2-17-7. CP waves.

A wave traveling normally out of the page is the resultant of two circularly polarized components $E_{right} = 5e^{j\omega t}$ and $E_{left} = 2e^{j\left(\omega t+90^\circ\right)}$ (V m^{-1}). Find (a) the axial ratio AR, (b) the tilt angle τ and (c) the hand of rotation (left or right).

Solution:



(a) $\mathrm{AR} = \dfrac{2+5}{2-5} = -7/3 = -2.33$ (*ans.*) [Note minus sign for RH (right-handed polarization)]

(b) From diagram, $\tau = -45^{\circ}$ (*ans.*)

(c) Since **E** rotates counterclockwise as a function of time, RH. (*ans.*)

2-17-8. EP wave.

A wave traveling normally out of the page (toward the reader) is the resultant of two linearly polarized components $E_x = 3\cos\omega t$ and $E_y = 2\cos(\omega t + 90^{\circ})$. For the resultant wave find (a) the axial ratio AR, (b) the tilt angle τ and (c) the hand of rotation (left or right).

Solution:

(a) AR = 3/2 = 1.5 (*ans.*)

(b) $\tau = 0^{\circ}$ (*ans.*)

(c) CW, LEP (*ans.*)

***2-17-9. CP waves.**

Two circularly polarized waves traveling normally out of the page have fields given by $E_{left} = 2e^{-j\omega t}$ and $E_{right} = 3e^{j\omega t}$ (V m^{-1}) (rms). For the resultant wave find (a) AR, (b) the hand of rotation and (c) the Poynting vector.

Solution:

(a) $\text{AR} = \dfrac{2+3}{2-3} = -5$ (*ans.*)

(b) REP (*ans.*)

(c) $\text{PV} = \dfrac{E_L^2 + E_R^2}{Z} = \dfrac{4+9}{377} = 0.034\ \text{Wm}^{-2} = 34\ \text{mWm}^{-2}$ (*ans.*)

2-17-10. EP waves.

A wave traveling normally out of the page is the resultant of two elliptically polarized (EP) waves, one with components $E_x = 5\cos\omega t$ and $E_y = 3\sin\omega t$ and another with components $E_r = 3e^{j\omega t}$ and $E_l = 4e^{-j\omega t}$. For the resultant wave, find (a) AR, (b) τ and (c) the hand of rotation.

Solution:

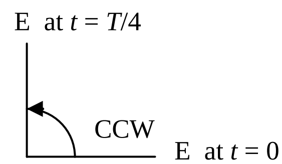
(a)

$$E_x = 5\cos\omega t + 3\cos\omega t + 4\cos\omega t = 12\cos\omega t$$
$$E_y = 3\sin\omega t + 3\sin\omega t - 4\sin\omega t = 2\sin\omega t$$

2-17-10. continued

$$\text{AR} = 12/2 = 6 \quad (ans.)$$

(b) Since E_x and E_y are in time-phase quadrature with E_x(max) > E_y(max), $\tau = 0^{\circ}$.
Or from (2-17-3), $\sin 2\tau = \tan 2\varepsilon / \tan\delta$, $\varepsilon = \tan^{-1}(1/\text{AR}) = 9.46^{\circ}$
but $\delta = 90^{\circ}$ so $\tan\delta = \infty$
Therefore $\tau = 0^{\circ}$ (*ans.*)



(c) At $t = 0$, $E_x = 12$, $E_y = 0$
At $t = T/4$ (ωt $= 90^{\circ}$), $E_x = 0$, $E_y = 2$
Therefore rotation is CCW, so polarization is right elliptical, REP (*ans.*)

***2-17-11. CP waves.**

A wave traveling normally out of the page is the resultant of two circularly polarized components $E_r = 2e^{j\omega t}$ and $E_l = 4e^{-j(\omega t + 45^{\circ})}$. For the resultant wave, find (a) AR, (b) τ and (c) the hand of rotation.

Solution:

(a) $$\text{AR} = \frac{E_1 + E_r}{E_1 - E_r} = \frac{4+2}{4-2} = \frac{6}{2} = 3 \quad (ans.)$$

(b) When $\omega t = 0$, $E_r = 2\angle 0^{\circ}$ and $E_1 = 4\angle -45^{\circ}$

When $\omega t = -22\tfrac{1}{2}^{\circ}$, $E_r = 2\angle -22\tfrac{1}{2}^{\circ}$ and $E_1 = 4\angle -22\tfrac{1}{2}^{\circ}$

so that $E_1 + E_r = E_{\max} = 6\ \angle -22\tfrac{1}{2}^{\circ}$ or $\tau = -22\tfrac{1}{2}^{\circ}$ (*ans.*)

Note that the rotation directions are opposite for E_r and E_1

so that for $-\omega t$, $E_r = 2\angle -\omega t$ but $E_1 = \angle +\omega t$

Also, τ can be determined analytically by combining the waves into an E_x and E_y component with values of

$$E_x = 5.60\angle{-30.4^\circ} \qquad \text{and} \qquad E_y = 2.95\angle{16.3^\circ}$$

from which $\delta = -46.7^\circ$

***2-17-11. continued**

Since from (a) AR = 3, ε can be determined and from (2-17-3), the tilt angle
$\tau = -22.5^\circ$ (*ans.*)

(c) $E_1 > E_r$ so rotation is CW (LEP) (*ans.*)

2-17-12. Circular-depolarization ratio.

If the axial ratio of a wave is AR, show that the circular-depolarization ratio of the wave is given by.

$$R = \frac{\mathrm{AR} - 1}{\mathrm{AR} + 1}$$

Thus, for pure circular polarization AR = 1 and R = 0 (no depolarization) but for linear polarization AR = ∞ and R = 1.

Solution:

Any wave may be resolved into 2 circularly-polarized components of opposite hand, E_r and E_1 for an axial ratio

$$\mathrm{AR} = \frac{E_{\max}}{E_{\min}} = \left| \frac{E_r + E_1}{E_r - E_1} \right|$$

from which the *circular depolarization ratio* $R = \dfrac{E_1}{E_r} = \dfrac{\mathrm{AR} - 1}{\mathrm{AR} + 1}$

Thus for pure circular polarization, AR = 1 and there is zero depolarization (R = 0), while for pure linear polarization AR = ∞ and the depolarization ratio is unity (R =1). When AR = 3, R = ½.

Chapter 3. The Antenna Family

3-4-1. Alpine-horn antenna.

Referring to Fig. 3-4a, the low frequency limit occurs when the open-end spacing > $\lambda/2$ and the high frequency limit when the transmission line spacing $d \approx \lambda/4$. If d = 2 mm and the open-end spacing = 1000 d, what is the bandwidth?

Solution:

D = opened end spacing, d = transmission line spacing

$$\text{Bandwidth} = \frac{\frac{\lambda_{\max}}{2}}{\frac{\lambda_{\min}}{2}} = \frac{D}{d} = 1000 \quad (ans.)$$

---

*3-4-2. Alpine-horn antenna.

If d = transmission line spacing, what open-end spacing is required for a 200-to-1 bandwidth?

Solution:

If d = transmission line spacing $= \lambda_{\min}/2$ and D = open-end spacing $= \lambda_{\max}/2$,

for 200-to-1 bandwidth, we must have $\dfrac{D}{d} = \dfrac{\frac{\lambda_{\max}}{2}}{\frac{\lambda_{\min}}{2}} = 200$, or $D = 200\ d$ (*ans.*)

---

*3-5-2. Rectangular horn antenna.

What is the required aperture area for an optimum rectangular horn antenna operating at 2 GHz with 16 dBi gain?

Solution:

From Fig. 3-5 for $f = 2\,\text{GHz}\ (\lambda = 0.15\,\text{m})$,

$$D = \frac{7.5wh}{\lambda^2} = 18\ \text{dBi} = 63.1, \qquad \therefore\ wh = \frac{63.1\lambda^2}{7.5} = 0.19\ \text{m}^2 \quad (ans.)$$

---

*3-5-3. Conical horn antenna.

What is the required diameter of a conical horn antenna operating at 3 GHz with 14 dBi gain?

Solution:

From Fig. 3-5 for $f = 3\,\text{GHz}\,(\lambda = 0.1\,\text{m})$,

$$D = \frac{6.5\pi r^2}{\lambda^2} = 12\text{ dBi} = 15.8, \qquad \therefore\ r = \sqrt{\frac{15.8\lambda^2}{6.5\pi}} = 0.09\text{ m}^2, \quad d = 2r = 0.18\text{ m} \quad (ans.)$$

3-7-2. Beamwidth and directivity

For most antennas, the half-power beamwidth (HPBW) may be estimated as HPBW = $\kappa\lambda/D$, where λ is the operating wavelength, D is the antenna dimension in the plane of interest, and κ is a factor which varies from 0.9 to 1.4, depending on the filed amplitude taper across the antenna. Using this approximation, find the directivity and gain for the following antennas: (a) circular parabolic dish with 2 m radius operating at 6 GHz, (b) elliptical parabolic dish with dimensions of 1 m × 10 m operated at 1 GHz. Assume κ = 1 and 50 percent efficiency in each case.

Solution:

From Fig. 3-9 for $f = 1600$ MHz ($\lambda = 0.1875$ m),

$$G = 17\text{ dBi} = 50 = D \quad \text{(for 100\% efficiency)}$$

(a) $D = \dfrac{15L}{\lambda} = 50,$ so $L = \dfrac{50}{15}\lambda = 3.33\lambda$

If spacing $= \lambda/\pi$, number of turns $= n = \dfrac{L}{\lambda/\pi} = 10.5 \approx 10$ (*ans.*)

(b) Turn diameter $= \lambda/\pi = 0.0596 \approx 6$ cm (*ans.*)

(c) Axial ratio $\text{AR} = \dfrac{2n+1}{2n} = \dfrac{21}{20} = 1.05$ (*ans.*)

Chapter 4. Point Sources

*4-3-1. Solar power

The earth receives from the sun 2.2 g cal min^{-1} cm^{-2}.
(a) What is the corresponding Poynting vector in watts per square meter?
(b) What is the power output of the sun, assuming that it is an isotropic source?
(c) What is the rms field intensity at the earth due to the sun's radiation, assuming all the sun's energy is at a single frequency?
Note: 1 watt = 14.3 g cal min^{-1}, distance earth to sun = 149 Gm.

Solution:

(a) $S = \dfrac{2.2\text{g cal min}^{-1}\text{cm}^{-2}}{14.3\text{ g cal min}^{-1}} = 0.1539\text{ W cm}^{-2} = 1539\text{ W m}^{-2}$ (*ans.*)

(b) $\text{P(sun)} = S \times 4\pi r^2 = 1539 \times 4\pi \times 1.49^2 \times 10^{22}\text{ W} = 4.29 \times 10^{26}\text{ W}$ (*ans.*)

(c) $S = E^2 / Z_o, \quad E = (SZ_o)^{1/2} = (1539 \times 377)^{1/2} = 762\text{ V m}^{-1}$ (*ans.*)

4-5-1. Approximate directivities.

(a) Show that the directivity for a source with at unidirectional power pattern given by $U = U_m \cos^n \theta$ can be expressed as $D = 2(n+1)$. U has a value only for $0° \le \theta \le 90°$. The patterns are independent of the azimuth angle ϕ. (b) Compare the exact values calculate from (a) with the approximate values for the directivities of the antennas found in Prob. 2-7-2 and find the dB difference from the exact values.

Solution:

(a) If $U = U_m \cos^n \theta$, $\quad D = \dfrac{4\pi}{2\pi\int_0^{\pi/2} \sin\theta \cos^n \theta d\theta} = \dfrac{2}{-\left.\dfrac{\cos^{n+1}\theta}{n+1}\right|_0^{\pi/2}} = 2(n+1)$ (*ans.*)

(b)

For n=1,	For n=2,	For n=3,
$D_{approx.} \approx 2.78 \Rightarrow 4.4$ dBi	$D_{approx.} \approx 4.94 \Rightarrow 6.9$ dBi	$D_{approx.} \approx 7.3 \Rightarrow 8.6$ dBi
$D_{exact} = 4 \Rightarrow 6.0$ dBi	$D_{exact} = 6 \Rightarrow 7.8$ dBi	$D_{exact} = 8 \Rightarrow 9.0$ dBi
$D_{exact} - D_{approx.} = 1.6$ dB	$D_{exact} - D_{approx.} = 0.9$ dB	$D_{exact} - D_{approx.} = 0.4$ dB

---

*4-5-2. Exact versus approximate directivities.

(a) Calculate the exact directivities of the three unidirectional antennas having power patterns as follows:

$$P(\theta,\phi) = P_m \sin\theta \sin^2\phi$$

$$P(\theta,\phi) = P_m \sin\theta \sin^3\phi$$

$$P(\theta,\phi) = P_m \sin^2\theta \sin^3\phi$$

$P(\theta,\phi)$ has a value only for $0 \le \theta \le \pi$ and $0 \le \phi \le \pi$ and is zero elsewhere.

(b) Compare the exact values in (a) with the approximate values found in Prob. 2-7-3.

Solution:

(a) $$D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\iint_{4\pi} P_n(\theta,\phi)\,d\Omega}, \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

For $P(\theta,\phi) = P_m \sin\theta \sin^2\phi$, $$D = \frac{4\pi}{\int_0^\pi\int_0^\pi \frac{P_m \sin\theta \sin^2\phi}{P_m}\sin\theta\,d\theta\,d\phi} = \frac{4\pi}{\int_0^\pi\int_0^\pi \sin^2\theta \sin^2\phi\,d\theta\,d\phi}$$

$$\int_0^\pi \sin^2\theta\,d\theta = \left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\Big|_0^\pi\right) = \frac{\pi}{2}, \qquad \therefore D = \frac{4\pi}{\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)} = \frac{16}{\pi} = 5.09 \quad (\textit{ans.})$$

Using the same approach, we find,

for $P(\theta,\phi) = P_m \sin\theta \sin^3\phi$, $$D = \frac{4\pi}{\int_0^\pi\int_0^\pi \sin^2\theta \sin^3\phi\,d\theta\,d\phi} = \frac{4\pi}{\left(\frac{\pi}{2}\right)\left(\frac{4}{3}\right)} = 6.0 \quad (\textit{ans.})$$

for $P(\theta,\phi) = P_m \sin^2\theta \sin^3\phi$, $$D = \frac{4\pi}{\int_0^\pi\int_0^\pi \sin^3\theta \sin^3\phi\,d\theta\,d\phi} = \frac{4\pi}{\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)} = 7.1 \quad (\textit{ans.})$$

(b) Tabulating, we have 5.1 vs. 3.8, 6.0 vs. 4.6, and 7.1 vs. 6.1 (*ans.*)

4-5-3. Directivity and minor lobes.

Prove the following theorem: if the minor lobes of a radiation pattern remain constant as the beam width of the main lobe approaches zero, then the directivity of the antenna approaches a constant value as the beam width of the main lobes approaches zero.

4-5-3. continued

Solution:

$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\Omega_M + \Omega_m}$$

where $\Omega_A =$ total beam area
$\Omega_M =$ main lobe beam area
$\Omega_m =$ minor lobe beam area

as $\Omega_M \to 0$, $\Omega_A \to \Omega_m$, so $D = 4\pi/\Omega_m$ (a constant) (*ans.*)

4-5-4. Directivity by integration.

(a) Calculate by graphical integration or numerical methods the directivity of a source with a unidirectional power pattern given by $U = \cos\theta$. Compare this directivity value with the exact value from Prob. 4-5-1. U has a value only for $0° \le \theta \le 90°$ and $0° \le \phi \le 360°$ and is zero else where.
(b) Repeat for a unidirectional power pattern given by $U = \cos^2\theta$.
(c) Repeat for a unidirectional power pattern given by $U = \cos^3\theta$.

Solution:

Exact values for (a), (b), and (c) are: 4, 6, and 8. (*ans.*)

4-5-5. Directivity.

Calculate the directivity for a source with relative field pattern $E = \cos 2\theta \cos\theta$.

Solution:

Assuming a unidirectional pattern, $(0 \le \theta \le \frac{\pi}{2})$, $D = 24$ (*ans.*)

Chapter 5. Arrays of Point Sources, Part I

5-2-4. Two-source end-fire array.

(a) Calculate the directivity of an end-fire array of two identical isotropic point sources in phase opposition, spaced $\lambda/2$ apart along the polar axis, the relative field pattern being given by

$$E = \sin\left(\frac{\pi}{2}\cos\theta\right)$$

where θ is the polar angle.
(b) Show that the directivity for an ordinary end-fire array of two identical isotropic point sources spaced a distance d is given by

$$D = \frac{2}{1 + (\lambda/4\pi d)\sin(4\pi d/\lambda)}.$$

Solution:

(a) $D = 2$ (*ans.*)

5-2-8. Four sources in square array.

(a) Derive an expression for $E(\phi)$ for an array of 4 identical isotropic point sources arranged as in Fig. P5-2-8. The spacing d between each source and the center point of the array is $3\lambda/8$. Sources 1 and 2 are in-phase, and sources 3 and 4 in opposite phase with respect to 1 and 2.
(b) Plot, approximately, the normalized pattern.

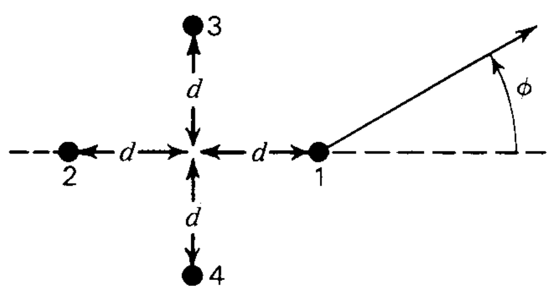


Figure P5-2-8. Four sources in square array.

Solution:

(a) $E_n(\phi) = \cos(\beta d \cos\phi) - \cos(\beta d \sin\phi)$ (*ans.*)

5-5-1. Field and phase patterns.

Calculate and plot the field and phase patterns of an array of 2 nonisotropic dissimilar sources for which the total field is given by

$$E = \cos\phi + \sin\phi \ \angle\psi$$

where $\psi = d\cos\phi + \delta = \frac{\pi}{2}\left(\cos\phi + 1\right)$

Take source 1 as the reference for phase. See Fig. P5-5-1.

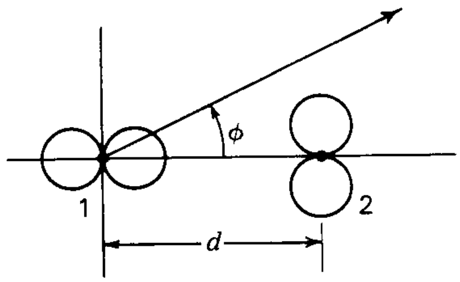


Figure P5-5-1. Field and phase patterns.

Solution:

See Figures 5-16 and 5-17.

5-6-5. Twelve-source end-fire array.

(a) Calculate and plot the field pattern of a linear end-fire array of 12 isotropic point sources of equal amplitude spaced $\lambda/4$ apart for the ordinary end-fire condition.
(b) Calculate the directivity by graphical or numerical integration of the entire pattern. Note that it is the power pattern (square of field pattern) which is to be integrated. It is most convenient to make the array axis coincide with the polar or z-axis of Fig. 2-5 so that the pattern is a function of θ.
(c) Calculate the directivity by the approximate half-power beamwidth method and compare with that obtained in (b).

Solution:

(b) $D = 17$ (*ans.*)

(c) $D = 10$ (*ans.*)

5-6-7. Twelve-source end-fire with increased directivity.

(a) Calculate and plot the pattern of a linear end-fire array of 12 isotropic point sources of equal amplitude spaced $\lambda/4$ apart and phased to fulfill the Hansen and Woodyard increased-directivity condition.
(b) Calculate the directivity by graphical or numerical integration of the entire pattern and compare with the directivity obtained in Prob. 5-6-5 and 5-6-6.
(c) Calculate the directivity by the approximate half-power beamwidth method and compare with that obtained in (b).

Solution:

(b) $D = 26$ (*ans.*)

(c) $D = 35$ (*ans.*)

5-6-9. Directivity of ordinary end-fire array.

Show that the directivity of an ordinary end-fire array may be expressed as

$$D = \frac{n}{1 + (\lambda/2\pi nd)\sum_{k=1}^{n-1}[(n-k)/k]\sin(4\pi kd/\lambda)}$$

Note that

$$\left[\frac{\sin(n\psi/2)}{(\psi/2)}\right]^2 = n + \sum_{k=1}^{n-1} 2(n-k)\cos 2k\frac{\psi}{2}$$

Solution: Change of variable.

It is assumed that the array has a uniform spacing *d* between the isotropic sources. The beam area

$$\Omega_A = \frac{1}{n^2}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right]^2 \sin\theta d\theta d\phi \qquad (1)$$

where $\theta =$ angle from array axis.

The pattern is not a function of ϕ so (1) reduces to

$$\Omega_A = \frac{2\pi}{n^2}\int_0^{\pi}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right]^2 \sin\theta d\theta \qquad (2)$$

where $\psi/2 = \pi d_\lambda(\cos\theta - 1)$ (2.1)

5-6-9. continued

Differentiating $$d\frac{\psi}{2} = \pi d_\lambda \sin\theta d\theta \qquad (3)$$

or $$\sin\theta d\theta = \frac{1}{\pi d_\lambda}\frac{\psi}{2} \qquad (4)$$

and introducing (4) in (2)

$$\Omega_A = \frac{2}{n^2 d_\lambda}\int_0^{2\pi d_\lambda}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right]^2 d\frac{\psi}{2} \qquad (5)$$

Note new limits with change of variable from θ to $\psi/2$.

When $\theta = 0$, $\psi/2 = 0$ and when $\theta = \pi$, $\psi/2 = 2\pi d_\lambda$.

Since $$\left[\frac{\sin(n\psi/2)}{(\psi/2)}\right]^2 = n + \sum_{k=1}^{n-1} 2(n-k)\cos(2k\psi/2) \qquad (6)$$

(5) can be expressed $$\Omega_A = \frac{2}{n^2 d_\lambda}\int_0^{2\pi d_\lambda}[n + \sum_{k=1}^{n-1} 2(n-k)\cos(2k\psi/2)]\, d\frac{\psi}{2} \qquad (7)$$

Integrating (7) $$\Omega_A = \frac{2}{n^2 d_\lambda}\left[n\frac{\psi}{2} + \sum_{k=1}^{n-1}\frac{2(n-k)}{2k}\sin(2k\psi/2)\right]_0^{2\pi d_\lambda} \qquad (8)$$

or $$\Omega_A = \frac{2}{n^2 d_\lambda}\left[2\pi n d_\lambda + \sum_{k=1}^{n-1}\frac{n-k}{k}\sin(4\pi k d_\lambda)\right] \qquad (9)$$

and $$D = \frac{4\pi}{\Omega_A} = \frac{2\pi n^2 d_\lambda}{2\pi n d_\lambda + \sum_{k=1}^{n-1}\frac{n-k}{k}\sin(4\pi k d_\lambda)} \qquad (10)$$

Therefore $$D = \frac{n}{1 + \frac{\lambda}{2\pi n d} + \sum_{k=1}^{n-1}\frac{n-k}{k}\sin(4\pi k d/\lambda)} \quad \text{q.e.d.} \qquad (11)$$

We note that when $d = \lambda/4$, or a multiple thereof, the summation term is zero and $D = n$ exactly.

This problem and the next one are excellent examples of integration with change of variable and change of limits.

The final form for D in (11) above is well adapted for a computer program.

5-6-10. Directivity of broadside array.

Show that the directivity of a broadside array may be expressed as

$$D = \frac{n}{1 + (\lambda/\pi nd)\sum_{k=1}^{n-1}[(n-k)/k]\sin(2\pi kd/\lambda)}$$

Solution:

The solution is similar to that for Prob. 5-6-9 with $\frac{\psi}{2} = \pi d_\lambda \cos\theta$ where $\theta = 0$, $\psi/2 = \pi d_\lambda$ and when $\theta = \pi$, $\psi/2 = -\pi d_\lambda$ so that (8) of Prob. 5-6-9 becomes

$$\Omega_A = \frac{2}{n^2 d_\lambda}\left[n\frac{\psi}{2} + \sum_{k=1}^{n-1}\frac{(n-k)}{k}\sin(2k\psi/2)\right]_{+\pi d_\lambda}^{-\pi d_\lambda}$$

$$\Omega_A = \frac{-2}{n^2 d_\lambda}\left[2\pi n d_\lambda + 2\sum_{k=1}^{n-1}\frac{n-k}{k}\sin(2\pi k d_\lambda)\right]$$

$$|\Omega_A| = \frac{4}{n^2 d_\lambda}\left[\pi n d_\lambda + \sum_{k=1}^{n-1}\frac{n-k}{k}\sin(2\pi k d_\lambda)\right]$$

$$D = \frac{4\pi}{\Omega_A} = \frac{\pi n^2 d_\lambda}{\pi n d_\lambda + \sum_{k=1}^{n-1}\frac{n-k}{k}\sin(2\pi k d_\lambda)} = \frac{n}{1 + \left(\frac{\lambda}{\pi n d}\right)\sum_{k=1}^{n-1}\frac{n-k}{k}\sin(2\pi k d/\lambda)} \qquad \text{q.e.d.}$$

Note that when $d = \lambda/2$, or a multiple thereof, the summation term is zero and $D = n$ exactly.

See application of the above relations to the evaluation of D and of the main beam area Ω_A of an array of 16 point sources in Prob. 16-6-7 (c) and (d).

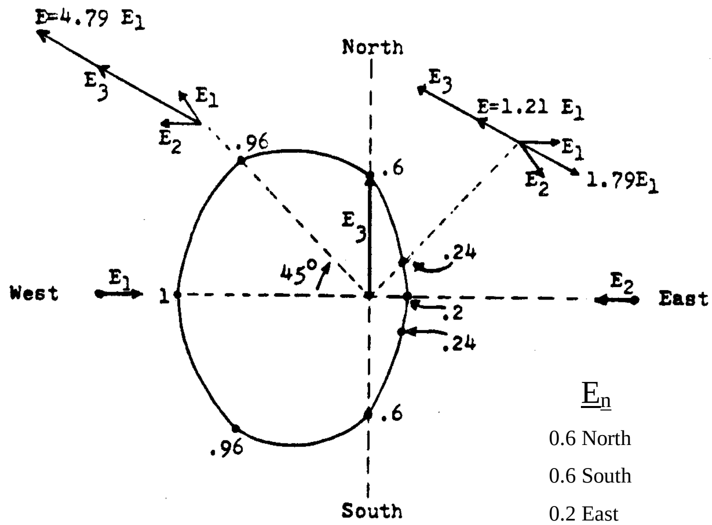
Chapter 5. Arrays of Point Sources, Part II

5-8-1. Three unequal sources.

Three isotropic in-line sources have $\lambda/4$ spacing. The middle source has 3 times the current of the end sources. If the phase of the middle source is 0°, the phase of one end source +90° and phase of the other end source -90°, make a graph of the normalized field pattern.

Solution:

Phasor addition



$\underline{E_n}$

0.6 North

0.6 South

0.2 East

1.0 West

0.24 North-East

0.96 North-West

5-8-7. Stray factor and directive gain.

The ratio of the main beam solid angle Ω_M to (total) beam solid angle Ω_A is called the *main beam efficiency*. The ratio of the minor-lobe solid angle Ω_m to the (total) beam solid angle Ω_A is called the *stray factor*. It follows that $\Omega_M/\Omega_A + \Omega_m/\Omega_A = 1$. Show that the *average directivity gain* over the minor lobes of a highly directive antenna is nearly equal to the stray factor. The directive gain is equal to the directivity multiplied by the normalized power pattern [$= D\, P_n(\theta,\phi)$], making it a function of angle with the maximum value equal to D.

Solution:

$$\text{Stray factor} = \frac{\Omega_m}{\Omega_A}$$

5-8-7. continued

where Ω_A = total beam area

Ω_M = main lobe beam area

Ω_m = minor lobe beam area

$$\frac{\Omega_m}{\Omega_A} = \frac{\iint\limits_{4\pi-\Omega_M} P_n(\theta,\phi)\,d\Omega}{\iint\limits_{4\pi} P_n(\theta,\phi)\,d\Omega}$$

Average directive gain over minor lobes = $DG_{av}(\text{minor}) = \dfrac{1}{4\pi-\Omega_M}\iint\limits_{4\pi-\Omega_M} DP_n(\theta,\phi)\,d\Omega$

where $D = 4\pi/\Omega_A$

Therefore $DG_{av}(\text{minor}) = \dfrac{1}{4\pi-\Omega_M}\dfrac{\iint\limits_{4\pi-\Omega_M} 4\pi P_n(\theta,\phi)\,d\Omega}{\Omega_A} = \dfrac{4\pi}{4\pi-\Omega_M}\dfrac{\Omega_m}{\Omega_A}$

If $\Omega_M << 4\pi$ (antenna highly directive),

$$DG_{av}(\text{minor}) \cong \frac{\Omega_m}{\Omega_A} \quad \text{(stray factor)} \quad \text{q.e.d.}$$

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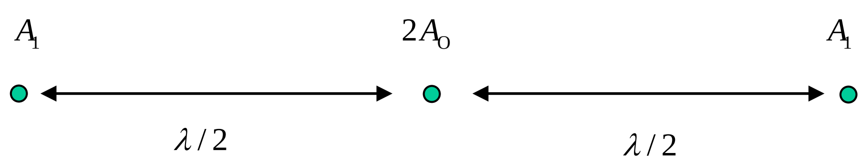
*5-9-2. Three-source array.

The center source of a 3-source array has a (current) amplitude of unity. For a sidelobe level 0.1 of the main lobe maximum field, find the Dolph-Tchebyscheff value of the amplitude of the end sources. The source spacing $d = \lambda/2$.

Solution: Let the amplitudes (currents) of the 3 sources be as in the sketch



$d = \lambda/2, \quad R = 10$

Let amplitude of center source $= 1 = 2A_o$

$$n-1 = 2, \qquad T_2(x) = 2x^2 - 1 = R$$

$$2x_o^2 - 1 = 10 \qquad 2x_o^2 = 11$$

$$x_o^2 = 5.5 \qquad x_o = \pm 2.345$$

5-9-2. continued

$$E_3 = 2A_o + 2A_1 \cos 2\frac{\psi}{2} = 2A_o + 2A_1\left(2\cos^2\frac{\psi}{2} - 1\right) = 2A_o + 2A_1(2w^2 - 1)$$

Let $w = x / x_o$ so

$$E_3 = 2A_o + 2A_1\left(2\frac{x^2}{x_o^2} - 1\right) = 2A_o + 4\frac{A_1}{x_o^2}x^2 - 2A_1 = 2x^2 - 1$$

$$E_3 = 0.728A_1x^2 + (A_o - 2A_1) = 2x^2 - 1$$

Therefore, $0.728A_1 = 2$ and $A_1 = 2.75$

$2A_o - 2A_1 = -1$ and $2A_o = 5.5 - 1 = 4.5$

Thus, normalizing $2A_o = 1$ and $A_1 = 2.75/4.5 = 0.61$ (*ans.*)

Amplitude distribution is 0.61 1.00 0.61
Pattern has 4 minor lobes. For center source, amplitude $= 1$
The side source amplitudes for different *R* values are:

R	8	10	12	15
A_1	0.64	0.61	0.59	0.57

5-9-4. Eight source D-T distribution.

(a) Find the Dolph-Tchebyscheff current distribution for the minimum beam width of a linear in-phase broadside array of eight isotropic sources. The spacing between the elements is $\lambda/4$ and the sidelobe level is to be 40 dB down. Take $\phi = 0$ in the broadside direction.
(b) Locate the nulls and the maxima of the minor lobes.
(c) Plot, approximately, the normalized field pattern ($0° \le \phi \le 360°$).
(d) What is the half-power beam width?

