As ωC_{gk} approaches zero, (14) reduces to

$$\frac{C_i'}{C_{gk}} = \frac{m(r_c g_m + 1) + (g_p + g_b) s r_c^2}{(r_c s + 1)^2}. \qquad (16)$$

Fig. 7 shows curves of C_i'/C_{gk} for the same tube and circuit parameters as those used for Figs. 4 and 6. Curves for other values of tube factors and load resistance are of the same form, but differ in the value of C_i'/C_{gk} approached at low values of ωC_{gk}. Examination of these curves or of (6) discloses that the effective input capacitance decreases with increase of r_c and of $1/g_b$, which is usually nearly equal to r_2. At values of ωC_{gk} below 10^{-4} mho, C_i' becomes very small for large values of r_c and r_2, and the total effective input capacitance approaches the value C_{gp}.

The tendency of a cathode-follower stage to oscillate can be prevented either by insuring that the shunt impedance of any oscillatory circuit shunting the input is low or by making the magnitude of the total negative input conductance of the tube small in one of the following ways:

(1) By using tubes of low transconductance (see (12)). This method is in general impractical because high transconductance is desirable in cathode-follower amplifiers for other reasons.

(2) By using a low value of cathode load resistance (see (12)). This method is feasible in some applications of cathode-follower amplifiers.

(3) By using low capacitance across the cathode load resistor (see Fig. 4). The load capacitance cannot, however, be reduced below the sum of the plate-cathode capacitance and the minimum circuit-wiring capacitance.

(4) By using a sufficiently low value of r_c so that the positive conductance resulting from r_c is equal to or nearly equal to the negative conductance resulting from C_{gk}. The principal objection to this method is that it increases the conductance at low frequencies to an excessively large positive value.

(5) By using resistance in series with the grid. Ordinarily a series resistance of less than 100 ohms is sufficient to prevent oscillation. This value is small enough so that frequency distortion resulting from voltage drop in the series resistance at high frequency is negligible. This method is obviously the most feasible.

Oscillation may also be prevented by insuring that the sum of the tube input susceptance and the susceptance of the input circuit is not zero, i.e., that resonance does not occur, in the frequency range in which the input conductance of the tube is negative. Since Fig. 6 shows that the magnitude of negative input conductance increases with frequency, it is apparent that this can be accomplished only by lowering the resonance frequency of the input circuit.

An Exponential Transmission Line Employing Straight Conductors*

WILBUR NORMAN CHRISTIANSEN†

Summary—An exponential transmission line is useful for impedance transformations over a wide band of radio frequencies.

It is shown that a four-wire line of the "side-connected" type employing uniform conductors, and in which the rates of taper change only once along the line, may be designed to approximate closely to an exponential line.

Design equations and charts are given which aid in determining the wire sizes, values of taper, and change in taper for building some of these transformers.

I. Introduction

In recent years the increasing use in short-wave radio communication of semiaperiodic antennas, principally of the rhombic type, has resulted in attention being given to impedance-transforming devices useful over a large range of radio frequencies.

Wide-band transformers can now be constructed to achieve any normally required impedance transformation in the useful frequency range of the antennas. These transformers are designed to have a high coefficient of coupling between primary and secondary windings, and this makes difficult their application where high radio-frequency voltages are present. Hence broad-band transformers using closely coupled coils are normally not used with transmitting antennas.

The useful impedance-transforming properties of the exponential horn in acoustics suggested[1] the analogous application of the principle to electrical transmission lines. Various types[2–5] of open-conductor exponential

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[1] C. W. Hansell, "Coupling devices for use in high-frequency circuits," Australian Patent 18,994/29, March, 1929.

[2] C. R. Burrows, "The exponential transmission line," *Bell Sys. Tech. Jour.*, vol. 17, pp. 555–573; October, 1938.

[3] M. S. Neiman, "Non-uniform lines with distributed constants," *Izvestiya Electroprom. Slab. Toka*, pp. 14–25; 1938.

[4] H. A. Wheeler, "Transmission lines with exponential taper," Proc. I.R.E., vol. 27, pp. 65–71; January, 1939.

[5] A. R. Volpert, "Lines with non-uniformly distributed parameters," *Elektrosyvaz*, pp. 40–65; 1940.

lines, and cable[6] have been described in the literature, all requiring more or less complicated shaping of the conductors.