Solution:
(a) 0.14, 0.42, 0.75, 1.00, 1.00, 0.75, 0.42, 0.14

(b) Max. at:
$\pm 21^o$, $\pm 27^o$, $\pm 36^o$, $\pm 48^o$, $\pm 61^o$, $\pm 84^o$, $\pm 96^o$, $\pm 119^o$, $\pm 132^o$, $\pm 144^o$, $\pm 153^o$, $\pm 159^o$
Nulls at:
$\pm 18^o$, $\pm 23^o$, $\pm 32^o$, $\pm 42^o$, $\pm 54^o$, $\pm 71^o$, $\pm 109^o$, $\pm 126^o$, $\pm 138^o$, $\pm 148^o$, $\pm 157^o$, $\pm 162^o$

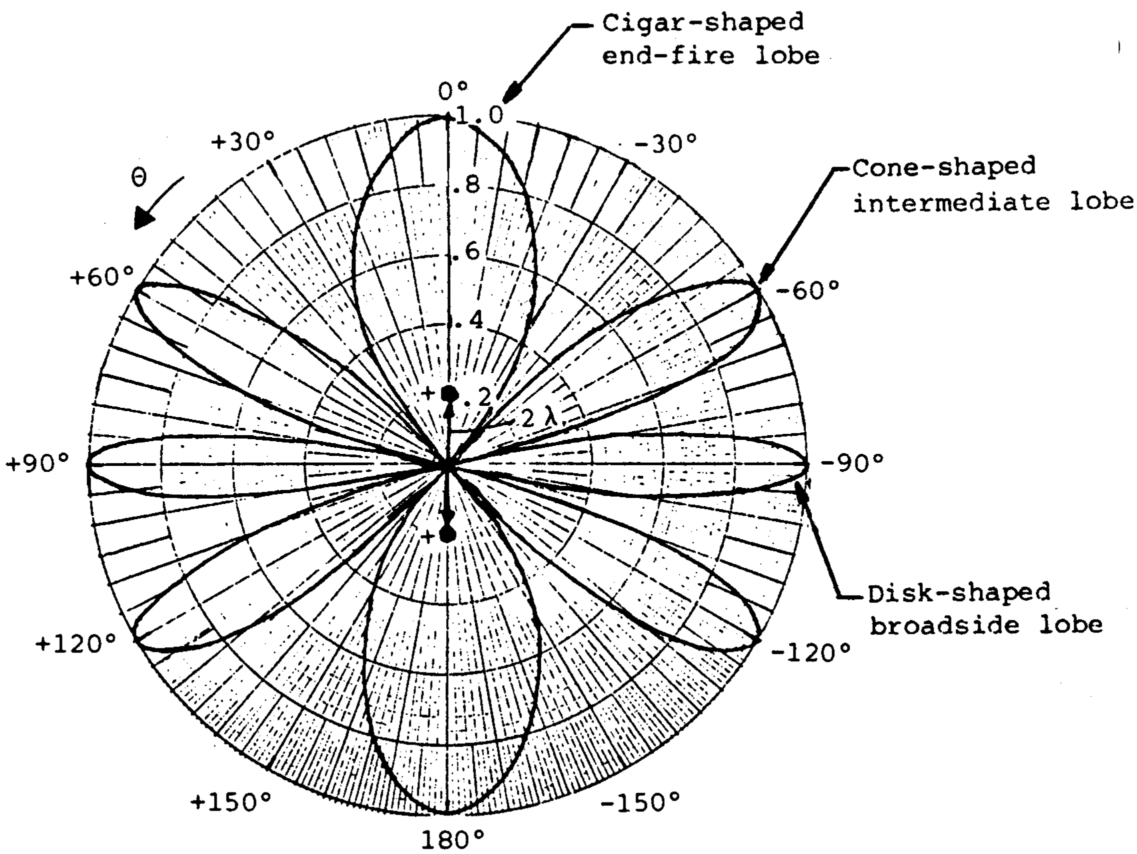
(d) HPBW $= 12^o$ (*ans.*)

***5-18-1. Two sources in phase.**

Two isotropic point sources of equal amplitude and same phase are spaced 2λ apart. (a) Plot a graph of the field pattern. (b) Tabulate the angles for maxima and nulls.

Solution:

(a) Power pattern $P_n = E_n^2$



Instructional comment to pass on to students:

The lobes with narrowest beam widths are broadside ($\pm 90^o$), while the widest beam width lobes are end-fire (0^o and 180^o). The four lobes between broadside and end-fire are intermediate in beam width. In three dimensions the pattern is a figure-of-revolution around the array axis (0^o and 180^o axis) so that the broadside beam is a flat disk, the end-fire lobes are thick cigars, while the intermediate lobes are cones. The accompanying figure is simply a cross section of the three-dimensional space figure.

5-18-2. Two sources in opposite phase.

Two isotropic sources of equal amplitude and opposite phase have 1.5λ spacing. Find the angles for all maxima and nulls.

Solution:

Maximum at: 0^o, 180^o, $\pm 70.5^o$, $\pm 109.5^o$, Nulls at: $\pm 48.2^o$, $\pm 90^o$, $\pm 131.8^o$

Chapter 6. The Electric Dipole and Thin Linear Antennas

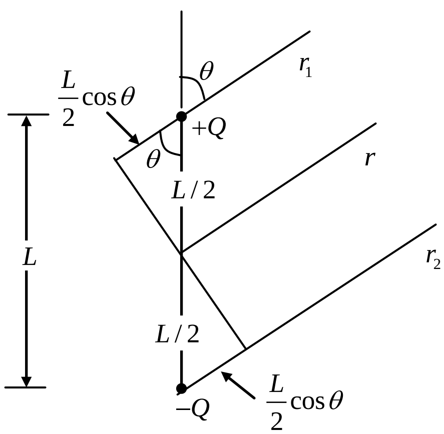
*6-2-1. Electric dipole.

(a) Two equal static electric charges of opposite sign separated by a distance L constitute a static electric dipole. Show that the electric potential at a distance r from such a dipole is given by

$$V = \frac{QL\cos\theta}{4\pi\varepsilon r^2}$$

where Q is the magnitude of each charge and θ is the angle between the radius r and the line joining the charges (axis of dipole). It is assumed that r is very large compared to L.
(b) Find the vector value of the electric field **E** at a large distance from a static electric dipole by taking the gradient of the potential expression in part (a).

Solution:



(a) $V(\text{at } r) = \dfrac{Q}{4\pi\varepsilon r_1} - \dfrac{Q}{4\pi\varepsilon r_2}$,

$r_1 = r - (L/2)\cos\theta, \qquad r_2 = r + (L/2)\cos\theta$

$$V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r-(L/2)\cos\theta} - \frac{1}{r+(L/2)\cos\theta}\right)$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{r+(L/2)\cos\theta - r + (L/2)\cos\theta}{r^2+(L/2)^2\cos^2\theta}\right)$$

For $r \gg L$, $\quad V = \dfrac{QL\cos\theta}{4\pi\varepsilon r^2} \qquad$ q.e.d.

(b) $$\mathbf{E} = -\nabla V = -\left(\hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\right) = \frac{-QL}{4\pi\varepsilon}\left(-\hat{\mathbf{r}}\frac{2\cos\theta}{r^3} - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\sin\theta}{r^2} + \hat{\boldsymbol{\phi}}\,0\right)$$

$$= \hat{\mathbf{r}}\frac{QL\cos\theta}{2\pi\varepsilon r^3} + \hat{\boldsymbol{\theta}}\frac{QL\sin\theta}{4\pi\varepsilon r^3}$$

or $\quad E_r = \dfrac{QL\cos\theta}{2\pi\varepsilon r^3}, \qquad E_\theta = \dfrac{QL\sin\theta}{4\pi\varepsilon r^3}, \qquad E_\phi = 0 \quad$ (*ans.*)

---

*6-2-2. Short dipole fields.

A dipole antenna of length 5 cm is operated at a frequency of 100 MHz with terminal current I_o = 120 mA. At time t = 1 s, angle θ = 45°, and distance r = 3 m, find (a) E_r, (b) E_θ, and (c) H_ϕ.

***6-2-2. continued**

Solution:

(a) From (6-2-12)

$$E_r = \frac{I_o l e^{j(\omega t - \beta r)} \cos\theta}{2\pi\varepsilon_o}\left(\frac{1}{cr^2} + \frac{1}{j\omega r^3}\right)$$

$$= (120\times10^{-3})(0.05)\frac{e^{j\left[(2\pi)100\times10^6(1) - \left(\frac{(2\pi)100\times10^6}{3\times10^8}\right)(3)\right]}}{2\pi(8.85\times10^{-12})}\cos 45^o\left(\frac{1}{3\times10^8(3^2)} + \frac{1}{j(2\pi)100\times10^6(3^3)}\right)$$

$$= 2.83\times10^{-2} - j(4.5\times10^{-3}) = 2.86\times10^{-2}\angle{-9^o}\ \text{V/m} \quad (ans.)$$

(b) From (6-2-13)

$$E_\theta = \frac{I_o l e^{j(\omega t - \beta r)} \sin\theta}{4\pi\varepsilon_o}\left(\frac{j\omega}{c^2 r} + \frac{1}{cr^2} + \frac{1}{j\omega r^3}\right) = 1.41\times10^{-2} + j(8.65\times10^{-2})$$

$$= 8.77\times10^{-2}\angle 81^o\ \text{V/m} \quad (ans.)$$

(c) From (6-2-15)

$$H_\phi = \frac{I_o l e^{j(\omega t - \beta r)} \sin\theta}{4\pi}\left(\frac{j\omega}{cr} + \frac{1}{r^2}\right) = 3.75\times10^{-5} + j(2.36\times10^{-4})$$

$$= 2.39\times10^{-4}\angle 81^o\ \text{A/m} \quad (ans.)$$

***6-2-4. Short dipole quasi-stationary fields.**

For the dipole antenna of Prob. 6-2-2, at a distance r = 1 m, use the general expressions of Table 6-1 to find (a) E_r, (b) E_θ, and (c) H_ϕ. Compare these results to those obtained using the quasi-stationary expressions of Table 6-1.

Solution:

Using the same approach for E_r, E_θ, and H_ϕ as in solution to Prob. 6-2-2, we find for $r = 1$ m,

$$E_r = 282\ \text{mV/m}$$

$$E_\theta = 242\ \text{mV/m}$$

$$H_\phi = 784\ \text{mA/m}$$

***6-2-4. continued**

Using quasi-stationary equations,

$$E_r = \frac{q_o l\cos\theta}{2\pi\varepsilon_o r^3} = \frac{I_o l\cos\theta}{j\omega(2\pi\varepsilon_o r^3)} = \frac{120\times10^{-3}(0.05\cos 45^\circ)}{j(2\pi)^2(100\times10^6)(8.85\times10^{-12})1^3} = 0 - j(121\times10^{-3})$$
$$= 121 \text{ mVm} \quad (\textit{ans.})$$

$$E_\theta = \frac{q_o l\sin\theta}{4\pi\varepsilon_o r^3} = \frac{I_o l\sin\theta}{j\omega(4\pi\varepsilon_o r^3)} = 0 - j(61\times10^{-3}) = 61 \text{ mV/m} \quad (\textit{ans.})$$

$$H_\phi = \frac{I_o l\sin\theta}{4\pi r^2} = 3.38\times10^{-4} = 338\ \mu\text{A/m} \quad (\textit{ans.})$$

***6-3-1. Isotropic antenna. Radiation resistance.**

An omnidirectional (isotropic) antenna has a field pattern given by $E = 10I/r$ (V m^{-1}), where I = terminal current (A) and r = distance (m). Find the radiation resistance.

Solution:

$$E = \frac{10I}{r} \quad \text{so} \quad S = \frac{E^2}{Z} = \frac{100I^2}{r^2 Z}$$

Let $P =$ power over sphere $= 4\pi r^2 S$, which must equal power $I^2 R$ to the antenna terminals. Therefore $I^2 R = 4\pi r^2 S$ and

$$R = \frac{1}{I^2}4\pi r^2\frac{100I^2}{r^2 120\pi} = \frac{400}{120} = 3.33\ \Omega \quad (\textit{ans.})$$

***6-3-2. Short dipole power.**

(a) Find the power radiated by a 10 cm dipole antenna operated at 50 MHz with an average current of 5 mA. (b) How much (average) current would be needed to radiate power of 1 W?

Solution:
(a)

$$P = \sqrt{\frac{\mu_o}{\varepsilon_o}}\frac{(\beta I_{av} l)^2}{12\pi} = 377\frac{\left(\frac{(2\pi)50\times10^6}{3\times10^8}(5\times10^{-3})0.1\right)^2}{12\pi} = 2.74\times10^{-6}\text{ W} = 2.74\ \mu\text{W} \quad (\textit{ans.})$$

***6-3-2. continued**

(b) For $P = 1$ W, $I_{av} = 5\times10^{-3}\left(\dfrac{1}{2.7\times10^{-6}}\right)^{1/2} = 3.0$ A (*ans.*)

6-3-4. Short dipole.

For a thin center-fed dipole $\lambda/15$ long find (a) directivity D, (b) gain G, (c) effective aperture A_e, (d) beam solid Ω_A and (e) radiation resistance R_r. The antenna current tapers linearly from its value at the terminals to zero at its ends. The loss resistance is 1 Ω.

Solution:

(a) $E_n(\theta) = \sin\theta$

$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\iint\limits_{4\pi}\sin^2\theta d\Omega} = \frac{4\pi}{\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\sin\theta d\theta d\Omega} = \frac{4\pi}{2\pi\int_0^{\pi}\sin^3\theta d\theta}$$

$$= \frac{4\pi}{2\pi\frac{4}{3}} = \frac{3}{2} = 1.5 \text{ or } 1.76 \text{ dBi} \quad (ans.)$$

(d) From (a), $\Omega_A = 8\pi/3 = 8.38$ sr (*ans.*)

(e) From (6-3-14), $R_r = 790\left(\dfrac{I_{av}}{I_o}\right)^2 L_\lambda^2 = 790\left(\dfrac{1}{2}\right)^2\left(\dfrac{1}{15}\right)^2 = 0.878\ \Omega$ (*ans.*)

(b) $G = kD = \dfrac{0.878}{0.878+1}\times1.5 = 0.70$ or -1.54 dBi (*ans.*)

(c) $A_e = kA_{em}$ where $A_{em} = \dfrac{\lambda^2}{\Omega_A} = \dfrac{3}{8\pi}\lambda^2$

Therefore $A_e = \dfrac{0.878}{0.878+1}\times\dfrac{3}{8\pi} = 0.058\ \lambda^2$ (*ans.*)

***6-3-5. Conical pattern.**

An antenna has a conical field pattern with uniform field for zenith angles (θ) from 0 to 60° and zero field from 60 to 180°. Find exactly (a) the beam solid angle and (b) directivity. The pattern is independent of the azimuth angle (ϕ).

Solution:

(a) $\Omega_A = \int_0^{360^\circ}\int_0^{60^\circ} d\Omega = 2\pi\int_0^{60^\circ}\sin\theta d\theta = -2\pi\cos\theta\Big|_0^{60^\circ} = \pi$ sr (*ans.*)

***6-3-5. continued**

(b) $D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\pi} = 4$ (*ans.*)

6-3-6. Conical pattern.

An antenna has a conical field pattern with uniform filed for zenith angles (θ) from 0 to 45° and zero field from 45 to 180°. Find exactly (a) the beam solid angle, (b) directivity and (c) effective aperture. (d) Find the radiation resistance if the E = 5 V m^{-1} at a distance of 50 m for a terminal current I = 2 A (rms). The pattern is independent of the azimuth angle (ϕ).

Solution:

(a) $\Omega_A = 2\pi \int_0^{45^\circ} \sin\theta d\theta = 1.84 \text{ sr}$ (*ans.*)

(b) $D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{1.84} = 6.83$ (*ans.*)

(c) $A_e = A_{em} = \frac{\lambda^2}{\Omega_A} = \frac{\lambda^2}{1.84} = 0.543\ \lambda^2$ (*ans.*)

(d) $I^2 R_r = \Omega_A r^2 \frac{E^2}{Z}$, $\quad R_r = \frac{1}{2^2} 1.84 \times 50^2 \frac{5^2}{377} = 76.3\ \Omega$ (*ans.*)

***6-3-7. Directional pattern in θ and ϕ.**

An antenna has a uniform field pattern for zenith angles (θ) between 45 and 90° and for azimuth (ϕ) angles between 0 and 120°. If E = 3 V m^{-1} at a distance of 500 m from the antenna and the terminal current is 5 A, find the radiation resistance of the antenna. E = 0 except within the angles given above.

Solution:

(a) $\Omega_A = \int_0^{120^\circ} \int_{45^\circ}^{90^\circ} \sin\theta d\theta d\phi = -\frac{2\pi}{3} \cos\theta \Big|_{45^\circ}^{90^\circ} = 1.48 \text{ sr}$ (*ans.*)

(c) $R_r = \frac{1}{I^2} \Omega_A r^2 \frac{E^2}{Z} = \frac{1}{5^2} 1.48 \times 500^2 \frac{3^2}{377} = 354\ \Omega$ (*ans.*)

---

*6-3-8. Directional pattern in θ and ϕ.

An antenna has a uniform field E = 2 V m^{-1} (rms) at a distance of 100 m for zenith angles between 30 and 60° and azimuth angles ϕ between 0 and 90° with E = 0 elsewhere. The antenna terminal current is 3 A (rms). Find (a) directivity, (b) effective aperture and (c) radiation resistance.

Solution:

(a) $$\Omega_A = \int_0^{90^\circ}\int_{30^\circ}^{60^\circ} \sin\theta d\theta d\phi = -\frac{\pi}{2}\cos\theta\Big|_{30^\circ}^{60^\circ} = 0.575 \text{ sr} \quad (ans.)$$

$$D = \frac{4\pi}{0.575} = 21.9 \quad (ans.)$$

(b) $$A_e = A_{em} = \frac{\lambda^2}{\Omega_A} = \frac{\lambda^2}{0.575} = 1.74\ \lambda^2 \quad (ans.)$$

(c) $$R_r = \frac{1}{I^2}\Omega_A r^2 \frac{E^2}{Z} = \frac{1}{3^2}0.575\times 100^2\frac{2^2}{377} = 6.78\ \Omega \quad (ans.)$$

---

*6-3-9. Directional pattern with back lobe.

The field pattern of an antenna varies with zenith angle (θ) as follows: E_n (=$E_{normalized}$) = 1 between θ = 0° and θ = 30° (main lobe), E_n = 0 between θ = 30° and θ = 90° and E_n = 1/3 between θ = 90° and θ = 180° (back lobe). The pattern is independent of azimuth angle (ϕ). (a) Find the exact directivity. (b) If the field equals 8 V m^{-1} (rms) for θ = 0° at a distance of 200 m with a terminal current I = 4 A (rms), find the radiation resistance.

Solution:

(a) $$\Omega_A = 2\pi\int_0^{30^\circ} \sin\theta d\theta + \frac{2\pi}{3^2}\int_{90^\circ}^{180^\circ} \sin\theta d\theta = 2\pi(0.134 + 0.111) = 2\pi(0.245)$$

$$D = \frac{4\pi}{2\pi(0.245)} = 8.16 \quad (ans.)$$

(b) $$R_r = \frac{1}{I^2}\Omega_A r^2 \frac{E^2}{Z} = \frac{1}{4^2}2\pi(0.245)200^2\frac{8^2}{120\pi} = 653\ \Omega \quad (ans.)$$

6-3-10. Short dipole.

The radiated field of a short-dipole antenna with uniform current is given by $|E| = 30\beta l(I/r)\sin\theta$, where l = length, I = current, r = distance and θ = pattern angle. Find the radiation resistance.

6-3-10. continued

Solution:

The current I given in the problem is a peak value, so we put $\underbrace{\frac{1}{2}I^2R_r}_{\text{Power input}} = \underbrace{\iint_{4\pi} Sr^2 d\Omega}_{\text{Power radiated}}$

where $S = \dfrac{E^2}{Z}$ and E is as given

so $$R_r = \frac{2}{I^2}2\pi\frac{30^2\beta^2 l^2 I^2}{r^2 120\pi}r^2\int_0^\pi \sin^3\theta d\theta = 80\pi^2(l/\lambda)^2 = 790(l/\lambda)^2\ \Omega \quad (ans.)$$

6-3-11. Relation of radiation resistance to beam area.

Show that the radiation resistance of an antenna is a function of its beam area Ω_A as given by

$$R_r = \frac{Sr^2}{I^2}\Omega_A$$

where S = Poynting vector at distance r in direction of pattern maximum
I = terminal current.

Solution:

Taking I as the rms value we set $\underbrace{I^2R_r}_{\text{Power input}} = \underbrace{Sr^2\Omega_A}_{\text{Power radiated}}$, therefore $R_r = \dfrac{Sr^2}{I^2}\Omega_A$ q.e.d.

***6-3-12. Radiation resistance.**

An antenna measured at a distance of 500 m is found to have a far-field pattern of $|E| = E_o(\sin\theta)^{1.5}$ with no ϕ dependence. If E_o = 1 V/m and I_o = 650 mA, find the radiation resistance of this antenna.

Solution:

From (6-3-5)

$$R_r = \frac{120\pi}{I_o^2}\int_s \frac{|E|^2}{Z_o^2}ds = \frac{120\pi}{(0.65)^2(377)^2}\int_0^{2\pi}\int_0^\pi \sin^2\theta r\sin\theta d\theta d\phi$$

$$= (6.28\times10^{-3})(500)2\pi\int_0^\pi \sin^4\theta d\theta$$

$$= 19.7\left[\frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32}\right]_0^\pi = 19.7\left(\frac{3\pi}{8}\right) = 23.2\ \Omega \quad (ans.)$$

---

*6-5-1. λ/2 antenna.

Assume that the current is of uniform magnitude and in-phase along the entire length of a *λ/2* thin linear element.
(a) Calculate and plot the pattern of the far field.
(b) What is the radiation resistance?
(c) Tabulate for comparison:

1. Radiation resistance of part (b) above
2. Radiation resistance at the current loop of a *λ/2* thin linear element with sinusoidal in-phase current distribution
3. Radiation resistance of a *λ/2* dipole calculated by means of the short dipole formula

(d) Discuss the three results tabulated in part (c) and give reasons for the differences.

Solution:

(a) $E_n(\theta) = \tan\theta \sin[(\pi/2)\cos\theta]$ (*ans.*)

(b) $R = 168\ \Omega$ (*ans.*)

(c) R [from (b)] = 168 Ω (*ans.*)
R (sinusoidal I) = 73 Ω (*ans.*)
R (short dipole) = 197 Ω (*ans.*)

(d) 168 Ω is appropriate for uniform current.
73 Ω is appropriate for sinusoidal current.
197 Ω assumes uniform current, but the short dipole formula does not take into account the difference in distance to different parts of the dipole (assumes $\lambda >> L$) which is not appropriate and leads to a larger resistance (197 Ω) as compared to the correct value of 168 Ω.

6-6-1. 2λ antenna.

The instantaneous current distribution of a thin linear center-fed antenna *2λ* long is sinusoidal as shown in Fig. P6-6-1.
(a) Calculate and plot the pattern of the far field.
(b) What is the radiation resistance referred to a current loop?
(c) What is the radiation resistance at the transmission-line terminals as shown?
(d) What is the radiation resistance *λ/8* from a current loop?

6-6-1. continued

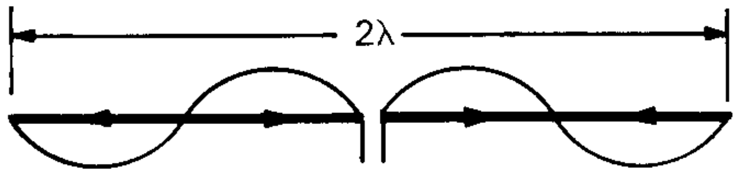


Figure P6-6-1. 2λ antenna.

Solution:

(a) $E_n(\theta) = \dfrac{\cos(2\pi\cos\theta) - 1}{\sin\theta}$ (*ans.*)

(b) $R(\text{at } I_{\max}) = 259\ \Omega$ (*ans.*)

(c) $R(\text{at terminals}) = \infty\ \Omega$ (*ans.*)

(d) $R(\text{at } \lambda/8 \text{ from } I_{\max}) = 518\ \Omega$ (*ans.*)

6-7-1. λ/2 antennas in echelon.

Calculate and plot the radiation-field pattern in the plane of two thin linear $\lambda/2$ antennas with equal in-phase currents and the spacing relationship shown in Fig. P6-7-1. Assume sinusoidal current distributions.

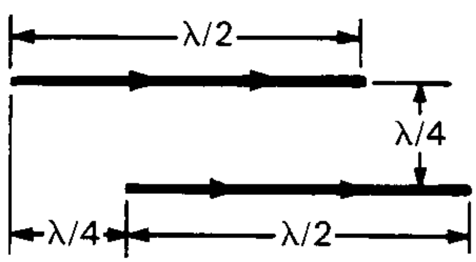


Figure P6-7-1. $\lambda/2$ antennas in echelon.

Solution:

$$E_n(\theta) = \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\cos\left(\frac{\sqrt{2}\pi}{4}\cos[(\pi/4)+\theta]\right)$$

---

*6-8-1. 1λ and 10λ antennas with traveling waves.

(a) Calculate and plot the far-field pattern in the plane of a thin linear element 1λ long, carrying a single uniform traveling wave for 2 cases of the relative phase velocity p = 1 and 0.5. (b) Repeat for the single case of an element 10λ long and p = 1.

Solution:

(a) From (6-8-5), $E_n(\phi) = \frac{\sin\phi}{1-p\cos\phi}[\sin\pi(\frac{1}{p}-\cos\phi)]$, patterns have 4 lobes.

(b) Pattern has 40 lobes.

6-8-2. Equivalence of pattern factors.

Show that the field pattern of an ordinary end-fire array of a large number of collinear short dipoles as given by Eq. (5-6-8), multiplied by the dipole pattern sin ϕ, is equivalent to Eq. (6-8-5) for a long linear conductor with traveling wave for p = 1.

Solution:

(1) Field pattern$=\dfrac{\sin\dfrac{n\psi}{2}}{\sin\dfrac{\psi}{2}}$ (5-6-8)

where $\psi = \beta d\cos\phi + \delta$

(2) Field pattern $=\sin\phi\dfrac{\sin[\dfrac{\omega b}{2pc}(1-p\cos\phi)]}{1-p\cos\phi}$ (6-8-5)

For ordinary end-fire, $\psi = \beta d(\cos\phi - 1)$

Also if d is small (1) becomes $\dfrac{\sin\left(\dfrac{\beta nd}{2}(1-\cos\phi)\right)}{\dfrac{\beta d}{2}(1-\cos\phi)}$

For larger n, $nd \cong b$. Also multiplying by the source factor $\sin\phi$ and taking the constant $\beta d/2 = 1$ in the denominator, (1) becomes

$$\sin\phi\frac{\sin\left(\dfrac{\beta d}{2}(1-\cos\phi)\right)}{1-\cos\phi}$$

6-8-2. continued

which is the same as (2) for $p = 1$

since $\dfrac{\omega b}{2pc} = \dfrac{2\pi fb}{2f\lambda} = \dfrac{\beta b}{2}$ q.e.d.

Note that for a given length *b*, the number *n* is assumed to be sufficiently large that *d* can be small enough to allow $\sin\psi/2$ in (1) to be replaced by $\psi/2$.

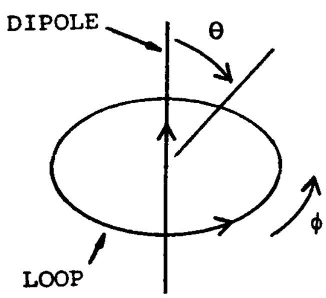
Chapter 7. The Loop Antenna

7-2-1. Loop and dipole for circular polarization.

If a short electric dipole antenna is mounted inside a small loop antenna (on polar axis, Fig. 7-3) and both dipole and loop are fed in phase with equal power, show that the radiation is everywhere circularly polarized with a pattern as in Fig. 7-7 for the 0.1λ diameter loop.

Solution:

Uniform currents are assumed.



$$E_\phi(\theta)(\text{loop}) = \frac{120\pi^2 IA\sin\theta}{r\lambda^2} \tag{1}$$

$$E_\theta(\theta)(\text{dipole}) = \frac{\text{j}60\pi IL\sin\theta}{r\lambda} \tag{2}$$

$$R_r(\text{loop}) = 320\pi^4\left(\frac{A}{\lambda^2}\right)^2\ \Omega \tag{3}$$

$$R_r(\text{dipole}) = 80\pi^2 L_\lambda^2 \tag{4}$$

For equal power inputs,

$$I_{\text{loop}}^2 R_r(\text{loop}) = I_{\text{dipole}}^2 R_r(\text{dipole})$$

$$\frac{I_{\text{loop}}^2}{I_{\text{dipole}}^2} = \frac{R_r(\text{dipole})}{R_r(\text{loop})} = \frac{80\pi^2 L_\lambda^2}{320\pi^4 (A/\lambda^2)^2} = \frac{L_\lambda^2}{4\pi^2 (A/\lambda^2)^2} \tag{5}$$

$$\frac{I_{\text{loop}}}{I_{\text{dipole}}} = \frac{L_\lambda}{2\pi(A/\lambda^2)} \tag{6}$$

Therefore

$$E_\phi(\theta)(\text{loop}) = \frac{120\pi^2 L_\lambda I_{\text{dipole}} A\sin\theta}{r\lambda^2\, 2\pi(A/\lambda^2)} = \frac{60\pi I_{\text{dipole}} L\sin\theta}{r\lambda} \tag{7}$$

which is equal in magnitude to $E_\theta(\theta)$ (dipole) but in time-phase quadrature (no *j*).

Since the 2 linearly polarized fields (E_ϕ of the loop and E_θ of the dipole) are at right angles, are equal in magnitude and are in time-phase quadrature, the total field of the loop-dipole combination is everywhere circularly polarized with a $\sin\theta$ pattern. q.e.d.

7-2-1. continued

Equating the magnitude of (1) and (2) (fields equal and currents equal) we obtain

$$\frac{L}{\lambda} = 2\pi \frac{A}{\lambda^2} \tag{8}$$

which satisfies (6) for equal loop and dipole currents. Thus (8) is a condition for circular polarization.

Substituting $A = \pi(d^2/4)$, where $d =$ loop diameter in (8) and putting $C = \pi d$

$$\frac{L}{\lambda} = 2\pi \frac{\pi d^2}{4\lambda^2} = \frac{1}{2}\frac{C^2}{\lambda^2} \tag{9}$$

we obtain
$$C_\lambda = (2L_\lambda)^{1/2} \tag{10}$$

as another expression of the condition for circular polarization.

Thus, for a short dipole $\lambda/10$ long, the loop circumference must be

$$C_\lambda = (2\times 0.1)^{1/2} = 0.45 \tag{11}$$

and the loop diameter
$$d = \frac{0.45\lambda}{\pi} = 0.14\lambda$$
or 1.4 times the dipole length. If the dipole current tapers to zero at the ends of the dipole, the condition for CP is

$$\frac{L}{\lambda} = 4\pi \frac{A}{\lambda^2} \tag{12}$$

and

$$C_\lambda = (L_\lambda)^{1/2} \tag{13}$$

For a $\lambda/10$ dipole the circumference must now be $C_\lambda = (0.1)^{1/2} = 0.316$ and the loop diameter $d = \dfrac{0.316\lambda}{\pi} \cong 0.1\lambda$ or approximately the same as the dipole length.

The condition of (10) is applied in the Wheeler-type helical antenna. See Section 8-22, equation (8-22-4) and Prob. 8-11-1.

7-4-1. The 3λ/4 diameter loop.

Calculate and plot the far-field pattern normal to the plane of a circular loop 3λ/4 in diameter with a uniform in-phase current distribution.

Solution:

$$C_\lambda = \pi \frac{3}{4} = 2.36$$

From (7-3-8) or Table 7-2, the E_ϕ pattern is given by

$$J_1(C_\lambda \sin\theta)$$

See Figure 7-6.

***7-6-1. Radiation resistance of loop.**

What is the radiation resistance of the loop of Prob. 7-4-1?

Solution:

From (7-6-13) for loop of any size

$$R_r = 60\pi^2 C_\lambda \int_0^{2C_\lambda} J_2(y)dy$$

where $C_\lambda = \pi\, 3/4 = 2.36, \quad 2C_\lambda = 4.71$

From (7-6-16), $\int_0^{2C_\lambda} J_2(y)dy = \int_0^{2C_\lambda} J_\mathrm{o}(y)dy - 2J_1(2C_\lambda)$

By integration of the $J_\mathrm{o}(y)$ curve from 0 to $2C_\lambda (= 4.71)$, $\int_0^{2C_\lambda} J_\mathrm{o}(y)dy = 0.792$

From tables (Jahnke and Emde), $J_1(2C_\lambda) = J_1(4.71) = -0.2816$

and $\int_0^{2C_\lambda} J_2(y)dy == 0.7920 + 2\times 0.2816 = 1.355$

Therefore $R_r = 60\pi^2 2.36 \times 1.355 = 1894\ \Omega$ (Round off to 1890 Ω) (*ans.*)

7-6-2. Small-loop resistance.

(a) Using a Poynting vector integration, show that the radiation resistance of a small loop is equal to $320\pi^4 (A/\lambda^2)^2\ \Omega$ where A = area of loop (m^2). (b) Show that the effective aperture of an isotropic antenna equals $\lambda^2/4\pi$.

Solution:

(a) $$R_r = \frac{Sr^2\Omega_A}{I^2} = \frac{E_{\max}^2 r^2 \Omega_A}{ZI^2}$$

From (7-5-2) and Table 7-2,

$$\left|E_\phi\right| = \frac{120\pi^2 IA}{r\lambda^2}\sin\theta = E_{\max}\sin\theta$$

$$\Omega_A = 2\pi\int_0^\pi \sin^2\theta\sin\theta d\theta = 2\pi\frac{4}{3} = \frac{8}{3}\pi$$

Therefore, $$R_r = \frac{120^2\pi^4 I^2 A^2}{r^2\lambda^4}\frac{r^2}{120\pi I^2}\frac{8\pi}{3} = 320\pi^4\left(\frac{A}{\lambda^2}\right)^2\ \Omega = 197 C_\lambda^4\ \ \Omega \quad \text{q.e.d.}$$

(b) $$\Omega_A = 4\pi, \qquad A_e = \frac{\lambda^2}{\Omega_A} = \frac{\lambda^2}{4\pi} \quad \text{q.e.d.}$$

7-7-1. The λ/10 diameter loop.

What is the maximum effective aperture of a thin loop antenna 0.1λ in diameter with a uniform in-phase current distribution?

Solution:

Ω_A is the same as for a short dipole $(= 8\pi/3$ sr). See Prob. 6-3-4a.

Therefore, $$A_{em} = \frac{\lambda^2}{\Omega_A} = \left(\frac{3}{8\pi}\right)\lambda^2 = 0.119\lambda^2 \quad (\textit{ans.})$$

7-8-1. Pattern, radiation resistance and directivity of loops.

A circular loop antenna with uniform in-phase current has a diameter d. What is (a) the far-field pattern (calculate and plot), (b) the radiation resistance and (c) the directivity for each of three cases where (1) $d = \lambda/4$, (2) $d = 1.5\lambda$ and (3) $d = 8\lambda$?

7-8-1. continued

Solution:

Since all the loops have $C_\lambda > 1/3$, the general expression for E_ϕ in Table 7-2 must be used. From Table 7-2 and Figures 7-10 and 7-11, the radiation resistance and directivity values are:

Diameter	C_λ	R_r	Directivity
$\lambda/4$	0.785	76 Ω	1.5
1.5λ	4.71	2340 Ω	3.82
8λ	25.1	14800 Ω	17.1

***7-8-2. Circular loop.**

A circular loop antenna with uniform in-phase current has a diameter *d*. Find (a) the far-field pattern (calculate and plot), (b) the radiation resistance and (c) the directivity for the following three cases: (1) $d = \lambda/3$, (2) $d = 0.75\lambda$ and (3) $d = 2\lambda$.

Solution:

See Probs. 7-4-1 and 7-8-1. Radiation resistance and directivity values are:

Diameter	C_λ	R_r	Directivity
$\lambda/3$	1.05	180 Ω	1.5
0.75λ	2.36	1550 Ω	1.2
2λ	6.28	4100 Ω	3.6

***7-9-1. The 1λ square loop.**

Calculate and plot the far-field pattern in a plane normal to the plane of a square loop and parallel to one side. The loop is 1λ on a side. Assume uniform in-phase currents.

*7-9-1. continued

Solution:

Pattern is that of 2 point sources in opposite phase. Referring to Case 2 of Section 5-2, we have for $d_r/2 = 2\pi(\lambda/2) = \pi$,

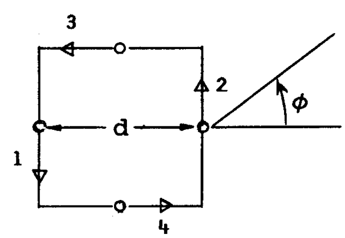
$$E_n(\phi) = \sin(\pi\cos\phi)$$

resulting in a 4-lobed pattern with maxima at $\phi = \pm 60^{\circ}$ and $\pm 120^{\circ}$ and nulls at $0^{\circ}, \pm 90^{\circ}$ and 180°.

7-9-2. Small square loop.

Resolving the small square loop with uniform current into four short dipoles, show that the far-field pattern in the plane of the loop is a circle.

Solution:



The field pattern $E(1,2)$ of sides 1 and 2 of the small square loop is the product of the pattern of 2 point sources in opposite phase separated by *d* as given by

$$\sin[(d_r/2)\cos\phi]$$

and the pattern of short dipole as given by $\cos\phi$

or $$E(1,2) = \cos\phi\sin[(d_r/2)\cos\phi]$$

For small *d* this reduces to $$E_n(1,2) = \cos^2\phi$$

The pattern of sides 3 and 4 is the same rotated through 90° or in terms of ϕ is given by

$$E_n(3,4) = \sin^2\phi$$

The total pattern in the plane of the square loop is then

$$E_n(\phi) = E_n(1,2) + E_n(3,4) = \cos^2\phi + \sin^2\phi = 1$$

Therefore $E(\phi)$ is a constant as a function of ϕ and the pattern is a circle. q.e.d.

Chapter 8. End-Fire Antennas: The Helical Beam Antenna and the Yagi-Uda Array, Part I

8-3-1. A 10-turn helix.

A right-handed monofilar helical antenna has 10 turns, 100 mm diameter and 70 mm turn spacing. The frequency is 1 GHz. (a) What is the HPBW? (b) What is the gain? (c) What is the polarization state? (d) Repeat the problem for a frequency of 300 MHz.

Solution:

(a) $$\lambda = \frac{3\times 10^8}{10^9} = 0.3 \text{ m} \qquad C = \pi(0.1) = 0.314$$

$$C_\lambda = \frac{0.314}{0.3} = 1.047 \qquad S_\lambda = \frac{0.07}{0.3} = 0.233$$

From (8-3-4)

$$\text{HPBW} = \frac{52^{\text{O}}}{C_\lambda (nS_\lambda)^{1/2}} = \frac{52^{\text{O}}}{1.047(10\times 0.233)^{1/2}} = 32.5^{\text{o}} \quad (ans.)$$

(b) From (8-3-7), $D \cong 12 C_\lambda^2 n S_\lambda = 30.7$ or 14.9 dBi (*ans.*)

If losses are negligible the gain = *D*.

(c) Polarization is RCP. (*ans.*)

(d) At 300 MHz, $\lambda = 3\times 10^8 / 300\times 10^6 = 1$ m, $C_\lambda = 0.314/1 = 0.314$.

This is too small for the axial mode which requires that $0.7 < C_\lambda < 1.4$.

From Table A-1, $D = \dfrac{41000}{32.5^2} = 38.8$ or 15.9 dBi or 1 dB higher.

The lower value is more realistic

8-3-2. A 30-turn helix.

A right-handed monofilar axial-mode helical antenna has 30 turns, $\lambda/3$ diameter and $\lambda/5$ turn spacing. Find (a) HPBW, (b) gain and (c) polarization state.

Solution:

(a) From (8-3-4), $$\text{HPBW} \cong \frac{52^{\text{o}}}{C_\lambda (nS_\lambda)^{1/2}} = \frac{52^{\text{o}}}{\frac{\pi}{3}(30\times 0.2)^{1/2}} = 20.3^{\text{o}} \quad (ans.)$$

(b) For zero losses, $G = D$

8-3-2. continued

From (8-3-7), $D \cong 12 C_\lambda^2 n S_\lambda = 12(\pi/3)^2 30 \times 0.2 = 79$ or 19 dBi (*ans.*)

(c) RCP (*ans.*)

8-3-3. Helices, left and right.

Two monofilar axial-mode helical antennas are mounted side-by-side with axes parallel (in the x direction). The antennas are identical except that one is wound left-handed and the other right-handed. What is the polarization state in the x direction if the two antennas are fed (a) in phase and (b) in opposite phase?

Solution:

Assuming that x is horizontal, (a) LHP (*ans.*) (b) LVP (*ans.*)

Chapter 8. The Helical Antenna: Axial and Other Modes, Part II

*8-8-1. An 8-turn helix.

A monofilar helical antenna has $\alpha = 12°$, $n = 8$, $D = 225$ mm. (a) What is p at 400 MHz for (1) in-phase fields and (2) increased directivity? (b) Calculate and plot the field patterns for $p = 1.0$, 0.9, and 0.5 and also for p equal to the value for in-phase fields and increased directivity. Assume each turn is an isotropic point source. (c) Repeat (b) assuming each turn has a cosine pattern.

Solution:

(a) The relative phase velocity for in-phase fields is given by (8-8-9) as

$$p = \frac{1}{\sin\alpha + \dfrac{\cos\alpha}{C_\lambda}}$$

The relative phase velocity for increased directivity is given by (8-8-12)

$$p = \frac{L_\lambda}{S_\lambda + \dfrac{2n+1}{2n}}$$

From the given value of frequency and diameter D, C_λ can be determined. Introducing it and the given values of α and n

$$p = 0.802 \text{ for in-phase fields}$$
$$p = 0.763 \text{ for increased directivity}$$

---

*8-11-1. Normal-mode helix.

(a) What is the approximate relation required between the diameter D and height H of an antenna having the configuration shown in Fig. P8-11-1, in order to obtain a circularly polarized far-field at all points at which the field is not zero. The loop is circular and is horizontal, and the linear conductor of length H is vertical. Assume D and H are small compared to the wavelength, and assume the current is of uniform magnitude and in phase over the system.

(b) What is the pattern of the far circularly polarized field?

***8-11-1. continued**

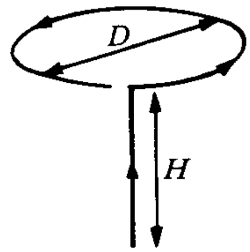


Figure P8-5-3. Normal mode helix.

Solution:

See solution to Prob. 7-2-1.

(a) $D_\lambda = (2H_\lambda)^{1/2} / \pi$ (*ans.*)

(b) $E = \sin\theta$ (*ans.*)

8-15-1. Design of quad-helix earth station antenna.

An array of four right-handed axial-mode helical antennas, shown in Fig. 8-54, can be used for communications with satellites. Determine (a) the best spacing based on the effective apertures of the helixes, (b) the directivity of the array. Assume the number of turns is 20 and the spacing between turns is 0.25 λ.

Solution:

(a) From (8-3-7) the directivity of each helix is

$$D \cong 12 \times (1.05)^2 \times 20 \times 0.25 = 66.15$$

$$A_e = \frac{66\lambda^2}{4\pi} = 5.26\lambda^2$$

The spacing is then $\sqrt{5.26}\lambda = 2.29\lambda$

(b) At 2.29λ spacing the effective aperture for the array is $5.26 \times 4 = 21.04\lambda^2$

so for the array

$$D = \frac{4\pi \times 21.04}{\lambda^2} = 264 \quad (24.2 \text{ dBi}) \quad (ans.)$$

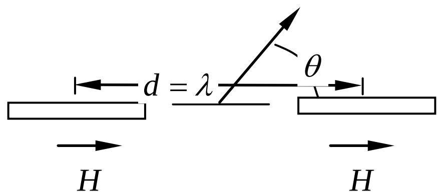
Chapter 9. Slot, Patch and Horn Antennas

9-2-1. Two $\lambda/2$ slots.

Two $\lambda/2$-slot antennas are arranged end-to-end in a large conducting sheet with a spacing of 1λ between centers. If the slots are fed with equal in-phase voltages, calculate and plot the far-field pattern in the 2 principal planes. Note that the *H* plane coincides with the line of the slots.

Solution:

Thin slots are assumed.



ϕ measured in plane perpendicular to page

The pattern in the *E plane* is a circle (*E* not a function of angle) or $E(\phi) = 1$ (*ans.*)

In the *H-plane* we have by pattern multiplication that the pattern is the product of 2 in-phase isotropic sources spaced 1λ and the pattern of a $\lambda/2$ slot. The pattern of the $\lambda/2$ slot is the same as for a $\lambda/2$ dipole but with E and H interchanged.

The pattern of the 2 isotropic sources is given by

$$|E| \text{ or } |H| = e^{-j(\beta d/2)\cos\theta} + e^{+j(\beta d/2)\cos\theta} = 2\cos[(\beta d/2)\cos\theta] = 2\cos(\pi\cos\theta)$$

The total normalized pattern in the *H*-plane is then

$$E_n(\theta) = \left|\cos(\pi\cos\theta)\frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\right| \quad (ans.)$$

---

*9-5-1. Boxed-slot impedance.

What is the terminal impedance of a slot antenna boxed to radiate only in one half-space whose complementary dipole antenna has a driving-point impedance of $Z = 150 + j0\ \Omega$? The box adds no shunt susceptance across the terminals.

Solution:

From (9-5-12) the impedance of an unboxed slot is $Z_s = \dfrac{35476}{R_d + jX_d}$

where R_d is the resistance and X_d is the reactance of the complementary dipole.

*9-5-1. continued

Thus,
$$Z_s = \frac{35476}{150 + j0} = 236.5\ \Omega$$

Boxing the slot doubles the impedance so $Z_s = 2 \times 236.5 = 473.0 \cong 473\ \Omega$ (*ans.*)

---

*9-5-2. Boxed slot.

The complementary dipole of a slot antenna has a terminal impedance $Z = 90 + j10\ \Omega$. If the slot antenna is boxed so that it radiates only in one half-space, what is the terminal impedance of the slot antenna? The box adds no shunt susceptance at the terminals.

Solution:

From (8-5-12) we have a boxed slot $Z_s = 2 \times \frac{35476}{90 + j10} = 779 - j87\ \Omega$ (*ans.*)

9-5-3. Open-slot impedance.

What dimensions are required of a slot antenna in order that its terminal impedance be 75 + $j0\ \Omega$? The slot is open on both sides.

Solution:

From (8-5-11),
$$Z_d = \frac{35476}{Z_s} = \frac{35476}{75} = 473\ \Omega$$

From Fig. 14-8 a center-fed cylindrical dipole with length-to-*diameter* ratio of ≈ 37 has a resistance at 4th resonance of $\approx 473\ \Omega$ (or twice that of a cylindrical stub antenna of a length-to-*radius* ratio of 37). The width of the complementary slot should be twice the dipole diameter, so it should have a length-to-width ratio of ≈ 181. At 4th resonance the dipole is $\approx 2\lambda$ long and the slot should be the same length. The pattern will be midway between those in Fig. 14-9 (right-hand column, bottom two patterns) but with *E* and *H* interchanged.

Nothing is mentioned in the problem statement about pattern so the question is left open as to whether this pattern would be satisfactory.

The above dimensions do not constitute a unique answer, as other shapes meeting the impedance requirement are possible.

9-7-1. 50 and 100 Ω patches.

What value of the patch length W results in (a) a 50 Ω and (b) a 100 Ω input resistance for a rectangular patch as in Fig. 9-22a?

Solution:

From (9-7-7.1), $$R_r = 90\left(\frac{\varepsilon_r^2}{\varepsilon_r - 1}\right)\left(\frac{L}{W}\right)^2$$

Solving for W

$$W = L\sqrt{\frac{90\varepsilon_r^2}{R_r(\varepsilon_r - 1)}}$$

Since $$L = \frac{0.49\lambda_o}{\sqrt{\varepsilon_r}}$$

$$W = 0.49 \times 9.49\sqrt{\frac{\varepsilon_r}{R_r(\varepsilon_r - 1)}}\lambda_o$$

With $\varepsilon_r = 2.27$

$$W = 4.65\sqrt{\frac{2.27}{1.27}}\frac{1}{\sqrt{R_r}}\lambda_o = 6.22\frac{1}{\sqrt{R_r}}\lambda_o$$

(a) For $R_r = 50\Omega$, $W = 0.88\lambda_o$

(b) For $R_r = 100\Omega$, $W = 0.62\lambda_o$

9-7-3. Microstrip line.

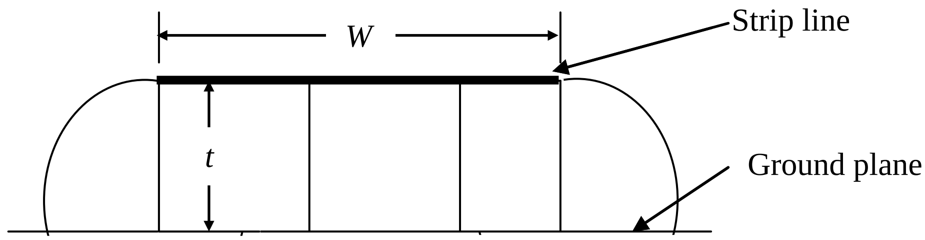
For a polystyrene substrate (ε_r = 2.7) what width-substrate thickness ratio results in a 50-Ω microstrip transmission line?

Solution:

From (9-7-4) (see Fig. 9-21),

$$Z_c = \frac{Z_o}{\sqrt{\varepsilon_r}[(W/t) + 2]} \quad \text{or} \quad \frac{W}{t} = \frac{Z_o}{Z_c\sqrt{\varepsilon_r}} - 2 = \frac{377}{50\sqrt{2.7}} - 2 = 2.6 \quad (\textit{ans.})$$

9-7-3. continued

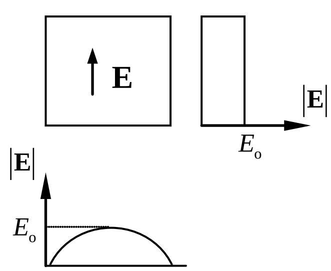


2.6 field cells under strip plus 2 fringing cells = 4.6 cells giving $Z_c == \dfrac{377}{\sqrt{2.7}\times 4.6} = 50\ \Omega$

***9-9-1. Optimum horn gain.**

What is the approximate maximum power gain of an optimum horn antenna with a square aperture 9λ on a side?

Solution:



Assuming a uniform E in the E-direction and cosine distribution in the H-direction, as in the sketches, and with phase everywhere the same, the aperture efficiency from (19-1-50) is

$$\varepsilon_{ap} = \frac{E_{av}^2}{(E^2)_{av}} = (2/\pi)^2 E_o^2 /(E_o^2/2) = 8/\pi^2 = 0.81$$

A more detailed evaluation of ε_{ap} for a similar distribution is given in the solution to Prob. 19-1-7.

Assuming no losses,

$$\text{Power gain} = D = \frac{4\pi A_e}{\lambda^2}$$

where $A_e = \varepsilon_{ap} A_{em} = \varepsilon_{ap} A_p = 0.81\times 10^2 \lambda^2 = 81\lambda^2$

and $D = 4\pi 81 = 1018$ or 30 dBi

The same gain is obtained by extrapolating the $a_{E\lambda}$ line in Fig. 9-29a to 10λ. However, this makes $a_{H\lambda} > a_{E\lambda}$ and not equal as in this problem.

***9-9-1. continued**

In an optimum horn, the length (which is not specified in this problem) is reduced by relaxing the allowable phase variation at the edge of the mouth by arbitrary amounts ($90^\circ = 2\pi \times 0.25$ rad in the E-plane and $144^\circ = 2\pi \times 0.4$ rad in the H-plane). This results in less gain than calculated above, where uniform phase is assumed over the aperture.

From (9-9-2), which assumes 60% aperture efficiency, the directivity of the 10λ square horn is

$$D = 7.5 \times A_p / \lambda^2 = 7.5 \times 10^2 = 750 \text{ or } 29 \text{ dBi}$$

To summarize: when uniform phase is assumed $(\varepsilon_{ap} = 0.81)$ as in the initial solution above, $D = 1018$ or 30 dBi but for an optimum (shorter) horn $(\varepsilon_{ap} = 0.6)$, $D = 750$ or 29 dBi.

9-9-2. Horn pattern.

(a) Calculate and plot the E-plane pattern of the horn of Prob. 9-9-1, assuming uniform illumination over the aperture.
(b) What is the half-power beamwidth and the angle between first nulls?

Solution:

(a) From (5-12-18) the pattern of a uniform aperture of length a is

$$E_n = \frac{\sin \frac{\psi'}{2}}{\frac{\psi'}{2}} = \frac{\sin(\pi a_\lambda \sin\theta)}{\pi a_\lambda \sin\theta} \qquad (1)$$

where a = aperture length = 10λ
θ = angle from broadside

(b) From Table 5-8, $\text{HPBW} = 50.8/10 = 5.08^\circ$ (*ans.*)

Introducing $5.08/2 = 2.54^\circ$ into (1) yields $E_n = 0.707$ which confirms that 5.08° is the true HPBW since $P_n = E_n^2 = 0.707^2 = 0.5$

Using (5-7-7) and setting $nd_\lambda = a_\lambda$ for a continuous aperture,

$$\text{BWFN} = 2\sin^{-1}(1/a_\lambda) = 2\sin^{-1}(1/10) = 11.48^\circ \quad (ans.)$$

9-9-2. continued

Setting $nd_\lambda = a_\lambda$ assumes n very large and d_λ very small, but we have not assumed that their product nd_λ is necessarily very large. If we had, we could write

$$\text{BWFN} = 2/a_\lambda \text{ rad}$$

and obtain

$$\text{BWFN} = 2/10 \text{ rad} = 11.46^{\circ}$$

for a difference of 0.02°.

9-9-3. Rectangular horn antenna.

What is the required aperture area for an optimum rectangular horn antenna operating at 2 GHz with 12 dBi gain?

Solution:

From (9-9-2) or Fig. 3-5b, $D \cong \dfrac{7.5A_p}{\lambda^2}$, $\quad A_p = \dfrac{D}{7.5}\lambda^2$

$$D = 10^{1.2} = 15.85, \qquad \lambda = 0.15 \text{ m}$$

$$A_p = \frac{15.85}{7.5} \times (0.15)^2 = 0.0475 \text{ m}^2$$

9-9-4. Conical horn antenna.

What is the required diameter of a conical horn antenna operating at 2 GHz with a 12 dBi gain?

Solution:

From Fig. 3-5b, $D \cong \dfrac{6.5\pi r^2}{\lambda^2}$

The diameter $d = 2r$ is $\quad d = 2\sqrt{\dfrac{D}{6.5\pi}}\lambda$, $\quad D = 10^{1.2} = 15.85$, $\quad \lambda = 0.15$ m

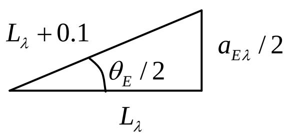
$$d = 2\sqrt{\frac{15.85}{6.5\pi}} \times 0.15 = 26.4 \text{ cm}$$

9-9-5. Pyramidal horn.

(a) Determine the length L, aperture a_H and half-angles in E and H planes for a pyramidal electromagnetic horn for which the aperture $a_E = 8\lambda$. The horn is fed with a rectangular waveguide with TE_{10} mode. Take $\delta = \lambda/10$ in the E plane and $\delta = \lambda/4$ in the H plane.
(b) What are the HPBWs in both E and H planes?
(c) What is the directivity?
(d) What is the aperture efficiency?

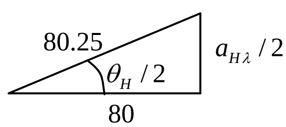
Solution:

(a) For a 0.1λ tolerance in the E-plane, the relation with dimensions in wavelengths is shown in the sketch.



From which $L_\lambda^2 + a_{E\lambda}^2/4 = L_\lambda^2 + 0.2L_\lambda + 0.01$

with $a_{E\lambda} = 8$ (given), $\quad L_\lambda = a_{E\lambda}^2/.8 = 80 \quad$ (*ans.*)



In the H-plane we have from the sketch that

$$a_{H\lambda}/2 = 6.33 \text{ and } a_{H\lambda} = 12.7$$
$$\theta_E/2 = \tan^{-1} 4/80 = 2.9^\circ \quad (\textit{ans.})$$
$$\theta_H/2 = \tan^{-1} 6.33/80 = 4.5^\circ \quad (\textit{ans.})$$

(c) If the phase over the aperture is uniform $\varepsilon_{ap} = 0.81$ (see solution to Probs. 19-1-7 and 9-9-1),

$$D = 4\pi \times 8 \times 12.7 \times 0.81 = 1034 \text{ or } 30.1 \text{ dBi}$$

However, the phase has been relaxed to $36^\circ = 2\pi \times 0.1$ rad in the E-plane and to $90^\circ = 2\pi \times 0.25$ rad in the H-plane, resulting in reduced aperture efficiency, so ε_{ap} must be less than 0.8. If the E-plane phase is relaxed to 90° and the H-plane phase to 144°, $\varepsilon_{ap} \approx 0.6$, which is appropriate for an optimum horn. Thus, for the conditions of this problem which are between an optimum horn and uniform phase, $0.6 < \varepsilon_{ap} < 0.8$. Taking $\varepsilon_{ap} \cong 0.7$,

$$D = 4\pi \times 8 \times 12.7 \times 0.7 = 894 \text{ or } 29.5 \text{ dBi} \quad (\textit{ans.})$$

(b) Assuming uniform phase in the E-plane,

$$(\text{HPBW})_E \cong \frac{50.8^\circ}{a_{E\lambda}} = \frac{50.8^\circ}{8} = 6.35^\circ \cong 6.4^\circ \quad (\textit{ans.})$$

and from the approximation

9-9-5. continued

$$D = \frac{41000}{(\text{HPBW})_E (\text{HPBW})_H} = \frac{41000}{6.4(\text{HPBW})_H} = 894$$

so $(\text{HPBW})_H \cong 7.2^\text{o}$

From Table 9-1 for an optimum horn,

$$\left.\begin{aligned} (\text{HPBW})_E &\cong \frac{56^\text{o}}{8} = 7^\text{o} \\ (\text{HPBW})_H &= \frac{67^\text{o}}{12.7} = 5.3^\text{o} \end{aligned}\right\} \quad (ans.)$$

The true $(\text{HPBW})_E$ for this problem is probably close to 6.4^o. While the true $(\text{HPBW})_H$ is probably close to 5.3^o.

(d) $\varepsilon_{ap} = 0.7$ from part (c). (*ans.*)

Chapter 10. Flat Sheet, Corner and Parabolic Reflector Antennas

10-2-1. Flat sheet reflector.

Calculate and plot the radiation pattern of a $\lambda/2$ dipole antenna spaced 0.15λ from an infinite flat sheet for assumed antenna loss resistance R_L = 0 and 5 Ω. Express the patterns in gain over a $\lambda/2$ dipole antenna in free space with the same power input (and zero loss resistance).

Solution:

From (10-2-1) the gain over a $\lambda/2$ reference dipole is given by

$$G_f(\phi) = 2\left(\frac{R_{11}}{R_{11}+R_L-R_{12}}\right)^{1/2}\left|\sin(S_r\cos\phi)\right| \qquad (1)$$

where,

$S =$ spacing of dipole from reflector

$\phi =$ angle from perpendicular to reflector

(See Fig. 10-2.)

Note that (1) differs from (10-2-1) in that $R_L = 0$ in the numerator under the square root sign since the problem requests the gain to be expressed with respect to a *lossless* reference antenna.

Maximum radiation is at $\phi = 0,$ so (1) becomes,

$$G_f(\phi) = 2\left(\frac{73.1}{73.1+R_L-29.4}\right)^{1/2}\sin(2\pi\times 0.15)$$

and for $R_L = 0$

$$G_f(\phi) = 2.09 \quad \text{or } 6.41\text{ dB } (= 8.56\text{ dBi}) \quad (\textit{ans.})$$

Note that R_{12} is for a spacing of $0.3\lambda(= 2\times 0.15\lambda)$. See Table 13-1.

Note that $R_L = 10\ \Omega$, $G_f(\phi) = 1.89$ or 5.52 dB (= 7.67 dBi) (*ans.*)

Note that $G_f(\phi)$ is the gain with respect to a reference $\lambda/2$ dipole and more explicitly can be written $G_f(\phi)[A/HW]$.

The loss resistance $R_L = 10\ \Omega$ results in about 0.9 dB reduction in gain with respect to a lossless reference dipole. If the reference dipole also has 10 Ω loss resistance, the gain reduction is about 0.3 dB.

10-2-1. continued

The above gains agree with those shown for $R_L = 0$ and extrapolated for $R_L = 10\,\Omega$ at $S = 0.15\lambda$ in Fig. 10-4. Note that in Fig. 10-4 an equal loss resistance is assumed in the reference antenna.

The pattern for $R_L = 0$ should be intermediate to those in Fig. 10-3 for spacings of 0.125λ (=λ/8) and 0.25λ (=λ/4). The pattern for $R_L = 10\,\Omega$ is smaller than the one for $R_L = 0$ but of the same shape (radius vector differing by a constant factor).

10-3-1. Square-corner reflector.

A square-corner reflector has a driven λ/2 dipole antenna space λ/2 from the corner. Assume perfectly conducting sheet reflectors of infinite extent (ideal reflector). Calculate and plot the radiation pattern in a plane at right angles to the driven element.

Solution:

From (10-3-6) the gain of a lossless corner reflector over a reference $\lambda/2$ dipole is given by

$$G_f(\phi) = 2\left(\frac{R_{11}}{R_{11} + R_{14} - 2R_{12}}\right)^{1/2} \left|[\cos(S_r \cos\phi) - \cos(S_r \sin\phi)]\right|$$

For $S = \lambda/2$ and maximum radiation direction $(\phi = 0^\circ)$ this becomes

$$G_f(\phi) = 4\left(\frac{73.1}{73.1 + 3.8 + 2\times 24}\right)^{1/2} = 3.06 \text{ or } 9.7 \text{ dB } (=11.9 \text{ dBi})$$

See Table 13-1 and Fig. 13-13 for the mutual resistance values for R_{14} at 1λ separation and R_{12} at 0.707λ separation. The above calculated gain agrees with the value shown by the curve in Fig. 10-11. The pattern should be identical to the one in Fig. 10-12a.

10-3-2. Square-corner reflector.

(a) Show that the relative field pattern in the plane of the driven λ/2 element of a square-corner reflector is given by

$$E = [1 - \cos(S_r \sin\theta)]\frac{\cos(90^\circ \cos\theta)}{\sin\theta}$$

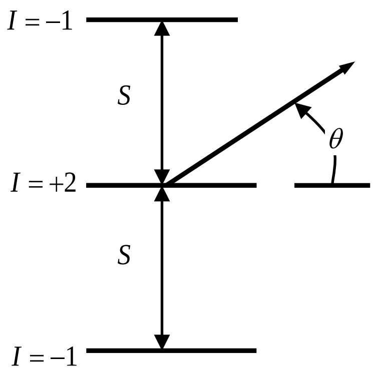
where θ is the angle with respect to the element axis. Assume that the corner-reflector sheets are perfectly conducting and of infinite extent.

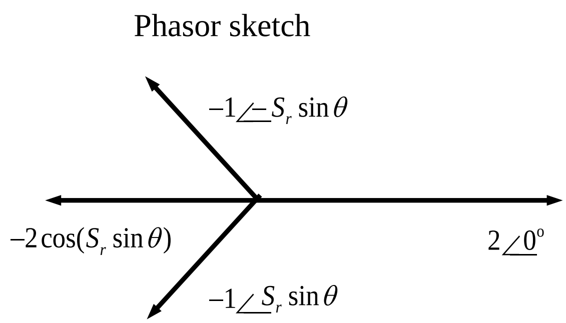
10-3-2. continued

(b) Calculate and plot the field pattern in the plane of the driven element for a spacing of $\lambda/2$ to the corner. Compare with the pattern at right angles (Prob. 10-3-1).

Solution:

(a) The pattern in the plane of the dipole (E plane) is that of an array of three $\lambda/2$ elements arranged as in the sketch with amplitudes 1:2:1 and phasing as indicated.





By pattern multiplication the pattern is the product of the pattern of an array of 3 isotropic sources with amplitudes and phasing $-1:+2:-1$ and the pattern of $\lambda/2$ dipole (6-4-4). Thus,

$$E = (2 - 1\angle S_r \sin\theta \quad -1\angle -S_r \sin\theta)\,\frac{\cos(90^\circ \cos\theta)}{\sin\theta}$$

or, see phasor sketch,

$$E = 2[1 - \cos(S_r \sin\theta)]\frac{\cos(90^\circ \cos\theta)}{\sin\theta}$$

Dropping the scale factor 2 yields the results sought, q.e.d.

***10-3-4. Square-corner reflector.**

(a) Calculate and plot the pattern of a 90° corner reflector with a thin center-fed $\lambda/2$ driven antenna spaced 0.35λ from the corner. Assume that the corner reflector is of infinite extent.
(b) Calculate the radiation resistance of the driven antenna.
(c) Calculate the gain of the antenna and corner reflector over the antenna alone. Assume that losses are negligible.

***10-3-4. continued**

Solution:

(a) From (10-3-6) the normalized field pattern for $S = 0.35\lambda$ is

$$E_n(\phi) = \frac{\left|[\cos(126^\circ \cos\phi) - \cos(126^\circ \sin\phi)]\right|}{1.588}$$

(b) $R_r = R_{11} + R_{14} - 2R_{12} = 73.1 - 24.8 + 25 = 73.3\ \Omega$ (*ans.*)

(c) From (10-3-6) for $\phi = 0$ and $S = 0.35\lambda$

$$G_f(\phi) = 2(73.1/73.3)^{1/2} \times 1.588 = 3.17 \text{ or } 10.0 \text{ dB } (=12.1 \text{ dBi}) \quad (\textit{ans.})$$

10-3-5. Square-corner reflector versus array of its image elements.

Assume that the corner reflector of Prob. 10-3-4 is removed and that in its place the three images used in the analysis are present physically, resulting in 4-element driven array.
(a) Calculate and plot the pattern of this array.
(b) Calculate the radiation resistance at the center of one of the antennas.
(c) Calculate the gain of the array over one of the antennas alone.

Solution:

(a) 4-lobed pattern as in Fig. 10-9 with shape of pattern of Prob. 10-3-4a.

(b) $R_r = 73.3\ \Omega$ (*ans.*)

(c) $G_f(\phi) = 1.59$ or 4.0 dB (= 6.1 dBi) (*ans.*)
since power is fed to all 4 elements instead of to only one (Power gain down by a factor of 4 or by 6 dB).

***10-3-6. Square-corner reflector array.**

Four 90° corner-reflector antennas are arranged in line as a broadside array. The corner edges are parallel and side-by-side as in Fig. P10-3-6. The spacing between corners is 1λ. The driven antenna in each corner is a $\lambda/2$ element spaced 0.4λ from the corner. All antennas are energized in phase and have equal current amplitude. Assuming that the properties of each corner are the same as if its sides were of infinite extent, what is (a) the gain of the array over a single $\lambda/2$ antenna and (b) the half-power beam width in the *H* plane?

***10-3-6. continued**

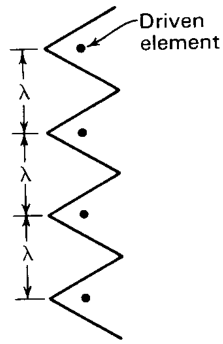


Figure P10-3-6. Square-corner reflector array.