The exponential line described in the present paper is of the balanced open-wire type. Its application is principally to the transformation required where a two-wire balanced line branches into a pair of lines. Such branching of feeders will occur, for example, where several rhombic antennas are arranged in an array and are to be fed from a common transmitter.

The object of the development to be described was to produce, for a 300- to 600-ohm transformation, an exponential line which would have convenient physical dimensions, would be easy to construct, and which would not require complicated shaping, or changes in the diameter of conductors.

II. Electrical Characteristics of an Exponential Line

An exponential transmission line has the capacitance and inductance per unit length of line varying in a manner such that

$$Z_{0x} = Z_{00} e^{\delta \cdot x} \tag{1}$$

where x is the distance from the origin of the point considered, Z_{0x} is the "nominal characteristic impedance" of the line at a point x along its length, and Z_{00} is the nominal characteristic impedance at $x=0$, i.e., at the end of the line. (The "nominal characteristic impedance" is equal to the characteristic impedance of a uniform transmission line having the same dimensions as the variable line at the point considered.)

From (1) it follows that

$$\delta = \log_e \frac{Z_{0l}/Z_{00}}{l} \tag{2}$$

where l is the physical length of the line and Z_{0l} refers to the remote end of the line. It has been demonstrated[2–4] that such a line behaves as an impedance-transforming high-pass filter with a cut-off frequency f_c, given by

$$f_c = \frac{1}{2\pi} \cdot v \cdot \frac{\delta}{2} = \frac{v \log_e Z_{0l}/Z_{00}}{4\pi l} \tag{3}$$

where v is the phase velocity of wave propagation along the line for very high frequencies, i.e., for frequencies where the change in Z_{0x} per wavelength is very small. It may be noted that v is assumed to be constant along the length of the exponential line. This implies that the line has low dissipation and unchanging dielectric.

For an open-wire line, (3) becomes approximately

$$f_c = \frac{55.0 \log_{10} Z_{0l}/Z_{00}}{l} \text{ megacycles} \tag{4}$$

where the unit of length is the meter. The useful impedance-transforming property of the exponential line appears as follows. If one end of the line is terminated in a load equal to Z_{0l}, then for frequencies much greater than f_c the driving-point impedance at the input to the line approximates very closely to Z_{00}, the nominal characteristic impedance at the input end of the line. As the frequency is decreased towards f_c, increasing deviations occur in the driving-point impedance from the value of Z_{00}.

From the analysis of Burrows[2] it may be shown that the frequencies at which these deviations occur are related mainly to the length of the line, while the magnitude of the deviations depends on the rate of line taper. For the line terminated with a resistance equal to Z_{0l}, it was shown by Burrows that the ratio of the driving point impedance Z_0', at the input to the line to the nominal characteristic impedance at that point is

$$\frac{Z_0'}{Z_{00}} = \frac{1+\sqrt{1-\nu^2}-j\nu-(1-\sqrt{1-\nu^2}-j\nu)e^{-2\gamma l\sqrt{1-\nu^2}}}{1+\sqrt{1-\nu^2}+j\nu-(1-\sqrt{1-\nu^2}+j\nu)e^{-2\gamma l\sqrt{1-\nu^2}}} \tag{5}$$

where $\nu = f_c/f$ and is less than unity, γ = the propagation constant of the line at frequencies very large compared with f_c, and for the line considered is approximately equal to $j\beta$, β being the phase-change coefficient, at such high frequencies.

On putting the exponentials into the trigonometric form, we obtain

$$\frac{Z_0'}{Z_{00}} = \frac{1+\sqrt{1-\nu^2}-(1-\sqrt{1-\nu^2})\cos 2\beta l\sqrt{1-\nu^2}+\nu\sin 2\beta l\sqrt{1-\nu^2}-j\{\nu-\nu\cos 2\beta l\sqrt{1-\nu^2}-(1-\sqrt{1-\nu^2})\sin 2\beta l\sqrt{1-\nu^2}\}}{1+\sqrt{1-\nu^2}-(1-\sqrt{1-\nu^2})\cos 2\beta l\sqrt{1-\nu^2}-\nu\sin 2\beta l\sqrt{1-\nu^2}+j\{\nu-\nu\cos 2\beta l\sqrt{1-\nu^2}+(1-\sqrt{1-\nu^2})\sin 2\beta l\sqrt{1-\nu^2}\}} \cdot \tag{6}$$

When $\nu=0$, $Z_0'/Z_{00}=1$.