Solution:

(a) From (10-3-6) the gain of one corner reflector with $S = 0.4\lambda$ is given by

$$G_f(\phi) = 2\left(\frac{73.1}{73.1 - 18.6 + 42}\right)^{1/2} \left|(\cos 144^\circ - \cos 0^\circ)\right|$$
$$= 2 \times 0.870 \times 1.81 = 3.15 \text{ or} \cong 10 \text{ dB (= 12.1 dBi)}$$

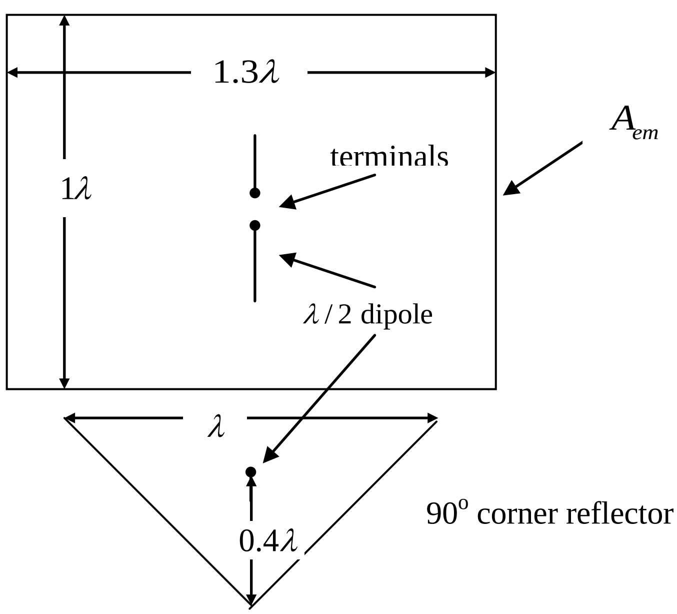
Under lossless conditions,

$$D = \left(G_f(\phi)\right)^2 \times 1.64 = 16.3$$

Thus, the maximum effective aperture of one corner is

$$A_{em} = \frac{D\lambda^2}{4\pi} \cong \frac{16.3\lambda^2}{4\pi} = 1.3\lambda^2$$

The effective aperture of a single corner may then be represented by a rectangle $1\lambda \times 1.3\lambda$ as in the sketch below.



***10-3-6. continued**

In an array of 4 reflectors as in Fig. P10-3-6 the edges of the apertures overlap 0.3λ so that the reflectors are too close. However, at the 1λ spacing the total aperture is $4\lambda \times 1\lambda = 4\lambda^2$ and the total gain of the array under lossless conditions is

$$G = D = \frac{4\pi A_{em}}{\lambda^2} = 4\pi \times 4 \cong 50 \text{ or } 17 \text{ dBi} \quad (ans.)$$

No interaction between corner reflectors has been assumed. With wider spacing $(=1.3\lambda)$ the expected gain $= 16.3 \times 4 = 65$ or 18 dBi.

(b) Assuming a uniform aperture distribution, the HPBW is given approximately from Table 5-8 by

$$\text{HPBW} = 50.8^{\circ} / L_{\lambda} = 50.8^{\circ} / 4 = 12.7^{\circ}$$

To determine the HPBW more accurately, let us use the total antenna pattern. By pattern multiplication it is equal to the product of an array of 4 in-phase isotropic point sources with 1λ spacing and the pattern of a single corner reflector as given by

$$E_n(\phi) = \frac{1}{4}\frac{\sin(4\pi\sin\phi)}{\sin(\pi\sin\phi)}\frac{1}{1.809}\left|[\cos(0.8\pi\cos\phi) - \cos(0.8\pi\sin\phi)]\right|$$

The ¼ is the normalizing factor for the array and 1/1.809 for the corner reflector. Thus, when $\phi = 0^{\circ}$, $E_n(\phi) = 1$. Note that ϕ must approach zero in the limit in the array factor to avoid an indeterminate result.

Half of the above approximate HPBW is $12.7^{\circ}/2 = 6.35^{\circ}$. Introducing it into the above equation yields $E_n(\phi) = 0.703$. For $\phi = 6.30^{\circ}$, $E_n(\phi) = 0.707$ as tabulated below.

ϕ	$E_n(\phi)$
6.35°	0.703
6.30°	0.707

Thus, $\quad \text{HPBW} = 2 \times 6.30 = 12.6^{\circ} \quad (ans.)$

The 4-source array factor is much sharper than the corner reflector pattern and largely determines the HPBW.

Returning to part (a) for the directivity, let us calculate its value with the approximate relation of (2-7-9) using the HPBW of part (b) for the *H*-plane and the HPBW of 78° for the *E*-plane from Example 6-4.1.

***10-3-6. continued**

Thus,

$$D \cong \frac{40000}{12.6 \times 7.8} = 40.7 \ \ (= 16 \text{ dBi})$$

as compared to $D \cong 50$ (= 17 dBi) as calculated in part (a).

Although the directivity of 16.3 for a single corner reflector should be accurate, since it is determined from the pattern via the impedances*, the directivity of 50 for the array of 4 corner reflectors involves some uncertainty (apertures overlapping). Nevertheless, the two methods agree within 1 dB.

---

*Assuming infinite sides

10-3-7. Corner reflector. λ/4 to the driven element.

A square-corner reflector has a spacing of λ/4 between the driven λ/2 element and the corner. Show that the directivity D = 12.8 dBi.

Solution:

For the case of no losses,

$$D = \left(G_f(\phi)\right)^2 \times 1.64, \ \text{ and for } \ S = \lambda/4 \ \text{ and } \ \phi = 0,$$

(10-3-6) becomes

$$G_f(\phi) = 2\left(\frac{73.1}{73.1 - 12.7 - 35}\right)^{1/2} = 3.39$$

Therefore, $$D = \left(G_f(\phi)\right)^2 \times 1.64 = 18.9 \ \text{ or } \ 12.8 \text{ dBi} \quad (\textit{ans.})$$

10-3-8. Corner reflector. λ/2 to the driven element.

A square-corner reflector has a driven λ/2 element λ/2 from the corner.
(a) Calculate and plot the far-field pattern in both principal planes.
(b) What are the HPBWs in the two principal planes?
(c) What is the terminal impedance of the driven element?
(d) Calculate the directivity in two ways: (1) from impedances of driven and image dipoles and (2) from HPBWs, and compare. Assume perfectly conducting sheet reflectors of infinite extent.

10-3-8. continued

Solution:

(a) From Prob. 10-3-2 the pattern in the *E*-plane is given by

$$E_n(\theta) = \frac{1}{2}[1 - \cos(\pi \sin\theta)]\frac{\cos(90^{\circ}\cos\theta)}{\sin\theta} \qquad (1)$$

From (10-3-6) the pattern in the *H*-plane is given by

$$E_n(\theta) = \frac{1}{2}\left|[\cos(\pi\cos\phi) - \cos(\pi\sin\phi)]\right| \qquad (2)$$

Note that $E_n(\theta)$ = maximum for $\theta = 90^{\circ}$ while $E_n(\phi)$ = maximum for $\phi = 0^{\circ}$.

(b) Assuming initially that $\text{HPBW}(\theta) \cong \text{HPBW}(\phi)$ and noting from Fig. 10-11 that for $S = \lambda/2$ the directivity is about 12 dBi, we have from (2-7-9) that

$$D \cong \frac{40000}{\text{HPBW}(\theta)^2} \cong 16 \text{ or } \text{HPBW}(\theta) \cong 50^{\circ} \quad \text{and} \quad \frac{\text{HPBW}(\theta)}{2} = \frac{50^{\circ}}{2} = 25^{\circ}$$

Introducing $\theta = 90^{\circ} - 25^{\circ} = 65^{\circ}$ in (1) yields $E_n(\theta)$ which is too high. By trial and error, we obtain $E_n(\theta) \cong 0.707$ when θ=34.5°.

Therefore, $\text{HPBW}(\theta) \cong 2 \times 34.5^{\circ} = 69^{\circ}$ (*ans.*)

Introducing $\phi = 25^{\circ}$ in (2) yields $E_n(\phi) = 0.60$ which is too low. By trial and error, we obtain $E_n(\phi) \cong 0.707$ when $\phi = 21^{\circ}$.

Therefore, $\text{HPBW}(\phi) \cong 2 \times 21^{\circ} = 42^{\circ}$ (*ans.*)

(c) The terminal impedance of the driven element is (see Prob. 10-3-1 solution),

$$R_T = 73.1 + 3.8 + 2 \times 24 = 125\,\Omega \quad (\textit{ans.})$$

From Prob. 10-3-1 solution,

$$D = \left(G_f(\phi)\right)^2 \times 1.64 = 15.4 \text{ (= 11.9 dBi) by impedance} \quad (\textit{ans.})$$

$$D \cong \frac{40000}{69^{\circ} \times 42^{\circ}} = 13.8 \text{ (= 11.4 dBi) by beam widths} \quad (\textit{ans.})$$

10-3-8. continued

The $D = 15.4$ value is, of course, more accurate since it is based on the pattern via the impedances. The two methods differ, however, by only 0.4 dB.

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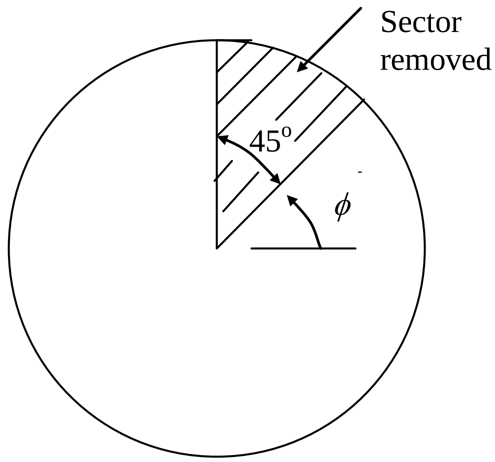
*10-7-2. Parabolic reflector with missing sector.

A circular parabolic dish antenna has an effective aperture of 100 m^2. If one 30° sector of the parabola is removed, find the new effective aperture. The rest of the antenna, including the feed, is unchanged.

Solution:



The full dish has an effective aperture $A_e = 100$ m^2. Assuming that the dish characteristics are independent of angle (ϕ), removing one 45° sector reduces the effective aperture to 7/8 of its original value provided the feed is modified and so as not to illuminate the area of the missing sector. However, the feed is not modified and, therefore, its efficiency is down to 7/8. Therefore, the net aperture efficiency is (7/8)^2 and the net effective aperture is

$$(7/8)^2 \times 100 = 76.6 \text{ m}^2 \quad (ans.)$$

Chapter 11. Broadband and Frequency-Independent Antennas

*11-2-2. The 2° cone.

Calculate the terminal impedance of a conical antenna of 2° total angle operating against a very large ground plane. The length *l* of the cone is $3\lambda/8$.

Solution:

From (11-2-2) for Z_k, and noting that when θ is small $(\theta < 20^{\circ})$

Then $$\cot\left(\frac{\theta}{4}\right) \cong \frac{4}{\theta}$$

or $$Z_k = 120 \ln \frac{4}{\theta}$$

and with (11-2-4) and (11-2-5) for Z_m and (11-2-3) for Z_i,

so $$Z_i = 270 + j350\Omega \quad (ans.)$$

11-5-1. Log spiral.

Design a planar log-spiral antenna of the type shown in Fig. 11-11 to operate at frequencies from 1 to 10 GHz. Make a drawing with dimensions in millimeters.

Solution:

High frequency limit = 10 GHz, $\lambda = 3\times10^{8} / 10\times10^{9} = 30$ mm

Low frequency limit = 1 GHz, $\lambda = 300$ mm

Take $\beta = 77.6^{\circ}$ (see Fig. 11-10)

From (11-5-5), $r =$ antiln $(\theta/\tan\beta)$ = antiln $(\theta/4.55)$

11-5-1. continued

θ	r	R
0 rad	1	1.5 mm
$\pi/2$	1.41	2.12
π	2.00	3.00
$3\pi/2$	2.82	4.23
2π	4.00	6.00
$5\pi/2$	5.66	8.50
3π	8.00	12.0
$7\pi/2$	11.3	17.0
4π	16.0	24.0
$9\pi/2$	22.6	34.0
5π	32.0	48.0
$11\pi/2$	45.2	68.0
6π	64.0	96.0

Spiral is like one in Fig. 11-10. If gap *d* at center is equal to $\lambda/10$ at high frequency limit, then gap should be 30/10 = 3 mm and radius *R* of actual spiral = 3/2 = 1.5 mm.

If *diameter* of spiral is $\lambda/2$ at low frequency limit, then the actual spiral radius should be $300/(2\times 2) =$ 75 mm.

This requires that $\theta = 4.55\ln(75/1.5) = 17.8 = 5.7\pi$

For good measure we make $\theta = 6\pi$. Thus, the spiral has 3 turns ($\theta = 6\pi$).

The table gives data for the actual spiral radius R in mm versus the angle θ in rad. The overall diameter of the spiral is $96\times 2 = 192$ mm which at 1 GHz is 192/300 = 0.64λ.

Calling the above spiral number 1, draw an identical spiral rotated through $\pi/2$ rad, a third rotated through π rad and a fourth rotated through $3\pi/2$ rad. Metalize the areas between spirals 1 and 4 and between spirals 2 and 3, leaving the remaining areas open. Connect the feed across the gap at the innermost ends of the spirals as in Fig. 11-11.

11-7-1. Log periodic.

Design an "optimum" log-periodic antenna of the type shown in Fig. 11-17 to operate at frequencies from 100 to 500 MHz with 11 elements. Give (a) length of longest element, (b) length of shortest element, and (c) gain.

Solution:

From Fig. 11-19 let us select the point where $\alpha = 15^{\circ}$ intersects the optimum design line which should result in an antenna with slightly more than 7 dBi gain. From the figure, k=1.195. The desired frequency ratio is 5 = F = 250/50. Thus, from (7) the required number of elements musts be at least equal to

$$n = \frac{\log F}{\log k} = \frac{\log 5}{\log 1.195} = 9.0$$

at 250 MHz, $\lambda = 1.2$ m, $\lambda / 2 = 0.6$ m (*ans.*)

at 50 MHz, $\lambda = 6$ m, $\lambda / 2 = 3$ m (*ans.*)

If element 1 is 0.6 in long, then element 10 (= n+1) is 0.6×1.195^{9} = 2.98 m or approximately 3 m as required.

Adding a director in front of element 1 and a reflector in back of element 10 brings the total number of elements to 12. (*ans.*)

The length ℓ_2 of any element with respect to the length ℓ, of the next shorter element is given by

$$\ell_2 / \ell_1 = k = 1.195 \quad (ans.)$$

From (11-7-5) (note geometry of Fig. 11-18), the spacing s between any two elements is related to the length ℓ of the adjacent shorter element by

$$s = \frac{\ell(k-1)}{2\tan\alpha} = \frac{0.195\ell}{2\tan 15^{\circ}} = 0.364\,\ell \quad (ans.)$$

[Note that (11-7-6) gives s with respect to adjacent *longer* elements.]

Finally, connect the elements as in Fig. 11-17.

11-7-2. Stacked LPs.

Two LP arrays like in the worked example of Sec. 11-7 are stacked as in Fig. 11-21a.
(a) Calculate and plot the vertical plane field pattern. Note that pattern multiplication cannot be applied.
(b) What is the gain?

Solution:

From the worked example of Sec. 11-7,

$$\alpha = 15^{\circ},\ k = 1.2,\ F = 4,\ n = 7.6\ \ (8) \text{ and } n+1 = 9$$

Consider that $\lambda_{\min} = 1$ m and $\lambda_{\max} = 4$ m.

Therefore, $\ell_1 = 0.5$ m and $\ell_{n+1} = \ell_1 k^n = 0.5 \times 1.2^8 = \ell_9 = 2$ m

From (11-7-6) the distance between elements 1 and 9 is

$$S = \sum_{n=0}^{n-2} \ell_1 \frac{(k-1)k^n}{2\tan\alpha} \quad \text{where } \ell_1 = 0.5 \text{ m}$$

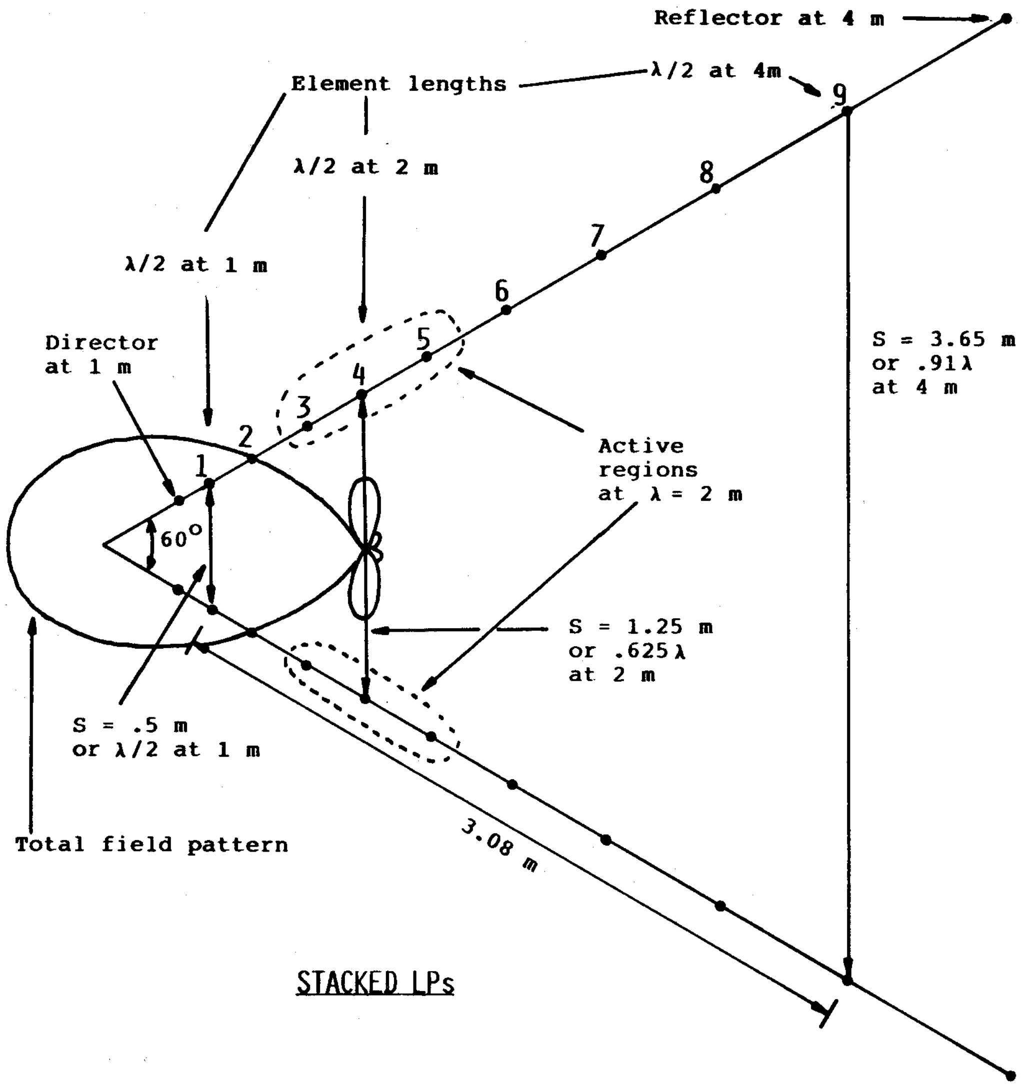
and

$$S = \sum_{n=0}^{7} \frac{0.2 \times 1.2^n}{4\tan 15^{\circ}} = 0.187+0.224+0.268+0.322+0.387+0.464+0.557+0.669 = 3.08 \text{ m}$$

The stacked LPs are shown in side view in the sketch below. Elements 1 through 9 are included in the calculation. Element 1 is $\lambda/2$ resonant at 1 m wavelength and element 9 is $\lambda/2$ resonant at 4 m wavelength. A director element is added ahead of element 1 and a reflector element is added after element 9 making a total of 11 elements.

In the 60° angle stacking arrangement the stacking distance is 0.5 m or $\lambda/2$ at 1 m wavelength and 3.65 m or 0.91λ at 4 m wavelength. At the geometric mean wavelength (2 m) the stacking distance is 1.25 m or 0.625λ.

11-7-2. continued



Let us calculate the vertical plane pattern at 2 m wavelength where elements 3, 4 and 5 of the upper and lower LPs are active. As an approximation, let us consider that the 3 active elements are a uniform ordinary end-fire array with spacing equal to the average of the spacing between elements 3 and 4 and between 4 and 5.

For element 4 we take $\ell_4 = \lambda/2$. Then from (11-7-6)

$$S_{45} = \frac{k-1}{4\tan\alpha} = \frac{1.2-1}{4\tan 15^{\circ}} = 0.187\lambda$$

11-7-2. continued

and

$$S_{34} = 0.187/1.2 = 0.155\lambda$$

$$S_{av} = \frac{0.155 + .187}{2} = 0.171\lambda$$

The end-fire array field pattern is given by $E = \frac{1}{3}\frac{\sin n\psi/2}{\sin\psi/2}$

where

$$\psi/2 = \frac{2\pi \times 0.171}{2}(\cos\phi - 1) = 30.8^{o}(\cos\phi - 1), \qquad n = 3$$

Each LP (end-fire array) has a broad cardiod-shaped pattern like the ones shown in Fig. 11-21a with one pattern directed up 30^{o} and the other down 30^{o}.

The total field pattern in the resultant of these patterns and a broadside array of 2 in-phase isotropic sources stacked vertically and spaced 0.625λ with pattern given by

$$E = \cos[(2\pi \times 0.625/2)\sin\phi] = \cos(112.5^{o}\sin\phi)$$

This pattern is shown in Fig. 16-11. Numerical addition of the LP patterns and multiplication of the resultant by the broadside pattern yields the total field pattern for $\lambda = 2$ m shown in the sketch. At $\lambda = 1$ m the up-and-down minor lobes disappear but the main beam is about the same. At $\lambda = 4$ m the main beam is narrower but the up-and-down minor lobes are larger.

(b) Each LP has a gain $\cong 7$ dBi. From the equation of Prob. 5-2-3, the directivity of 2 in-phase isotropic sources with 0.625λ spacing is 2.44 or 3.9 dBi. So for $\lambda = 2$ m, the gain may be as much as $7 + 3.9 \cong 10.9$ dBi (*ans.*)

At both $\lambda = 1$ m and $\lambda = 4$ m the gain may be less than this.

The HPBW in the vertical plane is about 47^{o} and in the horizontal plane about 76^{o}.
From the approximate directivity relation, we have, neglecting minor lobes,

$$D = \frac{41000}{47 \times 76} = 11.5 \text{ or } 10.6 \text{ dBi} \quad (\textit{ans.})$$

Actual directivity is probably □ 10 dBi.

Chapter 12. Antenna Temperature, Remote Sensing and Radar Cross Section

*12-2-1. Antenna temperature.

An end-fire array is directed at the zenith. The array is located over flat nonreflecting ground. If 0.9 Ω_A is within 45° of the zenith and 0.08 Ω_A between 45° and the horizon calculate the antenna temperature. The sky brightness temperature is 5 K between the zenith and 45° from the zenith, 50 K between 45° from the zenith and the horizon and 300 K for the ground (below the horizon). The antenna is 99 percent efficient and is at a physical temperature of 300K.

Solution:

$0.9\Omega_A$ is at 5 K

$0.08\Omega_A$ is at 50 K

Therefore $0.02\Omega_A$ is at 300 K

From (12-1-8),

$$T_A = \frac{1}{\Omega_A}[5\times 0.9\Omega_A + 50\times 0.08\Omega_A + 300\times 0.02\Omega_A] = 4.5+4+6 = 14.5 \text{ K} \quad (\textit{ans.})$$

---

*12-2-2. Earth-station antenna temperature.

An earth-station dish of 100 m^2 effective aperture is directed at the zenith. Calculate the antenna temperature assuming that the sky temperature is uniform and equal to 6 K. Take the ground temperature equal to 300 K and assume that 1/3 of the minor-lobe beam area is in the back direction. The wavelength is 75 mm and the beam efficiency is 0.8.

Solution:

From (12-1-8),

$$T_A = \frac{1}{\Omega_A}[6\times 0.8\Omega_A + 6\times\frac{2}{3}\times 0.2\Omega_A + 300\times\frac{1}{3}\times 0.2\Omega_A] = 4.8+0.8+20 = 25.6 \text{ K} \quad (\textit{ans.})$$

***12-3-4. Satellite TV downlink.**

A transmitter (transponder) on a Clarke orbit satellite produces an effective radiated power (ERP) at an earth station of 35 dB over 1 W isotropic.
(a) Determine the S/N ratio (dB) if the earth station antenna diameter is 3m, the antenna temperature 25 K, the receiver temperature 75 K and the bandwidth 30 MHz. Take the satellite distance as 36,000 km. Assume the antenna s a parabolic reflector (dish-type) of 50 percent efficiency. (See Example 12-3.1).
(b) If a 10-dB S/N ratio is acceptable, what is the required diameter of the earth station antenna?

Solution:

(a) Satellite ERP = 35 dB (over 1 W isotropic)

$$\text{ERP} = P_t D_t = P_t 4\pi A_{et} / \lambda^2, \qquad P_t A_{et} = \lambda^2 \text{ERP} / 4\pi$$

From (12-3-3),

$$\frac{S}{N} = \frac{P_t A_{et} A_{er}}{kT_{sys} r^2 \lambda^2 B} = \frac{\lambda^2 \text{ ERP } A_{er}}{4\pi k T_{sys} r^2 \lambda^2 B}, \qquad \text{ERP = 35 dB or 3162}$$

$$A_{er} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi 1.5^2 = 3.53 \text{ m}^2, \qquad T_{sys} = 25 + 75 = 100 \text{ K}$$

$$\frac{S}{N} = \frac{3162 \times 3.53}{4\pi \times 1.38 \times 10^{-23} \times 100 \times 3.6^2 \times 10^{14} \times 3 \times 10^7} = 16.6 \text{ or } 12.2 \text{ dB} \quad (\textit{ans.})$$

(b) If only 10 dB S/N ratio is acceptable, the dish aperture could be 2.2 dB $(= 12.2 - 10 \text{ dB})$ less or 60% of the $\pi\,1.5^2$ = 7.07 m^2 area specified in part (a).

Therefore acceptable area $= 7.07 \times 0.6 = 4.24 \text{ m}^2 = \pi r^2$

For a diameter $= 2r = 2(4.24/\pi)^{1/2} = 2.3$ m (*ans.*)

***12-3-5. System temperature.**

The digital output of a 1.4 GHz radio telescope gives the following values (arbitrary units) as a function of the sidereal time while scanning a uniform brightness region. The integration time is 14 s, with 1 s idle time for printout. The output units are proportional to power.

***12-3-5. continued**

Time	Output	Time	Output
31^m30^s	234	32^m45^s	229
31 45	235	33 00	236
32 00	224	33 15	233
32 15	226	33 30	230
32 30	239	33 45	226

If the temperature calibration gives 170 units for 2.9 K applied, find (a) the rms noise at the receiver, (b) the minimum detectable temperature, (c) the system temperature and (d) the minimum detectable flux density. The calibration signal is introduced at the receiver. The transmission line from the antenna to the receiver has 0.5 dB attenuation. The antenna effective aperture is 500 m^2. The receiver bandwidth is 7 MHz. The receiver constant $k'=2$.

Solution:

(a) Noise output readings 234, 235, etc. with respect to average value (231), are squared, averaged and then square rooted for root-mean-square (rms) noise value 4.71.

The rms noise at receiver is then

$$\frac{4.71}{170}\times 2.9 = 0.08 \text{ K} \qquad (\textit{ans.})$$

(b) Transmission line attenuation = 0.5 dB for efficiency of 0.89.

Therefore, $$\Delta T_{\min} = 0.08/0.89 = 0.09 \text{ K} \qquad (\textit{ans.})$$

(c) From (12-2-3),

$$T_{sys} = \frac{\Delta T_{\min}(\Delta f t)^{1/2}}{k'} = \frac{0.09(7\times 10^6 \times 14)^{1/2}}{2} = 445 \text{ K} \qquad (\textit{ans.})$$

(d) From (12-1-7),

$$\Delta S_{\min} = \frac{2k\Delta T_{\min}}{A_e} = \frac{2\times 1.38\times 10^{-23}\times 0.09}{500} = 497 \text{ mJy} \cong 500 \text{ mJy (rounded off)} \qquad (\textit{ans.})$$

12-3-6. System temperature.

Find the system temperature of a receiving system with 15 K antenna temperature, 0.95 transmission-line efficiency, 300 K transmission-line temperature, 75 K receiver first-stage temperature, 100 K receiver second-stage temperature and 200 K receiver third-stage temperature. Each receiver stage has 16 dB gain.

Solution:

From (12-2-1) and (12-2-2),

$$\begin{aligned} T_{sys} &= 15 + 300\left(\frac{1}{.95} - 1\right) + \frac{1}{.95}\left(75 + \frac{100}{40} + \frac{200}{40}\right) \\ &= 15 + 15.8 + 78.9 + 2.6 + 5.3 = 117.6 \text{ K} \quad (\textit{ans.}) \end{aligned}$$

---

*12-3-7. Solar interference to earth station.

Twice a year the sun passes through the apparent declination of the geostationary Clarke-orbit satellites, causing solar-noise interference to earth stations. A typical forecast notice appearing on U.S. satellite TV screens reads:

ATTENTION CHANNEL USERS:
WE WILL BE EXPERIENCING
SOLAR OUTAGES FROM
OCTOBER 15 TO 26
FROM 12:00 TO 15:00 HOURS

(a) If the equivalent temperature of the sun at 4 GHz is 50,000 K, find the sun's signal-to-noise ration (in decibels) for an earth station with a 3-m parabolic dish antenna at 4 GHz. Take the suns diameter as 0.5° and the earth-station system temperature as 100 K.
(b) Compare this result with that for the carrier-to-nose ratio calculated in Example 12-3.1 for a typical Clarke-orbit TV transponder.
(c) How long does the interference last? Note that the relation $\Omega_A = \lambda^2 / A_e$ gives the solid beam angle in steradians and not in square degrees.
(d) Why do the outages occur between October 15 and 26 and not at the autumnal equinox around September 20 when the sun is crossing the equator?
(e) How can satellite services work around a solar outage?

***12-3-7. continued**

Solution:

(a) Earth station $$S_{\min} = \frac{2kT_{sys}}{A_e}$$

$$S_{\text{sun}} = \frac{2k\Delta T_A}{A_e} = \frac{2k\Delta T_A \Omega_A}{A_e \Omega_A} = \frac{2kT_s \Omega_s}{\lambda^2}$$

Assuming 50% aperture efficiency as in Prob. 12-3-4,

$$\Omega_A = \frac{\lambda^2}{A_e} = \frac{0.075^2}{\frac{1}{2}\pi 1.5^2} = 1.59\times 10^{-3} \text{ sr} = 5.22 \text{ sq. deg.}$$

Therefore, $$\frac{S}{N} = \frac{S_{\text{sun}}}{S_{\min}} = \frac{T_s \Omega_s}{T_{sys}\Omega_A} = \frac{5\times 10^4 \times \pi (.25^{\circ})^2}{100\times 5.22} = 18.8 \text{ or } 12.7 \text{ dB} \quad (ans.)$$

sq. deg.

(b) From Prob. 12-3-4,

$12.7 - 12.2 = 0.5$ dB more than satellite carrier, resulting in degradation of TV picture quality. The phenomenon may be described as *noise jamming* by the sun.

(c) At half-power, solar noise will be reduced to only $3 - 0.5 = 2.5$ dB below carrier. Assuming low side-lobes, the solar interference should not last more than the time it takes the sun to drift between first nulls. Using this criterion we have

$$\text{Time} = \underbrace{4(\text{min/deg})}_{\text{Solar drift rate}} \times \text{BWFN (deg)}$$

From Table 15-1,

$\text{HPBW} = 66^{\circ}/D_{\lambda} = 66^{\circ}/(3/0.075) = 1.65^{\circ}$ and $\text{Time} \cong 4\times 2\times 1.65 = 13.2$ min (*ans.*)

Allowing for the angular extent of the sun (approx. $1/2^{\circ}$) increases the time by about 2 minutes.

***12-3-9. Critical frequency. MUF.**

Layers may be said to exist in the earth's ionosphere where the ionization gradient is sufficient to refract radio waves back to earth. [Although the wave actually may be bent gradually along a curved path in an ionized region of considerable thickness, a useful simplification for some situations is to assume that the wave is reflected as though from a horizontal perfectly conducting surface situated at a (*virtual*) *height h*.] The highest frequency at which this layer reflects a vertically incident wave back to the earth is called the *critical frequency* f_o. Higher frequencies at the vertical incidence pass through. For waves at oblique incidence (ϕ>0 in Fig. P12-3-9) the *maximum usable frequency* (MUF) for point-to-point communication on the earth is given by MUF = $f_o/\cos\phi$, where ϕ = angle of incidence. The critical frequency $f_o = 9\sqrt{N}$, where N = electron density (number m^{-3}). N is a function of solar irradiation and other factors. Both f_o and h vary with time of day, season, latitude and phase of the 11-year sunspot cycle. Find the MUF for (a) a distance d = 1.3 Mm by F_2-layer (h = 3 25 km) reflection with F_2-layer electron density $N = 6 \times 10^{11}$ m^{-3}; (b) a distance d = 1.5 Mm by F_2-layer (h = 275 km) reflection with $N = 10^{12}$ m^{-3}; and (c) a distance d = 1 Mm by sporadic E-layer (h = 100 km) reflection $N = 8 \times 10^{11}$ m^{-3}. Neglect earth curvature.

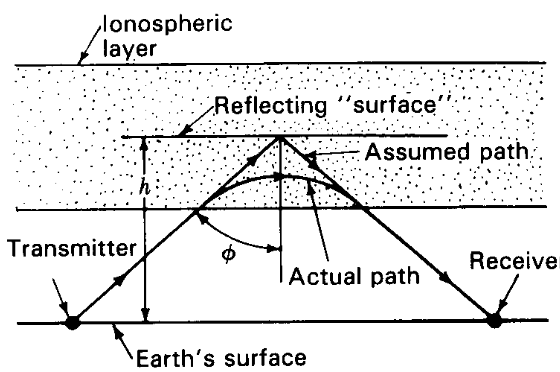


Figure P12-3-9. Communication path via reflection from ionospheric layer.

Solution:

MUF = Maximum Usable Frequency for communication via ionospheric reflection.

(a)

$$f_o = 9\sqrt{N} = 9(6\times10^{11})^{1/2} = 6.97 \text{ MHz}, \quad \phi = \tan^{-1}[(d/2)/h] = \tan^{-1}\left(\frac{6.5\times10^5}{3.25\times10^5}\right) = 63.43^{o}$$

$$\cos\phi = 0.477, \qquad \text{MUF} = f_o/\cos\phi = 6.97/0.447 = 15.6 \text{ MHz} \quad (ans.)$$

***12-3-9. continued**

(b) $f_o = 9(10^{12})^{1/2} = 9$ MHz, $\phi = \tan^{-1}\left(\dfrac{7.5\times 10^5}{2.75\times 10^5}\right) = 69.86^\circ$, $\cos\phi = 0.344$

$$\text{MUF} = 9/0.344 = 26.1 \text{ MHz} \quad (ans.)$$

(c) $f_o = 9(8\times 10^{11})^{1/2} = 8.05$ MHz, $\phi = \tan^{-1}\left(\dfrac{5\times 10^5}{10^5}\right) = 78.69^\circ$, $\cos\phi = 0.196$

$$\text{MUF} = 8.05\times 10^6/0.196 = 41.0 \text{ MHz}$$

12-3-10. mUF for Clarke-orbit satellites.

Stationary communication (relay) satellites are placed in the Clarke orbit at heights of about 36 Mm. This is far above the ionosphere, so that the transmission path passes completely through the ionosphere twice, as in Fig. 12-3-10. Since frequencies of 2 GHz and above are usually used the ionosphere has little effect. The high frequency also permits wide bandwidths. If the ionosphere consists of a layer 200 m thick between heights of 200 and 400 km with a uniform electron density $N = 10^{12}$ m^{-1}, find the lowest frequency (or *minimum usable frequency,* mUF) which can be used with a communication satellite (a) for vertical incidence and (b) for paths 30° from the zenith. (c) For an earth station on the equator, what is the mUF for a satellite 15° above the eastern or western horizon?

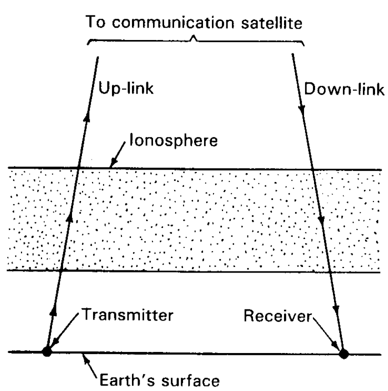


Figure P12-3-10. Communication path via geostationary Clarke-orbit relay satellite.

Solution:

mUF = minimum usable frequency for transmission through ionosphere

(a) $\text{mUF} = 9(N)^{1/2} = 9(10^{12})^{1/2} = 9$ MHz $\quad$ (*ans.*)

12-3-10. continued

Although MUF = mUF, whether it is one or the other, depends on the point of view. It is MUF for reflection and mUF for transmission. Below critical frequency, wave is reflected; above critical frequency, wave is transmitted.

(b) $\text{mUF} = 9\ \text{MHz}/\cos 30^{\circ} = 10.4\ \text{MHz}$ (*ans.*)

(c) $\text{mUF} = 9\ \text{MHz}/\cos 75^{\circ} = 34.8\ \text{MHz}$ (*ans.*)

A typical Clarke orbit satellite frequency is 4 GHz, which is 115 $(= 4\times 10^{9}/34.8\times 10^{6})$ times higher in frequency than the mUF, so that at 4 GHz, transmission through the ionosphere can occur at much lower elevation angles than 15°.

***12-3-11. Minimum detectable temperature.**

A radio telescope has the following characteristics: antenna noise temperature 50 K, receiver noise temperature 50 K, transmission-line between antenna and receiver 1 dB loss and 270 K physical temperature, receiver bandwidth 5 MHz, receiver integration time 5 s, receiver (system) constant $k' = \pi/\sqrt{2}$ and antenna effective aperture 500 m^2. If two records are averaged, find (a) the minimum detectable temperature and (b) the minimum detectable flux density.

Solution:
(a) For 1 dB loss, $\varepsilon_2 = 1/1.26 = 0.79$

From (12-2-1),

$$T_{sys} = 50 + 270\left(\frac{1}{.79} - 1\right) + \frac{50}{.79} = 50 + 71.8 + 63.3 = 185\ \text{K}$$

From (12-2-3),

$$\Delta T_{\min} = \frac{k' T_{sys}}{(\Delta f\ t\, n)^{1/2}} = \frac{\pi}{\sqrt{2}}\frac{185}{(5\times 10^{6}\times 5\times 2)^{1/2}} = 0.058\ \text{K} \quad \cong 0.06\ \text{K (rounded off)} \quad (\textit{ans.})$$

where n = number of records averaged

(b) From (12-1-7), $\Delta S_{\min} = \dfrac{2kT_{\min}}{A_e} = \dfrac{2\times 1.38\times 10^{-23}\times 0.058}{500} = 320\ \text{mJy}$ (*ans.*)

12-3-12. Minimum detectable temperature.

A radio telescope operates at 2650 MHz with the following parameters: system temperature 150 K, predetection bandwidth 100 MHz, postdetection time constant 5 s, system constant *k'* = 2.2 and effective aperture of antenna 800 m^2. Find (a) the minimum detectable temperature and (b) the minimum detectable flux density. (c) If four records are averaged, what change results in (a) and (b)?

Solution:

(a) $$\Delta T_{\min} = \frac{2.2 \times 50}{(10^8 \times 5)^{1/2}} = 0.015 \text{ K} \quad (ans.)$$

(b) $$\Delta S_{\min} = \frac{2 \times 1.38 \times 10^{-23} \times 0.015}{800} = 52 \text{ mJy} \quad (ans.)$$

(c) $$\Delta T_{\min} = \frac{0.015}{\sqrt{4}} = 0.008 \text{ K}$$

$$\Delta S_{\min} = \frac{52}{\sqrt{4}} = 26 \text{ mJy} \quad (ans.)$$

---

*12-3-13. Interstellar wireless link.

If an extraterrestrial civilization (ETC) transmits 10^6 W, 10 s pulses of right-hand circularly polarized 5 GHz radiation with a 100 m diameter dish, what is the maximum distance at which the ETC can be received with an SNR = 3. Assume the receiving antenna on the earth also has a 100 m diameter antenna responsive to right circular polarization, that both antennas (theirs and ours) have 50 percent aperture efficiency, and that the earth station has a system temperature of 10 K and bandwidth of 0.1 Hz.

Solution: (Note: Problem statement should specify S/N =3.)

From (12-3-3), $$\frac{S}{N} = \frac{P_t A_{er} A_{et}}{r^2 \lambda^2 B k T_{sys}}, \qquad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^9} = 0.06 \text{ m}$$

$$r^2 = \frac{P_t A_{er} A_{et}}{\lambda^2 B k T_{sys} (S/N)} = \frac{10^6 [\pi (50)^2 0.5][\pi (50)^2 0.5]}{(0.6)^2 (0.1)(1.38 \times 10^{-23})(10)(3)} = 1.03 \times 10^{38} \text{ m}^2$$

Therefore, $r = 1.02 \times 10^{19}$ m $\cong 1000$ light years (*ans.*)

***12-3-14. Backpacking penguin.**

This penguin (Fig. P12-3-14) participated in a study of Antarctic penguin migration habits. Its backpack radio with $\lambda/4$ antenna transmitted data on its body temperature and its heart and respiration rates. It also provided information on its location as it moved with its flock across the ice cap. The backpack operated at 100 MHz with a peak power of 1 W and a bandwidth of 1- kHz of tone-modulated data signals. If T_{sys} = 1000 K and SNR = 30 dB, what is the maximum range? The transmitting and receiving antennas are $\lambda/4$ stubs.

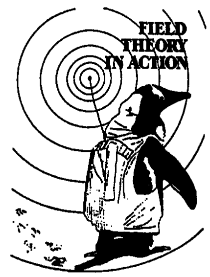


Figure P12-3-14. Antarctic backpacking penguin.

Solution:

From (12-3-3), $$\frac{S}{N}=\frac{P_t A_{er} A_{et}}{r^2\lambda^2 BkT_{sys}}, \qquad \lambda=\frac{c}{f}=\frac{3\times10^8}{100\times10^6}=3\text{ m}$$

$$r^2=\frac{P_t A_{er} A_{et}}{\lambda^2 BkT_{sys}(S/N)}=\frac{P_t\left(\frac{1.5\lambda^2}{4\pi}\right)\left(\frac{1.5\lambda^2}{4\pi}\right)}{\lambda^2 BkT_{sys}(S/N)}$$

$$=\frac{1\left(\frac{1.5(3)^2}{4\pi}\right)}{(3)^2(10\times10^3)(1.38\times10^{-23})(1000)(10^3)}=9.3\times10^{11}\text{ m}^2$$

Therefore, $r=9.6\times10^5\text{ m}\cong 960\text{ km}$ (*ans.*)

***12-3-17. Low earth orbit communications satellite.**

A communications satellite in low earth orbit (LEO) has r_{up} = 1500 km and r_{down} = 1000 km, with an uplink frequency of 14.25 GHz and a down link frequency of 12 GHz. Find the full-circuit *C/N* if the transmitting earth station ERP is 60dBW and the satellite ERP is 25 dBW. Assume the satellite receiver *G/T* is 5 dB/K and the earth-station *G/T* is 30 dB/K.

***12-3-17. continued**

Solution:

From Prob. 12-3-15, $\frac{C}{N} = \frac{P_t A_{et} A_{er}}{r^2 \lambda^2 k T_{sys}}$ and using $G = \frac{4\pi A_e}{\lambda^2}$, we have $\frac{C}{N} = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2 r^2 k T_{sys}}$

We now define effective radiated power = ERP = $P_t G_t$, Power loss $= L = (4\pi)^2 r^2 / \lambda^2$

This gives $$\frac{C}{N} = (\text{ERP})\left(\frac{1}{\text{L}}\right)\left(\frac{1}{k}\right)\left(\frac{G_r}{T_{sys}}\right)$$

For the uplink,

$$\lambda_{\text{up}} = \frac{c}{f_{\text{up}}} = \frac{3\times10^8}{14.25\times10^9} = 0.021\text{ m}, \quad \text{and} \quad \left(\frac{C}{N}\right)_{\text{up}} = (\text{ERP})_{\text{up}}\left(\frac{1}{L_{\text{up}}}\right)\left(\frac{1}{k}\right)\left(\frac{G_r(\text{sat})}{T_{sys}(\text{sat})}\right)$$

Working in dB, ERP_{up} = 60 dBW, $$L_{\text{up}} = 10\log\left(\frac{4\pi(1.5\times10^6)}{0.021}\right)^2 = 179.1\text{ dB}$$

$$k = 10\log(1.38\times10^{-23}) = -228.6\text{ dB}, \qquad \frac{G_r(\text{sat.})}{T_{sys}(\text{sat.})} = 5\text{ dB K}^{-1}$$

So $$\left(\frac{C}{N}\right)_{\text{up}} = 60 - 179.1 + 228.6 + 5 = 114.5\text{ dB}$$

Similarly, $$\left(\frac{C}{N}\right)_{\text{down}} = 25 - 174 + 228.6 + 30 = 109.6$$

From Prob. 12-3-1, $$\left(\frac{C}{N}\right)^{-1} = \left(\frac{C}{N}\right)_{\text{up}}^{-1} + \left(\frac{C}{N}\right)_{\text{down}}^{-1} = \frac{1}{10^{11.45}} + \frac{1}{10^{10.96}} = 1.45\times10^{-11}$$

$$\frac{C}{N} = \frac{1}{1.45\times10^{-11}} = 6.89\times10^{10} = 108.4\text{ dB} \quad (\textit{ans.})$$

***12-3-18. Direct broadcast satellite (DBS).**

Direct broadcast satellite services provide CD quality audio to consumers via satellites in geosynchronous orbit. The World Administrative Radio Conference (WARC) has established these requirements for such services.

***12-3-18. continued**

Frequency band	11.7 to 12.2 GHz (K_u band)
Channel bandwidth	27 MHz
Minimum power flux density	-103 dBW/m^2
Receiver figure of merit (*G/T*)	6 dB/K
Minimum carrier-to-noise ratio	14 dB

(a) Find the effective radiated power (ERP) over 1 W isotropic needed to produce the specified flux density at the Earth's surface from a DBS satellite in a 36,000 km orbit. (b) If the satellite has a 100 W transmitter and is operated at 12 GHz, what size circular parabolic dish antenna must be used to achieve the required ERP? Assume 50 percent efficiency. (c) Does a consumer receiver with circular 1 m dish antenna with 50 percent efficiency and system noise temperature of 1000 K meet the specified *G/T*? (d) By how much does the system specified in parts (a) through (c) exceed the required carrier-to-noise ratio?

Solution:

(a) Since $\mathrm{ERP} = P_t G_t$ and the power flux density $\psi(r) = \dfrac{P_t G_t}{4\pi r^2}$,

we have
$$-103 \text{ dB Wm}^{-2} = \frac{\mathrm{ERP}}{4\pi(3.6\times10^7)^2}$$

$$\therefore \mathrm{ERP} = 4\pi(3.6\times10^7)^2 10^{-10.3} = 8.16\times10^5 \text{ W} \quad (\textit{ans.})$$

(b)
$$G = k\frac{4\pi A}{\lambda^2} = \frac{\mathrm{ERP}}{P_t} = k\frac{4\pi(\pi)(d/2)^2}{\lambda^2}$$

$$d = \sqrt{\frac{\mathrm{ERP}(\lambda^2)4}{P_t 4\pi^2 k}} = \sqrt{\frac{8.16\times10^5(0.025)^2 4}{(100)(4)(3.14)^2(0.5)}} = 1.0 \text{ m} \quad (\textit{ans.})$$

(c) For a 1-m receive antenna and 1000 K noise temperature,

$$\frac{G}{T} = k\frac{4\pi A_e}{\lambda^2 T} = 0.5\frac{4\pi(\pi)(1/2)^2}{(0.03)^2(1000)} = 5.48 \text{ K}^{-1} = 7.4 \text{ dB K}^{-1} \quad \text{(yes)} \quad (\textit{ans.})$$

(d) The *C*/*N* for a channel bandwidth *B* is

$$\frac{C}{N} = \mathrm{ERP}\left(\frac{1}{L}\right)\left(\frac{1}{k}\right)\left(\frac{G}{T}\right)\left(\frac{1}{B}\right) = 8.16\times10^5\left(\frac{0.03}{4\pi(3.6\times10^7)}\right)^2\left(\frac{1}{1.38\times10^{-23}}\right)(5.5)\left(\frac{1}{2.7\times10^7}\right)$$
$$= 52.9 = 17.2 \text{ dB}$$

***12-3-18. continued**

Therefore C/N exceeds requirement (14 dB) by $17.2 - 14 = 3.2$ dB (*ans.*)

12-3-19. Simplified expression for *C/N.*

The expression for *C/N* provided in P12-3-15 may be simplified by making the following substitutions:

Effective isotropic radiated power = ERP = $P_t G_t$ (W)
Link path loss = $L_{\text{link}} = 4\pi r^2/\lambda^2$

The carrier-to noise ratio may then be written as

$$\frac{C}{N} = \text{ERP}\frac{1}{L_{\text{link}}}\frac{1}{k}\frac{G_{\text{r}}}{T_{\text{sys}}}$$

where $G_{\text{r}}/T_{\text{sys}}$ is the receive antenna gain divided by the system noise temperature. This ratio, referred to as "*G over T*," is commonly used as a figure of merit for satellite and earth station receivers. Find the *C/N* ratio for the uplink to a satellite at the Clarke orbit (r = 36,000 km) equipped with a 1 m parabolic dish antenna with efficiency of 50 percent and a receiver with noise temperature of 1500 K. Assume that the transmitting earth station utilizes a 1 kW transmitter and a 50 percent efficient 10 m dish antenna and operates at a frequency of 6 GHz.

Solution:

$$\frac{C}{N} = \text{ERP}\left(\frac{1}{L}\right)\left(\frac{1}{K}\right)\left(\frac{G_r}{T_{sys}}\right), \quad \lambda = 0.05 \text{ m}, \quad L = \left(\frac{4\pi r}{\lambda}\right)^2 = \left(\frac{4\pi(3.6\times10^7)}{0.05}\right)^2 = 8.18\times10^{19}$$

$$\text{ERP} = P_t G_t = 1000\left(0.5\frac{4\pi(\pi)(5)^2}{(0.05)^2}\right) = 1.97\times10^8 \text{ W}, \qquad \frac{G_r}{T_{sys}} = \frac{0.5\dfrac{4\pi(\pi)(5)^2}{(0.05)^2}}{1500} = 1.3 \text{ K}^{-1}$$

Therefore,
$$\frac{C}{N} = (1.97\times10^8)\left(\frac{1}{8.18\times10^{19}}\right)\left(\frac{1}{1.38\times10^{-23}}\right)(1.3)$$
$$= 2.3\times10^{11} = 113.5 \text{ dB} \quad (\textit{ans.})$$

12-3-22. Galileo's uncooperative antenna.

When the Galileo spacecraft arrived at Jupiter in 1995, ground controllers had been struggling for 3 years to open the spacecraft's 5 m high-gain communications dish, which was to operate at 10 GHz (X band). Unable to deploy this antenna because of prelaunch loss of lubricant, a low-directivity (G = 10 dB) S-band antenna operating at 2 GHz had to be used to relay all pictures and data from the spacecraft to the Earth. For a spacecraft transmit power of 20 W, distance to Earth of 7.6 x 10^{11} m, and 70 m dish with 50 percent efficiency at the receiving station, find (a) the maximum achievable data rate if the 5 m X-band antenna had deployed and (b) the maximum data rate using the 1 m S-band antenna.

Solution:

$$\frac{C}{N} = \text{ERP}\left(\frac{1}{L}\right)\left(\frac{1}{K}\right)\left(\frac{G_r}{T_{sys}}\right)$$

(a) For the 5-m X-band antenna, $\text{ERP} = P_t G_t = 20\left(0.5\frac{4\pi(\pi)(2.5)^2}{(0.03)^2}\right) = 2.74\times10^6 \text{ W}$

$$L = \left(\frac{4\pi r}{\lambda}\right)^2 = \left(\frac{4\pi(7.6\times10^{11})}{0.03}\right)^2 = 1.01\times10^{29}, \qquad \frac{G_r}{T_{sys}} = \frac{0.5\frac{4\pi(\pi)(35)^2}{(0.03)^2}}{500} = 5.37\times10^4 \text{ K}^{-1}$$

Therefore, $\frac{C}{N} = (2.74\times10^6)\left(\frac{1}{1.01\times10^{29}}\right)\left(\frac{1}{1.38\times10^{-23}}\right)(5.37\times10^4) = 1.05\times10^5$

The maximum data rate $M = 1.44\frac{C}{N} = 1.44(1.05\times10^5) \cong 152 \text{ kBits/s}$ (*ans.*)

***12-4-1. Antenna temperature with absorbing cloud.**

A radio source is occulted by an intervening emitting and absorbing cloud of unity optical depth and brightness temperature 100 K. The source has a uniform brightness distribution of 200 K and a solid angle of 1 square degree. The radio telescope has an effective aperture of 50 m^2. If the wavelength is 50 cm, find the antenna temperature when the radio telescope is directed at the source. The cloud is of uniform thickness and has an angular extent of 5 square degrees. Assume that the antenna has uniform response over the source and cloud.

***12-4-1. continued**

Solution:

From (12-4-1) and (12-1-6), $$\Delta T_A = \frac{\Omega_c}{\Omega_A} T_c (1 - e^{-\tau_c}) + \frac{\Omega_s}{\Omega_A} T_s e^{-\tau_c}$$

$$\Omega_c = 5/57.3^2 = 0.00152 \text{ sr}$$

$$\Omega_s = 1/57.3^2 = 0.00030 \text{ sr}$$

$$\Omega_A = \lambda^2 / A_{em} = 0.5^2 / 50 = 0.005 \text{ sr}$$

Therefore, $$\Delta T_A = \frac{.00152}{.005} 100(1 - e^{-1}) + \frac{.0003}{.005} 200 e^{-1} = 23.6 \text{ K} \quad (\textit{ans.})$$

12-4-3. Forest absorption.

An earth-resource satellite passive remote-sensing antenna directed at the Amazon River Basin measures a night-time temperature T_A = 21°C. If the earth temperature T_e = 27°C and the Amazon forest temperature T_f = 15°C, find the forest absorption coefficient τ_f.

Solution:

From (12-4-2), $$e^{-\tau_f} = \frac{\Delta T_A - T_f}{T_f - T_e} = \frac{294 - 288}{300 - 288} = 0.5 \quad \text{and} \quad \tau_f = 0.693 \quad (\textit{ans.})$$

***12-4-4. Jupiter signals.**

Flux densities of 10^{-20} W m^{-2} Hz^{-1} are commonly received from Jupiter at 20 MHz. What is the power per unit bandwidth radiated at the source? Take the earth-Jupiter distance as 40 light-minutes and assume that the source radiates isotropically.

Solution:

$$r = 40 \text{ light min } \times 60 \text{ s min}^{-1} \times 3 \times 10^8 \text{ m s}^{-1} = 7.20 \times 10^{11} \text{ m}$$

$$\frac{P_r}{\Delta f} = \frac{P_t}{\Delta f} \times \frac{1}{4\pi r^2}$$

or $$\frac{P_t}{\Delta f} = \frac{P_r}{\Delta f} \times 4\pi r^2 = 10^{-20} \times 4\pi \times 7.2^2 \times 10^{22} = 65.1 \text{ kW Hz}^{-1} \quad (\textit{ans.})$$

12-5-1. Radar detection

A radar receiver has a sensitivity of 10^{-12} W. If the radar antenna effective aperture is 1 m^2 and the wavelength is 10 cm, find the transmitter power required to detect an object with 5 m^2 radar cross section at a distance of 1 km.

Solution:

$$P_t = \frac{P_r 4\pi r^4 \lambda^2}{A^2\sigma} = \frac{10^{-12} \times 4\pi \times (10^{-3})^4 (0.1)^2}{(1)^2 \times 5} = 25 \text{ mW} \quad (ans.)$$

---

*12-5-3. RCS of electron.

The alternating electric field of a passing electromagnetic wave causes an electron (initially at rest) to oscillate (Fig. P12-5-3). This oscillation of the electron makes it equivalent to a short dipole antenna with D = 1.5. Show that the ratio of the power scattered per steradian to the incident Poynting vector is given by $(\mu_o e^2 \sin\theta / 4\pi m)^2$, where e and m are the charge and mass of the electron and θ is the angle of the scattered radiation with respect to the direction of the electric field **E** of the incident wave. This ratio times 4π is the radar cross section of the electron. Such reradiation is called *Thompson scatter*.

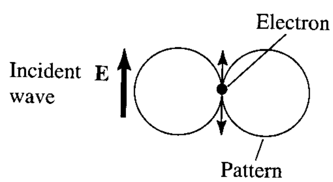


Figure P12-5-3.

Solution:

From (6-2-17), the magnitude of the electric dipole far-field is $|E_\theta| = \dfrac{\omega I_o l \sin\theta}{4\pi\varepsilon_o c^2 r}$

The force on the electron due to the incident field E_o is $F = eE_o = ma = m\omega^2 l$

So, $l = \dfrac{eE_o}{m\omega^2}$

Since $I_o = e\omega$, $\quad |E_\theta| = \dfrac{\omega(e\omega)\left(\dfrac{eE_o}{m\omega^2}\right)\sin\theta}{4\pi\varepsilon_o c^2 r}$

Thus, the scattered power density $\quad S_{scat} = \dfrac{4\pi r^2 |E_\theta|^2}{Z} = 4\pi\left(\dfrac{e^2 E_o \sin\theta}{4\pi\varepsilon_o c^2 m}\right)^2 \dfrac{1}{Z}$

***12-5-3. continued**

and since the incident power density $S_{inc} = \frac{E_o^2}{Z}$, the ratio of scattered to incident power is

$$\sigma = \frac{S_{scat}}{S_{inc}} = \frac{4\pi\left(\frac{e^2 E_o \sin\theta}{4\pi\varepsilon_o c^2 m}\right)\frac{1}{Z}}{\frac{E_o^2}{Z}} = \frac{1}{4\pi}\left(\frac{e^2\sin\theta}{\varepsilon_o c^2 m}\right)^2 = \frac{1}{4\pi}\left(\frac{e^2\sin\theta}{\varepsilon_o\left(1/\mu_o\varepsilon_o\right)m}\right)^2 = \frac{1}{4\pi}\left(\frac{\mu_o e^2\sin\theta}{m}\right)^2$$

For $\sin\theta = 1$, $\quad\sigma \cong 1\times10^{-28}\ \text{m}^2$ (*ans.*)

***12-5-5. Detecting one electron at 10 km.**

If the Arecibo ionospheric 300 m diameter antenna operates at 100 MHz, how much power is required to detect a single electron at a height (straight up) of 10 km with an SNR – 0 dB? See Fig. P12-5-5. The bandwidth is 1 Hz, T_{sys} = 100 K and the aperture efficiency = 50 percent.

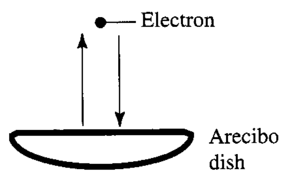


Figure P12-5-5.

Solution:

$$\frac{S}{N} = \frac{P_t G_t A_{er}\sigma}{(4\pi)^2 r^4 kTB}, \qquad \lambda = \frac{3\times10^8}{100\times10^6} = 3\text{ m}, \qquad A_{er} = 0.5\pi(150)^2 = 3.53\times10^4\ \text{m}^2$$

$$G_t = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi(3.53\times10^4)}{3^2} = 4.93\times10^4$$

Thus,

$$P_t = \frac{(4\pi)^2 r^4 kTB(S/N)}{G_t A_{er}\sigma} = \frac{(4\pi)^2(10^4)^4(1.38\times10^{-23})(100)(1)(1)}{(4.93\times10^4)(3.53\times10^4)(1\times10^{-28})} = 1.25\times10^{16}\ \text{W} \quad (\textit{ans.})$$

12-5-6. Effect of resonance on radar cross section of short dipoles.

(a) Calculate the radar cross section of a lossless resonant dipole ($Z_L = -jX_A$) with length = $\lambda/10$ and diameter = $\lambda/100$. (See Secs. 2-9, 14-12).

(b) Calculate the radar cross section of the same dipole from $\dfrac{34L^6}{\lambda^4\left[\ln(L/2b)-1\right]^2}$, where L is the dipole length and b is its radius.

(c) Compare both values with the maximum radar cross section shown in Fig. 12-9. Comment on results.

Solution:

(a) $$0.714\lambda^2 = \underbrace{4\times \overset{A_{em}\text{ (matched dipole, } Z_L = Z_A^*)}{0.119}}_{A_s = \sigma_t\text{ (resonant dipole, } Z_L = -jX_A)}\times \overset{\text{Directivity}}{1.5}\lambda^2 \qquad (\textit{ans.})$$

(b) $2\times 10^{-5}\lambda^2$ (*ans.*)

(c) $0.83\lambda^2$ (*ans.*)

Summary and comparison:

Case	σ	Condition
(a)	$0.714\lambda^2$	Resonant short dipole, length 0.1λ ($Z_L = -jX_A$)
(b)	$2\times 10^{-5}\lambda^2$	Non-resonant 0.1λ dipole ($Z_L = 0$)
(c)	$0.83\lambda^2$	Resonant 0.47λ dipole ($Z_L = 0$)

In (a) resonance is obtained by making $Z_L = -jX_A$. In (c) resonance is obtained ($X_A = 0$) by increasing the length to 0.47λ (with $Z_L = 0$, terminals short-circuited).

The cross section in (c) is larger than in (a) because the dipole is physically longer.

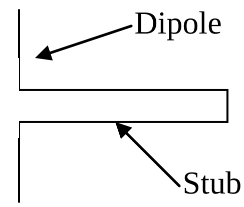
In (b) the dipole is non-resonant because $Z_L = 0$ (terminals short-circuited) and the length (0.1λ) is much less than the resonant length (0.47λ), resulting in a very small radar cross section.

It appears that if a short dipole (length $\le 0.1\lambda$) is resonated by making $Z_L = -jX_A$, its radar cross section $(0.714\lambda^2)$ approaches the cross section of a resonant $\lambda/2$ dipole (length =0.47 λ), regardless of how short it is, provided it is lossless.

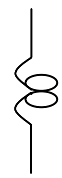
A short dipole may be resonated $(Z_L = -jX_A)$ by connecting a stub of appropriate length across its terminals,

12-5-6. continued

Thus,



or by connecting lumped inductance, thus



***12-5-12 Fastball velocity.**

A 20 GHz radar measures a Doppler shift of 6 kHz on a baseball pitcher's fastball. What is the fastball's velocity?

Solution:

$$\frac{\Delta f}{f} = \frac{2v}{c}, \qquad v = \frac{c\Delta f}{2f} = \frac{(3\times10^8)(6\times10^3)}{2(20\times10^9)} = 45 \text{ m/s} = 162 \text{ km/h} = 101 \text{ mi/h} \quad (\textit{ans.})$$

***12-5-13. Radar power for fastball measurement.**

To measure the velocity of the fastball of Prob. 12-5-12 with the 20 GHz radar at a distance of 100 m, what power is required for an SNR = 30 dB? The radar uses a conical horn with diameter = 8 cm and aperture efficiency ε_{ap} = 0.5. The ball diameter = 7 cm and it has a radar cross section (RCS) half that of a perfectly conducting sphere of the same diameter.

Solution:

$$\frac{S}{N} = \frac{P_t G_t G_r \sigma}{(4\pi)^3 r^4 kTB}, \qquad \lambda = \frac{3\times10^8}{20\times10^9} = 0.015 \text{ m},$$

$$G_t = G_r = kD = k\frac{6.5\pi r^2}{\lambda^2} = 0.5\frac{6.5\pi(0.04)^2}{(0.015)^2} = 72.6, \qquad \sigma = 0.5\pi(0.035)^2 = 1.9\times10^{-3} \text{ m}^2$$

***12-5-13. continued**

$$P_t = \frac{(4\pi)^3 r^4 kTB(S/N)}{G_t G_r \sigma} = \frac{(4\pi)^3 (100)^4 (1.38\times10^{-23})(600)(10^6)(10^3)}{(72.6)^2 (1.9\times10^{-3})} \approx 160 \text{ mW} \quad (\textit{ans.})$$

***12-5-14. Anticollision radar.**

To provide anticollision warnings, forward-looking radars on automobiles, trucks and other vehicles (see Fig. P12-5-14) can alert the driver of vehicles ahead that are decelerating too fast or have stopped. The brake light on the vehicle ahead may not be working or it may be obscured by poor visibility. To warn of clear-distance decrease rates of 9 m/s or more, what doppler shift must a 20 GHz radar be able to detect?

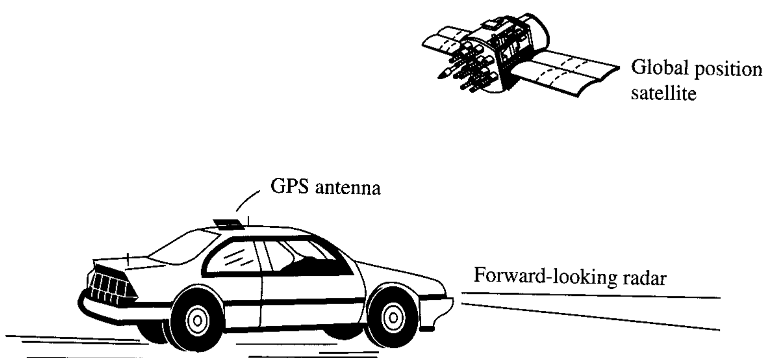


Figure P12-5-14.

Solution:

$$\frac{\Delta f}{f} = \frac{2v}{c}, \qquad \lambda = \frac{c}{f} = \frac{3\times10^8}{20\times10^9} = 0.015 \text{ m}, \quad v = 9 \text{ m/s},$$

$$\Delta f = \frac{2vf}{c} = \frac{2v}{\lambda} = \frac{2(9)}{0.015} = 1.2 \text{ kHz} \quad (\textit{ans.})$$

***12-5-18. Police radar.**

A pulsed speed measuring radar must be able to resolve the returns form two cars separated by 30 m. Find the maximum pulse width that can be used to prevent overlapping of the returns from the two vehicles.

Note that it takes $t = 2R/c$ seconds for a signal to travel from the transmitter to a target and return, where t is the time, R is the range and c is the velocity in the media. If τ is the pulse width, then it can be shown that the *range resolution* (the minimum range difference between objects for which the returns do not overlap in time) is given by

***12-5-18. continued**

$$\Delta R = R_2 - R_1 = \frac{c\tau}{2}$$

where the subscripts denote the different objects.