When

$$\sin 2\beta l\sqrt{1-\nu^2} = 0,$$

$$\text{i.e., } 2\beta l\sqrt{1-\nu^2} = n\pi, \quad n \text{ being an integer,} \tag{7}$$

Z_0'/Z_{00} is equal to unity or $(1-j\nu)/(1+j\nu)$, depending on whether n is even or odd. In either case $|Z_0'/Z_{00}|$ is equal to one.

The magnitude of the input impedance is, therefore, equal to the nominal characteristic impedance of the line at the input for frequencies

$$f = \{(nv/4l)^2 + f_c^2\}^{1/2}. \tag{8}$$

It approaches unity also for all values of βl as ν approaches zero, i.e., as f approaches infinity.

If f is large compared with f_c, (8) becomes approximately

$$f = n/4 \cdot v/l; \tag{9}$$

i.e., frequencies f are those for which the line is an integral number of quarter waves in length.

[6] E. Keutner, "Hochfrequenzkabel mit veränderlichem Wellenwiderstand," E.F.D. 62 Folge: pp. 3–9; March, 1943.

If f is large compared with f_c, we may use the approximation $\sqrt{1-\nu^2}=1$, and (6) then becomes

$$\frac{Z_0'}{Z_{00}} = \frac{2+\nu \sin 2\beta l - j\nu(1-\cos 2\beta l)}{2-\nu \sin 2\beta l + j\nu(1-\cos 2\beta l)} \tag{10}$$

and

$$|Z_0'/Z_{00}| = \left\{\frac{2+\nu^2(1-\cos 2\beta l)+2\nu \sin 2\beta l}{2+\nu^2(1-\cos 2\beta l)-2\nu \sin 2\beta l}\right\}^{1/2} \tag{11}$$

and this has maximum values of approximately $(1+\nu)$ when

$$2\beta l = \frac{4n+1}{2}\pi,$$

i.e., when

$$f = \frac{4n+1}{8}\cdot\frac{v}{l}. \tag{12}$$

Similarly, minimum values of approximately $1/(1+\nu)$ occur when

$$f = \frac{4n-1}{8}\cdot\frac{v}{l}. \tag{13}$$

The above calculations are for δ positive, i.e., the impedance Z_0' is considered at the low-impedance end of the line. For impedances at the high-impedance end, δ is negative, since the line is then convergent. The sign of ν is changed in (5), thereby inverting it. Hence the values of Z_0'/Z_{00} corresponding to the frequencies (12) and (13) are also inverted.

It was shown by Wheeler[4] that if the exponential line is placed between the elements of a half section of a constant-K low-pass filter, and in addition an M-derived half section is connected at each end of the system, it is possible to keep the impedance (resistance) deviations within 5 per cent of the required value for all frequencies 15 per cent or more above the cut-off frequency.

Where it is desired to limit the length of the exponential line, or where exact matching is required, such terminating sections have useful application. In many cases, however, it is simpler to use a line of such a length that the working frequency is always very high compared with the cut-off frequency of the line so that a purely resistive line termination may be used.

III. The Design of an Open-Wire Exponential Line for a 2-to-1 Impedance Transformation

(a) *Previous Designs*

The transformation from 600 to 300 ohms with an open-wire exponential line is made difficult by the fact that the construction of a two-wire line with the latter value of characteristic impedance involves the employment of inconvenient physical dimensions for the line, while if a multiple-wire line is used, the same difficulty is experienced at the high-impedance end.

If a two-wire line is designed to provide such a transformation, the ratio of wire separation d to the radius r must change from 150/1 to 12/1 over the length of the line. In many applications of such a line the wire separation cannot be reduced below a value of several inches if mechanical instability and the danger of dielectric breakdown are to be avoided. Hence, for $d/r=12$, tubing rather than wire must be used in the construction of the line.