Solution:

$$\Delta R = \frac{c\tau}{2} \qquad \text{or} \qquad \tau = \frac{2\Delta R}{c}$$

The pulse width must be less than $\tau = \dfrac{2(10)}{3\times 10^8} = 6.6\times 10^{-8}\ \text{s} = 66\ \text{ns}$ (*ans.*)

***12-5-20. Sea clutter.**

Search-and-rescue aircraft using radar to locate lost vessels must contend with backscatter from the surface of the ocean. The amplitude of these returns (know as *sea clutter*) depends on the frequency and polarization of the radar waveform, the size of the illuminated patch on the surface, the angle of incidence, and the sea state. The scattering geometry is shown in Fig. P12-5-20. To characterize sea cluster independently on the radar footprint on the surface, the scattering cross section of the ocean may be specified per unit area. This parameter, designated σ_o, has dimensions of square meters of dB above a square meter (dBsm). The total RCS of a patch of ocean surface is found by multiplying σ_o by the area of the patch. For area scattering, the radar equation is written

$$P_r = P_t \frac{A_e^2 \lambda^2}{4\pi r^4} \sigma_o A_{\text{patch}} .$$

For a pulsed radar with pulse width τ and 3 dB antenna beamwidth of θ rad, the illuminated area for low grazing angle is approximately $(c\tau/2)(r\theta)$. The radar equation may therefore be written as

$$P_r = P_t \frac{A_e^2 \lambda^2}{4\pi r^4} \sigma_o \left(\frac{c\tau}{2}\right)(r\theta) = P_r \frac{A_e^2 \lambda^2 \sigma_o c\tau\theta}{8\pi r^3}$$

(a) Determine the received power from sea clutter at a range of 10 km for a monostatic pulsed radar transmitting a 1 μs pulse with 1 kW peak power at a frequency of 6 GHz. Assume a 1.5 m circular dish antenna with 50 percent efficiency and sea state 4 (σ_o = -30 dBsm/m^2 at C band). (b) For a receiver bandwidth of 10 kHz and noise figure of 3.5 dB, find the receiver noise power. (c) If this radar used to search for a ship with RCS = 33

***12-5-20. continued**

dBsm under these conditions, what signal-to-noise and signal-to-clutter ratios can be expected?

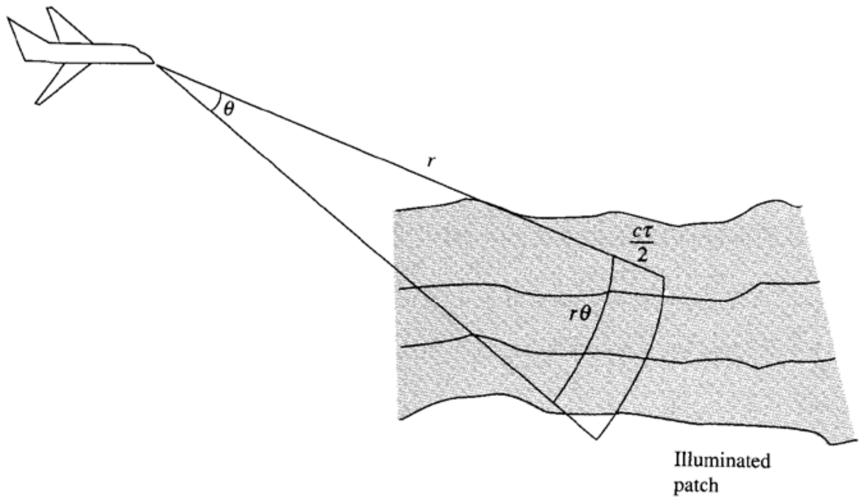


Figure P12-5-20. Sea search-and-rescue geometry.

Solution:

$$P_t = P_r \frac{A_e^2 \lambda^2 \sigma_o c\tau\theta}{8\pi r^3}, \qquad \lambda = \frac{3\times10^8}{6\times10^9} = 0.05 \text{ m}$$

(a) $$\theta_{3\text{ dB}} \cong \frac{\lambda}{D} = \frac{0.05}{1.5} = 0.033 \text{ radians}$$

$$P_r = (1000)\frac{[\pi(0.75)^2]^2(0.05)^2(10^{-30/10})(3\times10^8)(1\times10^{-6})(0.033)}{8\pi(10^4)^3} = 3.1\times10^{-15}\text{ W} \quad (\textit{ans.})$$

(b) The receiver noise power is

$$P_n = N_f kTB = (10^{3.5/10})(1.38\times10^{-23})(290)10^4 = 8.96\times10^{-17}\text{ W} \quad (\textit{ans.})$$

(c) For a ship with $\sigma = 33$ dB m^2 at $r = 10$ km.

$$P_r = P_t \frac{A_e^2\lambda^2\sigma}{4\pi r^4} = 1000\frac{[\pi(0.75)^2]^2(0.05)^2(10^{33/10})}{4\pi(10^4)^4} = 1.24\times10^{-3}\text{ W}$$

Thus, the signal-to-noise (*S*/*N*) and signal-to-clutter (*S*/*C*) ratios are:

$$\frac{S}{N} = \frac{1.24\times10^{-3}}{8.96\times10^{-17}} = 1384 = 31.4\text{ dB} \quad (\textit{ans.}), \qquad \frac{S}{C} = \frac{1.2\times10^{-3}}{3.1\times10^{-15}} = 40 = 16\text{ dB} \quad (\textit{ans.})$$

Chapter 13. Self and Mutual Impedances

*13-4-1. A 5λ/2 antenna.

Calculate the self-resistance and self-reactance of a thin, symmetrical center-fed linear antenna 5λ/2 long.

Solution:

From (13-5-2),

$$Z_{11} = 30[0.577 + \ln(2\pi n) - \mathrm{Ci}(2\pi n) + j\mathrm{Si}(2\pi n)], \qquad \text{where } n = 5$$

Since $2\pi n = 2\pi 5 = 10\pi >> 1$, we have from (13-3-18), $\quad \mathrm{Ci}(10\pi) = \dfrac{\sin(10\pi)}{10\pi} = 0$

And from (13-3-22),

$$\mathrm{Si}(10\pi) = \frac{\pi}{2} - \frac{\cos(10\pi)}{10\pi} = \frac{\pi}{2} - \frac{1}{10\pi} = 1.539$$

and $\quad R_{11} = 30[0.577 + \ln(10\pi) - 0] = 120.7\ \Omega \quad$ (*ans.*)

$\quad X_{11} = 30 \times 1.539 = 46.2\ \Omega \quad$ (*ans.*)

13-6-1. Parallel side-by-side λ/2 antennas.

Calculate the mutual resistance and mutual reactance for two parallel side-by-side thin linear λ/2 antennas with a separation of 0.15λ.

Solution:

From (13-7-6),

$$R_{21} = 30\{2\mathrm{Ci}(\beta d) - \mathrm{Ci}[\beta(\sqrt{d^2 + L^2} + L)] - \mathrm{Ci}[\beta(\sqrt{d^2 + L^2} - L)]\}$$

where $d = 0.15\lambda, \qquad L = 0.5\lambda$

and $\qquad R_{21} = 30[2\mathrm{Ci}(0.942) - \mathrm{Ci}(6.42) - \mathrm{Ci}(0.138)]$

From (13-3-16) or Ci table or from Fig. 13-5 and from (13-3-18), we have

$$R_{21} = 30(0.60 - 0 - 0.577 - \ln 0.138) = 60.1\,\Omega \quad (\textit{ans.}) \quad \text{(compare with Fig. 13-13)}$$

13-6-1. continued

From (13-7-7),

$$X_{21} = -30\{2\mathrm{Si}(\beta d) - \mathrm{Si}[\beta(\sqrt{d^2 + L^2} + L)] - \mathrm{Si}[\beta(\sqrt{d^2 + L^2} - L)]\}$$

and

$$X_{21} = -30[2\mathrm{Si}(0.942) - \mathrm{Si}(6.42) - \mathrm{Si}(0.138)]$$

From (13-3-20) or Si table or from Fig. 13-5 and from (13-3-21), we have

$$X_{11} = -30(1.8 - 1.42 - 0.138) = -7.3\ \Omega \quad (ans.) \quad \text{(compare with Fig. 13-13)}$$

***13-6-3. Three side-by-side antennas.**

Three antennas are arranged as shown in Fig. P13-6-3. The currents are of the same magnitude in all antennas. The currents are in-phase in (a) and (c), but the current in (b) is in anti-phase. The self-resistance of each antenna is 100 Ω, while the mutual resistances are: R_{ab} = R_{bc} = 40 Ω and R_{ac} = -10 Ω. What is the radiation resistance of each of the antennas? The resistances are referred to the terminals, which are in the same location in all antennas.

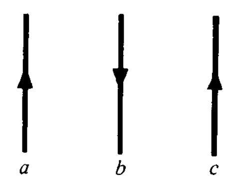


Figure P13-6-3. Three side-by-side antennas.

Solution:

$$R_a = R_s - R_{ab} + R_{ac} = 100 - 40 - 10 = 50\ \Omega$$
$$R_b = R_s - 2R_{ab} = 100 - 80 = 20\ \Omega \qquad (ans.)$$
$$R_c = R_a = 50\ \Omega$$

13-8-1. Two λ/2 antennas in echelon.

Calculate the mutual resistance and reactance of two parallel thin linear $\lambda/2$ antennas in echelon for the case where $d = 0.25\lambda$ and $h = 1.25\lambda$ (see Fig. 13-16).

Solution:

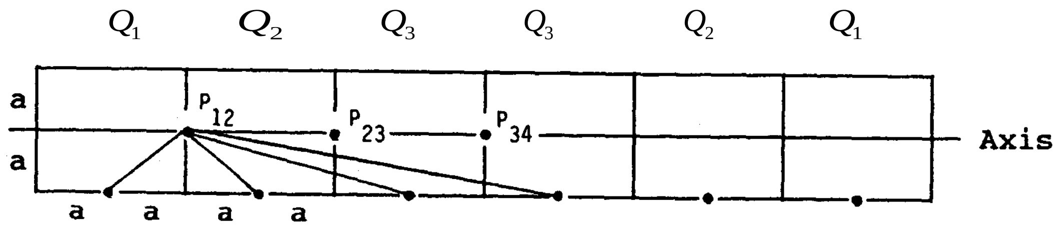
Use (13-9-1), (13-9-2), (13-3-16) and (13-3-20) where $h = 1.25\lambda$ and $d = 0.25\lambda$

Chapter 14. The Cylindrical Antenna and the Moment Method (MM)

14-10-1. Charge distribution.

Determine the electrostatic charge distribution on a cylindrical conducting rod with a length-diameter ratio of 6.

Solution:



Divide rod into 6 equal segments. By symmetry charges are as shown. Neglecting end faces, the potential at point P_{12} is

$$V(P_{12}) = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1}{\sqrt{2}a} + \frac{Q_2}{\sqrt{2}a} + \frac{Q_3}{\sqrt{10}a} + \frac{Q_3}{\sqrt{26}a} + \frac{Q_2}{\sqrt{50}a} + \frac{Q_1}{\sqrt{82}a}\right)$$

Writing similar expression for $V(P_{23})$ and $V(P_{34})$, equating them (since the potential is constant along the rod) and solving for the charges yields:

$$Q_1 : Q_2 : Q_3 = 1.582 : 1.062 : 1.000 \quad (ans.)$$

14-12-2. λ/10 dipole impedance.

Show that the convergence or true value of the self impedance Z_s of the dipole of Table 14-4 is 1.852-j1895 Ω.

Solution:

Using Z_s values of Table 14-4, plot R_s vs. N to suitable large scale and suppressed zero and note that R_s approaches a constant (convergence) value as N becomes large which should agree with Richmond's value given following (14-12-34). Calculate R_s for larger values of N than 7, if desired. Do same for X_s.

Chapter 15. The Fourier Transform Relation Between Aperture Distribution and Far-Field Pattern

*15-3-1. Pattern smoothing.

An idealized antenna pattern-brightness distribution is illustrated by the 1-dimensional diagram in Fig. P15-3-1. The brightness distribution consists of a point source of flux density S and a uniform source 2° wide, also of flux density S. The point source is 2° from the center of the 2° source. The antenna pattern is triangular (symmetrical) with a 2° beam width between zero points and with zero response beyond.
(a) Draw an accurate graph of the observed flux density as a function of angle from the center of the 2° source.
(b) What is the maximum ratio of the observed to the actual total flux density (2S)?

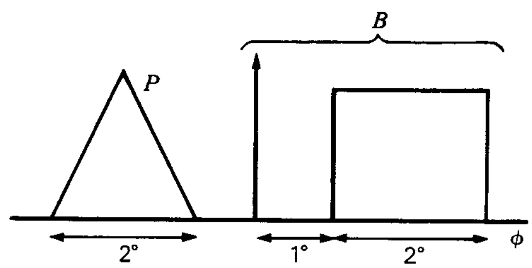


Figure P15-3-1. Pattern smoothing.

Solution:

Ratio = 1/2 (*ans.*)

15-6-1. Number of elements.

In Fig. 15-15 how many elements n have been assumed?

Solution:

From Fig. 15-15,

$$\frac{2}{nd_\lambda} = \frac{1}{4} \times \frac{1}{d_\lambda}$$

or

$$n = 8$$

Chapter 16. Arrays of Dipoles and of Apertures

*16-2-1. Two λ/2-element broadside array.

(a) Calculate and plot the gain of a broadside array of 2 side-by-side λ/2 elements in free space as a function of the spacing *d* for values of *d* from 0 to 2λ. Express the gain with respect to a single λ/2 element. Assume all elements are 100 percent efficient.
(b) What spacing results in the largest gain?
(c) Calculate and plot the radiation field patterns for λ/2 spacing. Show also the patterns of the λ/2 reference antenna to the proper relative scale.

Solution:

(a) From (16-2-27),

$$G_f(\phi)(A/HW) = [2R_{00}/(R_{11}+R_{12})]^{1/2} \cos[(d_r \cos\phi)/2]$$

In broadside direction $\phi = \pi/2$

so $$G_f(\phi)(A/HW) = [2R_{00}/(R_{11}+R_{12})]^{1/2}$$

where $R_{00} = R_{11} = 73.1\ \Omega$

and R_{12} is a function of the spacing as given in Table 13-1 (p. 453). A few values of the gain for spacings from 0 to 1λ are listed below:

Spacing λ	Gain over $\lambda/2$ reference
0.0	1.00
0.1	1.02
0.2	1.08
0.3	1.19
0.4	1.36
0.5	1.56
0.6	1.72
0.7	1.74
0.8	1.64
0.9	1.49
1.0	1.38
etc.	

Note that

$$D(\lambda/2) = 4\pi A_{em}(\lambda/2)/\lambda^2 = 4\pi(30/73.1\pi) = 1.64 \text{ or } 2.15 \text{ dBi}$$

***16-2-1. continued**

so D of 2 in-phase $\lambda/2$ elements at 0.67λ spacing is equal to $4.9+2.15=7.1$ dBi as above.

(b) By interpolation, the highest gain occurs for a spacing of about 0.67λ for which the gain is about 1.76 or 4.9 dB (= 7.1 dBi). At spacings over 1λ no gains exceed this.

16-3-1. Two λ/2-element end-fire array.

A 2-element end-fire array in free space consists of 2 vertical side-by-side λ/2 elements with equal out-of-phase currents. At what angles in the horizontal plane is the gain equal to unity:
(a) When the spacing is λ/2?
(b) When the spacing is λ/4?

Solution:

(a) From (16-3-18),

$$G_f(\phi)(A/HW) = [2R_{00}/(R_{11}-R_{12})]^{1/2}\sin[(d_r\cos\phi)/2]$$

When $d=\lambda/2$,

$$G_f(\phi)(A/HW) = [2R_{00}/(R_{11}-R_{12})]^{1/2}\sin[(\pi/2)\cos\phi)]$$

where $R_{00}=R_{11}=73.1\ \Omega$

and from Table 13-1, $R_{12}=-12.7\ \Omega$

so $$G_f(\phi)(A/HW) = 1.31\sin[(\pi/2)\cos\phi)]$$

For unit gain, $\sin[(\pi/2)\cos\phi)]=1/1.31=0.763$

or $\cos\phi = 49.8^{\circ}/90^{\circ}=0.553$ and $\phi=\pm 56^{\circ},\ \pm 124^{\circ}$ (*ans.*)

(b) When $\lambda/4$,

$$G_f(\phi)(A/HW) = [2R_{00}/(R_{11}-R_{12})]^{1/2}\sin[(\pi/4)\cos\phi]$$

where $R_{00}=R_{11}=73.1\ \Omega$

and from Table 13-1, $R_{12}=40.9\ \Omega$

16-3-1. continued

so $$G_f(\phi)(A/HW) = 2.13\sin[(\pi/4)\cos\phi]$$

For unit gain, $\sin[(\pi/4)\cos\phi] = 1/2.13 = 0.47$

$$\cos\phi = 28^{\circ}/45^{\circ} = 0.62 \text{ and } \phi = \pm 52^{\circ}, \pm 128^{\circ} \quad (\textit{ans.})$$

***16-3-2. Impedance and gain of 2-element array.**

Two thin center-fed $\lambda/2$ antennas are driven in phase opposition. Assume that the current distributions are sinusoidal. If the antennas are parallel and spaced 0.2λ,
(a) Calculate the mutual impedance of the antennas.
(b) Calculate the gain of the array in free space over one of the antennas alone.

Solution:

(a) This is a single-electron W8JK array.

From Sec. 13-7, $Z_{12} = 52 - j21\ \Omega$ (*ans.*)

(b) From (16-5-8) and assuming losses,

$$G_f(\phi)(\max)(A/HW) = \left(\frac{2\times 73}{73-52}\right)^{1/2}\sin 36^{\circ}$$
$$= 2.64\times 0.588 = 1.55 \text{ or } 3.8 \text{ dB } (= 6.0 \text{ dBi}) \quad (\textit{ans.})$$

16-4-3. Two-element array with unequal currents.

(a) Consider two $\lambda/2$ side-by-side vertical elements spaced a distance d with currents related by $I_2 = aI_1/\delta$. Develop the gain expression in a plane parallel to the elements and the gain normal to the elements, taking a vertical $\lambda/2$ element with the same power input as reference ($0 \le a \le 1$). Check that these reduce to Eq. (16-4-15) and Eq. (16-4-13) when $a = 1$.
(b) Plot the field patterns in both planes and also show the field pattern of the reference antenna in proper relative proportion for the case where $d = \lambda/4$, $a = ½$ and $\delta = 120°$.

Solution:

(a) $$G_f(\theta) = \{R_{11}/[R_{11}(1+a^2) + 2aR_{12}\cos\delta]\}^{1/2} \times (1 + a^2 + 2a\cos\psi)^{1/2}$$

16-4-3. continued

where $\psi = d_r \sin\theta + \delta$

$$G_f(\phi) = G_f(\theta) \text{ but with } \psi = d_r \cos\phi + \delta$$

***16-6-1. Impedance of D-T array.**

(a) Calculate the driving-point impedance at the center of each element of an in-phase broadside array of 6 side-by-side $\lambda/2$ elements spaced $\lambda/2$ apart. The currents have a Dolph-Tchebyscheff distribution such that the minor lobes have 1/5 the field intensity of the major lobe.
(b) Design a feed system for the array.

Solution:

(a) From Prob. 5-9-5 and the 6 sources have the distribution:

1	2	3	4	5	6
0.93	0.84	1.00	1.00	0.84	0.93

Normalizing the current for element 1, the distribution is

1.00	0.90	1.08	1.08	0.90	1.00

Using impedance data from Chap. 13 and assuming thin elements, the driving point impedance of element 1 is

$$R_1 = 73 + j43 + 0.9(-12 - j29) + 1.08(3 + j18) + 1.08(-2 - j12) + 0.9(1 + j10) - 1 - j3$$
$$= 73 - 10.8 + 3.2 - 2 + 0.9 - 1 + j(43 - 26 + 19.4 - 13 + 9 - 3) = 63 - j29\ \Omega = R_6 \quad (ans.)$$

In like manner,

$$R_2 = R_5 = 46 - j2\,\Omega, \quad (ans.) \qquad R_3 = R_4 = 53 - j10\,\Omega, \quad (ans.)$$

16-6-3. Square array.

Four isotropic point sources of equal amplitude are arranged at the corners of a square, as in Fig. P16-6-3. If the phases are as indicated by the arrows, determine and plot the far-field patterns.

16-6-3. continued

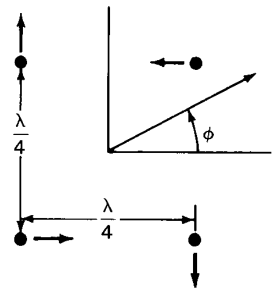


Figure 16-6-3. Square array.

Solution:

Pattern is a rounded square.

$$E_n = 1.00 \text{ at } \phi = 0^{\circ},\ \pm 90^{\circ},\ 180^{\circ}$$
$$E_n = 0.895 \text{ at } \phi = \pm 45^{\circ},\ \pm 135^{\circ}$$

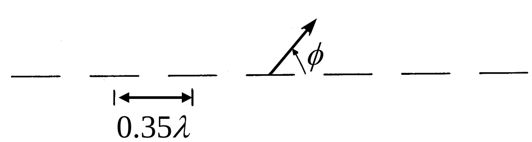
---

*16-6-4. Seven short dipoles. 4-dB angle.

A linear broadside (in-phase) array of 7 short dipoles has a separation of 0.35λ between dipoles. Find the angle from the maximum field for which the field is 4 dB (to nearest 0.1°).

Solution:

The dipoles are assumed to be aligned collinearly so that the pattern of a single dipole is proportional to $\sin\phi$ where ϕ is the angle from the array. Thus



Since the dipoles are in-phase, the maximum field is at $\phi = 90^{\circ}$ or $\phi(E_{\max}) = 90^{\circ}$.

The normalized pattern is given by

$$E_n = \frac{1}{n}\frac{\sin n\psi/2}{\sin\psi/2}\sin\phi \tag{1}$$

where $n = 7$

$$\psi = (2\pi/\lambda)\times 0.35\lambda\cos\phi, \qquad 4 = 20\log x, \quad x = 1.585$$

*16-6-4. continued

Therefore, $E_n(-4\text{ dB}) = 1/1.585 = 0.631$

Setting (1) equal to 0.631, n = 7 and solving (see note below) yields $\phi(-4\text{ dB}) = 78.3^{\circ}$

Angle from $\phi(E_{\max})$ is $90^{\circ} - 78.3^{\circ} = 11.7^{\circ}$ (*ans.*)

Note: Use trial and error to solve (1) for $\phi(-4\text{ dB})$ or calculate pattern with small increments in ϕ.

16-6-5. Square array.

Four identical short dipoles (perpendicular to page) are arranged at the corners of a square $\lambda/2$ on a side. The upper left and lower right dipoles are in the same phase while the 2 dipoles at the other corners are in the opposite phase. If the direction to the right (x direction) corresponds to $\phi = 0°$, find the angles ϕ for all maxima and minima of the field pattern in the plane of the page.

Solution:

Pattern maxima at $\phi = \pm 45^{\circ},\ \pm 135^{\circ}$ Pattern minima at $\phi = 0^{\circ},\ \pm 90^{\circ},\ 180^{\circ}$

---

*16-6-7. Sixteen-source broadside array.

A uniform linear array has 16 isotropic in-phase point sources with at spacing $\lambda/2$. Calculate exactly (a) the half-power beam width, (b) the level of the first sidelobe, (c) the beam solid angle, (d) the beam efficiency, (e) the directivity and (f) the effective aperture.

Solution:

(a) From (5-6-9),

$$E(\text{HP}) = 0.707 = \frac{1}{16}\frac{\sin(1440^{\circ}\cos\phi)}{\sin(90^{\circ}\cos\phi)}$$

By trial and error, $\phi = 86.82^{\circ}$ and $\text{HPBW} = 2(90^{\circ} - 86.82^{\circ}) = 6.36^{\circ} = 6^{\circ}22'$ (*ans.*)

(b) From (5-18-10),

K = 1 (first minor lobe)

$$E_{ML} \cong \frac{1}{16\sin[(2+1)\pi/32]} = 0.215 \text{ or -13.3 dB}$$

***16-6-7. continued**

This is only approximate (becomes exact only for very large n).
To determine the level more accurately, we find the approximate angle for the maximum of the first minor lobe from (5-18-5).

$$\phi_m \cong \cos^{-1}\frac{\pm(2K+1)}{2nd_\lambda} = \cos^{-1}\frac{\pm 3}{2\times 16\times \frac{1}{2}} = 79.2^\circ$$

Then from (5-6-9) we calculate E at angles close to 79.2° and find that E peaks at 79.7° with $E = 0.22012$ or -13.15 dB (*ans.*)

Although (5-18-5) locates the angle where the numerator of (5-18-5) is a maximum (= 1), the denominator is not constant. See discussion of Sec. 5-18 (p. 159) and also Fig. 5-47 (p. 100).

(c) From equation for D in Prob. 5-6-10, the summation term is zero for $d = \lambda/2$ so that D = 16 exactly.

Since $D = 4\pi/\Omega_A$, $\Omega_A = 4\pi/D = 4\pi/16 = \pi/4$ sr (*ans.*)

(d) $\text{HPBW} \cong 1/nd_\lambda = 1/(16\times 0.5) = 1/8$ rad in ϕ direction ,
BW in θ direction $= 2\pi$ rad

Therefore, $\Omega_M = 2\pi\times(1/8) = \pi/4$ sr and $\varepsilon_M = \Omega_M/\Omega_A = 1$ or 100%

This result is too large since with any minor lobes ε_M must be less than unity (or $\Omega_M < \Omega_A$).

For an exact evaluation, we have from Prob. 5-6-10 that

$$\Omega_M = \frac{2}{n^2 d_\lambda}\left[n\pi d_\lambda\cos\theta + \sum_{k=1}^{n-1}\frac{n-k}{k}\sin(2\pi k d_\lambda\cos\theta)\right]_{\theta_1}^{\theta_2}$$

where $\theta_1 = 90^\circ - \gamma_o$

$\theta_2 = 90^\circ + \gamma_o$

$\gamma_o =$ angle to first null

From (5-7-7), $\gamma_o = \sin^{-1}(1/nd_\lambda) = \sin^{-1}[1/(16\times 0.5)] = \sin^{-1}(1/8) = 7.18^\circ$
Therefore, $\theta_1 = 82.82^\circ$ and $\theta_2 = 97.18^\circ$

*16-6-7. continued

Thus

$$\Omega_M = \frac{4}{n^2 d_\lambda}\left[n\pi d_\lambda \cos\theta + \sum_{k=1}^{n-1}\frac{n-k}{k}\sin(2\pi k d_\lambda \cos\theta)\right]_{90^\circ}^{97.18^\circ}$$

$$\left|\Omega_M\right| = \frac{2}{n^2 d_\lambda}\left[0.125 n\pi d_\lambda + \sum_{k=1}^{n-1}\frac{n-k}{k}\sin(0.25\pi k d_\lambda)\right] = \frac{0.5\pi}{16} + \frac{4}{16^2\times 0.5}\left[\frac{15}{1}\sin(0.125\pi)\right.$$

$$+\frac{14}{2}\sin(0.25\pi)+\frac{13}{3}\sin(0.375\pi)+\frac{12}{4}\sin(0.5\pi)+\frac{11}{5}\sin(0.625\pi)+\frac{10}{6}\sin(0.75\pi)$$
$$+\frac{9}{7}\sin(0.875\pi)+\frac{8}{8}\sin(1.00\pi)+\frac{7}{9}\sin(1.25\pi)+\frac{6}{10}\sin(1.375\pi)+\frac{5}{11}\sin(1.5\pi)$$
$$\left.+\frac{4}{12}\sin(1.625\pi)+\frac{3}{13}\sin(1.75\pi)+\frac{2}{14}\sin(1.875\pi)+\frac{1}{15}\sin(2.00\pi)\right]$$

$\left|\Omega_M\right| = 0.0982 + 0.03125$ [5.740 + 4.950 + 4.003 + 3.000 + 2.033 + 1.179 + 0.492
+ 0 – 0.550 – 0.554 – 0.455 – 0.308 – 0.163 – 0.055 +0]
$= 0.0982 + 0.03125 \times 19.312 = 0.702$ sr $= \Omega_M$

$$\varepsilon_M = \Omega_M / \Omega_A = 0.702/(\pi/4) = 0.894 \quad (ans.)$$

By graphical integration (see Example 4-5.6 and Fig. 4-8b) ε_M was found to be approximately 0.90, in good agreement with the above result. The graphical integration took a fraction of the time of the above analytical integration and although less accurate, provided confidence in the result because it is much less susceptible to gross errors.

(e) As noted in (c), $D = 16$ (*ans.*)

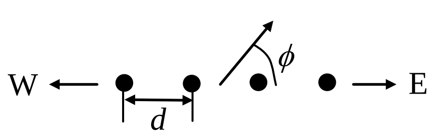
(f) From $D = 4\pi A_{em}/\lambda^2$, $A_{em} = D\lambda^2/4\pi = 16\lambda^2/4\pi = 1.27\lambda^2$ (*ans.*)

---

*16-8-6. Four-tower broadcast array.

A broadcast array has 4 identical vertical towers arranged in an east-west line with a spacing d and progressive phase shift δ. Find (a) d and (b) δ so that there is a maximum field at ϕ =45° (northeast) and a null at ϕ =90° (north). There can be other nulls and maxima, but no maximum can exceed the one at 45°. The distance d must be less than λ/2.

Solution:



***16-8-6. continued**

(a)

Null at $\phi = 90^\circ$ requires that $\delta = \pm 90^\circ$ or $\pm 180^\circ$

For maximum field (fields of all towers in phase) set

$$\psi = \beta d \cos\phi_{\max} + \delta = 0 \quad \text{and} \quad \delta = -90^\circ = -\pi/2 \text{ rad}$$

so $$d = -\frac{\delta}{\beta\cos\phi_{\max}} = \frac{\pi/2}{(2\pi/\lambda)\cos 45^\circ} = 0.354\lambda \quad (\textit{ans.})$$

If $\delta = -180^\circ = -\pi$ rad, $$d = \frac{\pi}{(2\pi/\lambda)\cos 45^\circ} = 0.707\lambda,$$ but this exceeds 0.5λ

(b) Therefore, $\delta = -90^\circ$ (*ans.*)

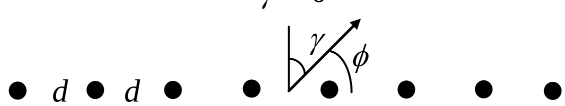
16-10-1. Eight-source scanning array.

A linear broadside array has 8 sources of equal amplitude and $\lambda/2$ spacing. Find the progressive phase shift required to swing the beam (a) 5°, (b) 10° and (c) 15° from the broadside direction. (d) Find BWFN when all sources are in phase.

Solution:

$\phi = 90^\circ$

$\gamma = 0^\circ$



Broadside is set at $\phi = 90^\circ$. Set $\psi = \beta d \cos\phi_{\max} + \delta = 0$

Therefore, $$\delta = -\frac{2\pi}{\lambda} d \cos\phi_{\max} = -\frac{2\pi}{\lambda}\frac{\lambda}{2}\cos 95^\circ = +15.7^\circ \quad (\textit{ans.})$$
$$\delta = -180^\circ \cos 85^\circ = -15.7^\circ$$

Thus, depending on whether δ is $+15.7^\circ$ or -15.7°, the beam is 5° left or right of broadside.

In the same way, we have

16-10-1. continued

(b) $\delta = \pm 31.3^{\circ}$ (*ans.*) for beam 10° left or right of broadside

(c) $\delta = \pm 46.6^{\circ}$ (*ans.*) for beam 15° left or right of broadside

(d) From (5-7-7) the angle of the first null from broadside, when the sources are in-phase $(\delta = 0)$, is given by the complementary angle

$$\gamma_{o} = 90 - \phi_{o} = \sin^{-1}(\lambda / nd) = \sin^{-1}(1/4) = 14.48^{\circ}$$

Therefore, $\text{BWFN} = 2 \times 14.48 = 28.96 \cong 29^{\circ}$ (*ans.*)

From the long broadside array equation (5-7-10),

$$\text{BWFN} \cong 2\lambda / nd = 1/2 \text{ rad} = 180^{\circ} / 2\pi = 28.65^{\circ}$$

The HPBW is a bit less than BWFN/2. For long broadside arrays, we have from Table 5-8 (p.155) that

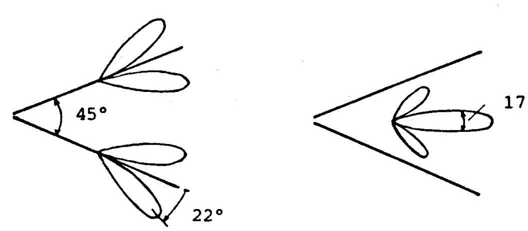
$$\text{HPBW} = 50.8^{\circ} / L_{\lambda} = 50.8 / 3.5 = 14.5^{\circ}$$

***16-16-1. Terminated V. Traveling wave.**

(a) Calculate and plot the far-field pattern of a terminated-V antenna with 5λ legs and 45° included angle.
(b) What is the HPBW?

Solution:

(a) The *field pattern* for each leg of the V is shown at the left and the *combined field pattern* at the right. Minor lobes are neglected except for the principal side lobe of the V.



(b) HPBW $\cong 17^{\circ}$ (*ans.*)

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*16-16-2. E-type rhombic.

Design a maximum *E*-type rhombic antenna for an elevation angle α = 17.5°.

Solution:

From Table 16-1 (p. 590) for a maximum *E* rhombic,

$$H_\lambda = 1/(4\sin\alpha) = 1/(4\sin 17.5^\circ) = 0.83 \quad (ans.)$$
$$\phi = 90^\circ - \alpha = 72.5^\circ \quad (ans.)$$
$$L_\lambda = 0.5/\sin^2\alpha = 5.5 \quad (ans.)$$

16-16-3. Alignment rhombic.