Burrows[2] constructed a two-wire exponential transmission line in which conductors with large radii were used at the low-impedance end of the line, the conductor radius being reduced at intervals along the line towards the high-impedance end. By this means the conductor spacing was kept at convenient values throughout the length of the line. Small discontinuities existed at the points where the conductor size was altered, but these were not serious as was shown by the fact that Burrows successfully approached the performance predicted by theory for the exponential line.

Another design for an exponential line for use where one feeder line branches into two was suggested by Neiman.[3] In this line the two 600-ohm-line pairs approach each other from a great distance in such a manner that the resultant four-wire line has an exponential characteristic. The pairs theoretically are required to coalesce to form a single 600-ohm pair at the high-impedance end of the line, but in practice a special two-wire section could be used to overcome this. The shaping of the line is done with tension spacers connecting the two pairs of lines. No change in the size of conductors is required with this line, except possibly for the section at the high-impedance end.

(b) *Design Employing Straight Conductors*

The exponential line to be described here[7] is based on the type of four-wire transmission line in which parallel connections are made between the wires of each adjacent

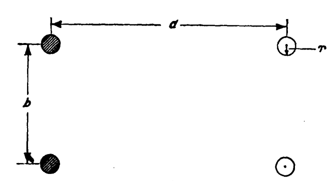

Fig. 1—Cross sections of a four-wire transmission line.

pair, instead of between diagonal wires as in the more commonly used four-wire transmission line. The arrangement of wires is shown in Fig. 1. If it is assumed that the spacings d and b are large compared with the wire radius r, and that dissipation in the line is negligible, then the characteristic impedance may be calculated from the expression

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{cC} \text{ ohms,} \tag{14}$$

[7] Australian Patent No. 121,850.

L and C being respectively the inductance (henries) and capacitance (farads) per centimeter length of line, and $c=3\times 10^{10}$ centimeters. The calculation of the capacitance per unit length of line may be done by the method of logarithmic potential to give

$$\frac{1}{C} = 2 \log_e \left(\frac{d\sqrt{b^2+d^2}}{br}\right) \text{ statfarad per centimeter}$$

$$= 2c^2 \log_e \left(\frac{d\sqrt{b^2+d^2}}{br}\right) \cdot 10^9 \text{ farad per centimeter.} \quad (15)$$

Therefore,

$$Z_0 = 138 \log_{10} \left(\frac{d\sqrt{b^2+d^2}}{br}\right). \quad (16)$$

For the production of an exponential line, d and b may both vary with the distance x along the line, while r preferably is fixed.

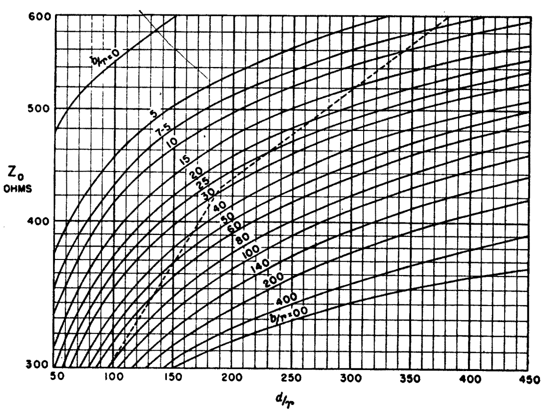

Fig. 2—Characteristic impedance of a four-wire (side-connected) transmission line for various values of conductor spacing.

It is obvious that the construction of an exponential line is greatly simplified if b and d can be arranged to vary linearly with x over appreciable ranges of variation of x, without causing Z_{0x} to depart appreciably from the exponential form. To investigate this, use is made of the graphical construction of Fig. 2, in which Z_0 is plotted on a logarithmic scale against d/r for various values of b/r. (For the ratio of $b/r=5$ it might be suspected that equation (16), which was based on the assumption that b and d were both very large compared with r, would no longer be accurate. However, a field plot for this spacing shows that the error in Z_0 resulting from the use of (16) is only about 1 per cent.)

If it is stipulated that d/r is to change linearly with the distance x along the feeder, then a straight line drawn on the graph represents a linear change of log Z_{0x} with respect to x, or $Z_{0x}=Ae^{Bx}$ (A and B being constants), which represents an exponential transmission line.

If, moreover, a straight line can be drawn on Fig. 2 so that the contours representing equal arithmetic intervals of b/r make equal intercepts on it, then the straight line on the graph represents an exponential transmission line in which b/r is related linearly to d/r and hence to x. Such a transmission line, therefore, would be constructed wholly of straight conductors.

It is found that a large number of lines can be drawn to fulfill approximately this condition over part of the impedance range of 300 to 600 ohms. Consideration must be given, however, to the use of the values of d/r and b/r that are convenient in practice.

The straight lines drawn on Fig. 2 represent a transformation from 300 to 600 ohms done in two sections, the division being made at the point where $Z_{0x}=\sqrt{Z_{00}Z_{0l}}$ $=424$ ohms approximately, which is the physical center of the exponential line.

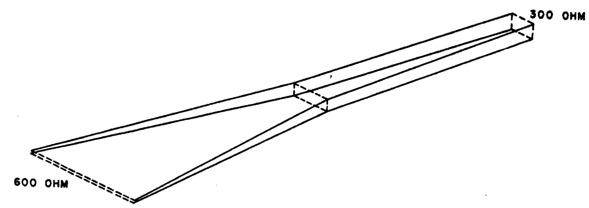

Fig. 3—Outline diagram of the exponential line described.

The configuration of the line is illustrated in Fig. 3. In Table I are shown the values of d/r and b/r at the ends and center of the exponential line. To illustrate the physical dimensions of a typical line, spacings are given for a line with conductors of No. 10 American Wire Gauge wire (0.102 inch).

TABLE I

Distance along feeder	$X=x/l$	0	0.5	1.0
Nominal characteristic impedance	Z_{ox}	300	424	600
Spacing	d/r	95	185	385
Spacing	b/r	80	30	6.5
For No. 10 A.W.G. wire	d	4.8″	9.4″	19.6″
For No. 10 A.W.G. wire	b	4.1″	1.53″	0.33″

In the exponential line represented by the straight lines on Fig. 2, b/r changes almost linearly. Hence we may approximate it with a feeder line in which b/r is actually linear, i.e., we may use straight conductors between the center and each end of the line. That the impedance variation along the resultant line follows very closely the exponential form is demonstrated in Fig. 4. The straight line represents an exponential change of impedance along the transmission line, while the points adjacent to the line are those calculated for a line having the form described above. The maximum departure of the impedance level along the "straight-wire" line from the corresponding values for a true exponential line is only one part in a hundred.

The exponential line described above requires to be supported only at the ends and the center. In a practical design, of course, it may be found necessary to use a greater number of supports.

Consider the design of a line to be used for impedance transformation in the range of frequencies from 6 to 20 megacycles. If terminating filter sections of the type indicated by Wheeler[4] are used, then 5 megacycles may be chosen as the cut-off frequency for the line. From (4) we find that $l=3.3$ meters, approximately. It should be noted, however, that with such a short length of line the size of conductors would have to be small; otherwise the conductor spacings would become comparable with the length of the line, and the transmission-line equations would no longer apply.

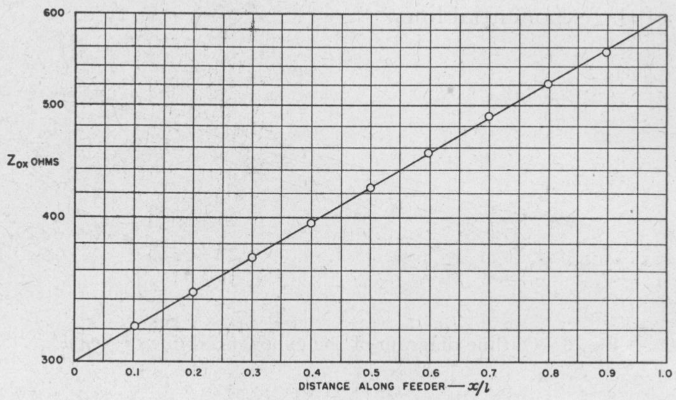

Fig. 4—Variation of nominal characteristic impedance along an ideal exponential line (full line) and "straight-wire exponential line" (small circles).