Design an alignment-type rhombic antenna for an elevation angle α = 17.5°.

Solution:

From Table 16-1 (p. 590) for an alignment rhombic,

$$H_\lambda = 1/(4\sin 17.5^\circ) = 0.83 \quad (ans.)$$
$$\phi = 90^\circ - \alpha = 72.5^\circ \quad (ans.)$$
$$L_\lambda = 0.371/\sin^2\alpha = 4.1 \quad (ans.)$$

---

*16-16-4. Compromise rhombic.

Design a compromise-type rhombic antenna for an elevation angle α = 17.5° at a height above ground of λ/2.

Solution:

From Table 16-1 (p. 590) for a compromise rhombic,

$$H_\lambda = 0.5 \quad (ans.)$$
$$\phi = 90^\circ - 17.5^\circ = 72.5^\circ \quad (ans.)$$
$$L_\lambda = \frac{\tan[(\pi L_\lambda)\sin^2 17.5^\circ]}{\sin 17.5^\circ}\left(\frac{1}{2\pi\sin 17.5^\circ} - \frac{0.5}{\tan(\pi\sin 17.5^\circ)}\right)$$

***16-16-4. continued**

or

$$\frac{L_\lambda}{\tan(16.3^{\text{o}} L_\lambda)} = 0.56$$

By trial and error, $L_\lambda = 5.14$ (*ans.*)

16-16-5. Compromise rhombic.

Design a compromise-type rhombic antenna for an elevation angle α = 17.5° with leg length of 3λ.

Solution:

From Table 16-1 (p. 590),

$$H_\lambda = 1/(4\sin 17.5^{\text{o}}) = 0.83 \quad (ans.)$$

$$\phi = \sin^{-1}\left(\frac{3 - 0.371}{3\cos 17.5^{\text{o}}}\right) = 67^{\text{o}} \quad (ans.)$$

***16-16-6. Compromise rhombic.**

Design a compromise-type rhombic antenna for an elevation angle α = 17.5° at a height above ground of λ/2 and a leg length of 3λ.

Solution:

From Table 16-1 (p. 590, bottom entry),

$$\frac{H_\lambda}{\sin\phi \tan\alpha \tan(2\pi H_\lambda \sin\alpha)} = \frac{1}{4\pi\psi} - \frac{L_\lambda}{\tan(\psi 2\pi L_\lambda)}$$

where $\psi = (1 - \sin\phi \cos\alpha)/2$

By trial and error, $\phi \cong 60^{\text{o}}$ (*ans.*)

Chapter 17. Lens Antennas

17-2-1. Dielectric lens.

(a) Design a plano-convex dielectric lens for 5 GHz with a diameter of 10λ. The lens material is to be paraffin and the *F* number is to be unity. Draw the lens cross section.
(b) What type of primary antenna pattern is required to produce a uniform aperture distribution?

Solution:

(a) $\lambda = 3\times 10^8/(5\times 10^9) = 0.06\text{ m} = 60\text{ mm}$

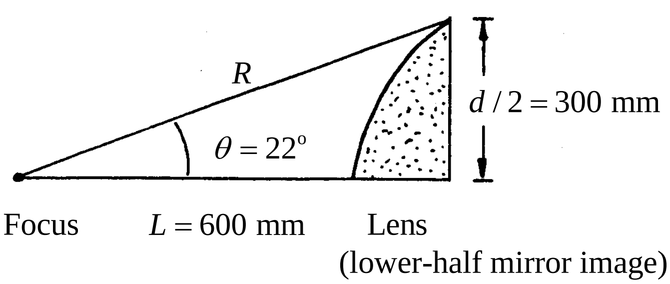
$F = 1$ so $L = d = 10\lambda = 600$ mm ($d =$ diameter)

$n = 1.4$ (see Table 17-1)

Therefore from (17-2-7),

$$R = \frac{(1.4-1)600}{1.4\cos\theta - 1}$$

θ	R	$R\sin\theta$
0°	600 mm	0 mm
10°	634	110
20°	761	260
22°	805	$\approx 300 = d/2$



(b) From (17-2-14), power density at edge of lens is

$$\frac{S_\theta}{S_o} = \frac{(1.4\cos 22^\circ - 1)^3}{(1.4-1)^2(1.4-\cos 22^\circ)} = 0.35 \text{ or } 4.6\text{ dB down}$$

17-2-1. continued

To reduce side lobes, this much or even more taper may be desirable. To obtain a uniform aperture distribution, as requested in the problem, requires a feed antenna at the focus with more radiation (up about 4.6 dB) at 22° off axis than on axis. This is difficult to achieve without unacceptable spillover unless the lens is enclosed in a conical horn, except that at edge locations where *E* is parallel to the edge, *E* must be zero. To reduce this effect a corrugated horn could be used.

17-3-1. Artificial dielectric.

Design an artificial dielectric with relative permittivity of 1.4 for use at 3 GHz when the artificial dielectric consists of (a) copper spheres, (b) copper disks, (c) copper strips.

Solution:

From Table 17-2,

(a) $\varepsilon_r(\text{sphere}) = 1 + 4\pi N a^3 = 1.4$

At 3 GHz, $\lambda = 3\times10^8\ \text{ms}^{-1}/3\times10^9\ \text{Hz} = 0.1\ \text{m} = 100\ \text{mm}$

For $a << \lambda$, take a (radius) $= 5$ mm from which

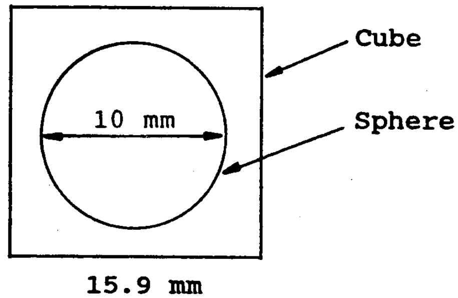
$$N = \frac{1.4-1}{4\pi a^3} = \frac{0.4}{4\pi(5\times10^{-3})^3} = 255,000\ \text{m}^{-3} \quad (\textit{ans.})$$

The dielectric volume per sphere = 1/255,000 $= 4\times10^{-6}\ \text{m}^3$

While the volume of each sphere is given by $(4/3)\pi a^3 = (4/3)\pi(5\times10^{-3})^3 = 5.2\times10^{-7}\ \text{m}^3$

Therefore, $$\frac{\text{volume of dielectric}}{\text{volume of sphere}} = \frac{4\times10^{-6}}{5.2\times10^{-7}} = 7.7$$

$(4\times10^{-6})^{1/3} = 1.59\times10^{-2} = 15.9\ \text{mm}$ = side of cube versus sphere diameter = $2\times5 = 10$ mm



Thus, there is $15.9 - 10 = 5.9$ mm between adjacent spheres in a cubical lattice so there is room for the spheres without touching, provided the lattice uniform.

17-3-1. continued

(b) ε_r(discs)=1+5.33Na^3, and taking a(radius) = 5 mm (diameter = 10 mm),

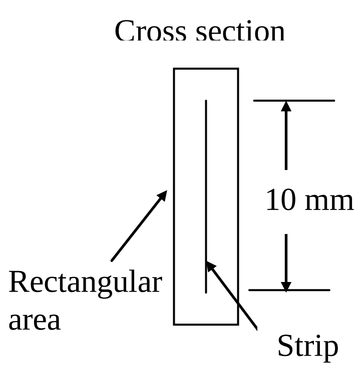
$$N = \frac{1.4-1}{5.33(5\times10^{-3})^3} = 600{,}000 \text{ m}^{-3} \quad (ans.)$$

The dielectric volume per disc $= 1/600{,}000 = 1.7\times10^{-6}$ m^3 for a cube side length of $(1.7\times10^{-6})^{1/3} \cong 12$ mm, so that there is $12-10 = 2$ mm minimum spacing between adjactent discs in a uniform lattice.

(c) ε_r(strips)=1+7.85Nw^2
Taking w (width) = 10 mm,

$$N = \frac{1.4-1}{7.85(10^{-2})^3} = 51{,}000 \text{ m}^{-2} \quad (ans.)$$

as viewed in cross section (see Fig. 17-8a). The square area per strip is then $1/51{,}000 = 2\times10^{-5}$ m^2 for a cross-sectional area side length $(2\times10^{-5})^{1/2} = 4.5$ mm.



This is less than the strip width. However, if the square is changed to a rectangle of the same area with side length ratio of 9 as in the sketch, the edges of the strips are separated by 3.5 mm and the flat sides by 1.5 mm.

The above answers are not unique and are not necessarily the best solutions.

***17-4-1. Unzoned metal-plate lens.**

Design an unzoned plano-concave E-plane type of metal plate lens of the unconstrained type with an aperture 10λ square for use with a 3 GHz line source 10λ long. The source is to be 20λ from the lens (F = 2). Make the index of refraction 0.6.
(a) What should the spacing between the plates be?
(b) Draw the shape of the lens and give dimensions.
(c) What is the bandwidth of the lens if the maximum tolerable path difference is λ/4?

***17-4-1. continued**

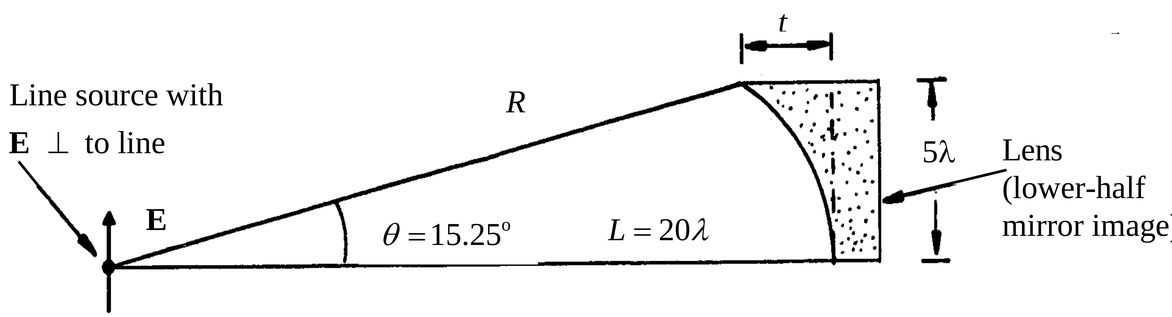
Solution:

$$\lambda = 3\times10^8/(3\times10^9) = 0.1 \text{ m} = 100 \text{ mm}$$
$$n = 0.6, \quad F = 2 \quad \text{so} \quad A = L/2 \quad \text{(Fig.17-13)}$$

(b) Expressing dimensions in λ, we have from (17-4-4)

$$R_\lambda = \frac{(1-n)L_\lambda}{1-n\cos\theta} = \frac{(1-0.6)20}{1-0.6\cos\theta} = \frac{8}{1-0.6\cos\theta}$$

θ	R_λ	$R_\lambda \sin\theta$
0°	20	0
10°	19.6	3.4
15.25°	19.0	5.0



(a) From (17-4-2), $n = [1-(\lambda_o/2b)^2]^{1/2}$ or $b = \lambda_o/2(1-n^2)^{1/2}$

For $n = 0.6$, $b = 0.625\lambda_o = 62.5$ mm (*ans.*)

(b) From (17-4-12), Bandwidth $= 2n\delta_\lambda/(1-n^2)t_\lambda$

$t_\lambda = L_\lambda - R_\lambda \cos\theta = 20 - 19\cos 15.25^\circ = 1.67$

Therefore, Bandwidth $= \dfrac{2\times0.6\times0.25}{(1-0.6^2)1.67} = 0.28$ or 28% (*ans.*)

Chapter 18. Frequency-Selective Surfaces and Periodic Structures. By Ben A. Munk

18-9-1. Unloaded tripole.

Determine the approximate length of the legs of an unloaded trislot operating at f = 15 GHz with
(a) No dielectric substrate.
(b) Dielectric substrate ε_r = 2.2 and thickness 0.50 mm located on both sides of the FSS (use arithmetic average of ε_r and ε_o for ε_{eff}).
(c) Determine D_x just short enough that no grating lobes are present when scanning in the xy – plane for any angle of incidence.

Solution:

(a) For one leg of the tripole, i.e., the monopole length

$$\frac{\lambda}{4} = \frac{2.0}{4} = 0.5 \text{ cm} \qquad (ans.)$$

(b) $\varepsilon_{\text{eff}} = 2.2$ since it is the same on both sides

$$\lambda_{\text{eff}} = \frac{2}{\sqrt{2.2}} = \frac{2}{1.48} = 1.35 \text{ cm}$$

$$\frac{\lambda_{\text{eff}}}{4} = \frac{1.35}{4} = 0.337 \text{ cm} \qquad (ans.)$$

(c) For the scattering case, (15-6-1) can be written as

$$\sin\eta_i + \sin\eta_s = \frac{m}{D_x / \lambda}$$

Since grating lobes start in the plane of the array, $\eta_i = 90^\circ = \eta_s$ and $m = 1$

So $\quad 2 = \lambda / D_x, \qquad D_x = \lambda / 2 = 2/2 = 1 \text{ cm} \qquad (ans.)$

18-9-2. Four-Legged loaded element.

Determine the approximate dimensions for a four legged loaded element operating at f = 15 GHz with
(a) No dielectric substrate.
(b) Dielectric substrate ε_r = 2.2 and thickness 0.50 mm located on only one side the FSS (estimate ε_{eff}).

18-9-2. continued

(c) Leave a separation of 1 mm between adjacent elements (rectangular grid); determine the lowest onset frequency for grating lobes for any angle of incidence.

Solution:

(a) For a loop type element, the size should be $\lambda_{\text{eff}} / 4$ across.

So $$\lambda / 4 = 2.0 / 4 = 0.50 \text{ cm} \quad (\textit{ans.})$$

(b) $$\varepsilon_{\text{eff}} = \frac{2.2 + 1.0}{2} = \frac{3.2}{2} = 1.6, \qquad \lambda_{\text{eff}} = 2 / \sqrt{1.6} = 2 / 1.265 = 1.58 \text{ cm}$$

$$\lambda_{\text{eff}} / 4 = 0.4 \text{ cm} \quad (\textit{ans.})$$

(c) As in Prob. 18-9-1, the condition we want to meet is

$$2 = \lambda / D_x \quad \text{or} \quad \lambda = 2D_x$$

With no dielectric, $$D_x = 0.5 + 0.1 = 0.6 \text{ cm}$$

so $$\lambda = 1.2 \text{ cm}, \qquad f = \frac{3 \times 10^8}{1.2 \times 10^{-2}} = 25 \text{ GHz} \quad (\textit{ans.})$$

With dielectric, $$D_x = 0.4 + 0.1 = 0.5 \text{ cm}$$

so $$\lambda = 1 \text{ cm}, \qquad f = \frac{3 \times 10^8}{1 \times 10^{-2}} = 30 \text{ GHz} \quad (\textit{ans.})$$

Chapter 19. Practical Design Considerations of Large Aperture Antennas

*19-1-3. Efficiency of rectangular aperture with partial taper.

Calculate the aperture efficiency and directivity of an antenna with rectangular aperture x_1y_1 with a uniform field distribution in the y direction and a cosine field distribution in the x direction (zero at edges, maximum at center) if $x_1 = 20\lambda$ and $y_1 = 10\lambda$.

Solution:

From Prob. 19-1-6 solution,

(a) $\varepsilon_{ap} = 0.81$ or 81% (*ans.*)

(b) $D = 4\pi \times 10 \times 20 \times 0.81 = 2036$ or 33 dBi (*ans.*)

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*19-1-4. Efficiency of rectangular aperture with full taper.

Repeat Prob. 19-1-3 for the case where the aperture field has a cosine distribution in both the x and y directions.

Solution:

From Prob. 19-1-7 solution,

(a) $\varepsilon_{ap} = 0.657 \cong 66\%$ (*ans.*)

(b) $D = 4\pi \times 10 \times 20 \times 0.657 = 1651$ or 32 dBi (*ans.*)

19-1-5. Efficiency of aperture with phase ripple.

A square unidirectional aperture (x_1y_1) is 10λ on a side and has a design distribution for the electric field which is uniform in the x direction but triangular in the y direction with maximum at the center and zero at the edges. Design phase is constant across the aperture. However, in the actual aperture distribution there is a plus-and-minus-30° sinusoidal phase variation in the x direction with a phase cycle per wavelength. Calculate (a) the design directivity, (b) the utilization factor, (c) the actual directivity, (d) the achievement factor, (e) the effective aperture and (f) the aperture efficiency.

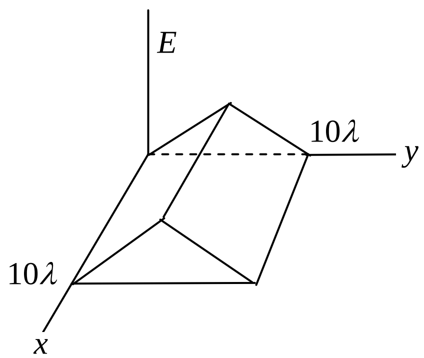
Solution:

Referring to Sec. 19-1,

19-1-5. continued

Let $$\underbrace{E'(x,y)_{\max}}_{\text{Design field}} = \underbrace{E(x,y)_{\max}}_{\text{Actual field}} = 1$$

Design: $$E'_{av} = \frac{1}{A_p}\iint E'(x,y)\,dxdy \qquad (1)$$



$$= \frac{2}{A_p}\int_0^{10\lambda}\int_0^{5\lambda}\frac{y}{5\lambda}\,dxdy = \frac{1}{2}$$

Note: This result can be deduced directly from the figure by noting that average height of triangle is ½ max.

(b) Utilization factor, k_u :

$$k_u = \frac{1}{\frac{1}{A_p}\iint\left(\frac{E'(x,y)}{E'_{av}}\right)\left(\frac{E'(x,y)}{E'_{av}}\right)^* dxdy} \qquad (2)$$

$$= \frac{1}{\frac{1}{A_p}\frac{2}{(1/2)^2}\int_0^{10\lambda}\int_0^{5\lambda}\left(\frac{y}{5\lambda}\right)^2 dxdy} = \frac{3}{4} \quad (ans.) \qquad (3)$$

Note that for in-phase fields (19-1-50) is a simplified form of (2) giving

$$\frac{E_{av}^2}{\left(E^2\right)_{av}} = \frac{(1/2)^2}{1/3} = 3/4 = k_u \text{ as in (3)} \qquad (4)$$

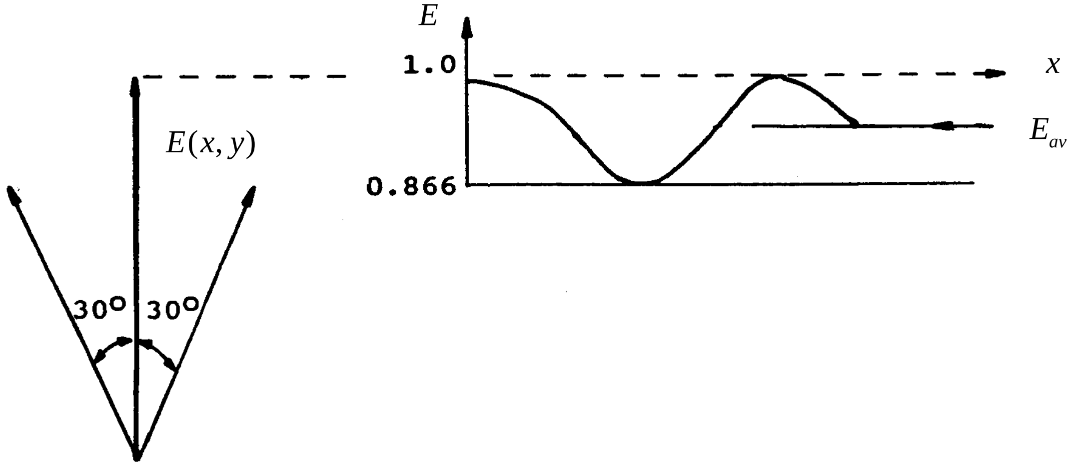
(a) Design directivity, D (design):

$$D(\text{design}) = \frac{4\pi}{\lambda^2}A_p k_u = \frac{4\pi}{\lambda^2}(100\lambda^2)(3/4) = 940 \quad (ans.) \qquad (5)$$

Turning attention now to the effect of the phase variation:

$$E_{av} = \frac{2}{A_p}\int_0^{10\lambda}\cos\left(\frac{\pi}{6}\sin\frac{2\pi x}{\lambda}\right)dx\int_0^{5\lambda}\frac{y}{5\lambda}dy = \frac{1}{2}(0.933) \qquad (6)$$

19-1-5. continued



Note that from figures above, $2E_{av} \cong \dfrac{1+0.866}{2} = 0.933$

(d) Achievement factor, k_a :

$$k_a = \frac{\dfrac{1}{A_p}\iint\left(\dfrac{E'(x,y)}{E'_{av}}\right)\left(\dfrac{E'(x,y)}{E'_{av}}\right)^* dxdy}{\dfrac{1}{A_p}\iint\left(\dfrac{E(x,y)}{E_{av}}\right)\left(\dfrac{E(x,y)}{E_{av}}\right)^* dxdy}$$

$$k_a = \frac{4/3}{\dfrac{1}{A_p}\dfrac{1}{[(1/2)0.933]^2}2\iint (y/5\lambda)^2\,dxdy} = 0.87 \quad (ans.) \tag{7}$$

where $E(x)E^*(x) = 1$

Note that gain loss due to total phase variation across aperture (*not* surface deviation) is from (19-2-3)

$$k_g = \cos^2\left(360^\circ\frac{\delta'}{\lambda}\right), \quad \text{where } \frac{\delta'}{\lambda} = \frac{30^\circ \times 0.707}{360^\circ} = \frac{21.2^\circ}{360^\circ}$$

or $\quad k_g = \cos^2 21.2^\circ = 0.87 = k_a$ as in (7)

19-1-5. continued

(c) Directivity:

$$D = \frac{4\pi A_p}{\lambda^2} k_u k_a = 4\pi \times 100 \times \frac{3}{4} \times 0.87 = 818 \quad (ans.)$$

(e) Effective aperture, A_e :

$$A_e = \frac{\lambda^2}{4\pi} D = A_p k_u k_a = 65.2\lambda^2 \quad (ans.)$$

(f) Aperture efficiency, ε_{ap} :

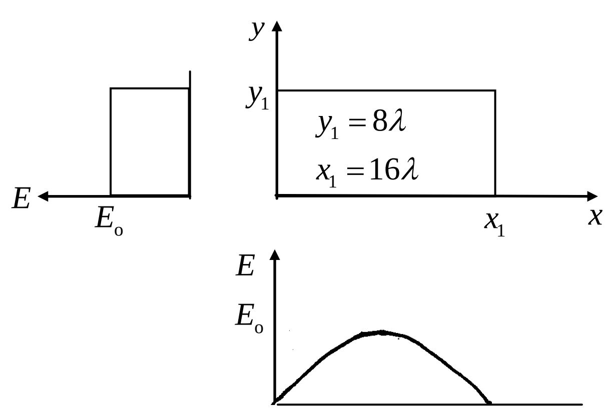
$$\varepsilon_{ap} = k_a k_u = 0.65 \quad (ans.)$$

Note: Although phase errors with small correlation distance $(\cong \lambda)$ as in Prob. 19-1-5 reduce the directivity and, hence, increase Ω_A, the HPBW is not affected appreciable. However, for larger correlation distances $(>> \lambda)$ the scattered radiation becomes more directive, causing the near side lobes to increase and ultimately the main beam and the HPBW may be affected.

***19-1-6. Rectangular aperture. Cosine taper.**

An antenna with rectangular aperture $x_1 y_1$ has a uniform field in the y direction and a cosine field distribution in the x direction (zero at edges, maximum at center). If $x_1 = 16\lambda$ and $y_1 = 8\lambda$, calculate (a) the aperture efficiency and (b) the directivity.

Solution:



$$E(x) = E_o \sin \frac{\pi x}{x_1}$$

***19-1-6. continued**

Although the taper in the *x*-direction is described as a cosine taper, let us represent it by a sine function as follows:

(a) From (19-1-50),
$$\varepsilon_{ap} = \frac{\left(E(x)_{av}\right)^2}{\left(E^2(x)\right)_{av}}$$

where
$$E(x)_{av} = \frac{1}{x_1}\int_0^{x_1} E(x)dx = \frac{E_o}{x_1}\int_0^{x_1} \sin\frac{\pi x}{x_1}dx = \frac{E_o}{x_1}\left(\frac{-x_1}{\pi}\right)\cos\frac{\pi x}{x_1}\Big|_0^{x_1} = \frac{2E_o}{\pi}$$

$$[E^2(x)]_{av} = \frac{1}{x_1}\int_0^{x_1} E^2(x)dx = \frac{E_o^2}{x_1}\int_0^{x_1}\sin^2\frac{\pi x}{x_1}dx = \frac{E_o^2}{2}$$

Therefore,
$$\varepsilon_{ap} = \frac{\left(\frac{2}{\pi}E_o\right)^2}{\frac{1}{2}E_o^2} = \frac{8}{\pi^2} = 0.811 \text{ or } \cong 81\%$$

(b)
$$A_e = \varepsilon_{ap}A_{em} = 0.81\times 8\lambda\times 16\lambda = 103.7\lambda^2$$

$$D = \frac{4\pi A_e}{\lambda^2} = (4\pi\times 103.7\lambda^2)/\lambda^2 = 1304 \text{ or } 31.2 \text{ dBi}$$

19-1-7. Rectangular aperture. Cosine tapers.

Repeat Prob. 19-1-6 for the case where the aperture field has a cosine distribution in both the *x* and *y* directions.

Solution:

Let the distribution be represented by

$$E(x,y) = E_o\sin\frac{\pi x}{x_1}\sin\frac{\pi x}{y_1}$$

19-1-7. continued

(a)
$$E(x,y)_{av}=\frac{1}{x_1y_1}\int_0^{x_1}\int_0^{y_1}E(x,y)dxdy=\frac{E_o}{x_1y_1}\int_0^{x_1}\sin\frac{\pi x}{x_1}dx\int_0^{y_1}\sin\frac{\pi y}{y_1}dy$$
$$=\frac{E_o}{x_1y_1}\left[-\frac{x_1}{\pi}\cos\frac{\pi x}{x_1}\right]_0^{x_1}\left[-\frac{y_1}{\pi}\cos\frac{\pi y}{y_1}\right]_0^{y_1}=\frac{4E_o}{\pi^2}$$

$$[E^2(x,y)]_{av}=\frac{1}{x_1y_1}\int_0^{x_1}\int_0^{y_1}E^2(x,y)dxdy=\frac{E_o^2}{x_1y_1}\int_0^{x_1}\sin^2\frac{\pi x}{x_1}dx\int_0^{y_1}\sin^2\frac{\pi y}{y_1}dy=\frac{E_o^2}{4}$$

Therefore,
$$\varepsilon_{ap}=\frac{\left(\frac{4}{\pi^2}E_o\right)^2}{\frac{1}{4}E_o^2}=\frac{16\times 4}{\pi^4}=0.657 \text{ or } \cong 66\%$$

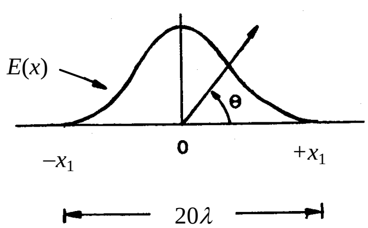
(b)
$$A_e=\varepsilon_{ap}A_{em}=0.657\times 8\lambda\times 16\lambda=84.1\ \lambda^2$$

$$D=\frac{4\pi A_e}{\lambda^2}=\frac{4\pi\times 84.1\lambda^2}{\lambda^2}=1057 \text{ or } 30.2 \text{ dBi}$$

***19-1-8. A 20λ line source. Cosine-squared taper.**

(a) Calculate and plot the far-field pattern of a continuous in-phase line source 20λ long with cosine-squared field distribution.
(b) What is the HPBW?

Solution:



The field along the line may be represented by
$$E(x)=\cos^2\frac{\pi x}{2x_1}$$

(a) The field pattern $E(\theta)$ is the Fourier transform of the distribution $E(x)$ along the line. Thus,
$$E(\theta)=\int_{-x_1}^{+x_1}E(x)e^{j(2\pi x/\lambda)\cos\theta}dx=\int_{-10_\lambda}^{+10_\lambda}\cos^2[(\pi/2)(x/10\lambda)]e^{j(2\pi x/\lambda)\cos\theta}dx$$

***19-1-8. continued**

Let $s = x/\lambda$ from which $dx = \lambda ds$

Then

$$E(\theta) = \lambda\int_{-10}^{+10}\cos^2(\pi s/20)e^{j2\pi s\cos\theta}ds = \lambda\int_{-10}^{+10}\frac{1+\cos(\pi s/10)}{2}e^{j2\pi s\cos\theta}ds$$

$$= \frac{\lambda}{2}\int_{-10}^{+10}e^{j2\pi s\cos\theta}ds + \frac{\lambda}{2}\int_{-10}^{+10}\cos(\pi s/10)e^{j2\pi s\cos\theta}ds$$

and

$$E_n(\theta) = \frac{\sin(20\pi\cos\theta)}{2\pi\cos\theta} + \frac{1}{2}\left[\frac{\sin(20\cos\theta+1)\pi}{[2\cos\theta+(1/10)]\pi} + \frac{\sin(20\cos\theta-1)\pi}{[2\cos\theta-(1/10)]\pi}\right]$$

$$= \frac{\sin(20\pi\cos\theta)}{2\pi\cos\theta}\left(1-\frac{4\cos^2\theta}{4\cos^2\theta-0.01}\right) \quad (ans.) \qquad (1)$$

(b) From graph or by trial and error from (1),

$$\text{HPBW} = 2(90^\circ - 87.9^\circ) = 4.2^\circ \quad (ans.)$$

From Table 4-3 for a 20λ *uniform* aperture,

$$\text{HPBW} = 50.8/L_\lambda = 50.8/20 = 2.5^\circ$$

Thus, the cosine-squared aperture distribution has nearly twice the HPBW of the uniform aperture, but its side lobes are much lower with first side lobe down 31 dB as compared to only 13 dB down for a 20λ uniform aperture distribution.

Chapter 21. Antennas for Special Applications

21-4-2. Horizontal dipole above imperfect ground.

Calculate the vertical plane field pattern broadside to a horizontal $\lambda/2$ dipole antenna $\lambda/4$ above actual homogeneous ground with constants $\varepsilon_r' = 12$ and $\sigma = 2 \times 10^{-3}\ \Omega^{-1}\ \text{m}^{-1}$ at (a) 100 kHz and (b) 100 MHz.

Solution:

$$\mu = \mu_o, \qquad \varepsilon_r' = 12, \qquad \sigma = 2\times10^{-3}\ \Omega^{-1}\text{m}^{-1}, \quad h = \lambda/4$$

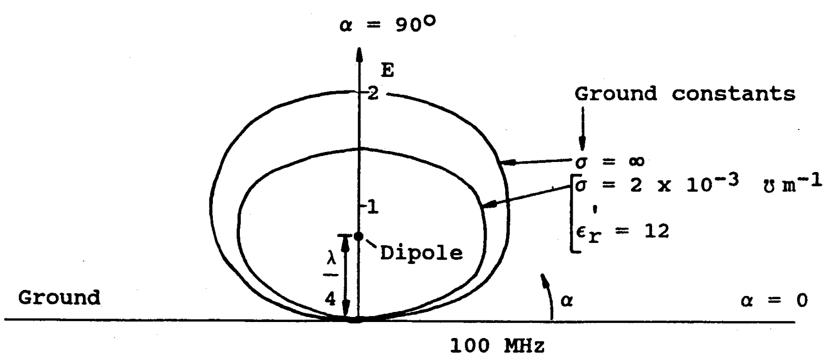
$$\rho_\perp = \frac{\sin\alpha - (\varepsilon_r - \cos^2\alpha)^{1/2}}{\sin\alpha + (\varepsilon_r - \cos^2\alpha)^{1/2}} \tag{1}$$

$$E_\perp = 1 + \rho_\perp \angle \underline{\cos(2\beta h \sin\alpha) + j\sin(2\beta h\sin\alpha)} \tag{2}$$

(b) $$\varepsilon_r'' = \frac{\sigma}{\omega\varepsilon_o} = \frac{2\times10^{-3}}{2\pi\, 10^8\ 8.85\times10^{-12}} = 0.36 \text{ at 100 MHz}$$

$$\varepsilon_r = \varepsilon_r' - j\varepsilon_r'' = 12 - j0.4 \cong 12$$

Introducing ε_r into (1), (1) into (2) and evaluating (2) as a function of α results in the pattern shown. The pattern for perfectly conducting ground $(\sigma = \infty)$ is also shown for comparison (same as pattern of 2 isotropic sources in phase opposition and spaced $\lambda/2$). For perfectly conducting ground the field doubles ($E = 2$) at the zenith ($\alpha = 90^\circ$), but with the actual ground of the problem, it is reduced to about 1.55 (down 2.2 dB) because of partial absorption of the wave reflected from the ground.



(a) At 100 kHz, $\varepsilon_r'' = 360$ and $\rho_\perp \cong -1$, so the pattern is approximately the same as for $\sigma = \infty$ in the sketch.

21-9-1. Square loop.

Calculate and plot the far-field pattern in the plane of a loop antenna consisting of four $\lambda/2$ center-fed dipoles with sinusoidal current distribution arranged to form a square $\lambda/2$ on a side. The dipoles are all in phase around the square.

Solution:

Squarish pattern with rounded edges.