If it is not desired to use filter sections at the line terminations, then the cut-off frequency must be made considerably lower than the lowest working frequency. Where impedance deviations of 10 per cent from the mean can be tolerated, then from (12) and (13) we see that for the above range of frequencies 0.6 megacycle may be taken as the frequency of cut-off. This gives $l=27.6$ meters, approximately.

IV. Experimental Line

A line of the type described above was constructed to allow an experimental check to be made of the system. The length of the line was 40 meters and the conductors used were of No. 12 Standard Wire Gauge (approximately No. 10 Brown and Sharpe) copper wire. Supports were spaced at 22-foot intervals, this being the spacing of poles used in associated uniform transmission lines. The insulated supports (see Fig. 5) were constructed to make possible easy adjustment of the wire spacings d and b. (The insulators shown in the photograph were not designed for horizontal mounting, but no better type was available when the line was being built.)

Short lengths of 300- and 600-ohm uniform transmission line were attached to the respective ends of the line.

For this line the cut-off frequency given by (4) is 0.415 megacycle. If the appropriate values for l and f_c are substituted in the expressions (9), (12), and (13), then the significant points on a curve of input impedance versus frequency are obtained. The calculated variation of input impedance of the line with frequency is shown in Fig. 6.

Measurements of input impedance were made at each end of the line, the remote end in each case being terminated with a carbon resistor of appropriate value.

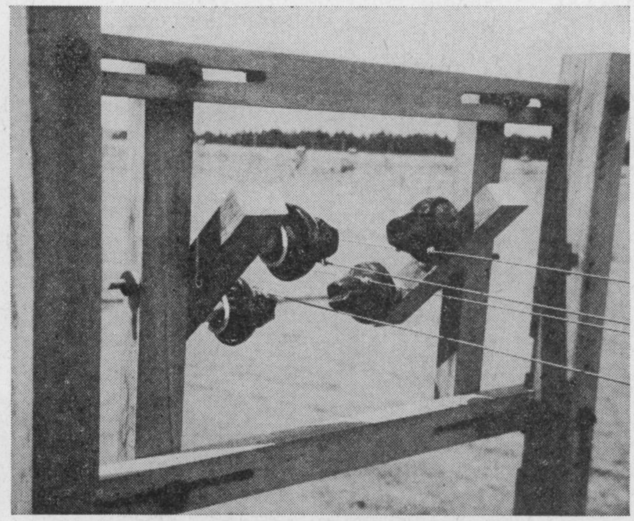

Fig. 5—Typical feeder pole used with an experimental line. Both horizontal and vertical separations of wires are variable.

For the measurements, a portable impedance meter was employed, this unit being composed of a radio-frequency oscillator, balanced amplifier, and tuned output circuit. Across the latter circuit is connected a pair of diodes, the rectified current from which actuates a

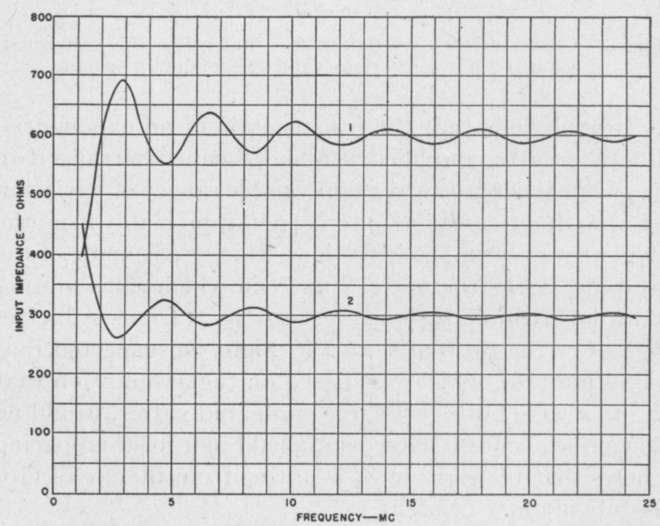

Fig. 6—Theoretical relationship between input impedance and frequency for an exponential line when a resistive termination is used. Length of the line is 40 meters, and the line is designed for a 2-to-1 impedance transformation. Curve 1 is for the high-impedance end of line; curve 2 for the low-impedance end.

meter, as in the ordinary vacuum-tube voltmeter. Across the tuned circuit may also be connected either the load to be measured or one of a number of fixed resistors. Measurements are made by setting the oscillator to the required frequency, tuning the output circuit, and, with an appropriate fixed resistor (of magnitude greater than the impedance to be measured) connected across the output circuit, adjusting the output from the oscillator until a full-scale deflection is seen on the diode-current meter. The fixed resistor is then replaced by the

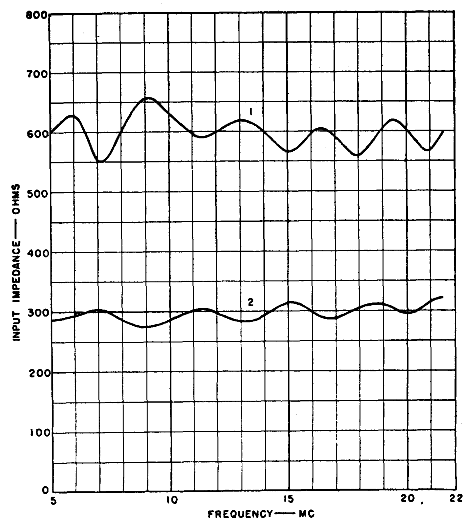

Fig. 7—Measured input impedance of an experimental line. Curve 1 is for the high-impedance end with the line terminated with a resistance of 300 ohms; curve 2 is for the low-impedance end with a termination of 600 ohms.

load to be measured. The reactive component of the latter (in terms of shunt reactance) is calculated from the change in capacitance required to tune the circuit when the external impedance is connected. The equivalent shunt resistance of the load is then read directly from the calibrated scale on the diode-current meter.

In Fig. 7 are shown the measured values of input impedance of the line. The impedances shown are resistive, since in all measurements the reactive component was small enough to be neglected. It is seen that the maximum deviation from the required transformation is 10 per cent. The deviations at the higher frequencies could, doubtless, have been reduced if more detailed attention had been given to the line terminations.

It may be noted in Fig. 7 that the short length of uniform transmission line attached to each end of the exponential line has caused the frequencies of maximum and minimum impedance to be displaced slightly from the positions shown in Fig. 6.

V. Conclusion

It has been shown that a four-wire line may be designed to provide a very close approximation to an exponential line in the range of impedances from 300 to 600 ohms. Experimental tests have confirmed that a satisfactory impedance transformation may be obtained with such a line. The employment of a single size of conductor throughout, the absence of elaborate shaping, and the convenient physical dimensions, are the useful features of the line. It has particular application to the problem of supplying power to multiple rhombic or other aperiodic antennas.

VI. Acknowledgment

The writer wishes to express his appreciation for the assistance given by the staff of the Rockbank Beam Receiving Station in the construction and testing of the experimental line.

Correspondence

Magnetic-Wire Response

The integral contained in equation (17) of Marvin Camras' paper on magnetic-wire record response in the August, 1946, issue of the Proceedings of the I.R.E. and Waves and Electrons can be found analytically.

Basset gives its value as $\int_0^\infty (\cos nx)/(\sqrt{x^2+1})dx = K_0(n)$, where K_0 is the modified Bessel function of the second kind of order zero.[1]

A table of $K_0(n)$ versus n is available in *British Association for the Advancement of Science, Mathematical Tables, vol. VI, Cambridge, 1937.* A graphical solution of the aforementioned integral is thus unnecessary.

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[1] A. B. Basset, "Hydrodynamics," vol. II, p. 18; Cambridge, 1888.

Demagnetizing Coefficient

In the issue of August, 1946, there was a paper by Mr. Camras[1] whose equation (17) expressed the demagnetizing coefficient D in terms of a certain integral, which he evaluated by graphical methods, presumably because he did not recognize it. In fact, however,

$$\int_0^\infty \frac{\cos nx}{\sqrt{x^2+1}} \cdot dx = K_0(n)$$

where K_0 is the modified Bessel function of the second kind which has been fairly completely tabulated. Evaluation of $D = \frac{1}{2}n^2K_0(n)$ shows that the author's Fig. 5 obtained by graphical methods is of good accuracy.

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[1] M. Camras, "Theoretical response from a magnetic-wire record," Proc. I.R.E., vol. 34, pp. 597–602; August, 1946.