Maximum-to-minimum field ratio = 1.14

***21-9-3. DF and monopulse.**

Many direction-finding (DF) antennas consist of small (in terms of λ) loops giving a figure-of-eight pattern as in Fig. P21-9-3a. Although the null is sharp the bearing (direction of transmitter signal) may have considerable uncertainty unless the S/N ratio is large. To resolve the 180° ambiguity of the loop pattern, an auxiliary antenna may be used with the loop to give a cardiod pattern with broad maximum in the signal direction and null in the opposite direction.

The maximum of a beam antenna pattern, as in Fig. P21-9-3b, can be employed to obtain a bearing with the advantage of a higher S/N ratio but with reduced pattern change per unit angle. However, if 2 receivers and 2 displace beams are used, as in Fig. P21-9-3c, a large power-pattern change can be combined with a high S/N ratio. An arrangement of this kind for receiving radar echo signals can give bearing information on a single pulse (*monopulse radar*). If the power received on beam 1 is P_1 and on beam 2 is P_2, then if $P_2 > P_1$ the bearing is to the right. If $P_1 > P_2$ the bearing is to the left and if $P_1 = P_2$ the bearing is on axis (boresight). (With 4 antennas, bearing information left-right *and* up-down can be obtained.)

(a) If the power pattern is proportional to $\cos^4 \theta$, as in Fig. P21-9-3c, determine P_2/P_1 if the interbeam (squint) angle $\alpha = 40°$ for $\Delta\theta = 5$ and 10°.

(b) Repeat for $\alpha = 50°$.

(c) Determine the P_0/P_1 of the single power pattern of Fig. P21-9-3b for $\Delta\theta = 5$ and 10° if the power pattern is also proportional to $\cos^4 \theta$.

(d) Tabulate the results for comparison and indicate any improvement of the double over the single beam.

***21-9-3. continued**

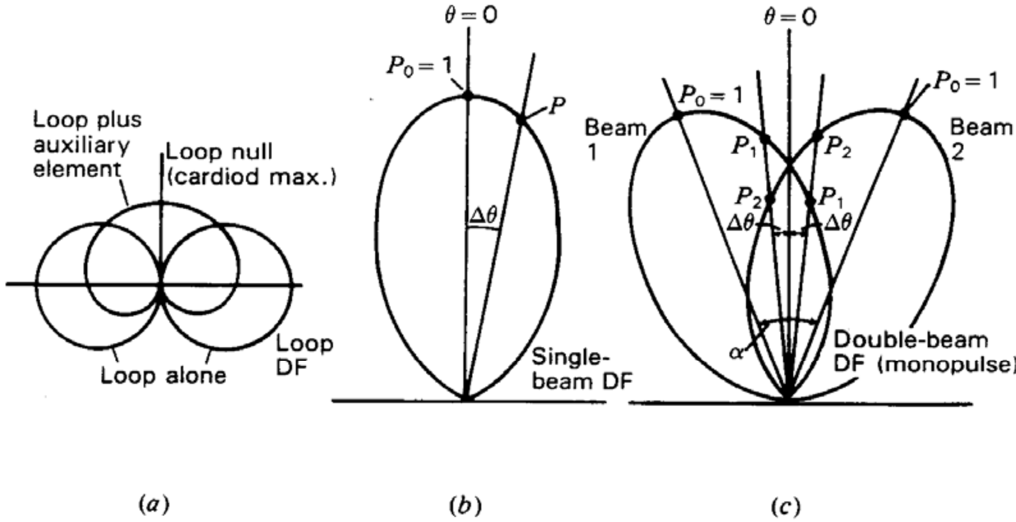


Figure P21-9-3. Direction finding: (a) with loop mull, (b) with beam maximum and (c) with double beam (monopulse).

Solution:

(a) $\alpha = 40^{\circ}$

$$\Delta\theta = 5^{\circ}, \qquad \frac{P_2}{P_1} = \frac{\cos^4(20^{\circ} - 5^{\circ})}{\cos^4(20^{\circ} + 5^{\circ})} = 1.290 \text{ or } 1.1 \text{ dB}$$

$$\Delta\theta = 10^{\circ}, \qquad \frac{P_2}{P_1} = \frac{\cos^4(20^{\circ} - 10^{\circ})}{\cos^4(20^{\circ} + 10^{\circ})} = 1.672 \text{ or } 2.2 \text{ dB}$$

(b) $\alpha = 50^{\circ}$

$$\Delta\theta = 5^{\circ}, \qquad \frac{P_2}{P_1} = \frac{\cos^4(25^{\circ} - 5^{\circ})}{\cos^4(25^{\circ} + 5^{\circ})} = 1.386 \text{ or } 1.4 \text{ dB}$$

$$\Delta\theta = 10^{\circ}, \qquad \frac{P_2}{P_1} = \frac{\cos^4(25^{\circ} - 10^{\circ})}{\cos^4(25^{\circ} + 10^{\circ})} = 1.933 \text{ or } 2.9 \text{ dB}$$

(c)

$$\alpha = 5^{\circ}, \qquad \frac{P_0}{P_1} = \frac{\cos^4 0^{\circ}}{\cos^4 5^{\circ}} = 1.015 \text{ or } 0.06 \text{ dB}$$

$$\alpha = 10^{\circ}, \qquad \frac{P_0}{P_1} = \frac{\cos^4 0^{\circ}}{\cos^4 10^{\circ}} = 1.063 \text{ or } 0.26 \text{ dB}$$

(d) Over 1 dB more at 5° and about 2 dB more at 10°.

***21-10-1. Overland TV for HP, VP and CP.**

(a) A typical overland microwave communications circuit for AM, FM or TV between a transmitter on a tall building and a distant receiver involves 2 paths of transmission, one direct path (length r_o) and one an indirect path with ground reflection (length $r_1 + r_2$), as suggested in Fig. P21-10-1. Let h_1 = 300 m and d = 5 km. For a frequency of 100 MHz calculate the ratio of the power received per unit area to the transmitted power as a function of the height h_2 of the receiving antenna. Plot these results in decibels as abscissa versus h_2 as ordinate for 3 cases with transmitting and receiving antennas both (1) vertically polarized, (2) horizontally polarized and (3) right-circularly polarized for h_2 values from 0 to 100 m. Assume that the transmitting antenna is isotropic and that the receiving antennas are also isotropic (all have the same effective aperture). Consider that the ground is flat and perfectly conducting.
(b) Compare the results for the 3 types of polarization, and show that circular polarization is best from the standpoint of both the noncriticalness of the height h_2 and the absence of echo or ghost signals. Thus, for horizontal or vertical polarization the direct and ground-reflected waves may cancel at certain heights while at other heights, where they reinforce, the images on the TV screen may be objectionable because the time difference via the 2 paths produces a double image (a direct image and its ghost).
(c) Extend the comparison of (b) to consider the effect of other buildings or structures that may produce additional paths of transmission.

Note that direct satellite-to-earth TV downlinks are substantially free of these reflection and ghost image effects.

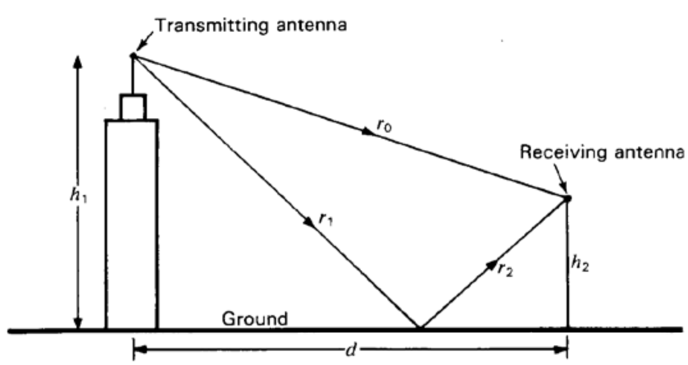


Figure P21-10-1. Overland microwave communication circuit.

Solution:

(a) and (b) answers in Appendix F, pg. 919-920.

(c) The effect of reflection from other buildings or structures (or from aircraft) can be minimized by the use of CP transmit and receive antennas of the same hand, particularly when these structures are many wavelengths in size and reflection is specular. Trouble-some reflections can be reduced by placing non-reflecting absorbers on the structure.

***21-12-1. Signaling to submerged submarines.**

Calculate the depths at which a 1 μV m^{-1} field will be obtained with E at the surface equal to 1 V m^{-1} at frequencies of 1, 10, 100 and 1000 kHz. What combination of frequency and antennas is most suitable?

Solution:

From Table A-6, take $\varepsilon_r' = 80$ and $\sigma = 4$ for sea water. At the highest frequency (1000 kHz), $\sigma >> \omega\varepsilon$, so that $\alpha = \sqrt{\omega\mu\sigma / 2}$ can be used at all four frequencies.

At 1 kHz,

$$\alpha = \sqrt{\frac{2\pi\times10^3\times4\pi\times10^{-7}\times4}{2}} = 0.13\ \text{Npm}^{-1}$$

Since

$$\frac{E}{E_o} = 10^{-6} = e^{-\alpha y}, \qquad y = \frac{6}{\alpha}\log e = \frac{13.8}{\alpha}$$

and

at 1 kHZ,	depth $y = 106$ m	
at 10 kHz,	$y = 35$ m	(*ans.*)
at 100 kHz,	$y = 11$ m	
at 1000 kHz	$y = 3.5$ m	

From the standpoint of frequency, 1 kHz gives greatest depth. However, from (21-2-3) the radiation resistance of a monopole antenna as a function of its height (h_p) is

$$R_r = 400\left(\frac{h_p}{\lambda}\right)^2\ \Omega$$

For $h_p = 300$ m at 1 kHz

$$R_r = 400\left(\frac{300}{3\times10^5}\right)^2 = 4\times10^{-4}\ \Omega \qquad (\text{or } 400\ \mu\Omega)$$

With such a small radiation resistance, radiation efficiency will be poor. At 10 kHz the radiation resistance is a hundred times greater. A practical choice involves a compromise of sea water loss, land (transmitting) antenna effective height, and submarine antenna efficiency as a function of the frequency.

*21-13-1. Surface-wave powers.

A 100-MHz wave is traveling parallel to a copper sheet ($|Z_c|$ = 3.7 x 10^{-3} Ω) with **E** (= 100 V m^{-1} rms) perpendicular to the sheet. Find (a) the Poynting vector (watts per square meter) parallel to sheet and (b) the Poynting vector into the sheet.

Solution:

(a) $$S_{\parallel \text{ to sheet}} = \frac{E_y^2}{Z_o} = \frac{100^2}{377} = 26.5 \text{ Wm}^{-2} \quad (ans.)$$

(b) $$S_{\text{into sheet}} = H^2 R_e Z_c = \frac{E^2}{Z_o^2} R_e Z_c = \left(\frac{100}{377}\right)^2 \frac{3.7\times10^{-3}}{\sqrt{2}} = 182 \ \mu\text{Wm}^{-2} \quad (ans.)$$

21-13-2. Surface-wave powers.

A 100-MHz wave is traveling parallel to a conducting sheet for which $|Z_c|$ = 0.02 Ω. If **E** is perpendicular to the sheet and equal to 150 V m^{-1} (rms), find (a) watts per square meter traveling parallel to the sheet and (b) watts per square meter into the sheet.

Solution:

(a) $$S_{\parallel \text{ to sheet}} = \frac{E_y^2}{Z_o} = \frac{150^2}{377} = 59.7 \text{ Wm}^{-2} \quad (ans.)$$

(b) $$S_{\text{into sheet}} = H^2 R_e Z_c = \frac{E^2}{Z_o^2} R_e Z_c = \left(\frac{150}{377}\right)^2 \frac{2\times10^{-2}}{\sqrt{2}} = 2.24 \text{ mWm}^{-2} \quad (ans.)$$

---

*21-13-3. Surface-wave power.

A plane 3-GHz wave in air is traveling parallel to the boundary of a conducting medium with **H** parallel to the boundary. The constants for the conducting medium are $\sigma = 10^7$ Ω^{-1} m^{-1} and $\mu_r = \varepsilon_r = 1$. If the traveling-wave rms electric field E = 75 mV m^{-1}, find the average power per unit area lost in the conducting medium.

Solution:

$$S_{\text{into sheet}} = H^2 R_e Z_c = \frac{E^2}{Z_o^2} R_e Z_c$$

$$R_e Z_c = \sqrt{\frac{\mu_o \omega}{2\sigma}} = \sqrt{\frac{4\pi\times10^{-7} 2\pi\times3\times10^9}{2\times10^7}} = 0.034 \ \Omega$$

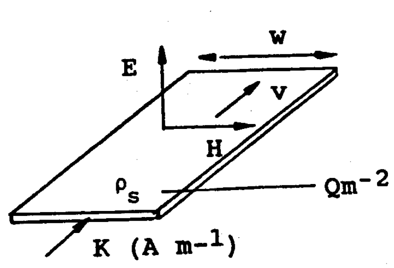
***21-13-3. continued**

Therefore, $$S_{\text{into sheet}} = \left(\frac{0.75}{377}\right)^2 0.034 = 1.35 \text{ nWm}^{-2} \quad (\textit{ans.})$$

21-13-4. Surface-wave current sheet.

A TEM wave is traveling in air parallel to the plane boundary of a conducting medium. Show that if $K = \rho_s v$, where K is the sheet-current density in amperes per meter, ρ_s is the surface charge density in coulombs per square meter and v the velocity of the wave in meters per second, it follows that $K = H$, where H is the magnitude of the **H** field of the wave.

Solution:



$$K = \rho_s v = \frac{\text{Q}}{\text{m}^2}\frac{\text{m}}{\text{s}} = \frac{\text{Q}}{\text{s}}\frac{1}{\text{m}} = \text{A m}^{-1} = \text{I m}^{-1}$$

By Amperes's law, integral of H around strip of width w equals current enclosed or

$$\oint \mathbf{H} \cdot d\mathbf{s} = I = wK$$

$$wH = wK \text{ and } H = K \quad (\text{note that } H \perp K) \quad (\textit{ans.})$$

---

*21-13-6. Coated-surface wave cutoff.

A perfectly conducting flat sheet of large extent has a dielectric coating (ε_r = 3) of thickness d = 5 mm. Find the cutoff frequency for the TM_o (dominant) mode and its attenuation per unit distance.

Solution:

$$\alpha = \frac{2\pi}{\lambda_\text{o}}\sqrt{3-1} = \frac{8.89}{\lambda_\text{o}} \text{ Np m}^{-1} \quad (\textit{ans.}) \qquad f_c = 0 \quad (\textit{ans.})$$

Chapter 23. Baluns, etc. By Ben A. Munk

23-3-1. Balun 200 Ω, antenna 70 Ω.

A Type III balun has the characteristic impedance equal to Z_{cp} = 200 Ω and the electrical length is equal to l_p = 7.5 cm. It is connected to an antenna with impedance Z_A = 70 Ω.
(a) Find the balun impedance jX_p at f = 500, 1000 and 1500 MHz.
(b) Calculate the parallel impedances $Z_A \parallel jX_p$ at 500 1000 and 1500 MHz and plot them in a Smith Chart normalized to Z_o = 50 Ω. Check that all these impedances lie on a circle with a diameter spanning over (0,0) and Z_A = 70 Ω. Alternatively, you may determine $Z_A \parallel jX_p$ graphically in a Smith Chart.
(c) Explain what effect it would have on the bandwidth if we changed Z_{cp} to 150 Ω or 250 Ω.

Solution:

(a) $\lambda_L = \dfrac{3\times10^8}{5\times10^8} = 60 \text{ cm},\qquad \dfrac{\ell_p}{\lambda_L} = \dfrac{7.5}{60} = 0.125$

$$\lambda_M = \frac{3\times10^8}{1\times10^9} = 30 \text{ cm},\qquad \frac{\ell_p}{\lambda_M} = \frac{7.5}{30} = 0.250$$

$$\lambda_H = \frac{3\times10^8}{1.5\times10^9} = 20 \text{ cm},\qquad \frac{\ell_p}{\lambda_H} = \frac{7.5}{20} = 0.375$$

From the Smith Chart, by moving the number of wavelengths around from the short (zero) position, it is found that

for $f = 500$ MHz, $\quad jX_p = j200\ \Omega$ (*ans.*)

for $f = 1000$ MHz, $\quad jX_p = j\infty\ \Omega$ (*ans.*)

for $f = 1500$ MHz, $\quad jX_p = -j200\ \Omega$ (*ans.*)

Alternatively, the transmission line equation can be used.

(b) $Z_A \parallel jX_p = \dfrac{Z_A jX_p}{Z_A + jX_p} = \dfrac{Z_A X_p^2 + jZ_A^2 X_p}{Z_A^2 + X_p^2}$

For mid frequency, $X_p = \infty,\qquad Z_A \parallel X_p = Z_A$

23-3-1. continued

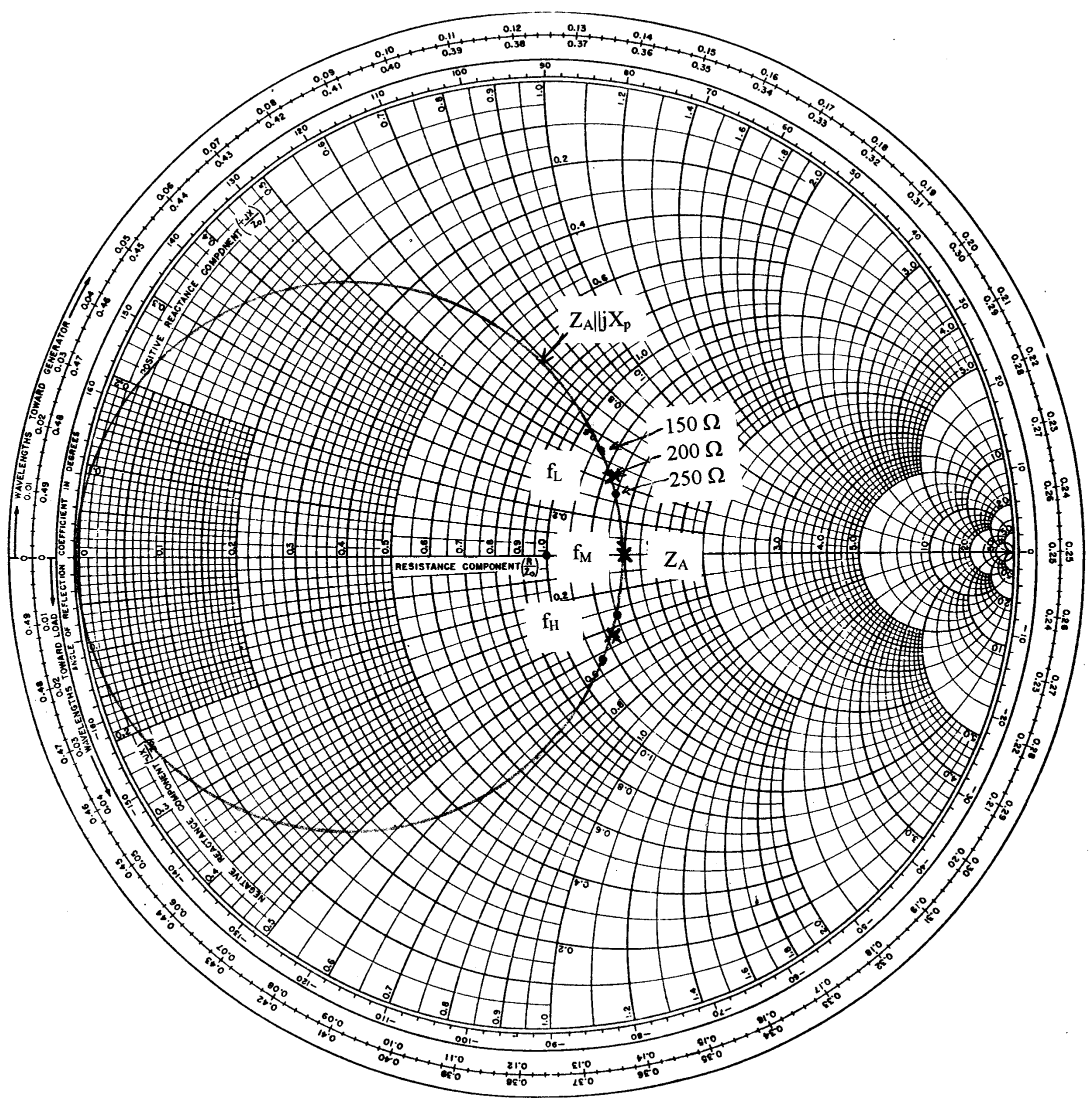
For low frequency, $X_p = 200$, $\quad Z_A \parallel jX_p = \dfrac{70 \times 200^2 + j70^2 \times 200}{70^2 + 200^2} = 62.36 + j21.83$

Normalized to $Z_o = 50$, $\quad \dfrac{Z_A \parallel jX_p}{Z_o} = 1.25 + j0.44$

Similarly, for high frequency, $X_p = -200$, $\quad Z_A \parallel jX_p = 62.36 - j21.83$

$$\frac{Z_A \parallel jX_p}{Z_o} = 1.25 - j0.44$$

See accompanying figure of Smith Chart



23-3-1. continued

To find these values by the Smith Chart, it is a matter of adding the values as admittances. This is accomplished by finding their position as impedances, projecting the values through the origin an equal distance, adding them, then projecting the added values an equal distance to the other side of the origin.

(c) For $Z_{cp} = 150\ \Omega$, it is found that $Z_A \; \pm jX_p = 57.48 \pm j26.82$

or normalized as $= 1.15 \pm j0.54$

For $Z_{cp} = 250\ \Omega$, it is found that $Z_A \; \pm jX_p = 64.91 \pm j18.18$

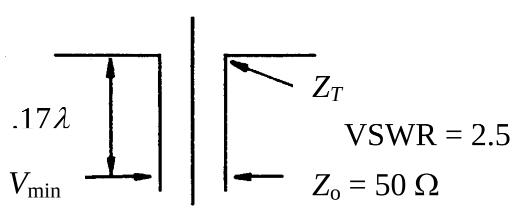
or normalized as $= 1.30 \pm j0.36$

It is easily seen that the 150 Ω value decreases the bandwidth and the 250 Ω value increases the bandwidth. Note: The closer the values are to the origin, the better the VSWR.

23-3-5 Stub impedance.

(a) What is the terminal impedance of a ground-plane mounted stub antenna fed with a 50-Ω air-filled coaxial line if the VSWR on the line is 2.5 and the first voltage minimum is 0.17λ from the terminals?
(b) Design a transformer so that the VSWR = 1.

Solution:



$$Z_m = Z_o \frac{Z_T + jZ_o \tan \beta x}{Z_o + jZ_T \tan \beta x} \tag{1}$$

where $Z_m =$ impedance on line at $V_{min} = R_m + j0$
$Z_o =$ line impedance $= 50 + j0\ \Omega$
$Z_T =$ stub antenna terminal impedance $= R_T + jX_T$

Rearranging (1) in terms of real and imaginary parts:

23-3-5 continued

$$R_m - R_T = \left(\frac{X_T R_m}{R_o}\right)\tan\beta x \quad \text{by equating reals,} \qquad (2)$$

and

$$\frac{R_T R_m}{R_o}\tan\beta x = X_T + R_o \tan\beta x \quad \text{by equating imaginaries} \qquad (3)$$

$$R_m = 50/2.5 = 20,\ R_o = 50,\ \ \tan\beta x = \tan(360^o \times .17) = 1.82$$

From (2), $\quad 20 - R_T = X_T \dfrac{20}{50}\times 1.82 = 0.728\ X_T$

From (3), $\quad R_T \dfrac{20}{50}\times 1.82 = X_T + 50\times 1.82, \qquad 0.728 R_T = X_T + 91$

From which, $\quad Z_T = R_T + jX_T = 56 - j50\ \Omega \quad$ (*ans.*)

Chapter 24. Antenna Measurements.
By Arto Lehto and Pertti Vainikainen

24-3-1. Uncertainty of pattern measurement due to reflected wave.

The level of a wave reflected from the ground is 45 dB below the level of the direct wave. How large of errors (in dB) are possible in the measurement of:
(a) main lobe peak;
(b) -13 dB sidelobe;
(c) -35 dB sidelobe?

Solution:

From Sec. 24-3b and since the reflected wave is -45 dB or $10^{-(45/20)} = 0.0056$,

(a) $1 - 0.0056 = 0.9944$ or -0.049 dB (*ans.*)

$1 + 0.0056 = 1.0056$ or $+0.049$ dB (*ans.*)

(b) -13 dB side lobes provides $10^{-(13/20)} = 0.2238$

so $$\frac{0.2238 - 0.0056}{0.2238} = 0.9749 \text{ or } -0.22 \text{ dB} \quad (\textit{ans.})$$

$$\frac{0.2238 + 0.0056}{0.2238} = 1.0251 \text{ or } +0.22 \text{ dB} \quad (\textit{ans.})$$

(c) -35 dB side lobes provides $10^{-(35/20)} = 0.0178$

so $$\frac{0.0178 - 0.0056}{0.0178} = 0.6838 \text{ or } -3.30 \text{ dB} \quad (\textit{ans.})$$

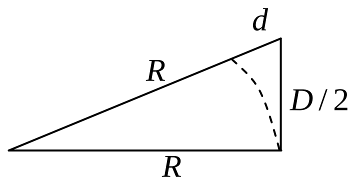
$$\frac{0.0178 + 0.0056}{0.0178} = 1.3162 \text{ or } +2.38 \text{ dB} \quad (\textit{ans.})$$

24-3-2. Range length requirement due to allowed phase curvature.

The maximum allowed phase curvature in the measurement of a very low-sidelobe antenna is 5°. The width of the antenna is 8 m and it operates at 5.3 GHz. Find the required separation between the source and AUT.

24-3-2. continued

Solution:



Similar to Fig. 24-5, let d be the distance causing the phase error.

Then $$(R+d)^2 = R^2 + \left(\frac{D}{2}\right)^2$$

$$R^2 + 2dR + d^2 = R^2 + \frac{D^2}{4}, \qquad R \cong \frac{D^2}{8d}$$

For a 5° phase error,

$$kd = \frac{2\pi}{\lambda} d = \frac{5}{180}\pi \text{ (rad)}$$

so, $$\frac{d}{\lambda} = \frac{5}{360} = \frac{1}{72}$$

Therefore, $$R = \frac{1}{8}\frac{D^2}{\lambda}72 = \frac{9D^2}{\lambda}$$

Since, $$\lambda = \frac{3\times 10^8}{5.3\times 10^9} = 0.0566 \text{ m}, \qquad R \geq \frac{9\times 64}{0.0566} = 10{,}176 \text{ m}$$

24-4-1. Design of elevated range.

Design an elevated range (range length, antenna heights, source antenna diameter) for the measurement of a 1.2 m reflector antenna operating at 23 GHz.

Solution:

$$\lambda = \frac{3\times 10^8}{2.3\times 10^{10}} = 0.013 \text{ m}$$

so, $$R \geq \frac{2D^2}{\lambda} = \frac{2\times(1.2)^2}{0.013} = 221 \text{ m} \quad (\textit{ans.})$$

24-4-1. continued

From combining requirements in (24-4-1) and (24-4-2)

$$H_R \cong 5 \times D = 5 \times 1.2 = 6 \text{ m} \quad (ans.)$$

and similarly for $H_T = H_R$

From (24-4-1), $$D_T \geq \frac{1.5\lambda R}{H_R} = \frac{1.5 \times 0.013 \times 221}{6} = 0.72 \text{ m} \quad (ans.)$$

24-4-2. Time required for near-field scanning.

Estimate the time needed for a planar near-field measurement of a 2 m antenna at 300 GHz. The sampling speed is 10 samples per second.

Solution:

$$\lambda = \frac{3 \times 10^8}{3 \times 10^{11}} = 0.001 \text{ m}, \qquad D = \frac{2 \text{ m}}{0.001 \text{ m}} = 2000\lambda$$

Sample at 2 per wavelength, so samples = 4000 per line per side

Total samples = $2 \times (4 \times 10^3)^2 = 32 \times 10^6$

$$t = \frac{32 \times 10^6}{10 \text{ samples/sec}} = 3.2 \times 10^6 \text{ sec} = 888 \text{ hrs } 54\text{min} \cong 37 \text{ days} \quad (ans.)$$

24-5-1. Power requirement for certain dynamic range.

The AUT has a gain of 40 dBi at 10 GHz. The gain of the source antenna is 20 dBi. The separation between the antennas is 200 m. The receiver sensitivity (signal level that is sufficient for measurement) is –105 dBm. Find the minimum transmitted power that is needed for a dynamic range of 60 dB.

Solution:

From (24-5-2) and since $$\left(\frac{\lambda}{4\pi R}\right)^2 = \left(\frac{0.03}{4\pi \times 200}\right)^2 = 1.42 \times 10^{-10} = -98 \text{ dB}$$

24-5-1. continued

$$\left(\frac{P_R}{P_T}\right)_{\text{dB}} = 40 \text{ dBi} + 20 \text{ dBi } - 98 \text{ dB} = -38 \text{ dB}$$

With -105 dBm needed at a minimum for the reception and a 60 dB dynamic range, then

$$P_t = -105 \text{ dBm} + 38 \text{ dB} + 60 \text{ dB} = -7 \text{ dBm}$$
$$P_t = 0.2 \text{ mW} \quad (\textit{ans.})$$

24-5-2. Gain measurement using three unknown antennas.

Three horn antennas, A, B, and C are measured in pairs at 12 GHz. The separation of antennas is 8 m. The transmitted power is +3 dBm. The received powers are -31 dBm, 36 dBm, and -28 dBm for antennas pairs AB, AC, and BC, respectively. Find the gains of the antennas.

Solution:

From (24-5-2), $$G_T G_R = \frac{P_T}{P_R}\left(\frac{\lambda}{4\pi R}\right)^{-2}, \qquad \lambda = \frac{3\times 10^8}{1.2\times 10^{10}} = 0.025 \text{ m}$$

$$\left(\frac{\lambda}{4\pi R}\right)^2 = \left(\frac{0.025}{4\pi 8}\right)^2 = 6.18\times 10^{-8} \text{ or } -72 \text{ dB}$$

then

$$G_A G_B = C_{AB} = -31 \text{ dBm} - 3 \text{ dBm} + 72 \text{ dB} = 38 \text{ dB}$$
$$G_A G_C = C_{AC} = -36 \text{ dBm} - 3 \text{ dBm} + 72 \text{ dB} = 33 \text{ dB}$$
$$G_B G_C = C_{BC} = -28 \text{ dBm} - 3 \text{ dBm} + 72 \text{ dB} = 41 \text{ dB}$$

$$\frac{G_B}{G_C} = \frac{C_{AB}}{C_{AC}}, \qquad G_B = \frac{C_{AB}}{C_{AC}} G_C, \qquad \left(\frac{C_{AB}}{C_{AC}}\right) G_C^2 = C_{BC}$$

So

$$G_C = \sqrt{\frac{C_{BC} C_{AC}}{C_{AB}}} = \frac{1}{2}(41 \text{ dB} + 33 \text{ dB} - 38 \text{ dB}) = 18 \text{ dBi} \quad (\textit{ans.})$$

$$G_A = \frac{C_{AC}}{G_C} = 33 \text{ dB} - 18 \text{ dB} = 15 \text{ dBi} \quad (\textit{ans.})$$

24-5-2. continued

$$G_B = \frac{C_{BC}}{G_C} = 41\text{ dB} - 18\text{ dB} = 23\text{ dBi} \quad (ans.)$$

24-5-3. Gain measurement using celestial radio source.

At 2.7 GHz the antenna temperature increases 50 K as a 20 m reflector is pointed to Cygnus A. Find the antenna gain and aperture efficiency.

Solution:

From (24-5-7), $$G = \frac{8\pi k \Delta T_A}{S\lambda^2} = \frac{8\pi \times 1.38\times 10^{-23} \times 50}{785\times 10^{-26} \times (0.111)^2} = 1.79\times 10^5 = 52.5\text{ dBi}$$

$$A_e = \frac{G\lambda^2}{4\pi} = \frac{1.79\times 10^5 \times (0.111)^2}{4\pi} = 175.5\text{ m}^2$$

For a 20 m circular reflector,

$$\varepsilon_{ap} = \frac{A_e}{A_p} = \frac{175.5}{\pi (10)^2} = 0.56 \text{ or } 56\%$$

24-5-4. Impedance in laboratory.

You try to measure the impedance of a horn antenna with 15 dBi gain at 10 GHz in a normal laboratory room by pointing the main lobe of the antenna perpendicularly towards a wall 2 m away. The power reflection coefficient of the wall is 0.3 and it can be assumed to cover practically the whole beam of the AUT. Estimate the uncertainty of the measurement of the reflection coefficient of the AUT due to the reflection of the wall.

Solution:

The normalized received power from the horn to the wall and back into the horn

$$\frac{P_R}{P_T} = \rho G_T G_R \left(\frac{\lambda}{4\pi R}\right)^2$$

$$\rho = 0.3 = -5\text{ dB}, \qquad \lambda = 0.03\text{ m}, \qquad \left(\frac{\lambda}{4\pi \text{R}}\right)^2 = \left(\frac{0.03}{4\pi \times 2 \times 2}\right)^2 = 3.6\times 10^{-7} = -64\text{ dB}$$

24-5-4. continued

$$\frac{P_R}{P_T} = -5\text{ dB} + 15\text{ dBi} + 15\text{ dBi} - 64\text{ dB} = -39\text{ dB}$$
$$= 0.000126 \quad \text{in power}$$
$$= 0.01122 \quad \text{in voltage}$$

So the uncertainty is about 1%.

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